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Bratteli diagrams in Borel dynamics

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Abstract. Bratteli–Vershik models have been very successfully applied to the study of various dynamical systems, in particular, in Cantor dynamics. In this paper, we study dynamics on the path spaces of *generalized* Bratteli diagrams that form models for non-compact Borel dynamical systems. Generalized Bratteli diagrams have countably infinite many vertices at each level; thus, the corresponding incidence matrices are also countably infinite. We emphasize differences (and similarities) between generalized and classical Bratteli diagrams. Our main results are as follows. (i) We utilize Perron–Frobenius theory for countably infinite matrices to establish criteria for the existence and uniqueness of tail-invariant path space measures (both probability and σ-finite). (ii) We provide criteria for the topological transitivity of the tail equivalence relation. (iii) We describe classes of stationary generalized Bratteli diagrams (hence Borel dynamical systems) that (a) do not support a probability tail-invariant measure and (b) are not uniquely ergodic with respect to the tail equivalence relation. (iv) We describe classes of generalized Bratteli diagrams which can or cannot admit a continuous Vershik map and construct a Vershik map which is a minimal homeomorphism of a (non-locally compact) Polish space. (v) We provide an application of the theory of stochastic matrices to analyze diagrams with positive recurrent incidence matrices.

Contents

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1. Introduction

This paper is dedicated to the study of discrete dynamical systems realized on the path space of *generalized Bratteli diagrams*. A generalized Bratteli diagram is a natural extension of the notion of classical (standard) Bratteli diagrams where each level has a countably infinite set of vertices. The structure of such diagrams is determined by a sequence of countably infinite incidence matrices.

In general terms, a Bratteli diagram is a certain combinatorial structure which encompasses the following: The diagram takes the form of a graph which is represented as a countable union of levels, and a specification of edges between levels. The level count in turn is indexed or labeled by non-negative integers, and edges link only levels with index count differing by 1. For the last decades, such diagrams (graphs) have come to serve as a powerful tool for the analysis of path space, representing infinite paths via the particular graphs under consideration. The countable index of the levels may represent discrete time in associated models of dynamics. Our aim here is to extend earlier results on dynamics via diagrams, and associated path space constructions, to the broader context of Borel dynamics, and then to study the corresponding measures on these generalized path spaces. Our analysis will entail an extended Perron–Frobenius theory, a tail equivalence relation, the construction of corresponding tail-invariant measures, and finally a study of generalized Vershik maps.

This wide framework of path space analysis is motivated by a variety of applications. In this paper, we identify new properties of stationary and non-stationary generalized Bratteli diagrams. Our main results contribute to the following five important questions: (i) identify the structure and the properties of Vershik maps and (ii) the tail equivalence relation associated with generalized Bratteli diagrams; (iii) existence and uniqueness of tail-invariant measures, finite and σ-finite. In this context, we present an analysis of (iv) Bratteli diagrams with positive recurrent incidence matrices; and (v) path space measures induced by stochastic matrices. These and other results can be found in Theorems 3.10, 3.12, 5.1, 7.2, 7.3, 8.7, and 8.13.

The existing literature on Bratteli diagrams, corresponding dynamical systems, invariant path space measures, and other areas used in the paper is very extensive. We give the references below discussing the most important ingredients of our work. The reader can find the main ideas in [20,45,48,51,53,57,70], and other fundamental works cited below.

Our results may be interesting for mathematicians studying infinite matrices as well as for experts in Markov chains and random walks on a countable set. If all incidence matrices of a generalized Bratteli diagram are 0-1 matrices, then the path space of such a diagram is a well-known object in the theory of random walks on a countable set. It is worth noting that our focus is on the study of the tail equivalence relation which gives the principal different kind of dynamics on the path space. Sections 7 and 8 contain several key theorems and examples about infinite matrices, their eigenvectors, and corresponding invariant measures.

Bratteli diagrams, incidence matrices, and path spaces. We recall that Bratteli diagrams are infinite-graded graphs that were named after Ola Bratteli who introduced them in his pioneering paper [20] on the classification of approximately finite C^*-algebras. In short, a Bratteli diagram is a countable graph $G = (V, E)$ where vertices $V = \bigcup_n V_n$ and edges $E = \bigcup_n E_n$ are divided into disjoint finite sets (levels) V_n and E_n. The edges from the set E_n exist only for some vertices from consecutive levels V_n and V_{n+1}. This set of edges E_n defines a $|V_{n+1}| \times |V_n|$ matrix F_n called the incidence matrix. Every F_n has non-negative integer entries. In this paper, we consider the generalized Bratteli diagrams satisfying the property of finite row sums for every incidence matrix F_n. This means that every row has finitely many non-zero entries. For a Bratteli diagram B, the path space X_B is formed by infinite sequences (e_i) of edges such that e_{i+1} begins at the vertex where e_i ends.

Bratteli diagrams and dynamical systems. For the last decades, Bratteli diagrams have been intensively studied and used in various areas of dynamics. The trend to use discrete structures, such as graphs and sequences of partitions, proved to be a very powerful tool in the theory of dynamical systems, (see e.g., [40] and the papers cited below). In the 1970s, Krieger and Vershik applied sequences of refining partitions to the study of ergodic automorphisms of a measure space [26, 54, 68–70]. In particular, Vershik proved that any ergodic automorphism of a measure space can be realized as a transformation acting on the path space indexed by a sequence of refining partitions, a prototype of a Bratteli diagram.

At the beginning of the 1990s, these ideas found their new applications in Cantor dynamics. Putnam [59] showed that, for every minimal homeomorphism φ of a Cantor set X, there exists a sequence of refining partitions into clopen sets that approximates the orbits of φ and the topology on X. Following this article, Herman–Putnam–Skau [48] proved that every minimal homeomorphism of a Cantor set can be realized as a homeomorphism φ_B (called a Vershik map) of a path space X_B of a Bratteli diagram B. In other words, Bratteli diagrams represent models of minimal Cantor dynamical systems. This remarkable result led to a breakthrough in Cantor dynamics based on the works by Giordano–Putnam–Skau [45], Glasner–Weiss [47], and others. It was proved that minimal homeomorphisms could be completely classified with respect to orbit equivalence. The articles by Forrest [38] and Durand–Host–Skau [33] answered the question about the role of stationary Bratteli diagrams: They represent substitution dynamical systems. Further applications of Bratteli diagrams in Cantor dynamics extended the class of homeomorphisms that can be realized as Vershik maps. Medynets [56] proved this fact for aperiodic homeomorphisms of a Cantor set (see also [6]). Recently, the papers by Downarowicz–Karpel [29] and Shimomura [62, 63] showed that any homeomorphism of a Cantor set could be represented as a Vershik map on a Bratteli diagram. There are several important classes of Bratteli diagrams that deserve special attention. Stationary non-simple Bratteli diagrams give models for aperiodic substitution systems [14]. Finite-rank Bratteli diagrams (i.e., $|V_n|$ is bounded), which, in particular, represent interval exchange transformations, were studied in [16]. Finite and infinite invariant measures were the focus of the papers [2, 13]. Eigenvalues of Cantor minimal systems were considered in a series of papers (see, e.g., [32] for references). We do not discuss all the interesting applications here. The reader can find more results about invariant measures, dynamics, and applications in [3, 30, 34, 43, 44, 46, 60, 64]. See also the surveys [11, 12, 28, 31] and the literature mentioned there. The reader who is interested in the classification of stationary Bratteli diagrams, various links to operator algebras, and K-theory can find more information in [20–23, 35, 36].

Generalized Bratteli diagrams versus standard Bratteli diagrams. Why do we need generalized Bratteli diagrams with countable levels? One obvious reason to study such diagrams is explained by the following result. Bezuglyi–Dooley–Kwiatkowski [5] proved that every aperiodic Borel automorphism of an uncountable standard Borel space admits a realization as a Vershik map on the path space of a *generalized Bratteli diagram*. A recent result in this direction was obtained in [10] where the authors proved that there is a wide class of substitution dynamical systems on infinite alphabets that can be realized as Vershik maps on stationary generalized Bratteli diagrams. We also refer our readers to related recent works [39, 55]. Among other possible applications of generalized Bratteli diagrams, we can mention Markov chains, random walks, iterated function systems [9], harmonic analysis on the path space of generalized Bratteli diagrams [7], etc.

It is clear that the variety of classes of generalized Bratteli diagrams is much wider than that of standard Bratteli diagrams. This fact suggests the possibility of now establishing key results in the wider context of all generalized Bratteli diagrams. Another obvious observation is that the case of countably infinite matrices requires new techniques and methods in comparison with finite matrices. In this paper, we consider mostly two classes of diagrams: stationary diagrams (when $F_n = F$) and bounded size diagrams (when all F_n are banded matrices with bounded row sums) (see Section 2 for definitions). These diagrams have the predictable behavior of infinite paths and, therefore, we can better understand their dynamical properties. It is an interesting problem to find out how much the structure of a generalized Bratteli diagram determines dynamics and invariant measures on the path space of a diagram.

The main results of this paper are concentrated on principal problems in dynamics on the path spaces of Bratteli diagrams. We consider the existence and uniqueness of measures invariant with respect to the tail equivalence relation and Vershik maps. We discuss the dynamical properties of the Vershik map, and how the structure of a Bratteli diagram affects the dynamics on the path space. The notions of isomorphic and order isomorphic generalized Bratteli diagrams are considered in this paper.

Extended Perron–Frobenius theory, the tail equivalence relation, and Vershik map. Our main emphasis in the current work is to point out the differences and similarities between the dynamics on path space of the classical Bratteli diagrams and the generalized ones. The tools used in this work are also significantly different. For example, considering stationary Bratteli diagrams, we work with Perron–Frobenius eigenpairs. For classical diagrams, the Perron–Frobenius theory covers all possible cases of stationary Bratteli diagrams. Moreover, using eigenvectors and eigenvalues of non-negative matrices, one can explicitly describe all finite and σ-finite invariant measures for simple and non-simple stationary Bratteli diagrams [15]. For infinite non-negative matrices, the Perron–Frobenius theory does not cover all cases of generalized Bratteli diagrams: There are stationary diagrams with incidence matrices that do not have a finite Perron eigenvalue. Another important circumstance is that the cases of recurrent and transient incidence matrices lead to essentially different results regarding the uniqueness of ergodic invariant measures. This means that one needs to use different techniques for finding invariant measures. We use the book by Kitchens [53] for references about the definitions and main results of the Perron–Frobenius theory. For the reader's convenience, we included in Appendix A the facts that are used in this paper.

One more essential distinction between standard and generalized Bratteli diagrams consists of the existence of invariant measures on the path spaces. For a standard diagram B, the path space X_B is a compact Cantor set, and every Vershik map φ_B is a homeomorphism of X_B. Therefore, the classical Bogoliubov–Krylov theorem guarantees the existence of a probability φ_B-invariant measure on X_B. By contrast, for a generalized Bratteli diagram, X_B is a Polish zero-dimensional space, and (X_B, φ_B) is a Borel dynamical system. There are then generalized Bratteli diagrams that do not support finite invariant measures. The following question is natural: For what classes of generalized Bratteli diagrams are there finite invariant measures? We note that minimal Cantor dynamics deals with probability invariant measures only. The settings for generalized Bratteli diagrams lead to the study of both finite and infinite invariant measures. We construct and describe classes of generalized Bratteli diagrams that do not support a probability invariant measure.

On every Bratteli diagram B, we can consider dynamical systems of two kinds: the tail equivalence relation $\mathcal{R}$ and a Borel dynamical system (X_B, φ_B) defined by a Vershik map φ_B. Then $\mathcal{R}$ is a countable Borel equivalence relation which is completely defined by the diagram. To define a Vershik map φ_B, we need to consider a partial ordering on the set of all edges E. The question about the continuity of a Vershik map was studied in Cantor dynamics [17, 18, 50, 56]. The result of Bezuglyi–Dooley–Kwiatkowski [5] shows that every aperiodic Borel automorphism of an uncountable standard Borel space admits a realization as a Vershik map on the path space of an ordered generalized Bratteli diagram such that the Vershik map is a homeomorphism. The left-to-right ordering on a simple (standard) Bratteli diagram always gives rise to a continuous Vershik map. But this is not the case for generalized Bratteli diagrams, even for diagrams with reasonable simple structure (see Section 3). In other words, there are irreducible generalized Bratteli diagrams such that the left-to-right order does not generate a continuous Vershik map. Moreover, in the class of generalized Bratteli diagrams with a unique infinite minimal path and a unique infinite maximal path, one can find diagrams for which both Vershik map and its inverse are discontinuous, or Vershik map is continuous but its inverse is not (see Section 4).

Infinite matrices, especially banded matrices, is the subject of great interest because of their applications in various areas of mathematics and mathematical physics. In this context, we mention [4, 24, 25].

The outline of the paper and main results. In Section 2, we provide basic definitions and discuss the properties of generalized Bratteli diagrams. We consider such notions as tail equivalence relation, Vershik map, and isomorphism of Bratteli diagrams. In Section 3, we focus mostly on the notion of *bounded size* generalized Bratteli diagrams (see Definition 3.1) and show that they form a natural class of diagrams that can be viewed as intermediate between classical (standard) and generalized Bratteli diagrams. We show that even diagrams with this natural structure can provide interesting examples that contrast the classical case. We find classes of bounded size generalized Bratteli diagrams such that, for the left-to-right ordering, the Vershik map is discontinuous (unlike the case of standard Bratteli diagrams); moreover, every infinite maximal path is a point of discontinuity. We also provide some classes of bounded size diagrams for which it is possible to prolong the Vershik map to a homeomorphism. In Section 4, we consider arbitrary generalized Bratteli diagrams and find conditions under which they have an order such that the Vershik map can be prolonged to a homeomorphism. In particular, we are interested in the orders such that the diagram does not possess infinite minimal and infinite maximal paths. We give sufficient conditions for a generalized Bratteli diagram to have such an order. We give an example of an ordered generalized Bratteli diagram with a non-locally compact path space such that the corresponding Vershik map is a minimal homeomorphism. In contrast to the case of standard Bratteli diagrams, there are examples of generalized Bratteli diagrams with a unique minimal and a unique maximal path such that the corresponding Vershik map cannot be prolonged to a homeomorphism. We show that in the class of ordered generalized Bratteli diagrams with a unique infinite minimal path and a unique infinite maximal path, one can find examples of diagrams such that (i) both the Vershik map φ_B and its inverse φ_B^{-1} are not continuous; (ii) the Vershik map φ_B is continuous but the inverse φ_B^{-1} is discontinuous; and (iii) both the Vershik map φ_B and its inverse φ_B^{-1} are continuous. Section 5 discusses some topological properties of generalized Bratteli diagrams. It is proved that, for an irreducible stationary generalized Bratteli diagram, the tail equivalence relation is topologically transitive. The case of non-stationary diagrams is also considered. It is proved that irreducible bounded size generalized Bratteli diagrams do not generate minimal tail equivalence relations. In Section 6, we discuss the tail-invariant measures on the path space of generalized Bratteli diagrams. We provide explicit examples of non-stationary generalized Bratteli diagrams that do not support full probability measures invariant with respect to the Vershik map, and examples of stationary generalized Bratteli diagrams which do not support finite tail-invariant measures, that is, the tail equivalence relations for such diagrams are compressible. In Section 7, we study finite and σ-finite invariant measures on stationary generalized Bratteli diagrams. We give an explicit description of these measures and prove their uniqueness. Several examples that illustrate these theorems are also given in Section 7. The detailed calculations and proofs for these examples are provided in Appendix B. Section 8 deals with applications of stochastic matrices, related to stationary and non-stationary generalized Bratteli diagrams, to the problem of the existence of invariant measures. We give examples of null-recurrent incidence matrices such that the invariant measure for the corresponding generalized Bratteli diagrams is not unique. We also show that, for positive recurrent matrices, one can control the growth of the heights of the Kakutani–Rokhlin towers in terms of the corresponding eigenvectors. In Section 9, we present further possible directions of research. In Appendix A, we provide a brief description of the Perron–Frobenius theory for infinite matrices.

2. Generalized Bratteli diagrams: Basic definitions and facts

This section contains the main definitions of the objects we consider in the paper. We discuss generalized Bratteli diagrams and their subclasses, the corresponding infinite matrices, the tail equivalence relation on the path space of a diagram, finite and σ-finite measures invariant with respect to tail equivalence relation, and orderings on the set of edges and corresponding Vershik maps.

2.1. Definition of generalized Bratteli diagrams

The notion of a generalized Bratteli diagram is a natural extension of the notion of a Bratteli diagram to the case when all levels in a generalized Bratteli diagram are countable. It was proved in [5] that any aperiodic Borel automorphism can be realized as a Vershik map on the path space of a generalized Bratteli diagram (see Theorem 2.17).

We will use the standard notation $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for the sets of numbers, and $|\cdot|$ denotes the cardinality of a set.

Definition 2.1. A *generalized Bratteli diagram* is a graded graph $B = (V, E)$ such that the vertex set V and the edge set E can be represented as partitions $V = \bigsqcup_{i=0}^{\infty} V_i$ and $E = \bigsqcup_{i=0}^{\infty} E_i$ satisfying the following properties:

- (i) The number of vertices at each level V_i, $i \in \mathbb{N}_0$ is countably infinite (in most cases, we will identify each V_i with $\mathbb{Z}$ or $\mathbb{N}$). The set V_i is called the i-th level of the diagram B. For all $i \in \mathbb{N}_0$, the set E_i of all edges between V_i and V_{i+1} is countable.
- (ii) For every edge $e \in E$, we define the range and source maps r and s such that $r(E_i) = V_{i+1}$ and $s(E_i) = V_i$ for $i \in \mathbb{N}_0$. In particular, we have $s^{-1}(v) \neq \emptyset$ for all $v \in V$, and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$.
- (iii) For every vertex $v \in V \setminus V_0$, we have $|r^{-1}(v)| < \infty$.

Remark 2.2. If the level V_0 consists of a single vertex and each set V_n is finite, then we get the usual definition of a Bratteli diagram which was first defined in [20]. In particular, these diagrams were used to model Cantor dynamical systems and classify them up to orbit equivalence (see [45, 48]).

Since all levels in a generalized Bratteli diagram are countable sets, we can use the integers $\mathbb{Z}$ (or natural numbers $\mathbb{N}$) to enumerate the vertices. When we index the vertices at each level by $\mathbb{Z}$, the generalized diagram is called *two-sided infinite*, and when the vertices are indexed by $\mathbb{N}$, we call it *one-sided infinite*.

To define the path space of a generalized Bratteli diagram, we consider a finite or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $s(e_i) = r(e_{i-1})$ which is called a finite or infinite path, respectively. Given a generalized Bratteli diagram $B = (V, E)$, we denote the set of infinite paths starting at V_0 by X_B and call it the *path space*. For a finite path $\overline{e} = (e_0, \dots, e_n)$, we denote $s(\overline{e}) = s(e_0), r(\overline{e}) = r(e_n)$. The set

$$[\overline{e}] := \{x = (x_i) \in X_B : x_0 = e_0, \dots, x_n = e_n\},$$

is called the *cylinder set* associated with $\overline{e}$.

The *topology* on the path space X_B is generated by cylinder sets. This topology coincides with the topology defined by the following metric on X_B: for $x = (x_i),\ y = (y_i)$, set

$$\mathrm{dist}(x, y) = \frac{1}{2^N}, \quad N = \min\{i \in \mathbb{N}_0 : x_i \neq y_i\}.$$

The path space X_B is a zero-dimensional Polish space and therefore a standard Borel space.

For a vertex $v \in V_m$ and a vertex $w \in V_n$, we will denote by $E(v, w)$ the set of all finite paths between v and w. Set $f^{(i)}_{v,w} = |E(v, w)|$ for every $w \in V_i$ and $v \in V_{i+1}$. In such a way, we associate with the generalized Bratteli diagram $B = (V, E)$ a sequence of non-negative countable infinite matrices (F_i), $i \in \mathbb{N}_0$, (called the *incidence matrices*) given by

$$F_i = (f^{(i)}_{v,w} : v \in V_{i+1}, w \in V_i), \quad f^{(i)}_{v,w} \in \mathbb{N}_0. \tag{2.1}$$

In this paper, a matrix $F = (f_{ij})$ is called *infinite* (or countably infinite) if its rows and columns are indexed by the same countably infinite set. Assuming that all matrices $F^n, n \in \mathbb{N}$ are defined (i.e., they have finite entries), we denote the entries of F^n by $f^{(n)}_{ij}$. Observe that this notation is similar to (2.1). It will be clear from the context whether $f^{(n)}_{ij}$ denotes the (i, j)-th entry of the matrix F_n (in a sequence of matrices $(F_n)_{n\in\mathbb{N}_0}$) or it denotes the (i, j)-th entry of the n-th power of the matrix F.

Remark 2.3. The structure of a generalized Bratteli diagram $B = (V, E)$ is completely determined by the sequence of incidence matrices (F_n), $n \in \mathbb{N}_0$. In this case, we write $B = B(F_n)$. For each $n \in \mathbb{N}_0$, the matrix F_n has at most finitely many non-zero entries in each row, and none of its rows or columns are entirely zero. A column of F_n may have a finite or infinite number of non-zero entries. Note that, X_B is locally compact if and only if every column of F_n has finitely many non-zero entries for all n.

Definition 2.4. Let $B = B(F_n)$ be a generalized Bratteli diagram. If $F_n = F$ for every $n \in \mathbb{N}_0$, then the diagram B is called *stationary.* We will write $B = B(F)$ in this case.

A generalized Bratteli diagram $B = (V, E)$, where all levels V_i are identified with a set V_0 (e.g., $V_0 = \mathbb{N}$ or $\mathbb{Z}$), is called *irreducible* if for any vertices $i, j \in V_0$ and any level V_n there exist $m > n$ and a finite path connecting $i \in V_n$ and $j \in V_m$. In other words, the (j, i)-entry of the matrix $F_{m-1} \cdots F_n$ is non-zero. Otherwise, the diagram is called *reducible*.

In particular, a generalized stationary Bratteli diagram is irreducible if and only if the corresponding incidence matrix is irreducible (see Appendix A for more definitions and results about infinite positive matrices).

Definition 2.5. Two paths $x = (x_i)$ and $y = (y_i)$ in X_B are called *tail equivalent* if there exists an $n \in \mathbb{N}_0$ such that $x_i = y_i$ for all $i \geq n$. This notion defines a *countable Borel equivalence relation* $\mathcal{R}$ on the path space X_B which is called the *tail equivalence relation*.

Remark 2.6. The set of generalized Bratteli diagrams contains various "exotic" examples. In this paper, we will consider the diagrams whose properties are natural from the point of view of dynamical systems. We will assume that the path space X_B of a generalized Bratteli diagram B has no isolated points, that is, for every infinite path $(x_0, x_1, x_2, \ldots) \in X_B$ and every $n \in \mathbb{N}_0$, there exists a level $m > n$ such that $|s^{-1}(r(x_m))| > 1$. Hence, the set X_B is uncountable. We will consider only such Bratteli diagrams for which the tail equivalence relation $\mathcal{R}$ is aperiodic. We will not consider cases where the Bratteli diagram is a disjoint union of two or more Bratteli diagrams, that is, the Bratteli diagram should be connected when considered as an undirected graph.

Definition 2.7. Given a generalized Bratteli diagram $B = (V, E)$ and a monotone increasing sequence $(n_k : k \in \mathbb{N}_0)$, $n_0 = 0$, we define a new generalized Bratteli diagram $B' = (V', E')$ as follows: The vertex sets are determined by $V'_k = V_{n_k}$, and the edge sets $E'_k = E_{n_k} \circ \cdots \circ E_{n_{k+1}-1}$ are formed by finite paths between the levels V'_k and V'_{k+1}. The diagram $B' = (V', E')$ is called a *telescoping* of the original diagram $B = (V, E)$.

Note that for a generalized stationary Bratteli diagram, after telescoping, we can always assume that the incidence matrix is aperiodic (see Appendix A).

2.2. Isomorphism of generalized Bratteli diagrams

Similarly to the case of standard Bratteli diagrams (see, e.g., [31]), we define the notion of isomorphic generalized Bratteli diagrams.

Definition 2.8. Two generalized Bratteli diagrams $B = (V, E)$ and $B' = (V', E')$ are called *isomorphic* if there exist two sequence of bijections $(g_n : V_n \to V'_n)_{n \in \mathbb{N}_0}$ and $(h_n : E_n \to E'_n)_{n \in \mathbb{N}_0}$ such that for every $n \in \mathbb{N}_0$, we have $g_n(V_n) = V'_n$ and $h_n(E_n) = E'_n$, and $s' \circ h_n = g_n \circ s$, $r' \circ h_n = g_n \circ r$.

To illustrate Definition 2.8, we consider the following example.

