Groups Geom. Dyn. 20 (2026), 1051–1069
DOI 10.4171/GGD/837

Small-cancellation groups with and without sigma-compact Morse boundary

Stefanie Zbinden

Abstract. We provide examples of classical $C'(1/6)$-small-cancellation groups which have non-σ-compact Morse boundary. These are first known examples of groups with non-σ-compact Morse boundary. Some $C'(1/6)$-small-cancellation groups do have σ-compact Morse boundary, so this property distinguishes quasi-isometry types of small-cancellation groups. In fact, we give a complete description of when Morse boundaries of $C'(1/6)$-groups have σ-compact Morse boundary. We also provide examples of $C'(1/6)$-groups where all Morse rays are strongly contracting.

1. Introduction

The Morse boundary is a quasi-isometry invariant which was introduced for CAT(0) groups in [4] and generalised for all finitely generated groups in [5].

The Morse boundary is neither compact nor metrisable for non-hyperbolic groups [7, 12], but for all known examples so far the Morse boundary is σ-compact, such as in the cases where it has been fully described [3, 16], and in all groups (quasi-isometric to a space) where all Morse rays are strongly contracting such as CAT(0) groups and coarsely Helly groups, including hierarchically hyperbolic groups [10, 13, 15].

We show that, in contrast, the Morse boundaries of $C'(1/6)$-small-cancellation groups exhibit a variety of behaviours. As suggested in [3], we show that there indeed exist $C'(1/6)$-small-cancellation groups with non-σ-compact Morse boundary. Further, we show that not all infinitely presented $C'(1/6)$-groups have non-σ-compact Morse boundary. In fact for some, being Morse is equivalent to being strongly contracting. We emphasise that, in particular, σ-compactness can be used to distinguish quasi-isometry types of small cancellation groups. While the theory of Morse boundaries is largely developed in analogy with Gromov boundaries, this is a genuinely new phenomenon unique to the Morse boundary.

Theorem A. *For each of the following properties, there exists an infinitely presented, finitely generated $C'(1/6)$-group $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ satisfying that property.*

(1) *The Morse boundary of G is non-σ-compact.*

Mathematics Subject Classification 2020: 20F65.
Keywords: Morse boundary, sigma-compact, strong contraction, small-cancellation.

(2) *Every Morse geodesic in* $\mathrm{Cay}(G, \mathcal{S})$ *is strongly contracting.*

(3) *The Morse boundary of G is* σ*-compact and there exists a Morse ray in* $\mathrm{Cay}(G, \mathcal{S})$ *which is not strongly contracting.*

The proof of the above theorem heavily relies on the work of [1] showing that being Morse is equivalent to being contracting and the work of [2] where they develop a tool to determine whether rays are Morse (or strongly contracting) in $C'(1/6)$-groups.

We further give a characterisation that we call increasing partial small-cancellation condition (IPSC) to show when $C'(1/6)$-groups have non-σ-compact Morse boundary. Roughly speaking, groups satisfy the IPSC, if there exist subwords of longer and longer relators which are a significant fraction of that relator and have very small intersection with other relators.

Theorem B. *The Morse boundary of a finitely generated* $C'(1/6)$*-group* $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ *is non-*σ*-compact if and only if G satisfies the IPSC.*

Theorem B is in some sense an upgraded version of Theorem A (1). The key to upgrade the proof is a technique used in the proof of Lemma 3.10 to construct geodesics in $C'(1/6)$-groups that contain desired subwords.

Outline. In Section 2, we recall background on small-cancellation. In Section 3, we introduce the IPSC and the $C'(1/f)$-small-cancellation condition for functions f. The latter is a small-cancellation condition where pieces of longer relators have to be a smaller portion of the relator than those of shorter relators. We show that all $C'(1/f)$-groups have non-σ-compact Morse boundary. The main idea behind the proof is that in $C'(1/f)$-groups, we can find sequences of geodesics which have very nice contraction properties for balls up to a certain size, but arbitrarily bad contraction properties for larger balls. Lastly, we show that satisfying the IPSC and having non-σ-compact Morse boundary is equivalent for $C'(1/6)$-groups. A key part of this proof is a strategy developed in the proof of Lemma 3.11, which allows us to construct geodesics that contain desired subwords. In Section 4, we construct a class of infinitely presented $C'(1/6)$-groups where being Morse is equivalent to being strongly contracting. We ensure this by making sure that the intersection of different relators is large enough. A more detailed outline of the strategy can be found at the beginning of Section 4. Lastly, in Section 5, we adapt the construction of Section 4 to get a group with σ-compact Morse boundary and where not all Morse rays are strongly contracting.

2. Preliminaries

Notation and conventions. For the rest of the paper, unless specified otherwise, $\mathcal{S}$ denotes a finite set of formal variables, $\mathcal{S}^{-1}$ its formal inverses and $\overline{\mathcal{S}}$ the symmetrised set $\mathcal{S} \cup \mathcal{S}^{-1}$. A word w over $\mathcal{S}$ (respectively $\overline{\mathcal{S}}$) is a finite sequence of elements in $\mathcal{S}$

(respectively $\overline{\mathcal{S}}$). We denote by w^- and w^+ the first and last letter of w, respectively. By abuse of notation, we sometimes allow words to be infinite.

Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a finitely generated group, $X = \mathrm{Cay}(G, \mathcal{S})$ its Cayley graph, p an edge path in X and v a word over $\overline{\mathcal{S}}$.

- By following p, we can read a word w over $\overline{\mathcal{S}}$. We say that p is labelled by w. We say a word w' is a subword of p if it is a subword of w.
- For any vertex $x \in X$, there is a unique edge path labelled by v and starting at x.

2.1. Small-cancellation

We will subsequently define most notions needed in our paper. For further background on small-cancellation, we refer to [11].

We say a word w over $\overline{\mathcal{S}}$ is *cyclically reduced* if it is reduced and all its cyclic shifts are reduced. Given a set $\mathcal{R}$ of cyclically reduced words, we denote by $\overline{\mathcal{R}}$ the cyclic closure of $\mathcal{R} \cup \mathcal{R}^{-1}$. If $\mathcal{R} = \{w\}$, we sometimes denote $\overline{\mathcal{R}}$ by $\overline{w}$.

Definition 2.1 (Piece). Let $\mathcal{S}$ be a finite set and let $\mathcal{R}$ be a set of cyclically reduced words over $\overline{\mathcal{S}}$. We say that p is *a piece* if there exists distinct words $r, r' \in \overline{\mathcal{R}}$ such that p is a prefix of both r and r'. We say that p is a *piece of a word* $r \in \overline{\mathcal{R}}$ if p is a piece and a subword of r.

Definition 2.2 ($C'(\lambda)$ condition). Let $\lambda > 0$ be a constant. We say that a set $\mathcal{R}$ of cyclically reduced words satisfies the $C'(\lambda)$*-small-cancellation condition* if for every word $r \in \overline{\mathcal{R}}$ and every piece p of r we have $|p| < \lambda |r|$.

