

Deformed Hermitian Yang–Mills equation on rational homogeneous varieties

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Abstract. In this paper, we show that the deformed Hermitian Yang–Mills (dHYM) equation on a rational homogeneous variety, equipped with any invariant Kähler metric, always admits a solution. Unlike the known existence results for projective varieties, our result shows that in the homogeneous setting no additional hypotheses on the phase angle (such as the supercritical condition) is necessary to solve the dHYM equation. Moreover, we describe the Lagrangian phase, with respect to any invariant Kähler metric, of every closed invariant $(1, 1)$ -form in terms of Lie theory. Additionally, we provide an explicit formula, in terms of Lie theory, for the slope of torsion-free coherent sheaves on rational homogeneous varieties. Using this formula, we establish a new criterion for the slope semistability of holomorphic vector bundles over these varieties by restricting the bundle to the generators of the associated cone of curves. Furthermore, we provide a new characterization of slope (semi)stability for holomorphic vector bundles over rational homogeneous varieties in terms of central charges defined by rational curves. As a byproduct, we introduce a constructive method to obtain non-trivial examples of Hermitian–Einstein metrics on certain holomorphic vector bundles over $\mathbb{P}(T_{\mathbb{P}^2})$ from solutions of linear Diophantine equations. Also, motivated by the problem of constructing line bundles with prescribed slope, we present some new insights that explore the interplay between elementary number theory and combinatorics via Schubert calculus.

1. Introduction

Let (X, ω) be a compact connected Kähler manifold, such that $\dim_{\mathbb{C}}(X) = n$, and let $[\psi] \in H^{1,1}(X, \mathbb{R})$. Motivated by mirror symmetry, the following problem was introduced in [33].

Problem 1.1. *Does there exist a smooth $(1, 1)$ -form $\chi \in [\psi]$, such that*

$$\operatorname{Im}(\omega + \sqrt{-1}\chi)^n = \tan(\hat{\Theta}) \operatorname{Re}(\omega + \sqrt{-1}\chi)^n, \quad (1.1)$$

where $\hat{\Theta}$ is an S^1 -valued topological constant (“phase angle”) depending on $[\omega], [\psi]$?

Equation (1.1) is known as the *deformed Hermitian Yang–Mills (dHYM) equation*. According to [42], it was discovered around the same time in [41, 44] using different points of view, for a detailed discussion about the physical origin of the dHYM equation and its relation to string theory and mirror symmetry, we suggest [15].

As it can be shown, (1.1) has an alternative (equivalent) formulation in terms of the notion of *Lagrangian phase* [33], more precisely, (1.1) is equivalent to the fully nonlinear elliptic equation

$$\Theta_\omega(\chi) := \sum_{j=1}^n \arctan(\lambda_j) = \widehat{\Theta} \pmod{2\pi}, \quad (1.2)$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\omega^{-1} \circ \chi$. In the above equation $\Theta_\omega(\chi)$ is called the Lagrangian phase of χ with respect to ω . The topological constant $\widehat{\Theta}$ can be obtained by integrating (1.1), i.e., it is given by the principal argument of the complex number

$$Z_{[\omega]} := \int_X \frac{(\omega + \sqrt{-1}\psi)^n}{n!},$$

which depends only on the classes $[\omega]$ and $[\psi]$. Thus, a necessary condition for the existence of a solution is that $Z_{[\omega]} \neq 0$. In [33], it was shown that solutions of the dHYM equation are unique, up to addition of a constant, see also [15, Lemma 2.3].

In view of the formulation provided by (1.2), since $\Theta_\omega(\chi)$ is real valued and $\widehat{\Theta}$ is S^1 -valued, we need to lift $\widehat{\Theta}$ to \mathbb{R} to study (1.2). Further, under the hypothesis of the supercritical condition, which means that the lifted angle $\widehat{\Theta}$ satisfies $\widehat{\Theta} > (n-2)\frac{\pi}{2}$, Collins–Jacob–Yau [13] showed that if there exists a supercritical C -subsolution, then the dHYM equation is solvable. Under the hypotheses of some curvature conditions and a hypercritical phase (i.e., $(n-1)\frac{\pi}{2} < \widehat{\Theta} < n\frac{\pi}{2}$), Jacob and Yau proved in [33] an existence result for (1.1). Also, other existence results were obtained in [51, 58] for some ranges of the phase angle $\widehat{\Theta}$ assuming the existence of a subsolution. As demonstrated in [13, 14, 16], it is possible to use algebraic geometry to study the problem related to the existence of solutions to the dHYM equation. More precisely, in the setting of Problem 1.1, given an analytic subvariety $Y \subseteq X$ of dimension p , one can define the *central charge*

$$Z_Y([\psi]) := - \int_Y e^{-\sqrt{-1}(\omega + \sqrt{-1}\psi)} = - \frac{(-\sqrt{-1})^p}{p!} \int_Y (\omega + \sqrt{-1}\psi)^p. \quad (1.3)$$

If $\dim_{\mathbb{C}}(X) = 2$, it has been shown in [13, Proposition 8.5] that a solution to the dHYM equation exists in $[\psi] \in H^{1,1}(X, \mathbb{R})$ if, and only if, for every curve $C \subset X$ we have

$$\operatorname{Im} \left(\frac{Z_C([\psi])}{Z_X([\psi])} \right) > 0.$$

In a more general setting, the following conjecture was introduced in [13].

Conjecture 1.2 (Collins–Jacob–Yau). *There exists a solution to the deformed Hermitian Yang–Mills equation in the class $[\psi]$ with phase angle $\widehat{\Theta} \in ((n-2)\frac{\pi}{2}, n\frac{\pi}{2})$ if and only if*

$$\operatorname{Im} \left(\frac{Z_Y([\psi])}{Z_X([\psi])} \right) > 0,$$

for all irreducible analytic subvarieties $Y \subsetneq X$.

In [10], Chen proved a uniform version of Conjecture 1.2. Then Chu–Lee–Takahashi [12] and Ballal [3] proved it independently for projective manifolds. Datar and Pingali [20] and Lin [42] proved the purely PDE version of the conjecture in some cases (for dimensions $n = 3$ and 4 among other cases). It is worth mentioning that Zhang provided an extremely important counterexample in [63], showing that Conjecture 1.2 is false as stated. As mentioned in [13], the complex number $Z_Y(-)$ resembles various notions of central charge appearing in stability conditions in several physical and mathematical theories, e.g., [24–26, 59]. In the particular case that $[\psi] = c_1(\mathbf{L})$, for some $\mathbf{L} \in \text{Pic}(X)$, it is conjectured [14] that the existence of a solution to the dHYM equation in the class $[\psi]$ should be equivalent to the Bridgeland stability [8] of the line bundle \mathbf{L} . For an introduction to Bridgeland stability, we suggest [43].

In this paper, we study the dHYM equation on projective manifolds defined by rational homogeneous varieties. In this setting, we show that the dHYM equation on a rational homogeneous variety, equipped with any invariant Kähler metric, always admits a solution. Unlike all known existence results for projective varieties (e.g., [3, 10, 12]), our result shows that in the homogeneous setting no additional hypotheses on the phase angle (such as the supercritical condition) is necessary to solve the dHYM equation. In particular, we describe the Lagrangian phase, with respect to any invariant Kähler metric, of every closed invariant $(1, 1)$ -form in terms of Lie theory. Further, we provide an explicit formula, in terms of Lie theory, for the slope of torsion-free coherent sheaves on rational homogeneous varieties. Using this formula, we derive a new criterion for slope semistability through restrictions of holomorphic vector bundles to the generators of the associated cone of curves. Moreover, we provide a new characterization, in terms of central charges defined by rational curves, for slope (semi)stability of holomorphic vector bundles over rational homogeneous varieties. In addition, we introduce a constructive method to obtain non-trivial Hermitian–Einstein metrics on holomorphic vector bundles over the Wallach flag manifold $\mathbb{P}(T_{\mathbb{P}^2})$ from solutions of linear Diophantine equations. In this last case, we describe explicitly all associated Hermitian Yang–Mills connections. Also, motivated by the problem of constructing line bundles with prescribed slope, we present some new insights that explore the interplay between elementary number theory and combinatorics via Schubert calculus.

1.1. Main results

The main contributions of this work can be summarized as follows:

- (1) In Theorem A, we prove that there is no obstruction to solving the dHYM equation on rational homogeneous varieties. In particular, we provide an explicit formula for the Lagrangian phase, with respect to any invariant Kähler metric, of every closed invariant $(1, 1)$ -form in terms of Lie theory.
- (2) In Theorem B, we provide an explicit formula, in terms of Lie theory, for the slope of torsion-free coherent sheaves on rational homogeneous varieties.

- (3) In Theorem C, we provide an explicit formula for certain central charges defined by divisors in rational homogeneous varieties. In particular, we derive a new criterion for slope semistability through restrictions of holomorphic vector bundles to the generators of the associated cone of curves.
- (4) As an application of our techniques, we classify all supercritical and hypercritical solutions of the dHYM equation on $(\mathbb{P}(T_{\mathbb{P}^2}), \omega_0)$, such that $\omega_0 \in c_1(\mathbb{P}(T_{\mathbb{P}^2}))$.
- (5) We introduce a constructive method to obtain non-trivial examples of Hermitian–Einstein metrics on certain holomorphic vector bundles over $\mathbb{P}(T_{\mathbb{P}^2})$ from solutions of linear Diophantine equations.
- (6) In Appendix A, we explore the interplay between elementary number theory and Schubert calculus. In particular, we prove in Theorem E that every integral Kähler class induces a decomposition on the Grothendieck group $K_0(X_P)$.

It is worth pointing out that the results mentioned in items (4) and (5) above allow us to construct the first explicit non-trivial example of deformed Hermitian Yang–Mills connection on a higher rank slope-unstable holomorphic vector bundle, see for instance [18]. The first non-trivial solution to the dHYM equation and Z-critical equation on a higher rank (semistable) holomorphic vector bundle was provided in [21].

In order to state precisely the results above, let us introduce some context. A rational homogeneous variety can be described as a quotient $X_P = G^{\mathbb{C}}/P$, where $G^{\mathbb{C}}$ is a semisimple complex algebraic group with Lie algebra $\mathfrak{g}^{\mathbb{C}} = \text{Lie}(G^{\mathbb{C}})$, and P is a parabolic Lie subgroup (Borel–Remmert [7]). Regarding $G^{\mathbb{C}}$ as a complex analytic space, without loss of generality, we may assume that $G^{\mathbb{C}}$ is a connected simply connected complex simple Lie group. Fixing a compact real form $G \subset G^{\mathbb{C}}$, and considering $X_P = G/G \cap P$ as a G -space, we are interested in the G -invariant solutions of (1.1). Given a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$ and a simple root system $\Delta \subset \mathfrak{h}^*$, up to conjugation, we have $P \subset G^{\mathbb{C}}$ completely determined by some $I \subset \Delta$, e.g., [1, Section 3.1]. Moreover, considering the associated fundamental weights $\varpi_{\alpha} \in \mathfrak{h}^*$, $\alpha \in \Delta$, it follows that

$$\text{Pic}(X_P) \cong H^{1,1}(X_P, \mathbb{Z}) \cong \Lambda_P := \bigoplus_{\alpha \in \Delta \setminus I} \mathbb{Z} \varpi_{\alpha},$$

see for instance [54, Theorem 6.4] (or Remark 2.20). From the aforementioned isomorphisms, we have a map

$$[\omega] \in H^{1,1}(X_P, \mathbb{Z}) \mapsto \lambda([\omega]) \in \Lambda_P. \quad (1.4)$$

Denoting by $\Phi = \Phi^+ \cup \Phi^-$ the root system associated with $\Delta \subset \mathfrak{h}^*$, and considering $\Phi_I^{\pm} := \Phi^{\pm} \setminus \{I\}^{\pm}$, we have the following result.

Theorem A. *Given a Kähler class $[\omega] \in \mathcal{K}(X_P)$, then for every $[\psi] \in H^{1,1}(X_P, \mathbb{R})$ we have*

$$\hat{\Theta} := \text{Arg} \int_{X_P} \frac{(\omega + \sqrt{-1}\psi)^n}{n!} = \sum_{\beta \in \Phi_I^+} \arctan \left(\frac{\langle \lambda([\psi]), \beta^{\vee} \rangle}{\langle \lambda([\omega]), \beta^{\vee} \rangle} \right) \pmod{2\pi}, \quad (1.5)$$

such that $\lambda([\psi]), \lambda([\omega]) \in \Lambda_P \otimes \mathbb{R}$. In particular, fixed the unique G -invariant representative $\omega_0 \in [\omega]$, there exists $\phi \in C^\infty(X_P)$, such that $\chi_\phi := \psi + \sqrt{-1}\partial\bar{\partial}\phi$ satisfies the deformed Hermitian Yang–Mills equation

$$\operatorname{Im}(\omega_0 + \sqrt{-1}\chi_\phi)^n = \tan(\hat{\Theta}) \operatorname{Re}(\omega_0 + \sqrt{-1}\chi_\phi)^n. \quad (1.6)$$

We notice that the sum on the right-hand side of (1.5) can be explicitly described in concrete cases through the coefficients of $\lambda([\psi]), \lambda([\omega]) \in \Lambda_P \otimes \mathbb{R}$ with respect to the basis $\{\varpi_\alpha \mid \alpha \in \Delta \setminus I\}$ and the Cartan matrix of $\mathfrak{g}^{\mathbb{C}}$ (e.g., [29]). Unlike the known existence results for (1.1) (e.g., [3, 10, 12, 13, 33, 51, 58]), the result of Theorem A shows that in the homogeneous setting no additional hypotheses on the phase angle (such as the supercritical condition) is necessary to solve the dHYM equation.

As a straightforward consequence of the above theorem and the results in [14, 15], we have the following corollary.

Corollary A. *In the setting of the previous theorem, given $[\psi] \in H^{1,1}(X_P, \mathbb{R})$, considering $\eta_\phi = \psi + \sqrt{-1}\partial\bar{\partial}\phi$, for each $\phi \in C^\infty(X_P)$, we have that the space of the almost calibrated $(1, 1)$ -forms*

$$\mathcal{H} := \{\phi \in C^\infty(X_P) \mid \operatorname{Re}(e^{-\sqrt{-1}\hat{\Theta}}(\omega_0 + \sqrt{-1}\eta_\phi)^n) > 0\}, \quad (1.7)$$

is non-empty ($\mathcal{H} \neq \emptyset$). In particular, the unique lift $\hat{\Theta}([\psi]) \in (-n\frac{\pi}{2}, n\frac{\pi}{2})$ of $\hat{\Theta}$ is given by

$$\hat{\Theta}([\psi]) = \sum_{\beta \in \Phi_I^+} \operatorname{Arg} \int_{\mathbb{P}_\beta^1} (\omega + \sqrt{-1}\psi), \quad (1.8)$$

such that $\mathbb{P}_\beta^1 = \overline{\exp(\mathfrak{g}_{-\beta})P} \subset X_P$, $\forall \beta \in \Phi_I^+$.

It is worth mentioning that the space \mathcal{H} described in (1.7) plays an important role in the study of mirror symmetry, see for instance [11, 16, 55, 56]. We observe that, since

$$\int_{\mathbb{P}_\beta^1} (\omega + \sqrt{-1}\psi) = \langle \lambda([\omega]), \beta^\vee \rangle + \sqrt{-1} \langle \lambda([\psi]), \beta^\vee \rangle, \quad \forall \beta \in \Phi_I^+,$$

and $\lambda([\psi]), \lambda([\omega]) \in \Lambda_P \otimes \mathbb{R}$ depend only on the cohomology classes involved, the unique lifted angle $\hat{\Theta}([\psi]) \in (-n\frac{\pi}{2}, n\frac{\pi}{2})$ coincides with the Lagrangian phase of the unique G -invariant representative in the class $[\psi]$. The corollary above shows that the unique lifted angle associated with some $[\psi] \in H^{1,1}(X_P, \mathbb{R})$ can be completely described in terms of the central charges $Z_{\mathbb{P}_\beta^1}([\psi])$, $\beta \in \Phi_I^+$, through the equation

$$\operatorname{Arg} \int_{\mathbb{P}_\beta^1} (\omega + \sqrt{-1}\psi) = \operatorname{Arg}(Z_{\mathbb{P}_\beta^1}([\psi])) - \frac{\pi}{2} \pmod{2\pi},$$

for every $\beta \in \Phi_I^+$. In the setting of Problem 1.1, we restrict our attention now to the case that $[\psi] = c_1(\mathbf{E})$, for some holomorphic vector bundle $\mathbf{E} \rightarrow X_P$. In the particular case that

\mathbf{E} is a line bundle, the dHYM equation is related with the Hermitian Yang–Mills (HYM) equation in the so-called large volume limit. More precisely, considering $(X_P, k\omega)$, for some positive integer $k \gg 0$, we have

$$\begin{aligned} \operatorname{Re}(\omega + \sqrt{-1}\psi)^n &= k^n \omega^n + O(k^{n-2}), \\ \operatorname{Im}(k\omega + \sqrt{-1}\psi)^n &= nk^{n-1}\omega^{n-1} \wedge \psi + O(k^{n-2}). \end{aligned}$$

Observing that $\tan(\hat{\Theta}_k)$ behaves like $O(1/k)$, where $\hat{\Theta}_k$ is the phase angle of ψ with respect to $k\omega$, the dHYM equation (1.6) for $k\omega$ takes the form

$$nk^{n-1}\omega^{n-1} \wedge \psi + O(k^{n-2}) = Ck^{n-1}\omega^n + O(k^{n-3}),$$

where C is a topological constant determined by $\tan(\Theta_k)$, see for instance [53]. Thus, the leading term in (1.6) has the form

$$n\omega^{n-1} \wedge \psi = C\omega^n,$$

and this equation coincides with the HYM equation if ψ is the curvature form of a Hermitian metric on \mathbf{E} . Based on this interplay, we investigate the relationship between slope (semi)stability and central charges defined by certain rational curves. Our main motivation is the possibility to detect the slope stability of vector bundles using central charges provided by irreducible subvarieties. In order to state our main results in this direction, let us recall some terminology. Fixed a Kähler class $\xi \in \mathcal{K}(X_P)$, the slope of a torsion-free coherent sheaf \mathcal{S} on X_P , with respect to ξ , is defined by

$$\mu_\xi(\mathcal{S}) := \frac{\int_{X_P} c_1(\mathcal{S}) \wedge \xi^{n-1}}{\operatorname{rank}(\mathcal{S})}.$$

From above, we say that a torsion-free coherent sheaf \mathcal{S} on X_P is ξ -(semi)stable, in the sense of Mumford–Takemoto, if

$$\mu_\xi(\mathcal{S}) \geq \mu_\xi(\mathcal{F}),$$

for every coherent subsheaf $0 \neq \mathcal{F} \subsetneq \mathcal{S}$. In order to provide a criterion for (semi)stability in terms of central charges, we prove firstly the following result.

Theorem B. *Let \mathcal{S} be a torsion-free coherent sheaf on X_P with $\operatorname{rank}(\mathcal{S}) = r$. Then, the slope of \mathcal{S} with respect to a Kähler class $\xi \in \mathcal{K}(X_P)$ is given by*

$$\mu_\xi(\mathcal{S}) = \frac{(n-1)!}{r} \left[\sum_{\beta \in \Phi_+^1} \frac{\langle \lambda(\mathcal{S}), \beta^\vee \rangle}{\langle \lambda(\xi), \beta^\vee \rangle} \right] \left[\prod_{\beta \in \Phi_+^1} \frac{\langle \lambda(\xi), \beta^\vee \rangle}{\langle \varrho^+, \beta^\vee \rangle} \right], \quad (1.9)$$

such that $\lambda(\mathcal{S}) \in \Lambda_P$, and $\lambda(\xi) \in \Lambda_P \otimes \mathbb{R}$. In particular, if $\mathbf{E} \rightarrow X_P$ is a holomorphic vector bundle whose restriction to each generator $[\mathbb{P}_\alpha^1]$, $\alpha \in \Delta \setminus I$, of the cone of curves $\operatorname{NE}(X_P)$ is semistable, then \mathbf{E} is ξ -semistable with respect to any Kähler class $\xi \in \mathcal{K}(X_P)$.