Example 2.9 (Isomorphic generalized Bratteli diagrams, Figure 1). Let $B(F)$ be a stationary generalized Bratteli diagram such that every level of B is identified with $\mathbb{Z}$, and the incidence matrix $F = (f_{ij})_{i,j\in\mathbb{Z}}$ has entries

$$f_{ij} = \begin{cases} 2, & \text{for } i = j, \\ 1, & \text{for } |i-j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, let $B'(F')$ be a stationary generalized Bratteli diagram such that every level of B' is identified with $\mathbb{N}_0$, and its incidence matrix $F' = (f'_{ij})_{i,j\in\mathbb{N}_0}$ has entries $f'_{00} = f'_{11} = 2$, $f'_{01} = 1$, $f'_{02} = 1$, $f'_{10} = 1$, $f'_{13} = 1$, $f'_{20} = 1$, $f'_{31} = 1$, and for $i, j \notin \{0, 1\}$:

$$f'_{ij} = \begin{cases} 2, & \text{for } i = j, \\ 1, & \text{for } |i-j| = 2, \\ 0, & \text{otherwise.} \end{cases}$$

The diagrams B and B' (elaborated in Figure 1) are isomorphic. Indeed, the bijections $(g_n : V_n \to V'_n)_{n\in\mathbb{N}_0}$ and $(h_n : E_n \to E'_n)_{n\in\mathbb{N}_0}$ that give the isomorphism are defined as follows: Since the diagrams are stationary, we set, for every $n \in \mathbb{N}_0$, $g_n = g : \mathbb{Z} \to \mathbb{N}_0$

$$g(n) = \begin{cases} 2n, & \text{if } n \geq 0, \\ -2n-1, & \text{if } n < 0. \end{cases}$$

The bijection $h_n : E_n \to E'_n$ is defined as follows: The two vertical edges in the diagram $B(F)$ with range $i \in V_n$ are mapped to the two vertical edges in the diagram $B'(F')$ with

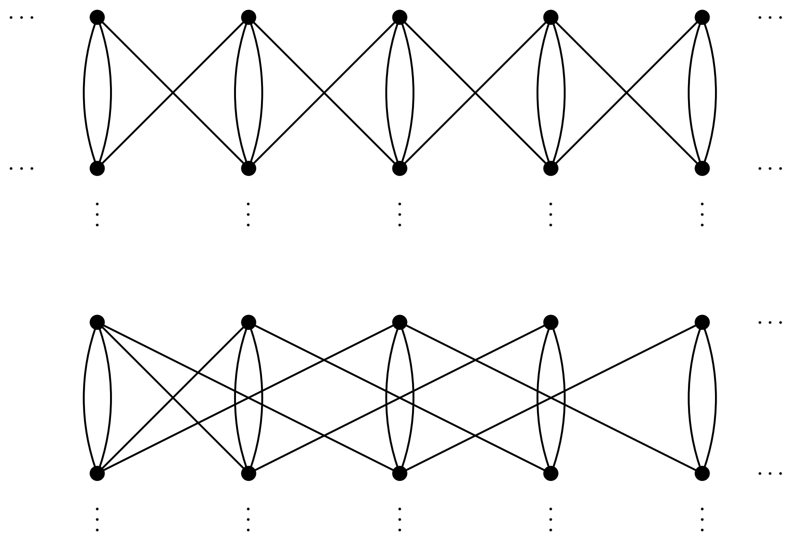

Figure 1. Isomorphic generalized Bratteli diagrams B and B'.

range $g(i) \in V_n$. For $i > 0$, the non-vertical edge with range i coming from left (respectively from the right) is mapped to the non-vertical edge with range $g(i) \in V'_n$ coming from the left (respectively from the right). For $i < 0$, the non-vertical edge with range i coming from left (respectively from the right) is mapped to the non-vertical edge with range $g(i) \in V'_n$ coming from the right (respectively from the left). For the non-vertical edges with range $0 \in V_n$, the mapping can be seen from Figure 1. It is easy to see that with the above bijections the two diagrams are isomorphic.

Remark 2.10. Note that, in general, isomorphism of generalized Bratteli diagrams does not preserve irreducibility. For instance, stationary diagram B from Example 2.9 is irreducible since starting from every vertex we eventually reach any other vertex for some sufficiently low level. It is easy to see that diagram B' is also irreducible. Define another stationary generalized Bratteli diagram $B'' = (V'', E'')$ isomorphic to B with every level identified with $\mathbb{Z}$. Consider the following sequence of bijections $g_n \colon V_n \to V''_n$ between vertices of B and B'' on level n:

$$g_n(i) = i + n.$$

Thus, the bijections g_n do not change the level V_0, shift all the vertices of level V_1 by 1 to the right and all vertices of level V_n by n to the right. For all n, the corresponding sequence of bijections $h_n \colon E_n \to E''_n$ between edges will map all edges outgoing from vertex $i \in V_n$ to the edges with the source $i + n \in V'_n$ and the range in the set $\{i + n, i + n + 1, i + n + 2\}$. In particular, the edge with source $i \in V_n$ and range $i - 1 \in V_{n+1}$ will be mapped to the vertical edge between $i + n \in V''_n$ and $i + n \in V''_{n+1}$. Thus, it will not be possible to reach from vertex $i \in V_n$ any vertex $j < i$ for any level $m > n$. The incidence matrix F'' of B'' is lower triangular (here we indicate the main diagonal of F'' with bold font):

$$F'' = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & \mathbf{1} & 0 & 0 & 0 & \dots \\ \dots & 2 & \mathbf{1} & 0 & 0 & \dots \\ \dots & 1 & 2 & \mathbf{1} & 0 & \dots \\ \dots & 0 & 1 & 2 & \mathbf{1} & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

To define a dynamical system on the path space of a generalized Bratteli diagram, we need to take a linear order $>$ on each (finite) set $r^{-1}(v)$, $v \in V \setminus V_0$. This order defines a partial order on the sets of edges E_i, $i = 0, 1, \dots$: Edges e, e' are comparable if and only if $r(e) = r(e')$. For a generalized Bratteli diagram $B = (V, E)$ equipped with partial order $>$, we define a partial lexicographical order on the set $E_k \circ \dots \circ E_l$ of all finite paths from V_k to V_{l+1} as follows: $(e_k, \dots, e_l) > (f_k, \dots, f_l)$ if and only if $e_i > f_i$ for some i with $k \le i \le l$, and $e_j = f_j$ for $i < j \le l$. Then any two paths from the (finite) set $E(V_0, v)$ of all paths connecting a vertex from V_0 and v are comparable with respect to the lexicographic order.

Definition 2.11. A generalized Bratteli diagram $B = (V, E)$ together with a partial order $>$ on E is called *an ordered generalized Bratteli diagram* $B = (V, E, >)$.

We call an infinite path $e = (e_0, e_1, \dots, e_i, \dots)$ *maximal (respectively minimal)* if every e_i has a maximal (respectively minimal) number among all elements from $r^{-1}(r(e_i))$. The same definition is used for finite maximal/minimal paths. Remark that there are unique minimal and maximal paths in the set $E(V_0, v)$ of all finite paths arriving at v for each $v \in V_i$, $i > 0$.

We note that, in contrast to standard Bratteli diagrams, there are orders on generalized Bratteli diagrams such that the sets $X_{\max}$ and $X_{\min}$ of maximal and minimal paths are empty (see [5] and Example 4.2). It is not hard to see that $X_{\max}$ and $X_{\min}$ are closed subsets of X_B.

Definition 2.12. For an ordered generalized Bratteli diagram $B = (V, E, >)$, we define a Borel transformation

$$\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min} \tag{2.2}$$

as follows. Given $x = (x_0, x_1, \dots) \in X_B \setminus X_{\max}$, let m be the smallest number such that x_m is not maximal. Let g_m be the successor of x_m in the finite set $r^{-1}(r(x_m))$. Then we set $\varphi_B(x) = (g_0, g_1, \dots, g_{m-1}, g_m, x_{m+1}, \dots)$ where $(g_0, g_1, \dots, g_{m-1})$ is the minimal path in $E(V_0, s(g_m))$. The map φ_B is a Borel bijection. Moreover, φ_B is a homeomorphism from $X_B \setminus X_{\max}$ onto $X_B \setminus X_{\min}$. If φ_B admits a bijective extension to the entire path space X_B, then we call the Borel transformation $\varphi_B : X_B \to X_B$ a *Vershik map*, and the Borel dynamical system (X_B, φ_B) is called a generalized *Bratteli–Vershik* system.

Remark 2.13. We collect here several facts about Vershik maps on generalized Bratteli diagrams.

(1) Let $B = (V, E, >)$ be an ordered generalized Bratteli diagram. Relation (2.2) defines φ_B uniquely as a map from $X_B \setminus X_{\max}$ onto $X_B \setminus X_{\min}$. We note that if φ_B can be prolonged to a Vershik map on X_B, then, in general, such an extension is not unique. Indeed, if $|X_{\min}| = |X_{\max}| > 1$ then we can choose arbitrary a Borel map from $X_{\max}$ onto $X_{\min}$ as a Vershik map acting on $X_{\max}$.

(2) There exist orders $>$ on X_B such that both sets $X_{\max}$ and $X_{\min}$ are empty. In this case, φ_B is uniquely determined according to (2.2) of Definition 2.12. Also, there exist orders $>$ on B such that $|X_{\max}| \neq |X_{\min}|$; in particular, one of these sets may be empty. In the latter case, $B(V, E, >)$ does not support a Vershik map.

(3) We note that for simple standard Bratteli diagrams, the left-to-right ordering always generates a Vershik homeomorphism. Indeed, in this case, all minimal edges start from the first vertex on each level, and all maximal edges start from the last vertex. This guarantees the uniqueness of an infinite minimal and infinite maximal path. For

irreducible generalized Bratteli diagrams, the left-to-right ordering does not necessarily produce a Borel Vershik automorphism (see Example 3.9) or a continuous Vershik map (see Theorem 3.10).

Definition 2.14. Two ordered generalized Bratteli diagrams $B = (V, E, >)$, $B' = (V', E', >')$ are called *order isomorphic* if they are isomorphic (see Definition 2.8) and, for all $n \in \mathbb{N}$, $v \in V_n$, and $e_1, e_2 \in r^{-1}(v)$, we have $e_2 > e_1$ if and only if $h_n(e_1) >' h_n(e_2)$ for $h_n(e_1), h_n(e_2) \in r^{-1}(g_n(v))$, where $g_n : V_n \to V'_n$ and $h_n : E_n \to E'_n$ are the bijections defined as in Definition 2.8.

We show that an order isomorphism of two generalized Bratteli diagrams implies an isomorphism of the respective generalized Bratteli–Vershik dynamical systems.

Theorem 2.15. *Let $B = (V, E, >)$ and $B' = (V', E', >')$ be order isomorphic generalized Bratteli diagrams. Assume that $|X_B(\max)| = |X_B(\min)|$ and $|X_{B'}(\max)| = |X_{B'}(\min)|$. Then the orders $>$ and $>'$ generate Vershik maps $\varphi_B : X_B \to X_B$ and $\varphi_{B'} : X_{B'} \to X_{B'}$ such that the generalized Bratteli–Vershik systems (X_B, φ_B) and $(X_{B'}, \varphi_{B'})$ are Borel isomorphic.*

Proof. We will construct two Vershik maps $\varphi_B : X_B \to X_B$ and $\varphi_{B'} : X_{B'} \to X_{B'}$ and define a Borel map $f : X_B \to X_{B'}$ such that, for $x \in X_B$,

$$f(\varphi_B(x)) = \varphi_{B'}(f(x)). \tag{2.3}$$

We use the concatenation of the maps $(h_n : E_n \to E'_n)_{n\in\mathbb{N}_0}$ given in Definition 2.8 to define f for $x = (x_0, x_1, \ldots) \in X_B$,

$$f(x) = (h_n(x_n))_{n\in\mathbb{N}_0} := (x'_n)_{n\in\mathbb{N}_0} := x' \in X_{B'}.$$

Since every h_n is a bijection, we see that $f : X_B \to X_{B'}$ is a well-defined bijective map. It follows from Definition 2.8 that $f(X_B(\max)) = X_{B'}(\max)$ and $f(X_B(\min)) = X_{B'}(\min)$.

By Definition 2.12, we have

$$\varphi_B : X_B \setminus X_B(\max) \to X_B \setminus X_B(\min), \quad \varphi_{B'} : X_{B'} \setminus X_{B'}(\max) \to X_{B'} \setminus X_{B'}(\min) \tag{2.4}$$

We show that, for $x \in X_B \setminus X_B(\max)$, the map $f : X_B \setminus X_B(\max) \to X_{B'} \setminus X_{B'}(\max)$ intertwines the Vershik maps φ_B and $\varphi_{B'}$. To see this, take $x = (x_0, x_1, \ldots) \in X_B \setminus X_B(\max)$ and find the smallest integer k such that x_k is not maximal. Let y_k be the successor of x_k in the finite set $r^{-1}(r(x_k))$. Then, by definition of the Vershik map, $\varphi_B(x) = (y_0, y_1, \ldots, y_{k-1}, y_k, x_{k+1}, \ldots)$ where $(y_0, y_1, \ldots, y_{k-1})$ is the minimal path in $E(V_0, r(y_{k-1}))$. By definition of f, we have

$$f(\varphi_B(x)) = (h_0(y_0), \ldots, h_{k-1}(y_{k-1}), h_k(y_k), h_{k+1}(x_{k+1}), \ldots),$$

where $(h_0(y_0), \ldots, h_{k-1}(y_{k-1}))$ is the minimal path with range $s'(h_k(y_k))$. We note that $h_k(y_k)$ is the successor of $h_k(x_k)$ in the set $(r')^{-1}(r'(h(y_k)))$.

We now compute $\varphi_{B'}(f(x))$:

$$\begin{aligned}\varphi_{B'}(f(x)) &= \varphi_{B'}(h_0(x_0), \dots, h_{k-1}(x_{k-1}), h_k(x_k), h_{k+1}(x_{k+1}), \dots) \\ &= (h_0(y_0), \dots, h_{k-1}(y_{k-1}), h_k(y_k), h_{k+1}(x_{k+1}), \dots)\end{aligned}$$

because the minimal path with range in $s'(h_k(y_k))$ is unique. Hence, $\varphi_{B'} f = f\varphi_B$ on $X_B \setminus X_B(\max)$.

It remains to show that the relation $\varphi_{B'} f = f\varphi_B$ can be extended to all X_B. By the condition of the theorem, φ_B can be extended to X_B, we keep the same notation φ_B for the extension. Since f implements a bijection between $X_B(\max)$ and $X_{B'}(\max)$ and between $X_B(\min)$ and $X_{B'}(\min)$, we can extend the definition of $\varphi_{B'}$ as follows: for $x \in X_{B'}(\max)$, we set

$$\varphi_{B'}(x) = f \circ \varphi_B \circ f^{-1}(x). \tag{2.5}$$

Then $\varphi_{B'} : X_{B'}(\max) \to X_{B'}(\min)$, and, taking into account the result proved above, we conclude that relation (2.5) holds for all x. ■

Now we mention an observation that will be of use later on.

Lemma 2.16. *Let $B = (V, E, >)$ be an ordered generalized Bratteli diagram. Then the sets $X_{\max}$ and $X_{\min}$ of maximal and minimal infinite paths are closed; in particular, they can be empty. The Vershik map $\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$ is a homeomorphism.*

Let $(X, \mathcal{B})$ be a standard Borel space. Recall that any two uncountable standard Borel spaces are Borel isomorphic. For a standard Borel space $(X, \mathcal{B})$, a one-to-one Borel map T of X onto itself is called a *Borel automorphism* of $(X, \mathcal{B})$. The following result shows that any aperiodic Borel automorphism of a standard Borel space can be realized as a Vershik map on a generalized Bratteli diagram.

Theorem 2.17 ([5]). *Let T be an aperiodic Borel automorphism acting on a standard Borel space $(X, \mathcal{B})$. Then there exist an ordered generalized Bratteli diagram $B = (V, E, >)$ and a Vershik map $\varphi_B : X_B \to X_B$ such that (X, T) is Borel isomorphic to (X_B, φ_B). Moreover, φ_B is a homeomorphism of the path space X_B.*

3. Generalized Bratteli diagrams of bounded size

In this section, we discuss generalized Bratteli diagrams of *bounded size* (see Definition 3.1). These diagrams are characterized by the fact that all incidence matrices are *banded*. Such diagrams have a locally compact path space and in our analysis, we consider them as an intermediate step between the classical Bratteli diagrams (i.e., having finitely many vertices at each level) and generalized Bratteli diagrams. Generalized Bratteli diagrams of bounded size present models for substitution dynamical systems on countably infinite alphabets (see [10]). We present more examples of generalized Bratteli diagrams of bounded size together with the corresponding substitution dynamical systems and invariant measures in Subsection 7.3. In Subsection 3.1, we prove statements about the structure of such diagrams which will be of use later on. In Subsection 3.2, we consider the Vershik map on the path space of a bounded size diagram defined by the *left-to-right order*. We show that, unlike the classical simple Bratteli diagrams, the left-to-right order on an irreducible generalized Bratteli diagram of bounded size does not necessarily give rise to a continuous Vershik map. We give conditions under which the left-to-right order can give rise to a continuous Vershik map and conditions under which every infinite maximal path is necessarily a point of discontinuity.

3.1. The structure of the generalized Bratteli diagrams of bounded size

This subsection is dedicated to studying the structure of generalized Bratteli diagrams $B = (V, E)$ of *bounded size*. Unless stated otherwise, we will identify the set of vertices at each level with integers, that is, for each $i \in \mathbb{N}_0$, we have $V_i = \mathbb{Z}$. Observe that such a diagram has a locally compact path space X_B, and the restriction of X_B to any cylinder set can be represented as the path space of a standard Bratteli diagram.

Definition 3.1. A generalized Bratteli diagram $B(F_n)$ is called of *bounded size* if there exists a sequence of pairs of natural numbers $(t_n, L_n)_{n\in\mathbb{N}_0}$ such that, for all $n \in \mathbb{N}_0$ and all $v \in V_{n+1}$,

$$s(r^{-1}(v)) \in \{v - t_n, \dots, v + t_n\} \quad \text{and} \quad \sum_{w\in V_n} f^{(n)}_{vw} = \sum_{w\in V_n} |E(w, v)| \le L_n. \tag{3.1}$$

If the sequence $(t_n, L_n)_{n\in\mathbb{N}_0}$ is constant, that is, $t_n = t$ and $L_n = L$ for all $n \in \mathbb{N}_0$, then we say that the diagram $B(F_n)$ is of *uniformly bounded size*.

Observe that, the condition $\sum_{w\in V_n} f^{(n)}_{v,w} \le L_n$ implies that the set $s(r^{-1}(v))$ is finite. Moreover, the cardinality $|s(r^{-1}(v))|$ is bounded as a function of v for every level V_n. The role of the first condition in (3.1) is to control the sources of edges arriving at $v \in V_{n+1}$.

Remark 3.2. We will use the following convention for bounded size Bratteli diagrams. For each $n \in \mathbb{N}_0$, the pair of natural numbers (t_n, L_n) are chosen to be the minimal possible. Also, for every $n \in \mathbb{N}_0$, it is assumed that $E(v - t_n, v)$ and $E(v + t_n, v)$ are non-empty for all $v \in V_{n+1}$. These assumptions are made to simplify our notation and calculations. Otherwise, we would have to use two sequences, (t_n^+), (t_n^-), where

$$t_n^+ = \max\{t : E(v + t, v) \neq \emptyset\}$$

and

$$t_n^- = \max\{t : E(v - t, v) \neq \emptyset\}.$$

The usage of two different sequences $(t_n^\pm)$ instead of (t_n) would affect computations but not the corresponding results.

Observe that the property of bounded size implies that the incidence matrices of the diagram are *banded* infinite matrices. Let $B = B(F_n)$ be a generalized Bratteli diagram of bounded size corresponding to a sequence $(t_n, L_n)_{n\in\mathbb{N}_0}$. Then all non-zero entries of F_n belong to a band of width $2t_n + 1$ along the main diagonal. Moreover, the sum of entries in every row of F_n is bounded by L_n. In other words, for every $n \in \mathbb{N}_0$ and $v \in V_{n+1}$, we have $f^{(n)}_{v,w} = 0$ if $|v - w| > t_n$ and $\sum_{w\in V_n} f^{(n)}_{v,w} \le L_n$. Also, by the above assumption, $E(v \pm t_n, v) \neq \emptyset$ (or $f^{(n)}_{v,v\pm t_n} > 0$) for all $v \in V_{n+1}, v \pm t_n \in V_n$, and $n \in \mathbb{N}_0$.

In [10], the authors studied substitution dynamical systems on a countably infinite alphabet $\mathcal{A}$ as Borel dynamical systems. It was proved that a substitution dynamical system on a countably infinite alphabet which is *left determined* (a generalization of the recognizability property to the countable case, see [37]) and has *bounded size* (see the definition below) admits a realization as a Vershik map on a stationary generalized Bratteli diagram.

Definition 3.3. Identifying $\mathcal{A}$ with $\mathbb{Z}$, we say that a substitution $\sigma : n \to \sigma(n)$, $n \in \mathbb{Z}$, is of *bounded size* if it is of bounded length and there exists a positive integer t (independent of n) such that for every $n \in \mathbb{Z}$, if $m \in \sigma(n)$, then $m \in \{n - t, \dots, n, \dots, n + t\}$.

Clearly, this definition is analogous to the definition of bounded size generalized Bratteli diagrams.

Theorem 3.4 ([10]). *Let σ be a bounded size left determined substitution on a countably infinite alphabet and (X_σ, T) be the corresponding subshift. Then there exist a stationary ordered generalized Bratteli diagram $B = (V, E, \ge)$ of bounded size and a Vershik map $\varphi\colon X_B \to X_B$ such that (X_σ, T) is Borel isomorphic to (X_B, φ).*

In the rest of the subsection, we discuss the structure of generalized Bratteli diagrams of bounded size.

Lemma 3.5. *Let $B = (V, E)$ be a generalized Bratteli diagram of bounded size. Let $n \in \mathbb{N}_0$, $v \in V_{n+1}$, and $E(V_0, v)$ be the set of all finite paths $\overline{e} = (e_0, \dots, e_n)$ such that $r(\overline{e}) = v$. Then*

$$s(E(V_0, v)) \subset \left\{ v - \sum_{i=0}^{n} t_i, \dots, v + \sum_{i=0}^{n} t_i \right\}$$

and

$$|E(V_0, v)| \le L_0 \cdots L_n.$$

Proof. We prove the lemma by induction. Case $n = 0$ follows from the definition. Suppose the statement of the lemma is true for $n = k$. Then for any $v \in V_{k+2}$,

$$s(r^{-1}(v)) \in \{v - t_{k+1}, \dots, v + t_{k+1}\}$$

and

$$s(E(V_0, v)) \subset \bigcup_{w \in \{v - t_{k+1}, \dots, v + t_{k+1}\}} s(E(V_0, w))$$
$$\subset \left\{ v - t_{k+1} - \sum_{i=0}^{k} t_i, \dots, v + t_{k+1} + \sum_{i=0}^{k} t_i \right\}$$
$$= \left\{ v - \sum_{i=0}^{k+1} t_i, \dots, v + \sum_{i=0}^{k+1} t_i \right\}.$$

We also have

$$|E(V_0, v)| = \sum_{w \in V_{k+1}} f_{vw}^{(k+1)} |E(V_0, w)| \le (L_0 \cdots L_k) \sum_{w \in V_{k+1}} f_{vw}^{(k+1)}$$
$$\le L_0 \cdots L_k \cdot L_{k+1}.$$

■

Corollary 3.6. *Let $B = (V, E)$ be a generalized Bratteli diagram of bounded size. Let $n \in \mathbb{N}_0$, $v \in V_{n+1}$, and $E(V_0, v)$ be the set of all finite paths $\overline{e} = (e_0, \dots, e_n)$ such that $r(\overline{e}) = v$. Then for every $m \le n$, we have*

$$s(E(V_m, v)) \subset \left\{ v - \sum_{i=m}^{n} t_i, \dots, v + \sum_{i=m}^{n} t_i \right\}.$$

Proof. The proof follows from an induction argument similar to that used in the proof of Lemma 3.5. ■

Assume that $B = (V, E)$ is a diagram of bounded size corresponding to a sequence $(t_n, L_n)_{n \in \mathbb{N}_0}$. Fix $v \in V_{n+1}$ and let $\overline{e} = (e_0, \dots, e_n)$ be a finite path with $r(\overline{e}) = v \in V_{n+1}$. By Lemma 3.5,

$$s(e_0) \in \left\{ v - \sum_{i=0}^{n} t_i, \dots, v + \sum_{i=0}^{n} t_i \right\}.$$

Lemma 3.7. *Let $B = (V, E)$ be a diagram of bounded size corresponding to a sequence $(t_n, L_n)_{n \in \mathbb{N}_0}$. Fix $v \in V_{n+1}$, and let $\overline{e} = (e_0, \dots, e_n)$ be a finite path with $r(\overline{e}) = v \in V_{n+1}$. Then for all infinite paths $x = (x_n)_{n \in \mathbb{N}_0} \in [\overline{e}]$ and all $m \ge 0$:*

$$r(x_{n+m}) \in \left\{ v - \sum_{i=1}^{m} t_{n+i}, \dots, v + \sum_{i=1}^{m} t_{n+i} \right\} \subset V_{n+m+1}. \tag{3.2}$$

Proof. We prove this lemma by induction. Fix $v \in V_{n+1}$, then (3.2) is trivially true for $m = 0$. By induction step, we assume that (3.2) holds for $m = k$, that is,

$$r(x_{n+k}) \in \left\{ v - \sum_{i=1}^{k} t_{n+i}, \dots, v + \sum_{i=1}^{k} t_{n+i} \right\} \subset V_{n+k+1}.$$

By Lemma 3.5, if $u \in V_{n+k+2}$, such that $E(v,u) \neq \emptyset$ then

$$v \in \left\{u - \sum_{i=0}^{k} t_{n+1+i}, \dots, u + \sum_{i=0}^{k} t_{n+1+i}\right\}.$$

This implies,

$$u \in \left\{v - \sum_{i=1}^{k+1} t_{n+i}, \dots, v + \sum_{i=1}^{k+1} t_{n+i}\right\}.$$

In other words,

$$r(x_{n+k+1}) \in \left\{v - \sum_{i=1}^{k+1} t_{n+i}, \dots, v + \sum_{i=1}^{k+1} t_{n+i}\right\} \subset V_{n+k+2}$$

as needed. ∎

3.2. Continuity of a Vershik map for generalized Bratteli diagrams of bounded size

In the case of classical Bratteli diagrams, that is, the diagrams with finitely many vertices at each level, it is a well-known fact that, for simple Bratteli diagrams, the left-to-right order always gives rise to a Vershik homeomorphism [31,48]. In this subsection, we show that this is not true for irreducible generalized Bratteli diagrams of bounded size.

Let $B = (V, E)$ be an irreducible generalized Bratteli diagram of bounded size with the corresponding sequence $(t_n, L_n)_{n \in \mathbb{N}_0}$. We will denote by ω a fixed partial order on E. To emphasize that a diagram B is ordered, we will write (B, ω). Let $X_{\max} = X_{\max}(\omega)$ and $X_{\min} = X_{\min}(\omega)$ denote the sets of infinite maximal and minimal paths, respectively, with respect to the corresponding order.