If $\mathcal{R}$ satisfies the $C'(\lambda)$-small-cancellation condition, we call the finitely generated group $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ a $C'(\lambda)$*-group*. If $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ is a $C'(\lambda)$-group, then the graph Γ defined as the disjoint union of cycle graphs labelled by the elements of $\mathcal{R}$ is a $Gr'(\lambda)$-labelled graph as defined in [9]. We can thus state and use the results of [2, 9] in the less general setting of groups satisfying the $C'(\lambda)$-small-cancellation condition.

Lemma 2.3 ([9, Lemma 2.15]). *Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a $C'(1/6)$-group. Let $r \in \mathcal{R}$ be a relator, Γ_0 a cycle graph labelled by r and let $f : \Gamma_0 \to \mathrm{Cay}(G, \mathcal{S})$ be a label-preserving graph homomorphism. Then f is an isometric embedding, and its image is convex.*

We call the image of such a label-preserving graph homomorphism an *embedded component*.

2.2. Disk diagrams

Definition 2.4. A *(disk) diagram* is a contractible, planar 2-complex. A disk diagram is

- *simple*, if it is homeomorphic to a disk.
- $\mathcal{S}$*-labelled*, if all edges are labelled by an element of $\overline{\mathcal{S}}$.

- a diagram *over* $\mathcal{R}$, if the boundary of any face is labelled by an element of $\overline{\mathcal{R}}$.

Let D be a disk diagram and let Π be a face of D. An *arc* is a maximal subpath of D whose interior vertices all have degree 2. An arc is an *interior arc* if its interior is contained in the interior of D and an *exterior arc* otherwise. Note that an exterior arc is contained in the boundary of D. The *interior degree* of a face Π is the number of interior arcs in its boundary, and the *exterior degree* of a face Π is the number of exterior arcs in its boundary. A face is an *interior face* if its exterior degree is 0 and *exterior face* otherwise.

Definition 2.5 (Combinatorial geodesic bigon). We say that a *combinatorial geodesic bigon* (D, γ_1, γ_2) is a simple diagram D whose boundary ∂D is a concatenation of γ_1 and γ_2 and such that the following conditions hold.

(1) Each boundary face whose exterior part is a single arc contained in one of the sides γ_i has interior degree at least 4.

(2) The boundary of each interior face consists of at least seven arcs.

Combinatorial geodesic bigons have been classified in [14] as follows.

Lemma 2.6 (Strebel's classification [14, Theorem 43]). *Let D be a combinatorial geodesic bigon. Either D consists of a single face or D has shape I_1 as depicted in Figure 1, where any but the left and rightmost faces are optional.*

We can use Strebel's classification to prove the following result, which is our main tool to show that certain paths we care about are in fact geodesics.

Lemma 2.7. *Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a $C'(1/6)$-group and let $X = Cay(G, \mathcal{S})$. Let γ be a path in X labelled by a reduced word w. If for every common subword u of both w and a relator r we have that*

$$|u| \leq \frac{|r|}{3}, \tag{2.1}$$

then γ is a geodesic.

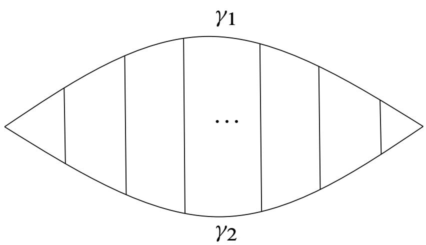

Figure 1. Shape I_1 of a combinatorial geodesic bigon.

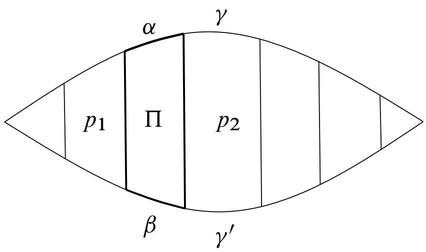

Figure 2. Diagram D in the proof of Lemma 2.7.

Proof. Assume that the statement does not hold and let γ be the shortest path which satisfies (2.1) but is not a geodesic. Let x and y be the start and endpoint of γ and let γ' be a geodesic, labelled by say v, from y to x. It is well known (see, e.g., [8, Lemma 2.13]) that there exists an $\overline{\mathcal{S}}$-labelled diagram D over $\mathcal{R}$ whose boundary is labelled by wv and where every interior arc is a piece. The minimality of γ implies that D is simple. The small-cancellation condition ensures that every interior face of D has degree at least 7. Furthermore, let Π be a face, labelled by say r, whose exterior part is a single arc λ contained either in γ or γ'. If λ is contained in γ, $|\lambda| \leq |r|/3$ by (2.1). On the other hand, if λ is contained in γ', then $|\lambda| \leq |r|/2$ because γ' is a geodesic. In either case, the small-cancellation condition ensures that Π has interior degree at least 4. Therefore, the diagram (D, γ, γ') is a combinatorial geodesic bigon. Furthermore, property (2.1) and the fact that γ' is a geodesic imply that D consists of at least two faces.

Strebel's classification (Lemma 2.6) yields that D has shape I_1 as depicted in Figure 1. In particular, as depicted in Figure 2, the boundary $\partial\Pi$ of any face Π of D can be divided into four (possibly) empty paths denoted by α, β, p_1 and p_2, where α is a subpath of γ, β is a subpath of γ' and p_1, p_2 are pieces. By (2.1) and the small-cancellation condition, $|\alpha| + |p_1| + |p_2| < 2|\partial\Pi|/3$ and hence $|\beta| > |\alpha|$. Since this holds for all faces, we have that $|\gamma| < |\gamma'|$, a contradiction to γ' being a geodesic. ■

2.3. Contraction and the Morse boundary

In this section we highlight some consequences of [1, 2]. For further background on the Morse boundary and contraction, we refer to [6] and [2], respectively.

Definition 2.8 (Intersection function). Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a $C'(1/6)$-group. Let γ be an edge path in $\mathrm{Cay}(G, \mathcal{S})$. The *intersection function* of γ is the function $\rho : \mathbb{N} \to \mathbb{R}_+$ defined by

$$\rho(t) = \max_{\substack{|r| \leq t \\ r \in \overline{\mathcal{R}}}} \big\{ |w| \mid w \text{ is a subword of } r \text{ and } \gamma \big\}.$$

Remark 2.9. In light of Lemma 2.3, the intersection function ρ of a geodesic γ is equal to the function ρ', defined via

$$\rho'(t) = \max_{|\Gamma_0| \leq t} \{|\Gamma_0 \cap \gamma|\},$$

where Γ_0 ranges over all embedded components.

The following lemma is a combination of [1, Theorem 1.4] and [2, Corollary 4.14 and Theorem 4.1]. It shows the relation between contraction, Morseness and the intersection function.

