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On the isoperimetric functions of a class of Artin groups

Arye Juhász

Abstract. We find a polynomial (n^6) isoperimetric function for Artin groups, the defining graph of which contains no edges labelled by 3. This in particular shows that even Artin groups have solvable word problem. We use small cancellation theory of relative extended presentations.

1. Introduction

Let $m, n \in \mathbb{N}$ ($\mathbb{N}$ – the natural numbers), $m \geq 0$, $n \geq 1$, and let Γ be a simple graph without loops, with vertex set $V = \{v_1, \dots, v_n\}$ and edge set $E = \{e_1, \dots, e_m\}$. Label the edges by natural numbers via a labelling function $\lambda : E \to \mathbb{N} \setminus \{1\}$. Denote $\lambda(e) = n_{ij}$, where the edge e connects v_i to v_j, $i \neq j$ ($n_{ij} = n_{ji}$), and let $n_{ij} = 0$ if v_i and v_j are not connected by an edge. For every such labelled graph corresponds a group presentation

$$\mathcal{P}(\Gamma) = \langle X \mid \mathcal{R} \rangle \tag{1.1}$$

such that

$X = \{x_v, v \in V\}$, $\mathcal{R} = \{R_e, e \in E\}$, denote x_{v_i} by x_i,

$R_e = (x_i x_j)^{\frac{1}{2} n_{ij}} (x_i^{-1} x_j^{-1})^{\frac{1}{2} n_{ij}}$, if n_{ij} is even and

$R_e = (x_i x_j)^{\frac{n_{ij}-1}{2}} x_i (x_i^{-1} x_j^{-1})^{\frac{n_{ij}-1}{2}} x_j^{-1}$, if n_{ij} is odd.

The group presented by $\mathcal{P}(\Gamma)$ is denoted by $A(\Gamma)$ and is called the *Artin group defined by* Γ.

Examples 1.

(1) $n_{ij} = 0$ for every i, j, $1 \leq i < j \leq n$.
Then $A(\Gamma)$ is the free group on X.

(2) $n_{ij} = 2$ for every i, j, $1 \leq i < j \leq n$.
Then $A(\Gamma)$ is the free abelian group on X.

(3) $n_{ij} \in \{0, 2\}$ for every i, j, $1 \leq i < j \leq n$.
Then $A(\Gamma)$ is called *right-angled Artin group*.

(4) $n_{ij} \in 2\mathbb{N}$ for every i, j, $1 \leq i < j \leq n$.
Then $A(\Gamma)$ is called an *even Artin group*.

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(5) $n_{ij} \neq 2$ for every i, j, $1 \leq i < j \leq n$.
Then $A(\Gamma)$ is said to be of *large type*.

To each Artin group $A(\Gamma)$, there corresponds a Coxeter group W_A, obtained by adding the relators x_i^2, $i = 1, \ldots, n$. An Artin group is said to be of *spherical type* if W_A is a finite group. The graphs of the spherical type Artin groups are classically known. An Artin group is said to be of *FC type* if it belongs to the smallest class of Artin groups which are closed under amalgamation along standard parabolic subgroups and contain all the spherical type Artin groups. Standard parabolic subgroups are the subgroups which are generated by subsets of X.

We recall isoperimetric functions. Let G be a group presented by $\mathcal{P} = \langle X \mid \mathcal{R} \rangle$. Let $W \in F(X)$, $F(X)$ be the free group freely generated by X, W cyclically reduced (i.e., WW contains neither xx^{-1} nor $x^{-1}x$, $x \in X$). Then W represents 1 if and only if

$$\begin{aligned} W &= C_1 \cdots C_{k_w}, \qquad && k_w \geq 1, \\ C_i &= f_i R_i^{\varepsilon_i} f_i^{-1}, \qquad && f_i \in F(X),\ R_i \in \mathcal{R} \text{ and } \varepsilon \in \{1, -1\}. \end{aligned} \tag{1.2}$$

A function $g : \mathbb{N} \to \mathbb{N}$ is an *isoperimetric function* for the presentation $\mathcal{P} = \langle X \mid \mathcal{R} \rangle$ if for every word W which represents 1 of the group G, k_W in (1.2) satisfies $k_W \leq g(|W|)$. Here $|W|$ denotes the length of W in $F(X)$.

The isoperimetric functions are known for spherical type, FC type, large type, right-angled Artin groups and some other classes of Artin groups. See [1,4,5].

Main Theorem. *Let A be an Artin group, $A = A(\Gamma)$. If $n_{ij} \neq 3$ for every n_{ij}, $1 \leq i < j \leq n$, then $f(n) = n^6$ is an isoperimetric function for $\mathcal{P}(\Gamma)$.*

We believe that "6" can be replaced by "2".

Corollary. *Let $A(\Gamma)$ be an Artin group. If $n_{ij} \neq 3$ for every i, j, $1 \leq i < j \leq n$, then A has solvable word problem. In particular, even Artin groups have solvable word problem.*

The corollary has been established recently also in [9], using a completely different method.