Lemma 3.8. *Let $B = (B, \omega)$ be an ordered generalized Bratteli diagram of bounded size where ω is the left-to-right partial order on E, then*

$$|X_{\max}| = |X_{\min}| = \aleph_0.$$

Proof. Let $(t_n, L_n)_{n \in \mathbb{N}_0}$ denote the sequence corresponding to the bounded size diagram (B, ω). Recall that for $n \in \mathbb{N}$ and $v \in V_n$, $E(V_0, v)$ denotes the set of all finite paths $\overline{e} = (e_0, \dots, e_n)$ such that $r(\overline{e}) = v$. Since ω is left-to-right ordering, the leftmost finite path and the rightmost finite path is the unique minimal and the unique maximal path, respectively, in the set $E(V_0, v)$. We denote them by $\overline{e}_{\min}$ and $\overline{e}_{\max}$. Since (B, ω) is of bounded size, there are vertices $u = v + t_{n+1}$ and $u' = v - t_{n+1}$ in V_{n+1} such that $e(v,u)$ is the minimal edge in the set $r^{-1}(u)$ and $e(v,u')$ is the maximal edge in the set $r^{-1}(u')$.

This observation implies that, for every $v \in V_n$, the finite minimal path $\overline{e}_{\min} \in E(V_0, v)$ and the finite maximal path $\overline{e}_{\max} \in E(V_0, v)$ have unique minimal and unique maximal extensions to the level $n + 1$. This proves that the sets $X_{\max}$ and $X_{\min}$ are countable. ∎

Observe that for a two-sided generalized Bratteli diagram of bounded size $B = (V, E)$ (i.e., each V_i is identified with $\mathbb{Z}$) with left-to-right ordering, there exist countably infinite "slanted" minimal paths which go "from left-to-right" and countably infinite "slanted" maximal paths which go "from right to left". Since by convention, each vertex has the rightmost and leftmost incoming edge corresponding to the parameters t_n, all infinite minimal paths go "parallelly" to each other; the same holds for maximal paths.

Note that for an ordered generalized Bratteli diagram (B, ω), the Vershik map $\varphi_B(\omega)$ (corresponding to ω) is a well-defined map from $X_B \setminus X_{\max}(\omega)$ to $X_B \setminus X_{\min}(\omega)$. Denote by $\Phi_B(\omega)$ the set of all possible extensions of $\varphi_B(\omega)$ to Vershik maps defined on the entire X_B. In general, $\Phi_B(\omega)$ can be empty. As mentioned in Lemma 2.16, the map $\varphi_B(\omega) : X_B \setminus X_{\max}(\omega) \to X_B \setminus X_{\min}(\omega)$ is continuous. In what follows, we discuss the question of whether $\varphi_B(\omega)$ can be extended to a *continuous* Vershik map on the entire path space X_B.

We first give a simple example where there is no Vershik map on a generalized Bratteli diagram; in other words, the map $\varphi_B(\omega) : X_B \setminus X_{\max}(\omega) \to X_B \setminus X_{\min}(\omega)$ cannot be extended to the entire path space in Borel fashion.

Example 3.9. Let $B = (B, \omega)$ be a one-sided generalized Bratteli diagram of bounded sized (i.e., each V_i is identified with $\mathbb{N}$) where ω is the left-to-right ordering. We considered such a diagram in Example 2.9 (although without the ordering, see the second diagram in Figure 1). Then it is easy to see that (B, ω) admits infinitely many minimal paths and no maximal paths in the diagram. Hence, the Vershik map $\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$ cannot be extended to a Borel bijection to X_B.

Now we show that there exists a class of generalized Bratteli diagrams of bounded size such that the left-to-right order does not give rise to a continuous Vershik map.

Theorem 3.10. *Let $B = (B, \omega)$ be a generalized Bratteli diagram of bounded size, with the corresponding sequence $(t_n, L_n)_{n \in \mathbb{N}_0}$ and left-to-right ordering ω. Assume that for all $N \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that*

$$t_{N+k} < \sum_{i=N}^{N+k-1} t_i. \tag{3.3}$$

Let $\varphi_B(\omega) \in \Phi(\omega)$ be any Vershik map on X_B corresponding to the order ω. Then $\varphi_B(\omega)$ is not continuous. Moreover, every infinite maximal path is a point of discontinuity.

Proof. Assume that the vertices of B are enumerated by $\mathbb{Z}$. By Lemma 3.8, there always exists a well-defined Vershik map $\varphi_B = \varphi_B(\omega)$ on X_B. To show that φ_B is not continuous, we will prove that any neighborhood of a maximal path contains two infinite paths, x' and x'', such that the distance between their images under the Vershik map is equal to 1.

Pick any $x = (x_n)_{n \in \mathbb{N}_0} \in X_{\max}$, we will show that in an arbitrarily small neighborhood of x there are paths $x' = (x'_n)_{n \in \mathbb{N}_0}, x'' = (x''_n)_{n \in \mathbb{N}_0} \in X_B \setminus X_{\max}$ such that

$d(\varphi_B(x'), \varphi_B(x'')) = 1$. For $\varepsilon > 0$, take any n such that $\frac{1}{2^{n-1}} < \varepsilon$. Let $x'_l = x_l$ for $l < n$ and x'_n be the minimal edge with the source in $r(x_{n-1})$. Define x'_k for $k > n$ in an arbitrary way such that $x' = (x'_l)$ form a path in X_B. Then $d(x, x') < \varepsilon$ and x' is not a maximal path. Let $y' = (y'_n) = \varphi_B(x')$. Then y'_n is the successor of x'_n and since the ordering is left-to-right, we have $s(y'_n) \geq r(x_{n-1}) = s(x'_n)$.

By (3.3), there exists $k \in \mathbb{N}$ such that

$$t_{n+k} < \sum_{i=n}^{n+k-1} t_i.$$

Set $x''_l = x_l$ for $0 \leq l \leq n + k - 1$. Let x''_{n+k} be the minimal edge with the source in $r(x_{n+k-1})$ and let x''_l for $l > n + k - 1$ be defined in an arbitrary way such that $x'' = (x''_l)$ form a path in X_B. Then $d(x, x'') < \varepsilon$ and x'' is not a maximal path. Let y''_{n+k} be the successor of x''_{n+k} and denote $y'' = (y''_l) = \varphi_B(x'')$. Since B is a diagram of bounded size, we have $s(y''_{n+k}) \leq r(x_{n+k-1}) + 2t_{n+k}$. Recall that we assume that the sets $E(v - t_n, v)$ and $E(v + t_n, v)$ are non-empty for all $v \in V_{n+1}$ and all n. This means that

$$r(x_{n+k-1}) = r(x_{n-1}) - \sum_{i=n}^{n+k-1} t_i.$$

Since $(y''_0, \ldots, y''_{n+k-1})$ is the finite minimal path and the sets $E(v - t_n, v)$, $E(v + t_n, v)$ are non-empty for every $v \in V_{n+1}$, we have

$$s(y''_n) = s(y''_{n+k}) - \sum_{i=n}^{n+k-1} t_i.$$

Thus,

$$\begin{aligned} s(y''_n) &= s(y''_{n+k}) - \sum_{i=n}^{n+k-1} t_i \\ &\leq r(x_{n+k-1}) + 2t_{n+k} - \sum_{i=n}^{n+k-1} t_i \\ &= r(x_{n-1}) + 2t_{n+k} - 2\sum_{i=n}^{n+k-1} t_i < r(x_{n-1}). \end{aligned}$$

Hence, $s(y'_n) > s(y''_n)$ and since all minimal paths in the diagram go "parallelly" to each other, we have $d(\varphi_B(x'), \varphi_B(x'')) = 1$. ∎

Remark 3.11. We note that the conditions of the theorem above hold for generalized Bratteli diagrams of uniformly bounded size. Indeed, since $t_n = t$ for all $n \in \mathbb{N}_0$, for every $N \in \mathbb{N}$ it is enough to take $k = 2$ and the condition (3.3) will be satisfied. Thus, even irreducible generalized Bratteli diagrams of uniformly bounded size with left-to-right ordering do not support a continuous Vershik map.

Now we show that there exists a class of generalized Bratteli diagrams of bounded size such that the left-to-right order does give rise to a continuous Vershik map. This class is dual to the class of diagrams considered in Theorem 3.10.

Theorem 3.12. *Let $B = (B, \omega)$ be a generalized Bratteli diagram of bounded size, with the corresponding sequence $(t_n, L_n)_{n \in \mathbb{N}_0}$ and left-to-right ordering ω. Let $L_n = 2$ for all $n \in \mathbb{N}$ and let there exist $N \in \mathbb{N}_0$ such that*

$$t_{N+k} = \sum_{i=N}^{N+k-1} t_i \tag{3.4}$$

for all $k \in \mathbb{N}$. Then there exists a continuous Vershik map corresponding to ω.

Proof. Clearly, the Vershik map is well defined on $X_B \setminus X_{\max}$. Define the Vershik map φ_B on the set of maximal paths as follows: For every vertex $w \in V_N$, let φ_B map the unique maximal path $x_{\max}^{(w)}$ passing through w to the unique minimal path $x_{\min}^{(w)}$ passing through w. Then φ_B is a bijection. Since every vertex $v \in V \setminus V_0$ has exactly two incoming edges, it is easy to verify the continuity of φ_B. Indeed, let $w \in V_N$ and x be any non-maximal path that coincides with $x_{\max}^{(w)}$ up to level $M \geq N$. Then

$$d(x_{\max}^{(w)}, x) = \frac{1}{2^M} \leq \frac{1}{2^N}.$$

Then it follows from the structure of the diagram that the distance

$$d(\varphi_B(x), x_{\min}^{(w)}) = \frac{1}{2^M}.$$

Thus, x will be mapped to the neighborhood of the corresponding $x_{\min}$ (see Figure 2 in Example 3.13). ∎

Now we give an example which illustrates Theorem 3.12.

Example 3.13. Let $B = (B, \omega)$ be a two-sided generalized Bratteli diagram of bounded size, with the corresponding sequence $(t_n, L_n)_{n \in \mathbb{N}_0}$ and left-to-right ordering ω. Suppose that

$$L_n = 2, \quad t_n = 2^{n-1} \quad \text{for } n \in \mathbb{N} \text{ and } t_0 = 1.$$

The diagram $B = (V, E)$ is shown in Figure 2. In other words, every vertex $v \in V \setminus V_0$ has exactly two incoming edges. Endow B with the left-to-right ordering. To define the Vershik map, we use the following rule: For every vertex $w \in V_0$, the maximal infinite path which starts at w is mapped to the minimal infinite path which starts at the same vertex w. Then it is easy to check that the corresponding Vershik map is a homeomorphism. Note that, in this case, for every $n \geq 1$

$$t_n = \sum_{i=0}^{n-1} t_i$$

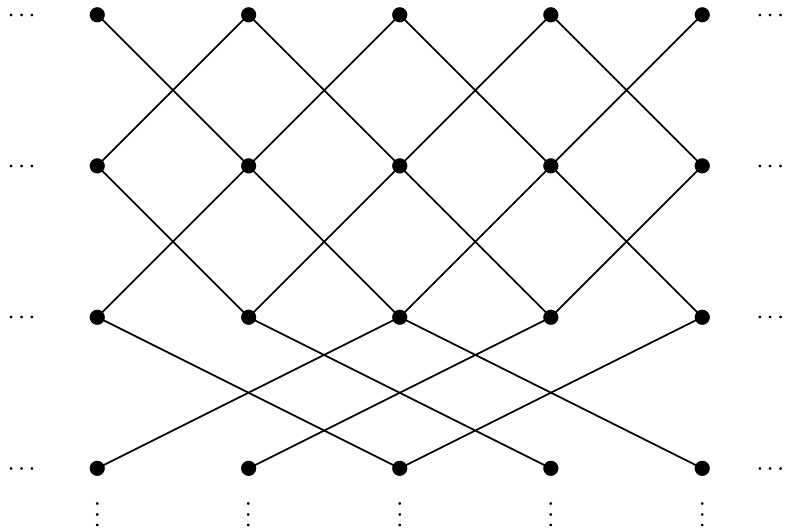

Figure 2. A continuous Vershik map (left-to-right ordering).

(compare with the result of Theorem 3.12). We remark also that the Bratteli diagram, constructed in this example, is reducible. Indeed, starting from the level V_2, all infinite paths that begin at even vertices will go through even vertices only.

4. Continuity of Vershik map for generalized Bratteli diagrams

The aim of this section is to find conditions under which a generalized Bratteli diagram admits an order such that the corresponding Vershik map can be prolonged to a homeomorphism. In particular, we are interested in orders ω such that there are no infinite minimal and no infinite maximal paths. The Vershik map $\varphi_B(\omega)$ corresponding to such an order is uniquely defined, and it is always a homeomorphism. Such orders were also used in Theorem 2.17 where generalized Bratteli diagrams corresponding to Borel dynamical systems were constructed using sequences of vanishing markers [5].

4.1. Ordered generalized Bratteli diagrams with no infinite minimal and maximal paths

Let B be a generalized Bratteli diagram such that, for every level $n \in \mathbb{N}$, the set of its vertices V_n is identified with $\mathbb{N}$. Note that this condition is not a restriction, since the set of vertices of any generalized Bratteli diagram can be enumerated this way (see Subsection 2.2 and Example 2.9). We call an edge e of such a diagram *slanted from right to left* if $r(e) < s(e)$, *slanted from left to right* if $r(e) > s(e)$, and *vertical* if $r(e) = s(e)$. We also say that a finite or infinite path is slanted from right to left if it consists of edges slanted from right to left.

Theorem 4.1. *Let B be a generalized Bratteli diagram such that the set of vertices on each level is identified with $\mathbb{N}$. Suppose that there exists a level N such that starting from that level every vertex has at least two incoming edges which are slanted from right to left. Then B admits an order such that the sets of infinite minimal and infinite maximal paths are empty.*

Proof. Put an arbitrary order on B up to level N. Starting from level N, we define an order such that, for every vertex, all minimal and maximal edges are slanted from right to left. Indeed, by the assumption, every vertex v beginning level N and below has at least two incoming edges slanted from right to left, say e_v and e'_v. We define an order ω such that a minimal edge and a maximal edge in $r^{-1}(v)$ are e_v and e'_v, respectively. Since all vertices are enumerated by natural numbers, we denote by v_1 the leftmost vertex in every level starting from N. Then the vertex v_1 of the diagram has no outgoing minimal edge and no outgoing maximal edge since all its outgoing edges are either vertical or slanted from left to right.

Let $x = (x_n)$ be any minimal or maximal path that passes through some vertex $w \in V_N$. Since for every $n \geq N$, all minimal and maximal edges are slanted from right to left, we obtain $r(x_n) < w + N - n$ and $r(x_{n+1}) < r(x_n)$ for every $n \geq N$. Since all levels of vertices are enumerated by natural numbers, every such path x must be finite. ∎

Example 4.2. The diagram in Figure 3 satisfies the conditions of Theorem 4.1 for $N = 0$. Indeed, define $B = B(F)$ to be a one-sided infinite generalized Bratteli diagram where $A = F^\top = (a_{ij})_{i,j\in\mathbb{N}}$ is defined by

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.1}$$

In other words, for all $j \in \mathbb{N}$,

$$a_{ij} = \begin{cases} 1 & \text{if } i = 1 \text{ or } i = j+1 \text{ or } i = j+2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Note that every vertex in $V \setminus V_0$ has exactly three incoming edges. Define a stationary order on B as follows: Let all the edges outgoing from the first vertex have label 1, and all other edges can be labeled by 0 and 2 in an arbitrary way. So it defines an ordered generalized Bratteli diagram. Then, for any vertex $v \in V_0$, any minimal or maximal path starting from this vertex will be slanted from right to left and finite (see Figure 3). Notice that the Vershik map corresponding to this diagram is a minimal homeomorphism. Indeed,

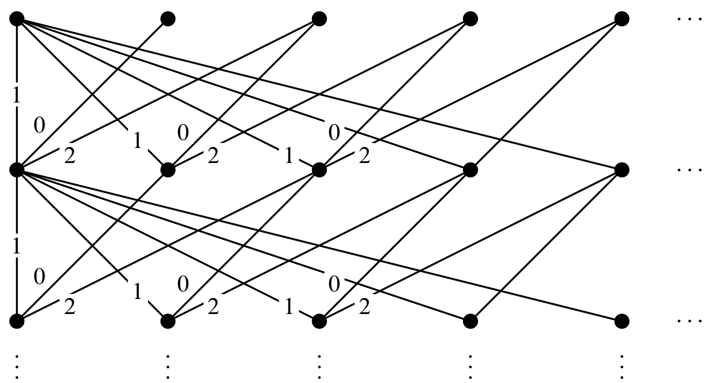

Figure 3. The Vershik map is a minimal homeomorphism, the sets of infinite minimal and maximal paths are empty.

every infinite path in X_B passes through the first vertex infinitely many times and the orbit of such path visits every cylinder set of X_B.

Example 4.3. The diagram in Figure 4 satisfies the conditions of the Theorem 4.1 for $N = 0$ after telescoping with respect to even levels. The incidence matrix of the diagram has the form

$$F = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 2 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The orders of E_0 and E_1 are presented in Figure 4. For every even i, the edges E_i are enumerated in the same way as E_0, and for every odd i, in the same way as E_1. Thus, after telescoping with respect to even-numbered levels, we obtain a stationary ordered Bratteli diagram such that all minimal and maximal edges are slanted from right to left.

Remark 4.4. Note that if every vertex of a generalized Bratteli diagram has an outgoing minimal edge and an outgoing maximal edge, then the sets of infinite minimal paths and infinite maximal paths are non-empty. The diagram in Example 4.3 has the property that every vertex has at least one extreme (minimal or maximal) outgoing edge, but this does not guarantee the existence of infinite maximal or infinite minimal paths. Moreover, the aforementioned property does not hold after telescoping with respect to even levels. After such a telescoping, the first vertex does not have any outgoing minimal or maximal edges.

Example 4.5. The diagram shown in Figure 5 does not satisfy conditions of Theorem 4.1 for any N, but still, there is an order for which there are no infinite minimal and infinite

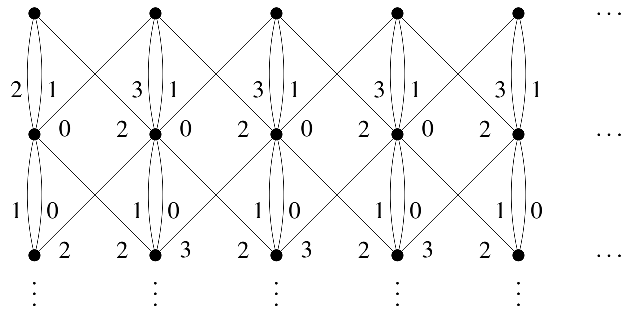

Figure 4. The Vershik map is a homeomorphism: After telescoping, all minimal and maximal edges are slanted from right to left.

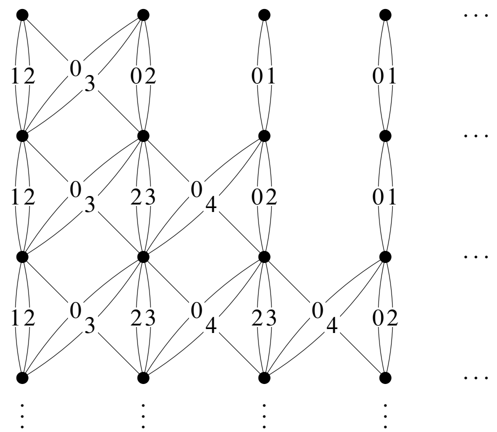

Figure 5. The sets of infinite minimal and maximal paths are empty (the labels are shown only for bow-shaped edges).

maximal paths. For every vertex $w \in V \setminus V_0$, if there are two incoming edges that are slanted from right to left, we enumerate them as minimal and maximal. Otherwise, we enumerate two vertical incoming edges as minimal and maximal. The remaining edges can be enumerated in arbitrary ways, for instance, from left to right. Then every minimal and maximal edge is either vertical or slanted from right to left and any minimal or maximal path eventually goes only through edges slanted from right to left.

This example shows that the assumption made in Theorem 4.1 is not necessary.

Remark 4.6. Not every generalized Bratteli diagram can be endowed with an order such that there are no infinite minimal and no infinite maximal paths. It is easy to provide examples of reducible generalized Bratteli diagrams such that for any order there exist infinite minimal and infinite maximal paths. We recall the definition of a *vertex subdiagram* which was used before for standard Bratteli diagrams but can also be defined in the same way for generalized ones. Let $B = (V, E)$ be a generalized Bratteli diagram. Let $\overline{W} = \{W_n\}_{n>0}$ be

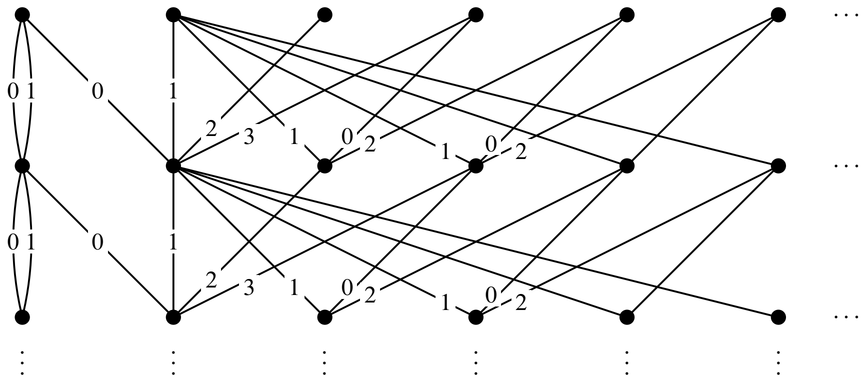

Figure 6. For the stationary order presented above, the Vershik map is a continuous bijection, but its inverse is discontinuous. For any order, there is at least one infinite minimal path and one infinite maximal path.

a sequence of proper, non-empty subsets $W_n \subset V_n$. Set $W_n' = V_n \setminus W_n$. The (vertex) subdiagram $\overline{B} = (\overline{W}, \overline{E})$ is a (standard or generalized) Bratteli diagram defined by the vertices $\overline{W} = \bigcup_{i \geq 0} W_n$ and the edges $\overline{E}$ that have their source and range in $\overline{W}$. In other words, the incidence matrix $\overline{F}_n$ of $\overline{B}$ is defined by those edges from B that have their source and range in vertices from W_n and W_{n+1}, respectively (see, e.g., [11]). Suppose that a path space of a generalized Bratteli diagram B has a compact subset that is invariant under the tail equivalence relation $\mathcal{R}$ and is represented by a standard Bratteli subdiagram $\overline{B}$ of B. Then for any order on B, the path space $X_{\overline{B}}$ has at least one minimal path and one maximal path. Since $X_{\overline{B}}$ is invariant under $\mathcal{R}$, there are no incoming edges to vertices of $\overline{B}$ from the vertices that do not lie in $\overline{B}$. Thus, the maximal and minimal paths which lie in $X_{\overline{B}}$ stay maximal and minimal also in X_B. For instance, in Figure 6, the subdiagram $\overline{B}$ consists of all paths which pass through the first vertex on each level. More generally, if a generalized Bratteli diagram B has n subdiagrams with compact $\mathcal{R}$-invariant path spaces for some $n \in \mathbb{N}$, then any order on B admits at least n infinite minimal and n infinite maximal paths.

4.2. Prolongation of the Vershik map

The goal of this subsection is to emphasize a sharp difference in the properties of Vershik maps for generalized and standard Bratteli diagrams.

Theorem 4.7. *There are stationary ordered generalized Bratteli diagrams with a unique infinite minimal and a unique infinite maximal paths such that*

(i) *both the Vershik map φ_B and its inverse φ_B^{-1} are not continuous;*

(ii) *the Vershik map φ_B is continuous but the inverse φ_B^{-1} is discontinuous;*

(iii) *both the Vershik map φ_B and its inverse φ_B^{-1} are continuous.*

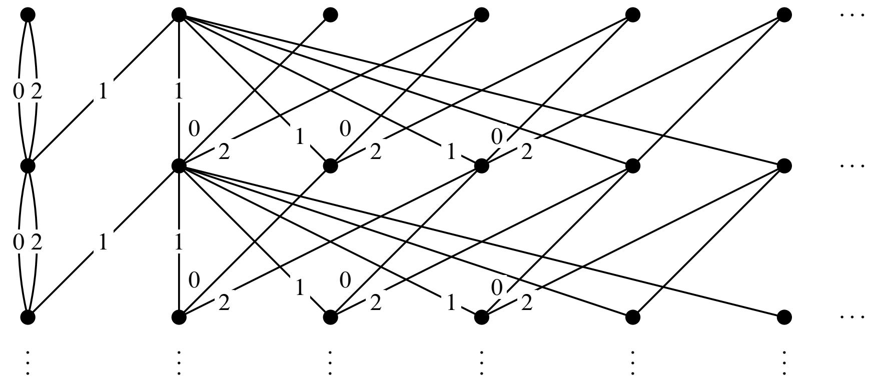

Figure 7. There is a unique infinite minimal path and a unique infinite maximal path, Vershik map φ_B is a Borel bijection, both φ_B and φ_B^{-1} are discontinuous.

The proof of this theorem is given in Examples 4.8 and 4.10 and Remark 4.9.

We recall that for standard Bratteli diagrams, it is obvious that if a diagram has a unique minimal infinite path $x_{\min}$ and a unique maximal infinite path $x_{\max}$, then the Vershik map which sends $x_{\max}$ to $x_{\min}$ is a homeomorphism. This fact follows from compactness of the path space. In the example below, we show that the result does not hold for generalized Bratteli diagrams.

Example 4.8. The stationary diagram in Figure 7 has a unique infinite minimal path $x_{\min}$ and a unique infinite maximal path $x_{\max}$ such that they pass through the first vertex on each level of the diagram. The second vertex of the diagram has no minimal or maximal outgoing edges and all maximal and minimal edges which start not at the first vertex are slanted from right to left, which guarantees that there are no other infinite minimal and maximal paths (see also Example 4.2). This means that the corresponding Vershik map φ_B sends $x_{\max}$ to $x_{\min}$ with necessity. We claim that φ_B is not a homeomorphism of X_B in this case. Indeed, let x be a non-maximal path that coincides with $x_{\max}$ for exactly n first edges, and then passes through a non-maximal edge. Then the image of x under the Vershik map φ_B lies in the cylinder set corresponding to the minimal path of length n which is slanted from right to left and ends in the second vertex of the diagram. Thus, the images of the non-maximal paths which lie in the neighborhood of the unique maximal path are not mapped into the neighborhood of the unique minimal path. Hence, the Vershik map φ_B is not continuous on X_B. Similarly, φ_B^{-1} is discontinuous.