Lemma 2.10 ([2, Theorem 4.1 and Corollary 4.14]). *Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a $C'(1/6)$-group. Let α be a geodesic in $X = \mathrm{Cay}(G, \mathcal{S})$ and let ρ be its intersection function. Then:*

(i) *The geodesic α is Morse if and only if ρ is sublinear.*

(ii) *The geodesic α is strongly contracting if and only if ρ is bounded.*

(iii) *If ρ is sublinear, α is M-Morse for some Morse gauge M only depending on ρ.*

(iv) *If α is M-Morse, then $\rho \leq \rho'$ for some sublinear function ρ' only depending on M.*

Proof. By [2, Corollary 4.14], (i) and (ii) hold. To get (iii) observe the following: Since α is a geodesic, it intersects every embedded component Γ_0, with say $|\Gamma_0| = \ell$, in a subsegment of length at most $\ell/2$. Thus for any $x \in \Gamma_0$ with $d(x, \Gamma_0 \cap \alpha) = r \leq \ell/8$, the projection of $B_{\Gamma_0}(x, r)$ in Γ_0 onto $\Gamma_0 \cap \alpha$ projects to a single point (namely one of the two endpoints of $\Gamma_0 \cap \alpha$). Therefore, $\alpha \cap \Gamma_0$ is (r, ρ')-contracting in Γ_0 for $\rho'(r) = \rho(8r)$. Now, [2, Theorem 4.1] implies that α is (r, ρ'')-contracting in X for some ρ'' only depending on α. By [1, Theorem 1.4], α is M-Morse for some Morse gauge M only depending on ρ'' (and hence only depending on ρ).

Lastly, we prove (iv): By [1, Theorem 1.4], α is (r, ρ')-contracting for some sublinear function ρ' only depending on M. We may assume the ρ' is increasing. Let Γ_0 be an embedded component which intersects α and let $x \in \Gamma_0$ be a point maximising $l = d(x, \Gamma_0 \cap \alpha)$. The projection of $B_{\Gamma_0}(x, l)$ onto $\Gamma_0 \cap \alpha$ (and, by Lemma 2.3, the projection of $B_X(x, l)$ onto α) has diameter $|\Gamma_0 \cap \alpha|$. Hence $\rho'(|\Gamma_0|) \geq \rho'(l) \geq |\Gamma_0 \cap \alpha|$. Since this is true for all embedded components Γ_0, we have that $\rho \leq \rho'$. ∎

Definition 2.11 (Contraction exhaustion). Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a $C'(1/6)$-group. A *contraction exhaustion* of $X = \mathrm{Cay}(G, \mathcal{S})$ is a sequence of sublinear functions $\rho_i : \mathbb{N} \to \mathbb{R}_+$ such that $\rho_i \leq \rho_{i+1}$ for all i and the following property holds. The intersection function ρ of any Morse geodesic ray γ in X satisfies $\rho \leq \rho_i$ for some i.

Lemma 2.12. *Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a $C'(1/6)$-group. A contraction exhaustion of* $\mathrm{Cay}(G, \mathcal{S})$ *exists if and only if $\partial_* G$ is σ-compact.*

Proof. The Morse boundary $\partial_* G$ is σ-compact if and only if there exists a sequence of Morse gauges $M_1 \leq M_2 \leq \cdots$ such that any Morse geodesic ray in $\mathrm{Cay}(G, \mathcal{S})$ is M_i-Morse for some i. Lemma 2.10 (iii) and (iv) conclude the proof. ∎

3. Criterion for non sigma-compact Morse boundaries

In this section we introduce the IPSC. We then prove Theorem 3.6, which states that $C'(1/6)$-groups have non-sigma-compact Morse boundary if and only if they satisfy the IPSC. We conclude the section by giving examples of $C'(1/6)$-groups that satisfy IPSC and hence have non-σ-compact Morse boundary.

Since the proof of Theorem 3.6 is quite technical, we start the section by, in Subsection 3.1, proving Lemma 3.3, which is a weaker result than Theorem 3.6, but illustrates the ideas of the proof without having to go into many of the technical details.

3.1. $C'(1/f)$-groups do not have σ-compact Morse boundary

We introduce $C'(1/f)$-groups and show that such groups do not have σ-compact Morse boundary. The proof illustrates the main ideas of the proof of Lemma 3.11.

Definition 3.1. We say that a non-decreasing function $f : \mathbb{N} \to \mathbb{R}_+$ is *viable* if $f(n) \geq 6$ for all n and $\lim_{n\to\infty} f(n) = \infty$.

Observe that a consequence of a function f being viable is that the function $\rho(n) = n/f(n)$ (or equivalently the function $\rho'(n) = \max\{\rho'(n-1), n/f(n)\}$) is sublinear.

Definition 3.2. Let f be a viable function. We say that a finitely generated group $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ is a $C'(1/f)$-*group* if $\mathcal{R}$ is infinite and for every piece p of a relator $r \in \overline{\mathcal{R}}$ we have that $|p| < |r|/f(|r|)$.

In other words, in $C'(1/f)$-groups, pieces of large relators have to be smaller fractions of their relators than pieces of smaller relators. Requiring that viable functions satisfy $f(n) \geq 6$ ensures that all $C'(1/f)$-groups are $C'(1/6)$-groups.

Lemma 3.3. *Let f be a viable function, let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a $C'(1/f)$-group and let $X = \mathrm{Cay}(G, \mathcal{S})$ be its Cayley graph. The Morse boundary $\partial_* G$ is not σ-compact.*

To do so, we assume that $\partial_* G$ is σ-compact and show that this leads to a contradiction. Note, we can and will assume that $\mathcal{R} = \overline{\mathcal{R}}$. By Lemma 2.12, there exists a contraction exhaustion $(\rho_i)_{i\in\mathbb{N}}$ of X. To get to a contradiction, we construct a geodesic ray γ which is Morse but whose intersection function ρ satisfies $\rho \not\leq \rho_i$ for all $i \in \mathbb{N}$.

Construction of the geodesic γ. Let $N \geq 6$ be an integer. For every integer $i \geq N$, let $k_i \geq n_i$ be integers such that $f(n_i) > 4i$ and $\rho_i(t) < t/(2i)$ for all $t \geq k_i$ (such an integer k_i exists since ρ_i is sublinear). Let $r_i = x_i y_i \in \mathcal{R}$ be a relator such that:

(1) $\overline{r_i} \neq \overline{r_{i-1}}$,

(2) $|r_i| \geq k_i$,

(3) $|r_i|/(2i) \leq |x_i| \leq |r_i|/i$,

(4) $x_{i-1}x_i$ is reduced.

Observe that such a relator r_i always exists; properties (1)–(3) can be satisfied by choosing any large enough relator r, while property (4) is satisfied by at least one relator in $\overline{r}$. Observe that $|x_i| \geq |r_i|/2i \geq |r_i|/f(|r_i|)$ and hence x_i is not a piece.

Define γ as the ray starting at the identity labelled by the word $\prod_{i=N}^{\infty} x_i$. It remains to show that γ is a geodesic, γ is Morse and its intersection function

$$\rho(t) = \max_{\substack{|r| \leq t \\ r \in \mathcal{R}}} \{|w| \mid w \text{ is a subword of } r \text{ and } \gamma\}$$

satisfies $\rho \not\lesssim \rho_i$ for all i. By Lemma 2.7, the path γ is a geodesic if $\rho(t) \leq t/3$ for all t. Further, Lemma 2.10 states that γ is Morse if ρ is sublinear. We therefore proceed to bound ρ.

