Groups Geom. Dyn. 20 (2026), 1099–1145
DOI 10.4171/GGD/838

The finiteness problem for groups generated by reset automata

Feyishayo Olukoya

Abstract. We provide sufficient conditions for when a group generated by a strongly synchronizing automaton (generalizations of reset automata) has infinite order.

1. Introduction

This article studies the finiteness problem in automata groups for a specialized class of automata which includes all *reset automata*. In particular, we give sufficient conditions for when these groups are infinite.

The automata we are interested in are *synchronizing* in the sense considered in [3]. We provide some informal definitions to make our results intelligible; formal definitions will be given in Section 2.

In this article an *automaton* (we will also interchangeably use the terms *transducer* and *Mealy-automaton*) is a machine with finitely many states, which on reading an input symbol from an alphabet of size n, writes a string from the same alphabet, possibly the empty string, and changes state awaiting the next input. In the case that each state of the transducer changes an input symbol to another symbol (and not a string), we say that the transducer is *synchronous* and thus each state of the transducers induces a transformation of the symbol set. In the case that this transformation is invertible, we say the transducer is *invertible*. It is a result in [9] that the inverse is also representable by a transducer. This article focuses only on synchronous transducers, as such we use the term transducers for this class of transducers.

We note that each state of a transducer induces a continuous map on the Cantor space $X^{\mathbb{N}}$ where X is the alphabet. Thus, for a transducer A we can consider the monoid of continuous maps on $X^{\mathbb{N}}$ generated by states of A. We call this monoid the *automaton semigroup generated by* A. In the case that A is an invertible transducer, the *automaton group generated by* A is the subgroup of the homeomorphisms of $X^{\mathbb{N}}$ generated by the states of A.

A transducer is said to be *synchronizing at level* k (see [3]) if there is a natural number $k \in \mathbb{N}$ such that for all strings of length k in the input alphabet, the state of the transducer

---

Mathematics Subject Classification 2020: 20F10 (primary); 20E08 (secondary).
Keywords: automaton groups, reset automata, synchronization, decision problems.

reached after processing any such string is independent of the starting state and depends only on the string processed. We should point out that there is a weaker notion of synchronization for automaton which occurs, for instance, in the Černý conjecture [20] and in the road colouring conjecture proved by Trahtman [19]. We are only concerned with the definition of synchronization given above. We call a transducer *synchronizing* if it is synchronizing at some level. If the transducer is invertible, with synchronizing inverse, then we say that the transducer is *bi-synchronizing*. For a synchronizing transducer A which is synchronizing at level k, we shall denote by $\mathrm{Core}(A)$ the sub-transducer consisting of those states reached after reading words of length k from any state of A.

We briefly define the automata we are interested in, with more detailed definitions provided in Section 2. The monoid $\mathcal{P}_n$ consists of core, bi-synchronizing transducers with alphabet set $\{0, 1, \dots, n-1\}$. The article [4] demonstrates that elements of $\mathcal{P}_n$ induce homeomorphisms of the full two-sided shift; therefore, they are invertible, and, although the inverse may not be contained in $\mathcal{P}_n$, it can be represented by a synchronizing (possibly non-synchronous) transducer. Hence, the term bi-synchronizing makes sense in $\mathcal{P}_n$. The monoid $\mathcal{P}_n$ is itself contained in the monoid $\widetilde{\mathcal{P}}_n$ which consists of core, synchronizing transducers. Let $\widetilde{\mathcal{H}}_n$ be the monoid consisting of elements of $\widetilde{\mathcal{P}}_n$ whose states induce permutations of the symbol set. The subgroup of $\widetilde{\mathcal{H}}_n$ consisting of bi-synchronizing elements we shall denote by $\mathcal{H}_n$. In [4], it is shown that $\mathcal{H}_n$ is isomorphic to the group of automorphisms of the one-sided shift on n letters.

This article is mainly concerned with groups or semigroups generated by automata in the monoid $\widetilde{\mathcal{H}}_n$, although some results also apply to $\mathcal{P}_n$ and $\widetilde{\mathcal{P}}_n$.

Our results are as follows.

Theorem 1.1. *Let $A \in \widetilde{\mathcal{H}}_n$. Then the automaton group generated by A is either finite or contains a free subsemigroup of rank at least* 2*.*

This has the following immediate consequence.

Corollary 1.2. *Let $A \in \widetilde{\mathcal{H}}_n$. Then the automaton group generated by A is either finite or has exponential growth.*

Corollary 1.2 is in the spirit of [11] which studies *bireversible automata* (a class of automata disjoint from the class of synchronizing automata).

Recall that the automaton semigroup generated by an element $A \in \widetilde{\mathcal{P}}_n$ is the subsemigroup of the monoid of continuous maps on the Cantor space $\{0, 1, \dots, n-1\}^{\mathbb{N}}$ generated by the states of A. On the other hand, as A is an element of the monoid $\widetilde{\mathcal{P}}_n$, we can ask if it is periodic in $\widetilde{\mathcal{P}}_n$, that is, if A generates a finite subsemigroup of $\widetilde{\mathcal{P}}_n$. On the one hand, we have a semigroup of continuous maps on $\{0, 1, \dots, n-1\}^n$ and on the other, a monogenic subsemigroup whose elements are the powers of A in $\widetilde{\mathcal{P}}_n$. The following question is therefore natural: How does the automaton semigroup generated by a transducer A in $\widetilde{\mathcal{P}}_n$ relate to the subsemigroup of $\widetilde{\mathcal{P}}_n$ generated by the transducer A?

The proposition below addresses this question.

Corollary 1.3. *Let $A \in \widetilde{\mathcal{P}}_n$ be synchronizing at level k. Then the automaton semigroup generated by A is finite if and only if the subsemigroup of $\widetilde{\mathcal{P}}_n$ generated by A is finite.*

For A an automaton, the *dual automaton* is the automaton with state set the alphabet set of A, alphabet set the state set of A, and, which has the natural dual transition and output functions (see Subsection 4.1 for definitions and details). We use the notation $A_m^{\vee}$ for the dual automaton of the mth power of A; we write $A^{\vee}$ for $A_1^{\vee}$. It is well known [1, 15] that the automaton semigroup generated by an automaton is related to the automaton semigroup generated by its dual automaton. In particular, one is finite if and only if the other is as well. We develop on this result in the class of synchronizing automata by structurally characterizing the dual automaton of an element of $\widetilde{\mathcal{P}}_n$ which generates a finite automaton semigroup.

Proposition 1.4. *Let A be an element of $\widetilde{\mathcal{P}}_n$ and suppose A is synchronizing at level k. Then the semigroup $\langle A \rangle$ generated by A is finite if and only if there is some $m \in \mathbb{N}$ such that $A_m^{\vee}$ is ω-equivalent to a transducer with r components $D_i 1 \leq i \leq r$. Moreover, for each component D_i there is a fixed pair of words $w_{i,1}, w_{i,2}$ (in the states of A) associated to D_i such that whenever we read any input from a state in the D_i, the output is of the form $w_{i,1}w_{i,2}^{l}v$ for $l \in \mathbb{N}$ and v a prefix of $w_{i,2}$ or has the form u for some prefix u of $w_{i,1}$. Moreover, the output depends only on the state in the component D_i from which the input is processed.*

There is also a connection between the growth rate of an automaton and the growth of the number of states of powers of the automaton. In particular, by exploiting this connection, Silva and Steinberg in [18] prove that the automaton group generated by a Cayley machine of an abelian group has exponential growth. This class of groups is interesting as they contain all lamplighter groups (we refer to Subsection 5.1 for details of the Cayley machine construction). We extend results of Silva and Steinberg in the following theorem.

Theorem 1.5. *Let G be a finite group and $\mathcal{C}(G)$ be the Cayley machine of G. Then for $n \in \mathbb{N}$, $\mathcal{C}(G)^n$ is connected, has $|G|^n$ states and contains no-pair of states which induce the same homeomorphism on $G^{\mathbb{N}}$. In particular, the automaton semigroup generated $\mathcal{C}(G)$ is free.*

We say a few words about the proof.

For an automaton $A \in \widetilde{\mathcal{P}}_n$ and for each natural number r bigger than the synchronizing level of A, we associate to A a finite graph $G_r(A)$ the *graph of bad pairs* (see Subsection 4.3). This graph encapsulates certain dynamics of the dual automaton of A when A generates an infinite subsemigroup of $\widetilde{\mathcal{P}}_n$. By exploiting this connection, we prove our results. Indeed, when A generates a finite subsemigroup of $\widetilde{\mathcal{P}}_n$, then for large enough $r \in \mathbb{N}$, $G_r(A)$ is empty. The following results tie combinatorial properties of the graph of bad pairs to the dynamics of the action of A on $X_n^{\mathbb{Z}}$ as a shift commuting homeomorphism.

Theorem 1.6. *Let $A \in \widetilde{\mathcal{H}}_n$ be synchronizing at level k, and let $r \geq k$. If the graph $G_r(A)$ of bad pairs has a circuit, then there is an eventually periodic word in $X_n^{\mathbb{Z}}$ on an infinite orbit under the action of A.*

Notice that if $A \in \mathcal{H}_n$ is such that $G_r(A)$ has a circuit, then A, in its action on $X_n^{\mathbb{Z}}$, has infinite order, and so the subgroup of $\mathcal{H}_n$ generated by A is infinite. In this case, we say that A is an element of infinite order. A corollary of the above result, via Corollary 1.3, is the following.

Corollary 1.7. *Let $A \in \widetilde{\mathcal{H}}_n$ be synchronizing at level k, and let $r \geq k$. If the graph $G_r(A)$ of bad pairs has a circuit, then the automaton group generated by A is infinite.*

It turns out that if the graph $G_r(A)$ has a circuit for some r, then A contains a free semigroup.

Proposition 1.8. *Let $A \in \widetilde{\mathcal{H}}_n$ be synchronizing at level k, and let $r \geq k$. If the graph $G_r(A)$ of bad pairs has a circuit, then A contains a free subsemigroup of rank at least* 2.

The proof of Theorem 1.1 now follows by studying the graphs $G_r(A)$ for sufficiently large $r \in \mathbb{N}$.

Silva and Steinberg in [18] study reset automata which fit in this context as level 1 synchronizing transducers. They show that the automaton group generated by an invertible level 1 synchronizing transducer is infinite if and only if it contains an element of infinite order, and in this case the group is locally finite-by-cyclic and so amenable. Thus, if the order problem is solvable in groups generated by reset automata, then the finiteness problem is also solvable in groups generated by reset automata (in fact, this is and if and only if by the results of [18]). While an early draft of this paper has been under review, the two papers [2, 8], demonstrating that the order problem in groups generated by automata is undecidable in general, were uploaded to the arXiv. However, the question remains open for reset automata. The results above address this question by providing sufficient conditions for when the group generated by a reset automaton is finite.

In [18], by making further assumptions the authors of that paper are also able to show that in the case where this group is infinite then it has exponential growth. Using Chou's classification of elementary amenable groups [6], a result of Rosset [16] and the result of Silva and Steinberg that groups generated by reset automata are locally finite-by-cyclic, it turns out that all groups generated reset automata are actually elementary amenable. Moreover, it is a result of Chou [6] that all finitely generated elementary amenable groups are either virtually nilpotent (and so of polynomial growth [10]) or contain a free semigroup on two generators. From this one may deduce that all finitely generated locally finite-by-infinite cyclic groups contain a free semigroup on two generators. Thus, we re-derive the result of Chou in the context of groups generated by reset automata. In particular, the class of automata groups generated by reset automata does not furnish examples of infinite Burnside groups or groups of intermediate growth: These groups are finite or they have exponential growth.

In the final section of the article, we give a construction that embeds direct sums of $\widetilde{\mathcal{P}}_n$ into $\widetilde{\mathcal{P}}_m$ for m large enough. This construction is useful for the following reasons. Firstly, by restricting to $\mathcal{H}_n$ and using a result in [5], we show that $\mathcal{H}_n$, for $n \geq 6$, contains the direct product of the free group on two generators. Secondly, we note a conjecture of Picantin [13] which implies that for $A \in \mathcal{H}_n$ an element of finite order, the smallest $m \in \mathbb{N}$ such that $A_m^{\vee}$ satisfies Proposition 1.4 is $m = |A| - 1$. The techniques of Section 7 enable us to construct examples of elements of $\mathcal{H}_n$ which attain this limit.

1.1. Outline of paper

In Section 2, we give a formal definition of the semigroup $\widetilde{\mathcal{P}}_n$ and present some of the basic properties of this group that we will need later on. In Section 3, we introduce some various useful properties of $\widetilde{\mathcal{P}}_n$ which, when restricted to $\widetilde{\mathcal{H}}_n$, begin to shed some light on when two elements commute or are conjugate to each other. In Section 4, we develop the techniques required to prove Theorem 1.1. In Section 5, we prove Theorem 1.1. In Section 6, we prove Theorem 1.6 and establish connections between the order of elements of $\mathcal{H}_n$ and properties of the graph of bad pairs of those elements. In Section 7, we show that direct-sums of copies of $\widetilde{\mathcal{H}}_n$ can be embedded in $\widetilde{\mathcal{H}}_m$ for m large enough.

2. Preliminaries

2.1. Transducers

Throughout the paper fix $X_n := \{0, \dots, n-1\}$, for $2 \leq n$ a natural number. For an arbitrary finite set X of symbols, we shall let X^* be the set of all finite strings, including the empty string (which we shall always denote by ε), and X^+ to be the set of all finite strings excluding the empty string. We call a word $\Gamma \in X_n^+$ which cannot be written as $\Gamma = (\gamma)^r$ for a strictly smaller word $\gamma \in X_n^+$ and $0 < r \in \mathbb{N}$ a *prime word*. For a natural number $k \in \mathbb{N}$ we shall let X^k be the set of all strings of length precisely k; $X^{\mathbb{N}}$ and $X^{\mathbb{Z}}$ shall denote the set of infinite and bi-infinite strings, respectively. We shall represent a point $x \in X^{\mathbb{N}}$ as a sequence $(x_i)_{i \in \mathbb{N}}$ which we will normally write as $x := x_0 x_1 x_2 \cdots$, likewise we shall represent a point $x \in X^{\mathbb{Z}}$ as a sequence $(x_i)_{i \in \mathbb{Z}}$, and we will normally write this as $x := \cdots x_{-1} x_0 x_1 \cdots$. Sometimes it shall be convenient to consider an element $x \in X_n^{\mathbb{Z}}$ as being composed of words in X_n^k for some natural number $k > 0$, and in this case we shall write

$$x := \cdots \Gamma_{-1} \dot{\Gamma}_0 \Gamma_1 \cdots ,$$

where Γ_i, $i \in \mathbb{Z}$ are all in X_n^k. The dot above Γ_0 is used to communicate that $x_0 \cdots x_{k-1} = \Gamma_0$. Then $x_k \cdots x_{2k-1} = \Gamma_1$ and so on. Taking the product topology on $X^{\mathbb{N}}$ and $X^{\mathbb{Z}}$ makes both of these spaces homeomorphic to Cantor space.

For a string $\Gamma \in X^*$, we shall use $|\Gamma|$ to denote the length (or size) of Γ; the empty string has length zero. We shall also use $|X|$ to denote the cardinality of the set X; if $i \in \mathbb{Z}$ then $|i|$ shall denote the absolute value of i. We shall let the context determine which meaning of $|\cdot|$ is being taken.

Definition 2.1. In our context, a *transducer* A is a tuple $A = \langle X_n, Q, \pi, \lambda \rangle$, where

(i) X_n is both the input and output alphabet.

(ii) Q is the set of states of A.

(iii) π is the *transition* function, and is a map:

$$\pi : X_n \sqcup \{\varepsilon\} \times Q \to Q.$$

(iv) λ is the *output* or *rewrite* function, and is a map:

$$\lambda : X_n \times Q \to X_n.$$

We take the convention that $\pi(\varepsilon, q) = q$ for any $q \in Q$ as this will allow single-state transducers to be synchronizing at level 0. Consequently, we also make utilize the convention that $\lambda(\varepsilon, q) = \varepsilon$. We note that usually λ is a map from which maps letters to strings; however, as we only consider synchronous (“letter-to-letter”) transducers, the restricted definition given suffices for our purposes. If $|Q| < \infty$, then we say the transducer A is finite. If the map $\lambda(\cdot, q) : X_n \to X_n$ is the identity map on X_n, then we say that q acts *locally as the identity*; if the map $\lambda(\cdot, q) : X_n \to X_n$ is a permutation of X_n, then we say that q acts *locally as a permutation.* If it is clear from the context then we may sometimes omit the prefix “locally.”

If we specify a state $q \in Q$ from which we start processing inputs, then we say A is *initialized at* q and shall denote this A_q. The transducer A_q is then called an initial transducer.

We can extend the domain of π and λ to $X_n^* \times Q$ using the rules below and induction:

$$\pi(\Gamma x, q) = \pi(x, \pi(\Gamma, q)) \tag{1}$$

$$\lambda(\Gamma x, q) = \lambda(\Gamma, q)\lambda(x, \pi(\Gamma, q)), \tag{2}$$

where $\gamma \in X_n^*$, $x \in X_n$ and $q \in Q$. Given a word in $\Gamma \in X_n^*$ and $q, p \in Q$ such that $\pi(\Gamma, q) = p$, then we shall say that we *read* Γ *from state* q *into* p, we shall sometimes supplement this by adding, *and the output is* Δ if $\Delta = \lambda(\Gamma, q)$.

Note that each state $q \in Q$ induces a continuous map from Cantor space $X_n^{\mathbb{N}}$ to itself. If this map is a homeomorphism, then we say that q is a *homeomorphism state*. Two states q_1 and q_2 are then said to be ω-equivalent if they induce the same continuous map. (This is can be checked in finite time.) A transducer, therefore, is called *minimal* if no two states are ω-equivalent. Two minimal transducers, $A = \langle X_n, Q_A, \pi_A, \lambda_A \rangle$ and $B = \langle X_n, Q_B, \pi_B, \lambda_B \rangle$, are said to be ω-equivalent if there is a bijection $f : Q_A \to Q_B$ such that q and $(q)f$ induce the same continuous map for $q \in Q_A$. In the case where A

and B are ω-equivalent, then we write $A =_\omega B$; otherwise, we write $A \neq_\omega B$. We note that all transducers have a unique (up-to ω-equivalence) minimal representative.

Given two transducers $A = \langle X_n, Q_A, \pi_A, \lambda_A \rangle$ and $B = \langle X_n, Q_B, \pi_B, \lambda_B \rangle$, the product $A * B$ shall be defined in the usual way. The set of states of $A * B$ will be $Q_A \times Q_B$, and the transition and rewrite functions, π_{A*B} and λ_{A*B} of $A * B$, are defined by the rules:

$$\pi_{A*B}(x, (p, q)) = (\pi_A(x, p), \pi_B(\lambda_A(x, p), q)) \tag{3}$$

$$\lambda_{A*B}(x, (p, q)) = \lambda_B(\lambda_A(x, p), q), \tag{4}$$

where $x \in X_n \sqcup \varepsilon$, $p \in Q_A$ and $q \in Q_B$.

A transducer $A = \langle X_n, Q_A, \pi_A, \lambda_A \rangle$ is said to be *invertible as a transducer* (or *invertible* when there is no ambiguity) if there is a transducer $B = \langle X_n, Q_B, \pi_B, \lambda_B \rangle$ and a bijection $f : Q_A \to Q_B$ such that every pair of states $(p, (q)f) \in Q_A \times Q_B$ of the transducer $A * B$ acts locally as the identity. The transducer B is called the *inverse transducer* of A (or *inverse* of A when there is no ambiguity). We note that A being invertible is equivalent to saying that every state of A is a homeomorphism state or alternatively that A possesses a homeomorphism state. In particular, if A is invertible as a transducer, then the transducer A^{-1} with state set $Q_{A^{-1}} = \{p^{-1} \mid p \in Q_A\}$ and transitions and outputs satisfying the following rule

$$\pi_{A^{-1}}(x, q^{-1}) = p^{-1} \quad \text{and} \quad \lambda_{A^{-1}}(x, q^{-1}) = y \quad \begin{array}{l} \text{if and only if } \pi_A(y, q) = p \\ \text{and } \lambda_A(y, q) = x \end{array}$$

for any $x \in X_n$ is (ω-equivalent to) the inverse of A. Note that if A is minimal and invertible, then the inverse transducer of A is also minimal.

As usual $A^i = A_1 * A_2 * \cdots * A_i$, where $A_j = A$ $1 \leq j \leq i$ and $i \in \mathbb{N}$, and $A^{-i} = (A^{-1})^i$. If $A := \langle X_n, Q_A, \lambda_A, \pi_A \rangle$, then we set $A^i = \langle X_n, Q_A^i, \lambda_{Ai}, \pi_{Ai} \rangle$.

If $A = \langle X_n, Q_A, \pi_A, \lambda_A \rangle$ is a transducer, then, as each state q of A induces a continuous function of $X_n^{\mathbb{Z}}$, we may consider the subsemigroup (or group in the case that A is invertible) of the endomorphisms of $X_n^{\mathbb{Z}}$ generated by the set $\{A_q \mid q \in Q_A\}$. We shall refer to this semigroup or group as the *automaton semigroup or automaton group generated by* A. We also consider the monogenic semigroup $\langle A \rangle = \{A^i \mid i \in \mathbb{N}\}$ or cyclic group $\langle A \rangle = \{A^i \mid i \in \mathbb{Z}\}$; we shall call these *the semigroup or group generated by* A. Where there is potential ambiguity we make it explicit that $\langle A \rangle$ refers either to the semigroup or to the group generated by A.

Definition 2.2. Given a non-negative integer k and an automaton $A = \langle X_n, Q, \pi, \lambda \rangle$, we say that A *is synchronizing at level* k if there is a map $\mathfrak{s} : X_n^k \to Q$, so that for all $q \in Q$ and any word $\Gamma \in X_I^k$ we have $\mathfrak{s}(\Gamma) = \pi(\Gamma, q)$. That is, the location in the automaton is determined by the last k letters read. We call $\mathfrak{s}$ the synchronizing map for A, the image of the map $\mathfrak{s}$ the *core of* A, and for a given $\Gamma \in X_n^k$, we call $\mathfrak{s}(\Gamma)$ the *state of* A *forced by* Γ. If A is invertible, and A^{-1} is synchronizing at some level $0 \leq l \in \mathbb{N}$, then we say that A *is bi-synchronizing at level* $\max(k, m)$. If A is synchronizing but not bi-synchronizing, then we shall say A is *one-way synchronizing*.