Notice that $\lambda(\mathcal{S}) := \lambda(c_1(\mathcal{S})) \in \Lambda_P$, see (1.4), and $\varrho^+ = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. The so-called restriction theorems for slope (semi)stability play an important role in the moduli theory of sheaves, see for instance [28, 39, 46, 47]. In fact, such theorems are useful tools used to reduce the study of sheaves to lower dimensions, e.g., [32, Theorem 7.3.1]. Besides the explicit formula (in terms of Lie theory) for the slope of torsion-free coherent sheaves on rational homogeneous varieties, the result above provides a new criterion to slope semistability through restrictions of locally free coherent sheaves to the generators of the underlying cone of curves. It is worth pointing out that the expression provided by (1.9) depends only on certain intersection numbers, namely,

$$\langle \lambda(\mathcal{S}), \beta^\vee \rangle = \langle c_1(\mathcal{S}), [\mathbb{P}_\beta^1] \rangle \quad \text{and} \quad \langle \lambda(\xi), \beta^\vee \rangle = \langle \xi, [\mathbb{P}_\beta^1] \rangle,$$

such that $\mathbb{P}_\beta^1 = \overline{\exp(\mathfrak{g}_{-\beta})P} \subset X_P$, $\forall \beta \in \Phi_I$. This fact leads us to investigate some aspects of intersection theory underlying the definition of certain central charges. In this setting, we have the following result.

Theorem C. *Fixed a G -invariant Kähler metric ω_0 on X_P , for every holomorphic vector bundle $\mathbf{E} \rightarrow X_P$ and every irreducible analytic subvariety $Y \subset X_P$, define*

$$Z_{[\omega_0]}(\mathbf{E}, Y) := - \int_Y e^{-\sqrt{-1}[\omega_0]} \text{ch}(\mathbf{E}).$$

Then, the following hold:

- (1) *In the particular case that $\mathbf{E} \in \text{Pic}(X_P)$ and $Y \in \text{Div}(X_P)$, we have*

$$\begin{aligned} & Z_{[\omega_0]}(\mathbf{E}, Y) \\ &= - \sum_{\alpha \in \Phi_I^+} \left[\prod_{\beta \in \Phi_I^+ \setminus \{\alpha\}} \left(\frac{\langle \lambda(\mathbf{E}), \beta^\vee \rangle}{\langle \lambda([\omega_0]), \beta^\vee \rangle} \frac{\langle \lambda_Y, \alpha^\vee \rangle}{\langle \lambda([\omega_0]), \alpha^\vee \rangle} - \sqrt{-1} \frac{\langle \lambda_Y, \alpha^\vee \rangle}{\langle \lambda([\omega_0]), \alpha^\vee \rangle} \right) \right] V_0, \end{aligned}$$

such that $V_0 = \text{Vol}(X_P, \omega_0)$, $\lambda(\mathbf{E}), \lambda_Y \in \Lambda_P$, and $\lambda([\omega_0]) \in \Lambda_P \otimes \mathbb{R}$. Moreover, for every $\mathbf{E} \in \text{Pic}(X_P)$, such that $c_1(\mathbf{E}) \neq 0$, we have

$$\frac{|Z_{[\omega_0]}(\mathbf{E}, X_P)|}{\|\text{ch}(\mathbf{E})\|} > 0, \quad (1.10)$$

where $\|\cdot\|$ is any norm on the finite-dimensional vector space $H^{\text{even}}(X_P, \mathbb{R})$.

- (2) *For every holomorphic vector bundle $\mathbf{E} \rightarrow X_P$, define*

$$\hat{\mu}_{[\omega_0]}(\mathbf{E}) := \sum_{\beta \in \Phi_I^+} \tan(\Theta_{\omega_0}(\mathbf{E}, \mathbb{P}_\beta^1)), \quad (1.11)$$

such that

$$\Theta_{\omega_0}(\mathbf{E}, \mathbb{P}_\beta^1) := \text{Arg}(Z_{[\omega_0]}(\mathbf{E}, \mathbb{P}_\beta^1)) - \frac{\pi}{2},$$

where $\mathbb{P}_\beta^1 = \overline{\exp(\mathfrak{g}_{-\beta})P} \subset X_P$, $\forall \beta \in \Phi_I^+$. Then, we have that \mathbf{E} is $[\omega_0]$ -(semi)stable

if, and only if,

$$\hat{\mu}_{[\omega_0]}(\mathbf{E}) \succeq \hat{\mu}_{[\omega_0]}(\mathcal{F}),$$

for every coherent subsheaf $0 \neq \mathcal{F} \subsetneq \mathbf{E}$.

(3) Given a holomorphic vector bundle $\mathbf{E} \rightarrow X_P$, if

$$\text{Arg}(Z_{[\omega_0]}(\mathbf{E}, \mathbb{P}_\beta^1)) \succeq \text{Arg}(Z_{[\omega_0]}(\mathcal{F}, \mathbb{P}_\beta^1)), \quad (1.12)$$

for every coherent subsheaf $0 \neq \mathcal{F} \subsetneq \mathbf{E}$ and every $\beta \in \Phi_I^+$, then \mathbf{E} is $[\omega_0]$ -(semi)stable.

From item (1) of the above theorem, we have a general formula for the central charges defined by pairs (Y, \mathbf{E}) , such that $Y \in \text{Div}(X_P)$ and $\mathbf{E} \in \text{Pic}(X_P)$, in terms of Lie theory. In view of the results provided in [27], it seems to be possible to obtain an explicit formula for all central charges defined by Schubert varieties and line bundles. The last statement of item (1) follows directly from Theorem A. In particular, it shows that the homogeneous representative of $c_1(\mathbf{E})$, for every $\mathbf{E} \in \text{Pic}(X_P)$, minimizes the functional

$$\chi \in c_1(\mathbf{E}) \mapsto \int_{X_P} r_{\omega_0}(\chi) \frac{\omega_0^n}{n!},$$

where $r_{\omega_0}(\chi) = |\det(\mathbb{1} + \sqrt{-1}(\omega_0^{-1} \circ \chi))|$, $\forall \chi \in c_1(\mathbf{E})$, see for instance [33]. It is worth mentioning that (1.10) is precisely the support property required in the definition of a Bridgeland stability condition, cf. [4, Section 2.2], [43, Section 5.2]. From item (2) of the above theorem, we obtain a new characterization for (semi)stability of holomorphic vector bundles over rational homogeneous varieties through the argument of certain central charges defined by rational curves. In order to prove item (2), we show that

$$\mu_{[\omega_0]}(\mathbf{E}) = -(n-1)! \left[\sum_{\beta \in \Phi_I^+} \tan(\Theta_{\omega_0}(\mathbf{E}, \mathbb{P}_\beta^1)) \right] \text{Vol}(X_P, \omega_0).$$

It follows from the above expression that the $[\omega_0]$ -slope of a holomorphic vector bundle over a rational homogeneous variety can be obtained through the standard notion of slope (inclination) of lines in the complex plane. The expression on the right-hand side of (1.11) also can be written in the following way

$$\hat{\mu}_{[\omega_0]}(\mathbf{E}) = - \sum_{\beta \in \Phi_I^+} \frac{\text{Re}(Z_{[\omega_0]}(\mathbf{E}, \mathbb{P}_\beta^1))}{\text{Im}(Z_{[\omega_0]}(\mathbf{E}, \mathbb{P}_\beta^1))}.$$

We observe that the above description of $\hat{\mu}_{[\omega_0]}(\mathbf{E})$ resembles the notion of Z -slope (or generalized slope) used to define stability conditions on abelian categories, cf. [43, Section 4]. In fact, considering the Grothendieck group of coherent sheaves $K_0(X_P)$, we have that

$$Z_{[\omega_0]}(-, \mathbb{P}_\beta^1): K_0(X_P) \rightarrow \mathbb{C}$$

defines an additive homomorphism explicitly given by

$$Z_{[\omega_0]}([\mathcal{E}], \mathbb{P}_\beta^1) = - \int_{\mathbb{P}_\beta^1} (c_1(\det(\mathcal{E})) - \text{rank}(\mathcal{E})\sqrt{-1}[\omega_0]),$$

for every $[\mathcal{E}] \in K_0(X_P)$ and every $\beta \in \Phi_I^+$. The criterion for $[\omega_0]$ -(semi)stability obtained through the inequalities given in (1.12) resembles the notion of stability for a given stability function in the setting of Bridgeland stability [8, 21] [43, Section 4]. From Kobayashi–Hitchin correspondence [22, 23, 61], we have the following differential-geometric counterpart of item (3) of the above theorem.

Corollary B. *Given a G -invariant Kähler metric ω_0 on X_P and a simple holomorphic vector bundle $\mathbf{E} \rightarrow X_P$, if*

$$\text{Arg}(Z_{[\omega_0]}(\mathbf{E}, \mathbb{P}_\beta^1)) > \text{Arg}(Z_{[\omega_0]}(\mathcal{F}, \mathbb{P}_\beta^1)), \quad (1.13)$$

for every coherent subsheaf $0 \neq \mathcal{F} \subsetneq \mathbf{E}$ and every $\beta \in \Phi_I^+$, then \mathbf{E} admits a Hermitian metric H solving the Hermitian Yang–Mills equation

$$\sqrt{-1}\Lambda_{\omega_0}F(H) = c\mathbb{1}_{\mathbf{E}}.$$

As an application of our main results, in Section 4, we classify all supercritical and hypercritical solutions of the dHYM equation on $(\mathbb{P}(T_{\mathbb{P}^2}), \omega_0)$, such that $\omega_0 \in c_1(\mathbb{P}(T_{\mathbb{P}^2}))$. In particular, we give a new example which illustrates that the “easier” direction of Conjecture 1.2 holds only in the supercritical case (cf. [10, Remark 1.10]). Furthermore, we introduce a constructive method to obtain non-trivial examples of Hermitian–Einstein metrics on certain holomorphic vector bundles over $\mathbb{P}(T_{\mathbb{P}^2})$. This method was explored by the author to construct in [18] the first explicit non-trivial example of deformed Hermitian Yang–Mills connection on a higher rank slope-unstable holomorphic vector bundle.

In Appendix A, we study the problem of constructing line bundles with prescribed slope. Our interest in this problem is motivated by the construction of polystable vector bundles through the Whitney sum of line bundles. Among other results, we provide a complete description for the Hodge–Riemann bilinear form

$$\mathcal{Q}_{\omega_0}: H^2(X_P, \mathbb{Z}) \times H^2(X_P, \mathbb{Z}) \rightarrow \mathbb{Z}, \quad \mathcal{Q}_{\omega_0}(a, b) := \int_{X_P} a \wedge b \wedge [\omega_0]^{n-2},$$

defined by an integral Kähler class $[\omega_0] \in \mathcal{K}(X_P)$ (e.g., [50, 62]), in terms of the Cartan matrix associated with the complex simple Lie algebra underlying X_P . From this description, fixed some integral Kähler class $[\omega_0] \in \mathcal{K}(X_P)$, and considering

$$\mathcal{A}_{[\omega_0]}(n) := \left\{ m \in \mathbb{Z}_{>0} \mid \begin{array}{l} 1 \leq m \leq n, \\ \exists \mathbf{E} \in \text{Pic}(X_P), \text{ s.t. } \mu_{[\omega_0]}(\mathbf{E}) = m \end{array} \right\},$$

for every $n \in \mathbb{Z}_{>0}$, we obtain the following result.

Theorem D. For every integral Kähler class $[\omega_0] \in \mathcal{K}(X_P)$, we have

$$\lim_{n \rightarrow +\infty} \frac{|\mathcal{A}_{[\omega_0]}(n)|}{n} = \frac{1}{\tau([\omega_0])},$$

such that $\tau([\omega_0]) := \gcd\{\deg_{\omega_0}(\mathcal{O}_\alpha(1)) \mid \alpha \in \Delta \setminus I\}$, where $\mathcal{O}_\alpha(1)$, $\alpha \in \Delta \setminus I$, are the generators of the Picard group $\text{Pic}(X_P)$.

The above result shows that the subset

$$\mathcal{A}_{[\omega_0]} = \{m \in \mathbb{Z}_{>0} \mid \mu_{[\omega_0]}(\mathbf{E}) = m, \text{ for some } \mathbf{E} \in \text{Pic}(X_P)\},$$

has asymptotic density (a.k.a. natural density)

$$d(\mathcal{A}_{[\omega_0]}) := \lim_{n \rightarrow +\infty} \frac{|\mathcal{A}_{[\omega_0]} \cap [1, n]|}{n} = \frac{1}{\tau([\omega_0])}.$$

In a suitable sense, $d(\mathcal{A}_{[\omega_0]})$ measures the proportion of natural numbers that belong to $\mathcal{A}_{[\omega_0]}$, i.e., the proportion of natural numbers which can be realized as the $[\omega_0]$ -slope of some holomorphic line bundle over X_P , for some integral Kähler class $[\omega_0] \in \mathcal{K}(X_P)$. For more details about natural density, we suggest [49] and references therein. Also as an application of the description of \mathcal{Q}_{ω_0} provided in Appendix A, we prove the following theorem.

Theorem E. Let X_P be a complex flag variety with Picard number $\rho(X_P) > 1$. Given an integral Kähler class $[\omega_0] \in \mathcal{K}(X_P)$, then we have

$$K_0(X_P) \cong SK_0(X_P) \oplus \text{Pic}_{\omega_0}^0(X_P) \oplus \tau([\omega_0])\mathbb{Z},$$

such that

- (1) $SK_0(X_P) := \ker(\det: K_0(X_P) \rightarrow \text{Pic}(X_P))$,
- (2) $\text{Pic}_{\omega_0}^0(X_P) := \{\mathbf{E} \in \text{Pic}(X_P) \mid \deg_{\omega_0}(\mathbf{E}) = 0\}$,
- (3) $\tau([\omega_0]) := \gcd\{\deg_{\omega_0}(\mathcal{O}_\alpha(1)) \mid \alpha \in \Delta \setminus I\}$,

where $\mathcal{O}_\alpha(1)$, $\alpha \in \Delta \setminus I$, are the generators of $\text{Pic}(X_P)$. Moreover, the generators of $\text{Pic}_{\omega_0}^0(X_P)$ are completely determined by the Hodge–Riemann bilinear form \mathcal{Q}_{ω_0} .

Considering $X_P = \mathbb{P}(T_{\mathbb{P}^2})$, in Appendix A we illustrate the results provided in Theorems D and E through explicit computations. As it was shown in [18], these explicit computations can be used to investigate problems related with stability conditions, for more details on stability conditions on Fano threefolds, we suggest [5, 36].

2. Generalities on flag varieties

In this section, we review some basic facts about flag varieties. For more details on the subject presented in this section, we suggest [1, 7, 30, 37].

2.1. The Picard group of flag varieties

Let $G^{\mathbb{C}}$ be a connected, simply connected, and complex Lie group with simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$. By fixing a Cartan subalgebra \mathfrak{h} and a simple root system $\Delta \subset \mathfrak{h}^*$, we have a decomposition of $\mathfrak{g}^{\mathbb{C}}$ given by

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where $\mathfrak{n}^- = \sum_{\alpha \in \Phi^-} \mathfrak{g}_{\alpha}$ and $\mathfrak{n}^+ = \sum_{\alpha \in \Phi^+} \mathfrak{g}_{\alpha}$, here we denote by $\Phi = \Phi^+ \cup \Phi^-$ the root system associated with the simple root system $\Delta \subset \mathfrak{h}^*$. Let us denote by κ the Cartan–Killing form of $\mathfrak{g}^{\mathbb{C}}$. From this, for every $\alpha \in \Phi^+$, we have $h_{\alpha} \in \mathfrak{h}$, such that $\alpha = \kappa(\cdot, h_{\alpha})$, and we can choose $x_{\alpha} \in \mathfrak{g}_{\alpha}$ and $y_{-\alpha} \in \mathfrak{g}_{-\alpha}$, such that $[x_{\alpha}, y_{-\alpha}] = h_{\alpha}$. From these data, we can define a Borel subalgebra¹ by setting $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$.

Remark 2.1. In the above setting, $\forall \phi \in \mathfrak{h}^*$, we also denote $\langle \phi, \alpha \rangle = \phi(h_{\alpha})$, $\forall \alpha \in \Phi^+$.

Now we consider the following result (see for instance [30, 37]).

Theorem 2.2. *Any two Borel subgroups are conjugate.*

From the result above, given a Borel subgroup $B \subset G^{\mathbb{C}}$, up to conjugation, we can always suppose that $B = \exp(\mathfrak{b})$. In this setting, given a parabolic Lie subgroup² $P \subset G^{\mathbb{C}}$, without loss of generality, we can suppose that

$$P = P_I, \quad \text{for some } I \subset \Delta,$$

where $P_I \subset G^{\mathbb{C}}$ is the parabolic subgroup which integrates the Lie subalgebra

$$\mathfrak{p}_I = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}(I)^-, \quad \text{with } \mathfrak{n}(I)^- = \sum_{\alpha \in (I)^-} \mathfrak{g}_{\alpha}.$$

By definition, it is straightforward to show that $P_I = N_{G^{\mathbb{C}}}(\mathfrak{p}_I)$, where $N_{G^{\mathbb{C}}}(\mathfrak{p}_I)$ is the normalizer in $G^{\mathbb{C}}$ of $\mathfrak{p}_I \subset \mathfrak{g}^{\mathbb{C}}$, see for instance [1, Section 3.1]. In what follows, it will be useful for us to consider the following basic chain of Lie subgroups

$$T^{\mathbb{C}} \subset B \subset P \subset G^{\mathbb{C}}.$$

For each element in the aforementioned chain of Lie subgroups we have the following characterization:

- $T^{\mathbb{C}} = \exp(\mathfrak{h})$ (complex torus);
- $B = N^+ T^{\mathbb{C}}$, where $N^+ = \exp(\mathfrak{n}^+)$ (Borel subgroup);
- $P = P_I = N_{G^{\mathbb{C}}}(\mathfrak{p}_I)$, for some $I \subset \Delta \subset \mathfrak{h}^*$ (parabolic subgroup).

¹A maximal solvable subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

²I.e., a Lie subgroup which contains some Borel subgroup.

Now let us recall some basic facts about the representation theory of $\mathfrak{g}^{\mathbb{C}}$, a detailed exposition of the subject can be found in [29]. For every $\alpha \in \Phi$, we set

$$\alpha^{\vee} := \frac{2}{\langle \alpha, \alpha \rangle} \alpha.$$

The fundamental weights $\{\varpi_{\alpha} \mid \alpha \in \Delta\} \subset \mathfrak{h}^*$ of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h})$ are defined by requiring that $\langle \varpi_{\alpha}, \beta^{\vee} \rangle = \delta_{\alpha\beta}$, $\forall \alpha, \beta \in \Delta$. We denote by

$$\Lambda^+ = \bigoplus_{\alpha \in \Delta} \mathbb{Z}_{\geq 0} \varpi_{\alpha},$$

the set of integral dominant weights of $\mathfrak{g}^{\mathbb{C}}$. Let V be an arbitrary finite dimensional $\mathfrak{g}^{\mathbb{C}}$ -module. By considering its weight space decomposition

$$V = \bigoplus_{\mu \in \Phi(V)} V_{\mu},$$

such that $V_{\mu} = \{v \in V \mid h \cdot v = \mu(h)v, \forall h \in \mathfrak{h}\} \neq \{0\}$, $\forall \mu \in \Phi(V) \subset \mathfrak{h}^*$, we have the following definition.

Definition 2.3. A highest weight vector (of weight λ) in a $\mathfrak{g}^{\mathbb{C}}$ -module V is a non-zero vector $v_{\lambda}^+ \in V_{\lambda}$, such that

$$x \cdot v_{\lambda}^+ = 0, \quad (\forall x \in \mathfrak{n}^+).$$

A weight $\lambda \in \Phi(V)$ associated with a highest weight vector is called highest weight of V .

From above, we consider the following standard results.

Theorem 2.4. Every finite dimensional irreducible $\mathfrak{g}^{\mathbb{C}}$ -module V admits a highest weight vector v_{λ}^+ . Moreover, v_{λ}^+ is the unique highest weight vector of V , up to non-zero scalar multiples.