Bounding the intersection function. Let w be a common subword of γ and a relator $r \in \mathcal{R}$. That is, $w = u_i x_{i+1} \cdots x_{j-1} u_j$ for some $i < j$ and (possibly empty) subwords u_i of x_i and u_j of x_j. By construction, x_{i+1} is not a piece, so either $j = i+1$ or $\overline{r} = \overline{r_{i+1}}$.

Case 1: $\overline{r} = \overline{r_{i+1}}$ (and $j \geq i+2$). By construction, $\overline{r_{i+1}} \neq \overline{r_{i+2}}, \overline{r_i}$. Thus $j = i+2$ and both u_i and u_j are pieces. Hence,

$$|w| = |u_i| + |u_j| + |x_{i+1}| < \frac{2|r_{i+1}|}{f(|r_{i+1}|)} + \frac{|r_{i+1}|}{i+1}.$$

In particular, $|w| < |r_{i+1}|/3 = |r|/3$.

Case 2: $j = i+1$. If $\overline{r} \neq \overline{r_i}$, then u_i is a piece of r and hence $|u_i| < |r|/f(|r|) \leq |r|/6$. If $\overline{r} = \overline{r_i}$, then $|x_i| \leq |r|/i$. Thus in either case $|u_i| \leq |r|/6$. Doing the same for u_j, one can show that $|w| < |r|/3$.

So indeed $\rho(t) \leq t/3$ for all t. Furthermore, summarising the two cases, we get that

$$|w| \leq \begin{cases} \dfrac{2|r_i|}{f(|r_i|)} + \dfrac{2|r_i|}{i} & \text{if } \overline{r} = \overline{r}_i \text{ for some } i \geq N, \\[2ex] \dfrac{2|r|}{f(|r|)} & \text{otherwise.} \end{cases}$$

Since f is increasing and unbounded, ρ is indeed sublinear. It remains to show that $\rho \not\lesssim \rho_i$ for all i or, equivalently, that $\rho \not\lesssim \rho_i$ for all $i \geq N$. Let $i \geq N$. Recall that $\rho_i(t) < t/(2i)$ for all $t \geq k_i$, so in particular $\rho_i(|r_i|) < |r_i|/(2i)$. On the other hand, $\rho(|r_i|) \geq |x_i| \geq |r_i|/(2i)$. So indeed $\rho \not\lesssim \rho_i$.

3.2. The general case

We first introduce the IPSC and strong IPSC. We then prove Theorem 3.6, the main result of this section.

Definition 3.4. Let f be a viable function, let x be a reduced word and let $\mathcal{R}$ be a set of cyclically reduced words. We say that *the pair* $(x, \mathcal{R})$ *satisfies the* $C'(1/f)$*-small-cancellation condition* if every common subword p of x and a relator $r \in \overline{\mathcal{R}}$ satisfies $|p| < |r|/f(|r|)$.

The following definition quantifies having sufficiently long subwords w of longer and longer relators that satisfy the $(w, \mathcal{R})$-small-cancellation condition.

Definition 3.5 (IPSC). Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a finitely generated group. We say that G satisfies the *IPSC* if for every sequence $(n_i)_{i \in \mathbb{N}}$ of positive integers, there exists a viable function f such that the following holds: For all $K \geq 0$ there exists $i \geq K$ and a relator $r = xy \in \overline{\mathcal{R}}$ satisfying:

(i) $|r| \geq n_i$,

(ii) $|x| \geq |r|/i$,

(iii) the pair $(x, \mathcal{R})$ satisfies the $C'(1/f)$-small-cancellation condition.

Theorem 3.6. *Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a $C'(1/6)$-group. The Morse boundary of G is non-σ-compact if and only if G satisfies the IPSC.*

Theorem 3.6 is a direct consequence of Lemmas 3.7, 3.10 and 3.11, which are stated and proven below.

Lemma 3.7. *Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a $C'(1/6)$-group. If the Morse boundary of G is not σ-compact, then G satisfies the IPSC.*

Proof. Assume that $\partial_* G$ is not σ-compact. Let $(n_i)_{i \in \mathbb{N}}$ be a sequence of positive integers. We may assume that $n_i < n_{i+1}$ for all i. Define the function $\rho : \mathbb{N} \to \mathbb{R}_+$ iteratively via $\rho(1) = 1$ and

$$\rho(t) = \begin{cases} t & \text{if } t < n_1, \\ \max\{t/i, \rho(t-1)\} & \text{if } n_i \leq t < n_{i+1}. \end{cases}$$

Furthermore, for all $k \in \mathbb{N}$ define the function $\rho_k : \mathbb{N} \to \mathbb{R}_+$ via $\rho_k(t) = \max\{\rho(t), k\}$. Observe that ρ and all ρ_k are sublinear.

Since $\partial_* G$ is not σ-compact, there exists a Morse geodesic ray γ whose intersection function ρ^* is sublinear but $\rho^* \not\preceq \rho_k$ for all k. Define the functions $f, f' : \mathbb{N} \to \mathbb{R}_+$ via

$$f'(t) = \min_{t' \geq t} \left\{ \frac{t'}{\rho^*(t')} \right\} - 1, \quad f(t) = \max\{6, f'(t)\}.$$

Observe that f is viable. Let K_0 be the smallest integer such that $f'(K_0) \geq 6$. Let $K \geq K_0$. Since $\rho^* \not\leq \rho_{n_K}$, there exists $t_0 \geq 1$ such that $\rho^*(t_0) > \rho_{n_K}(t_0)$. In particular, there exists a relator $r \in \overline{\mathcal{R}}$ of length $m \leq t_0$ and a common subword x of r and γ such that $|x| > \rho_{n_K}(t_0)$. We may assume that x is a prefix of r. Let j be such that $n_j \leq m < n_{j+1}$. It remains to show that $j \geq K$ and (ii) and (iii) from the definition of IPSC are satisfied for $r = xy$.

Since $\rho_{n_K}(t_0) \geq n_K$, we have indeed that $j \geq K$. Further, $\rho_{n_K}(m) \geq m/j$ and hence $|x| \geq |r|/j$. Lastly, we prove (i). Let r' be a relator with $|r'| \geq K_1$ and let w be a common subword of r' and x. Since x is a subword of γ, we have that $|w| \leq \rho^*(|r'|) < \frac{|r'|}{f(|r'|)}$. On the other hand, let r' be a relator with $|r'| < K_1$ and let w be a common subword of r' and x. Since $|r| \geq K_1$, w is a piece and hence $|w| < |r'|/6 = |r'|/f(|r'|)$. So indeed, the pair $(x, \mathcal{R})$ satisfies the $C'(1/f)$-small-cancellation condition. Hence, G satisfies the IPSC, which concludes the proof. ∎

Before we can state Lemmas 3.10 and 3.11, we need to introduce the strong IPSC.

Definition 3.8. Let w be a reduced word over $\overline{\mathcal{S}}$. We can write $w = s_1^{k_1} \cdots s_n^{k_n}$, where $s_i \in \overline{\mathcal{S}}$, $k_i > 0$ and $s_{i+1} \neq \{s_i, s_i^{-1}\}$. We say that the word $s_2^{k_2} \cdots s_{n-1}^{k_{n-1}}$ is the *interior* of w. In particular, if $n \leq 2$, the interior of w is empty.