The following remark describes a class of ordered generalized Bratteli diagrams with a unique infinite minimal path and a unique infinite maximal path such that the corresponding Vershik map is a homeomorphism.

Remark 4.9. Similar to the case of standard Bratteli diagrams, it is easy to see that if for every level n of a generalized Bratteli diagram B there is a unique vertex $v_{\min}^{(n)}$ such that all minimal edges of E_n start at $v_{\min}^{(n)}$, and a unique vertex $v_{\max}^{(n)}$ such that all maximal edges of E_n start at $v_{\max}^{(n)}$, then there is a unique infinite minimal path $x_{\min}$, a unique infinite maximal path $x_{\max}$, and the Vershik map which maps $x_{\max}$ to $x_{\min}$ is a homeomorphism of X_B.

The example below presents a stationary ordered generalized Bratteli diagram B with a unique infinite minimal and a unique infinite maximal path such that the Vershik map φ_B is a continuous bijection, but φ_B^{-1} is discontinuous.

Example 4.10. The stationary ordered diagram on Figure 6 has a unique infinite minimal path $x_{\min}$ and a unique infinite maximal path $x_{\max}$ passing through the first vertex on each level of the diagram. All minimal and maximal edges which do not end in the first vertex of the diagram are slanted from right to left; there are no outgoing minimal or maximal edges from the second vertex of the diagram. Thus, the infinite minimal and infinite maximal paths are unique. We define the Vershik map φ_B on these paths by setting $\varphi_B(x_{\max}) = x_{\min}$. Then it is easy to see that φ_B is a continuous bijection of X_B. We show that φ_B^{-1} is not continuous. Indeed, any non-maximal path from a neighborhood of $x_{\max}$ is mapped to a neighborhood of $x_{\min}$. Now, consider a path x which first pass through minimal edges and the first vertex of the diagram for n levels, but then goes once along the minimal edge to the second vertex and then along the edge e enumerated by 1 to the third vertex. We see that x lies in a neighborhood of $x_{\min}$, but this path x is not mapped to a neighborhood of $x_{\max}$ by φ_B^{-1}. Indeed, the edge e has a predecessor which is the edge e' labeled by 0 slanted from right to left. The source of e' is the fourth vertex of the diagram, and it is joined with V_0 by a finite maximal path slanted from right to left. Thus, the preimage of x does not belong to a neighborhood of $x_{\max}$.

Recall that some results concerning the prolongation of a Vershik map for generalized Bratteli diagrams of bounded size can also be found in Section 3.

5. Topological transitivity of the tail equivalence relation

In this section, we prove that the tail equivalence relation on the path space of a stationary generalized Bratteli diagram with an irreducible and aperiodic incidence matrix is topologically transitive. We also show that the irreducibility of the diagram does not imply the tail equivalence relation is minimal (see Theorem 5.4).

Theorem 5.1. *Let $B = (V, E)$ be a generalized stationary Bratteli diagram with an irreducible aperiodic incidence matrix $F = (f_{ij})_{i,j\in\mathbb{Z}}$. Then the tail equivalence relation $\mathcal{R}$ is topologically transitive.*

Proof. The idea of the proof is to show that the diagram has "vertical" paths with dense orbits in the path space. We identify vertices at each level with $\mathbb{Z}$. Fix a vertex $i \in \mathbb{Z}$.

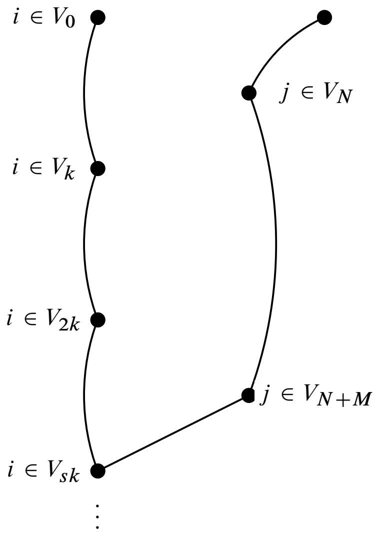

Figure 8. Illustration to the proof of Theorem 5.1. A diagram with "vertical" paths resulting in dense orbits in the path space.

Since F is irreducible, there exists $k \in \mathbb{N}$ such that $f_{ii}^{(k)} > 0$. In other words, there is an infinite path x in X_B which passes through the vertex i on levels $s \cdot k$ for all $s \in \mathbb{N}$. We will show that x is a transitive point, that is,

$$\overline{[x]_{\mathcal{R}}} = X_B.$$

Indeed, it is enough to show that the tail equivalence class of x intersects every cylinder set. Let $[\overline{e}] = (e_0, \dots, e_{N-1})$ be an arbitrary cylinder set. Denote $j = r(e_{N-1}) \in V_N$, we show that for $s \in \mathbb{N}$ large enough there is a path between $i \in V_{sk}$ and $j \in V_N$. Since the matrix F is irreducible, there exists $t \in \mathbb{N}$ such that $f_{ij}^{(t)} > 0$. By Lemma A.1, there exists $l = l(j) \in \mathbb{N}$ such that

$$f_{jj}^{(m)} > 0 \quad \text{for all } m \geq l.$$

Note that there exist $s \in \mathbb{N}$ and an integer $M \geq l$ such that there is a path of length t between $i \in V_{sk}$ and $j \in V_{sk-t}$ and a path of length M between $j \in V_{sk-t}$ and $j \in V_N$ (see Figure 8). To see this, choose s such that $sk - t - N > l$. Take $M = sk - t - N$. Hence, we obtain

$$f_{ij}^{(M+t)} \geq f_{ij}^{(t)} f_{jj}^{(M)} > 0.$$

Thus, there exists a path of length $M + t$ between vertex $j = r(e_N) \in V_N$ and vertex i on level V_{sk}. Therefore, $[x]_{\mathcal{R}}$ and $[\overline{e}]$ have a non-empty intersection. ∎

Remark 5.2. Theorem 5.1 focuses on *stationary* diagrams. In this remark, we consider a structural property of the diagram which implies the transitivity of the tail equivalence relation for non-stationary diagrams as well. Let $B(V, E)$ be an irreducible generalized

Bratteli diagram with a vertex $i \in \mathbb{Z}$ (recall that we identify vertices at each level with integers) such that for every level n there is an edge between $i \in V_n$ and $i \in V_{n+1}$. In other words, there exists an infinite vertical path $x \in X_B$ which passes through vertex i on each level of the diagram. Then, it can be easily proved (in the same manner as in the proof of Theorem 5.1) that the orbit $[x]_{\mathcal{R}}$ is dense in X_B, and hence the tail equivalence relation $\mathcal{R}$ is topologically transitive. More generally, it suffices to assume that there exists a sequence (n_k) such that there is a path between the vertex $i \in V_{n_k}$ and $i \in V_{n_{k+1}}$ for all k. Then after telescoping, we can use the above result.

It is natural to ask if there exist conditions on the structure of a generalized Bratteli diagram that would imply the minimality of the tail equivalence relation. To this effect, we show (see Theorem 5.4) that bounded size diagrams (see Definition 3.1) with irreducible incidence matrices contain proper closed subsets that are invariant under the tail equivalence relation and nowhere dense in the path space. In particular, this shows that the tail equivalence relation is *not minimal*. Section 4 provides an example of a minimal Vershik map on an ordered generalized Bratteli diagram which has no infinite minimal and no infinite maximal paths (see Example 4.2). Since the orbits of the Vershik map and the tail equivalence relation are the same for such a diagram, we obtain an example of a stationary generalized Bratteli diagram with an irreducible aperiodic incidence matrix such that the corresponding tail equivalence relation is minimal.

Let $B = (V, E)$ be a generalized Bratteli diagram of bounded size, with the corresponding sequence $(t_n, L_n)_{n \in \mathbb{N}_0}$. For $w \in V_0$, we define

$$Z_w^+ = \left\{ x = (x_n) \in X_B : s(x_0) \geq w \text{ and } r(x_n) \geq w + \sum_{i=0}^{n} t_i \text{ for all } n \in \mathbb{N}_0 \right\}.$$

Similarly, for $w \in V_0$ define

$$Z_w^- = \left\{ x = (x_n) \in X_B : s(x_0) \leq w \text{ and } r(x_n) \leq w - \sum_{i=0}^{n} t_i \text{ for all } n \in \mathbb{N}_0 \right\}.$$

Lemma 5.3. *Let $B = (V, E)$ be a generalized Bratteli diagram of bounded size, with the corresponding sequence $(t_n, L_n)_{n \in \mathbb{N}_0}$. Then, for every $w \in V_0$, the sets Z_w^+, Z_w^- are invariant with respect to the tail equivalence relation $\mathcal{R}$.*

Proof. Fix $x = (x_n) \in Z_w^+$ and consider an infinite path $y = (y_n) \in X_B$ which is tail equivalent to x. Thus, there exists $n \in \mathbb{N}$ such that $r(x_n) = r(y_n) = v$ for some $v \in V_{n+1}$. Since $x \in Z_w^+$, we have

$$v \geq w + \sum_{i=0}^{n} t_i.$$

By Corollary 3.6,

$$s(y_m) \in \left[v - \sum_{i=m}^{n} t_i, v + \sum_{i=m}^{n} t_i \right] \subset \left[w + \sum_{i=1}^{m-1} t_i, \infty \right)$$

for all $m \leq n$. In other words,

$$r(y_m) = s(y_{m+1}) \in \left[w + \sum_{i=1}^{m} t_i, \infty \right).$$

Since $x \in Z_w^+$ and x and y are tail equivalent, we also have

$$r(y_k) \in \left[w + \sum_{i=1}^{k} t_i, \infty \right)$$

for all $k \geq n$. Thus, $y \in Z_w^+$, this shows that Z_w^+ is invariant with respect to the tail equivalence relation $\mathcal{R}$. A similar argument shows that Z_w^- is also invariant with respect to the tail equivalence relation $\mathcal{R}$. ∎

We will call the sets Z_w^+, Z_w^- *slanting sets* for $w \in V_0$.

Theorem 5.4. *Let $B = (V, E)$ be a generalized Bratteli diagram of bounded size, with the corresponding sequence $(t_n, L_n)_{n \in \mathbb{N}_0}$. Then, for every $w \in V_0$, the sets Z_w^+, Z_w^- are closed nowhere dense sets with respect to the topology generated by cylinder sets. In particular, this shows (together with Lemma 5.3) that the tail equivalence relation $\mathcal{R}$ is not minimal.*

Proof. First, we prove that the set Z_w^+ is closed. Let Z_0 be a union of cylinder sets, on level V_0 which correspond to the vertices in the interval $[w, \infty) \subset V_0$. Let Z_n be a union of all cylinder sets corresponding to finite paths of length n which lie in Z_w^+. Then we have

$$Z_0 \supset Z_1 \supset \cdots \supset Z_n \supset \cdots,$$

each Z_n is closed and

$$Z_w^+ = \bigcap_{n=0}^{\infty} Z_n.$$

Hence, Z_w^+ is closed.

We show that Z_w^+ does not contain any cylinder set. Let $\overline{e} = (e_0, \ldots, e_n)$ be a finite path which lies in Z_w^+ and $v = r(e_n) \in V_{n+1}$. There exists m such that $v - \sum_{i=1}^{m} t_{n+i} < w + \sum_{i=1}^{m} t_{n+i}$. Since $E(u - t_n, u) \neq \emptyset$ for all n and for all $u \in V_{n+1}$, there is a finite path $(e_{n+1}, \ldots, e_{n+m})$ between $v \in V_{n+1}$ and $v - \sum_{i=1}^{m} t_{n+i} \in V_{n+m}$. Hence, the cylinder set generated by the path $(e_0, \ldots, e_{n+m})$ is a subset of $[\overline{e}]$ which does not belong to Z_w^+. Thus, Z_w^+ has empty interior. Since Z_w^+ is closed, it follows that Z_w^+ is nowhere dense. A similar argument shows that Z_w^- is also closed and nowhere dense. ∎

To end this section, we discuss the cardinality of the sets of the form Z_w^+ and Z_w^- for a bounded size generalized Bratteli diagram and provide conditions that guarantee that Z_w^+ and Z_w^- are countable sets.

Let $B = (V, E)$ be a bounded size generalized Bratteli diagram. For every $w \in V_0$, let Y_w^+ denote the set of all infinite paths which start at w and then pass through the rightmost possible vertex on each level, that is, for every $m \in \mathbb{N}$, the paths from Y_w^+ go through the vertex $w + \sum_{i=0}^{m-1} t_i$ on level m. Obviously, we have $Y_w^+ \subset Z_w^+$ for all $w \in V_0$. Analogously, let $Y_w^- \subset Z_w^-$ be the set of all infinite paths which start at w and then pass through the leftmost possible vertex on each level. Note that the sets Y_w^+, Y_w^- can be either finite or uncountable (then they are odometers). We can say that Y_w^+, Y_w^- are the "boundary" paths for Z_w^+, Z_w^-.

Proposition 5.5. *Let $B = (V, E)$ be a generalized Bratteli diagram of bounded size, with the corresponding sequence $(t_n, L_n)_{n \in \mathbb{N}_0}$. If the sets Y_w^+ are finite for all $w \in V_0$, then the sets Z_w^+ are countable for all $w \in V_0$. Otherwise, there exists an uncountable set Z_w^+. The same is true for Y_w^- and Z_w^-. In particular, if $|E(v + t_n, v)| = |E(v - t_n, v)| = 1$ for all $v \in V_{n+1}$ and $n \in \mathbb{N}$, then sets Z_w^+, Z_w^- are countable for all $w \in V_0$.*

Proof. Let Y_w^+ be finite for all $w \in V_0$. Fix any $w \in V_0$. Suppose $y = (y_n)_{n=0}^\infty \in Z_w^+$ and $s(y_0) = u \geq w$. Since B is of bounded size, for every $m \in \mathbb{N}_0$:

$$r(y_m) = u + \widetilde{t}_0 + \cdots + \widetilde{t}_m,$$

where $\widetilde{t}_i \in [-t_i, t_i]$ for $i = 1, \ldots, m$. Since $y \in Z_w^+$, we have

$$r(y_m) = u + \widetilde{t}_0 + \cdots + \widetilde{t}_m \in \left[w + \sum_{i=0}^m t_i, \infty\right).$$

Thus, for all $m \in \mathbb{N}_0$,

$$u + \sum_{i=0}^m \widetilde{t}_i \geq w + \sum_{i=0}^m t_i$$

and

$$u - w \geq \sum_{i=0}^m (t_i - \widetilde{t}_i).$$

Recall that $t_i \geq \widetilde{t}_i$, hence $t_i - \widetilde{t}_i \geq 0$. Thus, we have

$$u - w \geq \sum_{i=0}^\infty (t_i - \widetilde{t}_i),$$

which is possible only if there are finitely many non-zero elements among $(t_i - \widetilde{t}_i)$. Hence, the path y should go in the same direction as the "boundary" paths Y_w^+, except for finitely many deviations. Since all the sets Y_w^+ are finite, the number of such paths y is countable. If for some $w \in V_0$ the set Y_w^+ is uncountable, then the set $Z_w^+ \supset Y_w^+$ is also uncountable. The same proof works for Y_w^- and Z_w^-. ■

6. Tail-invariant measures for generalized Bratteli diagrams

In this section, we discuss *tail-invariant measures* on the path space of a generalized Bratteli diagram. We emphasize that in this paper, the term *measures* is used for non-atomic positive Borel measures. Moreover, we are mostly interested in *full measures*, that is, every cylinder set must be of positive measure. We consider both finite (probability) and σ-finite measures. In the case of σ-finite measures, we are interested in only those measures which take finite values on cylinder sets.

We describe every tail-invariant measure in terms of a sequence of positive vectors associated with vertices of each level (see Theorem 6.6). We give an explicit construction of ordered generalized Bratteli diagrams for which there exists no full probability measure invariant under the Vershik map, and the restriction of the tail equivalence relation onto the equivalence class of some non-empty clopen set is compressible (see Theorem 6.9). We also provide a class of generalized Bratteli diagrams such that there exists no tail-invariant (finite or σ-finite) measure with finite values on cylinder sets (see Proposition 6.11).

Definition 6.1. Let $B = (V, E)$ be a generalized Bratteli diagram and $\mathcal{R}$ the tail equivalence relation on the path space X_B (see Definition 2.5). A measure μ on X_B is called *tail-invariant* if, for any cylinder sets $[\overline{e}]$ and $[\overline{e}']$ such that $r(\overline{e}) = r(\overline{e}')$, we have $\mu([\overline{e}]) = \mu([\overline{e}'])$.

Remark 6.2. The theory of countable Borel equivalence relations is a key object in Borel dynamics; it has been considered from different points of view in numerous books and articles (see, e.g., [27,41,42,49,52] and the literature within). For any generalized Bratteli diagram B, the tail equivalence relation $\mathcal{R}$ is a countable Borel hyperfinite equivalence relation. This means that there exists a Borel automorphism $T : X_B \to X_B$ whose orbits coincide with the orbits of $\mathcal{R}$. Can we take a Vershik map φ_B for T? First, we note that the set of tail-invariant measures does not depend on an order on B. Second, the orbits of a Vershik map and the tail-invariant relation differ at the sets of maximal and minimal paths. In general, every measure μ that is invariant with respect to a Vershik map is also tail-invariant. Hence, if the sets of maximal and minimal paths have zero measure, then we can identify tail-invariant measures with measures invariant with respect to a Vershik map. This happens for generalized Bratteli diagrams of bounded size (see Lemma 3.8). As a rule, we will consider measures on the path space of (ordered) Bratteli diagrams with zero-measure sets of maximal and minimal paths. This property will allow us to use the notions of tail-invariant measures and that of φ_B-invariant measures interchangeably (see Section 6).

In what follows, we will use the following obvious fact: Suppose that a tail-invariant Borel measure μ on X_B takes finite values on all cylinder sets. Then μ is uniquely determined by its values on cylinder sets in X_B, that is, it can be extended uniquely to all Borel sets.

The definition below uses the notion of *Kakutani–Rokhlin towers* which is well studied in the context of Cantor dynamics. We refer the reader to [11, 45, 48, 60] where this notion is discussed.

Definition 6.3. Let $B = (V, E)$ be a generalized Bratteli diagram, for $w \in V_n, n \in \mathbb{N}$, denote

$$X_w^{(n)} = \{x = (x_i) \in X_B : r(x_{n-1}) = w\}.$$

The collection of all such sets forms a partition of X_B into *Kakutani–Rokhlin towers* corresponding to the vertices from V_n. Each finite path $\overline{e} = (e_0, \dots, e_{n-1})$ with $r(e_{n-1}) = w$ determines a "floor" of this tower

$$X_w^{(n)}(\overline{e}) = \{x = (x_i) \in X_B : x_i = e_i,\ i = 0, \dots, n-1\}$$

(we denoted this set by $[\overline{e}]$ above; the notation $X_w^{(n)}(\overline{e})$ indicates the position of $[\overline{e}]$ in the tower $X_w^{(n)}$). Clearly,

$$X_w^{(n)} = \bigcup_{\overline{e} \in E(V_0, w)} X_w^{(n)}(\overline{e}).$$

Thus, the set $X_w^{(n)}$ is a union of a finite number of cylinder sets and can be considered as a *tower associated with the vertex* $w \in V_n$.

Definition 6.4. For $v \in V_n$ and $v_0 \in V_0$, we set $h_{v_0,v}^{(n)} = |E(v_0, v)|$ and define

$$H_v^{(n)} = \sum_{v_0 \in V_0} h_{v_0,v}^{(n)}, \quad n \in \mathbb{N}.$$

Set $H_v^{(0)} = 1$ for all $v \in V_0$. This gives us the vector $H^{(n)} = (H_v^{(n)} : v \in V_n)$ associated with every level $n \in \mathbb{N}_0$. Since $H_v^{(n)} = |E(V_0, v)|$, we call $H_v^{(n)}$ the *height of the tower* $X_v^{(n)}$ corresponding to the vertex $v \in V_n$.

Remark 6.5. We have defined the vector $H^{(0)} = (H_v^{(0)} : v \in V_0)$ such that $H_v^{(0)} = 1$ for all v (see Definition 6.4). In fact, one can choose any finite values for $H_v^{(0)}$. The role of $H^{(0)}$ can be interpreted as the vector of heights of the Kakutani–Rokhlin towers between the vertices of V_0 and an imaginary level V_{-1} consisting of exactly one vertex.

Observe that

$$H_v^{(n+1)} = \sum_{w \in V_n} f_{vw}^{(n)} H_w^{(n)}, \quad v \in V_{n+1},$$

which immediately implies that

$$F_n H^{(n)} = H^{(n+1)} \quad \text{and} \quad F_n \cdots F_0 H^{(0)} = H^{(n+1)}, \quad n \in \mathbb{N}_0. \tag{6.1}$$

We consider here the problem of the existence of tail-invariant measures on the path space of a generalized Bratteli diagram B. Our main results are mostly related to Bratteli diagrams of bounded size. Note that every incidence matrix F_n defines a linear map from $\mathbb{R}^{V_n}$ to $\mathbb{R}^{V_{n+1}}$ (recall that we identify all V_n). Using Lemma 3.7, we see that, for every fixed $n \in \mathbb{N}_0$ and any $m \in \mathbb{N}$, we can define the sequence of convex sets

$$C_m^{(n)} = F_n^\top \cdots F_{n+m-1}^\top(\mathbb{R}_+^{V_{n+m}}),$$

where $F_i^\top$ stands for the transpose of F_i. The above relation is well defined because each $F_i^\top$ has rows with finitely many non-zero entries.

Set

$$C_\infty^{(n)} = \bigcap_{m=1}^{\infty} C_m^{(n)}.$$

In general, the set $C_\infty^{(n)}$ might be empty.

Given a Bratteli diagram B (generalized or classical), let $\mathcal{M}(B)$ denote the set of tail-invariant finite or σ-finite measures on the path space X_B which takes finite values on cylinder sets. In the following theorem, we assume that the set $\mathcal{M}(B)$ is not empty.

Theorem 6.6. *Let $B = (V, E)$ be a Bratteli diagram (generalized or classical) with the sequence of incidence matrices (F_n). Then:*

(1) *If $\mu \in \mathcal{M}(B)$, then for every $n \in \mathbb{N}_0$ the vector defined as follows*

$$p^{(n)} = (\mu(X_w^{(n)}(\overline{e})))_{w \in V_n} \tag{6.2}$$

satisfies $p^{(n)} \in C_\infty^{(n)}$ and

$$F_n^\top p^{(n+1)} = p^{(n)} \tag{6.3}$$

for all $n \geq 0$.

(2) *Conversely, suppose that $\{p^{(n)} = (p_w^{(n)})\}_{n \in \mathbb{N}_0}$ is a sequence of non-negative vectors such that $p^{(n)} \in C_\infty^{(n)}$ and $F_n^\top p^{(n+1)} = p^{(n)}$ for all $n \in N_0$. Then there exists a uniquely determined tail-invariant measure μ such that $\mu(X_w^{(n)}(\overline{e})) = p_w^{(n)}$ for $w \in V_n, n \in \mathbb{N}_0$.*

The *proof* of Theorem 6.6 is straightforward and can be found in [15] (for classical Bratteli diagrams) and [7] (for generalized Bratteli diagrams).

Remark 6.7. We stress that part (1) of Theorem 6.6 is true for any generalized Bratteli diagram whose path space admits a tail-invariant measure. This means also that, for this diagram, the cone $C_\infty^{(n)}$ is not empty for all $n \in \mathbb{N}_0$. In Proposition 6.11, we give an example of a bounded size diagram such that both sets $\mathcal{M}(B)$ and $C_\infty^{(n)}$ are empty.

Let $\mathcal{E}$ be a countable Borel equivalence relation on a standard Borel space X. It is a well-known fact that the existence of an $\mathcal{E}$-invariant probability Borel measure μ on X is determined by the property of $\mathcal{E}$ called *compressibility*. For a fixed $x \in X$, the set $\{y \in X : (x, y) \in \mathcal{E}\}$ is called the $\mathcal{E}$-class. It is said that $\mathcal{E}$ is *compressible* if there is an injective Borel map $f : X \to X$ such that for each $\mathcal{E}$-class L, $f(L) \subsetneq L$. A Borel set $A \subset X$ is compressible if the restriction of $\mathcal{E}$ onto A is compressible. We refer to [57, 58] where the following lemma is proved (see also [27]).

Lemma 6.8. *Let $\mathcal{E}$ be a countable equivalence relation on a standard Borel space X. The following are equivalent:*

(1) *$\mathcal{E}$ is not compressible.*

(2) *There is an $\mathcal{E}$-invariant probability measure.*

(3) *There is an $\mathcal{E}$-ergodic, $\mathcal{E}$-invariant probability measure.*

Now we give an explicit example of a generalized Bratteli diagram such that the restriction of the tail equivalence relation onto the equivalence class of a non-empty clopen set is compressible.

Theorem 6.9. *For the one-sided generalized Bratteli diagram $B = (B, \omega)$ with the left-to-right ordering ω shown in Figure 9, we have*

(1) *The set $X_{\max}$ is empty.*

(2) *The Vershik map $\varphi_B : X_B \to X_B \setminus X_{\min}$ is a homeomorphism.*

(3) *There exists a non-empty clopen set C such that its tail equivalence class $\mathcal{R}(C)$ is compressible.*

(4) *There is no probability φ_B-invariant measure on X_B that assigns positive values to all cylinder sets.*

Proof. The one-sided diagram in Figure 9 is a modified version of [29, Example 7.2]. We identify each vertex level V_i with $\mathbb{N}$. Similar to Example 3.9, it is easy to note that there are no infinite maximal paths in the diagram with respect to the left-to-right order ω. This shows (1). As a consequence, the Vershik map φ_B corresponding to ω is a homeomorphism from X_B to $X_B \setminus X_{\min}$. Hence we get (2).