Remark 2.3. We have the following observations about the core:

(i) It is an easy observation that for a synchronizing transducer the core of A is a synchronizing transducer in its own right. We shall denote this transducer by $\mathrm{Core}(A)$, and if $A = \mathrm{Core}(A)$ then we say that *A is core*.

(ii) For a synchronizing transducer A, $\mathrm{Core}(A)$ induces a continuous map from $X_n^{\mathbb{Z}}$ to itself. This follows since if $\mathrm{Core}(A)$ is synchronizing at level k with synchronizing map $\mathfrak{s}$, and given a bi-infinite string $(x_i)_{i\in\mathbb{Z}}$, then $\mathrm{Core}(A)(x_i) = \pi(x_i, \mathfrak{s}(x_{i-k} \cdots x_{i-1}))$. That is we look at the preceding k symbols to determine from which state the subsequent symbol is to be processed. Let f_A be the map induced on $X_n^{\mathbb{Z}}$ induced by $\mathrm{Core}(A)$, then the map f_A preserves indices and so is a well-defined map on $X_n^{\mathbb{Z}}$.

Before defining the monoid $\widetilde{\mathcal{P}}_n$, it is necessary to establish the following standard result.

Claim 2.4. Let $A = \langle X_n, Q_A, \pi_A, \lambda_A\rangle$ and $B = \langle X_n, Q_B, \pi_B, \lambda_B\rangle$ be synchronizing transducers. If A is synchronizing at level j and B is synchronizing at level k, then $A * B$ is synchronizing at level $j + k$.

Proof. We show that any word of length $j + k$ is synchronizing for $A * B$, where we recall the set of states of $A * B$ is the product set $Q_A \times Q_B$ (of course, not all of these states are in the core of $A * B$).

Let Γ be a word of length j and Δ a word of length k, both $\Gamma, \Delta \in X_n^*$. Suppose that Γ forces us into a state q_A of A. Let $\overline{\Gamma}$ be the length j prefix of $\Delta\Gamma$, and let $\overline{\Delta}$ be the complementary length k suffix. Let (p_A, p_B) any pair in $Q_A \times Q_B$. Then we have that $\pi_A(\Delta\Gamma, p_A) = q_A$. Let $\overline{q}_A$ be the state which we are in after processing $\overline{\Gamma}$. Let q_B be the state we are forced to after reading $\lambda_A(\overline{\Delta}, \overline{q}_A)$ from any state in B. Then $\lambda_A(\Delta\Gamma, p_A) = \lambda_A(\overline{\Gamma}, p_A)\lambda_A(\overline{\Delta}, \overline{q}_A)$. Therefore, $\pi_B(\lambda_A(\overline{\Gamma}, p_A)\lambda_A(\overline{\Delta}, \overline{q}_A), p_B) = q_B$, thus reading $\Delta\Gamma$ from any state pair of $Q_A \times Q_B$, the active state becomes (q_A, q_B). ■

Remark 2.5. From the above it follows that the set $\widetilde{\mathcal{P}}_n$ of core, synchronizing transducers forms a monoid [4].

With this in place, given $1 \leq n \in \mathbb{N}$, we can describe the monoid $\widetilde{\mathcal{P}}_n$ as the set of those continuous functions on $\{0, 1, \ldots, n-1\}^{\mathbb{Z}}$ which can be represented by finite, synchronizing, core transducers. (We mention core since we are restricting our attention only to those states in the core. The product is the automaton product, where after taking this product, one removes non-core states as they are irrelevant to the action. This is always possible by the above claim.) The submonoid $\mathcal{P}_n$ is the subset of $\widetilde{\mathcal{P}}_n$ consisting of those elements which induce homeomorphisms of $X_n^{\mathbb{Z}}$. The group $\mathcal{H}_n$ is the subset of $\mathcal{P}_n$ consisting of those transducers H for which there is a state q of H such that the initial transducer H_q is a homeomorphism of $X_n^{\mathbb{N}}$. Notice that all the states of H are homeomorphism states. It is a result in the forthcoming paper [4] that $\mathcal{H}_n$ is isomorphic to the group of automorphisms of the one-sided shift on n letters. Finally, define $\widetilde{\mathcal{H}}_n$ to be those elements of $\widetilde{\mathcal{P}}_n$ which have a homeomorphism state and which are synchronizing but not bi-synchronizing.

Given two elements $A, B \in \widetilde{\mathcal{P}}_n$, then we shall denote the minimal transducer representing the core of the product of A and B by $\min(\mathrm{Core}(A * B))$. Since the operations of minimizing and reducing to the core commute with each other, the order in which we perform these operations is irrelevant.

Most of our results shall be for $\mathcal{H}_n$ and $\widetilde{\mathcal{H}}_n$, though some of our results also apply to $\widetilde{\mathcal{P}}_n$ and $\mathcal{P}_n$. We give below a table listing these various groups and monoid together with their defining properties.

$\widetilde{\mathcal{P}}_n$	Core, synchronizing transducers
$\mathcal{P}_n$	Subset of $\widetilde{\mathcal{P}}_n$ which induce homeomorphisms of $X_n^{\mathbb{Z}}$
$\widetilde{\mathcal{H}}_n$	Subset of $\widetilde{\mathcal{P}}_n$ consisting of elements with a homeomorphism state
$\mathcal{H}_n$	Subset of $\mathcal{P}_n$ containing elements with a homeomorphism state.

The following should aid the reader in remembering which groups/monoids are denoted by which symbols: $\mathcal{H}_n$ and $\widetilde{\mathcal{H}}_n$ are those elements of $\mathcal{P}_n$ and $\widetilde{\mathcal{P}}_n$, respectively, with a homeomorphism state, thus $\mathcal{H}$ represents homeomorphism; a tilde above a symbol means that the corresponding set contains elements that do not induce homeomorphisms of $X_n^{\mathbb{Z}}$.

3. Properties of $\widetilde{\mathcal{P}}_n$

Let $A = \langle Q, X_n, \pi, \lambda \rangle$ be an element of $\widetilde{\mathcal{H}}_n$. If A represents an element of $\mathcal{H}_n$ there is a constant k so that both A and A^{-1} are synchronizing at level k. Note also that all words Γ of length at least k are synchronizing words for A and for A^{-1}.

We make the following claims about A which will be useful later on. The first two shall apply to all elements $A \in \widetilde{\mathcal{P}}_n$.

Claim 3.1. Let A and B be elements of $\widetilde{\mathcal{P}}_n$, and let $m \in \mathbb{N}\backslash\{0\}$ be minimal such that both A and B are synchronizing at level m. Then if $A \neq_\omega B$, there is a word Γ, $|\Gamma| = k \geq m$, and states p and q of A and B, respectively, such that:

(i) p is the state in A forced by Γ and q is the state in B forced by Γ.

(ii) p and q are not ω-equivalent.

Proof. Since $A \neq_\omega B$ they induce different homeomorphisms of $X_n^{\mathbb{Z}}$, and so there is a bi-infinite word $w = \cdots x_{-2}x_{-1}x_0x_1 \cdots$ which they process differently.

Let $w_1 = \cdots y_{-2}y_{-1}y_0y_1y_2 \cdots$ and $w_2 = \cdots z_{-2}z_{-1}z_0z_1z_2 \cdots$ be the outputs from A and B, respectively. Let $k \in \mathbb{N}\backslash\{0\}$ be such that A and B are synchronizing at level k. Note that $k \geq m$. Let $l \in \mathbb{N}$ be minimal such that $y_l \neq z_l$ or $y_{-l} \neq z_{-l}$. Then one of the words $x_{l-k} \cdots x_{l-2}x_{l-1}$ or $x_{-l-k} \cdots x_{-l-2}x_{-l-1}$ satisfies the premise of the claim. ■

Lemma 3.2. *Let* $A \in \widetilde{\mathcal{P}}_n$ *be such that* $\min \mathrm{Core}(A^i) \neq_\omega \min \mathrm{Core}(A^j)$ *for any pair* $i, j \in \mathbb{N}$*. Then for* $i \neq j \in \mathbb{N}$ *and two distinct states* u *and* v *of* A^i *and* A^j*, respectively, the initial transducers* A^i_u *and* A^j_v *are not* ω*-equivalent.*

Proof. Let A, i, j, u and v be as in the statement of the lemma above. Observe that since $\min \mathrm{Core}(A^i) \neq_\omega \min \mathrm{Core}(A^j)$, by Claim 3.1 there is a word Γ of size greater than or equal to the maximum of the minimum synchronizing levels of A and B such that the state of $\min \mathrm{Core}(A^i)$ forced by Γ is not ω-equivalent to the state $\min \mathrm{Core}(A^j)$ forced by Γ. Now since A^i and A^j are synchronizing, the initial transducers A^i_u and A^j_v are also synchronizing. Moreover, $\mathrm{Core}(A^i_u) =_\omega \min \mathrm{Core}(A^i)$, likewise $\mathrm{Core}(A^j_v) =_\omega \min \mathrm{Core}(A^j)$. Therefore, let Λ be a long enough word such that when read from the state u of A^i and state v of A^j the resultant state is in the core of A^i and A^j, respectively. Now let u' and v' be the states of A^i_u and A^j_v, respectively, reached after reading $\Lambda\Gamma$ in A^i_u and A^j_v. Then u' and v' are not ω-equivalent since $\mathrm{Core}(A^i_u) =_\omega \min \mathrm{Core}(A^i)$, and $\mathrm{Core}(A^j_v) =_\omega \min \mathrm{Core}(A^j)$. Therefore, there exists a word $\delta \in X_n^{\mathbb{N}}$ such that $(\delta)A^i_{u'} \neq (\delta)A^i_{v'}$; therefore, we have that $(\Lambda\Gamma\delta)A^i_u \neq (\Lambda\Gamma\delta)A^j_v$. The result now follows. ∎

We introduce some terminology for the claim below. Let X be a finite set and ρ a permutation of X. For any $a_1 \in X$, there is a sequence $(a_1 a_2 \cdots a_m)$ such that $(a_i)\rho = a_{i+1}$ for $1 \leq i < m$ and $(a_m)\rho = a_1$. We call such a sequence a *cycle*. Note that up to cyclically permuting the elements, the cycle containing a given element $a \in X$ is unique; therefore, we typically do not distinguish between a cycle $(a_1 a_2 \cdots a_m)$ and a cyclic permutation $(a_i a_{i+1} \cdots a_m a_1 \cdots a_m)$ of its elements for some $1 \leq i \leq m$. Another way of saying this is that for $a \in X$, a cycle $(a_1 a_2 \cdots a_m)$ containing a corresponds uniquely to the permutation γ of X which maps $a_i \to a_{i+1}$ for all $1 \leq i \leq m$, maps $a_m \to a_1$ and fixes every other point. Clearly, the map γ is invariant under cyclically permuting the elements of the cycle containing a. We do not distinguish between a cycle containing a and the permutation γ. The length of a cycle is the number of points in its support, that is, the number of points it moves. Let $\gamma_1, \ldots, \gamma_l$ be cycles such that every element of X is contained in exactly one of the γ_i. Then, we say that $\gamma_1\gamma_2 \cdots \gamma_l$ is an expression of ρ as a *product of (disjoint) cycles*. Conventionally, cycles of length 1 are omitted in such a decomposition. Two permutations ρ and ϕ are said to have the same cycle structure if there are cycle decompositions $\rho = \gamma_1\gamma_2 \cdots \gamma_l$ and $\phi = \eta_1\eta_2 \cdots \eta_l$ such that for $1 \leq i \leq l$, the length of γ_i is equal to the length of η_i.

Claim 3.3. Let A be a finite, invertible, transducer which is bi-synchronizing at level k. Then for any non-empty word Γ_1 there is a unique state $q_1 \in Q_A$ such that $\pi(\Gamma_1, q_1) = q_1$. Moreover, there is a cycle $(\Gamma_1 \cdots \Gamma_m)$ such that we have the following:

(i) Let q_i, $1 \leq i < m$ be such that $\pi(\Gamma_i, q_i) = q_i$, then $\Gamma_{i+1} = \lambda(\Gamma_i, q_i)$.

(ii) We have $\Gamma_1 = \lambda(\Gamma_m, q_m)$.

Proof. Throughout the proof let Γ be any non-empty word of length $j \geq 1$. We observe first that if there is a state q such that $\pi(\Gamma, q) = q$ then this state must be unique. Since if there was a state q' such that $\pi(\Gamma, q') = q'$ then $\pi(\Gamma^k, q) = q$ while $\pi(\Gamma^k, q') = q'$, and since Γ^k has length at least k we see it is a synchronizing word and so can conclude that $q = q'$.

To see that such a state q exists, consider again the word Γ^k. Since Γ is non-empty, $|\Gamma^k| \geq k$, so there is a unique state q such that $\Gamma^k = q$. Now consider the state p so that $\pi(\Gamma, q) = p$. Since $\pi(\Gamma^k, q) = q$ it is the case that $\pi(\Gamma^{k+1}, q) = p$, but Γ^k and Γ^{k+1} have the same length k suffix, so that $p = q$. In particular, we have $\pi(\Gamma, q) = q$.

We now free the symbol Γ. We want to show the map defined on X_n^j (words of length exactly j), by $\Gamma \mapsto \lambda(\Gamma, q)$, where $\pi(\Gamma, q) = q$, is a bijection (and so is decomposable into cycles as indicated in the statement of the claim).

To prove this map is injective, suppose there are two words, Γ and Δ, with associated states q and r, respectively, of length l, such that $\lambda(\Gamma, q) = \Gamma' = \lambda(\Delta, r)$. Now, as q is the state forced by Γ^k as above, while r is the state forced by Δ^k (again as above), we see that $\pi^{-1}((\Gamma')^k, q) = q$ while $\pi^{-1}((\Gamma')^k, r) = r$, but as $(\Gamma')^k$ is synchronizing for A^{-1} we must have that $q = r$, and then, by injectivity of A_q, that $\overline{\Gamma} = \overline{\Delta}$, so that in particular $\Gamma = \Delta$.

Therefore for each $j \in \mathbb{N}$, $j \geq 1$ the map induced by A, from the set of words of length j to itself, is injective. Therefore, as this set of words is finite, the map is actually a bijection and so can be expressed as a product of disjoint cycles (of words of length j). ■

Remark 3.4. Notice that we have only used the full bi-synchronizing condition in arguing invertibility. The existence and uniqueness of the state $q \in Q$ such that for $1 \leq j \in \mathbb{N}$ and $\Gamma \in X_n^j$, $\pi(\Gamma, q) = q$ holds for all elements of the monoid $\widetilde{\mathscr{P}}_n$.

We illustrate the above claim with the example below.

Example 3.5. Let C be the following transducer (Figure 1): It is easily verified that this transducer is bi-synchronizing at level 2. The sets $\{00, 10, 21\}$, $\{01, 11, 20\}$ and $\{02, 12, 22\}$ are, respectively, the set of words which force the states q_0, q_1 and q_2. The permutation of

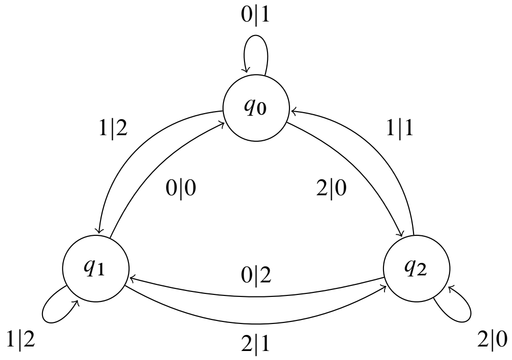

Figure 1. An example.

words of length 2 associated to this transducer in the manner described above is given by: (00 11 22)(10 20 12)(21 01 02). We make the following observation: The states forced by each element of a cycle is a cyclic permutation of $(q_0\ q_1\ q_2)$. We shall later see how such behaviour plays a role in understanding the order an element.

Remark 3.6. If we have found a permutation (as above) for a transducer A for words of length $j \geq 1$, then the cycle structure of this permutation will be present in all permutations associated to A for words of length mj, for $m \in \mathbb{N}\backslash\{0\}$. This is seen, for example, if $(\Gamma_1 \cdots \Gamma_l)$ is a cycle in the permutation associated to words of length j, then $(\Gamma_1\Gamma_1 \cdots \Gamma_l\Gamma_l)$ is a cycle in the level $2j$ permutation. This is because each Γ_i is processed from the state of A it forces and the output is Γ_{i+1}. Generalize in the obvious way for the permutation of words of length mj. For instance in the example above $(0\cdots0\ 1\cdots1\ 2\cdots2)$ will be present in the permutation of words of length $2m$ associated to C (where each $i\cdots i$ is of length $2m$, $i \in \{0, 1, 2\}$).

We establish some further notation. For $A \in \mathcal{H}_n$ bi-synchronizing at level k, and $1 \leq j \in \mathbb{N}$, let $\overline{A_j}$ represent the permutation of X_n^j indicated in Claim 3.3.

Remark 3.7. Observe that a similar proof to that given in Claim 3.3 will show that we can analogously associate to each element of $\widetilde{\mathcal{P}}_n\backslash\mathcal{P}_n$ a map from $X_n^j \to X_n^j$ for every $1 \leq j \in \mathbb{N}$. However, this map need not be invertible for every such $1 \leq j$ (we shall later see that for one-way synchronizing transducers there is some j, where the map so defined is not invertible). In light of this, for each $A = \langle X_n, Q, \pi, \lambda\rangle \in \widetilde{\mathcal{P}}_n$ and $1 \leq j \in \mathbb{N}$ let $\overline{A_j} : X_n^j \to X_n^j$ be the transformation given by $\Gamma \mapsto \lambda(\Gamma, q)$ where $q \in Q$ is the unique state such that $\pi(\Gamma, q) = q$. We observe that if $A \in \mathcal{P}_n$ then $\overline{A_j}$ is a permutation for every $j \in \mathbb{N}$. This is because $\mathcal{P}_n$ induces a homeomorphism of $X_n^{\mathbb{Z}}$. Since if for some $j \in \mathbb{N}$, $\overline{A_j}$ is not injective, then there are words Γ, Δ for which $(\Gamma)A_j = (\Delta)A_j$; this means that the bi-infinite strings $(\cdots\Gamma\dot{\Gamma}\Gamma\cdots)$ and $(\cdots\Delta\dot{\Delta}\Delta\cdots)$ are mapped to the same element of $X_n^{\mathbb{Z}}$ by A contradicting injectivity.

The following lemma shows that these maps behave well under multiplication.

Claim 3.8. Let $A = \langle X_n, Q_A, \pi_A, \lambda_A\rangle$ and $B = \langle Q_B, X_n, \pi_B, \lambda_B\rangle$ be elements of $\widetilde{\mathcal{P}}_n$. Let $A * B = \langle Q_A, S, \pi_{A*B}, \lambda_{A*B}\rangle$ be the core of the product of A and B, where $S \subset Q_A \times Q_B$ is the set of states in the core of $A * B$. Then $\overline{(A * B)_l} = \overline{A_l} * \overline{B_l}$.

Proof. Let Γ be a word of length l in X_n, and let $p \in Q_A$ be such that $\pi_A(\Gamma, p) = p$. Let $\Delta := \lambda_A(\Gamma, p)$, and let $q \in Q_B$ be such that $\pi_B(\Delta, q) = q$. Then (p, q) is a state of $A * B$ such that $\pi_{A*B}(\Gamma, (p, q)) = (p, q)$. If $\Lambda = \lambda_B(\Delta, q)$, then we have in $\overline{(A * B)_l}$ that $\Gamma \mapsto \Lambda$. However, $\overline{A} * \overline{B}$ sends Γ to Λ also. Since Γ was an arbitrary word of length l, this gives the result. ∎

Let $\tau_l : \widetilde{\mathcal{P}}_n \to Sym(X_n^l)$ be the map defined by $A \mapsto \overline{A_l}$ for every $l \in \mathbb{N}$. Below we demonstrate the usefulness of these maps.

Proposition 3.9. *Let* A *and* B *be elements of* $\widetilde{\mathcal{P}}_n$ *then the following hold:*

(i) A *and* B *commute if and only if for every* $l \geq 1 \overline{A_l}$ *and* $\overline{B_l}$ *commute.*

(ii) A *and* B *are conjugate by an invertible element of* $\widetilde{\mathcal{P}}_n$ *if and only if there is an invertible,* $h \in \widetilde{\mathcal{P}}_n$*, such that for every* $l \geq 1$ $\overline{h_l}^{-1}\overline{A_l}\,\overline{h_l} = \overline{B_l}$*.*

(iii) A *and* B *are equal if and only if for every* $l \geq 1 \overline{A_l} = \overline{B_l}$*.*

Proof. The forward direction in all cases follows by Claim 3.8 which shows that the map $\tau_l : \widetilde{\mathcal{P}}_n \to Sym(X_n^l)$ is a monoid homomorphism. We need only prove the reverse implications.

We proceed by contradiction.

For (i) suppose that $\overline{A_l}$ and $\overline{B_l}$ commute for every l, however, $\mathrm{Core}(B * A) \neq_\omega \mathrm{Core}(A * B)$. Let $m \in \mathbb{N}\backslash\{0\}$ be such that both $\mathrm{Core}(A * B)$ and $\mathrm{Core}(B * A)$ are bi-synchronizing at level m. Let Γ be a word of length m as in Claim 3.1 such that p is the state of $\mathrm{Core}(A * B)$ forced by Γ and q is the state of $\mathrm{Core}(B * A)$ forced by Γ.

Let λ_{AB} and λ_{BA} denote, respectively, the output function of $\mathrm{Core}(A * B)$ and $\mathrm{Core}(B * A)$. Since p is not ω-equivalent to q there is a word Δ, of length $l \geq 1$ say, such that $\Lambda := \lambda_{AB}(\Delta, p) \neq \lambda_{BA}(\Delta, q) =: \Xi$. This now means that in $\overline{\mathrm{Core}(A * B)_{l+m}}$, $\Delta\Gamma \mapsto \Lambda W_1$ and in $\overline{\mathrm{Core}(B * A)_{l+m}}$, $\Delta\Gamma \mapsto \Xi W_2$ (for some words W_1 and W_2 of length l). Therefore, we conclude that $\overline{\mathrm{Core}(A * B)_{l+m}} \neq \overline{\mathrm{Core}(B * A)_{l+m}}$ which is a contradiction.