Theorem 2.5. Let V and W be finite dimensional irreducible $\mathfrak{g}^{\mathbb{C}}$ -modules with highest weight $\lambda \in \mathfrak{h}^*$. Then, V and W are isomorphic.

Remark 2.6. We will denote by $V(\lambda)$ a finite dimensional irreducible $\mathfrak{g}^{\mathbb{C}}$ -module with highest weight $\lambda \in \mathfrak{h}^*$.

Theorem 2.7. In the above setting, the following hold:

- (1) If V is a finite dimensional irreducible $\mathfrak{g}^{\mathbb{C}}$ -module with highest weight $\lambda \in \mathfrak{h}^*$, then $\lambda \in \Lambda^+$.
- (2) If $\lambda \in \Lambda^+$, then there exists a finite dimensional irreducible $\mathfrak{g}^{\mathbb{C}}$ -module V , such that $V = V(\lambda)$.

From the above theorem, it follows that the map $\lambda \mapsto V(\lambda)$ induces a one-to-one correspondence between Λ^+ and the isomorphism classes of finite dimensional irreducible $\mathfrak{g}^{\mathbb{C}}$ -modules.

Remark 2.8. In what follows, it will be useful also to consider the following facts:

- (i) For all $\lambda \in \Lambda^+$, we have $V(\lambda) = \mathfrak{U}(\mathfrak{g}^{\mathbb{C}}) \cdot v_{\lambda}^+$, where $\mathfrak{U}(\mathfrak{g}^{\mathbb{C}})$ is the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$.
- (ii) The fundamental representations are defined by $V(\varpi_{\alpha})$, $\alpha \in \Delta$.
- (iii) For all $\lambda \in \Lambda^+$, we have the following equivalence of induced irreducible representations

$$\varrho: G^{\mathbb{C}} \rightarrow \mathrm{GL}(V(\lambda)) \iff \varrho_*: \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{gl}(V(\lambda)),$$

such that $\varrho(\exp(x)) = \exp(\varrho_*x)$, $\forall x \in \mathfrak{g}^{\mathbb{C}}$, notice that $G^{\mathbb{C}} = \langle \exp(\mathfrak{g}^{\mathbb{C}}) \rangle$.

Given a representation $\varrho: G^{\mathbb{C}} \rightarrow \mathrm{GL}(V(\lambda))$, for the sake of simplicity, we shall denote $\varrho(g)v = gv$, for all $g \in G^{\mathbb{C}}$ and all $v \in V(\lambda)$. Let $G \subset G^{\mathbb{C}}$ be a compact real form for $G^{\mathbb{C}}$. Given a complex flag variety $X_P = G^{\mathbb{C}}/P$, regarding X_P as a homogeneous G -space, that is, $X_P = G/G \cap P$, the following theorem allows us to describe all G -invariant Kähler structures on X_P through elements of representation theory.

Theorem 2.9 (Azad–Biswas [2]). *Let $\omega \in \Omega^{1,1}(X_P)^G$ be a closed invariant real $(1, 1)$ -form, then we have*

$$\pi^* \omega = \sqrt{-1} \partial \bar{\partial} \varphi,$$

where $\pi: G^{\mathbb{C}} \rightarrow X_P$ is the natural projection, and $\varphi: G^{\mathbb{C}} \rightarrow \mathbb{R}$ is given by

$$\varphi(g) = \sum_{\alpha \in \Delta \setminus I} c_{\alpha} \log(\|gv_{\varpi_{\alpha}}^+\|), \quad (\forall g \in G^{\mathbb{C}})$$

with $c_{\alpha} \in \mathbb{R}$, $\forall \alpha \in \Delta \setminus I$. Conversely, every function φ as above defines a closed invariant real $(1, 1)$ -form $\omega_{\varphi} \in \Omega^{1,1}(X_P)^G$. Moreover, ω_{φ} defines a G -invariant Kähler form on X_P if, and only if, $c_{\alpha} > 0$, $\forall \alpha \in \Delta \setminus I$.

Remark 2.10. It is worth pointing out that the norm $\|\cdot\|$ considered in the above theorem is a norm induced from some fixed G -invariant inner product $\langle \cdot, \cdot \rangle_{\alpha}$ on $V(\varpi_{\alpha})$, $\forall \alpha \in \Delta \setminus I$.

Remark 2.11. An important consequence of Theorem 2.9 is that it allows us to describe the local Kähler potential for any homogeneous Kähler metric in a quite concrete way. For some examples of explicit computations, we suggest [17, 19].

Remark 2.12. We observe that, if we have a parabolic Lie subgroup $P \subset G^{\mathbb{C}}$, such that $P = P_I$, for some $I \subset \Delta$, it follows that

$$\mathrm{Hom}(P_I, \mathbb{C}^{\times}) \cong \mathrm{Hom}(T(\Delta \setminus I)^{\mathbb{C}}, \mathbb{C}^{\times}), \quad \chi \mapsto \chi|_{T(\Delta \setminus I)^{\mathbb{C}}} \quad (2.1)$$

such that $T(\Delta \setminus I)^{\mathbb{C}} \subset T^{\mathbb{C}}$ is the complex torus

$$T(\Delta \setminus I)^{\mathbb{C}} = \exp \left\{ \sum_{\alpha \in \Delta \setminus I} a_{\alpha} h_{\alpha} \mid a_{\alpha} \in \mathbb{C} \right\},$$

see for instance [34, Part II, p. 169].

By means of the above theorem we can describe the unique G -invariant representative of each integral class in $H^2(X_P, \mathbb{Z})$. In fact, consider the associated P -principal bundle $P \hookrightarrow G^{\mathbb{C}} \rightarrow X_P$. By choosing a trivializing open covering $X_P = \bigcup_{i \in J} U_i$, in terms of Čech cocycles we can write

$$G^{\mathbb{C}} = \{(U_i)_{i \in J}, \psi_{ij}: U_i \cap U_j \rightarrow P\}.$$

Given $\varpi_\alpha \in \Lambda^+$, such that $\alpha \in \Delta \setminus I$, we consider the induced character $\vartheta_{\varpi_\alpha} \in \text{Hom}(P, \mathbb{C}^\times)$, such that $(d\vartheta_{\varpi_\alpha})_e = \varpi_\alpha$. From the homomorphism $\vartheta_{\varpi_\alpha}: P \rightarrow \mathbb{C}^\times$ one can equip \mathbb{C} with the structure of a P -space, such that $pz = \vartheta_{\varpi_\alpha}(p)^{-1}z$, $\forall p \in P$, and $\forall z \in \mathbb{C}$. Denoting by $\mathbb{C}_{-\varpi_\alpha}$ this P -space, we can form an associated holomorphic line bundle $\mathcal{O}_\alpha(1) = G^{\mathbb{C}} \times_P \mathbb{C}_{-\varpi_\alpha}$, which can be described in terms of Čech cocycles by

$$\mathcal{O}_\alpha(1) = \{(U_i)_{i \in J}, \vartheta_{\varpi_\alpha}^{-1} \circ \psi_{ij}: U_i \cap U_j \rightarrow \mathbb{C}^\times\}, \quad (2.2)$$

that is, $\mathcal{O}_\alpha(1) = \{g_{ij}\} \in \check{H}^1(X_P, \mathcal{O}_{X_P}^*)$, such that $g_{ij} = \vartheta_{\varpi_\alpha}^{-1} \circ \psi_{ij}$, $\forall i, j \in J$.

Remark 2.13. Throughout this paper we shall use the following notation

$$\mathcal{O}_\alpha(k) := \mathcal{O}_\alpha(1)^{\otimes k},$$

for every $k \in \mathbb{Z}$ and every $\alpha \in \Delta \setminus I$.

Given $\mathcal{O}_\alpha(1) \in \text{Pic}(X_P)$, such that $\alpha \in \Delta \setminus I$, as described above, if we consider an open covering $X_P = \bigcup_{i \in J} U_i$ which trivializes both $P \hookrightarrow G^{\mathbb{C}} \rightarrow X_P$ and $\mathcal{O}_\alpha(1) \rightarrow X_P$, by taking a collection of local sections $(s_i)_{i \in J}$, such that $s_i: U_i \rightarrow G^{\mathbb{C}}$, we can define $q_i: U_i \rightarrow \mathbb{R}^+$, such that

$$q_i := e^{-2\pi(\varphi_{\varpi_\alpha} \circ s_i)} = \frac{1}{\|s_i v_{\varpi_\alpha}^+\|^2}, \quad (2.3)$$

for every $i \in J$. Since $s_j = s_i \psi_{ij}$ on $U_i \cap U_j \neq \emptyset$, and $p v_{\varpi_\alpha}^+ = \vartheta_{\varpi_\alpha}(p) v_{\varpi_\alpha}^+$, for every $p \in P$, and every $\alpha \in \Delta \setminus I$, the collection of functions $(q_i)_{i \in J}$ satisfies $q_j = |\vartheta_{\varpi_\alpha}^{-1} \circ \psi_{ij}|^2 q_i$ on $U_i \cap U_j \neq \emptyset$. Hence, we obtain a collection of functions $(q_i)_{i \in J}$ which satisfies on the overlaps $U_i \cap U_j \neq \emptyset$ the following relation

$$q_j = |g_{ij}|^2 q_i, \quad (2.4)$$

such that $g_{ij} = \vartheta_{\varpi_\alpha}^{-1} \circ \psi_{ij}$, $\forall i, j \in J$. From this, we can define a Hermitian structure \mathbf{h} on $\mathcal{O}_\alpha(1)$ by taking on each trivialization $f_i: \mathcal{O}_\alpha(1)|_{U_i} \rightarrow U_i \times \mathbb{C}$ the metric defined by

$$\mathbf{h}(f_i^{-1}(x, v), f_i^{-1}(x, w)) = q_i(x) v \bar{w}, \quad (2.5)$$

for every $(x, v), (x, w) \in U_i \times \mathbb{C}$. The Hermitian metric above induces a Chern connection $\nabla \stackrel{\text{loc}}{=} d + \partial \log \mathbf{h}$ with curvature F_∇ satisfying (locally)

$$\frac{\sqrt{-1}}{2\pi} F_\nabla \stackrel{\text{loc}}{=} \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log (\|s_i v_{\varpi_\alpha}^+\|^2).$$

Therefore, by considering the closed G -invariant $(1, 1)$ -form $\Omega_\alpha \in \Omega^{1,1}(X_P)^G$, which satisfies $\pi^* \Omega_\alpha = \sqrt{-1} \partial \bar{\partial} \varphi_{\varpi_\alpha}$, where $\pi: G^{\mathbb{C}} \rightarrow G^{\mathbb{C}}/P = X_P$, and $\varphi_{\varpi_\alpha}(g) = \frac{1}{2\pi} \log \|g v_{\varpi_\alpha}^+\|^2$, $\forall g \in G^{\mathbb{C}}$, we have

$$\Omega_\alpha|_{U_i} = (\pi \circ s_i)^* \Omega_\alpha = \frac{\sqrt{-1}}{2\pi} F_\nabla|_{U_i},$$

i.e., $c_1(\mathcal{O}_\alpha(1)) = [\Omega_\alpha]$, $\forall \alpha \in \Delta \setminus I$.

Remark 2.14. In what follows, given $I \subset \Delta$, we shall denote $\Phi_I^\pm := \Phi^\pm \setminus \langle I \rangle^\pm$.

Remark 2.15. In order to perform some local computations we shall consider the open set $U^-(P) \subset X_P$ defined by the “opposite” big cell in X_P . This open set is a distinguished coordinate neighborhood $U^-(P) \subset X_P$ of $x_0 = eP \in X_P$ defined as follows

$$U^-(P) = B^- x_0 = R_u(P_I)^- x_0 \subset X_P, \quad (2.6)$$

where $B^- = \exp(\mathfrak{h} \oplus \mathfrak{n}^-)$, and

$$R_u(P_I)^- = \prod_{\alpha \in \Phi_I^+} N_\alpha^-, \quad (\text{opposite unipotent radical})$$

with $N_\alpha^- = \exp(\mathfrak{g}_{-\alpha})$, $\forall \alpha \in \Phi_I^+$, e.g., [38, Section 3], [1, Section 3.1]. It is worth mentioning that the opposite big cell defines a contractible open dense subset in X_P , thus the restriction of any vector bundle (principal bundle) over this open set is trivial.

The next lemma is fundamental for the ideas that will be developed later in this work.

Lemma 2.16. Consider $\mathbb{P}_\beta^1 = \overline{\exp(\mathfrak{g}_{-\beta}) x_0} \subset X_P$, such that $\beta \in \Phi_I^+$. Then,

$$\int_{\mathbb{P}_\beta^1} \Omega_\alpha = \langle \varpi_\alpha, \beta^\vee \rangle, \quad \forall \alpha \in \Delta \setminus I.$$

Proof. By definition, we have $\mathbb{P}_\beta^1 = \overline{N_\beta^- x_0}$, $\forall \beta \in \Phi_I^+$, thus

$$\int_{\mathbb{P}_\beta^1} \Omega_\alpha = \int_{N_\beta^- x_0} \Omega_\alpha.$$

Consider the local section $s_U: U^-(P) \rightarrow G^{\mathbb{C}}$, such that $s_U(gP) = g$, $\forall gP \in U^-(P)$, and the parameterization $u_\beta: \mathbb{C} \rightarrow N_\beta^- x_0$, such that

$$u_{-\beta}(z) = \phi_\beta \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} x_0, \quad \forall z \in \mathbb{C},$$

where ϕ_β is the natural Lie group isomorphism between $\mathrm{SL}_2(\mathbb{C})$ and the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{g}_\beta \oplus [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}] \oplus \mathfrak{g}_{-\beta}$, i.e.,

$$\phi_{\beta*}: \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto e_\beta, \quad \phi_{\beta*}: \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto f_{-\beta}, \quad \phi_{\beta*}: \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \frac{2}{\langle \beta, \beta \rangle} h_\beta,$$

such that $e_\beta \in \mathfrak{g}_\beta$, $f_{-\beta} \in \mathfrak{g}_{-\beta}$, and $[e_\beta, f_{-\beta}] = \frac{2}{\langle \beta, \beta \rangle} h_\beta$. From above, since $N_\beta^- x_0 \subset U^-(P)$, we have

$$\begin{aligned} \int_{N_\beta^- x_0} \Omega_\alpha &= \int_{\mathbb{C}} u_{-\beta}^* (\Omega_\alpha|_{U^-(P)}) = \int_{\mathbb{C}} u_{-\beta}^* \circ (\pi \circ s_U)^* \Omega_\alpha \\ &= \frac{\sqrt{-1}}{2\pi} \int_{\mathbb{C}} \partial \bar{\partial} \log \left\| \phi_\beta \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} v_{\varpi_\alpha}^+ \right\|^2. \end{aligned}$$

From the Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{C})$, we have

$$\phi_\beta \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \phi_\beta(k) \phi_\beta \begin{pmatrix} (1+|z|^2)^{\frac{1}{2}} & 0 \\ 0 & (1+|z|^2)^{-\frac{1}{2}} \end{pmatrix} \phi_\beta \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

such that $k \in \mathrm{SU}(2)$, so $\phi_\beta(k) \in G$. Since $\|\cdot\|$ is obtained from some G -invariant inner product on $V(\varpi_\alpha)$, and $\mathfrak{n}^+ \cdot v_{\varpi_\alpha}^+ = \{0\}$, we have

$$\int_{\mathbb{P}_\beta^1} \Omega_\alpha = \frac{\sqrt{-1}}{2\pi} \int_{\mathbb{C}} \partial \bar{\partial} \log \left\| (\vartheta_{\varpi_\alpha} \circ \phi_\beta) \begin{pmatrix} (1+|z|^2)^{\frac{1}{2}} & 0 \\ 0 & (1+|z|^2)^{-\frac{1}{2}} \end{pmatrix} \right\|^2.$$

Observing that $(\vartheta_{\varpi_\alpha} \circ \phi_\beta)|_{T_{\mathbb{C}}^1}$ is a character of the complex torus $T_{\mathbb{C}}^1 \subset \mathrm{SL}_2(\mathbb{C})$, such that

$$T_{\mathbb{C}}^1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \mid a \in \mathbb{C}^\times \right\},$$

it follows that

$$(\vartheta_{\varpi_\alpha} \circ \phi_\beta) \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^{(\mathrm{d}\vartheta_{\varpi_\alpha})e}(\phi_{\beta*} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = a^{\langle \varpi_\alpha, \beta^\vee \rangle}, \quad \forall a \in \mathbb{C}^\times.$$

Thus, we conclude that

$$\begin{aligned} \int_{\mathbb{P}_\beta^1} \Omega_\alpha &= \frac{\sqrt{-1} \langle \varpi_\alpha, \beta^\vee \rangle}{2\pi} \int_{\mathbb{C}} \partial \bar{\partial} \log(1+|z|^2) \\ &= \frac{\sqrt{-1} \langle \varpi_\alpha, \beta^\vee \rangle}{2\pi} \int_{\mathbb{C}} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} = \langle \varpi_\alpha, \beta^\vee \rangle. \quad \blacksquare \end{aligned}$$

From the ideas described above we have the following result.

Proposition 2.17. *Let X_P be a complex flag variety associated with some parabolic Lie subgroup $P = P_I$. Then, we have*

$$\mathrm{Pic}(X_P) = H^{1,1}(X_P, \mathbb{Z}) = H^2(X_P, \mathbb{Z}) = \bigoplus_{\alpha \in \Delta \setminus I} \mathbb{Z}[\Omega_\alpha]. \quad (2.7)$$

Proof. In order to introduce some ideas and some notation, let us sketch the proof. The last equality on the right-hand side of (2.7) follows from the following facts:

- (i) $\pi_2(X_P) \cong \pi_1(T(\Delta \setminus I)^{\mathbb{C}}) = \mathbb{Z}^{|\Delta \setminus I|}$, where $T(\Delta \setminus I)^{\mathbb{C}}$ is given as in Remark 2.12.
- (ii) Since X_P is simply connected, it follows that $H_2(X_P, \mathbb{Z}) \cong \pi_2(X_P)$ (Hurewicz's theorem).
- (iii) By taking the (Schubert) curves $\mathbb{P}_\beta^1 \hookrightarrow X_P$, such that

$$\mathbb{P}_\beta^1 = \overline{\exp(\mathfrak{g}_{-\beta})x_0} \subset X_P, \quad (2.8)$$

for all $\beta \in \Delta \setminus I$, where $x_0 = eP \in X_P$, it follows from Lemma 2.16 that

$$\langle c_1(\mathcal{O}_\alpha(1)), [\mathbb{P}_\beta^1] \rangle = \int_{\mathbb{P}_\beta^1} \boldsymbol{\Omega}_\alpha = \langle \varpi_\alpha, \beta^\vee \rangle = \delta_{\alpha\beta},$$

for every $\alpha, \beta \in \Delta \setminus I$. Hence, we obtain

$$\pi_2(X_P) = \bigoplus_{\alpha \in \Delta \setminus I} \mathbb{Z}[\mathbb{P}_\alpha^1], \quad \text{and} \quad H^2(X_P, \mathbb{Z}) = \bigoplus_{\alpha \in \Delta \setminus I} \mathbb{Z}c_1(\mathcal{O}_\alpha(1)).$$

Since $c_1(\mathcal{O}_\alpha(1)) \in H^{1,1}(X_P, \mathbb{Z})$, $\forall \alpha \in \Delta \setminus I$, it follows that $H^{1,1}(X_P, \mathbb{Z}) = H^2(X_P, \mathbb{Z})$. In order to conclude the proof, from the Lefschetz theorem on $(1, 1)$ -classes [31], and from the fact that $\text{rk}(\text{Pic}^0(X_P)) = 0$, we obtain $\text{Pic}(X_P) = H^{1,1}(X_P, \mathbb{Z})$. ■

Remark 2.18 (Kähler cone of X_P). It follows from (2.7) and Theorem 2.9 that the Kähler cone (e.g., [40]) of a complex flag variety X_P is given explicitly by

$$\mathcal{K}(X_P) = \bigoplus_{\alpha \in \Delta \setminus I} \mathbb{R}^+[\boldsymbol{\Omega}_\alpha].$$

Remark 2.19. Combining the above result with Lemma 2.16, we obtain the following

$$[\mathbb{P}_\beta^1] = \sum_{\alpha \in \Delta \setminus I} \langle \varpi_\alpha, \beta^\vee \rangle [\mathbb{P}_\alpha^1] \quad (\forall \beta \in \Phi_I^+).$$

Remark 2.20. In the above setting, we consider the weights of $P = P_I$ as being

$$\Lambda_P := \bigoplus_{\alpha \in \Delta \setminus I} \mathbb{Z}\varpi_\alpha.$$

From this, the previous result provides $\Lambda_P \cong \text{Hom}(P, \mathbb{C}^\times) \cong \text{Pic}(X_P)$, such that

- (1) $\lambda = \sum_{\alpha \in \Delta \setminus I} k_\alpha \varpi_\alpha \mapsto \prod_{\alpha \in \Delta \setminus I} \vartheta_{\varpi_\alpha}^{k_\alpha} \mapsto \bigotimes_{\alpha \in \Delta \setminus I} \mathcal{O}_\alpha(k_\alpha)$;
- (2) $\mathbf{E} \mapsto \vartheta_{\mathbf{E}} := \prod_{\alpha \in \Delta \setminus I} \vartheta_{\varpi_\alpha}^{\langle c_1(\mathbf{E}), [\mathbb{P}_\alpha^1] \rangle} \mapsto \lambda(\mathbf{E}) := \sum_{\alpha \in \Delta \setminus I} \langle c_1(\mathbf{E}), [\mathbb{P}_\alpha^1] \rangle \varpi_\alpha$.