To prove (4), we show that there exists a cylinder set $C \subset X_B$ such that $\varphi_B(X_B) = X_B \setminus C$. Let C be the cylinder set formed by all paths that begin at the leftmost vertex of V_0. The subdiagram corresponding to C is a tree and consists only of infinite minimal paths. Thus, φ_B maps continuously X_B to $X_B \setminus C$. The property

$$\varphi_B(X_B) = X_B \setminus X_{\min} \subset X_B \setminus C$$

shows that there does not exist any probability φ_B-invariant measure μ such that $\mu(C) > 0$. Denote by $\mathcal{R}(C)$ the tail equivalence class of C. Since every tail equivalence class L in $\mathcal{R}(C)$ contains a minimal path, we have $\varphi_B(L) \subsetneq L$ for every L. Thus, (3) is also proved. ∎

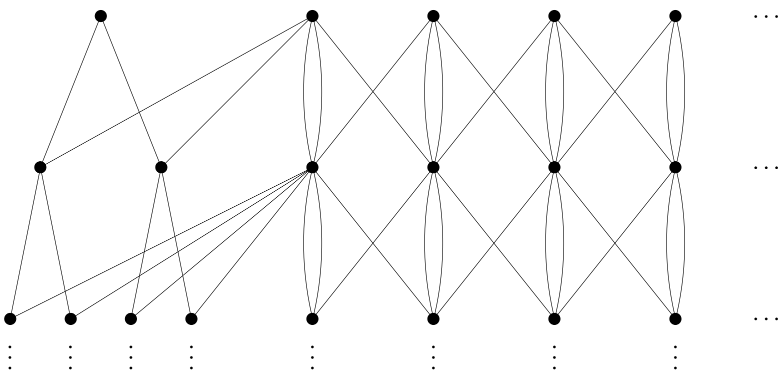

Figure 9. A diagram with the left-to-right ordering, and no infinite maximal paths.

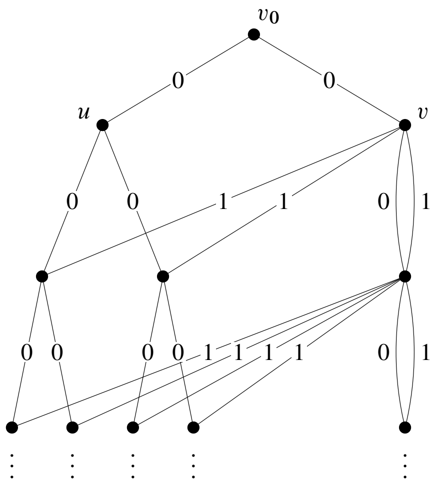

Figure 10. Illustration of the result in Theorem 6.9 via a standard Bratteli diagram.

Remark 6.10. Observe that for the classical (standard) Bratteli diagram shown in Figure 10, the conclusion (4) of Theorem 6.9 also holds. To see this, we extend the Vershik map to the entire path space by mapping the unique maximal path to the unique minimal path in the 2-odometer (i.e., the subdiagram corresponding to the vertex v in Figure 10).

Let C be the cylinder set defined by the edge $[v_0, u]$. Then it is easy to see that $\varphi_B(X_B) = X_B \setminus C$. Hence, every probability φ_B-invariant measure μ must satisfy the condition $\mu(C) = 0$ which implies (4). There is a unique probability invariant measure on X_B sitting on the minimal component of the tail equivalence relation, the 2-odometer corresponding to the vertex v.

Proposition 6.11 gives an example of a stationary generalized Bratteli diagram such that there is no tail-invariant measure with finite values on clopen sets. We emphasize that this example can be generalized to a class of stationary diagrams with similar property.

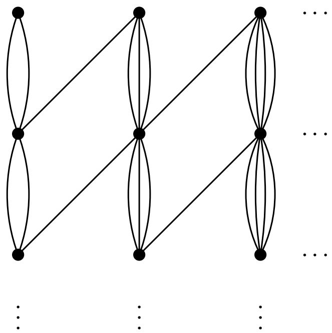

Figure 11. A Bratteli diagram with no finite ergodic invariant measure.

These diagrams have an incidence matrix of the form given by (6.4) where the diagonal entries form an increasing sequence of positive integers.

Proposition 6.11. *Let $B = B(F)$ be a one-sided generalized stationary Bratteli diagram as shown in Figure 11 and given by $\mathbb{N} \times \mathbb{N}$ incidence matrix:*

$$F = \begin{pmatrix} 2 & 1 & 0 & 0 & \dots \\ 0 & 3 & 1 & 0 & \dots \\ 0 & 0 & 4 & 1 & \dots \\ 0 & 0 & 0 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{6.4}$$

There does not exist any tail-invariant measure on X_B that assigns finite values to cylinder sets.

Proof. As above, we identify vertices at each level with natural numbers. For $k \in \mathbb{N}$, denote by C_k the cylinder set corresponding to the vertex k on level V_0, that is,

$$C_k = \{x \in X_B : s(x) = k\}, \quad k \in V_0.$$

Recall that $s : E \to V$ is the source map. Suppose that there exists a non-zero tail-invariant measure μ on X_B which assigns finite values to cylinder sets. Denote by

$$m = \min\{k \in \mathbb{N} : \mu(C_k) > 0\}.$$

By our assumption, the minimum exists. Normalize the measure μ such that $\mu(C_m) = 1$. Let $\overline{e} = (e_0, \dots, e_n)$ be a finite path of length $(n + 1)$ such that the range of $\overline{e}$ is the vertex labeled by m on level V_n, that is, $r(\overline{e}) = m \in V_n$. As before, we denote by $[\overline{e}]$ the corresponding cylinder set. It follows from the definition of F that the path space X_B contains countably many odometers: The set of vertical paths going through a vertex i is an $(i+1)$-odometer. By tail invariance of μ, all cylinder sets in the $(m+1)$-odometer have the same measure as the set $[\overline{e}]$ has:

$$\mu([\overline{e}]) = \frac{1}{(m+1)^n}.$$

Thus, we have

$$\mu(C_{m+1}) \geq \sum_{n=1}^{\infty} \frac{(m+2)^{n-1}}{(m+1)^n} = \infty,$$

and this is a contradiction. ∎

Remark 6.12. Let $B(F)$ be a stationary Bratteli diagram as in Proposition 6.11, and let μ be any tail-invariant measure. Recall that the vector $p^{(0)}$ (see (6.2)) consists of the values of the measure μ of cylinder sets corresponding to the level V_0. It follows from the proof of Proposition 6.11 that if, for some vertex $i \in V_0$, we have

$$0 < p_i^{(0)} < \infty,$$

then $p_j^{(0)} = 0$ for every $j < i$ and $p_j^{(0)} = \infty$ for every $j > i$. Since the diagram is stationary, the same property holds for every level V_n; $n \in \mathbb{N}_0$.

7. Uniqueness of tail-invariant measures for stationary Bratteli diagrams

7.1. Tail-invariant measures and Perron–Frobenius eigenvectors

In [7], the authors used the Perron–Frobenius (P-F) theory for infinite matrices to provide a description of tail-invariant measures on the path space of a class of stationary generalized Bratteli diagrams. This class consists of diagrams with irreducible, aperiodic, and recurrent incidence matrices. For the reader's convenience, we provide a brief description of the P-F theory for infinite matrices in Appendix A. The results formulated in Appendix A are mostly taken from Chapter 7 of the book [53]. The foundations of the P-F theory for infinite matrices were laid down in the 1960s in a series of articles by D. Vere-Jones [65–67]. In this section, we use results from Appendix A, in particular, Theorems A.4 and A.7 to prove a criterion for the uniqueness of the tail-invariant measure on the path space of a stationary generalized Bratteli diagram $B(F)$. Recall that a matrix is called countably infinite if its rows and columns are indexed by a countable set. If F is indexed by $\mathbb{N}$, then the diagram $B(F)$ is one-sided infinite, and if F is indexed by $\mathbb{Z}$, then the diagram is two-sided infinite. The main results in this section hold for both kinds of diagrams. We provide examples (see Subsection 7.3) of both kinds of diagrams.

We will keep the following notations: A is the transpose of the infinite incidence matrix F of the generalized Bratteli diagram $B(F)$. When it exists (see Theorem A.4), we will denote by λ the Perron eigenvalue and by $\xi = (\xi_v)$, $\eta = (\eta_v)$ the right and left eigenvectors for A, that is, $A\xi = \lambda\xi$ and $\eta A = \lambda\eta$. Note that all entries of ξ and η are positive.

In this section, we will work with an ordered stationary generalized Bratteli diagram $B(F) = B(V, E, >)$, where $>$ denotes a fixed order. We will assume that the order $>$ gives rise to a Vershik map on the path space of the diagram. As before we will denote the corresponding dynamical system by (X_B, φ_B). We recall the following result (proved in [8]) which gives an explicit formula for a tail-invariant measure μ on the path space of a stationary generalized Bratteli diagram.

Theorem 7.1 ([8, Theorem 2.20]). *Let $B(F) = B(V, E)$ be a stationary generalized Bratteli diagram such that the matrix $A = F^\top$ is infinite, irreducible, aperiodic, and recurrent. Let $\xi = (\xi_v)$ be a Perron–Frobenius right eigenvector for A, that is $A\xi = \lambda\xi$, $\xi_v > 0$.*

(1) *There exists a tail-invariant measure μ on the path space X_B, satisfying the following property: If $\overline{e}(w, v)$ is a finite path that begins at $w \in V_0$ and ends at $v \in V_n$, $n \in \mathbb{N}$, then*
$$\mu([\overline{e}(w, v)]) = \frac{\xi_v}{\lambda^n}, \tag{7.1}$$
where $[\overline{e}(w, v)]$ is the corresponding cylinder set.

(2) *The measure μ is finite if and only if the Perron eigenvector $\xi = (\xi_v)$ has the property $\sum_v \xi_v < \infty$.*

Now we show that if the incidence matrix of a stationary generalized Bratteli diagram is *positive recurrent* (in addition to the properties described in Theorem 7.1), then the invariant measure given by (7.1) is unique. We consider the cases of finite and σ-finite measures separately. In the σ-finite case (Theorem 7.3), we will work with the assumption that the dynamical system is conservative. Examples 7.16 and 7.9 illustrate Theorems 7.2 and 7.3, respectively.

Theorem 7.2. *Let $B(F) = B(V, E, >)$ be an ordered stationary generalized Bratteli diagram such that the matrix $A = F^\top$ is infinite, irreducible, aperiodic, and positive recurrent. Let $\xi = (\xi_v)$ be a Perron–Frobenius right eigenvector for A such that $\sum_{u \in V_0} \xi_u = 1$. Then the measure μ given in (7.1) is the unique ergodic probability φ_B-invariant measure that takes positive values on cylinder sets.*

Proof. We fix a vertex $w \in V_n$, and consider a cylinder set given by $[\overline{e}] = (e_0, \dots, e_{n-1})$ with $r(e_{n-1}) = w$. Then by (7.1), we have
$$\mu([\overline{e}]) = \frac{\xi_w}{\lambda^n}.$$
As proved in Theorem 7.1, the measure μ is probability and takes finite positive values on cylinder sets. Suppose that ν is a probability ergodic φ_B-invariant measure with positive

values on cylinder sets. Let $N \geq n$ and $v \in V_N$ be such that $E(w, v) \neq \emptyset$. Then, by the Birkhoff ergodic theorem,

$$\nu([\overline{e}]) = \lim_{N\to\infty} \frac{|E(w,v)|}{H_v^{(N)}},$$

where $H_v^{(N)}$ is the total number of finite paths with the range at vertex $v \in V_N$ (see Definition 6.4). Since A is positive recurrent, it follows from Theorem A.7 that, for any $v, w \in V$,

$$\lim_{N\to\infty} \frac{a_{wv}^{(N)}}{\lambda^N} = \xi_w \eta_v,$$

where $a_{wv}^{(N)}$ is the entry of A^N and $\eta = (\eta_v)$ is the left eigenvector of A normalized by the condition $\eta \cdot \xi = 1$. Thus, we obtain

$$\nu([\overline{e}]) = \lim_{N\to\infty} \frac{a_{wv}^{(N-n)}}{\sum_{u\in V_0} a_{uv}^{(N)}} = \frac{\xi_w \eta_v}{\lambda^n \sum_{u\in V_0} \xi_u \eta_v} = \frac{\xi_w}{\lambda^n} = \mu([\overline{e}]). \qquad \blacksquare$$

Now we show the uniqueness of the infinite σ-finite φ_B-invariant measure given by (7.1). We work with an additional assumption that the dynamical system (X_B, φ_B, μ) is conservative. Recall that a measure-preserving transformation T of a σ-finite measure space $(X, \mathcal{B}, \mu)$ is called *conservative* if for any $A \in \mathcal{B}$ with $\mu(A) > 0$ one has $\mu(A \cap T^{-n}(A)) > 0$ for some $n \in \mathbb{N}$ (see, e.g., [1]).

Theorem 7.3. *Let $B(F) = B(V, E, >)$ be an ordered stationary generalized Bratteli diagram such that the matrix $A = F^\top$ is infinite, irreducible, aperiodic, and positive recurrent. Let $\xi = (\xi_v)$ be a Perron–Frobenius right eigenvector for A such that $\sum_{u\in V_0} \xi_u = \infty$. Let μ be the σ-finite φ_B-invariant (given by (7.1)) such that (X_B, φ_B, μ) is conservative. Then μ is the unique (up to a constant multiple) σ-finite φ_B-invariant ergodic measure that takes positive values on cylinder sets.*

Proof. By Theorem 7.1, there exists an invariant σ-finite measure μ on the path space of a generalized Bratteli diagram with irreducible, aperiodic, and recurrent incidence matrix. Consider two cylinder sets $[\overline{e_1}], [\overline{e_2}] \subset X_B$ such that $r(\overline{e_1}) = v_1 \in V_{n_1}$ and $r(\overline{e_2}) = v_2 \in V_{n_2}$. Without loss of generality, assume that $n_2 > n_1$. Using (7.1), we calculate the ratio of their measures:

$$\frac{\mu([\overline{e_1}])}{\mu([\overline{e_2}])} = \frac{\xi_{v_1}/\lambda^{n_1}}{\xi_{v_2}/\lambda^{n_2}} = \frac{\xi_{v_1}}{\xi_{v_2}} \lambda^{(n_2-n_1)}. \tag{7.2}$$

Let m be a σ-finite ergodic measure on X_B. Now we apply Hopf's ratio ergodic theorem for m and find the ratio of measures of the same cylinder sets $[\overline{e_1}]$ and $[\overline{e_2}]$ (see [1] for references). For this, let $N > n_2$ and $w \in V_N$ be such that the sets $E(v_1, w)$ and $E(v_2, w)$ are non-empty. Since A is irreducible, we can choose such N using Lemma A.1 (a). Now, we apply Hopf's ratio ergodic theorem to obtain

$$\frac{m([\overline{e_1}])}{m([\overline{e_2}])} = \lim_{N\to\infty} \frac{|E(v_1,w)|}{|E(v_2,w)|} = \lim_{N\to\infty} \frac{a_{v_1 w}^{(N-n_1)}}{a_{v_2 w}^{(N-n_2)}} = \lim_{N\to\infty} \frac{a_{v_1 w}^{(N-n_1)}}{\lambda^{N-n_1}} \cdot \frac{\lambda^{N-n_2}}{a_{v_2 w}^{(N-n_2)}} \cdot \lambda^{n_2-n_1}.$$

Since A is positive recurrent, it follows from Theorem A.7 that, for any $v, w \in V$,

$$\lim_{N\to\infty} \frac{a_{vw}^{(N)}}{\lambda^N} = \xi_v \eta_w.$$

Therefore,

$$\frac{m([\overline{e_1}])}{m([\overline{e_2}])} = \frac{\xi_{v_1}\ \eta_w}{\xi_{v_2}\ \eta_w}\lambda^{n_2-n_1} = \frac{\xi_{v_1}}{\xi_{v_2}}\lambda^{n_2-n_1} = \frac{\mu([\overline{e_1}])}{\mu([\overline{e_2}])}. \tag{7.3}$$

Relation (7.3) shows that, for every cylinder set $[\overline{e}]$, the ratio

$$\frac{m([\overline{e}])}{\mu([\overline{e}])} = c$$

for some constant c. This means that $m = c\mu$, and the proof is complete. ∎

7.2. Generalized Bratteli diagrams with finite tail-invariant measures

Suppose that a matrix $A = (a_{ij})$ with $a_{ij} \in \mathbb{N}_0$ for all i, j is such that the Perron–Frobenius theorem holds: There exist a finite Perron eigenvalue λ and a non-negative right eigenvector $\xi = (\xi_i)$ such that $A\xi = \lambda\xi$. We give sufficient conditions on the matrix A that lead to the existence of a summable eigenvector ξ, that is, $\sum_i \xi_i < \infty$. By Theorem 7.1, these conditions will guarantee the existence of a finite tail-invariant measure.

Proposition 7.4. *Let $A = (a_{i,j} : i, j \in \mathbb{N})$ be a non-negative matrix such that there exists a positive eigenvector ξ corresponding to an eigenvalue λ. If there exists a row of A with finitely many zero entries, then the eigenvector $\xi = (\xi_i)$ is summable:*

$$\sum_{i\in\mathbb{N}} \xi_i < \infty.$$

Proof. Let the i-th row of A have finitely many zero entries:

$$a_{ij} = 0 \iff (j \in I,\ |I| < \infty).$$

From the equality

$$\sum_{j\in\mathbb{N}} a_{ij}\xi_j = \lambda\xi_i,$$

we have

$$\sum_{j\in\mathbb{N}} \xi_j = \sum_{j\in I} \xi_j + \sum_{j\notin I} \xi_j \le \sum_{j\in I} \xi_j + \sum_{j\in\mathbb{N}} a_{ij}\xi_j = \sum_{j\in I} \xi_j + \lambda\xi_i < \infty.$$ ∎

Note that the matrix A will also have a summable right eigenvector if there are two rows in A such that one row contains non-zero elements in even-numbered columns and the other one in odd-numbered columns. More generally, the following result holds.

Proposition 7.5. *Let A be an $\mathbb{N} \times \mathbb{N}$ matrix with non-negative integer entries such that $A\xi = \lambda\xi$ for $0 < \lambda < \infty$ and $\xi > 0$. For every $k \in \mathbb{N}$, let*

$$M_k = \{j \in \mathbb{N} : a_{kj} > 0\}.$$

Assume that there exists a finite collection of rows $\{k_1, \dots, k_p\}$ such that

$$\left| \mathbb{N} \setminus \bigcup_{t=1}^{p} M_{k_p} \right| < \infty.$$

Then the eigenvector $\xi = (\xi_i)$ is finite in the sense that

$$\sum_{i \in \mathbb{N}} \xi_i < \infty.$$

Proof. For each $t = 1, \dots, p$, we have

$$\sum_{j \in \mathbb{N}} a_{k_t j} \xi_j = \sum_{j \in M_{k_t}} a_{k_t j} \xi_j = \lambda \xi_{k_t}.$$

Consider the sum of these relations:

$$\sum_{t=1}^{p} \sum_{j \in M_{k_t}} a_{k_t j} \xi_j = \lambda \sum_{t=1}^{p} \xi_{k_t}.$$

Note that since $a_{k_t j} \geq 1$ for $j \in M_{k_t}$, we have

$$\sum_{t=1}^{p} \sum_{j \in M_{k_t}} a_{k_t j} \xi_j \geq \sum_{j \in \bigcup_{t=1}^{p} M_{k_p}} \xi_j.$$

Then we get

$$\begin{aligned} \sum_{j \in \mathbb{N}} \xi_j &= \sum_{j \in \mathbb{N} \setminus \bigcup_{t=1}^{p} M_{k_p}} \xi_j + \sum_{j \in \bigcup_{t=1}^{p} M_{k_p}} \xi_j \\ &\leq \sum_{j \in \mathbb{N} \setminus \bigcup_{t=1}^{p} M_{k_p}} \xi_j + \lambda \sum_{t=1}^{p} \xi_{k_t} < \infty, \end{aligned}$$

since the set $\mathbb{N} \setminus \bigcup_{t=1}^{p} M_{k_p}$ is finite. ∎

Remark 7.6. The converse of Proposition 7.5 is not true since there are banded matrices with probability right eigenvectors (see examples in Subsection 7.3).

7.3. Examples

In this subsection, we consider several classes of stationary generalized Bratteli diagrams that admit finite and σ-finite tail-invariant measures on their path spaces. The reader can find the proof of these results in Appendix B.

Example 7.7. For $a, b \in \mathbb{N}$, consider the generalized stationary Bratteli diagram $B(F_1)$ where $A_1 = F_1^\top$ is given by

$$A_1 = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & 2b & 0 & a & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 2b & 0 & a & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 2b & 0 & b & \mathbf{0} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 2b & a & \mathbf{b} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b} & \mathbf{a} & \mathbf{2b} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \mathbf{b} & 0 & 2b & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & a & 0 & 2b & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & a & 0 & 2b & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{7.4}$$

We use the bold font to indicate the 0-th row and 0-th column. Remark that this matrix is also considered in Example (8.4) from a different point of view. In [10], it was shown that for $a = b = 1$, one can model by $B(F_1)$ endowed with the left-to-right order a substitution dynamical system given by the so-called "one step forward, two steps back" substitution on $\mathbb{Z}$:

$$\begin{aligned} -1 &\mapsto -2\,-1\,0; \quad 0 \mapsto -1\,0\,1; \\ n &\mapsto (n-1)(n+1)(n+1) \quad \text{for } n \le -2; \\ n &\mapsto (n-1)(n-1)(n+1) \quad \text{for } n \ge 1. \end{aligned}$$

Proposition 7.8. *The stationary generalized Bratteli diagram $B(F_1)$, where $A_1 = F_1^\top$ as in (7.4), supports a tail-invariant measure μ given by (7.1). The measure μ is defined using the eigenvalue $\lambda = a + 2b$ and the corresponding right eigenvector ξ of A_1 given by*

$$\xi = \left(\ldots, \frac{1}{2^3}\left(\frac{a}{b}\right)^2, \frac{1}{2^2}\left(\frac{a}{b}\right), \frac{1}{2}, 1, \mathbf{1}, \frac{1}{2}, \frac{1}{2^2}\left(\frac{a}{b}\right), \frac{1}{2^3}\left(\frac{a}{b}\right)^2, \ldots\right)^\top.$$

(The 0-th entry of ξ is shown in bold font.) Moreover, μ is finite if and only if $a < 2b$.

We prove this result in Appendix B.

Example 7.9 (see also [19]). This example deals with a Bratteli diagram of different nature. This diagram is defined by a null-recurrent matrix of period 2 and has at least two infinite σ-finite measures which take finite positive values on cylinder sets.

Let a and b be natural numbers. Consider the generalized stationary Bratteli diagram $B(F_2)$ where $A_2 = F_2^{\top}$ is given by

$$A_2 = A_2(a,b) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & 0 & b & 0 & 0 & 0 & \cdots \\ \cdots & a & 0 & b & 0 & 0 & \cdots \\ \cdots & 0 & a & 0 & b & 0 & \cdots \\ \cdots & 0 & 0 & a & 0 & b & \cdots \\ \cdots & 0 & 0 & 0 & a & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{7.5}$$

Proposition 7.10. *Let $B(F_2)$ be the Bratteli diagram where $A_2 = F_2^{\top}$ is as in (7.5).*

(1) *If $a \neq b$, then there exist at least two tail-invariant infinite σ-finite measures on the path space X_B, μ and $\widetilde{\mu}$. According to (7.1), the measure μ is determined by $\lambda = a + b$ and $\xi = (\ldots, 1, 1, 1 \ldots)^{\top}$, and the measure $\widetilde{\mu}$ is determined by $\widetilde{\lambda} = 2\sqrt{ab}$ and $\widetilde{\xi} = (\widetilde{\xi}_n)^{\top}$ where $\widetilde{\xi}_n = \left(\frac{a}{b}\right)^{\frac{n}{2}}$, $n \in \mathbb{Z}$.*

(2) *If $a = b$, then $\mu = \widetilde{\mu}$.*
The matrix $A_2(a,b)$ is null recurrent for any $a, b \in \mathbb{N}$.

This matrix is also considered in Example 8.2 where we use the corresponding stochastic matrix to show that $A_2(a,b)$ is null recurrent for any $a, b \in \mathbb{N}$. We prove Proposition 7.10 in Appendix B.

Examples of one-sided infinite Bratteli diagrams. In this subsection, we consider examples of one-sided infinite stationary generalized Bratteli diagrams and find conditions under which infinite tri-diagonal matrices (indexed by $\mathbb{N}$) have eigenvectors with finite entry sum (hence admit finite invariant measure).

Let $A = (a_{ij})_{i,j \in \mathbb{N}}$ be a one-sided infinite tri-diagonal matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & \ldots \\ a_{21} & a_{22} & a_{23} & 0 & 0 & \ldots \\ 0 & a_{32} & a_{33} & a_{34} & 0 & \ldots \\ 0 & 0 & a_{43} & a_{44} & a_{45} & \ldots \\ 0 & 0 & 0 & a_{54} & a_{55} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We assume that A has an eigenvalue λ with the right eigenvector ξ, that is, $A\xi = \lambda\xi$. Our goal is to find conditions when the vector ξ is summable, $\sum_{i=1}^{\infty} \xi_i < \infty$. We denote by

$$\sigma_i = \sum_{j=1}^{\infty} a_{ij}$$

the sum of the i-th row of A. It turns out that an important class of examples comes from the matrices having equal column sum property together with an additional requirement on the row sum as described in the definition below.

Definition 7.11. We say that a one-sided infinite matrix $A = (a_{ij})_{i,j \in \mathbb{N}}$ is *balanced* if it satisfies following conditions:

(1) A has the property of equal column sum, that is, $\sum_{i=1}^{\infty} a_{ij} = c$ for every $j \in \mathbb{N}$; this automatically implies that $c = \lambda$ for every $j \in \mathbb{N}$.

(2) $\sigma_2 = \sigma_3 = \sigma_4 = \cdots$, and $\sigma_1 > \sigma_i$ for all $i > 1$.

Below we provide some examples of one-sided infinite stationary Bratteli diagrams with *balanced* incidence matrices.