Part (ii) proceeds in an analogous fashion. The forward implication is clear. For the reverse, suppose A and B are conjugate by an invertible element $h \in \widetilde{\mathcal{P}}_n$ but $\mathrm{Core}(A * h) \neq \mathrm{Core}(h * B)$. Let $m \in \mathbb{N}\backslash\{0\}$ be such that $\mathrm{Core}(A * h)$ and $\mathrm{Core}(h * B)$ are bi-synchronizing at level m. Let Γ be a word as in Claim 3.1 and let p be the state of $\mathrm{Core}(Ah)$ forced by Γ and q the state of $\mathrm{Core}(h * B)$ forced by Γ and p and q are not ω-equivalent. Now we are able to construct a word as in part (i) demonstrating that $\overline{\mathrm{Core}(A * h)_l} \neq \overline{\mathrm{Core}(h * B)_l}$ for some l yielding a contradiction.

Part (iii) follows from part (ii) with h the identity transducer. ■

The following is a corollary of Proposition 3.9.

Corollary 3.10. *Let* A *and* B *be elements of* $\widetilde{\mathcal{P}}_n$*, and let* $k \geq 1 \in \mathbb{N}$ *be such that both* A *and* B *are synchronizing at level* k *then the following hold:*

(i) $A = B$ *if and only if* $\overline{A}_{k+1} = \overline{B}_{k+1}$*.*

(ii) *Let* BA *and* AB *denote the minimal transducers representing the products* $\mathrm{Core}(A * B)$ *and* $\mathrm{Core}(B * A)$ *and* $l \geq 1 \in \mathbb{N}$ *be such that both* AB *and* BA *are synchronizing at level* l*. Then* $AB = BA$ *if and only if* $\overline{A}_{l+1}\overline{B}_{l+1} = \overline{B}_{l+1}\overline{A}_{l+1}$*.*

(iii) A *and* B *are conjugate in* $\widetilde{\mathcal{P}}_n$ *if and only if there is an invertible* $h \in \widetilde{\mathcal{P}}_n$ *such that* $h^{-1}Ah$ *(where this is the minimal transducer representing the product) is synchronizing at level* k *and* $\overline{h}_{k+1}^{-1}\overline{A}_{k+1}\overline{h}_{k+1} = \overline{B}_{k+1}$*.*

Proof. Throughout the proof all products indicated shall represent the minimal transducer under ω-equivalence representing the product.

Observe that parts (ii) and (iii) are consequences of part (i). Since for part (ii) AB and BA are synchronizing at level l; for part (iii) B and $h^{-1}Ah$ are synchronizing at level k (where h is the conjugator). Therefore, it suffices to prove only part (i).

The forward implication follows by Proposition 3.9, so we need only show the reverse implication. Let k be as in the statement of part (i) and assume that $\overline{A}_{k+1} = \overline{B}_{k+1}$. Denote by a triple (Ξ, u, v) for $\Xi \in X_n^k$ and u and v states of A and B, respectively, that u is the state of A forced by Ξ and v is the state of B forced by Ξ. Notice that for each such $\Xi \in X_n^k$ such a triple is unique.

Let $\Gamma \in X_n^k$ belong to a triple (Γ, p, q). Let $i \in X_n$ be arbitrary. Since $\overline{A}_{k+1} = \overline{B}_{k+1}$, we must have that $A_p(i) = B_q(i)$ since $\overline{A}_{k+1}(\Gamma i) = \overline{B}_{k+1}(\Gamma i)$.

Free the symbols Γ, p and q.

Now let $w = \cdots w_{-k} \cdots w_{-1} w_0 w_1 \cdots w_k \cdots$ be a bi-infinite word. We show that A and B process this word identically. Let w_i $i \in \mathbb{Z}$ denote the ith letter of w. Then the ith letter of $A(w)$ is $A_p(w_i)$ where p is the state of A forced by $\Gamma = w_{i-k} \cdots w_{i-1}$, the word of length k immediately to the left of w_i. Likewise, the ith letter of $B(w)$ is $B_q(w_i)$ where q is the state of B forced by Γ. Therefore, (Γ, p, q) is an allowed triple. However, from above we know that $A_p(w_i) = B_q(w_i)$. Since $i \in \mathbb{Z}$ was arbitrary, $A(w) = B(w)$, and $A = B$ since w was arbitrary and A and B are assumed minimal. ∎

Remark 3.11. Corollary 3.10 part (i) demonstrates that the group $\mathcal{H}_n$ is residually finite. This is because given $A \in \mathcal{H}_n$ a non-identity element that is synchronizing at level k, the map sending $B \in \mathcal{H}_n$ to $\overline{B}_{k+1}$ in the symmetric group on n^{k+1} points is a homomorphism that maps A to a non-trivial element.

Remark 3.12. Corollary 3.10 part (ii) demonstrates that if $B \in \widetilde{\mathcal{P}}_n$ is synchronizing at level $j \geq 1 \in \mathbb{N}$, and $A \in \widetilde{\mathcal{P}}_n$ is synchronizing at level $k \geq 1 \in \mathbb{N}$, then B commutes with A if and only if $\overline{A}_{j+k+1}$ commutes with $\overline{B}_{j+k+1}$ by Claim 2.4.

Remark 3.13. In order to restate Corollary 3.10 part (iii) for a non-invertible $h \in \widetilde{\mathcal{P}}_n \backslash \mathcal{P}_n$ showing that the equation $\overline{A}_{k+1}\overline{h}_{k+1} = \overline{h}_{k+1}\overline{B}_{k+1}$ holds might no longer suffice. Instead, we might have to check that $\overline{A}_{j+1}\overline{h}_{j+1} = \overline{h}_{j+1}\overline{B}_{j+1}$ where $j \in \mathbb{N}$ is a level such that $\mathrm{Core}(A * h)$ and $\mathrm{Core}(B * h)$ are synchronizing at level j.

The result distinguishes between elements of $\mathcal{H}_n$ and $\widetilde{\mathcal{H}}_n$. However, we require the following definitions first.

Definition 3.14. Let $\Gamma = \gamma_0\gamma_2 \cdots \gamma_k - 1$ be a word in X_n^k for some natural number $k > 0$. Define the ith rotation of Γ to be the word: $\Gamma' = \gamma_{k-i}\gamma_{k-i+1} \cdots \gamma_0\gamma_1 \cdots \gamma_{k-i-1}$.

Remark 3.15. One can think of Γ as decorating a circle divided into k intervals (counting from zero), and Γ' is the result of rotating the circle clockwise by i. Then the 0th rotation of Γ is simply Γ.

Proposition 3.16. *Let* $A = \langle X_n, Q_A, \pi_A, \lambda_A\rangle$ *be an element of* $\widetilde{\mathcal{H}}_n \backslash \mathcal{H}_n$ *with synchronizing level* k *and let* $A^{-1} = \langle X_n, Q_{A^{-1}}, \pi_{A^{-1}}, \lambda_{A^{-1}}\rangle$ *the inverse of* A. *Then there is an* $l \in \mathbb{N}$ *with* $0 < l \leq k(|Q_A|^2 + 1)$ *such that* $\overline{A}_l$ *is not a permutation. In particular, the action of* A *on* $X_n^{\mathbb{Z}}$ *is non-injective. Moreover, there exist words* Δ *and* Λ *in* X_n^+ *such that* Δ *is not a cyclic rotation of* Λ *and the bi-infinite strings* $(\cdots\Delta\Delta\cdots)$ *and* $(\cdots\Lambda\Lambda\cdots)$ *have the same image under* A.

Proof. Suppose A is synchronizing at level k. Since A^{-1} is not synchronizing it follows that $|Q_A| = |Q_{A^{-1}}| > 1$. Moreover, there is a pair of states (r_1, r_2) such that there is an infinite set W_1 of words $w_i \in X_n^+$ for which $\pi_{A^{-1}}(w_i, r_1) \neq \pi_{A^{-1}}(w_i, r_2)$. This follows since A^{-1} is not synchronizing at level l for any $l \in \mathbb{N}$. Therefore for each $l \in \mathbb{N}$, there is a pair of states (r_1^l, r_2^l) and a word $w_l \in X_n^l$ such that $\pi_{A^{-1}}(w_l, r_1^l) \neq \pi_{A^{-1}}(w_l, r_2^l)$. Since A is a finite automaton there is a pair of states (r_1, r_2) such that for infinitely many $l \in \mathbb{N}$, $(r_1^l, r_2^l) = (r_1, r_2)$; therefore, taking $W_1 := \{w_l \mid l \in \mathbb{N} \text{ and } (r_1^l, r_2^l) = (r_1, r_2)\}$, (r_1, r_2) and W_1 satisfy the conditions.

Now since W_1 is infinite, by an argument similar to that above, there is a pair of states (s_1, s_2) such that $\pi_{A^{-1}}(w_i, r_1) = s_1$ and $\pi_{A^{-1}}(w_i, r_2) = s_2$ and $s_1 \neq s_2$ for infinitely many $w_i \in W_1$. Let W_2 denote the set of words w_i such that $\pi_{A^{-1}}(w_i, r_1) = s_1$ and $\pi_{A^{-1}}(w_i, r_2) = s_2$.

Let $w_i \in W_2$ be such that $|w_i| \geq k(|Q_A|^2 + 1)$. Now since $s_1 \neq s_2$, then for any prefix φ of w_i we must have $\pi_{A^{-1}}(\varphi, r_1) \neq \pi_{A^{-1}}(\varphi, r_2)$. Moreover, since $|w_i| \geq k(|Q_A|^2 + 1)$ there are prefixes φ_1 and φ_2 of w_i such that $||\varphi_1| - |\varphi_2|| = jk \leq k(|Q_A|^2 + 1)$ $(j \in \mathbb{N}\backslash\{0\})$ satisfying $\pi_{A^{-1}}(\varphi_1, r_1) = \pi_{A^{-1}}(\varphi_2, r_1) = p^{-1}$ and $\pi_{A^{-1}}(\varphi_1, r_2) = \pi_{A^{-1}}(\varphi_2, r_2) = q^{-1}$ with $p^{-1} \neq q^{-1}$, and $p^{-1}, q^{-1} \in Q_{A^{-1}}$.

Assume φ_1 is a prefix of φ_2 and let v be the such that $\varphi_1 v = \varphi_2$. By construction, v satisfies $\pi_{A^{-1}}(v, p) = p$ and $\pi_{A^{-1}}(v, q) = q$ such that $p^{-1} \neq q^{-1}$. Let $\Lambda = \lambda_{A^{-1}}(v, p^{-1})$ and $\Delta = \lambda_{A^{-1}}(v, q^{-1})$. Since A is synchronizing at level k, $\Lambda \neq \Delta$, otherwise $p = q$ and $|\Lambda| = |\Delta|$.

Therefore, in A we have $\pi_A(\Lambda, p) = p$ and $\pi_A(\Delta, q) = q$; moreover, $\lambda_A(\Lambda, p) = \lambda_A(\Delta, q) = v$. This shows that $\overline{A}_\Lambda$ is not a permutation of $X_n^{|\Lambda|}$. We now make the assumption that Λ and Δ are the smallest words such that $\pi_A(\Lambda, p) = p$ and $\pi_A(\Delta, q) = q$; moreover, $\lambda_A(\Lambda, p) = \lambda_A(\Delta, q)$. Let $v \in X_n^{|\Lambda|}$ be such that $\lambda_A(\Lambda, p) = \lambda_A(\Delta, q) = v$.

In order to show that A represents a non-injective map on $X_n^{\mathbb{Z}}$ observe that the bi-infinite strings $(\cdots\Lambda\Lambda\cdots)$ and $(\cdots\Delta\Delta\cdots)$ are mapped to the bi-infinite string $(\cdots vv\cdots)$ under A. Therefore, taking $(\cdots\Theta\dot{\Theta}\Theta\cdots)$ for $\Theta \in X_n^+$ to represent the element $y \in X_n^{\mathbb{Z}}$ defined by $y_{j|\Theta|}y_{j|\Theta|+1}\cdots y_{j|\Theta|+|\Theta|-1} := \Theta$ for any $j \in \mathbb{Z}$, we see that $(\cdots\Lambda\dot{\Lambda}\Lambda\cdots)$ and $(\cdots\Delta\dot{\Delta}\Delta\cdots)$ are distinct elements of $X_n^{\mathbb{Z}}$ which have the same image under A. This shows A is non-injective.

To conclude the proof we now need to argue that there exist words Λ' and Δ' which are not cyclic rotations of each other such that $(\cdots\Lambda'\dot{\Lambda}'\Lambda'\cdots)$ and $(\cdots\Delta'\dot{\Delta}'\Delta'\cdots)$ are mapped by A to the same place.

Suppose that Λ is a cyclic rotation of Δ, since we are done otherwise.

Since $\pi_A(\Lambda, p) = p$ we must have that v is equal to a non-trivial cyclic rotation of itself. This is the case if and only if v is equal to some power of a third word ν strictly smaller than v (see, for instance, [17, Theorem 1.2.9]). In fact, if $v = v'v'' = v''v'$ then both v'' and v' are powers of this word ν.

We may assume that ν is a prime word (i.e., it cannot be written as a power of a strictly smaller word). Let $r \in \mathbb{N}$ be such that $\nu^r = v$. Notice that $r|\nu| = |v| = |\Lambda|$.

First suppose that there is word $u \in X_n^{|\nu|}$ such that $A_{|\nu|}(u) = \nu$ and u^r is a rotation of Λ. If a non-trivial suffix $u_1 \neq u$ of u is a prefix of Λ, then since $\lambda_A(\Lambda, p) = \nu^r = v$, we must have that ν is equal to a non-trivial cyclic rotation of itself contradicting that ν is a prime word. Therefore, $\Lambda = u^r$. However, since $A_{|\Lambda|}(\Delta) = A_{|\Lambda|}(\Lambda)$ and Δ is a cyclic rotation of Λ then u^r is also a cyclic rotation of Δ. Therefore, by the same argument, we must have that $\Delta = u^r$. However, this now implies that $\Delta = \Lambda$ yielding a contradiction since we assumed that $\Delta \neq \Lambda$.

Now since $|\nu| < |v|$, then either there is a word u, such that $|u| = |\nu|$ for which $\overline{A}_{|\nu|}(u) = \nu$ or $\overline{A}_{|\nu|}$ is not surjective from $X_n^{|\nu|}$ to itself, and so it is also not injective (since $X_n^{|\nu|}$ is finite). If the latter occurs, then there are strictly smaller distinct words Λ' and Δ' and states p' and q' such that $\pi_A(\Lambda', p') = p'$ and $\pi_A(\Delta', q') = q'$, so that $\lambda_A(\Lambda', p') = \lambda_A(\Delta', q')$. Notice that since $A \in \mathcal{H}_n$ all its states are homeomorphism states; therefore, p' and q' cannot be equal or A would have a non-homeomorphism state. However, this is a contradiction since we assumed that Λ and Δ were the smallest such words. Therefore, there is a word u so that $|u| = |\nu|$ and $\overline{A}_{|\nu|}(u) = \nu$. Notice that u^r cannot be a rotation of Λ by an argument above. Moreover, the bi-infinite sequences $(\cdots \dot{u}^r u^r \cdots)$ and $(\cdots \dot{\Lambda}\Lambda \cdots)$ are mapped by A to the same bi-infinite string $(\cdots \dot{v} v \cdots)$. ■

Remark 3.17. Let A be an element of $\widetilde{\mathcal{H}}_n \backslash \mathcal{H}_n$ which is invertible as an automaton, then A represents a surjective map from the Cantor space $X_n^{\mathbb{Z}}$ to itself. In particular, as a consequence of the proposition above an element $A \in \widetilde{\mathcal{H}}_n$ is injective on $X_n^{\mathbb{Z}}$ if and only if it is a homeomorphism if and only if it is bi-synchronizing.

Proof. Our argument shall proceed as follows: We shall make use of the well-known results that the continuous image of a compact topological space is compact, and that a compact subset of a Hausdorff space is closed. This means it suffices to argue that the image of A is dense in $X_n^{\mathbb{Z}}$.

Let $k \in \mathbb{N}$ be the minimal synchronizing level for A.

Notice that since A is invertible as an automaton, each state of A defines an invertible map from $X_n^{\mathbb{N}}$ to itself. Therefore, given an element $y \in X_n^{\mathbb{Z}}$, let p be a state of A and fix an index $i \in \mathbb{Z}$, then defining $z := y_i y_{i+1} y_{i+1} \cdots$ in $X_n^{\mathbb{N}}$, there exists $x \in X_n^{\mathbb{N}}$ such that the initial automaton $A_p : X_n^{\mathbb{N}} \to X_n^{\mathbb{N}}$ maps x to z.

Now let y, p, z and x be as in the previous paragraph, and let $\Gamma \in X_n^k$ be a word such that the state of A forced by Γ is p. Let $u \in X_n^{\mathbb{Z}}$ be defined by $u_i u_{i+1} \cdots := x$, $u_{i-k} u_{i-k+1} \cdots u_{i-1} := \Gamma$ and $u_j := 0$ for all $j < i - k$.

If $w \in X_n^{\mathbb{Z}}$ is the image of u under A, then $w_i w_{i+1} \cdots = z$. Therefore for any $y \in X_n^{\mathbb{Z}}$ and any open neighbourhood U of y, $A(X_n^{\mathbb{Z}}) \cap U$ is non-empty. ■

Remark 3.18. Given an element, A, of $\widetilde{\mathcal{H}}_n$, Proposition 3.16 gives an algorithm for determining if $A \in \mathcal{H}_n$ or if $A \in \widetilde{\mathcal{H}}_n \backslash \mathcal{H}_n$, since we have only to check if $\overline{A}_j$ is a permutation for all $1 \leq j \leq kM(A)$, where k is the synchronizing level of A and $M(A)$ is quadratic in the states of A.

Remark 3.19. It is a consequence of the proof of the proposition above that for $A \in \mathcal{P}_n$, $\overline{A}_l$ maps prime words to prime words for every $l \in \mathbb{N}$. This is because if $(\Gamma)\overline{A}_l = (\gamma)^r$ for Γ a prime word, for $|\gamma| < |\Gamma|$ and $r \in \mathbb{N}$. Then either $\overline{A}_l : X_n^{|\gamma|}$ is not surjective and so it is not injective either, or there is a word $\delta \in X_n^{\gamma}$ such that $(\delta)\overline{A}_l = \gamma$. Since Γ is a prime word it follows in either case, as in the proof of Proposition 3.16, that A does not induce a homeomorphism of $X_n^{\mathbb{Z}}$. An alternative proof of this fact can be found in [3].

Proposition 3.9 indicates that if two elements A and B in $\mathcal{H}_n$ are such that $\overline{A}_j$ and $\overline{B}_j$ have the same d-cycle structure for all $j \in \mathbb{N}$ then A and B are likely to be conjugate. This, however, need not be the case, as will be seen below. First we make the following definition.

Definition 3.20 (Rotation). Let $A \in \mathcal{P}_n$ and let $l \in \mathbb{N}$. Given a prime word $\Gamma \in X_n^l$ let C be the cycle of $\overline{A}_l$ containing Γ. Notice that C consists only of prime words by Remark 3.19. Let $1 \leq s \leq \text{length}(C)$ be minimal in $\mathbb{N}$ such that $(\Gamma)\overline{A}_l^s$ is a rotation of Γ. Let $0 \leq i < l$ be such that $\overline{A}_l^s(\Gamma)$ is the ith rotation of Γ, then we say that C has minimal *rotation* i of Γ. We call the triple $(\text{length}(C), s, i)_\Gamma$ *the triple associated to* C *for* Γ.

Lemma 3.21. *Let* $C \in \overline{A}_l$ *be a cycle with associated triple* $(\text{length}(C), s_C, r_C)_{\Gamma_0}$ *for* Γ_0 *a prime word belonging to* C*. Then we have the following:*

(i) *For any other word* Γ *belonging to* C *we have*

$$(\text{length}(C), s_C, r_C)_{\Gamma_0} = (\text{length}(C), s'_C, r'_C)_\Gamma.$$

(ii) $\text{Length}(C) = o \cdot s_C$ *where* o *is the order of* r_C *in the additive group* $\mathbb{Z}_l$*, if* $r_C = 0$ *then take* $o = 1$.

Proof. Let $C = (\Gamma_0 \cdots \Gamma_j)$ and let $(\text{length}(C), s_C, r_C)_{\Gamma_0}$ be the triple associated to C for Γ_0, where Γ_0 is a prime word. Then s_C is minimal such that Γ_{s_C} is the r_Cth rotation of Γ_0. Now since Γ_1 is the output of the unique loop of A labelled by Γ_0, then Γ_{s_C+1} is also an r_Cth rotation of Γ_1. This is because the unique loop of A labelled by Γ_{s_C+1} is the r_Cth rotation of the loop labelled by Γ_0. We can now replace C with the cycle $(\Gamma_1 \cdots \Gamma_j \Gamma_1)$ and repeat the argument, until we have covered all rotations of C. This shows that the triple $(\text{length}(C), s_C, r_C)_{\Gamma_1}$ is independent of the choice of Γ_1.

For the second part of the lemma, first observe that if $s_C = \text{length}(C)$, then $r_C = 0$ and we are done. Therefore, we may assume that $1 \leq s_c < \text{length}(C)$.

Now observe that by minimality of s_C and the above argument, $\Gamma_{s_C+s_C}$ is the $2r_C$th rotation of Γ_0; moreover, no Γ_k for $s_C < k < 2s_C$ is a rotation of Γ_0. Notice that r_C has finite order in the additive group $\mathbb{Z}_l$. Let o be the order of r_C. Then Γ_{os_C} is the or_Cth rotation of Γ which is just Γ. Moreover, by minimality of s_C, and repetitions of the argument in the previous paragraph, o is minimal such that $\Gamma_{os_C} = \Gamma_0$. However, by the first part of the lemma, we must also have $\overline{A}_l^{os_C}(\Gamma_k) = \Gamma_k$ $1 \leq k \leq j$. Minimality now ensures that $os_C = j$. ■

As a consequence of the remark above for a given cycle $C \in \overline{A}_l$, we shall simply refer to $(\text{length}(C), s_C, r_C)$ as *the triple associated to* C.