Thus, for every torsion-free coherent sheaf \mathcal{S} on X_P we define $\lambda(\mathcal{S}) \in \Lambda_P$, such that

$$\lambda(\mathcal{S}) := \sum_{\alpha \in \Delta \setminus I} \langle c_1(\mathcal{S}), [\mathbb{P}_\alpha^1] \rangle \varpi_\alpha, \quad (2.9)$$

where $c_1(\mathcal{S}) = c_1(\det(\mathcal{S}))$. More generally, $\forall \xi \in H^{1,1}(X_P, \mathbb{R})$, we have $\lambda(\xi) \in \Lambda_P \otimes \mathbb{R}$, such that

$$\lambda(\xi) := \sum_{\alpha \in \Delta \setminus I} \langle \xi, [\mathbb{P}_\alpha^1] \rangle \varpi_\alpha. \quad (2.10)$$

Remark 2.21 (Cone of curves of X_P). It is worth observing that the cone of curves $\text{NE}(X_P)$ of a flag variety X_P is generated by the rational curves $[\mathbb{P}_\alpha^1] \in \pi_2(X_P)$, $\alpha \in \Delta \setminus I$, see for instance [60, Section 18.3] and references therein.

Proposition 2.22. *Let X_P be a flag variety and let ω_0 be a G -invariant Kähler metric on X_P . Then, for every closed G -invariant real $(1, 1)$ -form ψ , the eigenvalues of the endomorphism $\omega_0^{-1} \circ \psi$ are given by*

$$\mathbf{q}_\beta(\omega_0^{-1} \circ \psi) = \frac{\langle \lambda([\psi]), \beta^\vee \rangle}{\langle \lambda([\omega_0]), \beta^\vee \rangle}, \quad \beta \in \Phi_I^+. \quad (2.11)$$

Proof. Given a closed G -invariant real $(1, 1)$ -form $\omega \in \Omega^{1,1}(X_P)^G$, it follows from Theorem 2.9 that $\omega = \omega_\varphi$, such that

$$\varphi(g) = \sum_{\alpha \in \Delta \setminus I} c_\alpha \log(\|g v_{\varpi_\alpha}^+\|), \quad (\forall g \in G^{\mathbb{C}})$$

where $c_\alpha \in \mathbb{R}$, for all $\alpha \in \Delta \setminus I$. Let \mathcal{H}_φ be the Hermitian form induced by ω_φ on the holomorphic tangent bundle $T^{1,0}X_P$, that is,

$$\mathcal{H}_\varphi(Y, Y) := -\sqrt{-1}\omega_\varphi(Y, \bar{Y}),$$

for all $Y \in T^{1,0}X_P$. By considering the coordinate neighborhood $U^-(P) \subset X_P$ of $x_0 \in X_P$ defined by the opposite big cell (see (2.6)), we obtain a suitable basis for $T_{x_0}^{1,0}X_P$, given by $Y_\beta^* = \frac{\partial}{\partial z}|_{z=0} \exp(zy_{-\beta})x_0$, $\beta \in \Phi_I^+$. The vectors Y_β^* , $\beta \in \Phi_I^+$, are orthogonal relative to any $(T^{\mathbb{C}} \cap G)$ -invariant Hermitian form. Moreover, we have

$$\mathcal{H}_\varphi(Y_\beta^*, Y_\beta^*) = \sum_{\alpha \in \Delta \setminus I} \frac{c_\alpha}{2} \langle \varpi_\alpha, \beta^\vee \rangle,$$

for every $\beta \in \Phi_I^+$, see for instance [2]. From this, considering the associated dual basis θ_β , $\beta \in \Phi_I^+$, at $x_0 \in X_P$, we have

$$\omega_\varphi = \sum_{\beta \in \Phi_I^+} \frac{\sqrt{-1}}{2} \left(\sum_{\alpha \in \Delta \setminus I} \frac{c_\alpha}{2} \langle \varpi_\alpha, \beta^\vee \rangle \right) \theta_\beta \wedge \bar{\theta}_\beta.$$

On the other hand, from Proposition 2.7, a straightforward computation shows that

$$\omega_\varphi = \sum_{\alpha \in \Delta \setminus I} \pi c_\alpha \Omega_\alpha,$$

such that $c_\alpha = \frac{\langle [\omega_\varphi], [\mathbb{P}_\alpha^1] \rangle}{\pi}$, $\forall \alpha \in \Delta \setminus I$. Joining the above descriptions, we conclude that the eigenvalues of the endomorphism associated with ω_φ at $x_0 \in X_P$ are given by

$$\mathbf{q}_\beta(\omega_\varphi) = - \sum_{\alpha \in \Delta \setminus I} \frac{\langle [\omega_\varphi], [\mathbb{P}_\alpha^1] \rangle}{4\pi\sqrt{-1}} \langle \varpi_\alpha, \beta^\vee \rangle = - \frac{\langle \lambda([\omega_\varphi]), \beta^\vee \rangle}{4\pi\sqrt{-1}}, \quad (2.12)$$

for every $\beta \in \Phi_I^+$, such that $\lambda([\omega_\varphi]) \in \Lambda_P \otimes \mathbb{R}$ (see (2.10)). Since ω_φ is G -invariant, it follows that the eigenvalues of the endomorphism $\omega_\varphi: T^{1,0}X_P \rightarrow T^{1,0}X_P$ are constant. Therefore, given some G -invariant Kähler metric ω_0 , for every closed G -invariant real $(1, 1)$ -form ψ , it follows that the eigenvalues of the endomorphism

$$\omega_0^{-1} \circ \psi: T^{1,0}X_P \rightarrow T^{1,0}X_P$$

are given by

$$\mathbf{q}_\beta(\omega_0^{-1} \circ \psi) = \frac{\mathbf{q}_\beta(\psi)}{\mathbf{q}_\beta(\omega_0)} = \frac{\langle \lambda([\psi]), \beta^\vee \rangle}{\langle \lambda([\omega_0]), \beta^\vee \rangle}, \quad (2.13)$$

for every $\beta \in \Phi_I^+$, which concludes the proof. \blacksquare

Remark 2.23. In the setting of the last proposition, since $n\psi \wedge \omega_0^{n-1} = \Lambda_{\omega_0}(\psi)\omega_0^n$, such that $n = \dim_{\mathbb{C}}(X_P)$, and $\Lambda_{\omega_0}(\psi) = \text{tr}(\omega_0^{-1} \circ \psi)$, it follows that

$$\Lambda_{\omega_0}(\mathbf{\Omega}_\alpha) = \sum_{\beta \in \Phi_I^+} \frac{\langle \varpi_\alpha, \beta^\vee \rangle}{\langle \lambda([\omega_0]), \beta^\vee \rangle}, \quad (2.14)$$

for every $\alpha \in \Delta \setminus I$. In particular, for every $\mathbf{E} \in \text{Pic}(X_P)$, we have a Hermitian structure \mathbf{h} , such that the curvature F_∇ of the Chern connection $\nabla \stackrel{\text{loc}}{=} d + \partial \log(\mathbf{h})$, satisfies

$$\frac{\sqrt{-1}}{2\pi} \Lambda_{\omega_0}(F_\nabla) = \sum_{\beta \in \Phi_I^+} \frac{\langle \lambda(\mathbf{E}), \beta^\vee \rangle}{\langle \lambda([\omega_0]), \beta^\vee \rangle}.$$

From this, we have that ∇ is a Hermitian Yang–Mills (HYM) connection. Notice that

$$c_1(\mathbf{E}) = \sum_{\alpha \in \Delta \setminus I} \langle \lambda(\mathbf{E}), \alpha^\vee \rangle [\mathbf{\Omega}_\alpha],$$

for every $\mathbf{E} \in \text{Pic}(X_P)$

Remark 2.24. In the setting of the proof of Proposition 2.22, if we consider

$$\zeta_\beta := \sqrt{\frac{\langle \lambda([\omega_0]), \beta^\vee \rangle}{2\pi}} \theta_\beta, \quad \beta \in \Phi_I^+,$$

we obtain from the previous result the following description

$$\omega_0 = \sum_{\beta \in \Phi_I^+} \frac{\sqrt{-1}}{2} \zeta_\beta \wedge \bar{\zeta}_\beta \quad \text{and} \quad \psi = \sum_{\beta \in \Phi_I^+} \frac{\sqrt{-1}}{2} \mathbf{q}_\beta(\omega_0^{-1} \circ \psi) \zeta_\beta \wedge \bar{\zeta}_\beta,$$

for every closed G -invariant real $(1, 1)$ -form $\psi \in \Omega^{1,1}(X_P)$.

2.2. The first Chern class of flag varieties

In this subsection, we shall review some basic facts related to the Ricci form of G -invariant Kähler metrics on flag varieties. Let X_P be a complex flag variety associated with some parabolic Lie subgroup $P = P_I \subset G^{\mathbb{C}}$. By considering the identification $T_o^{1,0}X_P \cong \mathfrak{m} \subset \mathfrak{g}^{\mathbb{C}}$, such that

$$\mathfrak{m} = \sum_{\alpha \in \Phi_I^-} \mathfrak{g}_{\alpha},$$

we can realize $T^{1,0}X_P$ as being a holomorphic vector bundle, associated with the holomorphic principal P -bundle $P \hookrightarrow G^{\mathbb{C}} \rightarrow X_P$, such that

$$T^{1,0}X_P = \{(U_i)_{i \in J}, \underline{\text{Ad}} \circ \psi_{ij}: U_i \cap U_j \rightarrow \text{GL}(\mathfrak{m})\},$$

where $\underline{\text{Ad}}: P \rightarrow \text{GL}(\mathfrak{m})$ is the isotropy representation. From this, we obtain

$$\mathbf{K}_{X_P}^{-1} = \det(T^{1,0}X_P) = \{(U_i)_{i \in J}, \det(\underline{\text{Ad}} \circ \psi_{ij}): U_i \cap U_j \rightarrow \mathbb{C}^{\times}\}. \quad (2.15)$$

Since the character $\det \circ \underline{\text{Ad}} \in \text{Hom}(P, \mathbb{C}^{\times})$ is completely determined by its restriction to the torus $T(\Delta \setminus I)^{\mathbb{C}}$, see (2.1), observing that

$$\det \underline{\text{Ad}}(\exp(\mathfrak{t})) = e^{\text{tr}(\text{ad}(\mathfrak{t})|_{\mathfrak{m}})} = e^{-(\delta_P, \mathfrak{t})},$$

$\forall \mathfrak{t} \in \text{Lie}(T(\Delta \setminus I)^{\mathbb{C}})$, such that $\delta_P = \sum_{\alpha \in \Phi_I^+} \alpha$, and denoting $\vartheta_{\delta_P}^{-1} = \det \circ \underline{\text{Ad}}$, it follows that

$$\vartheta_{\delta_P} = \prod_{\alpha \in \Delta \setminus I} \vartheta_{\frac{\delta_P, \alpha^{\vee}}{w_{\alpha}}} \implies \mathbf{K}_{X_P}^{-1} = \bigotimes_{\alpha \in \Delta \setminus I} \mathcal{O}_{\alpha}(\ell_{\alpha}), \quad (2.16)$$

such that $\ell_{\alpha} = \langle \delta_P, \alpha^{\vee} \rangle$, $\forall \alpha \in \Delta \setminus I$. In particular, notice that $\lambda(\mathbf{K}_{X_P}^{-1}) = \delta_P$, see (2.9). If we consider the invariant Kähler metric $\rho_0 \in \Omega^{1,1}(X_P)^G$, described by

$$\rho_0 = \sum_{\alpha \in \Delta \setminus I} 2\pi \langle \delta_P, \alpha^{\vee} \rangle \Omega_{\alpha}, \quad (2.17)$$

it follows that

$$c_1(X_P) = \left[\frac{\rho_0}{2\pi} \right]. \quad (2.18)$$

By the uniqueness of G -invariant representative of $c_1(X_P)$, we have

$$\text{Ric}(\rho_0) = \rho_0,$$

i.e., $\rho_0 \in \Omega^{1,1}(X_P)^G$ defines a G -invariant Kähler–Einstein metric on X_P (cf. [45]).

Remark 2.25. Given any G -invariant Kähler metric ω on X_P , we have $\text{Ric}(\omega) = \rho_0$. Thus, it follows that the smooth function $\frac{\det(\omega)}{\det(\rho_0)}$ is constant. From this, we obtain

$$\text{Vol}(X_P, \omega) = \frac{1}{n!} \int_{X_P} \omega^n = \frac{\det(\rho_0^{-1} \circ \omega)}{n!} \int_{X_P} \rho_0^n.$$

Since $\det(\rho_0^{-1} \circ \omega) = \frac{1}{(2\pi)^n} \prod_{\beta \in \Phi_I^+} \frac{\langle \lambda([\omega]), \beta^\vee \rangle}{\langle \delta_P, \beta^\vee \rangle}$ and $\frac{1}{n!} \int_{X_P} c_1(X_P)^n = \prod_{\beta \in \Phi_I^+} \frac{\langle \delta_P, \beta^\vee \rangle}{\langle \varrho^+, \beta^\vee \rangle}$, we conclude that³

$$\text{Vol}(X_P, \omega) = \prod_{\beta \in \Phi_I^+} \frac{\langle \lambda([\omega]), \beta^\vee \rangle}{\langle \varrho^+, \beta^\vee \rangle}, \quad (2.19)$$

where $\varrho^+ = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. Combining the above formula with the ideas introduced in Remark 2.23 we obtain the following expression for the degree of a holomorphic torsion-free coherent sheaf \mathcal{S} on X_P with respect to some G -invariant Kähler metric ω on X_P :

$$\begin{aligned} \deg_\omega(\mathcal{S}) &= \int_{X_P} c_1(\mathcal{S}) \wedge [\omega]^{n-1} \\ &= (n-1)! \left[\sum_{\beta \in \Phi_I^+} \frac{\langle \lambda(\mathcal{S}), \beta^\vee \rangle}{\langle \lambda([\omega]), \beta^\vee \rangle} \right] \left[\prod_{\beta \in \Phi_I^+} \frac{\langle \lambda([\omega]), \beta^\vee \rangle}{\langle \varrho^+, \beta^\vee \rangle} \right], \end{aligned} \quad (2.20)$$

such that $\lambda(\mathcal{S}) = \lambda(\det \mathcal{S}) \in \Lambda_P$, and $\lambda([\omega]) \in \Lambda_P \otimes \mathbb{R}$, see (2.9) and (2.10). It is worth pointing out that, as far as the author is aware, the explicit formula for the degree of a torsion-free coherent sheaf on a complex flag variety with respect to an arbitrary G -invariant Kähler metric ω provided above is new in the literature. Similar results for integral Kähler classes and line bundles on flag varieties can be found in [9].

Remark 2.26. As in the case of holomorphic line bundles, one can attach to each divisor class $[D] \in \text{Cl}(X_P)$ a weight $\lambda_D := \lambda(\mathcal{O}(D)) \in \Lambda_P$. From this, we obtain the following expression for the degree of a divisor $D \in \text{Div}(X_P)$ with respect to some G -invariant Kähler metric ω on X_P :

$$\deg_\omega(D) = \int_D \omega^{n-1} = (n-1)! \left[\sum_{\beta \in \Phi_I^+} \frac{\langle \lambda_D, \beta^\vee \rangle}{\langle \lambda([\omega]), \beta^\vee \rangle} \right] \left[\prod_{\beta \in \Phi_I^+} \frac{\langle \lambda([\omega]), \beta^\vee \rangle}{\langle \varrho^+, \beta^\vee \rangle} \right], \quad (2.21)$$

such that $\lambda([\omega]) \in \Lambda_P \otimes \mathbb{R}$. Notice that, in the particular case that $D \in \text{Div}(X_P)$ is very ample, one can choose $[\omega] = c_1(\mathcal{O}(D))$ so that $\lambda_D = \lambda([\omega])$. Thus, in this last case we obtain the well-known formula for the degree of the induced projective embedding $X_P \hookrightarrow \mathbb{P}(V(\lambda_D))$, e.g., [9], [60, Example 18.13]. It is worth mentioning that, as far as the author is aware about, the explicit formula for the degree of an arbitrary divisor with respect to an arbitrary G -invariant Kähler metric on a flag variety as above is new in the literature, cf. [6], [60, Example 18.13].

3. Proof of main results

In this section, we shall prove the main results presented in the introduction.

³Cf. [2].

3.1. Theorem A

Proof of Theorem A. Since $\widehat{\Theta}$ depends only on the cohomology classes $[\omega]$ and $[\psi]$, we can consider G -invariant representatives $\omega_0 \in [\omega]$ and $\chi \in [\psi]$. From this, at $x_0 = eP \in X_P$, we have

$$\omega_0 = \sum_{\beta \in \Phi_I^+} \frac{\sqrt{-1}}{2} \zeta_\beta \wedge \bar{\zeta}_\beta \quad \text{and} \quad \chi = \sum_{\beta \in \Phi_I^+} \frac{\sqrt{-1}}{2} \mathbf{q}_\beta(\omega_0^{-1} \circ \chi) \zeta_\beta \wedge \bar{\zeta}_\beta,$$

such that

$$\mathbf{q}_\beta(\omega_0^{-1} \circ \chi) = \frac{\langle \lambda([\chi]), \beta^\vee \rangle}{\langle \lambda([\omega_0]), \beta^\vee \rangle},$$

for every $\beta \in \Phi_I^+$, see Proposition 2.22 and Remark 2.24. Hence, we obtain

$$(\omega_0 + \sqrt{-1}\chi)^n = \prod_{\beta \in \Phi_I^+} (1 + \sqrt{-1}\mathbf{q}_\beta(\omega_0^{-1} \circ \chi)) \omega_0^n = r_{\omega_0}(\chi) e^{\sqrt{-1}\Theta_{\omega_0}(\chi)} \omega_0^n, \quad (3.1)$$

such that

- (a) $r_{\omega_0}(\chi) = \prod_{\beta \in \Phi_I^+} \sqrt{(1 + \mathbf{q}_\beta(\omega_0^{-1} \circ \chi))^2}$;
- (b) $\Theta_{\omega_0}(\chi) = \sum_{\beta \in \Phi_I^+} \arctan(\mathbf{q}_\beta(\omega_0^{-1} \circ \chi)) \bmod 2\pi$.