The strong IPSC is similar to the IPSC, but condition (ii) is replaced with the stronger version (ii*), which allows more control on the interior of x. This control will be crucial in the proof of Lemma 3.11.

Definition 3.9 (Strong IPSC). Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a finitely generated group. We say that G satisfies the *strong IPSC* if for every sequence $(n_i)_i$ of integers, there exists a viable function f such that the following holds: For all $K \geq 0$ there exists $i \geq K$ and a relator $r = xy \in \overline{\mathcal{R}}$ satisfying:

(i) $|r| \geq n_i$,

(ii*) $|x'| \geq |r|/i$, where x' denotes the interior of x,

(iii) the pair $(x, \mathcal{R})$ satisfies the $C'(1/f)$-small-cancellation condition.

Lemma 3.10. *Let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a $C'(1/6)$-group satisfying the IPSC. Then G satisfies the strong IPSC.*

Proof. For every generator $s \in \overline{\mathcal{S}}$, define the function $\rho_s : \mathbb{N} \to \mathbb{R}_+$ via

$$\rho_s(t) = \max_{\substack{r \in \overline{\mathcal{R}} \\ |r| \leq t}} \{k \mid s^k \text{ is a subword of } r\}.$$

Observe that the function ρ_s need not be sublinear. Let $\mathcal{S}_1$ be the set of generators $s \in \overline{\mathcal{S}}$ for which ρ_s is sublinear, and let $\mathcal{S}_2$ be the set of generators $s \in \overline{\mathcal{S}}$ for which ρ_s is not sublinear. Since $\mathcal{S}$ is finite, the function $\rho : \mathbb{N} \to R_+$ defined via $\rho(t) = \max_{s \in \mathcal{S}_1}\{\rho_s(t)\}$ is sublinear.

Let $(n_i)_{i\in\mathbb{N}}$ be a sequence of positive integers. For all $i \geq 1$ choose an integer n'_i such that $n'_i \geq n_{3i}$, $n'_i \geq 3i^2$ and such that $\rho(t) \leq \frac{t}{3i}$ for all $t \geq n'_i$. Since ρ is sublinear, we can always find such an integer n'_i.

Next we use that G satisfies the IPSC. Let f be the viable function corresponding to the sequence $(n'_i)_i$. Note that f is unbounded. Let $s \in \mathcal{S}_2$. Since ρ_s is not sublinear, there exists a constant D_s such that $\rho_s(D_s) > D_s/f(D_s)$. In particular, if $(s^k, \mathcal{R})$ satisfies the $C'(1/f)$-small-cancellation condition, for some $s \in \mathcal{S}_2$, then $k < D_s$. Let $D = \max_{s\in\mathcal{S}_2}\{D_s\}$.

Let $K \geq D$. Since $\langle \mathcal{S} \mid \mathcal{R}\rangle$ satisfies the IPSC, there exists an integer $i \geq K$ and a relator $r \in \overline{\mathcal{R}}$ with $r = xy$ and such that:

(1) $|r| \geq n'_i \geq n_{3i}$,

(2) $|x| \geq |r|/i$,

(3) the pair $(x, \mathcal{R})$ satisfies the $C'(1/f)$-small-cancellation condition.

Observe the following: To prove that G satisfies the strong IPSC, it now suffices to show that $|x'| \geq |r|/(3i)$, where x' is the interior of x. Write $x = s_s^k x' s_e^l$ for some $s_s, s_e \in \overline{\mathcal{S}}$. If $s_e \in \mathcal{S}_1$, then $k \leq \rho(|r|)$ by the definition of ρ. By the definition of n'_i, we have additionally $\rho(|r|) \leq |r|/(3i)$. If $s_e \in \mathcal{S}_2$, we have $k \leq D$ as explained above. Since $i \geq D$ and $n'_i \geq 3i^2$, we have in both cases $k \leq |r|/(3i)$. The same holds for l. Thus $|x'| \geq |r|/(3i)$, implying that $\langle \mathcal{S} \mid \mathcal{R}\rangle$ indeed satisfies the strong IPSC. ∎

Lemma 3.11. *Let $G = \langle \mathcal{S} \mid \mathcal{R}\rangle$ be a $C'(1/6)$-group which satisfies the strong IPSC. Then the Morse boundary of G is not σ-compact.*

Proof. We prove this by contradiction. Assume that $\partial_* G$ is σ-compact and let $(\rho_i)_{i\in\mathbb{N}}$ be a contraction exhaustion of $X = \mathrm{Cay}(G, \mathcal{S})$. As in the proof of Lemma 3.3, we construct a Morse geodesic γ whose intersection function ρ satisfies $\rho \not\lesssim \rho_i$ for all i, contradicting the assumption that $\partial_* G$ is σ-compact. However, before we can start with the construction of γ, we need to do some technical setup.

Constructing the x_i. Choose any relator $r_0 \in \overline{\mathcal{R}}$ with $|r_0| \geq 42$. For $i \geq 1$, choose $n_i \geq 6i|r_0|$ such that $\rho_i(t) < t/i$ for all $t \geq n_i$ (such an integer always exists since ρ_i is sublinear). Since $\langle \mathcal{S} \mid \mathcal{R}\rangle$ satisfies the strong IPSC, there exists a viable function f and a sequence of relators r_i with prefix $a_i^{j_i} x_i b_i^{l_i}$ and a sequence of indices $k_i \geq i$ such that:

(1) $a_i, b_i \in \overline{\mathcal{S}}$, $l_i, j_i > 0$, and x_i is the interior of $a_i^{j_i} x_i b_i^{l_i}$,

(2) $|r_i| \geq n_{k_i}$,

(3) $|x_i| \geq |r_i|/k_i$,

(4) the pair $(a_i^{j_i} x_i b_i^{l_i}, \mathcal{R})$ satisfies the $C'(1/f)$-small-cancellation condition.

Unlike in the proof of Lemma 3.3, we cannot guarantee that the words x_i are not pieces or that $x_i x_{i+1}$ is reduced. Unfortunately, in Lemma 3.3, this was a crucial step in showing that γ is a Morse geodesic. Thus it is not enough to define γ as the path labelled by the word $\Pi_{i\geq 1}^{\infty} x_i$. Instead, we have to do some surgery on the words x_i.

Performing surgery on the x_i. Recall that w^- (w^+) denotes the first (last) letter of a word w. Let q be a subword of the relator r_0 such that $|q| \leq |r_0|/3$ and $|q| - 6 \geq |r_0|/6$. With this, $12|q| \leq |x_i|$ for all i. Write $q = s_1 s_2 s_3 q' s_4 s_5 s_6$ for some letters $s_j \in \overline{S}$ for $1 \leq j \leq 6$. Observe that $|q'| \geq |r_0|/6$ and hence q' is not a piece. In the following, we show that for each $i \geq 1$ we can choose words u_i, v_i, w_i, z_i such that:

(1) for $i \geq 2$, $q_i = z_{i-1} q' u_i$ is a subword of q,
(2) for $i \geq 1$, $y_i = v_i x_i w_i$ is a subword of $a_i^{j_i} x_i b_i^{l_i}$,
(3) the word $w = \prod_{i=1}^{\infty} y_i q_{i+1}$ is reduced,
(4) for $i \geq 2$, neither $q_i y_i^-$ nor $y_{i-1}^+ q_i$ is a subword of any relator.