Definition 3.22 (Spectrum). Let $A \in \mathscr{P}_n$, and let $k \in \mathbb{N}$. For each triple (L_C, S_C, T_C) associated to a cycle of prime words in the cycle structure of $\overline{A}_k$, let d_C denote the multiplicity with which it occurs as we consider all such triples associated to the cycles of $\overline{A}_k$. Then define $\mathrm{Sp}_k(A) := \{(k, d_C, (L_C, S_C, T_C))\}$ as C runs over all cycles of $\overline{A}_k$. Define $\mathrm{Sp}(A) := \bigcup_{k\in\mathbb{N}} \mathrm{Sp}_k(A)$.

Theorem 3.23. *Let $A \in \mathscr{P}_n$, and let $k \in \mathbb{N}$, then* $\mathrm{Sp}_k(A)$ *is a conjugacy invariant of A in $\mathscr{P}_n$.*

Proof. Let C be a cycle in the cycle structure of $\overline{A}_k$ and let (L_C, S_C, T_C) be its associated triple. Let $J \in \mathscr{P}_n$ be arbitrary and invertible.

That L_C is preserved under conjugation by J follows from Proposition 3.9, and standard results about permutation groups.

That S_C is preserved under conjugation is a consequence of the fact that $J \in \mathscr{P}_n$. To see this first suppose that $C = (\Gamma_1 \cdots \Gamma_j)$ for some $j \in \mathbb{N}$. Let $\Delta_i = (\Gamma_i)\overline{J}_k$. Then $(\Delta_1 \cdots \Delta_j)$ is a cycle of $\overline{J}_k^{-1}\overline{A}_k\overline{J}_k$. Since Δ_i is the output of the unique loop of J labelled by Γ_i $(1 \leq i \leq j)$, and since S_C is minimal so that Γ_{S_C} is a rotation of Γ_1, then S_C is also the minimal position so that Δ_{S_C} is a rotation of Δ_1.

That T_C is preserved under conjugation is once more a consequence of the fact that $J \in \mathscr{P}_n$. Let Γ_i and Δ_i for $1 \leq i \leq j$ be as in the previous paragraph. Since Γ_{S_C} is the T_Cth rotation of Γ_1, then as Δ_1 is the output of the unique loop of J labelled by Γ_1, Δ_{S_C} is the T_Cth rotations of Δ_1. ■

Corollary 3.24. *Let $A \in \mathscr{P}_n$, then* $\mathrm{Sp}(A)$ *is a conjugacy invariant of A in $\mathscr{P}_n$.*

It is possible to construct elements $M \in \mathscr{H}_n$ for which $\mathrm{Sp}(M) \neq \mathrm{Sp}(M^{-1})$. Clearly, $\overline{M}_j$ and $\overline{(M^{-1})}_j$ have the same cycle structure for all $j \in \mathbb{N}$.

4. The structure of the dual automaton for elements of $\widetilde{\mathscr{P}}_n$

In this section introduce the tools needed to understand the growth rates of automaton semigroups generated by automata in the group $\widetilde{\mathscr{H}}_n$. As we will see, and has been touched upon in the introduction, the techniques we introduce are also helpful in determining whether or not an element of $\mathcal{H}_n$ has infinite order. However, as the techniques are applicable to $\widetilde{\mathcal{P}}_n$ and will be relevant in later sections, we work in this monoid for majority of this section restricting our attention to $\mathcal{H}_n$ only when necessary. It is standard in the literature to tackle the order problem and growth by investigating the structure of the dual automaton, see for instance [1, 12], and this is what we do below.

4.1. Splits and implications

Let $A = \langle X, Q, \pi, \lambda \rangle$ be a transducer and let $k \in \mathbb{N}$.

We form the level k dual,

$$A_k^\vee = \langle X_k^\vee, Q_k^\vee, \pi_k^\vee, \lambda_k^\vee \rangle,$$

of A as follows. The state set $Q_k^\vee$ of $A_k^\vee$ is the set of all words of length k in the input alphabet X. This dual automaton has its input alphabet equal to its output alphabet and they are both equal to $X_k^\vee := Q$ the set of states of A. The transition function $\pi_k^\vee$ is defined as follows: for states $q, q' \in Q$, and $\Gamma, \Gamma' \in Q_k^\vee$, we have

(1) $\pi_k^\vee(q, \Gamma) = \Gamma'$ if and only if $\lambda(\Gamma, q) = \Gamma'$, and

(2) $\lambda_k^\vee(q, \Gamma) = q'$ if and only if $\pi(\Gamma, q) = q'$.

We observe that $A_{k+1}^\vee = A_k^\vee * A_1^\vee$. For suppose that Γi a word of length $k+1$ is a state in the dual, and q is any state symbol of A such that after reading q from Γi in $A_{k+1}^\vee$ we are in state Δj and the output is p. Then in A we have $\pi(\Gamma i, q) = p$ and $\lambda(\Gamma i, q) = \Delta j$. We can break up this transition into two steps. Suppose $\pi(\Gamma, q) = p'$, then we have $\lambda(\Gamma, q) = \Delta$, $\lambda(i, p') = j$ and $\pi(i, p') = p$. Hence, in $A_k^\vee$ we read q from Γ and transition to Δ and p' is the output p'. Moreover, in $A_1^\vee$ we read p' from i and transition to j with output p. Therefore, the state (Γ, i) of $A_k^\vee * A_1^\vee$ is such that we read q from this state and transition to the state (Δ, j) and the output produced is p.

The following definition gives a tool which connects the synchronizing level of powers of an element of $\widetilde{\mathcal{P}}_n$ to a property of the dual automaton. First we introduce some notation.

Let $A \in \widetilde{\mathcal{P}}_n$ be a transducer and $t \in Q_A$ be a state of A. Let $k \in \mathbb{N}$ be the synchronizing length of A, then denote by W_t the synchronizing words for the state t. That is, W_t is the set of all words $\Gamma \in X_n^*$ of length at least k such that $\pi_A(\Gamma, q) = t$ for some (and so any) state of A.

Definition 4.1 (Splits). Let A be an element of $\widetilde{\mathcal{P}}_n$, which is synchronizing level k. Then, for a natural number r greater than or equal to k, we say that $A_r^\vee$ *splits* if there is a word $\Gamma \in X_n^r$, elements $(p_1, p_2, \ldots, p_l), (q_1, q_2, \ldots, q_l), (s_1, s_2, \ldots, s_l) \in Q_A^l$, where $\Gamma \in W_{s_1}$, and distinct elements $t_1, t_2 \in Q_A$, such that the sequences $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$ and $\Lambda_1, \Lambda_2, \ldots, \Lambda_l$ defined by $\Gamma_1 = \lambda_A(\Gamma, p_1)$, $\Lambda_1 = \lambda_A(\Gamma, q_1)$ and for $1 < i \leq l$, $\Gamma_i = \lambda_A(\Gamma_{i-1}, p_i)$ and $\Lambda_i = \lambda_A(\Lambda_{i-1}, q_i)$, satisfy: $\Gamma_i, \Lambda_i \in W_{s_{i+1}}$ for all $1 \leq i \leq l-1$, $\Gamma_l \in W_{t_1}$ and $\Lambda_l \in W_{t_2}$. In other words, Figure 2 depicting the transitions in $A_r^\vee$ at the state Γ is valid.

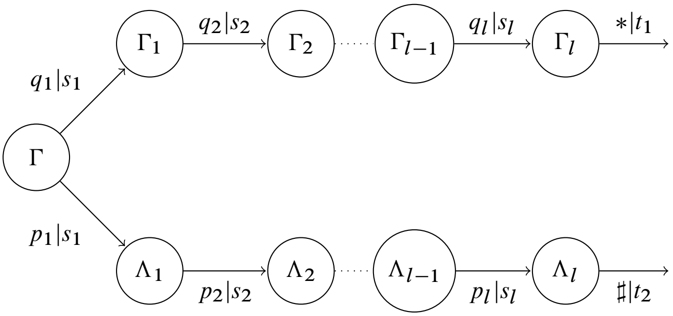

Figure 2. A split; the symbols $*$ and $\sharp$ represent arbitrary elements of Q_A.

We further say that the l-tuples $(p_1, \ldots, p_l)$ and $(q_1, \ldots, q_l)$ *split* $A_r^\vee$. We shall call the set $\{p_1, q_1\}$ the *top of the split*, the set $\{t_1, t_2\}$ the *bottom of the split* and the triple $((q_1, \ldots, q_l), (p_1, \ldots, p_l), \Gamma)$ a *split* of $A_r^\vee$.

Definition 4.2. Let A be an element of $\widetilde{\mathscr{P}}_n$, with synchronizing level k. Let $r \geq k$ and let $((q_1, \ldots, q_l), (p_1, \ldots, p_l), \Gamma)$ be a split of $A_r^\vee$ for $\Gamma \in X_n^r$ and $(q_1, \ldots, q_l), (p_1, \ldots, p_l) \in Q_A^l$. Let $\{t_1, t_2\}$ be the bottom of this split. Then we say that *the bottom of the split* $((q_1, \ldots, q_l), (p_1, \ldots, p_l), \Gamma)$ *depends only on the top* if for any other tuples $U_1, U_2 \in Q_A^{l-1}$ we have that $((q_1, U_1), (p_1, U_2), \Gamma)$ is also a split with bottom $\{t_1, t_2\}$ and, for any $u, u' \in Q$, $\pi_{\mathrm{Al}}(\Gamma, (p_1, \ldots, p_l, u)) = \pi_{\mathrm{Al}}(\Gamma, (p_1, U_2, u'))$ and $\pi_{\mathrm{Al}}(\Gamma, (q_1, \ldots, q_l, u)) = \pi_{\mathrm{Al}}(\Gamma, (q_1, U_2, u'))$. The last condition means that if $\lambda_{\mathrm{Al}}(\Gamma, (q_1, \ldots, q_l)) \in W_{t_1}$ then so also is $\lambda_{\mathrm{Al}}(\Gamma, (q_1, U_1))$ and likewise for $(p_1, \ldots, p_l)$, (P_1, U_2) and W_{t_2}.

Definition 4.3. For a transducer A, we define the r *splitting length of* A (for r greater than or equal to the minimal synchronizing length) to be minimal l such that there is a pair of l-tuples of states which split $A_r^\vee$. If there is no such pair then we set the r splitting length of A to be ∞.

Remark 4.4. Let A be a transducer with minimal r splitting length $l < \infty$, by minimality of l it follows that for a given pair in $Q^l \times Q^l$ which splits $A_r^\vee$, then the bottom of the split depends only on the top. Therefore, the top and bottom of the split have cardinality two. In particular, for any split whose bottom depends only on its top, the top and bottom of the split both have cardinality 2.

Remark 4.5. Let A be a transducer such that the minimal r splitting length of A is infinite for some r then the minimal $r + 1$ splitting length of A is also infinite.

The following lemma demonstrates that for $A \in \widetilde{\mathscr{P}}_n$ an $r > 2$ the r splitting length of A is bigger than the $r - 1$ splitting length of A.

Lemma 4.6. *Let $A \in \widetilde{\mathscr{P}}_n$ be synchronizing at level k, and suppose that the mk splitting length of A is finite for $m \in \mathbb{N}$, $m > 0$, then the $(m+1)k$ splitting length of A is strictly greater than the mk splitting length of A.*

Proof. Let A, m and k be as in the statement of the lemma. Suppose that A has mk splitting length l. It suffices to show that for any word $\Gamma \in X_n^{(m+1)k}$, and any $l+1$-tuple P in Q_A^{l+1}, the output of P through Γ depends only on Γ.

We set up the following notation to simplify the discussion that follows. Let $A^j := \langle X_n, Q_A^j, \lambda_j, \pi_j \rangle$ and let $A_j^\vee := \langle Q_A, X_n^j, \lambda_j^\vee, \pi_j^\vee \rangle$ for $j \in \mathbb{N}$. For a word $\gamma \in X_n^k$ let q_γ denote the state of A forced by Γ.

Now since A has mk splitting length l, it follows that for any $P := (p_1, \ldots, p_l)$ and $T := (t_1, \ldots, t_l)$ in Q_A^l and $\Gamma \in X_n^{mk}$ we have that $\lambda_{mk}^\vee(P, \Gamma) = \lambda_{mk}^\vee(T, \Gamma)$. By definition of the dual, $\lambda_{mk}^\vee(P, \Gamma) = \pi_l(\Gamma, P)$.

Now let $\gamma \in X_n^k$ be arbitrary and let $p \in Q_A$ and $P \in Q_A^l$ also be arbitrary. Consider $\lambda_{(m+1)k}^\vee(Pp, \Gamma\gamma)$, we have

$$\begin{aligned}\lambda_{(m+1)k}^\vee(Pp, \Gamma\gamma) &= \pi_{l+1}(\Gamma\gamma, Pp)\\ &= \pi_l(\gamma, \pi_l(\Gamma, P))\pi_1(\lambda_l(\gamma, \pi_l(\Gamma, P)), \pi_1(\lambda_l(\Gamma, P), p)).\end{aligned}$$

The following observations conclude the proof of the lemma. Since A is synchronizing at level k that the suffix $\pi_1(\lambda_l(\gamma, \pi_l(\Gamma, P)), \pi_1(\lambda_l(\Gamma, P), p))$ depends only on $\lambda_l(\gamma, \pi_l(\Gamma, P))$. However, since $A_{mk}^\vee$ has minimal splitting length l we have that $\pi_l(\Gamma, P)$ depends only on Γ. Therefore, we have that $\lambda_{(m+1)k}^\vee(Pp, \Gamma\gamma)$ depends only on $\Gamma\gamma$. ■

Remark 4.7. It follows from the lemma above that if $A \in \widetilde{\mathscr{P}}_n$ is synchronizing at level k then the mk splitting length of A, if it is finite, is at least m for $m \in \mathbb{N}$, $m > 0$.

The following lemma shows that the minimal splitting length is connected with the synchronizing level of powers of a transducer.

Lemma 4.8. *Let A be a transducer with synchronizing level less than or equal to k, then if A has k splitting length l, then* $\min(\mathrm{Core}(A^{l+1}))$ *has minimal synchronizing level $m \geq k+1$.*

Proof. Let l be as in the statement of the lemma. Now consider $\mathrm{Core}(A^l)$. The states of $\mathrm{Core}(A^l)$ will consist of all length l outputs of $A_k^\vee$. Moreover, by choice of l, $\mathrm{Core}(A^l)$ is also synchronizing at level k.

Let Γ be a word for which there is a split $((q_1, \ldots, q_l), (p_1, \ldots, p_l), \Gamma)$ of the minimal length l giving rise to Figure 3, where $\Gamma_l \in W_{t_1}$ and $\Lambda_l \in W_{t_2}$ for distinct states t_1 and t_2.

We now consider $B := \mathrm{Core}(A \times \mathrm{Core}(A^l))$. It is easy to see that there are states in B of the form (p_1, P), (q_1, Q) for appropriate $P, Q \in \mathrm{Core}(A^l)$. Therefore, in B when we have read Γ through (p_1, P), we are in state $(s_1, \ldots, s_l, t_1)$, and when we have read Γ through state (q_1, Q) we go to state $(s_1, \ldots, s_l, t_2)$. Since $t_1 \neq t_2$ these states are not ω equivalent. This concludes the proof. ■

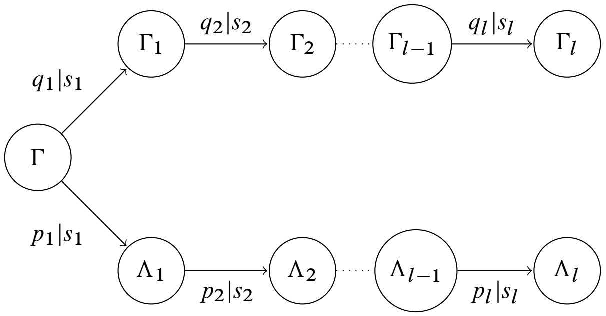

Figure 3. A minimal split.

4.2. Equivalence of finiteness hypothesis

Now suppose that $A \in \widetilde{\mathscr{P}}_n$ and the subsemigroup $\langle A\rangle_{\widetilde{\mathscr{P}}_n} \leq \widetilde{\mathscr{P}}_n$ of $\widetilde{\mathscr{P}}_n$ generated by A is finite. Notice that if $A \in \widetilde{\mathscr{H}}_n$ and the semigroup $\langle A\rangle_{\widetilde{\mathscr{P}}_n}$ is finite then A is in fact an element of $\mathscr{H}_n$. The next result demonstrates that in the case where the semigroup $\langle A\rangle_{\widetilde{\mathscr{P}}_{\mathrm{arg1}}}$ is finite, there must be some $j \in \mathbb{N}$, $j > 0$ for which the j splitting length of A is infinite.

Lemma 4.9. *Let $A \in \widetilde{\mathscr{P}}_n$ be synchronizing at level k. Suppose that the subsemigroup of $\widetilde{\mathscr{P}}_n$ generated by A is finite, and that j is the maximum of the minimal synchronizing level of the elements of $\langle A\rangle_{\widetilde{\mathscr{P}}_n}$. Then A has infinite j splitting length.*

Proof. This is a consequence of Lemma 4.8. Since if A has j splitting length l, then by Lemma 4.8 $\min(\mathrm{Core}(A^{l+1}))$ has minimal synchronizing level $j+1$, which is a contradiction. ■

Remark 4.10. The above means that we can partition $A_j^{\vee}$ into components $D_1, \ldots, D_i$ and for each component there is a pair of words $W_{i,1}$ and $W_{i,2}$ in the states of A such that for any input into a state of a component of D_i, the only possible output has the form $u(W_{i,2})^l v$ where u is a possibly empty suffix of $W_{i,1}W_{i,2}$, and v is a possibly empty prefix of $W_{i,2}$. From this it follows that A^i is synchronizing at level j for any $i \in \mathbb{N}$.

Below we illustrate Lemma 4.9 with some examples.

Example 4.11. Consider the transducer C from Example 3.5 which, for convenience, is reproduced in Figure 4. This is a transducer of order 3; in particular, it is a conjugate of the single state transducer which can be identified with the permutation $(0, 1, 2)$.

This transducer, as noted before, is bi-synchronizing at the second level. The level 3 dual has 27 nodes, and so we have chosen not to illustrate it. However, utilizing either the AAA package or the AutomGrp package [14] in GAP [7], together with (in AutomGrp) the function “MinimizationOfAutomaton( )” which returns an ω-equivalent automaton, applied to the third power of the dual automaton, we get the following result.

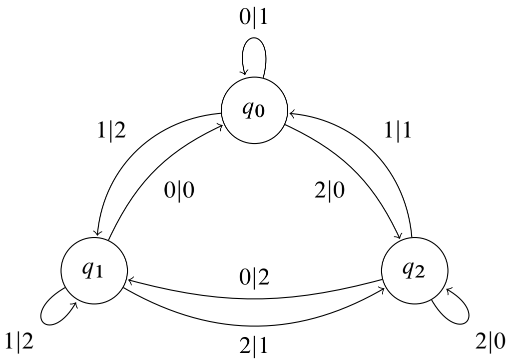

Figure 4. An element of order 3 in $\mathcal{H}_n$.

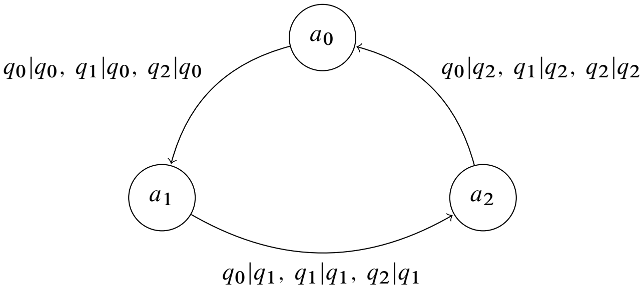

Figure 5. The level 3 dual of C.

Since the original transducer C has order 3, we can see from its level 3 dual in Figure 5 that the states in the core will be cyclic rotations of (q_0, q_1, q_2) all of which are locally identity.

We illustrate another example below, but now with an element of order 2.

Example 4.12. Consider the automaton of order 2 given in Figure 6. This automaton is synchronizing on the first level. We give the dual in Figure 7.

It is easy to see that the states 0 and 1 are ω-equivalent, and can be identified to a single node that produces q_0 for all inputs. The states in the core of the square of the original automaton will be (q_0, q_0) and (q_1, q_1).

Corollary 4.13. *Let $A \in \widetilde{\mathcal{P}}_n$ be synchronizing at level k. Then the semigroup $\langle A \rangle$ generated by A is finite if and only if the automaton semigroup generated by A is finite if and only if the subsemigroup of $\widetilde{\mathcal{P}}_n$ generated by A is finite.*

Proof. Throughout this proof we write A^i, $i \in \mathbb{N}$, for the minimal representative of A^i.

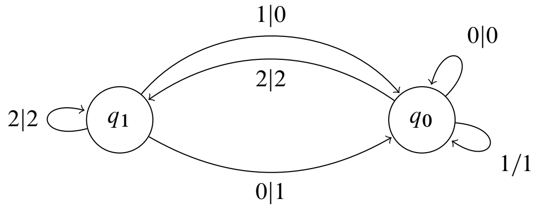

Figure 6. An element of order 2.

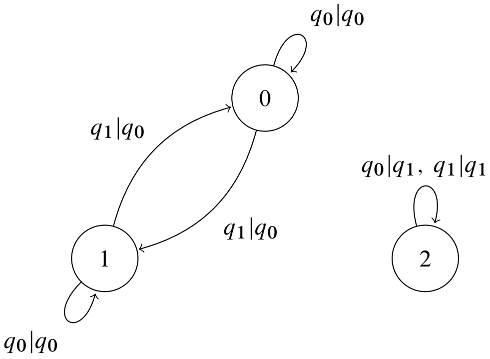

Figure 7. The level 1 dual.

First suppose that the automaton semigroup generated by A is finite. By definition, then there is a maximum $N \in \mathbb{N}$ such that any element of the automaton semigroup generated by A is ω-equivalent to some initial transducer A^i_q for $1 \le i \le N$ and $q \in Q_{A^i}$. Let $j \in \mathbb{N}$ be minimal such that A^i is synchronizing at level j for all $1 \le i \le j$.