Our basic method is small cancellation theory of extended (relative) presentations via van Kampen and Howie diagrams (see [6, Lemma 1.2.2, Remark 1.2.3]). (Recall that a van Kampen $\mathcal{R}$-diagram over $F(X)$ is a connected, simply connected labelled planar 2-complex, labelled by elements of $F(X)$ such that the labels of the boundary of the 2-cells are elements of $\mathcal{R}$. We say that the corresponding 2-cell (region) *realises* the element of $\mathcal{R}$.)

The original version of this article appeared in [6] *and has been approved by the referee. The present article is a shortened version of it.*

Outline of the proof of the Main Theorem. In order to find an isoperimetric function for the presentation $\mathcal{P}$, we have to find an upper bound on the number k_w in (1.2) in terms of the length $|W|$ of W for every W representing 1 in the group defined by $\mathcal{P}$. By the basic

theorem of van Kampen diagrams, for every cyclically reduced W representing 1, there is a van Kampen diagram M with boundary label W such that the number of regions in M is at most k_W. See [10, Chapter V]. So our problem can be considered as a counting problem in van Kampen diagrams, and this is our approach. We use the following simple basic principles, together with the well-known result in Proposition 1. Thus let S be a finite set. In order to count the elements of S, we subdivide S into subsets $S_1, \dots, S_m$ such that we know m and know the maximal possible number n of elements in S_i, $i = 1, \dots, m$. Then S has at most mn elements. In our case, Proposition 1 provides the first approximation to the numbers m and n, in terms of the length of W.

Proposition 1. *Let M be a reduced van Kampen $\mathcal{R}$-diagram over $F(X)$. Let W be a boundary label of M, cyclically reduced. Assume M satisfies conditions* C(4) *and* T(4). *Suppose that each piece has length* 1. *Then each of the following holds:*

(a) *For each region Δ of M, we have $|\partial\Delta| \leq |W|$.*

(b) *The number $|M|$ of regions Δ in M is at most $|W|^2$.*

Hence if our diagram M would be a C(4) and T(4) diagram, then by part (b) of the proposition, the number of regions in M would be bounded by $|W|^2$. Being this not the case, we first modify our approach. The core idea is not to try to estimate the number of all the regions via a result like Proposition 1 but to use that result for the estimation of a part of the regions only, namely, $|\mathrm{Reg}_{4^+}(M)|$, the number of regions of M having boundary cycle of length at least 8, and then relate $|\mathrm{Reg}_2(M)|$, the number of remaining regions, to $|\mathrm{Reg}_{4^+}(M)|$.

In this direction, we have the following.

Theorem 1. *Let M be a van Kampen $\mathcal{R}$-diagram, $\mathcal{R}$ given by* (1.1). *Assume*

$$\mathcal{R} = \{R_e \mid e \in E\}, \quad \lambda(e) \neq 3 \quad \textit{for every } e \in E. \tag{1.3}$$

Then there is a van Kampen diagram M' with the same boundary label as M such that $|\mathrm{Reg}_{4^+}(M')| \leq |\partial M|^4$.

For the proof, see [6, pp. 5–11,105].

Now, we relate the number of regions in $\mathrm{Reg}_2(M)$ to the number of regions in $\mathrm{Reg}_{4^+}(M)$. First, let us state a definition.

Definition 1 (a-Bands). Let $D_1, D_2 \in \mathrm{Reg}_2(M)$, and let $\mathrm{Supp}(D_1) = \{a, x\}$, $\mathrm{Supp}(D_2) = \{a, y\}$ ($x = y$ is not excluded). Say that D_1 and D_2 are *a-neighbours* if $\theta := \partial D_1 \cap \partial D_2$ contains a subpath θ_0 with label $a^{\pm 1}$. (We consider D_i as a neighbour of itself, $i = 1, 2$.) The transitive closure of "a-neighbourhood" is an equivalence relation. For $D \in \mathrm{Reg}_2(M)$ with $\mathrm{Supp}(D) = \{a, x\}$, denote by $\widehat{L}_a(D)$ the a-equivalence class of D. Let $L_a(D)$ be the

subdiagram of M generated by $\widehat{L}_a(D)$. Call $L_a(D)$ the *a-band generated by* D. Denote by $\mathrm{Str}(M)$ the collection of all the bands in M. Clearly,

$$\mathrm{Reg}_2(M) = \bigcup_{L \in \mathrm{Str}(M)} \mathrm{Reg}(L).$$

We make the observation that every band either starts on the boundary of a region in $\mathrm{Reg}_{4+}(M)$ or on the boundary of M. The same is true for the ends of bands. Hence we can estimate the number of bands in terms of the number of regions from $\mathrm{Reg}_{4+}(M)$. Next, we show that the number of regions in a band is bounded by the number of bands. Relying on Theorem 1, this easily implies that $|\mathrm{Reg}_2(M)|$ is bounded by $|\partial M|^6$. See [6, pp. 10–12].

The proof of the Main Theorem is reduced to the proofs of Theorems A–C and some results on the way to their proofs. In the rest of the work, we concentrate on their proofs. We assume that $\mathcal{R}$ is given by (1.3).

See Definition 2.1.4 for $\mathcal{M}_3(W)$.