We show how to choose the words u_i and v_i, the words w_i and z_i can be chosen analogously. In the following, λ denotes the empty word.

Case 1: $s_4 = s_5$.

$$u_i = s_4, \quad v_i = \begin{cases} a_i^{j_i} & \text{if } a_i \notin \{s_4, s_4^{-1}\}, \\ \lambda & \text{otherwise.} \end{cases}$$

Case 2: $s_4 \neq s_5 = s_6$.

$$u_i = s_4 s_5, \quad v_i = \begin{cases} a_i^{j_i} & \text{if } a_i \notin \{s_5, s_5^{-1}\}, \\ \lambda & \text{otherwise.} \end{cases}$$

Case 3: $s_4 \neq s_5 \neq s_6$

$$u_i = \begin{cases} s_4 & \text{if } a_i \notin \{s_4^{-1}, s_5\} \\ s_4 s_5 & \text{if } a_i \in \{s_4^{-1}, s_5\} \text{ but } a \notin \{s_5^{-1}, s_6\} \\ s_4 & \text{if } s_4 = s_6^{-1} \text{ and } x_i'^- = s_5^{-1} \\ s_4 s_5 & \text{if } s_4 = s_6^{-1} \text{ and } x_i'^- \neq s_5^{-1}. \end{cases}, \quad v_i = \begin{cases} a_i^{j_i} & \text{if } s_4 \neq s_6^{-1}, \\ \lambda & \text{otherwise.} \end{cases}$$

Properties (1)–(3) follow immediately from the choice. Furthermore, the choice assures that if $u_i = s_4$, then $y_i^- \neq s_5$ and if $u_i = s_4 s_5$, then $y_i^- \neq s_6$. Since q' (and hence q_i) is not a piece, this implies property (4).

Constructing γ. Let γ be the path starting at the identity and labelled by $w = \prod_{i=1}^{\infty} y_i q_{i+1}$ and let ρ be its intersection function. If $\rho(t) \leq t/3$ for all t, then γ is a geodesic by Lemma 2.7. If in addition, ρ is sublinear, then γ is Morse by Lemma 2.10. So next, we show that these two conditions indeed hold.

Let w be a common subword of a relator r and γ. By (4), we have one of the following: $w = u y_i v$, $w = uz$ or $w = zv$, where u is a suffix of q_i, z a subword of y_i and v a prefix of q_{i+1}.

Case 1: $\bar{r} = \bar{r}_0$. We have that $|r_0| < |y_i|$ and hence $w = uz$ (or $w = zv$). If z is empty, then by the choice of q, $|w| \leq |r_0|/3$. If z is not empty, then u (respectively v) is a piece by (4). Since $\bar{r}_0 \neq \bar{r}_i$, the word z is a piece as well and hence $|w| \leq |r_0|/3$.

Case 2: $\bar{r} \neq \bar{r}_0$. Recall that the pair $(y_i, \mathcal{R})$ satisfies the $C'(1/f)$-small-cancellation condition and $|y_i| > 12|q|$. Thus if $w = uy_iv$, then $|y_i| < |r|/f(|r|)$ and hence $|w| < |r|/f(|r|) + 2|q| \leq |r|/3$. On the other hand, if $w = uz$, then u is a piece and thus $|w| = |u| + |z| < |r|/6 + |r|/f(|r|) < |r|/3$. Also, $|w| < |r|/f(|r|) + |q|$. We can prove the same if $w = zv$.

In any case, $|w| \leq |r|/3$, and $|w| \leq |r|/f(|r|) + 2|q|$. The former shows that indeed $\rho(t) \leq t/3$ for all t. Since f is unbounded, the latter shows that ρ is indeed sublinear.

To conclude the proof, it suffices to show that $\rho \not\precsim \rho_i$ for all i. By construction of the sequence $(n_i)_i$, we have that $\rho_i(t) < t/i$ for all $t \geq n_i$. In particular, $\rho_i(|r_i|) < |r_i|/k_i$. Since y_i is a subword of γ, we have that $\rho(|r_i|) \geq |y_i| \geq |r_i|/k_i$. Thus indeed $\rho \not\precsim \rho_i$. ∎

3.3. Examples of groups with non-σ-compact Morse boundary

We reprove Lemma 3.3 as a corollary of Theorem 3.6. Then we show that for at least some viable functions, the class of $C'(1/f)$-groups is not empty. In fact, it contains some of the standard examples of small-cancellation groups.

Corollary 3.12. *Let f be a viable function and let $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ be a $C'(1/f)$-group. Then the Morse boundary $\partial_* G$ is not σ-compact.*

Proof. By Theorem 3.6, it is enough to show that G satisfies the IPSC. Let $(n_k)_k$ be a sequence of integers. We can choose a sequence $(r_k)_k$ of relators such that $|r_k| > \max\{k, n_k, |r_{k-1}|\}$ for all k. Define the function $f' : \mathbb{N} \to \mathbb{R}_+$ via

$$f'(n) = \begin{cases} f(n) & \text{if } n < |r_{12}|, \\ \min\left\{ f(n), \dfrac{k}{2} \right\} & \text{if } |r_k| \leq n < |r_{k+1}| \text{ for some } k \geq 12. \end{cases}$$

Let $k \geq 12$ and let x be a prefix of a relator r_k satisfying $2|r_k|/k > |x| \geq |r_k|/k$. The pair $(x, \mathcal{R})$ satisfies the $C'(1/f')$-small-cancellation condition. Thus G indeed satisfies the IPSC. ∎

Example 3.13. Let $\mathcal{S} = \{x, y\}$. For $i \geq 14$ define

$$r_i = \prod_{j=0}^{2i} xy^{i^2+j}.$$

Note that $|r_i| \geq i^3$. Let $\mathcal{R} = \{r_i\}_{i \geq 14}$ and let $f : \mathbb{N} \to \mathbb{R}_+$ be defined as follows:

$$f(i) = \begin{cases} 6 & \text{if } i < 14, \\ \left\lceil \dfrac{i^3}{2(i+1)^2} \right\rceil & \text{otherwise.} \end{cases}$$

We show below that the group $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ is a $C'(1/f)$-group. As a consequence, its Morse boundary is non-σ-compact by Corollary 3.12 and its existence implies Theorem A (1). Any piece p of a relator $r \in \overline{r_i}$ has the form y^k, y^{-k}, $y^k x y^{k'}$ or $y^{-k} x^{-1} y^{-k'}$ for some $0 \leq k, k' \leq (i+1)^2 - 1$. Hence $|p| < 2(i+1)^2 \leq |r|/f(|r|)$. Thus $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$ is indeed a $C'(1/f)$-group.