Let $m \ge \mathbb{N}$ and consider A^m. Let $q \in Q_{A^m}$, then since A^m_q is an element of the automaton semigroup generated by A, it is ω-equivalent to A^i_p for some $1 \le i \le N$ and $p \in Q_{A^i}$. However, since $\mathrm{Core}(A^m)$ and $\mathrm{Core}(A^i)$ are connected, then it follows that $\mathrm{Core}(A^i)$ is ω-equivalent to $\mathrm{Core}(A^m)$. Moreover, since for any $\gamma \in X_n^j$, $\pi_{A^i}(\gamma, p)$ is a state of $\mathrm{Core}(A^m)$, it follows, by minimality of A^m, that for any $\gamma \in X_n^j$, $\pi_{A^m}(\gamma, q)$ is a state in $\mathrm{Core}(A^m) =_\omega \mathrm{Core}(A^i)$.

By definition of j, it follows that any state of A^m is a distance at most j from $\mathrm{Core}(A^m)$. Now as m was arbitrarily chosen, we see that for any $m \in \mathbb{N}$, $\mathrm{Core}(A^m) =_\omega \mathrm{Core}(A^i)$ for some $1 \le i \le N$; moreover, every state of A^m is a distance at most j from $\mathrm{Core}(A^m)$. Thus, as there are only finitely many transformations of the set X_n^j, then the semigroup generated by A is finite.

It is clear that if the semigroup generated by A is finite, the automaton semigroup generated by A is finite. This follows since an element of the automaton semigroup generated by A is ω-equivalent to A^i_q for some $i \in \mathbb{N}$ and some $q \in Q_{A^i}$.

If the semigroup generated by A is finite, then it follows that there is an $N \in \mathbb{N}$ such that $\mathrm{Core}(A^j) = \mathrm{Core}(A^i)$ for some $1 \leq i \leq N$ and so the subsemigroup of $\widetilde{\mathscr{P}}_n$ generated by A is finite.

Suppose the subsemigroup of $\widetilde{\mathscr{P}}_n$ generated by A is finite. It follows that the set $\{\mathrm{Core}(A^i) \mid i \in \mathbb{N}\}$ is finite (up-to-ω-equivalence). Now Lemma 4.9 implies there is a $j \in \mathbb{N}$ such that $A_j^{\vee}$ has infinite j splitting length. From this one deduces that for any $i \in \mathbb{N}$, A^i is synchronizing at level at most j. The result now follows as above since, for any $i \in \mathbb{N}$, any state of A^i is at a distance of at most j from its core, and the set of cores is finite. ■

For a transducer of finite order, A, as above, we have the following result about the semigroup $\langle A^{\vee} \rangle$.

Theorem 4.14. *Let $A \in \widetilde{\mathscr{P}}_n$ be synchronizing at level k and suppose that the semigroup $\langle A \rangle$ is finite and let j be the maximum of the minimal synchronizing levels of the elements of $\langle A \rangle$. Then $A_j^{\vee} = (A^{\vee})^j$ is a zero in $\langle A^{\vee} \rangle$, the semigroup generated by $A^{\vee}$.*

Proof. It suffices to show that $A_j^{\vee}$ is a right zero of the semigroup since the semigroup $\langle A^{\vee} \rangle$ is commutative.

Our strategy is to show that for any state Γ of $A_j^{\vee}$ and any state x of $A^{\vee}$, that the state Γx of $A_{j+1}^{\vee}$ is ω-equivalent to a state of $A_j^{\vee}$. To this end let $\Gamma \in X_n^j$ be a word of length $j+1$. By Lemma 4.6 and Remark 4.10, there is a pair of words W_{1,Γ_1} and W_{2,Γ_1} such that any input read from Γ_1 has output of the form $W_1(W_2)^l v$ for $l \in \mathbb{N}$ and v a prefix of W_2, otherwise the output is a prefix of W_1. Let $\gamma \in X_n^j$ be the length j suffix of Γ_1. Observe that the outputs of the state γ of $A_j^{\vee}$ must also all be of the form $W_1(W_2)^l v$ for $l \in \mathbb{N}$ and v a prefix of W_2, otherwise the output is a prefix of W_1 and the output depends only on the length of the input word. Therefore, we must have that Γ and γ are ω-equivalent.

On the other hand, given a word $\gamma \in X_n^j$, then a similar argument demonstrates that the state $x\gamma$ for any $x \in X_n^j$ is ω-equivalent to γ. ■

The next lemma observes that Lemma 4.9 gives a complete characterization of elements of $\mathscr{H}_n$ with finite order.

Proposition 4.15. *Let A be an element of $\widetilde{\mathscr{P}}_n$ and suppose A is synchronizing at level k. Then the semigroup $\langle A \rangle$ generated by A is finite if and only if there is some $m \in \mathbb{N}$ such that the following holds:*

(i) *$A_m^{\vee}$ is a zero of the semigroup $\langle A^{\vee} \rangle$.*

(ii) *$A_m^{\vee}$ is ω-equivalent to a transducer with r components D_i $1 \leq i \leq r$. For each component D_i there is a fixed pair of words $w_{i,1}, w_{i,2}$ (in the states of A) associated to D_i such that whenever we read any input from a state in the D_i, the output is of the form $w_{i,1} w_{i,2}^l v$ for $l \in \mathbb{N}$ and v a prefix of $w_{i,2}$ or has the form u for some prefix u of $w_{i,1}$. Moreover, the output depends only on the state in the component D_i from which the input is processed.*

Proof. $\Rightarrow$: This direction follows from Lemma 4.9, Remark 4.10 and Theorem 4.14.

$\Leftarrow$: Assume that $A_m^\vee$ has r components and let $w_{i,1}$ and $w_{i,2}$ $1 \le i \le r$ be the pair of words in the states of A associated with each component D_i. To see that the semigroup $\langle A \rangle$ is finite, observe that the assumptions that $A_m^\vee$ is a zero of the semigroup $\langle A^\vee \rangle$ and that the output depends only on which state in the component D_i of $A_m^\vee$ we begin processing inputs means that A^l is synchronizing at level m for all $l \in \mathbb{N}$. Therefore, the set $\{A^l \mid l \in \mathbb{N}\}$ is finite, since there are only finitely many automata which are synchronizing at level l. ■

Remark 4.16. In the case where A is an element of $\mathcal{H}_n$ in the above proposition, then each component D_i is a strongly connected component. In particular, one of $w_{i,1}$ or $w_{i,2}$ will be the empty string for any component D_i.

Remark 4.17. Given a transducer $A \in \widetilde{\mathcal{P}}_n$ which is synchronizing at level k, by increasing the alphabet size to n^k, one can identify A with an element of A' of $\widetilde{\mathcal{P}}_{n^k}$. It is an easy exercise to verify that $(A_k^\vee)^\vee = A'$. Therefore $A'^\vee = A_k^\vee$, hence by Proposition 4.15, A' has finite order if and only if A has finite order. In order to simplify calculations, we shall often assume that our transducer if bi-synchronizing at level 1.

4.3. The graph of bad pairs

On the surface Lemma 4.15 appears to reduce the order problem in $\mathcal{H}_n$ to an equivalent problem of deciding whether the semigroup generated by the dual has a zero. However, using the above lemmas (in particular, Lemma 4.9), we are able to attach a graph to elements of $\widetilde{\mathcal{P}}_n$ which, as we demonstrate, is useful for determining when an element of $\mathcal{H}_n$ has finite order. This graph is also our key combinatorial tool for demonstrating that the automaton semigroup generated by an automaton in $\widetilde{\mathcal{H}}_n$ is either finite or has exponential growth rate. Once more we work in $\widetilde{\mathcal{P}}_n$ restricting to $\mathcal{H}_n$ as required.

Definition 4.18 (Bad pairs). Let $A \in \widetilde{\mathcal{P}}_n$ be a transducer which is synchronizing at level k, and let $r \ge k$. Let l be the minimal splitting length of $A_r^\vee$. Let $\mathcal{B}_r$ be the set of tops of those pairs $((q_1, \ldots, q_m), (p_1, \ldots, p_m))$ of m tuples, $m \ge l$, which split $A_r^\vee$ and for which there is a split $((q_1, \ldots, q_m), (p_1, \ldots, p_m), \Gamma)$ such that the bottom of the split depends only on the top. Then we call $\mathcal{B}_r$ the set of *bad pairs* associated to $A_r^\vee$. Notice that if $B \in \mathcal{B}_r$ then $B \subset Q$ and $|B| = 2$. Furthermore, observe that by minimality of l, $\mathcal{B}_r$ contains the tops of all splits consisting of a pair of l tuples and a word in X_n^r. Let $B_r \subset \mathcal{B}_r$ be this subset. We call B_r the *minimal bad pairs* associated to $A_r^\vee$.

Definition 4.19 (Graph of bad pairs). For a transducer $A \in \widetilde{\mathcal{P}}_n$, and for r greater than or equal to the minimal synchronizing level, such that $A_r^\vee$ has minimal splitting length l, form a directed graph $G_r(A)$ associated to $A_r^\vee$ as follows:

(i) The vertex set of $G_r(A)$ is the set $\mathcal{B}_r$ of bad pairs.

(ii) Two elements $\{x_1, x_2\}$, and $\{y_1, y_2\}$ of $\mathcal{B}_r$ are connected by an arrow going from $\{x_1, x_2\}$ into $\{y_1, y_2\}$, if there are pairs $(T_1, T_2) \in Q^m \times Q^m$, for some $m \geq l$, splitting $A_r^\vee$, with top $\{x_1, x_2\}$ and bottom $\{y_1, y_2\}$ and such that the bottom depends only on the top.

We call $G_r(A)$ the *graph of bad pairs* associated to $A_r^\vee$. By Remark 4.4, this definition makes sense.

The graph $G_r(A)$ possesses an interesting subgraph $\overline{G}_r(A)$ whose vertices are elements of B_r, the set of minimal bad pairs, with an edge from $\{x_1, x_2\}$ to $\{y_1, y_2\}$, $\{x_1, x_2\}, \{y_1, y_2\} \in B_r$ if there is a split of minimal length l, with top $\{x_1, x_2\}$ and bottom $\{y_1, y_2\}$. We call $\overline{G}_r(A)$ the *minimal graph of bad pairs.*

Remark 4.20. There is a much larger graph containing $G_r(A)$ which we do not consider here. The vertex set of this graph is all subsets of Q_A of size 2, and there is a directed edge between two such vertices if there is a split of $A_r^\vee$ with top the initial vertex and bottom the terminal vertex. The reason we do not consider this larger graph is due to the existence of elements of finite order whose dual at the synchronizing level splits (see Example 6.8). This means that the larger graphs contain information which is not carried by powers of the transducers.

The following results link graph theoretic properties of $G_r(A)$ and the order of A when $A \in \widetilde{\mathcal{H}}_n$. All of these results apply also to the minimal graph of bad pairs $\overline{G}_r(A)$; indeed, typically the information given by $G_r(A)$ can already be seen in $\overline{G}_r(A)$.

Lemma 4.21. *Let $A \in \widetilde{\mathcal{H}}_n$ be a transducer, and suppose that k is the minimal synchronizing level of A. Let $r \geq k$ and let $G_r(A)$ be the graph of bad pairs associated to $A_r^\vee$. If $G_r(A)$ is non-empty and contains a circuit, that is, there is a vertex which we can leave and return to, then A has infinite order.*

Proof. Let l be the minimal splitting length of $A_r^\vee$. The proof will proceed as follows: For every $m \geq 1$, we construct a word, $w(rm)$, of length rm, such that there are two distinct elements of Q^{ml+1} which have different outputs when processed through $w(rm)$ (in $A_{rm}^\vee$). This will contradict A having finite order, since by Lemma 4.9, if A has finite order, then there will be a j such that any two sequences of states of any length will have the same output when processed through a word of length rj (see Remark 4.10).

Since $G_r(A)$ is non-empty, and has a circuit, there exists a circuit: $\{x_1, y_1\} \to \{x_2, y_2\} \to \cdots \to \{x_j, y_j\} \to \{x_1, y_1\}$. For $i \in \mathbb{N}$ let $A^i = \langle X_n, Q^i, \pi_i, \lambda_i \rangle$.

Now for $m = 1$, since $\{x_1, y_1\}$ is a vertex of $G_r(A)$ with at least one incoming edge, there is a state Γ_1 of $A_r^\vee$, and a pair $(S_1, T_1) \in Q^{l_1} \times Q^{l_1}$ (for $l_1 \geq l$) such that (S_1, T_1, Γ_1) is a split of $A_r^\vee$ with bottom $\{x_1, y_1\}$ and such that the bottom depends only on the top. We may assume that the top of this split is $\{x_j, y_j\}$. Therefore for any $p \in Q$, the output of $S_1 p$ when processed through Γ_1 is not equal to the output of $T_1 p$ when processed through Γ_1. However, the output of S_1 and T_1 is equal when processed through Γ_1 since

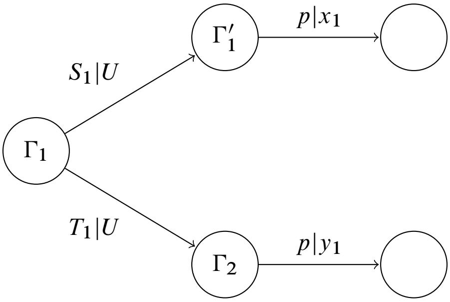

Figure 8. Stage 1 of construction.

the bottom of the split depends only on its top. Hence, Figure 8 is valid, for appropriate $U \in Q^{l_1}$.

Now since $\{x_1, y_1\}$ is connected to $\{x_2, y_2\}$ there is a word $\Lambda_1 \in X_n^r$ such that there is a pair $(S_2, T_2) \in Q^{l_2} \times Q^{l_2}$ and (S_2, T_2, Λ_1) is a split with top $\{x_1, y_1\}$ and bottom $\{x_2, y_2\}$ such that the bottom depends only on the top. Let Λ'_1 be the word of length r such that $\lambda_{l_1}(\Lambda'_1, U) = \Lambda_1$ (such a word exists since A is invertible). Since the bottom of the split depends only on the top, there is $V \in Q^{l_2}$ such that for any $P \in Q^{l_2-1}$ we have $\pi_{l_2}(\Lambda_1, (x_1, P)) = V = \pi_{l_2}(\Lambda_1, (y_1, P))$. Let V' be the state of A^{l_1} such that $\pi_{l_1}(\Lambda'_1, U) = V'$. Then let $w(k2) = \Gamma_1\Lambda'_1$. Now by Remark 4.4, we have the following transition for some $P \in Q^{l_2}$ and $p \in Q$. Now we can iterate the above process, since $\{x_2, y_2\}$ is a vertex of $G_r(A)$ with an outgoing edge to another vertex of $G_r(A)$, and the output of S_1P and T_1P when processed through $\Gamma_1\Lambda'_1$ is the same.

Label the levels of Figure 9 by $1, 2, 3, 4$. We grow our word from bottom to top. Let Δ_1 be the word such that there is a pair (S_3, T_3) in $Q^{l_3} \times Q^{l_3}$ and (S_3, T_3, Δ_1) is a split of $A_r^{\vee}$ with top $\{x_2, y_2\}$ and bottom $\{x_3, y_3\}$ such that the bottom depends only on the top. Attach Δ_1 to right end of both words representing the states of level 3. There is a word Δ'_1 such that $\lambda_{l_1+l_2}(\Delta'_1, V'V) = \Delta_1$. Hence, $W(r3) = \Gamma_1\Lambda'_1\Delta'_1$. Moreover, since the bottom of the split (S_3, T_3, Δ_1) depends only on its top, we see that $A_{3r}^{\vee}$ has a split of

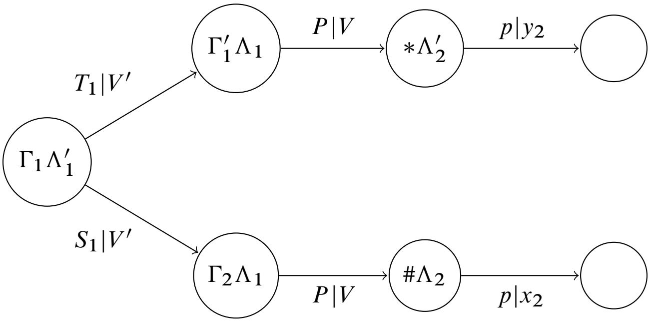

Figure 9. Stage 2 of construction.

length $l_1 + l_2 + l_3$ with bottom $\{x_3, y_3\}$ whose bottom depends only on its top. We repeat the above process and in this way construct the words $w(rm)$ demonstrating that $A^{\vee}_{rm}$ has a split with bottom $\{x_i, y_i\}$ where $1 \leq i \leq j$, and $i \equiv m \mod j$ such that the bottom depends only on the top. ■

Remark 4.22. The proof above actually demonstrates that if $A \in \widetilde{\mathcal{H}}_n$ is such that for some r bigger than or equal to its synchronizing level, the graph $G_r(A)$ of bad pairs has a circuit, then there is an $r' > r$, depending on the length of the circuit in $G_r(A)$, such that $G_{r'}(A)$ has a loop.

4.4. Finding circuits in the graph of bad pairs

We are interested in finding circuits (and so loops) in the graph of bad pairs. The following give some handle on how to do this.

Let $A \in \widetilde{\mathcal{H}}_n$, and let $r \in \mathbb{N}$ be greater than or equal to the minimal synchronizing level of A. Let $l \in \mathbb{N}$ be the minimal splitting length of $A^{\vee}_r$.

To each state Γ of $A^{\vee}_r$, we associate a transformation σ_Γ of the set Q_A of states of A as follows. For each state $q \in Q_A$ and for any $l-1$ tuple $S \in Q^{l-1}$, there is a unique state $p \in Q_A$, such that if Δ is the output when Γ is processed through qS in A^l, then $\Delta \in W_p$. Since l is the minimal splitting length of $A^{\vee}_r$, p is independent of which $l-1$ tuple S we chose. Therefore, define σ_Γ such that $q \overset{\sigma_\Gamma}{\mapsto} p$.

For $j \in \mathbb{N}$ let $\mathfrak{S}_{r,j}$ be the set of all products of length j of elements of the set $\{\sigma_\Gamma \mid \Gamma \in X_n^r\}$. We have the following result.

Proposition 4.23. *$A \in \widetilde{\mathcal{H}}_n$, and let $r \in \mathbb{N}$ be greater than or equal to the minimal synchronizing level of A. Let $l \in \mathbb{N}$ be the minimal splitting length of $A^{\vee}_r$. Then $\mathfrak{S}_{r,|Q_A|^2+1}$ contains a transformation of Q_A which is not a right zero if and only if the graph $\overline{G}_r(A)$ of minimal bad pairs contains a circuit.*

Proof. Let A, r and l be as in the statement of the proposition.

Now $\mathfrak{S}_{r,|Q_A|^2+1}$ contains a transformation of Q_A which is not a right zero if and only if there is a product $\sigma_{\Gamma_1}\sigma_{\Gamma_2}\cdots\sigma_{\Gamma_{|Q_A|^2+1}}$, for $\Gamma_i \in X_n^r$, $1 \leq i \leq |Q_A|^2 + 1$, whose image set has size at least 2. This occurs if and only if there are $p_0, q_0 \in Q_A$ which map to distinct elements under $\sigma_{\Gamma_1}\sigma_{\Gamma_2}\cdots\sigma_{\Gamma_{|Q_A|^2+1}}$. Let $p_i := (p_0)\sigma_{\Gamma_1}\sigma_{\Gamma_2}\cdots\sigma_{\Gamma_i}$ and $q_i := (q_0)\sigma_{\Gamma_1}\sigma_{\Gamma_2}\cdots\sigma_{\Gamma_i}$ for $1 \leq i \leq |Q_A|^2 + 1$. Notice that $p_i \neq q_i$ since p_0 and q_0 have distinct images under $\sigma_{\Gamma_1}\sigma_{\Gamma_2}\cdots\sigma_{\Gamma_{|Q_A|^2+1}}$.

By the pigeonhole principle, there exists $1 \leq i, j \leq |Q_A|^2 + 1$ such that $\{p_i, q_i\} := \{p_j, q_j\}$.

This implies that in the graph $\overline{G}_r(A)$ of minimal bad pairs we have: $\{p_i, q_i\} \to \{p_{i+1}, q_{i+1}\} \to \cdots \to \{p_j, q_j\} \to \{p_i, q_i\}$. This follows by definition of the σ_Δ, $\Delta \in X_n^r$ and of the graph $\overline{G}_r(A)$.

Now suppose the graph $\overline{G}_r(A)$ contains a circuit. Let $j \in \mathbb{N}$ be the length of the circuit, and let $\{p_i, q_i\}1 \leq i \leq j$ be the vertices on the circuit.

Let $1 \leq i < j$ be arbitrary. Now an edge $\{p_i, q_i\} \to \{p_{i+1}, q_{i+1}\}$ corresponds to the existence of some $\Gamma_i \in X_n^r$ and $S_i, T_i \in Q_A^{l-1}$ such that $(p_i S_i, q_i T_i, \Gamma_i)$ is a split of $A_r^{\vee}$ with bottom $\{p_{i+1}, q_{i+1}\}$. It then follows that the product $\sigma_{\Gamma_1} \cdots \sigma_{\Gamma_j}$ maps $\{p_1, q_1\} \to \{p_1, q_1\}$. This means that $\mathfrak{S}_{r,|Q_A|^2+1}$ contains a transformation of Q_A which is not a right zero. ■

Remark 4.24. The above now implies that if $A \in \widetilde{\mathcal{H}}_n$ has infinite order, but none of its graph of minimal bad pairs, $\overline{G}_r(A)$, for $r \in \mathbb{N}$ greater than or equal to the minimal synchronizing length, has a loop, then $\mathfrak{S}_{r,|Q_A|^2+1}$ consists entirely of right zeroes.

The following algebraic condition also implies that the graph of bad pairs has a loop.

We have already seen that given an element $A \in \widetilde{\mathcal{P}}_n$ we can associate a transformation $\overline{A}_j$ of the set X_n^j to A. We shall now introduce a new transformation, which is defined only for elements of $\widetilde{\mathcal{H}}_n$.

Definition 4.25. Let $H \in \widetilde{\mathcal{H}}_n$, and let $j \in \mathbb{N}$, we shall define a transformation $\underline{H}_j$ of X_n^j by

$$\Gamma \mapsto (\Gamma)q_{\Gamma}^{-1},$$

where q_{Γ} is the unique state of H forced by Γ, and $(\Gamma)q_{\Gamma}^{-1}$ is the unique element of X_n^j such that $\lambda_H((\Gamma)q_{\Gamma}^{-1}, q_{\Gamma}) = \Gamma$. If j is zero, then $\underline{H}_j$ is simply the identity map on the set containing the empty word.