Since the eigenvalues $\mathbf{q}_\beta(\chi)$ and $\mathbf{q}_\beta(\omega_0)$ are constant, $\forall \beta \in \Phi_I^+$, we conclude that

$$\int_{X_P} \frac{(\omega_0 + \sqrt{-1}\chi)^n}{n!} = r_{\omega_0}(\chi) e^{\sqrt{-1}\Theta_{\omega_0}(\chi)} \text{Vol}(X_P, \omega_0) \neq 0.$$

Therefore, we have

$$\text{Arg} \int_{X_P} \frac{(\omega + \sqrt{-1}\psi)^n}{n!} = \sum_{\beta \in \Phi_I^+} \arctan \left(\frac{\langle \lambda([\chi]), \beta^\vee \rangle}{\langle \lambda([\omega_0]), \beta^\vee \rangle} \right) \bmod 2\pi. \quad (3.2)$$

Since $\lambda([\chi]) = \lambda([\psi])$ and $\lambda([\omega_0]) = \lambda([\omega])$, we obtain the desired equality. For the second part, we just need to observe that

$$\chi = \psi + \sqrt{-1}\partial\bar{\partial}\phi, \quad \text{for some } \phi \in C^\infty(X_P).$$

Since (1.6) is equivalent to $\Theta_{\omega_0}(\chi) = \widehat{\Theta} \bmod 2\pi$, we conclude the proof. \blacksquare

3.2. Theorem B

Proof of Theorem B. By definition, given a Kähler class $\xi = [\omega]$, for every torsion-free coherent sheaf \mathcal{S} on X_P , we have

$$\mu_{[\omega]}(\mathcal{S}) = \frac{\int_{X_P} c_1(\mathcal{S}) \wedge [\omega]^{n-1}}{r} = \frac{\text{deg}_\omega(\mathcal{S})}{r}.$$

Since $c_1(\mathcal{S}) = \sum_{\alpha \in \Delta \setminus I} \langle c_1(\mathcal{S}), [\mathbb{P}_\alpha^1] \rangle [\mathbf{\Omega}_\alpha]$, we obtain the following

$$\begin{aligned} \mu_{[\omega]}(\mathcal{S}) &= \sum_{\alpha \in \Delta \setminus I} \frac{\langle c_1(\mathcal{S}), [\mathbb{P}_\alpha^1] \rangle}{r} \int_{X_P} \mathbf{\Omega}_\alpha \wedge \omega^{n-1} \\ &= \sum_{\alpha \in \Delta \setminus I} \frac{\langle c_1(\mathcal{S}), [\mathbb{P}_\alpha^1] \rangle}{nr} \Lambda_{\omega_0}(\mathbf{\Omega}_\alpha) \int_{X_P} \omega^n. \end{aligned}$$

From (2.14) and (2.19), we have

$$\mu_{[\omega]}(\mathcal{S}) = \sum_{\alpha \in \Delta \setminus I} \frac{\langle c_1(\mathcal{S}), [\mathbb{P}_\alpha^1] \rangle}{nr} \sum_{\beta \in \Phi_I^+} \frac{\langle \varpi_\alpha, \beta^\vee \rangle}{\langle \lambda([\omega]), \beta^\vee \rangle} n! \left[\prod_{\beta \in \Phi_I^+} \frac{\langle \lambda([\omega]), \beta^\vee \rangle}{\langle \varrho^+, \beta^\vee \rangle} \right].$$

Observing that $\lambda(\mathcal{S}) = \sum_{\alpha \in \Delta \setminus I} \langle c_1(\mathcal{S}), [\mathbb{P}_\alpha^1] \rangle \varpi_\alpha$, and rearranging the above expression, we conclude that

$$\mu_{[\omega]}(\mathcal{S}) = \frac{(n-1)!}{r} \left[\sum_{\beta \in \Phi_I^+} \frac{\langle \lambda(\mathcal{S}), \beta^\vee \rangle}{\langle \lambda([\omega]), \beta^\vee \rangle} \right] \left[\prod_{\beta \in \Phi_I^+} \frac{\langle \lambda([\omega]), \beta^\vee \rangle}{\langle \varrho^+, \beta^\vee \rangle} \right].$$

Here we observe the following. Given a torsion-free coherent sheaf \mathcal{S} on X_P , it follows that every coherent subsheaf $\mathcal{F} \subset \mathcal{S}$ is torsion-free. Therefore, we can use the above formula to compute the slope of every coherent subsheaf of a torsion-free coherent sheaf.

Suppose now that $\mathbf{E} \rightarrow X_P$ is a holomorphic vector bundle whose restriction to each generator $[\mathbb{P}_\alpha^1]$, $\alpha \in \Delta \setminus I$, of the cone of curves $\text{NE}(X_P)$ is semistable. Considering the inclusion $\iota_\alpha: \mathbb{P}_\alpha^1 \hookrightarrow X_P$, $\alpha \in \Delta \setminus I$, since $\iota_\alpha^* \mathbf{E}$ is a holomorphic vector bundle, for every coherent subsheaf $0 \neq \mathcal{F} \subsetneq \mathbf{E}$, such that $\text{rank}(\mathcal{F}) = s < r$, we have that $\iota_\alpha^* \mathcal{F} \subset \iota_\alpha^* \mathbf{E}$ is a coherent subsheaf. Since $\iota_\alpha^* \mathbf{E}$ is semistable, for every $\alpha \in \Delta \setminus I$, we have

$$\frac{c_1(\iota_\alpha^* \mathbf{E})}{r} \geq \frac{c_1(\iota_\alpha^* \mathcal{F})}{s} \implies \frac{1}{r} \int_{\mathbb{P}_\alpha^1} c_1(\mathbf{E}) \geq \frac{1}{s} \int_{\mathbb{P}_\alpha^1} c_1(\mathcal{F}),$$

that is,

$$\frac{\langle c_1(\mathbf{E}), [\mathbb{P}_\alpha^1] \rangle}{r} \geq \frac{\langle c_1(\mathcal{F}), [\mathbb{P}_\alpha^1] \rangle}{s},$$

for every $\alpha \in \Delta \setminus I$. Since $\langle \varpi_\alpha, \beta^\vee \rangle \geq 0$, $\forall \alpha \in \Delta \setminus I$ and $\forall \beta \in \Phi_I^+$, it follows that

$$\frac{\langle \lambda(\mathbf{E}), \beta^\vee \rangle}{r} - \frac{\langle \lambda(\mathcal{F}), \beta^\vee \rangle}{s} = \sum_{\alpha \in \Delta \setminus I} \left(\frac{\langle c_1(\mathbf{E}), [\mathbb{P}_\alpha^1] \rangle}{r} - \frac{\langle c_1(\mathcal{F}), [\mathbb{P}_\alpha^1] \rangle}{s} \right) \langle \varpi_\alpha, \beta^\vee \rangle \geq 0.$$

From above, given any G -invariant Kähler metric ω on X_P , since $\langle \lambda([\omega]), \beta^\vee \rangle > 0$, for every $\beta \in \Phi_I^+$, we conclude that

$$\frac{1}{s} \left[\sum_{\beta \in \Phi_I^+} \frac{\langle \lambda(\mathcal{F}), \beta^\vee \rangle}{\langle \lambda([\omega]), \beta^\vee \rangle} \right] \leq \frac{1}{r} \left[\sum_{\beta \in \Phi_I^+} \frac{\langle \lambda(\mathbf{E}), \beta^\vee \rangle}{\langle \lambda([\omega]), \beta^\vee \rangle} \right] \implies \mu_{[\omega]}(\mathcal{F}) \leq \mu_{[\omega]}(\mathbf{E}).$$

Therefore, \mathbf{E} is $[\omega]$ -semistable. Since every Kähler class admits a G -invariant representative, we obtain the desired result. \blacksquare

3.3. Theorem C

Proof of Theorem C. In order to prove item (1), let us recall some basic facts. Given $\mathbf{E} \in \text{Pic}(X_P)$, we have

$$\text{ch}(\mathbf{E}) = e^{c_1(\mathbf{E})} = \sum_{k=0}^{\infty} \frac{c_1(\mathbf{E})^k}{k!}.$$

Thus, fixed a G -invariant Kähler metric ω_0 on X_P , given $Y \in \text{Div}(X_P)$ and $\chi \in c_1(\mathbf{E})$, we have

$$Z_{[\omega_0]}(\mathbf{E}, Y) = - \int_Y e^{-\sqrt{-1}([\omega_0] + \sqrt{-1}c_1(\mathbf{E}))} = - \frac{(-\sqrt{-1})^{n-1}}{(n-1)!} \int_Y (\omega_0 + \sqrt{-1}\chi)^{n-1}.$$

Since the integral above on the right-hand side depends only on the cohomology classes involved, we can suppose that $\chi \in c_1(\mathbf{L})$ is also G -invariant. Moreover, let us consider the unique G -invariant representative $\omega_Y \in c_1(\mathcal{O}(Y))$. Let ζ_1, \dots, ζ_n be an orthonormal coframe w.r.t. ω_0 of $T_{x_0}^* X_P$, where $x_0 = eP$, such that

$$\omega_0 = \sum_{\beta \in \Phi_I^+} \frac{\sqrt{-1}}{2} \zeta_\beta \wedge \bar{\zeta}_\beta, \quad \chi = \sum_{\beta \in \Phi_I^+} \frac{\sqrt{-1}}{2} \mathbf{a}_\beta \zeta_\beta \wedge \bar{\zeta}_\beta, \quad \omega_Y = \sum_{\beta \in \Phi_I^+} \frac{\sqrt{-1}}{2} \mathbf{b}_\beta \zeta_\beta \wedge \bar{\zeta}_\beta,$$

where $\mathbf{a}_\beta = \frac{\langle \lambda([\chi]), \beta^\vee \rangle}{\langle \lambda([\omega_0]), \beta^\vee \rangle}$, $\mathbf{b}_\beta = \frac{\langle \lambda([\omega_Y]), \beta^\vee \rangle}{\langle \lambda([\omega_0]), \beta^\vee \rangle}$, $\beta \in \Phi_I^+$, are, respectively, the eigenvalues of $\omega_0^{-1} \circ \chi$ and $\omega_0^{-1} \circ \omega_Y$ at x_0 (see Remark 2.24). From these data, we obtain at $x_0 = eP$ the following

$$\omega_Y \wedge (\omega_0 + \sqrt{-1}\chi)^{n-1} = \sum_{\alpha \in \Phi_I^+} (n-1)! \prod_{\beta \in \Phi_I^+ \setminus \{\alpha\}} (1 + \sqrt{-1}\mathbf{a}_\beta) \mathbf{b}_\alpha \frac{\omega_0^n}{n!}.$$

From above, it follows that

$$\begin{aligned} - \frac{(-\sqrt{-1})^{n-1}}{(n-1)!} \int_Y (\omega_0 + \sqrt{-1}\chi)^{n-1} &= - \frac{(-\sqrt{-1})^{n-1}}{(n-1)!} \int_{X_P} \omega_Y \wedge (\omega_0 + \sqrt{-1}\chi)^{n-1} \\ &= - (-\sqrt{-1})^{n-1} \int_{X_P} \frac{\omega_Y \wedge (\omega_0 + \sqrt{-1}\chi)^{n-1}}{(n-1)!} \\ &= - \sum_{\alpha \in \Phi_I^+} \prod_{\beta \in \Phi_I^+ \setminus \{\alpha\}} (\mathbf{a}_\beta \mathbf{b}_\alpha - \sqrt{-1} \mathbf{b}_\alpha) V_0, \end{aligned}$$

such that $V_0 := \text{Vol}(X_P, \omega_0)$. Since $\lambda([\chi]) = \lambda(\mathbf{E})$ and $\lambda([\omega_Y]) = \lambda_Y$, we conclude that

$$\begin{aligned} Z_{[\omega_0]}(\mathbf{E}, Y) &= - \sum_{\alpha \in \Phi_I^+} \left[\prod_{\beta \in \Phi_I^+ \setminus \{\alpha\}} \left(\frac{\langle \lambda(\mathbf{E}), \beta^\vee \rangle}{\langle \lambda([\omega_0]), \beta^\vee \rangle} \frac{\langle \lambda_Y, \alpha^\vee \rangle}{\langle \lambda([\omega_0]), \alpha^\vee \rangle} - \sqrt{-1} \frac{\langle \lambda_Y, \alpha^\vee \rangle}{\langle \lambda([\omega_0]), \alpha^\vee \rangle} \right) \right] V_0. \end{aligned}$$

The proof of the second statement of item (1) goes as follows. Given $\mathbf{E} \in \text{Pic}(X_P)$, and considering the unique G -invariant representative $\chi \in c_1(\mathbf{E})$, it follows that

$$\begin{aligned} Z_{[\omega_0]}(\mathbf{E}, X_P) &= -\frac{(-\sqrt{-1})^n}{n!} \int_{X_P} (\omega_0 + \sqrt{-1}\chi)^n \\ &= -(-\sqrt{-1})^n r_{\omega_0}(\chi) e^{\sqrt{-1}\Theta_{\omega_0}(\chi)} \text{Vol}(X_P, \omega_0), \end{aligned}$$

see for instance (3.1). Thus, since

$$r_{\omega_0}(\chi) = \prod_{\beta \in \Phi_I^+} \sqrt{(1 + \mathbf{q}_\beta(\omega_0^{-1} \circ \chi)^2)},$$

see for instance (2.11), we have

$$|Z_{[\omega_0]}(\mathbf{E}, X_P)| = r_{\omega_0}(\chi) \text{Vol}(X_P, \omega_0) > 0.$$

For item (2), since

$$\text{ch}(\mathbf{E}) = r + c_1(\mathbf{E}) + \frac{1}{2}(c_1(\mathbf{E})^2 - 2c_2(\mathbf{E})) + \dots$$

for every holomorphic vector bundle $\mathbf{E} \rightarrow X_P$, with $\text{rank}(\mathbf{E}) = r$, e.g., [35], it follows that

$$Z_{[\omega_0]}(\mathbf{E}, \mathbb{P}_\beta^1) = -\int_{\mathbb{P}_\beta^1} (c_1(\mathbf{E}) - r\sqrt{-1}[\omega_0]) = \int_{\mathbb{P}_\beta^1} (-c_1(\mathbf{E}) + r\sqrt{-1}[\omega_0]),$$

for every $\mathbb{P}_\beta^1 = \overline{\exp(\mathfrak{g}_{-\beta})P} \subset X_P$, $\beta \in \Phi_I^+$. Since

$$[\mathbb{P}_\beta^1] = \sum_{\alpha \in \Delta \setminus I} \langle \varpi_\alpha, \beta^\vee \rangle [\mathbb{P}_\alpha^1] \quad (\forall \beta \in \Phi_I^+),$$

see Remark 2.19, we obtain

$$\begin{aligned} Z_{[\omega_0]}(\mathbf{E}, \mathbb{P}_\beta^1) &= \sum_{\alpha \in \Delta \setminus I} \langle \varpi_\alpha, \beta^\vee \rangle \int_{\mathbb{P}_\alpha^1} (-c_1(\mathbf{E}) + r\sqrt{-1}[\omega_0]) \\ &= -\langle \lambda(\mathbf{E}), \beta^\vee \rangle + r\sqrt{-1} \langle \lambda([\omega_0]), \beta^\vee \rangle, \end{aligned} \quad (3.3)$$

for every $\mathbb{P}_\beta^1 = \overline{\exp(\mathfrak{g}_{-\beta})P} \subset X_P$, $\beta \in \Phi_I^+$. Therefore, we have

$$\text{Arg}(Z_{[\omega_0]}(\mathbf{E}, \mathbb{P}_\beta^1)) = \underbrace{\arctan\left(\frac{\langle \lambda(\mathbf{E}), \beta^\vee \rangle}{r\langle \lambda([\omega_0]), \beta^\vee \rangle}\right)}_{\Theta_{\omega_0}(\mathbf{E}, \mathbb{P}_\beta^1)} + \frac{\pi}{2}, \quad (3.4)$$

for every $\mathbb{P}_\beta^1 = \overline{\exp(\mathfrak{g}_{-\beta})P} \subset X_P$, $\beta \in \Phi_I^+$. Since

$$-\frac{\langle \lambda(\mathbf{E}), \beta^\vee \rangle}{r\langle \lambda([\omega_0]), \beta^\vee \rangle} = \frac{\text{Re}(Z_{[\omega_0]}(\mathbf{E}, \mathbb{P}_\beta^1))}{\text{Im}(Z_{[\omega_0]}(\mathbf{E}, \mathbb{P}_\beta^1))} = \cot\left(\Theta_{\omega_0}(\mathbf{E}, \mathbb{P}_\beta^1) + \frac{\pi}{2}\right) = -\tan(\Theta_{\omega_0}(\mathbf{E}, \mathbb{P}_\beta^1)),$$

for every $\beta \in \Phi_I^+$, from Theorem B, it follows that

$$\mu_{[\omega_0]}(\mathbf{E}) = -(n-1)! \left[\sum_{\beta \in \Phi_I^+} \tan(\Theta_{\omega_0}(\mathbf{E}, \mathbb{P}_\beta^1)) \right] V_0 = (n-1)! \hat{\mu}_{[\omega_0]}(\mathbf{E}) V_0,$$

such that $V_0 = \text{Vol}(X_P, \omega_0)$. From the above expression we conclude the proof of item (2).

In order to prove item (3), we proceed as follows. Given a holomorphic vector bundle $\mathbf{E} \rightarrow (X_P, \omega_0)$, such that $r = \text{rank}(\mathbf{E})$, suppose that

$$\text{Arg}(Z_{[\omega_0]}(\mathbf{E}, \mathbb{P}_\beta^1)) \succeq \text{Arg}(Z_{[\omega_0]}(\mathcal{F}, \mathbb{P}_\beta^1)),$$

for every coherent subsheaf $0 \neq \mathcal{F} \subsetneq \mathbf{E}$ and every $\beta \in \Phi_I^+$. From (3.4), we have

$$\Theta_{\omega_0}(\mathbf{E}, \mathbb{P}_\beta^1) \succeq \Theta_{\omega_0}(\mathcal{F}, \mathbb{P}_\beta^1),$$

for every coherent subsheaf $0 \neq \mathcal{F} \subsetneq \mathbf{E}$ and every $\beta \in \Phi_I^+$. Since $\tan(-)$ is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$, it follows that

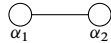
$$\hat{\mu}_{[\omega_0]}(\mathbf{E}) = \sum_{\beta \in \Phi_I^+} \tan(\Theta_{\omega_0}(\mathbf{E}, \mathbb{P}_\beta^1)) \succeq \sum_{\beta \in \Phi_I^+} \tan(\Theta_{\omega_0}(\mathcal{F}, \mathbb{P}_\beta^1)) = \hat{\mu}_{[\omega_0]}(\mathcal{F}),$$

for every coherent subsheaf $0 \neq \mathcal{F} \subsetneq \mathbf{E}$. Thus, from item (2), it follows that \mathbf{E} is $[\omega_0]$ -(semi)stable. \blacksquare

4. Applications and final comments

In what follows, we provide a detailed and constructive example to illustrate our main results. It is worth pointing out that the results presented in this final section are of independent interest and were designed for studying the dHYM equation on higher rank holomorphic vector bundles. Further applications of the ideas introduced in this section can be found in [18].

Example 4.1. Consider the complex Lie group $G^{\mathbb{C}} = \text{SL}_3(\mathbb{C})$. In this case, the structure of the associated Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ can be completely determined by means of its Dynkin diagram



Fixed the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_3(\mathbb{C})$ of diagonal matrices, we have the associated simple root system given by $\Delta = \{\alpha_1, \alpha_2\}$, such that

$$\alpha_j(\text{diag}(d_1, d_2, d_3)) = d_j - d_{j+1}, \quad j = 1, 2.$$

$\forall \text{diag}(d_1, d_2, d_3) \in \mathfrak{h}$. The set of the positive roots in this case is given by

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\}.$$

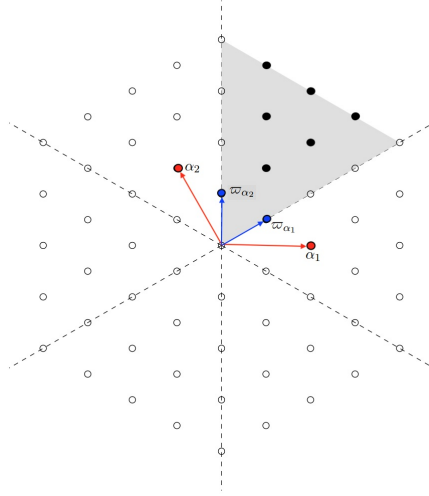


Figure 1. Simple roots, dominant integral elements, and fundamental weights for $\mathfrak{sl}_3(\mathbb{C})$.