Example 7.12. Fix $b, c, \alpha \in \mathbb{N}$ such that $\alpha > 1$. Consider the generalized diagram $B(F_3)$ where $A_3 = F_3^\top$ is given by

$$A_3 = \begin{pmatrix} b+\alpha c & \alpha c & 0 & 0 & 0 & \dots \\ c & b & \alpha c & 0 & 0 & \dots \\ 0 & c & b & \alpha c & 0 & \dots \\ 0 & 0 & c & b & \alpha c & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{7.6}$$

Proposition 7.13. *The stationary generalized Bratteli diagram $B(F_3)$, where $A_3 = F_3^\top$ is defined in (7.6), supports a tail-invariant measure μ defined by (7.1) using the eigenvalue $\lambda = b + c + \alpha c$ and the corresponding right eigenvector $\xi = (\xi_n)_{n \in \mathbb{N}}$ for A_3 such that*

$$\xi = \left(\xi_1, \frac{\xi_1}{\alpha}, \frac{\xi_1}{\alpha^2}, \dots, \frac{\xi_1}{\alpha^{n-1}}, \dots\right)^\top,$$

where ξ_1 can be chosen to be any positive integer.

Proposition 7.13 is proved in Appendix B.

Example 7.14. Fix $b \in \mathbb{N}_0$, $r \in \mathbb{N}$ and take integers α and β such that $|\alpha|, |\beta| < r$. Denote by

$$q_1 = \frac{r-\alpha}{r+\beta}, \quad q_2 = \frac{r-\beta}{r+\alpha}. \tag{7.7}$$

Consider the one-sided infinite stationary Bratteli diagram $B(F_4)$ where $A_4 = F_4^\top$ is given by

$$A_4 = F_4^\top = \begin{pmatrix} (b+r+\alpha) & r+\beta & 0 & 0 & 0 & \dots \\ r-\alpha & b & r+\alpha & 0 & 0 & \dots \\ 0 & r-\beta & b & r+\beta & 0 & \dots \\ 0 & 0 & r-\beta & b & r+\beta & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{7.8}$$

Proposition 7.15. *For the stationary Bratteli diagram $B(F_4)$ defined by the matrix given in (7.4), the following statements hold:*

(1) *For the eigenvalue $\lambda = b + 2r$, the right eigenvector $\xi = (\xi_n)$ ($A\xi = \lambda\xi$) for A_4 has the entries*

$$x_1 = 1, \quad \xi_{2n} = \frac{(r-\alpha)^n\,(r-\beta)^{n-1}}{(r+\alpha)^{n-1}(r+\beta)^n}, \quad \xi_{2n+1} = \frac{(r-\alpha)^n(r-\beta)^n}{(r+\alpha)^n(r+\beta)^n}. \tag{7.9}$$

(2) *If r, α, β are chosen such that $q_1q_2 < 1$ (see (7.7)), then the diagram $B(F_4)$ supports a finite tail-invariant measure μ determined as in (7.1).*

(3) *If r, α, β are chosen such that $q_1q_2 \geq 1$, the tail-invariant measure is σ-finite.*

The proof of Proposition 7.15 is given in Appendix B.

We finish this section with two examples of generalized Bratteli diagrams of different types: They are not of bounded size.

Example 7.16 (Renewal subshift). Consider the one-sided infinite generalized diagram $B(F_5)$ where $A_5 = F_5^\top = (a_{ij})_{i,j\in\mathbb{N}}$ is defined by

$$A_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{7.10}$$

In other words,

$$a_{ij} = \begin{cases} 1 & \text{if } i = 1 \text{ or } i = j+1, \\ 0 & \text{otherwise.} \end{cases} \tag{7.11}$$

We remark that the matrix A_5 is balanced (see Definition 7.11).

Proposition 7.17. *The stationary generalized Bratteli diagram $B(F_5)$ corresponding to the renewal shift in Example 7.16 supports a unique probability ergodic tail-invariant measure μ. The measure μ is defined by (7.1) using the Perron eigenvalue $\lambda = 2$ and the corresponding right eigenvector $\xi = \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right)^\top$ of A_5. The matrix A_5 is positive recurrent.*

The proof follows from an application of Theorem 7.2 and is given in Appendix B.

Rest of the examples in this section consist of generalized Bratteli diagrams with incidence matrices that are not balanced.

Example 7.18 (Pair renewal shift, see [61]). Consider the one-sided infinite generalized diagram $B(F_6)$ where $A_6 = F_6^\top = (a_{ij})_{i,j\in\mathbb{N}}$ is given by

$$a_{ij} = \begin{cases} 1, & \text{if } i = 1 \text{ and } j \in \mathbb{N}, \\ 1, & \text{if } i = 2 \text{ and } j = 2n \text{ for } n \in \mathbb{N}, \\ 1, & \text{if } i = n+1 \text{ and } j = n \text{ for } n \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases} \tag{7.12}$$

Note that A_6 does not have the equal column sum property and is not balanced. Explicitly, the matrix A_6 is

$$A_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This infinite matrix is a modified version of A_5, and the corresponding shift space is called the pair renewal shift (see [61]).

Proposition 7.19. *The stationary generalized Bratteli diagram $B(F_6)$ corresponding to the pair renewal shift in Example 7.18 supports a unique up constant multiple finite tail-invariant measure μ given by (7.1). The measure μ is defined by the Perron eigenvalue $\lambda = 1 + \sqrt{2}$ and the corresponding right eigenvector $\xi = (\xi_n)_{n\in\mathbb{N}}$ of A_6 which is given by*

$$\xi_1 = \frac{1}{1+\sqrt{2}}, \quad \xi_n = \frac{2}{(1+\sqrt{2})^n}, \quad n \geq 2.$$

The matrix A_6 is positive recurrent.

The proof of Proposition 7.19 is given in Appendix B.

Example 7.20. Consider the one-sided infinite generalized diagram $B(F_7)$ where $A_7 = F_7^\top = (a_{ij})_{i,j\in\mathbb{N}}$ is defined by

$$A_7 = \begin{pmatrix} c_0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ c_1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ c_2 & 0 & 0 & 1 & 0 & 0 & \cdots \\ c_3 & 0 & 0 & 0 & 1 & 0 & \cdots \\ c_4 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{7.13}$$

Here, $c_k \in \mathbb{N}$ for all $k \geq 0$.

Proposition 7.21. *The stationary generalized Bratteli diagram $B(F_7)$ corresponding to the matrix defined in (7.13) supports a σ-finite infinite tail-invariant measure μ given by (7.1) if there exists constant $C \in \mathbb{N}$ such that for every $k \in \mathbb{N}_0$, $c_k < C$. The measure μ is defined by the eigenvalue $\lambda \leq C + 1$ and the corresponding right eigenvector $\xi = (\xi_k)_{k\in\mathbb{N}}$ of A_7 which is given by*

$$\xi_{k+1} = \lambda^{k+1} - \sum_{i=0}^{k} c_i \lambda^{k-i}, \quad k \in \mathbb{N}. \tag{7.14}$$

Moreover, $\eta \cdot \xi < \infty$ if and only if

$$\sum_{k=1}^{\infty} \frac{k c_k}{\lambda^{k+1}} < \infty,$$

where η is the left eigenvector corresponding to λ.

The proof of Proposition 7.21 is given in Appendix B.

8. Stochastic matrices in Bratteli diagrams

In this section, we consider several stochastic matrices and discuss the relations between them. Also, we consider the properties of the corresponding generalized Bratteli diagrams.

8.1. Stochastic matrices and measures

Let $B = (V, E)$ be a stationary generalized Bratteli diagram with infinite incidence matrix F. Let $A = F^{\top}$. Assume that A is irreducible, has a finite Perron eigenvalue λ, and that A admits a positive right eigenvector ξ for λ:

$$A\xi = \lambda\xi.$$

Define the matrix $P = (p_{w,v} : w, v \in V_0)$ as follows:

$$p_{w,v} = \frac{a_{w,v}\, \xi_v}{\lambda \xi_w}. \tag{8.1}$$

Clearly, the matrix P is row stochastic, that is,

$$\sum_{v\in V} p_{w,v} = 1.$$

Hence, P can be considered as a Markov matrix that gives the probability to get from $w \in V_0$ to $v \in V_1$ along any edge from $E(v, w)$. The matrix P is called also a probability transition kernel. Denote $P^n = (p^{(n)}_{w,v})$ and $A^n = (a^{(n)}_{w,v})$. By induction, we have

$$p^{(n)}_{w,v} = \frac{a^{(n)}_{w,v}\, \xi_v}{\lambda^n \xi_w}, \quad v, w \in V_0, \ n \in \mathbb{N}.$$

In particular,

$$p_{vv}^{(n)} = \frac{1}{\lambda^n} a_{v,v}^{(n)}. \tag{8.2}$$

From (8.2), it easily follows that the spectral radius of P is 1, and the corresponding right eigenvector consists of all ones.

The following result was proved by Thiago Raszeja.[1]

Proposition 8.1. *Let P be the stochastic matrix defined in (8.1) by a matrix A. Then P is recurrent (null recurrent, positive recurrent) or transient if and only if A is recurrent (null recurrent, positive recurrent) or transient. In particular, P is recurrent if and only if $\sum_n p_{w,w}^{(n)} = \infty$ for all $w \in V_0$.*

Example 8.2. Consider the matrix $A = A(a, b)$ with two positive integer parameters a, b:

$$A = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & 0 & b & 0 & 0 & 0 & \cdots \\ \cdots & a & 0 & b & 0 & 0 & \cdots \\ \cdots & 0 & a & 0 & b & 0 & \cdots \\ \cdots & 0 & 0 & a & 0 & b & \cdots \\ \cdots & 0 & 0 & 0 & a & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{8.3}$$

As shown in [19], the spectral radius (Perron eigenvalue) of A is $\lambda_A = 2\sqrt{ab}$. It can also be checked, using Proposition 8.1, that A is null recurrent for any $a, b \in \mathbb{N}$.

Proposition 8.3. *Let A be as in (8.3) with $a \neq b$. Then the corresponding stationary generalized Bratteli diagram B admits at least two tail-invariant measures.*

Proof. For $\lambda = a + b$, we define a σ-finite invariant measure μ using the eigenvector $\xi = (\dots, 1, 1, \dots)$: for $\overline{e} = (e_0, \dots, e_{n-1})$, $\mu([\overline{e}]) = (a + b)^{-n}$.

To get another tail-invariant measure m, we solve the equation $A\tau = \lambda_A \tau$. Omitting computations, we find that

$$\tau = \left(\left(\frac{a}{b}\right)^{i/2} : i \in \mathbb{Z}\right).$$

Therefore, the measure m of the cylinder set $[\overline{e}]$ is

$$m([\overline{e}]) = \frac{1}{(2b)^n} \left(\frac{a}{b}\right)^{\frac{i-n}{2}},$$

where $r(\overline{e}) = r(e_{n-1}) = i$. ∎

[1]We are thankful to Thiago for the permission to include this statement in the paper.

Example 8.4 (see Example 7.7). One can define stochastic matrix P using $\lambda = a + 2b$ as follows:

$$p_{0,0} = p_{-1,-1} = \frac{a}{a+2b}; \quad p_{0,-1} = p_{0,1} = p_{-1,-2} = p_{-1,0} = \frac{b}{a+2b};$$

for $k \geq 1$, we have

$$p_{k,k-1} = \frac{2b}{a+2b}; \quad p_{k,k+1} = \frac{a}{a+2b};$$

for $k \leq -2$,

$$p_{k,k-1} = \frac{a}{a+2b}; \quad p_{k,k+1} = \frac{2b}{a+2b}.$$

All other entries of P are zero.

Notice that if $a < 2b$, the random walk on $\mathbb{Z}$ corresponding to P is positive recurrent. Indeed, for $k \geq 1$, the probability to walk from k to $k+1$ is less than the probability to walk from k to $k-1$. For $k \leq -2$, the probability to walk from k to $k-1$ is less than the probability to walk from k to $k+1$. This means that the random walk approaches $\{-1, 0\}$ with a higher probability than escapes to infinity. Since the inverse of the Perron eigenvalue λ_P for A_1 is the convergence radius of the series from Definition A.3, we obtain that λ_P is greater than or equal to $a + 2b$.

8.2. Stationary diagrams with positive recurrent incidence matrix

Let $A = F^\top$ be an infinite matrix that determines a stationary generalized Bratteli diagram B. Suppose that there exists a Perron–Frobenius eigenpair (λ, ξ), $A\xi = \lambda\xi$. Let $P = (p_{w,v})_{w,v\in V}$ be the stochastic matrix corresponding to the matrix A as defined in (8.1). Using the definition of a tail-invariant measure as in (7.1), note that P can also be defined as

$$p_{w,v} = a_{w,v} \frac{\mu_v^{(n+1)}}{\mu_w^{(n)}}, \quad \text{where } \mu_v^{(n)} = \frac{\xi_v}{\lambda^n}. \tag{8.4}$$

This formula remains true for non-stationary Bratteli diagrams B defined by a sequence of incidence matrices (F_n) and the corresponding sequence of transpose matrices $A_n = F_n^\top$. We suppose that there exists a sequence of positive vectors $(\mu^{(n)})$ such that $A_n \mu^{(n+1)} = \mu^{(n)}$. Recall that such a sequence generates a tail-invariant measure. In this case, we define the sequence of row stochastic matrices $\widetilde{P}_n = (\widetilde{p}^{(n)}_{w,v})_{w,v\in V}$:

$$\widetilde{p}^{(n)}_{w,v} = a^{(n)}_{w,v} \frac{\mu_v^{(n+1)}}{\mu_w^{(n)}}. \tag{8.5}$$

Observe that if $B = (V, E)$ is stationary, then $\widetilde{P}_n = P$ for each $n \in \mathbb{N}$.

Another way to realize row stochastic matrices is by using the height vectors $H^{(n)} = (H_v^{(n)} : v \in V_n)$ as in Definition 6.4 and (6.1). Recall that $F_n H^{(n)} = H^{(n+1)}$ for $n \in \mathbb{N}$. Thus, we can define $\widetilde{F}_n = (\widetilde{f}^{(n)}_{v,w})$, $w \in V_n$, $v \in V_{n+1}$ as follows:

$$\widetilde{f}^{(n)}_{v,w} = f^{(n)}_{v,w} \frac{H_w^{(n)}}{H_v^{(n+1)}} \quad w \in V_n,\ v \in V_{n+1}. \tag{8.6}$$

The sequence of matrices $\widetilde{F}_n$ consists of row stochastic matrices. Let the clopen set $X_v^{(n)}$ be as in Definition 6.3, then for any tail-invariant measure μ on X_B, we have

$$\mu(X_v^{(n)}) = \mu_v^{(n)} H_v^{(n)} =: \widetilde{q}_v^{(n)}. \tag{8.7}$$

Setting $\widetilde{q}^{(n)} := (\widetilde{q}_v^{(n)})_{v \in V_n}$, we observe that

$$\widetilde{q}^{(n+1)} \widetilde{F}_n = \widetilde{q}^{(n)}. \tag{8.8}$$

In what follows, we will focus on the case of *positive recurrent* incidence matrices.

Let $B = (V, E)$ be a stationary generalized Bratteli diagram such that the matrix $A = F^{\top}$ is irreducible, aperiodic, and positive recurrent. Let ξ and η denote the right and left positive eigenvectors, respectively, corresponding to the Perron value λ. Moreover, we assume that the right eigenvectors $\xi = (\xi_v)_{v \in V_n}$ is summable, that is, $\sum_{v \in V_n} \xi_v < \infty$. Without loss of generality, we assume that $\sum_{v \in V_n} \xi_v = 1$. According to Theorem 7.1, the right eigenvector ξ defines a tail-invariant measure. We discuss here the role of the left eigenvector η (see Theorem 8.7).

We define a sequence of vectors $(\nu^{(n)})_{n \in \mathbb{N}_0}$ as follows: $\nu^{(0)} = \xi$, and $\nu^{(n+1)} = \nu^{(n)} P$ for all $n \in \mathbb{N}_0$, where P is the row stochastic matrix as defined in (8.1). Note that $(\nu^{(n)})$ is a sequence of probability vectors. To see this, we note that $\nu^{(0)} = \xi$ is a probability vector, and by induction,

$$\begin{aligned}
\sum_{v \in V_n} \nu_v^{(n+1)} &= \sum_{v \in V_n} \sum_{w \in V_n} \nu_w^{(n)} p_{w,v} \\
&= \sum_{v \in V_n} \sum_{w \in V_n} \nu_w^{(n)} \frac{a_{w,v}\, \xi_v}{\lambda\, \xi_w} \\
&= \sum_{w \in V_n} \nu_w^{(n)} \frac{\lambda\, \xi_w}{\lambda\, \xi_w} = \sum_{w \in V_n} \nu_w^{(n)} = 1.
\end{aligned}$$

Lemma 8.5. *For every $v \in V_n$, we have*

$$\nu_v^{(n)} = \frac{\xi_v}{\lambda^n} \sum_{w \in V_0} a_{w,v}^{(n)},$$

where $a_{w,v}^{(n)}$ ($a_{w,v}^{(1)} = a_{w,v}$) denotes the (w, v)-th *entry of the matrix A^n.*

Proof. The proof is by induction. We recall that $\sum_{w \in V_0} a_{w,v} < \infty$. Next,

$$\nu_v^{(1)} = \sum_{w \in V_0} \nu_w^{(0)} P_{w,v} = \sum_{w \in V_0} \xi_w\, a_{w,v} \frac{\xi_v}{\lambda\, \xi_w} = \frac{\xi_v}{\lambda} \sum_{w \in V_0} a_{w,v}.$$

Check the induction step:

$$\begin{aligned}
\nu_v^{(n+1)} = \sum_{w\in V_0} \nu_w^{(n)} P_{w,v} &= \sum_{w\in V_0} \left(\frac{\xi_w}{\lambda^n} \sum_{s\in V_0} a_{s,w}^{(n)} \right) p_{w,v} \\
&= \sum_{w\in V_0} \left(\frac{\xi_w}{\lambda^n} \sum_{s\in V_0} a_{s,w}^{(n)} \frac{a_{w,v}\,\xi_v}{\lambda\,\xi_w} \right) \\
&= \sum_{s\in V_0} \frac{\xi_v}{\lambda^{n+1}} \sum_{w\in V_0} a_{s,w}^{(n)}\, a_{w,v} \\
&= \frac{\xi_v}{\lambda^{n+1}} \sum_{s\in V_0} a_{s,v}^{(n+1)}.
\end{aligned}$$

■

Proposition 8.6. *Let A be a positive recurrent infinite matrix.*

(1) *Then, for every vertex $v \in V_0$,*

$$\mu(X_v^{(n)}) = \nu_v^{(n)},$$

where μ is the tail-invariant measure defined in (7.1).

(2) *Assume that the right eigenvector ξ is probability, then for every $v \in V_0$,*

$$\nu_v^{(n)} \to \xi_v\,\eta_v \quad n \to \infty.$$

Proof. (1) Note that by (8.7), we have

$$\mu(X_v^{(n)}) = \mu_v^{(n)} H_v^{(n)} = \sum_{w\in V_0} a_{w,v}^{(n)}\,\mu_v^{(n)} = \sum_{w\in V_0} a_{w,v}^{(n)} \frac{\xi_v}{\lambda^n} = \nu_v^{(n)}.$$

(2) Theorem A.7 states that if A is positive recurrent and the left and right eigenvectors are normalized by the condition $\eta\cdot\xi = 1$, then for every $v, w \in V_0$,

$$\lim_{n\to\infty} \frac{a_{w,v}^{(n)}}{\lambda^n} = \xi_w \eta_v.$$

It follows from Lemma 8.5 that

$$\lim_{n\to\infty} \nu_v^{(n)} = \lim_{n\to\infty} \frac{\xi_v}{\lambda^n} \sum_{w\in V_0} a_{w,v}^{(n)} = \lim_{n\to\infty} \xi_v \sum_{w\in V_0} \frac{a_{w,v}^{(n)}}{\lambda^n} = \xi_v \sum_{w\in V_0} \xi_w \eta_v = \xi_v\,\eta_v.$$

We used here the fact that the vector ξ is probability. ■

Theorem 8.7. *Let $B(F)$ be a stationary generalized Bratteli diagram and the matrix $A = F^\top$. Suppose that A is irreducible, aperiodic, and positive recurrent. Let λ be the Perron eigenvalue of A and let $\xi = (\xi_i)$, $\eta = (\eta_i)$ be the corresponding right and left eigenvectors normalized such that $\sum_{v\in V_0} \xi_v = 1$ and $\eta\cdot\xi = 1$. Then, for every $v, w \in V_0$,*

$$\frac{H_w^{(n)}}{H_v^{(n+1)}} \to \frac{\eta_w}{\lambda\cdot\eta_v} \text{ as } n \to \infty.$$

Proof. Since B is a stationary Bratteli diagram, we identify all levels V_n with $\mathbb{Z}$. It follows from (8.7) and (8.8) that, for every $n \in \mathbb{N}$,

$$\mu(X_v^{(n+1)})\widetilde{F} = \mu(X_v^{(n)}), \tag{8.9}$$

By definition of $\widetilde{F}$, we get that, for every $w \in \mathbb{Z}$,

$$\sum_{v\in\mathbb{Z}} \mu(X_v^{(n+1)})\, f_{v,w} \frac{H_w^{(n)}}{H_v^{(n+1)}} = \mu(X_w^{(n)}). \tag{8.10}$$

Taking the limit in (8.10) as $n \to \infty$ and using Proposition 8.6, we obtain that

$$\lim_{n\to\infty}\left(\sum_{v\in Z} \mu(X_v^{(n+1)}) f_{v,w} \frac{H_w^{(n)}}{H_v^{(n+1)}}\right) = \lim_{n\to\infty}\left(\mu(X_w^{(n)})\right) = \xi_w\, \eta_w. \tag{8.11}$$

This implies that the series is convergent:

$$\sum_{v\in V_0} \lim_{n\to\infty}\left(\mu(X_v^{(n+1)})\, f_{v,w} \frac{H_w^{(n)}}{H_v^{(n+1)}}\right) = \xi_w\, \eta_w < \infty, \quad w \in \mathbb{Z}. \tag{8.12}$$

Hence, the limit

$$\lim_{n\to\infty}\left(\mu(X_v^{(n+1)}) f_{v,w} \frac{H_w^{(n)}}{H_v^{(n+1)}}\right)$$

exists. Proposition 8.6 states that the limit $\lim_{n\to\infty}\mu(X_v^{(n+1)})$ exists for every v. We conclude therefore that

$$L_{v,w} := \lim_{n\to\infty} \frac{H_w^{(n)}}{H_v^{(n+1)}} < \infty.$$

The proved facts allow us to rewrite (8.12) as follows:

$$\sum_{v\in V_0} \xi_v\, \eta_v\, f_{v,w}\, L_{v,w} = \xi_w\, \eta_w.$$

Multiplying in the above relation both sides by $\frac{\lambda}{\eta_w}$, we get

$$\sum_{v\in V_0} \xi_v\, f_{v,w}\left(L_{v,w}\frac{\lambda\eta_v}{\eta_w}\right) = \lambda\xi_w.$$

Since

$$\sum_{v\in V_0} \xi_v\, f_{v,w} = \lambda\xi_w,$$

we obtain that $L_{v,w}\frac{\lambda\eta_v}{\eta_w} = 1$. In other words,

$$L_{v,w} = \lim_{n\to\infty}\frac{H_w^{(n)}}{H_v^{(n+1)}} = \frac{\eta_w}{\lambda\eta_v},$$

as needed. ∎

8.3. Stochastic matrices and measures for non-stationary generalized Bratteli diagrams

Suppose that a generalized Bratteli diagram is defined by the sequence of incidence matrices (F_n) and $A_n = F_n^\top$. Assume that there exists a probability tail-invariant measure μ. According to Theorem 6.6, this measure is completely determined by the sequence of non-negative vectors $(\mu^{(n)})$ such that $A_n\mu^{(n+1)} = \mu^{(n)}$. Simultaneously, we have the sequence $(H^{(n)})$ which satisfies the condition $F_n H^{(n)} = H^{(n+1)}$. Since μ is a probability measure, the sequences $(H^{(n)})$ and $(\mu^{(n)})$ satisfy the equality:

$$\langle \mu^{(n)}, H^{(n)} \rangle := \sum_{v \in V_n} \mu_v^{(n)} H_v^{(n)} = 1, \quad n \in \mathbb{N}_0. \tag{8.13}$$

Let

$$\|\mu^{(n)}\|_\infty = \sup_{v \in V_n} \mu_v^{(n)}, \quad n \in \mathbb{N}_0.$$

Denote

$$\widehat{\mu}^{(n)} = \frac{\mu^{(n)}}{\|\mu^{(n)}\|_\infty}, \quad \widehat{H}^{(n)} = \frac{H^{(n)}}{\langle \widehat{\mu}^{(n)}, H^{(n)} \rangle}.$$

It can be checked directly that $\langle \widehat{\mu}^{(n)}, \widehat{H}^{(n)} \rangle = 1$.

Lemma 8.8. *Let*

$$\lambda_n = \frac{\|\mu^{(n)}\|_\infty}{\|\mu^{(n+1)}\|_\infty}.$$

Then, for $n \in \mathbb{N}_0$,

(1) $\lambda_n > 1$,

(2) $A_n \widehat{\mu}^{(n+1)} = \lambda_n \widehat{\mu}^{(n)}$,

(3) $F_n \widehat{H}^{(n)} = \lambda_n \widehat{H}^{(n+1)}$.

Proof. We have for all $n \in \mathbb{N}_0$

$$A_n \widehat{\mu}^{(n+1)} = \frac{1}{\|\mu^{(n+1)}\|_\infty} A_n \mu^{(n+1)} = \frac{\mu^{(n)}}{\|\mu^{(n+1)}\|_\infty} = \frac{\|\mu^{(n)}\|_\infty}{\|\mu^{(n+1)}\|_\infty} \widehat{\mu}^{(n)} = \lambda_n \widehat{\mu}^{(n)}.$$

To see that $\lambda_n > 1$, we take $\varepsilon > 0$ and find $v_0 \in V_{n+1}$ such that

$$\|\mu^{(n+1)}\|_\infty - \varepsilon < \mu_{v_0}^{(n+1)}.$$

Let w be a vertex in V_n such that $E(w, v_0) \neq \emptyset$. Since $|s^{-1}(w)| > 1$, we see that $\mu_{v_0}^{(n+1)} < \mu_w^{(n)} \leq \|\mu^{(n)}\|_\infty$. Using the fact that the set $r^{-1}(v_0)$ is finite, we conclude that $\|\mu^{(n+1)}\|_\infty < \|\mu^{(n)}\|_\infty$ as desired.