4. Strongly contracting Morse boundary

In this section, we construct a class of examples of $C'(1/6)$-groups $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$, where $|\mathcal{R}|$ is infinite and every geodesic Morse ray in $X = \mathrm{Cay}(G, \mathcal{S})$ is strongly contracting. Every geodesic Morse ray being strongly contracting implies that the Morse boundary of G is σ-compact and hence the examples constructed here stand in contrast to the examples of Section 3.

In light of Lemma 2.10, we have to ensure that for these examples, whenever a geodesic has sublinear intersection function, it has to have bounded intersection function. We ensure this by making sure that the overlaps in relators are 'large enough' while not violating the $C'(1/6)$-small-cancellation condition.

More precisely, we start with a certain set of relators and say that they are 'level 1' relators. We then construct the rest of the relators inductively by level. During the construction, we make sure that relators of level $i+1$ are concatenations of subwords of level i relators. Doing this the right way, 'long' subwords of a relator contain a substantial fraction of a lower level relator, which can be used to show that every geodesic ray is either not Morse or strongly contracting.

Definition 4.1. We say that a word $w = s_1 \cdots s_n$ is *non-periodic* if $|\overline{w}| = 2n$. That is, none of the cyclic shifts of w or their inverses are equal to w.

4.1. Construction

Let $N \geq 28$, let L be a multiple of N and let $\mathcal{S}$ be a finite set of formal variables. Let r_1^1, r_2^1 and r_3^1 be distinct non-periodic words over $\mathcal{S}$ of length L such that $\mathcal{R}_1 = \{r_1^1, r_2^1, r_3^1\}$ satisfies the $C'(1/(4N))$-small-cancellation condition. For example, $\mathcal{S} = \{s_1, s_2, \ldots, s_{4N}\}$, and $r_i^1 = s_{Ni+1} s_{Ni+2} \cdots s_{Ni+N}$ for $1 \leq i \leq 3$. We think of $\mathcal{R}_1$ as the set of level 1 relators.

Remark 4.2. It is important that r_1^1, r_2^1 and r_3^1 are words over $\mathcal{S}$ instead of words over $\overline{\mathcal{S}}$. Because of this, they are reduced and cyclically reduced. The same holds for all words constructed in the remainder of this section.

Convention. For the rest of this section, unless stated otherwise, $1 \leq i, i', i'' \leq 3$ and $1 \leq l, l', l'' \leq N$ are integers. Furthermore, we think of i, i and i'' as integers modulo 3 and l, l' and l'' as integers modulo N. So if $i = 3$, then $i + 1$ is equal to 1.

Write $r_i^1 = y_{(i,1)}^1 y_{(i,2)}^1 \cdots y_{(i,N-1)}^1 y_{(i,N)}^1$ for subwords $y_{(i,l)}^1$ of length $|r_i^1|/N$. For $k \geq 1$ iteratively define

$$y_{(i,l)}^{k+1} = \prod_{j=1}^{N} y_{(i,j)}^k y_{(i+1,l)}^k.$$

In other words, if $a = y_{(i,l)}^k$ and $b_j = y_{(i+1,j)}^k$, then

$$y_{(i,l)}^{k+1} = \prod_{j=1}^{N} a b_j.$$

For $k \geq 2$ define

$$r_i^k = \prod_{j=1}^{N} y_{(i,j)}^k, \tag{4.1}$$

and define $\mathcal{R} = \bigcup_{i=1}^{3} \bigcup_{k=1}^{\infty} r_i^k$. We think of relators r_i^k as relators of level k. With this definition we assure the following: While a particular word $y_{(i,l)}^k$ appears several times in the definition of the y^{k+1}, any word $y_{(i,l)}^k y_{(i',l')}^k$ appears at most once in the definition of the y^{k+1}.

This property is a key step in showing that $\mathcal{R}$ satisfies the $C'(1/6)$-small-cancellation condition, while ensuring that if a geodesic has a large common subword with relator r_*^k, then it also has a relatively large common subword with a relator $r_*^{k'}$ for some $k' < k$.

Lemma 4.3. *Let $1 \leq k' < k$. We can write $y_{(i,l)}^k$ as the product*

$$y_{(i,l)}^k = \prod_{j=1}^{m} y_{(i_j,l_j)}^{k'},$$

for some integer $m \geq 2$ such that:

(1) *$i_1 \neq i_2$ and $i_{m-1} \neq i_m$,*

(2) *if $i_l = i_{l+1}$, then $i_{l-1} \neq i_l$ and $i_{l+2} \neq i_{l+1}$.*

Proof. If $k' = k-1$, this follows from the definition of $y_{(i,l)}^k$. Otherwise it follows by induction on k. ∎

Lemma 4.4. *Let $1 \leq k' < k$ we can write r_i^k as the concatenation*

$$r_i^k = \prod_{j=1}^{m} y_{(i_j,l_j)}^{k'},$$

for some integer $m \geq 2$ such that:

(1) $i_1 \neq i_2$ *and* $i_{m-1} \neq i_m$,

(2) *if* $i_l = i_{l+1}$, *then* $i_{l-1} \neq i_l$ *and* $i_{l+2} \neq i_{l+1}$.

Proof. This is an immediate consequence of the definition of r_i^k and Lemma 4.3. ∎

The following lemma explores how a word $y^k_{(i,l)}$ can be a subword of a product of terms $y^k_{(i',l')}$ and is a key ingredient in showing that $\mathcal{R}$ satisfies the $C'(1/6)$-small-cancellation condition.

Lemma 4.5. *Let* $k \geq 1$ *and let* $w = \prod_{j=1}^m y^k_{(i_j,l_j)}$ *be a word. If* $y^k_{(i,l)}$ *is a subword of* w*, then there exists an index* j *such that* $(i_j, l_j) = (i, l)$*. Furthermore, if* w *can be written as* $uy^k_{(i,l)}v$*, then* $u = \prod_{j=1}^{n-1} y^k_{(i_j,l_j)}$*, for some* $1 \leq n \leq m$ *with* $(i_n, l_n) = (i, l)$*.*

Proof. If $a = y^1_{(i,l)}$ is a subword of w, then either $w = uav$, or there exists an index $1 \leq j < m$ such that $y^k_{(i_j,l_j)} y^k_{(i_{j+1},l_{j+1})} = u'pqv'$ for some words u', v', p, q with $pq = a$. It suffices to show that either u' is empty and $(i, l) = (i_j, l_j)$ or v' is empty and $(i, l) = (i_{j+1}, l_{j+1})$. We first show this for $k = 1$ and then by induction on k.

If $k = 1$ and the statement above does not hold, then p and q are both pieces. Hence $L/N = |a| = |p| + |q| < L/(2N)$, a contradiction.