Considering the Cartan–Killing form⁴ $\kappa(X, Y) = \text{tr}(\text{ad}(X) \text{ad}(Y))$, $\forall X, Y \in \mathfrak{sl}_3(\mathbb{C})$, it follows that $\alpha_j = \kappa(\cdot, h_{\alpha_j})$, $j = 1, 2, 3$, such that⁵

$$h_{\alpha_1} = \frac{1}{6}(E_{11} - E_{22}), \quad h_{\alpha_2} = \frac{1}{6}(E_{22} - E_{33}), \quad h_{\alpha_3} = \frac{1}{6}(E_{11} - E_{33}),$$

here we consider the matrices E_{ij} as being the elements of the standard basis of $\mathfrak{gl}_3(\mathbb{C})$. Moreover, we have the following relation between simple roots and fundamental weights (Figure 1):

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} w_{\alpha_1} \\ w_{\alpha_2} \end{pmatrix}, \quad \begin{pmatrix} w_{\alpha_1} \\ w_{\alpha_2} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix},$$

here we consider the Cartan matrix $C = (C_{ij})$ of $\mathfrak{sl}_3(\mathbb{C})$ given by

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad C_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}, \quad (4.1)$$

for more details on the above subject, see for instance [29]. Fixing the standard Borel subgroup $B \subset \text{SL}_3(\mathbb{C})$, i.e.,

$$B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \text{SL}_3(\mathbb{C}) \right\},$$

⁴In this case, we have $\kappa(X, Y) = 6 \text{tr}(XY)$, $\forall X, Y \in \mathfrak{sl}_3(\mathbb{C})$, see for instance [52, Chapter 10, Section 4].

⁵Notice that $\langle \alpha_j, \alpha_j \rangle = \alpha_j(h_{\alpha_j}) = \frac{1}{3}$, $\forall j = 1, 2, 3$.

we consider the flag variety obtained from $I = \emptyset$, i.e., the homogeneous Fano threefold given by the Wallach space $\mathbb{P}(T_{\mathbb{P}^2}) = \mathrm{SL}_3(\mathbb{C})/B$. In this particular case, we have the following:

- (i) $H^2(\mathbb{P}(T_{\mathbb{P}^2}), \mathbb{R}) = H^{1,1}(\mathbb{P}(T_{\mathbb{P}^2}), \mathbb{R}) = \mathbb{R}[\mathbf{\Omega}_{\alpha_1}] \oplus \mathbb{R}[\mathbf{\Omega}_{\alpha_2}]$;
- (ii) $H_2(\mathbb{P}(T_{\mathbb{P}^2}), \mathbb{Z}) = \pi_2(\mathbb{P}(T_{\mathbb{P}^2})) = \mathbb{Z}[\mathbb{P}_{\alpha_1}^1] \oplus \mathbb{Z}[\mathbb{P}_{\alpha_2}^1]$.

Let ω_0 be the unique $\mathrm{SU}(3)$ -invariant Kähler metric on $\mathbb{P}(T_{\mathbb{P}^2})$, such that

$$[\omega_0] = c_1(\mathbb{P}(T_{\mathbb{P}^2})).^6$$

Since $\lambda(\mathbf{K}_{\mathbb{P}(T_{\mathbb{P}^2})}^{-1}) = \delta_B = 2(\varpi_{\alpha_1} + \varpi_{\alpha_2})$, from (2.18), it follows that

$$\omega_0 = \langle \delta_B, \alpha_1^\vee \rangle \mathbf{\Omega}_{\alpha_1} + \langle \delta_B, \alpha_2^\vee \rangle \mathbf{\Omega}_{\alpha_2} = 2(\mathbf{\Omega}_{\alpha_1} + \mathbf{\Omega}_{\alpha_2}),$$

in particular, we have $\lambda([\omega_0]) = \delta_B$. Given any $[\psi] \in H^{1,1}(\mathbb{P}(T_{\mathbb{P}^2}), \mathbb{R})$, from Theorem A, we have

$$\widehat{\Theta} = \mathrm{Arg} \int_{\mathbb{P}(T_{\mathbb{P}^2})} (\omega_0 + \sqrt{-1}\psi)^3 = \sum_{j=1}^3 \arctan \left(\frac{\langle \lambda([\psi]), \alpha_j^\vee \rangle}{\langle \delta_B, \alpha_j^\vee \rangle} \right) \pmod{2\pi},$$

notice that, in this particular case, $I = \emptyset$, so $\Phi_I^+ = \Phi^+ = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\}$. Therefore, if we suppose that $[\psi] = s_1[\mathbf{\Omega}_{\alpha_1}] + s_2[\mathbf{\Omega}_{\alpha_2}]$, for some $s_1, s_2 \in \mathbb{R}$, by considering the Cartan matrix $C = (C_{ij})$ of $\mathfrak{sl}_3(\mathbb{C})$ (see (4.1)), we obtain the following:

- (1) $\langle \delta_B, \alpha_1^\vee \rangle = \langle \delta_B, \alpha_2^\vee \rangle = 2$ and $\langle \delta_B, \alpha_3^\vee \rangle = 4$;
- (2) $\langle \lambda([\psi]), \alpha_1^\vee \rangle = s_1$, $\langle \lambda([\psi]), \alpha_2^\vee \rangle = s_2$, $\langle \lambda([\psi]), \alpha_3^\vee \rangle = s_1 + s_2$.

From above, we conclude that $\widehat{\Theta}([\psi]) \in (-\frac{3\pi}{2}, \frac{3\pi}{2})$ is given by

$$\widehat{\Theta}([\psi]) = \arctan \left(\frac{s_1}{2} \right) + \arctan \left(\frac{s_2}{2} \right) + \arctan \left(\frac{s_1 + s_2}{4} \right). \quad (4.2)$$

From (3.1), given an arbitrary $\mathrm{SU}(3)$ -invariant $(1, 1)$ -form $\chi = s_1 \mathbf{\Omega}_{\alpha_1} + s_2 \mathbf{\Omega}_{\alpha_2}$, we have the following concrete expression for its Lagrangian phase w.r.t. ω_0 :

$$\Theta_{\omega_0}(\chi) = \arctan \left(\frac{s_1}{2} \right) + \arctan \left(\frac{s_2}{2} \right) + \arctan \left(\frac{s_1 + s_2}{4} \right).$$

The $(1, 1)$ -classes $[\psi] = s_1[\mathbf{\Omega}_{\alpha_1}] + s_2[\mathbf{\Omega}_{\alpha_2}]$ which satisfy the inequality

$$\pi < \arctan \left(\frac{s_1}{2} \right) + \arctan \left(\frac{s_2}{2} \right) + \arctan \left(\frac{s_1 + s_2}{4} \right) < \frac{3\pi}{2},$$

⁶It is worth pointing out that there is nothing special with this choice. In fact, all the computations presented in this example work for an arbitrary choice of $\mathrm{SU}(3)$ -invariant (integral) Kähler class on $\mathbb{P}(T_{\mathbb{P}^2})$.

define the hypercritical solutions of the dHYM equation on $(\mathbb{P}(T_{\mathbb{P}^2}), \omega_0)$. If we consider, for instance, $[\psi] = 4([\Omega_{\alpha_1}] + [\Omega_{\alpha_2}])$, since

$$\pi < 3 \arctan(2) < \frac{3\pi}{2},$$

we have that $\chi = 4(\Omega_{\alpha_1} + \Omega_{\alpha_2}) \in [\psi]$ defines a hypercritical solution to the dHYM on $(\mathbb{P}(T_{\mathbb{P}^2}), \omega_0)$. In this particular case, following [11], we have that the Riemannian manifold with non-positive sectional curvature defined by the space of calibrated $(1, 1)$ -forms

$$\mathcal{H} = \{\Upsilon \in [\psi] \mid \operatorname{Re}(e^{\sqrt{-1}\hat{\Theta}([\psi])}(\omega_0 + \sqrt{-1}\Upsilon)^n) > 0\},$$

has a well-defined metric structure, and its completion is a CAT(0) geodesic metric space. The $(1, 1)$ -classes $[\psi] = s_1[\Omega_{\alpha_1}] + s_2[\Omega_{\alpha_2}]$ which satisfy the inequality

$$\frac{\pi}{2} < \arctan\left(\frac{s_1}{2}\right) + \arctan\left(\frac{s_2}{2}\right) + \arctan\left(\frac{s_1 + s_2}{4}\right) < \frac{3\pi}{2},$$

define the supercritical solutions of the dHYM equation on $(\mathbb{P}(T_{\mathbb{P}^2}), \omega_0)$. From the above condition, it is straightforward to produce solutions of the dHYM equation which do not satisfy the supercritical condition. For instance, we can take the unique solution χ of the dHYM equation in the class $-([\Omega_{\alpha_1}] + [\Omega_{\alpha_2}])$. In this last case, we obtain

$$-\frac{3\pi}{2} < \Theta_{\omega_0}(\chi) = 3 \arctan\left(-\frac{1}{2}\right) < \frac{\pi}{2}.$$

Consider now the following central charges associated with the aforementioned solution χ :

- (a) $Z_{\mathbb{P}(T_{\mathbb{P}^2})}([\chi]) = -\int_{\mathbb{P}(T_{\mathbb{P}^2})} e^{-\sqrt{-1}(\omega_0 + \sqrt{-1}\chi)} = -\frac{(-\sqrt{-1})^3}{3!} \int_{\mathbb{P}(T_{\mathbb{P}^2})} (\omega_0 + \sqrt{-1}\chi)^3;$
- (b) $Z_{\mathbb{P}_{\alpha_1}^1}([\chi]) = -\int_{\mathbb{P}_{\alpha_1}^1} e^{-\sqrt{-1}(\omega_0 + \sqrt{-1}\chi)} = \sqrt{-1} \int_{\mathbb{P}_{\alpha_1}^1} (\omega_0 + \sqrt{-1}\chi).$

Computing the above integrals, we obtain

- (c) $\operatorname{Arg}(Z_{\mathbb{P}(T_{\mathbb{P}^2})}([\chi])) = \frac{3\pi}{2} + 3 \arctan(-\frac{1}{2});$
- (d) $\operatorname{Arg}(Z_{\mathbb{P}_{\alpha_1}^1}([\chi])) = \frac{\pi}{2} + \arctan(-\frac{1}{2}).$

Hence, it follows that $\operatorname{Arg}(Z_{\mathbb{P}(T_{\mathbb{P}^2})}([\chi])) > \operatorname{Arg}(Z_{\mathbb{P}_{\alpha_1}^1}([\chi]))$, in other words, we have

$$\operatorname{Im}\left(\frac{Z_{\mathbb{P}_{\alpha_1}^1}([\chi])}{Z_{\mathbb{P}(T_{\mathbb{P}^2})}([\chi])}\right) = \left|\frac{Z_{\mathbb{P}_{\alpha_1}^1}([\chi])}{Z_{\mathbb{P}(T_{\mathbb{P}^2})}([\chi])}\right| \sin\left(-\pi - 2 \arctan\left(-\frac{1}{2}\right)\right) < 0.$$

As in [10, Remark 1.10], the example above shows that the “easier” direction of Collins–Jacob–Yau’s conjecture (Conjecture 1.2) holds only in the supercritical case.

By keeping the previous notation, let $\mathbf{E} \in \operatorname{Pic}(\mathbb{P}(T_{\mathbb{P}^2}))$ be some holomorphic line bundle. Since $\operatorname{Pic}(\mathbb{P}(T_{\mathbb{P}^2}))$ is generate by $\mathcal{O}_{\alpha_1}(1)$ and $\mathcal{O}_{\alpha_2}(1)$, without loss of generality, we can suppose that

$$\mathbf{E} = \mathcal{O}_{\alpha_1}(a) \otimes \mathcal{O}_{\alpha_2}(b),$$

for some $a, b \in \mathbb{Z}$. From above, it follows that $\lambda(\mathbf{E}) = a\varpi_{\alpha_1} + b\varpi_{\alpha_2}$. As before, considering the unique $SU(3)$ -invariant Kähler metric ω_0 on $\mathbb{P}(T_{\mathbb{P}^2})$, such that $[\omega_0] = c_1(\mathbb{P}(T_{\mathbb{P}^2}))$, we have

$$Z_{\mathbb{P}_{\alpha_j}^1}(\mathbf{E}) = - \int_{\mathbb{P}_{\alpha_j}^1} e^{-\sqrt{-1}[\omega_0]} \text{ch}(\mathbf{E}) = -\langle \lambda(\mathbf{E}), \alpha_j^\vee \rangle + \sqrt{-1} \langle \delta_B, \alpha_j^\vee \rangle,$$

$\forall j = 1, 2, 3$, recall that $\lambda([\omega_0]) = \delta_B$. From Theorem B and item (2) of Theorem C, respectively, we have

$$(1) \quad \mu_{[\omega_0]}(\mathbf{E}) = 2! \left[\sum_{j=1}^3 \frac{\langle \lambda(\mathbf{E}), \alpha_j^\vee \rangle}{\langle \delta_B, \alpha_j^\vee \rangle} \right] \left[\prod_{j=1}^3 \frac{\langle \delta_B, \alpha_j^\vee \rangle}{\langle \varrho^+, \alpha_j^\vee \rangle} \right] = 12(a + b);$$

$$(2) \quad \hat{\mu}(\mathbf{E}) = - \left[\sum_{j=1}^3 \frac{\text{Re}(Z_{\mathbb{P}_{\alpha_j}^1}(\mathbf{E}))}{\text{Im}(Z_{\mathbb{P}_{\alpha_j}^1}(\mathbf{E}))} \right] = \sum_{j=1}^3 \frac{\langle \lambda(\mathbf{E}), \alpha_j^\vee \rangle}{\langle \delta_B, \alpha_j^\vee \rangle} = \frac{3}{4}(a + b).$$

In the above computation, for item (1), we have used that $\delta_B = 2\varrho^+$, notice that

$$\text{Vol}(\mathbb{P}(T_{\mathbb{P}^2}), \omega_0) = \prod_{j=1}^3 \frac{\langle \delta_B, \alpha_j^\vee \rangle}{\langle \varrho^+, \alpha_j^\vee \rangle} = 8,$$

see (2.19). The computations above show that, in general, it is more simple to compute $\hat{\mu}(\mathbf{E})$ than $\mu_{[\omega_0]}(\mathbf{E})$.

In what follows, we present a constructive method to obtain non-trivial examples of Hermitian Yang–Mills structures on certain holomorphic vector bundles over $\mathbb{P}(T_{\mathbb{P}^2})$. Fixed an integer number $m \in \mathbb{Z}$, one can seek for solutions of the linear Diophantine equation

$$\mu_{[\omega_0]}(\mathbf{E}) = 12(a + b) = m, \quad \mathbf{E} \in \text{Pic}(\mathbb{P}(T_{\mathbb{P}^2})), \quad (4.3)$$

in order to construct example of polystable holomorphic vector bundles through Whitney sums. The Diophantine equation above can be solved if, and only if, $12|m$ (e.g., [48, Chapter 5]), so let us suppose that $m = 12k$, for some $k \in \mathbb{Z}$. From this, the previous equation becomes

$$a + b = k.$$

Therefore, given a particular solution $a_0, b_0 \in \mathbb{Z}$, for the last equation above, we have an infinite number of solutions explicitly given by $a(s) := a_0 + s$ and $b(s) := b_0 - s$, $s \in \mathbb{Z}$. For every $s \in \mathbb{Z}$, let us define $\mathbf{E}(s) := \mathcal{O}_{\alpha_1}(a(s)) \otimes \mathcal{O}_{\alpha_2}(b(s))$, such that $a(s) := a_0 + s$ and $b(s) := b_0 - s$. By construction, we have

$$\mu_{[\omega_0]}(\mathbf{E}(s)) = 12(a(s) + b(s)) = 12k = m.$$

Given $s_1, \dots, s_r \in \mathbb{Z}$, $r \in \mathbb{Z}_{>0}$, we can define

$$\mathbf{E}(s_1, \dots, s_r) := \bigoplus_{j=1}^r \mathbf{E}(s_j).$$

If $r = 2$, since $\mathbf{E}(s)$ is $[\omega_0]$ -stable for every $s \in \mathbb{Z}$, and $\mu_{[\omega_0]}(\mathbf{E}(s_1)) = \mu_{[\omega_0]}(\mathbf{E}(s_2))$, it follows that $\mathbf{E}(s_1, s_2)$ is $[\omega_0]$ -semistable, see for instance [35]. Observing that

$$\mu_{[\omega_0]}(\mathbf{E}(s_1, \dots, s_r)) = \frac{1}{r} \sum_{j=1}^r \mu_{[\omega_0]}(\mathbf{E}(s_j)) = m,$$

for every $r \in \mathbb{Z}_{>0}$, by an inductive argument, one can conclude that $\mathbf{E}(s_1, \dots, s_r)$ is a direct sum of stable vector bundles of the same $[\omega_0]$ -slope, i.e., $\mathbf{E}(s_1, \dots, s_r)$ is $[\omega_0]$ -polystable, for all $s_1, \dots, s_r \in \mathbb{Z}$ and all $r \in \mathbb{Z}_{>0}$. From Kobayashi–Hitchin correspondence [22, 23, 61], we conclude that $\mathbf{E}(s_1, \dots, s_r)$ is Hermite–Einstein, for all $s_1, \dots, s_r \in \mathbb{Z}$ and all $r \in \mathbb{Z}_{>0}$. Moreover, in this case, we can describe the associated Hermite–Einstein structure as follows. At first, we observe the following, given $s \in \mathbb{Z}$, we have $c_1(\mathbf{E}(s)) = [\Omega(s)]$, such that

$$\Omega(s) = a(s)\Omega_{\alpha_1} + b(s)\Omega_{\alpha_2}.$$

Thus, by construction, we obtain

$$m = \mu_{[\omega_0]}(\mathbf{E}(s)) = \int_{\mathbb{P}(T_{\mathbb{P}^2})} \Omega(s) \wedge \omega_0^2 = \frac{1}{3} \Lambda_{\omega_0}(\Omega(s)) \text{Vol}(\mathbb{P}(T_{\mathbb{P}^2}), \omega_0) 3!.$$

Since $\text{Vol}(\mathbb{P}(T_{\mathbb{P}^2}), \omega_0) = 8$, we have

$$\Lambda_{\omega_0}(\Omega(s)) = \frac{m}{16}.$$

Therefore, given $\mathbf{E} = \mathbf{E}(s_1, \dots, s_r)$, as before, we can take a Hermitian structure H on \mathbf{E} , such that the curvature $F(H)$ of the associated Chern connection $\nabla^H \stackrel{\text{loc}}{=} d + H^{-1} \partial H$ satisfies

$$\frac{\sqrt{-1}}{2\pi} F(H) = \text{diag} \{ \Omega(s_1), \dots, \Omega(s_r) \}.$$

From above, we obtain

$$\sqrt{-1} \Lambda_{\omega_0}(F(H)) = 2\pi \text{diag} \{ \Lambda_{\omega_0}(\Omega(s_1)), \dots, \Lambda_{\omega_0}(\Omega(s_r)) \} = \frac{m\pi}{8} \mathbb{1}_{\mathbf{E}}.$$

Hence, we conclude that $\nabla^H \stackrel{\text{loc}}{=} d + H^{-1} \partial H$ is a Hermitian Yang–Mills connection. We can still go one step further to describe the Hermitian Yang–Mills instanton ∇^H in a quite explicitly way. In fact, given an open set $U \subset \mathbb{P}(T_{\mathbb{P}^2})$ which trivializes both $\mathbf{E} \rightarrow \mathbb{P}(T_{\mathbb{P}^2})$ and $B \hookrightarrow \text{SL}_3(\mathbb{C}) \rightarrow \mathbb{P}(T_{\mathbb{P}^2})$, and considering fiber coordinates (w_1, \dots, w_r) in $\mathbf{E}|_U$, we can construct H by gluing the local Hermitian structures

$$H_U = \sum_{j=1}^r \frac{w_j \bar{w}_j}{\|s_U v_{\omega_{\alpha_1}}^+\|^{2a(s_j)} \|s_U v_{\omega_{\alpha_2}}^+\|^{2b(s_j)}},$$

for some local section $s_U: U \subset \mathbb{P}(T_{\mathbb{P}^2}) \rightarrow \text{SL}_3(\mathbb{C})$, here we consider $\|\cdot\|$ defined by some fixed $\text{SU}(3)$ -invariant inner product on $V(\varpi_{\alpha_k})$, $k = 1, 2$. From this, we have

$$\nabla^H|_U = d + \text{diag} \{ A_U^{(1)}, \dots, A_U^{(r)} \},$$

such that

$$A_U^{(j)} = -\partial \log (\|s_U v_{\varpi_{\alpha_1}}^+\|^{2a(s_j)} \|s_U v_{\varpi_{\alpha_2}}^+\|^{2b(s_j)}), \quad \forall j = 1, \dots, r.$$

In particular, consider $U = U^-(B)$, such that

$$U^-(B) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_2 & z_3 & 1 \end{pmatrix} B \mid z_1, z_2, z_3 \in \mathbb{C} \right\} \quad (\text{opposite big cell}),$$

see Remark 2.15. By taking the local section $s_U: U^-(B) \rightarrow \mathrm{SL}_3(\mathbb{C})$, such that $s_U(nB) = n$, $\forall nB \in U^-(B)$, and observing that

$$V(\varpi_{\alpha_1}) = \mathbb{C}^3 \quad \text{and} \quad V(\varpi_{\alpha_2}) = \bigwedge^2(\mathbb{C}^3),$$

where $v_{\varpi_{\alpha_1}}^+ = e_1$, and $v_{\varpi_{\alpha_2}}^+ = e_1 \wedge e_2$, fixed $\|\cdot\|$ defined by the standard $\mathrm{SU}(3)$ -invariant inner product on \mathbb{C}^3 and $\bigwedge^2(\mathbb{C}^3)$, we obtain

$$A_U^{(j)} = -\partial \log \left[\left(1 + \sum_{i=1}^2 |z_i|^2 \right)^{a(s_j)} \left(1 + |z_3|^2 + \left| \det \begin{pmatrix} z_1 & 1 \\ z_2 & z_3 \end{pmatrix} \right|^2 \right)^{b(s_j)} \right],$$

for each $j = 1, \dots, r$. From above we obtain infinitely many explicit examples of Hermitian Yang–Mills connections.