For the third relation, we compute using (8.13)

$$\begin{aligned} F_n \widehat{H}^{(n)} &= \frac{H^{(n+1)}}{\langle \widehat{\mu}^{(n)}, H^{(n)} \rangle} \\ &= \frac{\|\mu^{(n)}\|_\infty}{\langle \mu^{(n)}, H^{(n)} \rangle} H^{(n+1)} \\ &= \frac{\|\mu^{(n)}\|_\infty}{\langle \mu^{(n+1)}, H^{(n+1)} \rangle} H^{(n+1)} \\ &= \frac{\|\mu^{(n)}\|_\infty}{\|\mu^{(n+1)}\|_\infty} \frac{1}{\langle \widehat{\mu}^{(n+1)}, H^{(n+1)} \rangle} H^{(n+1)} \\ &= \lambda_n \widehat{H}^{(n+1)}. \end{aligned}$$

∎

We summarize the above discussion in the following theorem.

Theorem 8.9. *Let B be a generalized Bratteli diagram and μ a tail-invariant measure on the path space X_B. Then there exist two sequences of positive vectors $(\widehat{\mu}^{(n)})$ and $(\widehat{H}^{(n)})$ such that for all $n \in \mathbb{N}_0$*

$$\langle \widehat{\mu}^{(n)}, \widehat{H}^{(n)} \rangle = 1,$$

and

$$A_n \widehat{\mu}^{(n+1)} = \lambda_n \widehat{\mu}^{(n)}, \quad A_n^\top \widehat{H}^{(n)} = \lambda_n \widehat{H}^{(n+1)}.$$

Remark 8.10. Theorem 8.9 remains true if instead of $H^{(0)} = (1, 1, \ldots)$ one takes an arbitrary sequence $t^{(0)}$ of positive integers $t_v^{(0)}$, $v \in V_0$, then the sequences $t^{(n)}$ are determined automatically by the relation $F_{n-1} \cdots F_1 F_0 t^{(0)}, n \in \mathbb{N}$.

Theorem 8.9 can be used to construct a sequence $(\widehat{P}_n)$ of row stochastic matrices.

Lemma 8.11. *Let D_n be the diagonal matrix whose non-zero entries are $\widehat{\mu}_v^{(n)}$, $v \in V_n$. Then the matrix*

$$\widehat{P}_n = \frac{1}{\lambda_n} D_n^{-1} A_n D_{n+1}, \quad n \in N_0 \tag{8.14}$$

is stochastic.

Proof. Indeed, for $w \in V_n, v \in V_{n+1}$, one has

$$\begin{aligned} \sum_{v \in V_{n+1}} \widehat{p}_{w,v}^{(n)} &= \sum_{v \in V_{n+1}} \frac{1}{\lambda_n \widehat{\mu}_w^{(n)}} a_{w,v}^{(n)} \widehat{\mu}_v^{(n+1)} \\ &= \frac{1}{\lambda_n \widehat{\mu}_w^{(n)}} \sum_{v \in V_{n+1}} a_{w,v}^{(n)} \widehat{\mu}_v^{(n+1)} \\ &= \frac{1}{\lambda_n \widehat{\mu}_w^{(n)}} \lambda_n \widehat{\mu}_w^{(n)} = 1. \end{aligned}$$

∎

We note that the entries of $\widehat{P}_n$ can be written in two ways:

$$\widehat{p}^{(n)}_{w,v} = \frac{a^{(n)}_{w,v}\widehat{\mu}^{(n+1)}_v}{\lambda_n \widehat{\mu}^{(n)}_w} = \frac{a^{(n)}_{w,v}\mu^{(n+1)}_v}{\mu^{(n)}_w}.$$

It can be easily seen that, for a stationary generalized Bratteli diagram, the stochastic matrices $\widehat{P}_n$ coincide with the matrix P defined in (8.1).

Similarly to the case of finite Bratteli diagrams, we can produce a sequence of stochastic incidence matrices for any generalized Bratteli diagram B with incidence matrices (F_n). For given B and (F_n), compute the sequence of vectors $(H^{(n)})$ as in Lemma 6.1. Then define entries of a new matrix $G_n = (g^{(n)}_{v,w})$ as follows:

$$g^{(n)}_{v,w} := \frac{f^{(n)}_{v,w} H^{(n)}_w}{H^{(n+1)}_v}, \quad v \in V_{n+1},\ w \in V_n,\ n \in \mathbb{N}. \tag{8.15}$$

Lemma 8.12. *Let μ be a tail-invariant measure on a generalized Bratteli diagram $B = (V, E)$. Then the matrix G_n, with entries defined by (8.15), is stochastic and satisfies the relation*

$$C_n s^{(n+1)} = s^{(n)}, \quad n \in \mathbb{N},$$

where $C_n = G_n^\top$ and $s^{(n)}$ is the probability vector with entries $(\mu(X^{(n)}_v) : v \in V_n)$.

Proof. The fact that G_n is stochastic follows from the relation $F_n H^{(n)} = H^{(n+1)}$.

To check the other statement of the lemma, we first recall that $s^{(n)}_v = \mu^{(n)}_v H^{(n)}_v$ and then compute

$$\begin{aligned}
(C_n s^{(n+1)})_v &= \sum_{w \in V_{n+1}} g^{(n)}_{w,v} s^{(n+1)}_w \\
&= \sum_{w \in V_{n+1}} f^{(n)}_{w,v} \frac{H^{(n)}_v}{H^{(n+1)}_w} \mu^{(n+1)}_w H^{(n+1)}_w \\
&= H^{(n)}_v \sum_{w \in V_{n+1}} f^{(n)}_{w,v} \mu^{(n+1)}_w \\
&= H^{(n)}_v \mu^{(n)}_v = s^{(n)}_v.
\end{aligned}$$

We used here relation (6.3) of Theorem 6.6. ∎

The main result of this subsection is as follows.

Theorem 8.13. *Let B be a generalized Bratteli diagram with incidence matrices F_n and μ be a probability tail-invariant measure on B. Let $A_n = F_n^\top$. Then there exist a sequence of numbers $\lambda_n > 1$, a sequence of vectors $\widehat{\mu}^{(n)} = (\widehat{\mu}^{(n)}_v)$, $v \in V_n$, and a sequence of vectors $\widehat{H}^{(n)} = (\widehat{H}^{(n)}_v)$, $v \in V_n$, such that for every $n \in \mathbb{N}$*

(1) $\langle \widehat{\mu}^{(n)}, \widehat{H}^{(n)} \rangle = 1$,

(2) $A_n \widehat{\mu}^{(n+1)} = \lambda_n \widehat{\mu}^{(n)}$,

(3) $F_n \widehat{H}^{(n)} = \lambda_n \widehat{H}^{(n+1)}$.

Conversely, if there exist a sequence of non-negative infinite integer matrices A_n with finite column sums, a sequence of numbers $\lambda_n > 1$, a sequence of non-negative infinite vectors $\widehat{\mu}^{(n)} = (\widehat{\mu}_v^{(n)})$, and a sequence of positive infinite vectors $\widehat{H}^{(n)} = (\widehat{H}_v^{(n)})$ with $\widehat{H}^{(0)} = H^{(0)}$ which satisfy conditions (1)–(3) above, then a generalized Bratteli diagram B defined by incidence matrices $F_n = A_n^{\top}$ possesses a probability tail-invariant measure, for which the measures of cylinder sets are defined by the sequence of vectors

$$p^{(0)} = \widehat{\mu}^{(0)} \quad \textit{and} \quad p^{(n)} = \frac{1}{\lambda_0 \cdots \lambda_{n-1}} \widehat{\mu}^{(n)}, \quad \textit{for } n \geq 1, \tag{8.16}$$

where $p^{(n)} = (\mu(X_w^{(n)}(\overline{e})) : w \in V_n)$.

Proof. The "if" part of the theorem is proved in Lemma 8.8. To prove the "only if" part, we first notice that for all $n \geq 0$:

$$A_n p^{(n+1)} = \frac{1}{\lambda_0 \cdots \lambda_n} A_n \widehat{\mu}^{(n+1)} = \frac{1}{\lambda_0 \cdots \lambda_n} \lambda_n \widehat{\mu}^{(n)} = p^{(n)}.$$

Hence by Theorem 6.6, the measure μ defined by (8.16) is a tail-invariant measure on B. Since $\widehat{H}^{(0)} = H^0$ and $\langle \widehat{\mu}^{(0)}, \widehat{H}^{(0)} \rangle = 1$, we obtain that μ is a probability measure. ■

9. Open problems

This section contains several open problems. We have not tried to create a comprehensive list of problems that would cover all possible directions. We focus here on the existence of Vershik maps and probability tail-invariant measures. These areas are well studied in the case of standard Bratteli diagrams. We refer to the literature mentioned in Section 1. It would be interesting to understand which of these results (or their analogs) can be proved in the context of generalized Bratteli diagrams.

(1) Find necessary and sufficient conditions for an aperiodic irreducible infinite non-negative integer matrix with finite Perron eigenvalue to have a right eigenvector $\xi = (\xi_v)$ with $\sum_{v \in V_0} \xi_v < \infty$. It follows from our results given in Section 7 that such conditions will imply the existence of a finite tail-invariant measure for the corresponding stationary generalized Bratteli which takes positive values on cylinder sets. We present some sufficient conditions of this kind in Subsections 7.2 and 7.3.

(2) Find conditions on incidence matrices of generalized Bratteli diagrams which allow determining the number of ergodic tail-invariant measures. In particular, it is important to know when a generalized Bratteli diagram is uniquely ergodic. This problem was discussed in many papers on Cantor dynamics (see, e.g., [11,

13, 31, 60] for the case of standard Bratteli diagrams). As a part of this problem, it would be interesting to consider the cases of null-recurrent and/or transient incidence matrices. In Example 8.2, we present two different tail-invariant measures. If a generalized Bratteli diagram is not uniquely ergodic, how can one determine the support of ergodic measures?

(3) Most results of this paper are related to the case of irreducible Bratteli diagrams. How can we describe tail-invariant measures on reducible (stationary) generalized Bratteli diagrams? We mention Proposition 6.11 to illustrate what may happen in this case. For the standard Bratteli diagrams, we refer to [15] where the method of measure extension played an important role. This remark motivates the following problem: Find conditions under which a measure on a generalized Bratteli diagram is an extension of a measure from a subdiagram (see [2]).

(4) Let B be a stationary generalized Bratteli diagram such that the corresponding incidence matrix is aperiodic irreducible and does not have a finite Perron eigenvalue (we give examples of such matrices in Appendix A). Can such a diagram possess a finite tail-invariant measure? In particular, can a stationary generalized Bratteli diagram with the incidence matrix

$$F = \begin{pmatrix} 2 & 1 & 0 & 0 & \dots \\ 1 & 3 & 1 & 0 & \dots \\ 0 & 1 & 4 & 1 & \dots \\ 0 & 0 & 1 & 5 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

possess a probability tail-invariant measure which takes positive values on cylinder sets?

(5) If B is a generalized Bratteli diagram, then one can consider various partial orders on the set of edges. For standard Bratteli diagrams, we know that some orders generate continuous Vershik maps. On the other hand, there are diagrams on which it is impossible to define a Vershik map (see details in [17, 56]). Is it true that any generalized Bratteli diagram B can be endowed with an order which generates a Borel Vershik map? In particular, is there an order without infinite maximal and minimal paths? Are there algebraic conditions on incidence matrices that guarantee the existence of a Vershik map? The reader can find more information in [17, 18] for standard Bratteli diagrams.

(6) For an ordered generalized Bratteli diagram, find conditions, under which the corresponding Vershik map is a homeomorphism (we have partly answered this question in Subsection 3.2).

A. Perron–Frobenius theory for infinite matrices

For the benefit of the readers, in this appendix, we provide some definitions and results from the Perron–Frobenius theory of infinite matrices which are of direct relevance to the proofs in the body of our paper. As mentioned before, these results are due to Vere-Jones (see [65–67]). The formulations of statements and definitions in this section are taken from the book by Kitchens [53, Chapter 7].

Recall that a matrix $A = (a_{ij})$ is called *infinite* (or countably infinite) if its rows and columns are indexed by the same countably infinite set. Let $A = (a_{ij})_{i,j\in\mathbb{Z}}$ be a real, non-negative, infinite matrix. We enumerate the rows and columns of A by $\mathbb{Z}$ to make this material closer to two-sided generalized Bratteli diagrams. Note that the results in this section are independent of the fact that we enumerate the rows and columns by $\mathbb{Z}$ or by $\mathbb{N}$. As before, we denote by $a_{ij}^{(n)}$ the (i,j)-th entry of A^n, $n \in \mathbb{N}$, whenever it exists, that is, whenever it is finite. The matrix A is called *irreducible* if for every pair $i, j \in \mathbb{Z}$ there is $n > 0$ such that $a_{ij}^{(n)} > 0$. Denote

$$p(i) = \gcd\{n : a_{ii}^{(n)} > 0\}, \quad i \in \mathbb{Z}.$$

Then $p(i)$ is called the *period of index i*. For an irreducible matrix A, the periods of all indices are the same and called the *period of A*. An irreducible matrix with period one is called *aperiodic*.

Lemma A.1 ([53]). *Let A be a real, non-negative, irreducible, aperiodic, infinite matrix. Fix $i \in \mathbb{Z}$. Then*

(a) *there exists $k \in \mathbb{N}$ such that $a_{ii}^{(n)} > 0$ for all $n \geq k$,*

(b) *there exists*
$$\lambda = \lim_{n\to\infty} \sqrt[n]{a_{ii}^{(n)}} = \sup_{n\in\mathbb{N}} \sqrt[n]{a_{ii}^{(n)}} \leq \infty, \tag{A.1}$$
where the value of λ does not depend on i.

If the value λ in Lemma A.1 is finite, then it is called the *Perron eigenvalue* of A. If A is a finite non-negative irreducible and aperiodic matrix, then λ coincides with the usual Perron eigenvalue of A. If the matrix A is periodic, then one defines the Perron eigenvalue of A as

$$\lambda = \limsup_{n\to\infty} \sqrt[n]{a_{ii}^{(n)}}.$$

The next example gives a banded matrix A with infinite spectral radius given by formula (A.1).

Example A.2. Let $A = (a_{ij})$ be a non-negative integer $\mathbb{Z} \times \mathbb{Z}$ matrix with the entries

$$\begin{cases} a_{0,0} = 1, & \\ a_{m,m} = |m|^{|m|}, & \text{for } m \in \mathbb{Z} \setminus \{0\}, \\ a_{m,m-1} = a_{m,m+1} = 1, & \text{for } m \in \mathbb{Z}, \\ a_{i,j} = 0 & \text{for } |i-j| > 1,\ i, j \in \mathbb{Z}. \end{cases}$$

We prove straightforwardly that the Perron value λ of A is infinite. Fix $m \in \mathbb{N}$. First we show by induction that $a^{(n)}_{m,m+n} \geq 1$ for all $n \in \mathbb{N}$. To see this, observe that $a_{k,k+1} = 1 \geq 1$ for all $k \in \mathbb{N}$. By induction step,

$$a^{(n+1)}_{m,m+n+1} \geq a^{(n)}_{m,m+n}\ a_{m+n,m+n+1} \geq 1.$$

Similarly, $a^{(n)}_{m+n,m} \geq 1$ for all $n \in \mathbb{N}$. Then

$$a^{(n+1)}_{m,m+n} \geq a^{(n)}_{m,m+n} a_{m+n,m+n} \geq (m+n)^{(m+n)}.$$

It follows that

$$a^{(2n+1)}_{mm} \geq a^{(n+1)}_{m,m+n} a^{(n)}_{m+n,m} \geq (m+n)^{(m+n)}.$$

Therefore,

$$\sqrt[2n+1]{a^{(2n+1)}_{mm}} \geq ((m+n)^{(m+n)})^{\frac{1}{2n+1}} \to \infty$$

as $n \to \infty$. Thus, the Perron value λ is infinite.

Definition A.3. Let A be a real, non-negative, irreducible, aperiodic, infinite matrix with a finite Perron value λ and $i \in \mathbb{Z}$.

(i) A is called *recurrent* if
$$\sum_{n=0}^{\infty} \frac{a^{(n)}_{ii}}{\lambda^n} = \infty,$$
where $A^0 = I$ is an infinite identity matrix.

(ii) A is called *transient* if
$$\sum_{n=0}^{\infty} \frac{a^{(n)}_{ii}}{\lambda^n} < \infty.$$

Moreover, the convergence of the series does not depend on the choice of i.

For a real, non-negative, irreducible, aperiodic infinite matrix A, we define the following generating function:

$$T^A_{w,v}(z) = \sum_{i=0}^{\infty} a^{(i)}_{w,v} z^i,$$

where $a_{w,v}^{(0)} = \delta_{w,v}$. The radius of convergence of $T_{w,v}^A(z)$ is λ^{-1}. Hence, A is recurrent if and only if

$$T_{w,w}^A(\lambda^{-1}) = \infty,$$

and A is transient if and only if

$$T_{w,w}^A(\lambda^{-1}) < \infty.$$

Theorem A.4 (Generalized Perron–Frobenius theorem [53]). *Let A be a real, non-negative, irreducible, aperiodic, recurrent, infinite matrix. Let $\lambda < \infty$ be a Perron eigenvalue of A. Then*

(i) *there exist strictly positive eigenvectors η, ξ such that $\eta A = \lambda\eta$, $A\xi = \lambda\xi$;*

(ii) *η and ξ are unique up to constant multiples.*

We will call ξ and η the left and the right Perron eigenvectors of A.

Example A.5. There are banded matrices that do not have finite Perron values. Here is an example of such a matrix:

$$A = \begin{pmatrix} n_1 & 1 & 0 & 0 & \dots \\ 1 & n_2 & 1 & 0 & \dots \\ 0 & 1 & n_3 & 1 & \dots \\ 0 & 0 & 1 & n_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to see that if the sequence (n_i) is unbounded, then the solution of $Ax = \lambda x$ does not exist for positive vectors x.

We set $l_{w,v}(0) = 0$ and for a real, non-negative, irreducible, aperiodic infinite matrix A define $l_{w,v}(1) = a_{w,v}$ and

$$l_{w,v}(n+1) = \sum_{i \neq w} l_{w,i}(n) a_{i,v}.$$

Then $l_{w,v}(n)$ is the number of paths of length n from vertex w to vertex v which do not return to w at any time prior to n. Define the corresponding generating function

$$L_{w,v}(z) = \sum_{n=1}^{\infty} l_{w,v}(n) z^n.$$

The matrix A is called *positive recurrent* if

$$\sum_{n=1}^{\infty} n l_{w,w}^{(n)}(\lambda^{-n}) < \infty,$$

and A is called *null recurrent* if

$$\sum_{n=1}^{\infty} n l_{w,w}^{(n)}(\lambda^{-n}) = \infty.$$

Proposition A.6 ([53]). *Let A be a real, non-negative, irreducible, aperiodic, recurrent, infinite matrix. Let η and ξ be the left and right Perron eigenvectors of A. If*

$$\eta \cdot \xi = \sum_i \eta_i \xi_i < \infty,$$

then A is positive recurrent. Otherwise, A is null recurrent.

Theorem A.7 ([53]). *Let A be a real, non-negative, irreducible, aperiodic, recurrent, infinite matrix, and let η and ξ be the left and right Perron eigenvectors of A. If A is positive recurrent and $\eta \cdot \xi = 1$, then*

$$\lim_{n\to\infty} \frac{A^n}{\lambda^n} = \xi\eta.$$

The next fact explains why we were interested in measures taking positive values on cylinder sets.

Proposition A.8. *Let A be a real, non-negative, irreducible, and aperiodic infinite matrix such that there exist $\lambda > 0$ and a non-zero non-negative vector $x = (x_i)_{i\in\mathbb{Z}}$ for which*

$$Ax \le \lambda x. \tag{A.2}$$

Then if $x_j > 0$ for some $j \in \mathbb{Z}$, it follows that $x_i > 0$ for all $i \in \mathbb{Z}$.

Proof. By formula (A.2), we have

$$\sum_j a_{ij} x_j \le \lambda x_i \quad \text{for all } i \in \mathbb{Z}.$$

Thus, we have

$$a_{ij} x_j \le \lambda x_i$$

for all i, j. We show by induction that

$$a_{i_0 i_1} \cdots a_{i_{n-1} i_n} x_{i_n} \le \lambda^n x_{i_0}.$$

Indeed, suppose the above inequality is true for some n, we prove it for $n + 1$:

$$a_{i_0 i_1} \cdots a_{i_{n-1} i_n} (a_{i_n i_{n+1}} x_{i_{n+1}}) \le a_{i_0 i_1} \cdots a_{i_{n-1} i_n} (\lambda x_{i_n}) \le \lambda(\lambda^n x_{i_0}) = \lambda^{n+1} x_{i_0}.$$

Let $x_j > 0$, fix any x_i. Since A is irreducible, there exists n such that

$$a_{ij}^{(n)} > 0,$$

where $A^n = (a_{ij}^{(n)})$. Thus, there exists $i_1, \dots, i_{n-1}$ such that

$$a_{ii_1} \cdots a_{i_{n-1}j} > 0.$$

Hence,

$$\lambda^n x_i \geq a_{ii_1} \cdots a_{i_{n-1}j} x_j > 0. \qquad \blacksquare$$

The next proposition gives bounds for a positive eigenvalue of an infinite matrix that has a non-negative corresponding eigenvector.

Proposition A.9. *Let A be a real, non-negative, irreducible, and aperiodic infinite matrix with the uniformly bounded sum of elements in each column. Suppose there exist $0 < \lambda < \infty$ and a non-zero non-negative vector $x = (x_i)_{i \in \mathbb{Z}}$ for which $Ax = \lambda x$. If*

$$\sum_i x_i < \infty,$$

then

$$\inf_j \sum_i a_{ij} \leq \lambda \leq \sup_j \sum_i a_{ij}.$$

In particular, if the sum of entries in every column of A is c, then $\lambda = c$.

Proof. We have

$$\sum_j a_{ij} x_j = \lambda x_i$$

and

$$\sum_i \sum_j a_{ij} x_j = \lambda \sum_i x_i.$$

Let

$$m = \inf_j \sum_i a_{ij}, \quad M = \sup_j \sum_i a_{ij}.$$

Then,

$$m \sum_j x_j \leq \lambda \sum_i x_i \leq M \sum_j x_j$$

and

$$m \leq \lambda \leq M.$$

In particular, if $m = M = c$, then $\lambda = c$. $\blacksquare$

B. Examples of Bratteli diagrams supporting invariant measures

In this appendix, we give the proofs of the propositions formulated in Subsection 7.3 for some classes of stationary generalized Bratteli diagrams. Most of them satisfy the conditions of Theorem 7.1. This means that, for such diagrams, there exist tail-invariant measures (finite or σ-finite) given by (7.1).

We will follow the notations of Appendix A. Recall that $A = (a_{ij})_{i,j\in\mathbb{Z}}$ is an integer-valued, non-negative, and countably infinite matrix.

Proof of Proposition 7.8. We first note that A_1 has the equal column sum property. Hence, there is an eigenvalue equal to the column sum $\lambda = a + 2b$. Denote by $\xi = (\xi_n)_{n\in\mathbb{Z}}$ the right eigenvector for A_1, that is, $A_1\xi = (a + 2b)\xi$. Then $b\xi_{-1} + a\xi_0 + 2b\xi_1 = (a + 2b)\xi_0$. We choose $\xi_{-1} = \xi_0 = 1$ and find that $\xi_1 = \frac{1}{2}$. Analogously, the equality

$$b\xi_0 + 2b\xi_2 = b + 2b\xi_2 = (a + 2b)\xi_1 = \frac{a}{2} + b$$

gives $\xi_2 = \frac{a}{4b}$. By induction, we prove that

$$\xi_n = \frac{a^{n-1}}{2^n b^{n-1}}$$

for each $n \in \mathbb{N}$. Indeed, we have

$$a\xi_{n-1} + 2b\xi_{n+1} = \frac{a^{n-1}}{2^{n-1}b^{n-2}} + 2b\xi_{n+1} = (a + 2b)\xi_n = \frac{a^n + 2a^{n-1}b}{2^n b^{n-1}}.$$

Thus,

$$\xi_{n+1} = \frac{a^n + 2a^{n-1}b - 2a^{n-1}b}{2^{n+1}b^n} = \frac{a^n}{2^{n+1}b^n}.$$

To find ξ_{-n}, we use the same equations. Therefore,

$$\xi = \left(\dots, \frac{1}{2^3}\left(\frac{a}{b}\right)^2, \frac{1}{2^2}\left(\frac{a}{b}\right), \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2^2}\left(\frac{a}{b}\right), \frac{1}{2^3}\left(\frac{a}{b}\right)^2, \dots\right)^{\top}$$

is the right eigenvector for A_1. Clearly, the tail-invariant measure μ, which is determined by (7.1), is finite if and only if

$$\sum_{n\in\mathbb{Z}} \xi_n < \infty \iff a < 2b.$$

Since $\eta = (\dots, 1, 1, 1, \dots)$ is the left eigenvector for A_1, the condition $a < 2b$ is equivalent to the property $\eta \cdot \xi < \infty$. ■

Proof of Proposition 7.10. (1) For $a \neq b$, calculations show that A_2 has eigenvalues $\lambda = a + b$ and $\widetilde{\lambda} = 2\sqrt{ab}$. We note that $\widetilde{\lambda}$ is the spectral radius of the matrix A_2 (see also Example 8.2). The eigenvectors corresponding to λ and $\widetilde{\lambda}$ are given below:

$$A_2\,\xi = \lambda\,\xi, \quad \lambda = a + b, \quad \xi = (\xi_n) = (\dots, 1, 1, 1\dots)^{\top}, \quad \sum_{n\in\mathbb{Z}} \xi_n = \infty$$

and

$$A_2 \widetilde{\xi} = \widetilde{\lambda} \widetilde{\xi}, \quad \widetilde{\lambda} = 2\sqrt{ab}, \quad \widetilde{\xi} = (\widetilde{\xi}_n), \quad \widetilde{\xi}_n = \left(\frac{a}{b}\right)^{\frac{n}{2}}, \quad \sum_{n\in\mathbb{Z}} \widetilde{\xi}_n = \infty.$$

We apply (7.1) to obtain two σ-finite invariant measures μ and $\widetilde{\mu}$ on the path space X_B, corresponding to λ and $\widetilde{\lambda}$, respectively.