Remark 4.26. Given an element $H \in \widetilde{\mathcal{H}}_n$ and a $j \in \mathbb{N}$ such that $j > 1$, then $\underline{H}_j$ is not injective in general. One can check that for the transducer B of Figure 11, $\underline{B}_1$ is not injective. The map ϕ_j from $\widetilde{\mathcal{H}}_n$ to the full transformation semigroup on X_n^j which maps H to $\underline{H}_j$ is not a homomorphism.

Lemma 4.27. *Let $H \in \widetilde{\mathcal{H}}_n$ and let $j \in \mathbb{N}$ be greater than or equal to the minimal synchronizing level of H, then $\underline{H}_j$ is not injective if and only if $H_j^{\vee}$ has a split of length 1 such that the top and bottom of the split are equal.*

Proof. ($\Rightarrow$) Suppose that, for $j \in \mathbb{N}$ and H as in the statement of the lemma, $\underline{H}_j$ is not injective. This means that there are two distinct elements Γ and Δ of X_n^j such that $(\Gamma)\underline{H}_j = (\Delta)\underline{H}_j$. Let $\Lambda := (\Gamma)\underline{H}_j$. Let q_{Γ} and q_{Δ} be the states of H forced by Γ and Δ, respectively. Then by definition of $\underline{H}_j$, we have $\lambda(\Lambda, q_{\Gamma}) = \Gamma$ and $\lambda(\Lambda, q_{\Delta}) = \Delta$.

Observe that as consequence it must be the case that $q_{\Gamma} \neq q_{\Lambda}$. Therefore, it follows that we have the following split in $H_j^{\vee}$ (Figure 10).

($\Leftarrow$) Suppose that $A_j^{\vee}$ has a split of length 1 such that the bottom of the split is equal to its top. This means that there exist Γ, Δ and Λ in X_n^j so that if q_{Δ} is the state of H forced by Δ and q_{Γ} is the state of H forced by Γ, then we have $\lambda(\Lambda, q_{\Delta}) = \Delta$ and $\lambda(\Lambda, q_{\Gamma}) = q_{\Gamma}$. This now means that $\Lambda = (\Gamma)\underline{H}_j = (\Delta)\underline{H}_j$ and $\underline{H}_j$ is not injective. ■

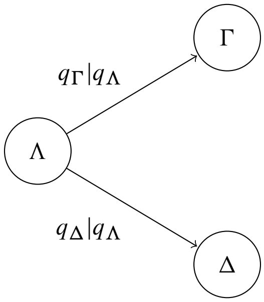

Figure 10. Split in $A_j^\vee$ with top equal to bottom.

5. The automaton group generated by a synchronizing transducer is either finite or has exponential growth

In this section, we prove Theorem 1.1 by linking existence of circuits in the graph of bad pairs to the existence of free subsemigroups in the automaton semigroup generated by an element of $\widetilde{\mathcal{H}}_n$. We begin with the following proposition.

Proposition 5.1. *Let $A \in \widetilde{\mathcal{H}}_n$ be an element of infinite order. Then either there is a $j \in \mathbb{N}$ such that the minimal graph of bad pairs $\overline{G}_j(A)$ has a loop or the automaton semigroup generated by A contains a free semigroup of rank at least* 2*.*

Proof. We may assume, by changing the alphabet size that A is bi-synchronizing at level 1.

Since A has infinite order then for each $j \in \mathbb{N}$, $(A^\vee)^j$ splits. Fix $j \in \mathbb{N}$ and let Top_j be the set of pairs of states $\{p_1, p_2\}$ such that there exist $(p_1, s_1, \ldots, s_r)$ and $(p_2, s'_1, \ldots, s'_r)$ which split $(A^\vee)^j$ where r is the minimal splitting length of $(A^\vee)^j$. By definition, Top_j is the set of tops of minimal length splits of $A_j^\vee$. Analogously, for fixed $j \in \mathbb{N}$ let Bottom_j be the set of bottoms of minimal length splits. That is, Bottom_j consists of sets $\{t_1, t_2\}$ such that there exists a split of minimal length $(\Gamma, (s_1, \ldots, s_r), (s'_1, \ldots, s'_r))$ of $A_j^\vee$ with bottom $\{t_1, t_2\}$.

Since A has finitely many states, there exist an infinite subset $\mathcal{J}' \subset \mathbb{N}$ and a fixed set of pairs $\{t_1, t_2\}$ such that $\{t_1, t_2\} \in \mathrm{Bottom}_j$ for all $j \in \mathcal{J}'$. Now consider the set of tops of all splits of $A_j^\vee$, $j \in \mathcal{J}'$ with bottom $\{t_1, t_2\}$. Since $|\mathcal{J}'| = \infty$ and $\{t_1, t_2\} \in \mathrm{Bottom}_j$ for all $j \in \mathcal{J}'$, there exist an infinite subset $\mathcal{J} \subset \mathcal{J}'$ and a fixed set of pairs $\{p_1, p_2\}$ such that $\{p_1, p_2\} \in \mathrm{Top}_j$ for all $j \in \mathcal{J}$ and there exist splits of $A_j^\vee$ with top $\{p_1, p_2\}$ and bottom $\{t_1, t_2\}$ for all $j \in \mathcal{J}$.

If $\{p_1, p_2\} = \{t_1, t_2\}$ we are done, since $\overline{G}_j(A)$ has a loop for any $j \in \mathcal{J}$. Therefore, assume that this is not the case. Under this assumption, we have two cases to consider.

Case 1: Suppose that there are $i, i' \in \mathbb{N}$, $i, i' \geq 1$ and $S_1, S'_1 \in Q^i$ and $S_2, S'_2 \in Q^{i'}$ such that (p_1, S_1) is ω-equivalent to (t_1, S'_1) and (p_2, S_2) is ω-equivalent to (t_2, S'_2). We may assume that $i = i'$ by padding out one of the pairs (p_i, S_i) and (t_i, S'_i), $i = 1, 2$.

Let $j \in \mathcal{J}$ be such that $j > i + 1$. Consider $(A^{\vee})^j$, it has minimal splitting length, r, greater than or equal to j. Now making use of the construction of the fixed set of pairs $\{p_1, p_2\}$, there exist $(p_1, s_1, \ldots, s_{r-1})$, $(p_2, s'_1, \ldots, s'_{r-1})$ elements of Q^r and Γ a state of $(A^{\vee})^j$ such that $((p_1, s_1, \ldots, s_{r-1}), (p_2, s'_1, \ldots, s'_{r-1}), \Gamma)$ is a split of $(A^{\vee})^j$ with bottom $\{t_1, t_2\}$. Hence, by minimality of r, it now follows that $((p_1, S_1, s_{i+1}, \ldots, s_{r-1}), (p_2, S_2, s'_{i+2}, \ldots, s'_{r-1}), \Gamma)$ is also a split of $(A^{\vee})^j$ with bottom (t_1, t_2). However, this now implies, again by minimality of r and since (t_1, S'_1) and (t_2, S'_2) are ω-equivalent to (p_1, S_1) and (p_2, S_2), respectively, that $((t_1, S'_1, s_{i+1}, \ldots, s_{r-1}), (t_2, S_2, s'_{i+2}, \ldots, s'_{r-1}), \Gamma)$ is also a split of $(A^{\vee})^j$ with bottom (t_1, t_2). Therefore, $(A^{\vee})^j$ has a loop.

Case 2: We assume that Case 1 does not hold, that is for all $i, i' \in \mathbb{N}$ there does not exist a choice of $S_1, S'_1 \in Q^i$ and $S_2, S'_2 \in Q^{i'}$ such that $(p_1, S1)$ is ω-equivalent to (t_1, S'_1) and (p_2, S_2) is ω-equivalent to (t_2, S'_2). We may also assume that none of the graph of bad pairs $\overline{G}_j(A)$ has a loop for any j greater than the minimal synchronizing level of A, since otherwise we are done.

The latter assumption implies that $\mathfrak{S}_{j,|Q_A|^2+1}$ consists of transformations with image size 1 by Remark 4.24. However, since $(A^{\vee})^j$ splits for every $j \in \mathbb{N}$, then for j larger than the minimal synchronizing level, there are elements $\Gamma \in X_n^j$, such that σ_Γ has image size at least 2. Fix an arbitrary such Γ. Since $\sigma_\Gamma^{|Q_A|^2+1}$ has image size 1, then there is a state $p \in Q_A$ such that $(p)\sigma_\Gamma = p$. Therefore, there is a pair of states $p_1, p_2 \in Q_A$ such that there is a split of $(A^{\vee})^j$ with top $\{p_1, p_2\}$, and bottom $\{t_1, p_2\}$.

The above argument now implies that we may choose $\{p_1, p_2\}$ and $\{t_1, t_2\}$ above so that $p_2 = t_2$, and there exists $\Gamma \in X_n^j$, $j \in \mathcal{J}$, such that $(p_1)\sigma_\Gamma = t_1$ and $(p_2)\sigma_\Gamma = p_2$.

Now since Case 1 does not hold, and $p_2 = t_2$, therefore it follows that for any $m \in \mathbb{N}\backslash\{0\}$ and any $S_1, S_2 \in Q^m$ that $p_1 S_1$ is not ω-equivalent to $t_1 S_2$. We now argue that the subsemigroup $\langle p_1, t_1 \rangle$ of $\mathcal{S}(A)$ (the automaton semigroup generated by A) is free.

Now as A has infinite order, it follows that $\mathrm{Core}(A^i) \neq_\omega \mathrm{Core}(A^j)$ for any $i \neq j \in \mathbb{N}$. Therefore, given two words v and w in $\langle p_1, t_1 \rangle$ such that v and w are ω-equivalent, it follows that $|v| = |w|$.

Therefore, consider the case of words $v, w \in \langle p_1, t_1 \rangle$ such that $|v| = |w|$. Suppose $v = v_1 \cdots v_l$ and $w = w_1 \cdots w_l$, where $|v| = |w| = l$. Let $1 \leq i \leq l$ be the minimal index so that $v_i \neq w_i$. We may assume that $v_i = p_1$ and $w_i = t$. Therefore, $v = v_1 \cdots v_i p_1 v_{i+2} \cdots v_l$ and $w = v_1 \cdots v_i t_1 w_{i+2} \cdots w_l$. Hence, v is ω-equivalent to w if and only if $p_1 v_{i+2} \cdots v_r$ is ω-equivalent to $t_1 w_{i+2} \cdots w_r$. However, by assumption this is not the case. Therefore, given any two distinct words in $\{p_1, t_1\}^*$, they represent distinct automorphisms of the n-ary rooted tree hence we conclude that $\langle p_1, t_1 \rangle$ is a free semigroup. ■

Corollary 5.2. *Let $A \in \widetilde{\mathcal{H}}_n$ be an element of infinite order. Then either there is a $j \in \mathbb{N}$ such that the graph of bad pairs $G_j(A)$ has a loop, otherwise the automaton semigroup generated by A contains a free semigroup of rank at least* 2.

In the proposition below, we introduce a condition on the graph of bad pairs $G_j(A)$ which guarantees the existence of free subsemigroups of certain rank in the automaton semigroup generated by an element of $\widetilde{\mathcal{H}}_n$. This condition at first glance appears to be very strong; however, in Subsection 5.1, we consider a large class of examples which satisfy the hypothesis of the proposition. In particular, whenever the graph of bad pair has a loop, the hypothesis is immediately satisfied.

Proposition 5.3. *Let $A \in \widetilde{\mathcal{H}}_n$ and suppose that A is synchronizing at level k and is minimal. Let $G_j(A)$ be the graph of bad pairs for some $j \geq k \in \mathbb{N}$. Suppose there is a subset $\mathcal{S}$ of the set of states of A, such that the following things hold:*

(i) $|\mathcal{S}| \geq 2$.

(ii) *The set $\mathcal{S}(2)$ of two element subsets of $\mathcal{S}$ is a subset of the vertices of $G_j(A)$.*

(iii) *For each element of $\mathcal{S}(2)$ there is a vertex accessible from it which belongs to a circuit.*

Then the automaton semigroup generated by A contains a free semigroup of rank at least $|\mathcal{S}|$. In particular, the automaton semigroup generated by A has exponential growth.

Proof. First observe that since $G_j(A)$ is assumed to have a circuit, by Lemma 4.21 A has infinite order. Now let U and V be distinct non-empty words in $\mathcal{S}^*$, if $|U| \neq |V|$ then since A has infinite order A_U cannot be ω-equivalent to A_V by Lemma 3.2. Therefore, we may assume that $|U| = |V|$.

Let $U = u_1 \cdots u_r$ and $V = v_1 \cdots v_r$ and let $1 \leq i \leq r$ be the minimal index so that $u_i \neq v_i$. If $i = r$ then we are done, since $U = Sq$ and $V = Sp$ (or vice versa) for some $S \in \mathcal{S}^{r-1}$ and $q \neq p \in \mathcal{S}$.

Therefore, assume that $i \leq r$ and that $U = SqT_1$ and $V = SpT_2$ for $S \in \mathcal{S}^{i-1}$, $T_1, T_2 \in \mathcal{S}^{r-i}$ and $q, p \in \mathcal{S}$. If U and V are not ω-equivalent, we are done. Therefore, assume that U and V are ω-equivalent.

Since $p, q \in \mathcal{S}(2)$, there is a path in $G_j(A)$ from $\{p, q\}$ to a vertex which belongs to a circuit. Therefore, we may assume that there is path in the graph $G_j(A)$ as follows:

$$\{p, q\} := \{p_0, q_0\} \to \{p_1, q_1\} \to \cdots \to \{q_l, p_l\} \to \{q_{l+1}, p_{l+1}\},$$

where for each $\{p_a, q_a\}$, $0 \leq a \leq l$ there is a split of length m_a with top $\{p_a, q_a\}$ such that the bottom depends only on the top, and the bottom is $\{p_{a+1}, q_{a+1}\}$ and $\{p_{l+1}, q_{l+1}\}$ is a vertex on a circuit in $G_j(A)$. Notice that $m_a \geq 1$ for all $1 \leq a \leq l$. Therefore, by travelling along this circuit in $G_j(A)$ as long as required, we may also assume that $m_0 + m_1 + \cdots + m_l + 1 \geq r - i + 1$.

By appending a common suffix to U and V, thus preserving ω-equivalence, if necessary we may further assume that $r - i + 1 = |qT_1| = |pT_2|$ is equal to $m_1 + m_1 \cdots + m_l + 1$. Redefining T_1 and T_2 we assume that $U = SqT_1t_1$ and $V = SpT_2t_2$ where $|qT_1| = |pT_2| = m_0 + m_1 \cdots + m_l + 1$ and t_1 and t_2 are possibly distinct elements of Q_A. Since $|qT_1| = |pT_2| = m_0 + m_1 \cdots + m_l + 1$, write $qT_1 = R_1 R_2 \cdots R_l$ and

$pT_2 = P_1P_2 \cdots P_l$ where $R_a, P_a \in Q_A^{m_a}$ for $1 \leq a \leq l$; moreover, R_1 begins with q and P_1 begins with p.

Since $\{q, p\}$ is a vertex of $G_j(A)$, there is a word Γ of length j belonging to a split of length m_0, whose bottom depends only on the top $\{q, p\}$, and with bottom $\{q_1, p_1\}$. Let Λ be the word such that the output when processed through A_S is Γ. Let S_Γ be the state of A^{m_0} such that $\pi_{A^{m_0}}(\Gamma, P_1) = \pi_{A^{m_0}}(\Gamma, Q_1)$. Such an S_Γ exists by definition of what it means for the bottom of a split to depend only on its top (see Definition 4.2). Then we have, on reading Λ through A_U and A_V, respectively, that we transition to the states $S'S_\Gamma Q'_1 Q'_3 \cdots Q'_l t'_1$ and $S'S_\Gamma P'_1 P'_3 \cdots P'_l t'_2$. Moreover, Q'_1 begins with q_1 and P'_1 begins with p_1. Once more $Q'_a \in Q_A^{m_a}$ for $1 \leq a \leq l$.

Since, by assumption, each $\{p_a, q_a\}$ for $1 \leq a \leq l$ has an outgoing edge corresponding to a split of length m_a whose bottom, $\{p_{a+1}, q_{a+1}\}$, depends only on its top, we can now repeat the argument of the above paragraph until the last letters of the final pair of state are a vertex of $G_r(A)$. Therefore, we are in the situation that $i = r$ at which point we conclude that the final pair of states is not ω-equivalent.

Now since U and V are ω-equivalent, then the final pair of states should also be ω-equivalent, since we read the same word from A_U and A_V into this pair. This yields the desired contradiction. Therefore, we conclude that A_U and A_V are not ω-equivalent.

The above now means that the semigroup $\langle A_p \mid p \in \mathcal{S} \rangle$ satisfies no relations and so is a free semigroup. In fact, this argument actually demonstrates that for any word $W \in Q^*$ (Q being the set of states of A), the semigroup $\langle A_{Wp} \mid p \in \mathcal{S} \rangle$ is a free semigroup. ■

Corollary 5.4. *Let $A \in \widetilde{\mathcal{H}}_n$ and suppose that A is synchronizing at level k and is minimal. As usual let $G_j(A)$ be the graph of bad pairs for some $j \geq k \in \mathbb{N}$. Suppose there is a subset $\mathcal{S}$ of set of states of A, such that the following things hold:*

(i) $|\mathcal{S}| \geq 2$.

(ii) *The set, $\mathcal{S}(2)$, of two element subsets of $\mathcal{S}$ is a subset of the vertices of $G_j(A)$.*

(iii) *For each element of $\mathcal{S}(2)$ there is a vertex accessible from it which belongs to a circuit.*

Then the automaton semigroup generated by A has exponential growth.

There are a few ways of extending the argument. One can also show that for a subset $\mathcal{S} \subset Q$ satisfying the conditions of the proposition, and for any vertex on a path from a vertex of G_k to a vertex accessible from $\mathcal{S}(2)$, then the pair of states making up this vertex generate a free semigroup. Notice that if the graph of bad pairs has a circuit, then the conditions of the proposition are satisfied.

Corollary 5.5. *Let $A \in \widetilde{\mathcal{H}}_n$ then the automaton semigroup generated by A is either finite or contains a free semigroup of rank at least* 2.

Proof. This follows from Propositions 5.1 and 5.3 and Corollary 4.13. ■

Theorem 5.6. *Let $A \in \widetilde{\mathcal{H}}_n$ then the automaton semigroup generated by A is either finite or has exponential growth.*

Proof. This follows from Corollary 5.5 and standard results in the literature on the growth rates of groups and semigroups and the fact that the automaton semigroup generated by A contains a free semigroup. ■

5.1. The growth rate of Cayley machines

In this section, we show that for a finite group G, the automaton semigroup generated by the Cayley machine, $\mathcal{C}(G)$, has growth rate, $|G|^n$. To this end, we begin by describing the construction of the Cayley machine.

Let M be a finite monoid (e.g., a finite group), then one can form the automaton $\mathcal{C}(M) := \langle M, M, \pi, \lambda \rangle$ called its *Cayley machine*, with input and output alphabet, M and state set M. The transition and rewrite function satisfy the following rules for $l, m \in M$:

(1) $\pi(l, m) := ml$

(2) $\lambda(l, m) := ml$.

In each case ml is the evaluation of the product of m and l in the monoid M. If M is a finite group G, then by Cayley's theorem no two states of $\mathcal{C}(G)$ are ω-equivalent, and the functions $\pi(\centerdot, m) : M \to M$ and $\lambda(\centerdot, m) : M \to M$ are bijections. Hence, $\mathcal{C}(G)$ is reduced and invertible. It is not hard to see that $(\mathcal{C}(G))^{-1}$ is synchronizing at level 1 (or is a reset automaton).

Remark 5.7. With a little work it can be shown that $(\mathcal{C}(G))^{-1}$ satisfies the conditions of Proposition 5.3 where, in this case, $\mathcal{S} = G$. This shows that the automaton semigroup generated by $\mathcal{C}(G)$ is free. Silva and Steinberg give a proof of this in [18].

We have the following lemma for synchronizing transducers.

Lemma 5.8. *Let $A = \langle X_n, Q, \pi, \lambda \rangle \in \widetilde{\mathcal{P}}_n$ be a transducer, which is synchronizing at level k. Furthermore, assume that for every $\Gamma \in X_n^k$ and for all states $q \in Q$, there is a state $p \in Q$ such that $\lambda(\Gamma, p) \in W_q$. Then under this condition, A has the property that for all $m \in \mathbb{N}$,* $\mathrm{Core}(A^m) = A^m$.

Proof. We may assume, by increasing the alphabet size, that A is synchronizing at level 1.

We proceed by induction on m. For $m = 1$ it holds that $A = \mathrm{Core}(A)$ by assumption that $A \in \widetilde{\mathcal{P}}_n$.

Assume $\mathrm{Core}(A^j) = A^j$ for all $j \leq m - 1$.

Consider $A^{m-1} = \mathrm{Core}(A^{m-1})$. Fix an arbitrary state $b_1 \cdots b_{m-1} \in A^{m-1}$. There is a state $a_1 \cdots a_{m-2}a_{m-1}$ and letters x and y in X_n such that

$$a_1 \cdots a_{m-2}a_{m-1} \xrightarrow{x|y} b_1 \cdots b_{m-1}.$$

Let y' be the output when x is read from $a_1 \cdots a_{m-2}$. Notice that since A is synchronizing at level 1 the state of A forced by y' must be b_{m-1}. By assumption, for every state $q \in Q$ there is a state p such that $\lambda(y', p) \in W_q$. Therefore, given an arbitrary $q \in Q$, by setting $a_{m-1} := p$ we may assume that $y \in W_q$; moreover, the inductive hypothesis guarantees that $a_1 \cdots a_{m-1}$ is a state of $\mathrm{Core}(A^{m-1})$.

Observe that A^{m-1} is synchronizing at level $m-1$ and so there is a word Δ of length $m-1$ labelling a loop based at $a_1 \cdots a_{m-2}a_{m-1}$. Let Λ be the output of this loop. Then reading Δx in $(A)^{m-1}$ from the state $a_1 \cdots a_{m-2}a_{m-1}$ the output is Λy. Now, the state of A forced by Λy is q; therefore, reading Δx through any state $a_1 \cdots a_{m-2}a_{m-1}s$ for any $s \in Q$, the active state becomes $b_1 \cdots b_{m-1}q$.