A. Line bundles with prescribed slope

Fixed some integral Kähler class $[\omega_0] \in \mathcal{K}(X_P)$ (Remark 2.18), and fixed some integer number $m_0 \in \mathbb{Z}$, in this appendix we investigate the problem related to the solvability of the equation

$$\mu_{[\omega_0]}(\mathbf{E}) = m_0, \quad \mathbf{E} \in \mathrm{Pic}(X_P). \quad (\text{A.1})$$

The main purpose is to describe the constraints which should be imposed on ω_0 and m_0 to ensure solvability of (A.1). As we shall see, the investigation of this problem leads us to a fruitful interaction between elementary number theory and Schubert calculus.

A.1. Proof of Theorem D

Proposition A.1. *Given $m_0 \in \mathbb{Z}$ and an integral Kähler class $[\omega_0] \in \mathcal{K}(X_P)$, then the equation*

$$\mu_{[\omega_0]}(\mathbf{E}) = m_0, \quad \mathbf{E} \in \mathrm{Pic}(X_P),$$

has a solution if, and only if, $\tau([\omega_0]) \mid m_0$, such that

$$\tau([\omega_0]) := \mathrm{gcd} \left\{ \int_{X_P} c_1(\mathcal{O}_\alpha(1)) \wedge [\omega_0]^{n-1} \mid \alpha \in \Delta \setminus I \right\},$$

where $\mathcal{O}_\alpha(1)$, $\alpha \in \Delta \setminus I$, are the generators of $\mathrm{Pic}(X_P)$.

Proof. Given a flag variety X_P , fixed an integral Kähler class $[\omega_0] \in \mathcal{K}(X_P)$, and fixed some $m_0 \in \mathbb{Z}$, we seek for solutions of the equation

$$\mu_{[\omega_0]}(\mathbf{E}) = m_0, \quad \mathbf{E} \in \text{Pic}(X_P).$$

Considering $\mathbf{E} = \bigotimes_{\alpha \in \Delta \setminus I} \mathcal{O}_\alpha(x_\alpha)$, such that $x_\alpha \in \mathbb{Z}$, $\forall \alpha \in \Delta \setminus I$, the equation above turns out to be the linear equation

$$\sum_{\alpha \in \Delta \setminus I} \deg_{\omega_0}(\mathcal{O}_\alpha(1))x_\alpha = m_0, \quad (\text{A.2})$$

such that $\deg_{\omega_0}(\mathcal{O}_\alpha(1)) = \int_{X_P} c_1(\mathcal{O}_\alpha(1)) \wedge [\omega_0]^{n-1}$, $\forall \alpha \in \Delta \setminus I$. Since $c_1(\mathcal{O}_\alpha(1))$, $\alpha \in \Delta \setminus I$, and $[\omega_0]$ are integral classes, it follows that

$$\deg_{\omega_0}(\mathcal{O}_\alpha(1)) = \int_{X_P} c_1(\mathcal{O}_\alpha(1)) \wedge [\omega_0] \wedge [\omega_0]^{n-2} = \mathcal{Q}_{\omega_0}(c_1(\mathcal{O}_\alpha(1)), [\omega_0]) \in \mathbb{Z},$$

$\forall \alpha \in \Delta \setminus I$, where $\mathcal{Q}_{\omega_0}: H^2(X_P, \mathbb{Z}) \times H^2(X_P, \mathbb{Z}) \rightarrow \mathbb{Z}$ is the Hodge–Riemann bilinear form associated with the underlying polarized Hodge structure (e.g., [50, 62]). Thus, it follows that (A.2) is equivalent to the linear Diophantine equation

$$\sum_{\alpha \in \Delta \setminus I} \mathcal{Q}_{\omega_0}(c_1(\mathcal{O}_\alpha(1)), [\omega_0])x_\alpha = m_0.$$

Since the linear Diophantine equation above admits a solution if, and only if,

$$\gcd \{ \mathcal{Q}_{\omega_0}(c_1(\mathcal{O}_\alpha(1)), [\omega_0]) \mid \alpha \in \Delta \setminus I \} \mid m_0,$$

see for instance [48, Chapter 5], [49, Section 1.6], we obtain the desired result. \blacksquare

Remark A.2. From the above result, we have that $\tau([\omega_0]) \mid \mu_{[\omega_0]}(\mathbf{E})$, $\forall \mathbf{E} \in \text{Pic}(X_P)$.

Remark A.3. Notice that, by the generalized Bezout’s identity [49, Section 1.2], we have

$$\tau([\omega_0])\mathbb{Z} = \bigoplus_{\alpha \in \Delta \setminus I} \mathcal{Q}_{\omega_0}(\boldsymbol{\Omega}_\alpha, \omega_0)\mathbb{Z},$$

here we have used the fact that

$$\mathcal{Q}_{\omega_0}(\boldsymbol{\Omega}_\alpha, \omega_0) := \int_{X_P} \boldsymbol{\Omega}_\alpha \wedge \omega_0^{n-1} = \mathcal{Q}_{\omega_0}(c_1(\mathcal{O}_\alpha(1)), [\omega_0]),$$

for every $\alpha \in \Delta \setminus I$.

Remark A.4. In the setting of Proposition A.1, given a prime number $p \in \mathbb{Z}$, the equation

$$\mu_{[\omega_0]}(\mathbf{E}) = p, \quad \mathbf{E} \in \text{Pic}(X_P),$$

is solvable if, and only if, $\tau([\omega_0]) = p$ or $\tau([\omega_0]) = 1$.

Remark A.5. Notice that $\mathcal{O}_\alpha(1) = \mathcal{O}(D_\alpha)$, for every Schubert divisor $D_\alpha \in \text{Div}(X_P)$, $\alpha \in \Delta \setminus I$. Thus, fixed an integral Kähler class $[\omega_0] \in \mathcal{K}(X_P)$, we have

$$\deg_{\omega_0}(\mathcal{O}_\alpha(1)) = \int_{D_\alpha} \omega_0^{n-1} = \langle [\omega_0^{n-1}], [D_\alpha] \rangle.$$

For every integer n and every prime number $p \in \mathbb{Z}$, let $v_p(n)$ be the greatest integer, such that $p^{v_p(n)} \mid n$. From the fundamental theorem of arithmetic, it follows that

$$\tau([\omega_0]) = \prod_{p \mid \tau([\omega_0])} p^{\min\{v_p(\langle [\omega_0^{n-1}], [D_\alpha] \rangle) \mid \alpha \in \Delta \setminus I\}},$$

see for instance [49, Theorem 1.11]. Hence, the prime factorization of $\tau([\omega_0])$ is completely determined by the intersection numbers $\langle [\omega_0^{n-1}], [D_\alpha] \rangle$, $\alpha \in \Delta \setminus I$.

Now we are in position to prove Theorem D.

Proof of Theorem D. From Proposition A.1, we have

$$\mathcal{A}_{[\omega_0]}(n) = \{m \in \mathbb{Z}_{>0} \mid 1 \leq m \leq n, \tau([\omega_0]) \mid m\},$$

$\forall n \in \mathbb{Z}_{>0}$. Given $n \in \mathbb{Z}_{>0}$, from the division algorithm, there exist unique integers $t(n)$ and $r(n)$, such that

$$n = \tau([\omega_0])t(n) + r(n), \quad 0 \leq r(n) < \tau([\omega_0]).$$

Therefore, for every $n \in \mathbb{Z}_{>0}$, we have

$$|\mathcal{A}_{[\omega_0]}(n)| = t(n) = \frac{n - r(n)}{\tau([\omega_0])}.$$

Since $0 \leq r(n) < \tau([\omega_0])$, it follows that

$$0 \leq \frac{1}{\tau([\omega_0])} - \frac{|\mathcal{A}_{[\omega_0]}(n)|}{n} < \frac{1}{n}.$$

Hence, $\frac{|\mathcal{A}_{[\omega_0]}(n)|}{n} \rightarrow \frac{1}{\tau([\omega_0])}$ as $n \rightarrow +\infty$. ■

Remark A.6. The above result shows that the subset

$$\mathcal{A}_{[\omega_0]} = \{m \in \mathbb{Z}_{>0} \mid \mu_{[\omega_0]}(\mathbf{E}) = m, \text{ for some } \mathbf{E} \in \text{Pic}(X_P)\},$$

has asymptotic density (e.g., [49, Chapter 16])

$$d(\mathcal{A}_{[\omega_0]}) := \lim_{n \rightarrow +\infty} \frac{|\mathcal{A}_{[\omega_0]} \cap [1, n]|}{n} = \frac{1}{\tau([\omega_0])}.$$

Remark A.7 (Hodge–Riemann bilinear form). In the setting of Proposition A.2, if we consider $\Delta \setminus I = \{\alpha_1, \dots, \alpha_\rho\}$, where ρ is the Picard number of X_P , we have

$$\deg_{\omega_0}(\mathbf{E}) = \mathcal{Q}_{\omega_0}(c_1(\mathbf{E}), [\omega_0]) = \sum_{i,j=1}^{\rho} \langle c_1(\mathbf{E}), [\mathbb{P}_{\alpha_i}^1] \rangle \langle [\omega_0], [\mathbb{P}_{\alpha_j}^1] \rangle \underbrace{\mathcal{Q}_{\omega_0}(\boldsymbol{\Omega}_{\alpha_i}, \boldsymbol{\Omega}_{\alpha_j})}_{(\mathcal{Q}_{\omega_0})_{ij}},$$

such that

$$(\mathcal{Q}_{\omega_0})_{ij} = \int_{X_P} \boldsymbol{\Omega}_{\alpha_i} \wedge \boldsymbol{\Omega}_{\alpha_j} \wedge \omega_0^{n-2},$$

$\forall i, j = 1, \dots, \rho$. From [57, Lemma 4.7], we have

$$(\mathcal{Q}_{\omega_0})_{ij} = (n-2)! (\Lambda_{\omega_0}(\boldsymbol{\Omega}_{\alpha_i}) \Lambda_{\omega_0}(\boldsymbol{\Omega}_{\alpha_j}) - \langle \boldsymbol{\Omega}_{\alpha_i}, \boldsymbol{\Omega}_{\alpha_j} \rangle_{\omega_0}) \text{Vol}(X_P, \omega_0),$$

$\forall i, j = 1, \dots, \rho$. By following Remark 2.24, it follows that

$$\Lambda_{\omega_0}(\boldsymbol{\Omega}_{\alpha}) = \sum_{\beta \in \Phi_+^+} \frac{\langle \varpi_{\alpha}, \beta^{\vee} \rangle}{\langle \lambda([\omega_0]), \beta^{\vee} \rangle}, \quad \langle \boldsymbol{\Omega}_{\alpha}, \boldsymbol{\Omega}_{\gamma} \rangle_{\omega_0} = \sum_{\beta \in \Phi_+^+} \frac{\langle \varpi_{\alpha}, \beta^{\vee} \rangle}{\langle \lambda([\omega_0]), \beta^{\vee} \rangle} \frac{\langle \varpi_{\gamma}, \beta^{\vee} \rangle}{\langle \lambda([\omega_0]), \beta^{\vee} \rangle},$$

$\forall \alpha, \gamma \in \Delta \setminus I$. Moreover, from (2.19), we have

$$\text{Vol}(X_P, \omega_0) = \prod_{\beta \in \Phi_+^+} \frac{\langle \lambda([\omega_0]), \beta^{\vee} \rangle}{\langle \varrho^+, \beta^{\vee} \rangle}.$$

Therefore, the Hodge–Riemann bilinear form $\mathcal{Q}_{\omega_0}: H^2(X_P, \mathbb{Z}) \times H^2(X_P, \mathbb{Z}) \rightarrow \mathbb{Z}$ is completely determined by the relation between fundamental weights and simple roots provided by the Cartan matrix of $\mathfrak{g}^{\mathbb{C}}$. Moreover, fixed an integral Kähler class $\omega_0 = \sum_{\alpha \in \Delta \setminus I} s_{\alpha} \boldsymbol{\Omega}_{\alpha}$, and fixed $m_0 \in \mathbb{Z}$, the equation $\mu_{[\omega_0]}(\mathbf{E}) = m_0$, $\mathbf{E} \in \text{Pic}(X_P)$, is equivalent to

$$\left(\int_{\mathbb{P}_{\alpha_1}^1} c_1(\mathbf{E}) \cdots \int_{\mathbb{P}_{\alpha_{\rho}}^1} c_1(\mathbf{E}) \right) \begin{pmatrix} (\mathcal{Q}_{\omega_0})_{11} & \cdots & (\mathcal{Q}_{\omega_0})_{1\rho} \\ \vdots & \ddots & \vdots \\ (\mathcal{Q}_{\omega_0})_{\rho 1} & \cdots & (\mathcal{Q}_{\omega_0})_{\rho\rho} \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_{\rho} \end{pmatrix} = m_0, \quad \mathbf{E} \in \text{Pic}(X_P). \quad (\text{A.3})$$

It is worth pointing out that, if $\mathbf{F} \in \text{Pic}(X_P)$ satisfies $\mu_{[\omega_0]}(\mathbf{F}) = m_0$, then all solutions of (A.3) can be written as $\mathbf{F} \otimes \mathbf{G}$, where $\mathbf{G} \in \text{Pic}(X_P)$ is a solution of the associated homogeneous problem $\mu_{[\omega_0]}(\mathbf{E}) = 0$, $\mathbf{E} \in \text{Pic}(X_P)$.

Let us illustrate the ideas developed above by means of an example.

Example A.8. Let $X_P = \mathbb{P}(T_{\mathbb{P}^2})$ as in Example 4.1. Given some integral Kähler metric $\omega_0 = s_1 \boldsymbol{\Omega}_{\alpha_1} + s_2 \boldsymbol{\Omega}_{\alpha_2}$, a straightforward computation shows us that

- (1) $(\mathcal{Q}_{\omega_0})_{11} = 1! \left[\left(\frac{1}{s_1} + \frac{1}{s_1+s_2} \right)^2 - \left(\frac{1}{s_1^2} + \frac{1}{(s_1+s_2)^2} \right) \right] \frac{s_1 s_2 (s_1+s_2)}{2} = s_2$;
- (2) $(\mathcal{Q}_{\omega_0})_{22} = 1! \left[\left(\frac{1}{s_2} + \frac{1}{s_1+s_2} \right)^2 - \left(\frac{1}{s_2^2} + \frac{1}{(s_1+s_2)^2} \right) \right] \frac{s_1 s_2 (s_1+s_2)}{2} = s_1$;
- (3) $(\mathcal{Q}_{\omega_0})_{12} = (\mathcal{Q}_{\omega_0})_{21} = 1! \left[\left(\frac{1}{s_1} + \frac{1}{s_1+s_2} \right) \left(\frac{1}{s_2} + \frac{1}{s_1+s_2} \right) - \frac{1}{(s_1+s_2)^2} \right] \frac{s_1 s_2 (s_1+s_2)}{2} = s_1 + s_2$.

Therefore, for an arbitrary integral class Kähler metric $\omega_0 = s_1 \Omega_{\alpha_1} + s_2 \Omega_{\alpha_2}$, we have

$$\mathcal{Q}_{\omega_0} = \begin{pmatrix} s_2 & s_1 + s_2 \\ s_1 + s_2 & s_1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}). \quad (\text{A.4})$$

From above, the general equation associated with the problem $\mu_{[\omega_0]}(\mathbf{E}) = m_0$, such that $m_0 \in \mathbb{Z}$, and $\mathbf{E} \in \mathrm{Pic}(\mathbb{P}(T_{\mathbb{P}^2}))$, can be explicitly written as follows:

$$(s_1 s_2 + s_2(s_1 + s_2))x_1 + (s_1 s_2 + s_1(s_1 + s_2))x_1 = m_0, \quad (\text{A.5})$$

where $\mathbf{E} = \mathcal{O}_{\alpha_1}(x_1) \otimes \mathcal{O}_{\alpha_2}(x_2) \in \mathrm{Pic}(\mathbb{P}(T_{\mathbb{P}^2}))$, cf. (A.3). Notice that, in the particular case that $s_1 = s_2 = 2$, we recover (4.3). From Proposition A.1, we conclude that the linear Diophantine equation (A.5) can be solved if, and only if,

$$\tau([\omega_0]) = \mathrm{gcd}\{s_1 s_2 + s_2(s_1 + s_2), s_1 s_2 + s_1(s_1 + s_2)\} | m_0.$$

As in Example 4.1, the above ideas allow us to construct several explicit non-trivial examples of Hermitian Yang–Mills instantons. In addition, from Theorem D, for every integral Kähler class $[\omega_0] \in \mathcal{K}(\mathbb{P}(T_{\mathbb{P}^2}))$, such that $\omega_0 = s_1 \Omega_{\alpha_1} + s_2 \Omega_{\alpha_2}$, we have the natural density of $\mathcal{A}_{[\omega_0]}$ given by

$$d(\mathcal{A}_{[\omega_0]}) = \lim_{n \rightarrow +\infty} \frac{|\mathcal{A}_{[\omega_0]} \cap [1, n]|}{n} = \frac{1}{\tau(s_1, s_2)}, \quad (\text{A.6})$$

such that $\tau(s_1, s_2) := \mathrm{gcd}\{s_1 s_2 + s_2(s_1 + s_2), s_1 s_2 + s_1(s_1 + s_2)\}$. Notice that, since $\tau(s_1, s_2)$ is symmetric in the variables s_1 and s_2 , we can easily produce examples of integral Kähler classes $[\omega_1], [\omega_2] \in \mathcal{K}(\mathbb{P}(T_{\mathbb{P}^2}))$, satisfying $\tau([\omega_1]) = \tau([\omega_2])$, with $[\omega_1] \neq [\omega_2]$.

A.2. Proof of Theorem E

In order to prove Theorem E, we introduce some basic results concerned with primitive $(1, 1)$ -forms on flag varieties.