(2) When $a = b$, we obtain the eigenvalue $\lambda = \widetilde{\lambda} = 2a$ and $\xi = \widetilde{\xi} = (\dots, 1, 1, 1\dots)^\top$, hence $\mu = \widetilde{\mu}$.

Direct computations or application of Proposition 8.1 show that matrix A is null recurrent. ■

Proof of Proposition 7.13. Note that the tri-diagonal matrix A_3 is a balanced matrix (see Definition 7.11) with $\lambda = b + c + \alpha c$, $\sigma_1 = b + 2\alpha c$ and $\sigma_i = b + c + \alpha c$ for $i > 1$. Let $\xi = (\xi_n)_{n\in\mathbb{N}}$ be the corresponding right eigenvector. Then the first entry of the equation

$$A_3\xi = (b + c + \alpha c)\xi$$

implies that $(b + \alpha c)\xi_1 + \alpha c\, \xi_2 = (b + c + \alpha c)\xi_1$. Hence, we get $\xi_2 = \frac{\xi_1}{\alpha}$.

Then, we apply the relation

$$c\, \xi_{n-1} + b\, \xi_n + \alpha c\, \xi_{n+1} = (b + c + \alpha c)\xi_n$$

for $n > 2$, and by induction, we can easily obtain that $\xi_n = \frac{\xi_1}{\alpha^{n-1}}$. Since $\alpha > 1$, we have $\sum_{n\in\mathbb{Z}} \xi_n < \infty$. Finally, we observe that $\eta A = \lambda\eta$ where $\eta = (\dots, 1, 1, 1, \dots)$. This means that $\eta \cdot \xi < \infty$. ■

Proof of Proposition 7.15. (1) Note that the tri-diagonal matrix A_4 given by (7.8) is a balanced matrix (see Definition 7.11) with the eigenvalue $\lambda = b + 2r$, $\sigma_1 = b + 2r + \alpha + \beta$ and $\sigma_i = b + 2r$ for $i > 1$. We show that the right eigenvector ξ corresponding to λ has entries as in (7.9). To see this, we set $\xi_1 = 1$ and find from the equation $(b + r + \alpha)x_1 + (r + \beta)x_2 = (b + 2t)x_1$ that $x_2 = \frac{r-\alpha}{r+\beta}$. Fix $n > 0$ and assume that ξ_i satisfies equations in (7.9) for all $i \le 2n - 1$. We note that, using the equality

$$(r - \beta)\xi_{2n-2} = (r + \alpha)\xi_{2n-1},$$

one can find ξ_{2n} from the relation

$$(r - \beta)\xi_{2n-2} + b\, \xi_{2n-1} + (r + \beta)\xi_{2n} = (b + 2r)\xi_{2n-1}.$$

Then,

$$(r + \alpha)\xi_{2n-1} + (r + \beta)\xi_{2n} = 2r\, \xi_{2n-1}$$

and

$$\xi_{2n} = \frac{(r - \alpha)\, \xi_{2n-1}}{(r + \beta)} = \frac{(r - \alpha)^n\, (r - \beta)^{n-1}}{(r + \alpha)^{n-1}(r + \beta)^n}.$$

A similar calculation gives the formula for ξ_{2n+1}:

$$\xi_{2n+1} = \frac{(r-\alpha)^n (r-\beta)^n}{(r+\alpha)^n (r+\beta)^n}.$$

(2) Now we find the conditions under which the eigenvector ξ is summable, that is, $\sum_{i\in\mathbb{N}} \xi_i < \infty$. Recall that the parameters q_1 and q_2 of the matrix A_4 have been defined in (7.7).

Suppose that $q_1 q_2 < 1$. Then we can write $\xi_{2n+1} = (q_1\, q_2)^n$, and $\xi_{2(n+1)} = q_1\, (q_1\, q_2)^n$. This means that the sum of all entries of ξ can be found as follows:

$$\begin{aligned}\sum_{i=0}^{\infty} \xi_i &= \sum_{n=0}^{\infty} (q_1\, q_2)^n + \sum_{n=0}^{\infty} q_1\, (q_1\, q_2)^n \\ &= \frac{1}{(1-q_1 q_2)} + \frac{q_1}{(1-q_1 q_2)} \\ &= \frac{2r+\beta-\alpha}{(r+\beta)} < \infty.\end{aligned}$$

It is obvious that $\eta = (\ldots, 1, 1, 1, \ldots)$ is the left eigenvector corresponding to λ, and we have $\eta \cdot \xi < \infty$.

(3) If $q_1 q_2 \geq 1$, then the series $\sum_{i=0}^{\infty} \xi_i$ diverges and the tail-invariant measure μ given by (7.1) is σ-finite. ■

Proof of Proposition 7.17. It is obvious that the matrix A_5 has the equal column sum property and the eigenvalue $\lambda = 2$. One can check by definition that λ is the Perron eigenvalue and the matrix A_5 is positive recurrent. Then the left and right eigenvectors are

$$\eta = (1, 1, 1, \ldots)$$

and

$$\xi = \left(1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right)^{\top}.$$

Since the matrix A_5 is positive recurrent, it follows from Theorem 7.2 that the Bratteli diagram supports the unique (up to constant multiple) finite ergodic invariant measure which takes positive values on cylinder sets, and this measure is generated by ξ and λ. It is easy to see that any probability tail-invariant measure on $B(F_5)$ should have positive values on cylinder sets. It follows from the fact that for every non-negative n, there is an edge between the first vertex on level n and every vertex on level $n+1$. Thus, the probability ergodic tail-invariant measure for $B(F_5)$ is unique. ■

Proof of Proposition 7.19. By definition of the matrix A_6, the entries of A_6 are

$$a_{1,n} = a_{2,2n} = a_{n+1,n} = 1, \quad \forall\, n \in \mathbb{N}$$

and zero otherwise. Hence, the equation $A_6\xi = \lambda\xi$ is equivalent to the system

$$\begin{cases} \sum_{k=1}^{\infty} \xi_k = \lambda \xi_1, \\ \xi_1 + \sum_{k=1}^{\infty} \xi_{2k} = \lambda \xi_2, \\ \xi_n = \lambda \xi_{n+1}, \qquad n \geq 2. \end{cases} \tag{B.1}$$

Direct calculations show that the Perron eigenvalue λ is $1 + \sqrt{2}$, and the corresponding probability right eigenvector $\xi = (\xi_n)_{n\in\mathbb{N}}$ is given by

$$\xi_1 = \frac{1}{1+\sqrt{2}}; \quad \xi_n = \frac{2}{(1+\sqrt{2})^n}, \quad n \geq 2,$$

and the matrix A_6 is recurrent (see [61]). The left eigenvector is $\eta = (1, \lambda - 1, 1, \lambda - 1, \ldots)$. Thus, the matrix A_6 is positive recurrent, and we can again apply Theorem 7.2 to conclude that the invariant measure given by (7.1) is uniquely ergodic. ■

Proof of Proposition 7.21. Let η and ξ be left and right eigenvectors corresponding to eigenvalue λ. Then $\eta A_7 = \lambda\eta$ implies

$$\sum_{k=0}^{\infty} c_k \eta_k = \lambda\, \eta_0, \quad \eta_0 = \lambda\, \eta_1, \ldots, \eta_k = \lambda\, \eta_{k+1}, \ldots$$

Setting $\eta_0 = 1$, we get $\eta = \left(1, \frac{1}{\lambda}, \ldots, \frac{1}{\lambda^k}, \ldots\right)$ and

$$\sum_{k=0}^{\infty} \frac{c_k}{\lambda^{k+1}} = 1. \tag{B.2}$$

If $c_k \leq C$ for all $k \in \mathbb{N}_0$, then we deduce from (B.2) that $1 \leq \frac{C}{\lambda} \cdot \frac{1}{1-1/\lambda}$ or $\lambda \leq C + 1$. In particular, $\lambda = C + 1$ if all $c_k = C$.

Now we calculate the right eigenvector $\xi = (\xi_k)_{k=0}^{\infty}$. Observe that $A_7\xi = \lambda\xi$ implies the following relations

$$c_o\, \xi_0 + \xi_1 = \lambda\, \xi_0, \ldots, c_k\, \xi_0 + \xi_{k+1} = \lambda\, \xi_k, \ldots$$

Hence, setting $\xi_0 = 1$, we have

$$\xi_{k+1} = \lambda^{k+1} - c_0\, \lambda^k - c_1\lambda^{k-1} - \cdots - c_{k-1}\lambda - c_k, \quad k \in \mathbb{N}_0,$$

which proves (7.14). We check that $\xi_k > 0$ for all $k \in \mathbb{N}_0$. Indeed,

$$\xi_{k+1} = \lambda^{k+1} - \lambda^{k+1}\left(\frac{c_0}{\lambda} + \frac{c_1}{\lambda^2} + \cdots + \frac{c_k}{\lambda^{k+1}}\right) > \lambda^{k+1}\left(1 - \sum_{i=0}^{\infty} \frac{c_i}{\lambda^{i+1}}\right) = 0$$

as follows from (B.2). The eigenvector $\xi = (\xi_k)$ and λ define the tail-invariant measure μ according to Theorem 7.1. Clearly, this measure is infinite.

We can also find conditions under which $\xi \cdot \eta < \infty$. We calculate

$$\begin{aligned}\sum_{k=1}^{\infty} \xi_k \, \eta_k &= \sum_{k=1}^{\infty} \frac{1}{\lambda^k}\Big(\lambda^k - \sum_{j=0}^{k-1} c_j \lambda^{k-j-1}\Big) \\ &= \sum_{k=1}^{\infty}\Big(1 - \sum_{j=0}^{k-1} \frac{c_j}{\lambda^{j+1}}\Big) \\ &= \sum_{k=1}^{\infty} \sum_{j \geq 1} \frac{c_j}{\lambda^{j+1}} \\ &= \sum_{k=1}^{\infty} \frac{k \cdot c_k}{\lambda^{k+1}}.\end{aligned}$$

This proves the second statement of Proposition 7.21. ∎

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References

[1] J. Aaronson, *An introduction to infinite ergodic theory*. Math. Surveys Monogr. 50, American Mathematical Society, Providence, RI, 1997 Zbl 0882.28013 MR 1450400

[2] M. Adamska, S. Bezuglyi, O. Karpel, and J. Kwiatkowski, Subdiagrams and invariant measures on Bratteli diagrams. *Ergodic Theory Dynam. Systems* **37** (2017), no. 8, 2417–2452 Zbl 1382.37005 MR 3719266

[3] M. Amini, G. A. Elliott, and N. Golestani, The category of ordered Bratteli diagrams. *Canad. J. Math.* **73** (2021), no. 1, 1–28 Zbl 1458.37008 MR 4201532

[4] N. Avni, J. Breuer, and B. Simon, Periodic Jacobi matrices on trees. *Adv. Math.* **370** (2020), article no. 107241, 42 pp. Zbl 1512.47056 MR 4103777

[5] S. Bezuglyi, A. H. Dooley, and J. Kwiatkowski, Topologies on the group of Borel automorphisms of a standard Borel space. *Topol. Methods Nonlinear Anal.* **27** (2006), no. 2, 333–385 Zbl 1136.37002 MR 2237460

[6] S. Bezuglyi, A. H. Dooley, and K. Medynets, The Rokhlin lemma for homeomorphisms of a Cantor set. *Proc. Amer. Math. Soc.* **133** (2005), no. 10, 2957–2964 Zbl 1134.37303 MR 2159774

[7] S. Bezuglyi and P. E. T. Jorgensen, Harmonic analysis invariants for infinite graphs via operators and algorithms. *J. Fourier Anal. Appl.* **27** (2021), no. 2, article no. 34, 46 pp. Zbl 1462.05261 MR 4244297

[8] S. Bezuglyi and P. E. T. Jorgensen, Harmonic analysis on graphs via Bratteli diagrams and path-space measures. *Dissertationes Math.* **574** (2022), 1–74 Zbl 1493.37016 MR 4388381

[9] S. Bezuglyi and P. E. T. Jorgensen, IFS measures on generalized Bratteli diagrams. In *Recent developments in fractal geometry and dynamical systems*, pp. 123–145, Contemp. Math. 797, American Mathematical Society, Providence, RI, 2024 Zbl 07840362 MR 4730766

[10] S. Bezuglyi, P. E. T. Jorgensen, and S. Sanadhya, Substitution-dynamics and invariant measures for infinite alphabet-path space. *Adv. in Appl. Math.* **156** (2024), article no. 102687, 39 pp. Zbl 1540.37026 MR 4716594

[11] S. Bezuglyi and O. Karpel, Bratteli diagrams: structure, measures, dynamics. In *Dynamics and numbers*, pp. 1–36, Contemp. Math. 669, American Mathematical Society, Providence, RI, 2016 Zbl 1376.37005 MR 3546661

[12] S. Bezuglyi and O. Karpel, Invariant measures for Cantor dynamical systems. In *Dynamics: topology and numbers*, pp. 259–295, Contemp. Math. 744, American Mathematical Society, Providence, RI, 2020 Zbl 1447.37004 MR 4062569

[13] S. Bezuglyi, O. Karpel, and J. Kwiatkowski, Exact number of ergodic invariant measures for Bratteli diagrams. *J. Math. Anal. Appl.* **480** (2019), no. 2, article no. 123431, 49 pp. Zbl 1425.37014 MR 4000100

[14] S. Bezuglyi, J. Kwiatkowski, and K. Medynets, Aperiodic substitution systems and their Bratteli diagrams. *Ergodic Theory Dynam. Systems* **29** (2009), no. 1, 37–72 Zbl 1169.37004 MR 2470626

[15] S. Bezuglyi, J. Kwiatkowski, K. Medynets, and B. Solomyak, Invariant measures on stationary Bratteli diagrams. *Ergodic Theory Dynam. Systems* **30** (2010), no. 4, 973–1007 Zbl 1205.37009 MR 2669408

[16] S. Bezuglyi, J. Kwiatkowski, K. Medynets, and B. Solomyak, Finite rank Bratteli diagrams: structure of invariant measures. *Trans. Amer. Math. Soc.* **365** (2013), no. 5, 2637–2679 Zbl 1278.37010 MR 3020111

[17] S. Bezuglyi, J. Kwiatkowski, and R. Yassawi, Perfect orderings on finite rank Bratteli diagrams. *Canad. J. Math.* **66** (2014), no. 1, 57–101 Zbl 1353.37023 MR 3150704

[18] S. Bezuglyi and R. Yassawi, Orders that yield homeomorphisms on Bratteli diagrams. *Dyn. Syst.* **32** (2017), no. 2, 249–282 Zbl 1430.37010 MR 3638433

[19] J. Bobok and H. Bruin, Constant slope maps and the Vere-Jones classification. *Entropy* **18** (2016), no. 6, article no. 234, 27 pp. MR 3530057

[20] O. Bratteli, Inductive limits of finite dimensional C^*-algebras. *Trans. Amer. Math. Soc.* **171** (1972), 195–234 Zbl 0264.46057 MR 0312282

[21] O. Bratteli, P. E. T. Jørgensen, K. H. Kim, and F. Roush, Non-stationarity of isomorphism between AF algebras defined by stationary Bratteli diagrams. *Ergodic Theory Dynam. Systems* **20** (2000), no. 6, 1639–1656 Zbl 0982.46042 MR 1804950

[22] O. Bratteli, P. E. T. Jorgensen, K. H. Kim, and F. Roush, Decidability of the isomorphism problem for stationary AF-algebras and the associated ordered simple dimension groups. *Ergodic Theory Dynam. Systems* **21** (2001), no. 6, 1625–1655 Zbl 1007.46046 MR 1869063

[23] O. Bratteli, P. E. T. Jorgensen, K. H. Kim, and F. Roush, Computation of isomorphism invariants for stationary dimension groups. *Ergodic Theory Dynam. Systems* **22** (2002), no. 1, 99–127 Zbl 1067.46060 MR 1889566

[24] J. S. Christiansen, B. Simon, and M. Zinchenko, Finite gap Jacobi matrices, III. Beyond the Szegő class. *Constr. Approx.* **35** (2012), no. 2, 259–272 Zbl 1238.42009 MR 2891229

[25] J. S. Christiansen, B. Simon, and M. Zinchenko, Finite gap Jacobi matrices: a review. In *Spectral analysis, differential equations and mathematical physics: a festschrift in honor of Fritz Gesztesy's 60th birthday*, pp. 87–103, Proc. Sympos. Pure Math. 87, American Mathematical Society, Providence, RI, 2013 Zbl 1319.47028 MR 3087900

[26] A. Connes and W. Krieger, Measure space automorphisms, the normalizers of their full groups, and approximate finiteness. *J. Funct. Anal.* **24** (1977), no. 4, 336–352 Zbl 0369.28013 MR 0444900

[27] R. Dougherty, S. Jackson, and A. S. Kechris, The structure of hyperfinite Borel equivalence relations. *Trans. Amer. Math. Soc.* **341** (1994), no. 1, 193–225 Zbl 0803.28009 MR 1149121

[28] T. Downarowicz and O. Karpel, Dynamics in dimension zero: a survey. *Discrete Contin. Dyn. Syst.* **38** (2018), no. 3, 1033–1062 Zbl 1394.37023 MR 3808986

[29] T. Downarowicz and O. Karpel, Decisive Bratteli–Vershik models. *Studia Math.* **247** (2019), no. 3, 251–271 Zbl 1430.37011 MR 3937447

[30] T. Downarowicz and A. Maass, Finite-rank Bratteli–Vershik diagrams are expansive. *Ergodic Theory Dynam. Systems* **28** (2008), no. 3, 739–747 Zbl 1171.37007 MR 2422014

[31] F. Durand, Combinatorics on Bratteli diagrams and dynamical systems. In *Combinatorics, automata and number theory*, pp. 324–372, Encyclopedia Math. Appl. 135, Cambridge University Press, Cambridge, 2010 Zbl 1272.37006 MR 2759109

[32] F. Durand, A. Frank, and A. Maass, Eigenvalues of minimal Cantor systems. *J. Eur. Math. Soc. (JEMS)* **21** (2019), no. 3, 727–775 Zbl 1416.54018 MR 3908764

[33] F. Durand, B. Host, and C. Skau, Substitutional dynamical systems, Bratteli diagrams and dimension groups. *Ergodic Theory Dynam. Systems* **19** (1999), no. 4, 953–993 Zbl 1044.46543 MR 1709427

[34] F. Durand and D. Perrin, *Dimension groups and dynamical systems—substitutions, Bratteli diagrams and Cantor systems*. Cambridge Stud. Adv. Math. 196, Cambridge University Press, Cambridge, 2022 Zbl 1493.37002 MR 4354542

[35] E. G. Effros, *Dimensions and C^*-algebras*. CBMS Reg. Conf. Ser. Math. 46, Conference Board of the Mathematical Sciences, Washington, DC, 1981 Zbl 0475.46050 MR 0623762

[36] E. G. Effros, D. E. Handelman, and C. L. Shen, Dimension groups and their affine representations. *Amer. J. Math.* **102** (1980), no. 2, 385–407 Zbl 0457.46047 MR 0564479

[37] S. Ferenczi, Substitution dynamical systems on infinite alphabets. *Ann. Inst. Fourier (Grenoble)* **56** (2006), no. 7, 2315–2343 Zbl 1147.37007 MR 2290783

[38] A. H. Forrest, K-groups associated with substitution minimal systems. *Israel J. Math.* **98** (1997), 101–139 Zbl 0891.46042 MR 1459849

[39] D. Frettlöh, A. Garber, and N. Mañibo, Catalan numbers as discrepancies for a family of substitutions on infinite alphabets. *Indag. Math. (N.S.)* **35** (2024), no. 5, 890–913 Zbl 07940186 MR 4793395

[40] J.-M. Gambaudo and M. Martens, Algebraic topology for minimal Cantor sets. *Ann. Henri Poincaré* **7** (2006), no. 3, 423–446 Zbl 1090.37006 MR 2226743

[41] S. Gao, *Invariant descriptive set theory*. Pure Appl. Math. (Boca Raton) 293, CRC Press, Boca Raton, FL, 2009 Zbl 1154.03025 MR 2455198

[42] S. Gao and S. Jackson, Countable abelian group actions and hyperfinite equivalence relations. *Invent. Math.* **201** (2015), no. 1, 309–383 Zbl 1388.03044 MR 3359054

[43] T. Giordano, D. Gonçalves, and C. Starling, Bratteli–Vershik models for partial actions of $\mathbb{Z}$. *Internat. J. Math.* **28** (2017), no. 10, article no. 1750073, 17 pp. Zbl 1378.37016 MR 3699170

[44] T. Giordano, H. Matui, I. F. Putnam, and C. F. Skau, Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems. *Invent. Math.* **179** (2010), no. 1, 119–158 Zbl 1293.37004 MR 2563761

[45] T. Giordano, I. F. Putnam, and C. F. Skau, Topological orbit equivalence and C^*-crossed products. *J. Reine Angew. Math.* **469** (1995), 51–111 Zbl 0834.46053 MR 1363826

[46] R. Gjerde and Ø. Johansen, Bratteli–Vershik models for Cantor minimal systems: applications to Toeplitz flows. *Ergodic Theory Dynam. Systems* **20** (2000), no. 6, 1687–1710 Zbl 0992.37008 MR 1804953

[47] E. Glasner and B. Weiss, Weak orbit equivalence of Cantor minimal systems. *Internat. J. Math.* **6** (1995), no. 4, 559–579 Zbl 0879.54046 MR 1339645

[48] R. H. Herman, I. F. Putnam, and C. F. Skau, Ordered Bratteli diagrams, dimension groups and topological dynamics. *Internat. J. Math.* **3** (1992), no. 6, 827–864 Zbl 0786.46053 MR 1194074

[49] S. Jackson, A. S. Kechris, and A. Louveau, Countable Borel equivalence relations. *J. Math. Log.* **2** (2002), no. 1, 1–80 Zbl 1008.03031 MR 1900547

[50] J. Janssen, A. Quas, and R. Yassawi, Bratteli diagrams where random orders are imperfect. *Proc. Amer. Math. Soc.* **145** (2017), no. 2, 721–735 Zbl 1368.37021 MR 3577873

[51] A. S. Kechris, *Classical descriptive set theory*. Grad. Texts in Math. 156, Springer, New York, 1995 Zbl 0819.04002 MR 1321597

[52] A. S. Kechris and B. D. Miller, *Topics in orbit equivalence*. Lecture Notes in Math. 1852, Springer, Berlin, 2004 Zbl 1058.37003 MR 2095154

[53] B. P. Kitchens, *Symbolic dynamics: one-sided, two-sided and countable state Markov shifts*. Universitext, Springer, Berlin, 1998 Zbl 0892.58020 MR 1484730

[54] W. Krieger, On ergodic flows and the isomorphism of factors. *Math. Ann.* **223** (1976), no. 1, 19–70 Zbl 0332.46045 MR 0415341

[55] N. Mañibo, D. Rust, and J. J. Walton, Substitutions on compact alphabets. 2022, arXiv:2204.07516v2, to appear in *J. Lond. Math. Soc. (2)*

[56] K. Medynets, Cantor aperiodic systems and Bratteli diagrams. *C. R. Math. Acad. Sci. Paris* **342** (2006), no. 1, 43–46 Zbl 1081.37004 MR 2193394

[57] M. G. Nadkarni, On the existence of a finite invariant measure. *Proc. Indian Acad. Sci. Math. Sci.* **100** (1990), no. 3, 203–220 Zbl 0733.28009 MR 1081705

[58] M. G. Nadkarni, *Basic ergodic theory*. 3rd edn., Texts Read. Math. 6, Hindustan Book Agency, Delhi; distributed outside Asia by Henry Helson, Berkeley, CA, 2013 Zbl 1270.28013 MR 1373195

[59] I. F. Putnam, The C^*-algebras associated with minimal homeomorphisms of the Cantor set. *Pacific J. Math.* **136** (1989), no. 2, 329–353 Zbl 0631.46068 MR 0978619

[60] I. F. Putnam, *Cantor minimal systems*. Univ. Lecture Ser. 70, American Mathematical Society, Providence, RI, 2018 Zbl 1396.37002 MR 3791491

[61] T. C. Raszeja, *Thermodynamic formalism for generalized countable Markov shifts*. Ph.D. thesis, Institute of Mathematics and Statistics, University of São Paulo, São Paulo, 2020

[62] T. Shimomura, Bratteli–Vershik models and graph covering models. *Adv. Math.* **367** (2020), article no. 107127, 54 pp. Zbl 1441.37015 MR 4080580

[63] T. Shimomura, Refinement of Bratteli–Vershik models. [v1] 2020, [v5] 2023, arXiv:2010.02617v5

[64] R. Treviño, Flat surfaces, Bratteli diagrams and unique ergodicity à la Masur. *Israel J. Math.* **225** (2018), no. 1, 35–70 Zbl 1405.37034 MR 3805642

[65] D. Vere-Jones, Geometric ergodicity in denumerable Markov chains. *Quart. J. Math.* **13** (1962), no. 1, 7–28 Zbl 0104.11805 MR 0141160

[66] D. Vere-Jones, Ergodic properties of nonnegative matrices. I. *Pacific J. Math.* **22** (1967), no. 2, 361–386 Zbl 0171.15503 MR 0214145

[67] D. Vere-Jones, Ergodic properties of nonnegative matrices. II. *Pacific J. Math.* **26** (1968), no. 3, 601–620 Zbl 0351.60069 MR 0236745

[68] A. M. Vershik, Uniform algebraic approximation of shift and multiplication operators (in Russian). *Dokl. Akad. Nauk SSSR* **259** (1981), no. 3, 526–529. English translation: *Soviet Math. Dokl.* **24** (1981), 97–100 Zbl 0484.47005 MR 0625756

[69] A. M. Veršik, A metric invariant of automorphisms of a measure space connected with uniform approximation and sequences of partitions (in Russian). *Dokl. Akad. Nauk SSSR* **209** (1973), no. 1, 15–18. English translation: *Soviet Math. Dokl.* **14** (1973), 297–301 Zbl 0289.28014 MR 0322139

[70] A. M. Veršik, A theorem on periodical Markov approximation in ergodic theory. In H. Michel (ed.), *Ergodic theory and related topics: Proceedings of the Conference held in Vitte/Hiddensee (GDR), October 19–23, 1981*, pp. 195–206, Math. Res. 12, De Gruyter, Boston, 1982 Zbl 0517.47005 MR 0730788

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