Assume the statement holds for $k - 1$. Define $b = y^k_{(i_j,l_j)}$ and $c = y^k_{(i_{j+1},l_{j+1})}$. Recalling their definitions, we can write the words a, b and c as a product of words of the form $y^{k-1}_{(i',l')}$. The term $y^{k-1}_{(i,l)}$ appears N times in the product of a, and unless $(i', l') = (i, l)$ appears at most once in the product of $y^k_{(i',l')}$. Applying the statement for $k - 1$ and bc, we get that b or c contains $y^{k-1}_{(i,l)}$ multiple times, implying that $(i_j, l_j) = (i, l)$ or $(i_{j+1}, l_{j+1}) = (i, l)$. Moreover, the terms $y^{k-1}_{(i+1,1)}$ and $y^{k-1}_{(i+1,N)}$ appear exactly once (and hence at a unique spot) when writing $y^k_{(i,l)}$ as a product, so we actually have either $(i_j, l_j) = (i, l)$ and u' is the empty word or $(i_{j+1}, l_{j+1}) = (i, l)$ and v' is the empty word, which shows that the statement holds for k. ∎

Instead of showing that $\mathcal{R}$ satisfies the $C'(1/6)$-small-cancellation condition, we show a slightly stronger result, which we need for the next section.

Lemma 4.6. *The set of relators* $\mathcal{R}$ = *satisfies the* $C'(1/7)$*-small-cancellation condition.*

Proof. It suffices to show that the common prefix p of a pair of distinct relators $r, r' \in \overline{\mathcal{R}}$ has length less than $|r|/7$ and $|r'|/7$. Since all the r_i^k are words over $\mathcal{S}$, we may assume that r and r' are cyclic shifts of relators r_i^k and $r_{i'}^{k'}$ for some $k' \geq k$.

Case 1: $k = k'$. Let us assume by contradiction that $|p| \geq |r|/7$. Recall that $r_i^k = y^k_{(i,1)} y^k_{(i,2)} \cdots y^k_{(i,N)}$. Since $N \geq 28$, there exists an integer l such that $y^k_{(i,l)}$ is a subword of $|p|$. Applying Lemma 4.5, we get a contradiction. Hence $|p| < |r|/7 = |r'|/7$.

Case 2: $k' > k$. Assume by contradiction that $|p| \geq |r|/7$. Since $N \geq 28$, there exists an integer l such that $y^k_{(i,l)} y^k_{(i,l+1)} y^k_{(i,l+2)}$ is a subword of p. Write $r^{k'}_{i'}$ as in Lemma 4.4. By Lemma 4.5, there exists an integer l' such that $i_{l'} = i_{l'+1} = i_{l'+2}$, a contradiction to Lemma 4.4 (2). Hence $|p| < |r|/7 < |r'|/7$. ∎

Proposition 4.7. *Every Morse geodesic ray in $X = \mathrm{Cay}(G, \mathcal{S})$ is strongly contracting.*

Proof. Assume that γ is a Morse geodesic ray which is not strongly contracting. In light of Lemma 2.10, its intersection function

$$\rho(x) = \max_{|r| \leq x} \{|w| \mid w \text{ is a subword of } r \text{ and } \gamma\},$$

is sublinear but unbounded. Let $k \geq 1$. Since ρ is unbounded, there exists a relator $r \in \overline{\mathcal{R}}$ and a common subword w of r and γ of length at least $|r^k_1|$. In light of Lemma 4.4, there exist (i, l) such that $y^k_{(i,l)}$ is a subword of w or w^{-1}, implying that either r^k_i or $(r^k_i)^{-1}$ have a common subword with γ of length at least $|r^k_i|/N$. Consequently, $\rho(|r^k_i|) \geq |r^k_i|/N$. Since this holds for all k, ρ cannot be sublinear, a contradiction. ∎

5. Sigma-compact but not all Morse rays are strongly contracting

In this section we modify the group constructed in the previous section. The modified group is a $C'(1/6)$-group which has σ-compact Morse boundary but not all Morse rays in its Cayley graph are strongly contracting. This rounds out the set of examples we give and shows that there are $C'(1/6)$-groups whose Morse boundary is σ-compact but not all Morse rays are strongly contracting.

We use the notation from the previous section. Let $\mathcal{S}' = \mathcal{S} \cup \{a\}$ for some formal variable $a \notin \mathcal{S}$. For $1 \leq i \leq 2$ define $\tilde{r}^k_i = r^k_i$ and for $i = 3$ define

$$\tilde{r}^k_3 = \begin{cases} r^k_3 a^k & \text{if } k \leq |r^k_3|/42, \\ r^k_3 & \text{otherwise.} \end{cases} \tag{5.1}$$

Define $\mathcal{R}' = \bigcup_{k \geq 1} \bigcup_{i=1}^{3} \{\tilde{r}^k_i\}$.

Lemma 5.1. *The set $\mathcal{R}'$ satisfies the $C'(1/6)$-small-cancellation condition.*

Proof. Let p be a piece of a relator $r \in \overline{\mathcal{R}}'$ which is a cyclic shift of a relator $\tilde{r}^k_i$. Then either $p = qa^j$ or $p = a^j q$ for some $0 \leq j \leq k$ and subword q which, in $\mathcal{R}$, is a piece of r^k_i. By (5.1), $j \leq |r|/42$ and since $\mathcal{R}$ satisfies the $C'(1/7)$-small-cancellation condition, $|q| < |r^k_i|/7$. Hence $|p| < |r|/6$. ∎

Proposition 5.2. *The group $G = \langle \mathcal{S}' \mid \mathcal{R}' \rangle$ has σ-compact Morse boundary but not every Morse ray in $X = \mathrm{Cay}(G, \mathcal{S}')$ is strongly contracting.*

Proof. We first show that there exists a geodesic Morse ray in X which is not strongly contracting. Define the sublinear function $\rho_0(x) = \log_N(x) + 2$. Let γ be the ray in X starting at the identity, where every edge on γ is labelled by a. Let w be a common subword of a relator $r \in \overline{\widetilde{r}_i^k}$. By (5.1), we have that $|w| \leq |r|/42$. Hence γ is a geodesic by Lemma 2.7. Further, if $i \neq 3$, then w is the empty word. Otherwise $|w| \leq k$. By construction, $|r_i^k| = (2N)^{k-1}|r_i^1|$. Thus, $|w| \leq \rho_0(|r|)$. This shows that the intersection function ρ of γ satisfies $\rho \leq \rho_0$, implying that γ is Morse by Lemma 2.10. Further, the intersection function ρ is unbounded, and hence, γ is not strongly contracting by Lemma 2.10.

Next we show that $\partial_* X$ is σ-compact. Morse precisely, for every integer $j \geq 1$ define $\rho_j(x) = \rho_0(x) + 2j$. We show that $\rho_j(x)$ is a contraction exhaustion of $\partial_* X$. Let γ be a Morse geodesic ray and let ρ be its intersection function. The proof of Proposition 4.7 shows that there exists some bound D such that every subword of γ which is also a subword of a relator $r \in \mathcal{R}$ has length at most D. Thus for any common subword w of γ and a relator $r \in \overline{\widetilde{r}_i^k}$, we have that $|w| \leq 2D + k \leq \rho_D(|r|)$. Hence $\rho \leq \rho_D$, which concludes the proof. ∎

Acknowledgements. I want to thank Matthew Cordes and my supervisor Alessandro Sisto for helpful discussions.

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Received 30 August 2023.

Stefanie Zbinden
Department of Mathematics, Heriot-Watt University, Colin MacLaurin Building, First Gait, Edinburgh EH14 4AS, UK; sz2020@hw.ac.uk