Remark A.9 (Primitive 2-forms). Let X_P be a complex flag variety with Picard number $\rho(X_P) > 1$. Fixed some integral Kähler class $[\omega_0] \in \mathcal{K}(X_P)$, and considering Λ_{ω_0} , i.e., the dual of the associated Lefschetz operator, we can describe explicitly the primitive submodule

$$H_{\omega_0}^2(X_P, \mathbb{Z})_{\mathrm{prim}} := \ker(\Lambda_{\omega_0}: H^2(X_P, \mathbb{R}) \rightarrow \mathbb{R}) \cap H^2(X_P, \mathbb{Z}),$$

in terms of the Cartan matrix of the complex simple Lie algebra underlying X_P . In fact, keeping the notation of Remark A.7, given $[\psi] \in H^2(X_P, \mathbb{Z})$, we have

$$\Lambda_{\omega_0}([\psi]) = \sum_{\alpha \in \Delta \setminus I} \Lambda_{\omega_0}([\Omega_{\alpha}])x_{\alpha},$$

such that $x_{\alpha} = \langle \lambda([\psi]), [\mathbb{P}^1_{\alpha}] \rangle$, $\forall \alpha \in \Delta \setminus I$. Since

$$(n-1)! \Lambda_{\omega_0}([\Omega_{\alpha}]) \mathrm{Vol}(X_P, \omega_0) = \mathcal{Q}_{\omega_0}(\Omega_{\alpha}, \omega_0),$$

$\forall \alpha \in \Delta \setminus I$, we have

$$\Lambda_{\omega_0}([\psi]) = 0 \iff \sum_{\alpha \in \Delta \setminus I} \frac{\mathcal{Q}_{\omega_0}(\mathbf{\Omega}_\alpha, \omega_0)}{\tau([\omega_0])} x_\alpha = 0.$$

Denoting $q_\alpha(\omega_0) = \frac{\mathcal{Q}_{\omega_0}(\mathbf{\Omega}_\alpha, \omega_0)}{\tau([\omega_0])}$, $\forall \alpha \in \Delta \setminus I$, and taking some $\gamma \in \Delta \setminus I$, we obtain a \mathbb{Z} -basis for $H_{\omega_0}^2(X_P, \mathbb{Z})_{\text{prim}}$ by setting

$$\xi_\alpha := -q_\alpha(\omega_0)[\mathbf{\Omega}_\gamma] + q_\gamma(\omega_0)[\mathbf{\Omega}_\alpha], \quad (\text{A.7})$$

for all $\forall \alpha \in \Delta \setminus I$, such that $\alpha \neq \gamma$. From this, we have

$$H_{\omega_0}^2(X_P, \mathbb{Z})_{\text{prim}} = \bigoplus_{\alpha \in \Delta \setminus I, \alpha \neq \gamma} \mathbb{Z} \xi_\alpha.$$

By construction, we obtain a \mathcal{Q}_{ω_0} -orthogonal decomposition

$$H^2(X_P, \mathbb{Z}) = \mathbb{Z}[\omega_0] \oplus H_{\omega_0}^2(X_P, \mathbb{Z})_{\text{prim}}.$$

If we consider the extension $\mathcal{Q}_{\omega_0}: H^2(X_P, \mathbb{Q}) \times H^2(X_P, \mathbb{Q}) \rightarrow \mathbb{Q}$, then we can apply the Gram–Schmidt process w.r.t. \mathcal{Q}_{ω_0} on the basis ξ_α , $\alpha \in \Delta \setminus I$, $\alpha \neq \gamma$, in order to obtain a complete decomposition of $H^2(X_P, \mathbb{Q})$ as a direct sum of \mathcal{Q}_{ω_0} -orthogonal \mathbb{Q} -subspaces. From (A.7), we can describe explicitly a set of generators for the kernel of the homomorphism $\mu_{[\omega_0]}: \text{Pic}(X_P) \rightarrow \mathbb{Z}$. In fact, denoting

$$\text{Pic}_{\omega_0}^0(X_P) = \{\mathbf{E} \in \text{Pic}(X_P) \mid \mu_{[\omega_0]}(\mathbf{E}) = 0\},$$

since $\text{Pic}_{\omega_0}^0(X_P) \cong H_{\omega_0}^2(X_P, \mathbb{Z})_{\text{prim}}$, we have that

$$\mathcal{O}_\gamma(-q_\alpha(\omega_0)) \otimes \mathcal{O}_\alpha(q_\gamma(\omega_0)), \quad \forall \alpha \in \Delta \setminus I, \alpha \neq \gamma,$$

define a set of generators for $\text{Pic}_{\omega_0}^0(X_P)$. Moreover, consider the finitely generated subgroup $H_{\omega_0} \subset \text{Hom}(P, \mathbb{C}^\times)$, such that

$$H_{\omega_0} := \{\vartheta_{\varpi_\gamma}^{-q_\alpha(\omega_0)} \vartheta_{\varpi_\alpha}^{q_\gamma(\omega_0)} \mid \alpha \in \Delta \setminus I, \alpha \neq \gamma\},$$

recall that $\text{Hom}(P, \mathbb{C}^\times) = \text{Hom}(T(\Delta \setminus I)^{\mathbb{C}}, \mathbb{C}^\times)$, and $(d\vartheta_{\varpi_\alpha})_e = \varpi_\alpha$, $\forall \alpha \in \Delta$. The subgroup H_{ω_0} constructed above completes the following commutative diagram:

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H_{\omega_0} & \xrightarrow{\iota} & \text{Hom}(P, \mathbb{C}^\times) & \xrightarrow{\hat{\mu}_{[\omega_0]}} & \tau([\omega_0])\mathbb{Z} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \text{Pic}_{\omega_0}^0(X_P) & \xrightarrow{\iota} & \text{Pic}(X_P) & \xrightarrow{\mu_{[\omega_0]}} & \tau([\omega_0])\mathbb{Z} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & \end{array}$$

In the top line of the above diagram we consider $\hat{\mu}_{[\omega_0]}: \text{Hom}(P, \mathbb{C}^\times) \rightarrow \tau([\omega_0])\mathbb{Z}$, such that

$$\hat{\mu}_{[\omega_0]}(\vartheta) := (n-1)! \left[\sum_{\beta \in \Phi_I^+} \frac{\langle (d\vartheta)_e, \beta^\vee \rangle}{\langle \lambda([\omega_0]), \beta^\vee \rangle} \right] \left[\prod_{\beta \in \Phi_I^+} \frac{\langle \lambda([\omega_0]), \beta^\vee \rangle}{\langle \varrho^+, \beta^\vee \rangle} \right].$$

$\forall \vartheta \in \text{Hom}(P, \mathbb{C}^\times)$. Thus, from Theorem B, we have $\hat{\mu}_{[\omega_0]}(\vartheta) = \mu_{[\omega_0]}(\mathbf{E}_\vartheta)$, where $\mathbf{E}_\vartheta \in \text{Pic}(X_P)$ is the holomorphic line bundle defined by $\vartheta \in \text{Hom}(P, \mathbb{C}^\times)$. Notice that Proposition A.1 ensures that both $\hat{\mu}_{[\omega_0]}$ and $\mu_{[\omega_0]}$ are surjective homomorphisms.

Example A.10. As before, consider the case that $X_P = \mathbb{P}(T_{\mathbb{P}^2})$. Given some integral Kähler class $[\omega_0] = s_1[\Omega_{\alpha_1}] + s_2[\Omega_{\alpha_2}]$, it follows from (A.4) that

$$(1) \quad \mathcal{Q}_{\omega_0}(\Omega_{\alpha_1}, \omega_0) = (1 \ 0) \begin{pmatrix} s_2 & s_1+s_2 \\ s_1+s_2 & s_1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = s_1s_2 + s_2(s_1 + s_2);$$

$$(2) \quad \mathcal{Q}_{\omega_0}(\Omega_{\alpha_2}, \omega_0) = (0 \ 1) \begin{pmatrix} s_2 & s_1+s_2 \\ s_1+s_2 & s_1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = s_1s_2 + s_1(s_1 + s_2).$$

Let us denote $q_j(\omega_0) = \frac{\mathcal{Q}_{\omega_0}(\Omega_{\alpha_j}, \omega_0)}{\tau([\omega_0])}$, $j = 1, 2$. From above, we have that

$$\xi = -q_2(\omega_0)[\Omega_{\alpha_1}] + q_1(\omega_0)[\Omega_{\alpha_2}],$$

generates $H_{\omega_0}^2(\mathbb{P}(T_{\mathbb{P}^2}), \mathbb{Z})_{\text{prim}}$. Hence, we have the \mathcal{Q}_{ω_0} -orthogonal decomposition

$$H^2(\mathbb{P}(T_{\mathbb{P}^2}), \mathbb{Z}) = \mathbb{Z}[\omega_0] \oplus \mathbb{Z}\xi.$$

In this particular case, we have

$$H_{\omega_0} = \left\{ \vartheta \frac{-nq_2(\omega_0)}{\varpi_{\alpha_1}} \vartheta \frac{nq_1(\omega_0)}{\varpi_{\alpha_2}} \mid n \in \mathbb{Z} \right\} = \left\langle \vartheta \frac{-q_2(\omega_0)}{\varpi_{\alpha_1}} \vartheta \frac{q_1(\omega_0)}{\varpi_{\alpha_2}} \right\rangle.$$

If we consider $\omega_0 \in c_1(\mathbb{P}(T_{\mathbb{P}^2}))$ as in Example 4.1, we obtain $q_1(\omega_0) = q_2(\omega_0) = 1$. Thus, it follows that

$$H_{\omega_0} = \left\{ \vartheta \frac{-n}{\varpi_{\alpha_1}} \vartheta \frac{n}{\varpi_{\alpha_2}} \mid n \in \mathbb{Z} \right\} = \left\langle \vartheta \frac{-1}{\varpi_{\alpha_1}} \vartheta \frac{1}{\varpi_{\alpha_2}} \right\rangle.$$

In particular, we obtain

$$\text{Pic}_{\omega_0}^0(\mathbb{P}(T_{\mathbb{P}^2})) = \left\{ \mathcal{O}_{\alpha_1}(-n) \otimes \mathcal{O}_{\alpha_2}(n) \mid n \in \mathbb{Z} \right\}.$$

The subgroups H_{ω_0} and $\text{Pic}_{\omega_0}^0(\mathbb{P}(T_{\mathbb{P}^2}))$ described above complete the following commutative diagram:

$$\begin{array}{ccccccc} & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H_{\omega_0} & \xrightarrow{\iota} & \text{Hom}(B, \mathbb{C}^\times) & \xrightarrow{\hat{\mu}_{[\omega_0]}} & 12\mathbb{Z} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \text{Pic}_{\omega_0}^0(\mathbb{P}(T_{\mathbb{P}^2})) & \xrightarrow{\iota} & \text{Pic}(\mathbb{P}(T_{\mathbb{P}^2})) & \xrightarrow{\mu_{[\omega_0]}} & 12\mathbb{Z} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & \end{array}$$

Building on the above ideas, we can construct examples of $[\omega_0]$ -polystable holomorphic vector bundles over $\mathbb{P}(T_{\mathbb{P}^2})$ in the following way: Given some $\mathbf{F}_0 \in \text{Pic}(\mathbb{P}(T_{\mathbb{P}^2}))$, take a collection of distinct elements $\mathbf{G}_1, \dots, \mathbf{G}_r \in \text{Pic}_{\omega_0}^0(\mathbb{P}(T_{\mathbb{P}^2}))$, and define

$$\mathbf{E} := \bigoplus_{j=1}^r (\mathbf{F}_0 \otimes \mathbf{G}_j).$$

By construction, we have that \mathbf{E} is $[\omega_0]$ -polystable. Notice that $\mu_{[\omega_0]}(\mathbf{E}) = \mu_{[\omega_0]}(\mathbf{F}_0)$. In summary, for every $m_0 \in 12\mathbb{Z}$ and every integer $r > 0$, there exists a $[\omega_0]$ -polystable holomorphic vector bundle \mathbf{E} over $\mathbb{P}(T_{\mathbb{P}^2})$, such that $\text{rank}(\mathbf{E}) = r$ and $\mu_{[\omega_0]}(\mathbf{E}) = m_0$.

From the ideas introduced in the above remark we can prove Theorem E.

Proof of Theorem E. Given $\mathbf{E} \in \text{Pic}(X_P)$, we have $\mathbf{E} = \bigotimes_{\alpha \in \Delta \setminus I} \mathcal{O}_{\alpha}(n_{\alpha})$, such that $n_{\alpha} \in \mathbb{Z}$, $\forall \alpha \in \Delta \setminus I$. Thus, if we define

$$\mathcal{E} := \bigoplus_{\alpha \in \Delta \setminus I} \mathcal{O}_{\alpha}(n_{\alpha}),$$

it follows that $[\mathcal{E}] \in K_0(X_P)$ and $\det([\mathcal{E}]) = \mathbf{E}$. Hence, $\det: K_0(X_P) \rightarrow \text{Pic}(X_P)$ is a surjective homomorphism. Denoting $SK_0(X_P) := \ker(\det: K_0(X_P) \rightarrow \text{Pic}(X_P))$, we have the short exact sequence of abelian groups

$$1 \longrightarrow SK_0(X_P) \xrightarrow{\iota} K_0(X_P) \xrightarrow{\det} \text{Pic}(X_P) \longrightarrow 1.$$

Since $\text{Pic}(X_P)$ is a free abelian group, the above short exact sequence of abelian groups splits, i.e.,

$$K_0(X_P) \cong SK_0(X_P) \oplus \text{Pic}(X_P).$$

From Remark A.9, we also have the following short exact sequence of abelian groups

$$1 \longrightarrow \text{Pic}_{\omega_0}^0(X_P) \xrightarrow{\iota} \text{Pic}(X_P) \xrightarrow{\deg_{\omega_0}} \tau([\omega_0])\mathbb{Z} \longrightarrow 1.$$

Since $\tau([\omega_0])\mathbb{Z} \subset \mathbb{Z}$ is also a free abelian group, the above short exact sequence of abelian groups also splits, thus

$$\text{Pic}(X_P) \cong \text{Pic}_{\omega_0}^0(X_P) \oplus \tau([\omega_0])\mathbb{Z}.$$

Combining the above facts, we obtain

$$K_0(X_P) \cong SK_0(X_P) \oplus \text{Pic}(X_P) \cong SK_0(X_P) \oplus \text{Pic}_{\omega_0}^0(X_P) \oplus \tau([\omega_0])\mathbb{Z}. \quad (\text{A.8})$$

As we have seen, fixed some $\gamma \in \Delta \setminus I$, it follows that

$$\text{Pic}_{\omega_0}^0(X_P) = \langle \mathcal{O}_{\gamma}(-q_{\alpha}(\omega_0)) \otimes \mathcal{O}_{\alpha}(q_{\gamma}(\omega_0)) \mid \alpha \in \Delta \setminus I, \alpha \neq \gamma \rangle,$$

such that $q_{\alpha}(\omega_0) = \frac{\mathcal{Q}_{\omega_0}(\Omega_{\alpha}, \omega_0)}{\tau([\omega_0])}$, $\forall \alpha \in \Delta \setminus I$, i.e., the generators of $\text{Pic}_{\omega_0}^0(X_P)$ are completely determined by the Hodge–Riemann bilinear form \mathcal{Q}_{ω_0} . \blacksquare

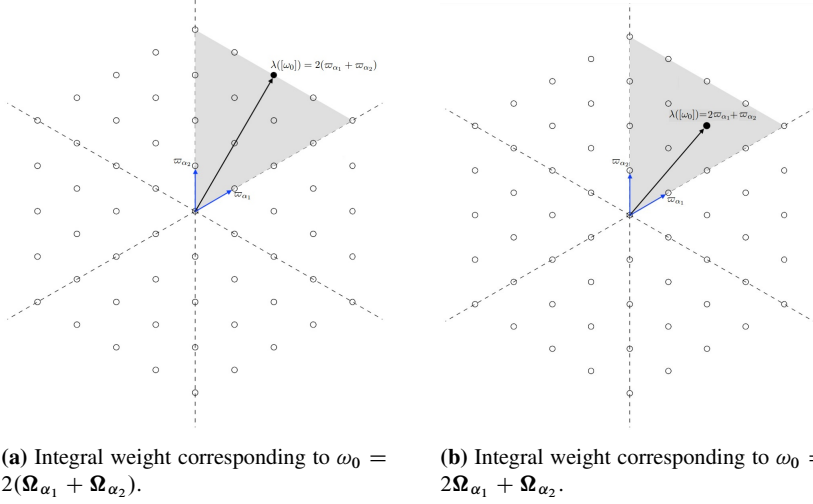


Figure 2. Integral weights defined by integral Kähler classes.

Remark A.11. In the setting of the above theorem, the decomposition provided in (A.8) also can be obtained from the following split exact sequences of abelian groups:

$$1 \longrightarrow \det^{-1}(\text{Pic}_{\omega_0}^0(X_P)) \xrightarrow{\iota} K_0(X_P) \xrightarrow{\deg_{\omega_0} \circ \det} \tau([\omega_0])\mathbb{Z} \longrightarrow 1$$

and

$$1 \longrightarrow SK_0(X_P) \xrightarrow{\iota} \det^{-1}(\text{Pic}_{\omega_0}^0(X_P)) \xrightarrow{\det} \text{Pic}_{\omega_0}^0(X_P) \longrightarrow 1.$$

Let us illustrate the results provided by the last theorems.

Example A.12. As before, consider $X_P = \mathbb{P}(T_{\mathbb{P}^2})$. Given $\omega_0 \in c_1(\mathbb{P}(T_{\mathbb{P}^2}))$, such that $\omega_0 = 2(\Omega_{\alpha_1} + \Omega_{\alpha_2})$, it follows from the previous computations (see Example A.10) that

$$K_0(\mathbb{P}(T_{\mathbb{P}^2})) \cong SK_0(\mathbb{P}(T_{\mathbb{P}^2})) \oplus \langle \mathcal{O}_{\alpha_1}(-1) \otimes \mathcal{O}_{\alpha_2}(1) \rangle \oplus 12\mathbb{Z}.$$

Let us consider the Kähler class $[\omega_0] \in \mathcal{K}(\mathbb{P}(T_{\mathbb{P}^2}))$, such that $\omega_0 = 2\Omega_{\alpha_1} + \Omega_{\alpha_2}$. From Example A.10, we obtain

$$\mathcal{Q}_{\omega_0}(\Omega_{\alpha_1}, \omega_0) = 5 \quad \text{and} \quad \mathcal{Q}_{\omega_0}(\Omega_{\alpha_2}, \omega_0) = 8.$$

Therefore, we have $\tau([\omega_0]) = 1$, so

$$K_0(\mathbb{P}(T_{\mathbb{P}^2})) \cong SK_0(\mathbb{P}(T_{\mathbb{P}^2})) \oplus \langle \mathcal{O}_{\alpha_1}(-8) \otimes \mathcal{O}_{\alpha_2}(5) \rangle \oplus \mathbb{Z}.$$

Furthermore, from (A.6), it follows that:

- (1) $d(\mathcal{A}_{[\omega_0]}) = \frac{1}{12}$, if $\omega_0 = 2(\Omega_{\alpha_1} + \Omega_{\alpha_2})$;
- (2) $d(\mathcal{A}_{[\omega_0]}) = 1$, if $\omega_0 = 2\Omega_{\alpha_1} + \Omega_{\alpha_2}$.

As it can be seen, if we consider $\omega_0 = 2\Omega_{\alpha_1} + \Omega_{\alpha_2}$, then the equation $\mu_{[\omega_0]}(\mathbf{E}) = m_0$, $\mathbf{E} \in \text{Pic}(\mathbb{P}(T_{\mathbb{P}^2}))$, can be solved for every $m_0 \in \mathbb{Z}$.

Acknowledgments. The author would like to thank the anonymous reviewer for their careful reading of the manuscript and for the valuable comments that helped improve this paper.

Funding. E. M. Correa was supported by São Paulo Research Foundation FAPESP grants 2022/10429-3 and 2025/18843-1.

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Communicated by Eckhard Meinrenken

Received 7 October 2024; revised 26 March 2026.

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