The above paragraph now implies that $b_1, \ldots, b_{m-1}q$ is a state of $\mathrm{Core}(A^m)$, since A^m is synchronizing at level m, hence the state of A^m forced by Δx is $b_1, \ldots, b_{m-1}q$. Therefore for any $q \in Q$, $b_1 \cdots b_{m-1}q$ is a state of $\mathrm{Core}(A^m)$. Moreover, $b_1 \cdots b_{m-1}$ was arbitrary, so we conclude that $\mathrm{Core}(A^m) = A^m$ as required. ■

In our next result we apply Lemma 5.8 to the transducer $(\mathscr{C}(G))^{-1}$ for a finite group G by showing that $(\mathscr{C}(G))^{-1}$ satisfies the condition of the lemma.

Theorem 5.9. *Let G be a finite group, then $|(\mathscr{C}(G))^n| = |G|^n$, hence the automaton $\mathscr{C}G$ has growth rate $|G|^n$. Moreover, every state of $\mathscr{C}(G)^n$ is accessible from every other state.*

Proof. Since, either by Remark 5.7 or a result in [18], the automaton semigroup generated by $\mathscr{C}(G)$ is free it suffices to show that $(\mathscr{C}(G)^{-1})$ satisfies the conditions of Lemma 5.8.

Since the states of $(\mathscr{C}(G))^{-1}$ are in bijective correspondence with the states of $\mathscr{C}(G)$, we shall let g' be the state of $(\mathscr{C}(G))^{-1}$ corresponding to the state g of $\mathscr{C}(G)$.

Let $g, h \in G$. We shall show that there is a state m' of $(\mathscr{C}(G))^{-1}$ such that

$$\lambda'_G(g, m') = h$$

(here λ'_G represents the rewrite function of $(\mathscr{C}(G))^{-1}$).

By definition of $\mathscr{C}(G)$, it suffices to take $m' = (gh^{-1})'$. ■

6. Infinite order and periodic witnesses

Section 5 gives a sufficient condition for determining when an element of $\widetilde{\mathscr{H}}_n$ has infinite order, although it does not produce a witness. In this section, we show that when the graph of bad pairs contains a circuit for an element of $\widetilde{\mathscr{H}}_n$ then we can construct an eventually periodic element on an infinite orbit under the action of the transducer. We also explore some other conditions which are sufficient for an element $A \in \mathscr{H}_n$ to have infinite order.

We start with the case where the graph of bad pairs contains a loop and proves the general case by reducing to the loop case.

Proposition 6.1. *Let $A \in \widetilde{\mathcal{H}}_n$ be synchronizing at level k, and let $r \geq k$. If the graph $G_r(A)$ of bad pairs has a loop, then there is an eventually periodic word in $X_n^{\mathbb{Z}}$ on an infinite orbit under the action of A.*

Proof. Let (p, q) be a vertex of G_r with a loop. Furthermore, assume that $m \in \mathbb{N}$ is the minimum splitting length of $A_r^{\vee}$. Let $\Gamma_1 \in X_n^r$, and $P, Q \in Q_A^{l-1}$ be such that (pP, qQ, Γ_1) is a split with top and bottom equal to $\{p, q\}$ such that the bottom depends only on the top.

Let $A^l = \langle X_n Q^l, \lambda_l, \pi_l \rangle$, then since (pP, qQ, Γ_1) is a split with top and bottom equal to $\{p, q\}$ such that the bottom depends only on the top, there is a state S_0 such that $\pi_l(pP, \Gamma_1) = S_0 = \pi_l(qQ, \Gamma_1)$ for any P and Q in Q_A^{l-1}. Therefore, let $S_1 := \pi_l(\Gamma_1, S_0)$. Let $\Gamma_2 \in X_n^r$ be the unique word such that $\lambda_l(\Gamma_2, S_1) = \Gamma_1$. Assume that Γ_i is defined and S_i is equal to $\pi_l(\Gamma_i, S_{i-1})$, then let Γ_{i+1} be the unique word in X_n^r such that $\lambda_l(\Gamma_{i+1}, S_i) = \Gamma_i$. Eventually we find there are $i \leq j \in \mathbb{N}$ such that $\Gamma_i = \Gamma_{j+1}$.

Suppose that $\lambda_l(\Gamma_1, pP) = \Delta$ and $\lambda_l(\Gamma_1, qP) = \Lambda$. Consider the bi-infinite word:

$$\cdots \Delta \dot{\Delta} \Gamma_1 \cdots \Gamma_{i-1} (\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots ,$$

where "$\dot{\Delta}$" indicates that Δ starts at the zero position. There are two cases to be considered.

Case 1: $\Delta \in W_p$ and $\Lambda \in W_q$. We consider how powers of A^l act on this word. Since $\Delta \in W_p$ and for any $T \in pQ^{l-1}$, $\lambda_l(\Gamma_1, T) \in W_p$, the bottom of the split (pP, qQ, Γ_1) depends only on the top, we must have that

$$\begin{aligned} &\cdots \Delta \dot{\Delta} \Gamma_1 \cdots \Gamma_{i-1} (\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots \\ &\quad \xrightarrow{A^l} \cdots \dot{*}_0 \; \Delta_1 \Gamma_1 \cdots \Gamma_{i-1} (\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots \end{aligned}$$

Now since $\Delta_1 \in W_p$, we can repeat the above

$$\begin{aligned} &\cdots *_0 \; \Delta_1 \Gamma_1 \cdots \Gamma_{i-1} (\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots \\ &\quad \xrightarrow{A^l} \cdots \dot{*}_0' \; *_1 \Delta_2 \Gamma_1 \cdots \Gamma_{i-1} (\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots \end{aligned}$$

Therefore, after applying A^l t times for some $t \in \mathbb{N}$ we see that from the position i onwards the output is of the form $\Delta_r \Gamma_1 \cdots \Gamma_{i-1} (\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots$, and $\Delta_m \in W_p$. Therefore, if $\Gamma_i \neq \Gamma_1$, $\cdots \Delta \dot{\Delta} \Gamma_1 \cdots \Gamma_{i-1} (\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots$ is on an infinite orbit under the action of A^l. This follows for if $t, t' \in \mathbb{N}$ such that $t \neq t'$, then we have

$$\begin{aligned} &(\cdots \Delta \dot{\Delta} \Gamma_1 \cdots \Gamma_{i-1} (\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots) A^{l*t} \\ &\quad \neq (\cdots \Delta \dot{\Delta} \Gamma_1 \cdots \Gamma_{i-1} (\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots) A^{l*t'}. \end{aligned}$$

Otherwise,

$$\begin{aligned} &\cdots \Delta \dot{\Delta} \Gamma_1 \cdots \Gamma_{i-1} (\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots \\ &\quad = (\cdots \Delta \dot{\Delta} \Gamma_1 \cdots \Gamma_{i-1} (\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots) A^{l*|t-t'|}. \end{aligned}$$

However, by minimality of i, we have that $\Gamma_1 \neq \Gamma_t$ for $1 < t \leq j$, yielding a contradiction.

If $\Gamma_i = \Gamma_1$, then our original word $\cdots \Delta \dot{\Delta} \Gamma_1 \cdots \Gamma_{i-1}(\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots$ becomes

$$\cdots \Delta \dot{\Delta} (\Gamma_1 \cdots \Gamma_{i-1})(\Gamma_1 \cdots \Gamma_{i-1}) \cdots$$

Notice that the infinite word corresponding to the coordinates $\mathbb{N} \backslash \{0\}$ is periodic with period $i - 1$. Now if $\Gamma_{i-1} \notin W_p$, Then for any $t \in \mathbb{N}$ such that $t > 0$ we have

$$(\cdots \Delta \dot{\Delta} (\Gamma_1 \cdots \Gamma_{i-1})(\Gamma_1 \cdots \Gamma_{i-1}) \cdots) A^{lt(i-1)} \neq \cdots \Delta \dot{\Delta} (\Gamma_1 \cdots \Gamma_{i-1})(\Gamma_1 \cdots \Gamma_{i-1}) \cdots$$

since $\Delta_{t(i-1)} \neq \Gamma_{t(i-1)} = \Gamma_{i-1}$. Therefore for any $t, t' \in \mathbb{N}$ such that $t \neq t'$, we must have that

$$\begin{aligned}(\cdots \Delta \dot{\Delta} (\Gamma_1 \cdots \Gamma_{i-1})(\Gamma_1 \cdots \Gamma_{i-1}) \cdots) A^{lt(i-1)} \\ \neq (\cdots \Delta \dot{\Delta} (\Gamma_1 \cdots \Gamma_{i-1})(\Gamma_1 \cdots \Gamma_{i-1}) \cdots) A^{lt'(i-1)}.\end{aligned}$$

If $\Gamma_i = \Gamma_1$ and $\Gamma_{i-1} \in W_p$, then consider the bi-infinite word

$$\cdots \Lambda \dot{\Lambda} \Gamma_1 \cdots \Gamma_{i-1}(\Gamma_1 \cdots \Gamma_{i-1}),$$

since $q \neq p$ and $\Lambda \in W_q$ and $\Gamma_{i-1} \in W_p$ we have $\Gamma_{t(i-1)} = \Gamma_{i-1} \neq \Lambda_{t(i-1)}$ for any $m \in \mathbb{N}$. Here, $\Lambda_{t(i-1)}$ is defined analogously to $\Delta_{t(i-1)}$. Therefore, by the argument above, $\cdots \Lambda \dot{\Lambda} \Gamma_1 \cdots \Gamma_{i-1}(\Gamma_1 \cdots \Gamma_{i-1}) \cdots$ is on an infinite orbit under the action of A^l.

Case 2: We assume now that $\Delta \in W_q$ and $\Lambda \in W_p$. As in the previous case we consider how powers of A^l act on the word $\cdots \Delta \dot{\Delta} \Gamma_1 \cdots \Gamma_{i-1}(\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots$

Making an argument similar to Case 1, we have that

$$\begin{aligned}&\cdots \Delta \dot{\Delta} \Gamma_1 \cdots \Gamma_{i-1}(\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots \\ &\xrightarrow{A^l} \cdots \dot{*}_0 \, \Delta_1 \Gamma_1 \cdots \Gamma_{i-1}(\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots\end{aligned}$$

However, in this case $\Delta_1 \in W_p$. Applying A^l again we have

$$\begin{aligned}&\cdots \dot{*}_0 \, \Delta_1 \Gamma_1 \cdots \Gamma_{i-1}(\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots \\ &\xrightarrow{A^l} \cdots \dot{*}_0 \, *_1 \Delta_2 \Gamma_1 \cdots \Gamma_{i-1}(\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots,\end{aligned}$$

where $\Delta_2 \in W_q$. Therefore, given $t \in \mathbb{N}$ we know that after applying A^l t times, the resulting word from the tth position onwards is of the form

$$\Delta_t \Gamma_1 \cdots \Gamma_{i-1}(\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j) \cdots$$

where $\Delta_t \in W_q$ if t is even, and $\Delta_t \in W_p$ if t is odd.

By considering the bi-infinite word, $\cdots \Lambda\dot{\Lambda}\Gamma_1 \cdots \Gamma_{i-1}(\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j)\cdots$, and similarly defining the Λ_t's $t \in \mathbb{N}$, we see that after applying A^l t-times to this word, the output from the tth position onwards is of the form $\Lambda_t\Gamma_1 \cdots \Gamma_{i-1}(\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j)\cdots$ where $\Lambda_t \in W_q$ if t is odd, and $\Lambda_t \in W_p$ if t is even.

Now we go through the subcases as in Case 1. If $\Gamma_i \neq \Gamma_1$, then the argument proceeds exactly as in Case 1.

Hence, consider the case $\Gamma_i = \Gamma_1$. Again we split into subcases. Now either $\Gamma_{i-1} \in W_p \sqcup W_q$ or it is disjoint from these two sets. We assume $\Gamma_{i-1} \in W_q$ (the other case is proved analogously). Since $\Lambda_{2t(i-1)} \in W_p$, for $t \in \mathbb{N}$ then by similar arguments to Case 1 above we conclude that $\cdots \Lambda\dot{\Lambda}\Gamma_1 \cdots \Gamma_{i-1}(\Gamma_i \cdots \Gamma_j)(\Gamma_i \cdots \Gamma_j)\cdots$ is on an infinite orbit under the action of A^{2l} and so under the action of A^l.

If $\Gamma_{i-1} \cap (W_p \sqcup W_q) = \emptyset$, then

$$\cdots \Delta\dot{\Delta}\Gamma_1 \cdots \Gamma_{i-1}(\Gamma_1 \cdots \Gamma_{i-1})\cdots$$

and

$$\cdots \Lambda\dot{\Lambda}\Gamma_1 \cdots \Gamma_{i-1}(\Gamma_1 \cdots \Gamma_{i-1})\cdots$$

are on infinite orbits under the action of A^l by repeating the argument in Case 1. ■

Remark 6.2. In the proof above, notice that we can replace the left infinite portion of the witness corresponding to the coordinates $\{\dots, -r-3, -r-2, -r-1\}$ by any infinite word in $X_n^{\mathbb{N}}$ (indexed from right to left).

Corollary 6.3. *Let $A \in \widetilde{\mathcal{H}}_n$ be synchronizing at level k, and let $r \geq k$. If the graph $G_r(A)$ of bad pairs has a circuit, then there is an eventually periodic element in $X_n^{\mathbb{Z}}$ on an infinite orbit under the action of A.*

Proof. This is a consequence of the proposition above and Remark 4.22. ■

For the next results, we recall the bits of notation $\mathfrak{S}_{r,|Q_A|^2+1}$ and $\underline{A}_j$ for a transducer $A \in \widetilde{\mathcal{H}}_n$ and $j \in \mathbb{N}$, given in Subsection 4.4.

Corollary 6.4. *$A \in \widetilde{\mathcal{H}}_n$, and let $r \in \mathbb{N}$ be greater than or equal to the minimal synchronizing level of A. Let $l \in \mathbb{N}$ be the minimal splitting length of $A_r^{\vee}$. If $\mathfrak{S}_{r,|Q_A|^2+1}$ contains a transformation of Q_A which is not a right zero then A has infinite order. Moreover, there is an eventually periodic element in $X_n^{\mathbb{Z}}$ on an infinite orbit under the action of A.*

Proof. This is a consequence of Corollary 6.3 and Proposition 4.23. ■

Corollary 6.5. *Let $H \in \widetilde{\mathcal{H}}_n$ and let $j \in \mathbb{N}$ be greater or equal to the minimal synchronizing level of H. If $\underline{H}_j$ is not injective, then H has infinite order and there is an eventually periodic element in $X_n^{\mathbb{Z}}$ on an infinite orbit under the action of H.*

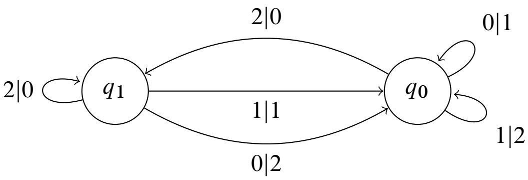

Figure 11. An element of infinite order in $\mathcal{H}_n$.

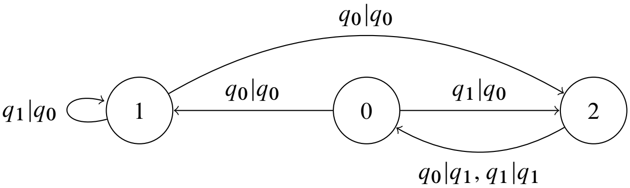

Figure 12. The level 1 dual of B.

Proof. This is a consequence of Corollary 6.3 and Lemma 4.27. ∎

Below is an example of a transducer B whose graph of bad pairs at level 1 $G_1(B)$ satisfies the conditions of Lemma 4.21; we also construct a witness as in Proposition 6.1 which demonstrates that the transducer has infinite order.

Example 6.6. Let B be the transducer in Figure 11. Its dual is given by Figure 12. From this it is easy to see that the pair $\{q_0, q_1\}$ is a vertex of G_1 and there is a directed edge with initial and terminal vertex $\{q_0, q_1\}$. Therefore, the conditions of Lemma 4.21 are satisfied and B has infinite order. Going through the construction in the proof of Proposition 6.1, we see that $\cdots 11\dot{1}(02)(02)\cdots$ is on an infinite orbit under the action of B.

Example 6.7. The transducer A shown in Figure 13 demonstrates that though the graph of bad pairs may contain a circuit at some level, the minimal graph of bad pairs at the same level may not.

It is easy to see that this transducer is bi-synchronizing at level 3 using the minimization procedure outlined in [3], or by direct computation in GAP. The graph $G_3(A)$ of bad pairs has a loop at the vertex $\{q_1, q_2\}$. The minimal graph of bad pairs $\overline{G}_3(A)$ is as shown in Figure 14: Here, the state pair (q_2, q_3) is not a minimal bad pair, and in fact reads any word of length 3 into a pair of the form (p, p) (it acts like a sink through which we escape the minimal bad pairs).

Example 6.8. The transducer $H \in \mathcal{H}_5$ shown in Figure 15 is an element of finite order whose dual at its minimal bi-synchronizing level splits. However, the next power of its dual is the zero of the semigroup generated by the dual. This means that the splits can be fixed by taking powers of the dual.

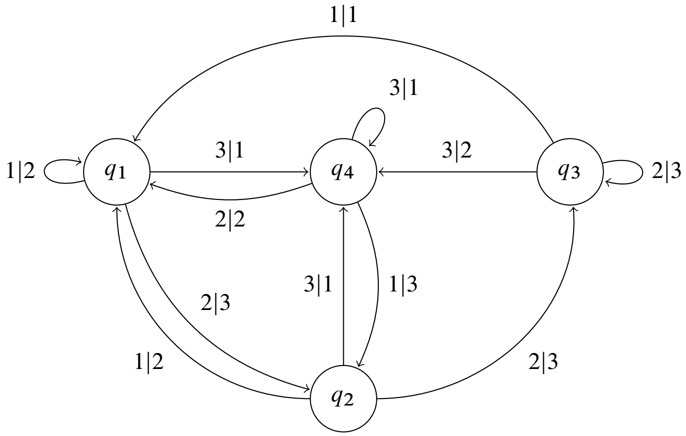

Figure 13. An element of infinite order whose minimal graph of bad pairs has no circuits.

7. Some embedding results

In this section, we show that given $n \in \mathbb{N}$ and an increasing sequence of non-zero natural numbers $d_1 \leq d_2 \leq \cdots \leq d_l$ such that $\sum_{i=1}^{l} d_i = n$, the semigroup $\widetilde{\mathcal{P}}_n$ contains a subsemigroup isomorphic to $\widetilde{\mathcal{P}}_{d_1} \times \widetilde{\mathcal{P}}_{d_2} \times \cdots \times \widetilde{\mathcal{P}}_{d_l}$. As a corollary, we conclude that $\mathcal{H}_n$ contains a subgroup isomorphic to $\mathcal{H}_{d_1} \times \mathcal{H}_{d_2} \times \cdots \times \mathcal{H}_{d_l}$ and so, via results in [5], that $\mathcal{H}_n$ contains direct product of free groups when $n \geq 6$. We also apply our techniques to the following conjecture of Picantin ([13]) which can be stated as follows: For a torsion element $A \in \mathcal{H}_n$, $(A^{\vee})^{(|A|-1)}$ is the zero of the semigroup generated by the dual. We construct examples of elements of $\mathcal{H}_n$ which attain this limit.

7.1. Embedding direct sums of $\widetilde{\mathcal{P}}_m$ in $\widetilde{\mathcal{P}}_n$ for n large enough

Let $n \in \mathbb{N}$ and let d_i $1 \leq i \leq l$ be an increasing sequence of non-zero natural numbers such that $\sum_{i=1}^{l} d_i = n$. Let $X_0 := \{0, 1, \ldots, d_1 - 1\}$ and for $2 \leq i \leq l$ let $X_i :=$

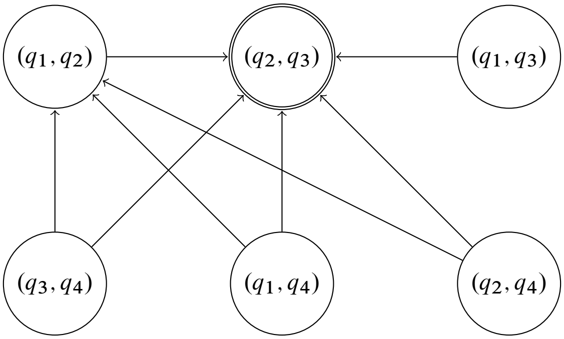

Figure 14. The graph $\overline{G}_3(A)$ of minimal bad pairs.

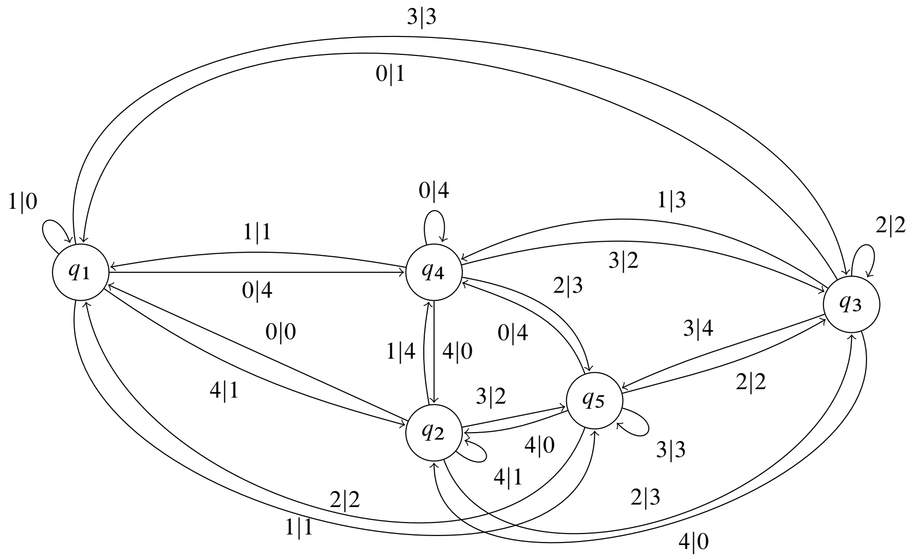

Figure 15. An element of finite order whose graph of bad pairs splits.

$\left\{\sum_{j=1}^{i-1} d_j, \sum_{j=1}^{i-1} d_j + 1, \dots, \sum_{j=1}^{i} d_j - 1\right\}$. Furthermore, let A_i $1 \leq i \leq l$ be synchronizing transducers on the alphabet X_i. We now describe how to form $\bigsqcup_{i=1}^{l} A_i \in \widetilde{\mathcal{P}}_n$.

For each $i \in \mathbb{N}$ let $\overline{A_i}$ denote the transformation of the words of length 1 induced by A_i as in Remark 3.7. The transducer $\bigsqcup_{i=1}^{l} A_i := \langle X_n, \bigsqcup_{i=1}^{l} Q_{A_i}, \pi_{\sqcup}, \lambda_{\sqcup} \rangle$ will consists of the disjoint union of copies of the A_i which are connected in a specific way. Fix a j such that $1 \leq j \leq l$, and consider the copy of A_j in $\bigsqcup_{i=1}^{l} A_i$. Then the copy of A_j in $\bigsqcup_{i=1}^{l} A_i \in \widetilde{\mathcal{P}}_n$ transitions precisely as A_j does when restricted to X_j; we now describe how A_j acts for inputs not in X_j.

Let $1 \leq i \leq l$ be arbitrary, then for any $x_i \in X_i$, there is a unique state $q_{x_i} \in Q_{A_i}$ such that $\pi_{A_i}(x_i, q_{x_i}) = q_{x_i}$ and $\lambda_{A_i}(x_i, q_{x_i}) = (x_i)\overline{A_i}$. Therefore, in $\bigsqcup_{i=1}^{l} A_i \in \mathcal{P}_n$ we set $\pi_{\sqcup}(x_i, A_j) = q_{x_i}$ and $\lambda_{\sqcup}(x_i, A_j) = (x_i)\overline{A_i}$. Hence, we have now described how the copy of A_j acts on all inputs in $\bigsqcup_{i \neq j, 1 \leq i \leq l} X_i$.

Repeating the above for each A_j, $1 \leq j \leq l$, we now have that $\bigsqcup_{i=1}^{l} A_i$ is connected, and all states are defined on $X_n := \{0, 1, \dots, n-1\}$.

It is not hard to see, using the construction of the map $\pi_{\sqcup}$ and the fact that the A_i's are synchronizing, that $\bigsqcup_{i=1}^{l} A_i \in \widetilde{\mathcal{P}}_n$ is also synchronizing. Indeed, if A_i has minimum synchronizing level k_i, then $\bigsqcup_{i=1}^{l} A_i$ is synchronizing at level $\max\{k_i + 1 \mid 1 \leq i \leq l\}$. Therefore, the following theorem is valid.

Theorem 7.1. *Let $n \in \mathbb{N}$ and let d_i, $1 \leq i \leq l$, be an increasing sequence of non-zero natural numbers such that $\sum_{i=1}^{l} d_i = n$. Let $X_1 := \{0, 1, \dots, d_1 - 1\}$ and for $2 \leq i \leq l$ let $X_i := \left\{\sum_{j=1}^{i-1} d_j - 1, \sum_{j=1}^{i-1} d_j, \dots, \sum_{j=1}^{i} d_j - 1\right\}$. Furthermore, let A_i, $1 \leq i \leq l$, be synchronizing transducers on the alphabet X_i. Then $\bigsqcup_{i=1}^{l} A_i$ is an element of $\widetilde{\mathcal{P}}_n$.*

Moreover, if k_i *is the minimal synchronizing level of each* A_i*, then the synchronizing level of* $\bigsqcup_{i=1}^{l} A_i$ *is at most* $\max_{1 \leq i \leq l}\{k_i\} + 1$.

Remark 7.2. For elements $A_i \in \mathscr{P}_{d_i}$ acting on the alphabet X_i, if at least one of the A_i's does not possess a homeomorphism state, then $\bigsqcup_{k=1}^{l} A_k$ does not represent a homeomorphism of $X_n^{\mathbb{Z}}$.

We have the following result.

Theorem 7.3. *Let* $n \in \mathbb{N}$ *and let* $d_i 1 \leq i \leq l$ *be an increasing sequence of non-zero natural numbers such that* $\sum_{i=1}^{l} d_i = n$*. Let* $X_1 := \{0, 1, \ldots, d_1 - 1\}$ *and for* $2 \leq i \leq l$ *let* $X_i := \{\sum_{j=1}^{i-1} d_j, \sum_{j=1}^{i-1} d_j, \ldots, \sum_{j=1}^{i} d_j - 1\}$*. By an abuse of notation, let* $\widetilde{\mathscr{P}}_{d_i}$ *denote the monoid of synchronizing transducers on the alphabet* X_i*. Then the map* $\phi : \bigoplus_{i=1}^{l} \widetilde{\mathscr{P}}_{d_i} \to \widetilde{\mathscr{P}}_n$*,* $(A_1, \ldots, A_l) \mapsto \bigsqcup_{i=1}^{l} A_i$ *is a monomorphism.*

Proof. That this map is injective follows from that the fact that the action of each A_i on $X_i^{\mathbb{Z}}$ is replicated exactly when we restrict $\bigsqcup_{i=1}^{l} A_i$ to $X_i^{\mathbb{Z}}$. Therefore, we need only prove that ϕ is a homomorphism.

Let $(A_1, \ldots, A_l)$ and $(B_1, \ldots, B_l)$ be elements of $\phi : \bigoplus_{i=1}^{l} \widetilde{\mathscr{P}}_{d_i}$, and let $(C_1, \ldots, C_l)$ be their product, hence we can say that $C_i = \mathrm{Core}(A_i * B_i)$. Therefore, we show that $D := \mathrm{Core}\big(\bigsqcup_{i=1}^{l} A_i * \bigsqcup_{i=1}^{l} B_i\big) = \bigsqcup_{i=1}^{l} C_i$.

First notice that for $q_i \in Q_{A_i}$ and $p_j \in Q_{B_j}$ the pair (q_i, p_j) is not a state of D. This is because for any word $\Gamma \in X_n^*$ such that the state of $\bigsqcup_{r=1}^{l} A_r$ forced by Γ is q_i, then Γ must have a non-empty suffix in X_i^*, and hence so also must its output through any state of $\bigsqcup_{r=1}^{l} A_r$ by construction. Therefore, the output of Γ through any state of $\bigsqcup_{r=1}^{l} B_r$ will synchronize to a state in Q_{B_i}. Therefore, the states of D are precisely a subset of $\bigsqcup_{i=1}^{l} Q_{A_i} \times Q_{B_i}$.

Now since the states of D intersecting $Q_{A_i} \times Q_{B_i}$ arising from the transducer product $A_i * B_i$ form precisely the sub-transducer C_i, therefore to conclude the proof it suffices (by the injectivity of ϕ) to show two things. Firstly, that for $j \neq i$ all states of $A_j \times B_j$ act on X_i precisely as $\overline{C_i}_1 = \overline{A_i}_1 \times \overline{B_i}_1$ (the final equality follows from Claim 3.8). Secondly, that all states (q_j, p_j) of $A_j \times B_j$ read an $x_i \in X_i$ into the unique state of C_i with a loop labelled by x_i.

The first part follows from the following observation. By construction for any $j \neq i$, the copy of A_j in $\bigsqcup_{i=1}^{l} A_i$ acts on X_i precisely as $\overline{A_i}_1$ does, similarly in $\bigsqcup_{i=1}^{l} B_i$. Now by Claim 3.8, $\overline{C_i}_1 := \overline{A_i}_1 \times \overline{B_i}_1$. Therefore, the first part is proved.

For the second part consider the following. Notice that for any $j \neq i$ and for any state q_j, a state of the copy of A_j in $\bigsqcup_{i=1}^{l} A_i$, and for any $x_i \in X_i$ we have that $\pi_{\sqcup A}(x_i, q_j) = q_{x_i}$, where $\pi_{\sqcup A}$ is the transition function of $\bigsqcup_{i=1}^{l} A_i$, and q_{x_i} is the unique state of A_i such that $\pi_{A_i}(x_i, q_{x_i}) = q_{x_i}$. An analogous statement holds for $\bigsqcup_{i=1}^{l} B_i$. Therefore, given $(q_j, p_j) \in A_j \times B_j$, and $x_i \in X_i$, we have $\pi_{\sqcup D}(x_i, q_j, p_j) = (\pi_{\sqcup A}(x_i, q_j), \pi_{\sqcup B}((x_i)\overline{A_i}_1, p_j))$; however, this is simply the state $(q_{x_i}, p_{(x_i)\overline{A_i}_1})$ of

$A_i \times B_i$. However, since by definition of $\overline{A_i}_1$, $(x_i)\overline{A_i}_1 = \lambda_{A_i}(x_i, q_{x_i})$, then $(q_{x_i}, p_{(x_i)\overline{A_i}_1})$ is precisely the unique state of C_i with a loop labelled by x_i. ■

Remark 7.4. It is straightforward to see from the above that ϕ maps $\bigoplus_{i=1}^{l} \mathcal{H}_{d_i}$ to a subgroup of $\mathcal{H}_n$ since $\bigoplus_{i=1}^{l} \mathcal{H}_{d_i}$ is a subgroup of $\widetilde{\mathcal{P}}_n$.

Corollary 7.5. *Let $n \in \mathbb{N}$ and let $d_i 1 \leq i \leq l$ be an increasing sequence of non-zero natural numbers such that $\sum_{i=1}^{l} d_i = n$. Let $X_1 := \{0, 1, \ldots, d_1 - 1\}$ and for $2 \leq i \leq l$ let $X_i := \left\{\sum_{j=1}^{i-1} d_j - 1, \sum_{j=1}^{i-1} d_j, \ldots, \sum_{j=1}^{i} d_j - 1\right\}$. By an abuse of notation, let $\widetilde{\mathcal{P}}_{d_i}$ denote the monoid of synchronizing transducers on the alphabet X_i. Given $(A_1, \ldots, A_l) \in \bigoplus_{i=1}^{l} \widetilde{\mathcal{P}}_{d_i}$, $\bigsqcup_{i=1}^{l} A_i$ has finite order if and only if A_i's has finite order for each i. Moreover, the order of $\bigsqcup_{i=1}^{l} A_i$ is precisely the lowest common multiple of the orders of the A_i.*

Proof. This is a consequence of Theorem 7.3. ■

It is a result by Boyle, Franks and Kitchens [5] that for $n \geq 3$, $\mathcal{H}_n$ contains free groups. Therefore, we have the following corollary.

Corollary 7.6. *Let $n \geq 3$ and let m and l be natural numbers such that $n = 3m + l$ where $0 \leq l \leq 3$. Then $\mathcal{H}_n$ contains a subgroup isomorphic to $\Pi_{i=1}^{m} F_2$ where F_2 is the free group on two generators.*

Notice that since $F_2 \times F_2$ has undecidable subgroup membership problem, it follows that for $n \geq 6$, $\mathcal{H}_n$ has undecidable subgroup membership problem.

Remark 7.7. We can modify the construction above. Let n, d_i and A_i, $1 \leq i \leq l$, be as before. For each A_i fix a permutation σ_i of X_i and an element $S_i \in Q_{A_i}^{d_i}$. Then we may form the transducer $\bigsqcup_{i=1,\sigma_i,S_i}^{l} A_i = \left\langle X_n, \bigsqcup_{i=1}^{l} A_i, \lambda_{\sqcup}, \pi_{\sqcup}\right\rangle$. For a given $j \in \{1, \ldots, l\}$, the copy of A_j in $\bigsqcup_{i=1,\sigma_i,S_i}^{l} A_i$ is precisely A_j when restricted to X_j. However, for $i \neq j$, and any state q_j of A_j, then given any $x_i \in X_i$ we have $\lambda_{\sqcup}(x_i, q_j) = (x_i)\sigma_i$; $\pi_{\sqcup}(x_i, q_j)$ is the entry of S_i corresponding to the position of x_i when the elements of X_i are ordered according to the natural ordering induced from $\mathbb{N}$. Then once more the resulting element of $\widetilde{\mathcal{P}}_n$ is synchronizing and has finite order if and only if all the A_i's have finite order.

7.2. On the difference between the synchronizing and bi-synchronizing level

Using the techniques developed above, we shall now construct a class of examples of finite order elements which are all synchronizing at level 1, but whose inverses are synchronizing at the maximum possible level for the given number of states. A side effect of the construction is that the alphabet size increases with the gap in the size of the synchronizing and bi-synchronizing level.

Our base transducer B is the transducer in Figure 16 to the left. Let A be the transducer on the right.

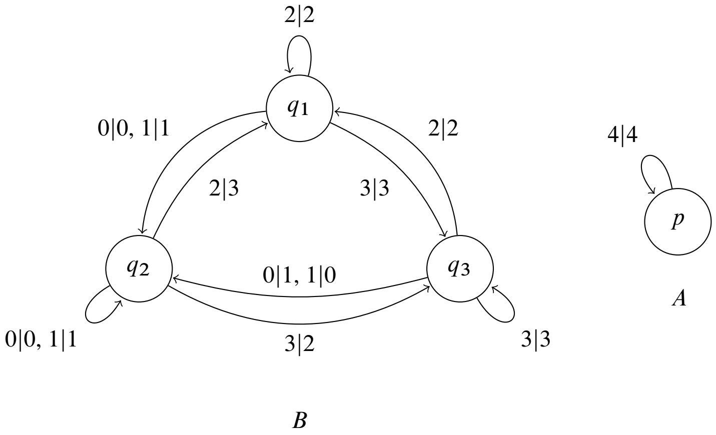

Figure 16. The base transducer B and an element of $\mathcal{H}_1$.

Notice that B is synchronizing at level 1 but bi-synchronizing at level 2. It is not hard to see that a transducer with j states is synchronizing at level at most $j-1$. Therefore, B^{-1} attains the maximum synchronizing level for a 3-state transducer. One can check that B has order 4.

Now we attach A to B using the construction described in Remark 7.7. Let σ_2 be any permutation of $\{0, 1, 2, 3\}$ that maps 0 to 3. There is only one permutation of $\{4\}$; form $\bigsqcup_{i=1,\sigma_i,S_i}^{2} C_i = \langle X_n, \bigsqcup_{i=1}^{2} C_i, \lambda_\sqcup, \pi_\sqcup \rangle$ where $C_1 = B$, $C_2 = A$, and σ_1 is the identity map. The resulting transducer is as shown in Figure 17 to the left.

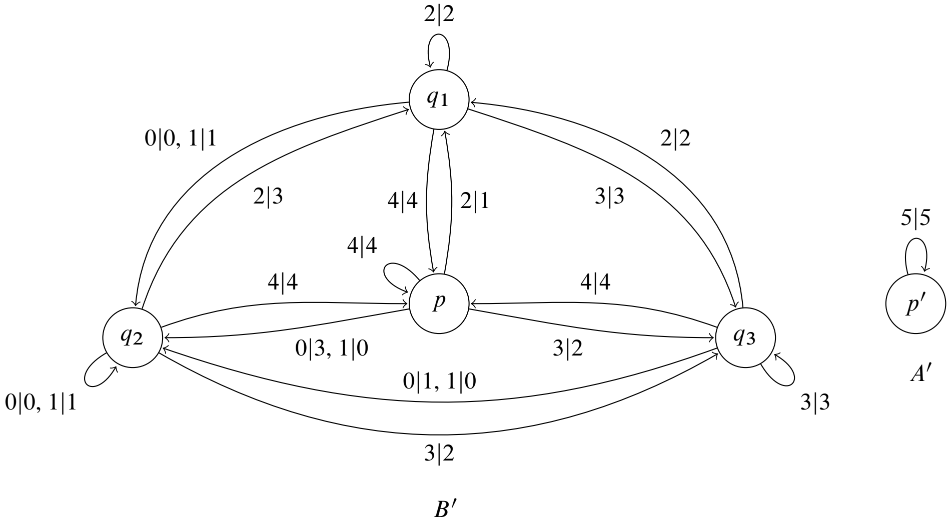

Figure 17. The resulting transducer B' which is a merge of B and A and an element A' of $\mathcal{H}_1$.

Since all the states B map $\{0,1\} \to \{0,1\}$, p is not ω-equivalent to any state of $B' := \bigsqcup_{i=1,\sigma_i,S_i}^{2} C_i$. Moreover, once can verify that B' is bi-synchronizing at level 3 and synchronizing at level 1.

Now since all the states of B' fix 4, we can repeat the process. Let A' be the transducer to the right of Figure 17, and let σ_2' be any permutation of $\{0,1,2,3,4\}$ that maps 4 to 3. Then, as before, the transducer $B'' := \bigsqcup_{i=1,\sigma_i',S_i}^{2} C_i = \langle X_n, \bigsqcup_{i=1}^{2} C_i, \lambda_\sqcup, \pi_\sqcup \rangle$, where $C_1 = B'$, $C_2 = A'$, and σ_1' is the identity map and is bi-synchronizing at level 4 and synchronizing at level 1. We may continue on in this way.

Notice that since the initial transducers B and A have finite order, then, by Remark 7.7 all the transducers B', B'' and so on have finite order.

Acknowledgements. The author would like to thank Collin Bleak for bringing this problem to his attention and for helpful comments on numerous versions of the paper. The author would also like to thank Laurent Bartholdi for helpful conversations, in particular for drawing attention to the results of Ching Chou and Shmuel Rosset. He also gratefully acknowledges the financial support of the Carnegie Trust.

References

[1] A. Akhavi, I. Klimann, S. Lombardy, J. Mairesse, and M. Picantin, On the finiteness problem for automaton (semi)groups. *Internat. J. Algebra Comput.* **22** (2012), no. 6, article no. 1250052, 26 pp. Zbl 1280.20038 MR 2974106

[2] L. Bartholdi and I. Mitrofanov, The word and order problems for self-similar and automata groups. *Groups Geom. Dyn.* **14** (2020), no. 2, 705–728 Zbl 1496.20056 MR 4118634

[3] C. Bleak, P. Cameron, Y. Maissel, A. Navas, and F. Olukoya, *The further chameleon groups of Richard Thompson and Graham Higman: automorphisms via dynamics for the Higman–Thompson groups* $G_{n,r}$, Mem. Amer. Math. Soc. 301, no. 1510, American Mathematical Society, Providence, RI, 2024 MR 4808712

[4] C. Bleak, P. J. Cameron, and F. Olukoya, Automorphisms of shift spaces and the Higman–Thompson groups: the one-sided case. *Discrete Anal.* **2021** (2021), article no. 15, 35 pp. Zbl 1497.20039 MR 4317278

[5] M. Boyle, J. Franks, and B. Kitchens, Automorphisms of one-sided subshifts of finite type. *Ergodic Theory Dynam. Systems* **10** (1990), no. 3, 421–449 Zbl 0695.54029 MR 1074312

[6] C. Chou, Elementary amenable groups. *Illinois J. Math.* **24** (1980), no. 3, 396–407 Zbl 0439.20017 MR 0573475

[7] GAP Group, GAP – Groups, Algorithms, and Programming, version 4.7.8, 2015, http://www.gap-system.org visited on 20 September 2024

[8] P. Gillibert, An automaton group with undecidable order and Engel problems. *J. Algebra* **497** (2018), 363–392 Zbl 1427.20040 MR 3743185

[9] R. I. Grigorchuk, V. V. Nekrashevich, and V. I. Sushchanskiĭ, Automata, dynamical systems, and groups (in Russian). *Proc. Steklov Inst. Math.* **231** (2000), 128–203 English Translation: *Tr. Mat. Inst. Steklova* **231** (2000), 134–214 Zbl 1155.37311 MR 1841755

[10] M. Gromov, Groups of polynomial growth and expanding maps. *Publ. Math. Inst. Hautes Etudes Sci.* **53** (1981), no. 1, 53–73 Zbl 0474.20018 MR 0623534

[11] I. Klimann, To infinity and beyond. In *45th International Colloquium on Automata, Languages, and Programming*, pp. 131:1–131:12, Leibniz Int. Proc. Inform. (LIPIcs) 107, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018, article no. 131 Zbl 1499.68210 MR 3830062

[12] I. Klimann, M. Picantin, and D. Savchuk, A connected 3-state reversible Mealy automaton cannot generate an infinite Burnside group. In I. Potapov (ed.), *Developments in language theory*, pp. 313–325, Lecture Notes in Comput. Sci. 9168, Springer, Cham, 2015 Zbl 1386.68106 MR 3440681

[13] P. Matthieu, *Automates, (semi)groupes et dualités*, pp. 110, Habilitation à diriger des recherches en mathématiques et en informatique, Université Paris Diderot, 2017

[14] Y. Muntyan and D. Savchuk, AutomGrp, automata groups, version 1.2.4, 2014, http://finautom.sourceforge.net/ visited on 20 September 2024, GAP package

[15] V. Nekrashevych, *Self-similar groups*. Math. Surveys Monogr. 117, American Mathematical Society, Providence, RI, 2005 Zbl 1087.20032 MR 2162164

[16] S. Rosset, A property of groups of non exponential growth. *Proc. Amer. Math. Soc.* **54** (1976), 24–26 Zbl 0324.20032 MR 0387420

[17] M. V. Sapir, *Combinatorial algebra: syntax and semantics*. With contributions by Victor S. Guba and Mikhail V. Volkov, Springer Monogr. Math., Springer, Cham, 2014 Zbl 1319.05001 MR 3243545

[18] P. V. Silva and B. Steinberg, On a class of automata groups generalizing lamplighter groups. *Internat. J. Algebra Comput.* **15** (2005), no. 5–6, 1213–1234 Zbl 1106.20028 MR 2197829

[19] A. N. Trahtman, The road coloring problem. *Israel J. Math.* **172** (2009), 51–60 Zbl 1175.05058 MR 2534238

[20] M. V. Volkov, Synchronizing automata and the Černý conjecture. In C. Martín-Vide, F. Otto, and H. Fernau (eds.), *Language and automata theory and applications*, pp. 11–27, Lecture Notes in Comput. Sci. 5196, Springer, Berlin, Heidelberg, 2008 Zbl 1156.68466 MR 2540309

Received 19 September 2023.

Feyishayo Olukoya
School of Computer Science and Mathematics, Keele University, Staffordshire, Keele ST5 5AA, UK; fo55@st-andrews.ac.uk