Theorem A. *Let M be a van Kampen $\mathcal{R}$-diagram over F with cyclically reduced boundary label W. If $M \in \mathcal{M}_3(W)$, then $\widetilde{\mathbb{M}}^t$ satisfies the small cancellation conditions* C(4) *and* T(4) *for every $t \in T(M)$.*

Theorem B. *Let M be a van Kampen $\mathcal{R}$-diagram over F with cyclically reduced boundary label W. If $M \in \mathcal{M}_3(W)$, then for every region D in* $\mathrm{Reg}(M)$ *realising a relation $R(a,t)$, $\|W\|_t \geq n(a,t)$ and $\|W\|_a \geq n(a,t)$, a and t in T.*

The main theorem obtained from the proof of Theorem 1 and the main result is Theorem C, which is a variant of Greendlinger's lemma.

Theorem C (Greendlinger's lemma). *Let $A(\Gamma)$ be an Artin group given by* (1.3). *Let $M_0 \in \mathcal{M}_3(W)$ (see Definition 2.1.4) be a connected, simply connected minimal $\mathcal{R}$-diagram over F with cyclically reduced boundary label W, $W \neq 1$ in $A(\Gamma_2)$.*

Let $W = W_1 W_2^{-1}$ be reduced as written. Then there is a van Kampen diagram M, $M \in \mathcal{M}_3(W)$, with boundary cycle ω, such that ω decomposes into $\omega = \omega_1\omega_2^{-1}$, ω_i labelled with W_i, $i = 1, 2$ and one of the following holds:

(a) *There is a boundary region $K \in \mathrm{Reg}_{4+}(\mathbb{M})$ with $\partial K \cap \omega_i$ connected, $\|\partial K \cap \omega_i\| \geq n(K)$, for $i = 1$ or $i = 2$, where $\mathrm{Reg}_{4+}(\mathbb{M}) = \{K \in \mathrm{Reg}(\mathbb{M}) \mid n(K) \geq 4\}$. Moreover, $|\xi| \leq |\partial K \cap \omega_i|$, where ξ is the complement of $\partial K \cap \omega_i$ on ∂K.*

(b) *There are boundary regions $D_1, D_2 \in \mathrm{Reg}_2(M)$ with $\partial D_i \cap \omega_j$ connected, $|\partial D_i \cap \omega_j| \geq 2$, for $j = 1$ or for $j = 2$.*

(c) *There is a region $D \in \mathrm{Reg}_2(M)$ with $\partial D \cap \omega_j$ connected, $|\partial D \cap \omega_j| \geq 2$, and there is a region $K \in \mathrm{Reg}_{4+}(\mathbb{M})$ with $\partial K \cap \omega_j$ connected, $\|\partial D \cap \omega_j\| \geq 2$, for $j = 1$ or for $j = 2$.*

Overview of the present work. The goals of the work are

(1) to estimate the number of regions in $\mathrm{Reg}_{4^+}(M)$,

(2) to estimate the number of regions in $\mathrm{Reg}_2(M)$.

The idea behind (1) is to count the number of regions in $\widetilde{\mathbb{M}}^t$, $t \in T(M)$ (for notations, see [6, the three constructions on pp. 6, 7, 11]) where due to Theorem A, conditions C(4) and T(4) are satisfied, hence we may do this, and then to show that the number of regions in $\widetilde{\mathbb{M}}^t$ and $\mathrm{Reg}_{4^+,t}\,\mathbb{M}$ is the same, via a natural mapping ψ_t of diagrams which sends a region Δ in $\mathbb{M}_t$ to a uniquely defined region $\widetilde{\Delta}$ in $\widetilde{\mathbb{M}}^t$. For Ψ_t, see [6, pp. 79–81]. One of the problems is that $\widetilde{\mathbb{M}}^t$ is obtained from $\mathbb{M}_t$ by shrinking edges and identifying vertices; hence in principle, $\widetilde{\Delta}$ may shrink to an edge or a vertex. Thus our task is to show that ψ_t (and ψ_t^{-1}) does not cause deformation and collapse of regions. One of the classical methods to avoid deformations in diagrams is by Greendlinger's lemma. It roughly says that a diagram with at least two regions has at least two Greendlinger regions, the boundary of which contains a big portion of the relators. See [6, Definition 1.3.1]. In general terms, it shows that if there is a kind of deformation in certain subdiagrams, then certain events are simultaneously unavoidable and forbidden, which of course is absurd. Hence no deformation occurs. For example, consider a reduced diagram in which every region has boundary label U^n, U cyclically reduced, not proper power, $n \geq 7$. Suppose that the boundary of a region E is not simple closed. (This is the deformation which we would like to avoid.) Then there is a loop in the boundary of the region which surrounds a disc Q, the boundary of which is labelled by a subword of the cyclic word U^n. Suppose that in Q, we have Greendlinger's lemma. Then Q has a boundary region D with $\partial D \cap \partial Q$ connected such that $|\partial D \cap \partial Q| \geq 3|U|$. Hence D and E necessarily have a common (unavoidable) piece of length at least $3|U|$. But such a word is forbidden, because due to U not being a proper power, this would mean that D cancels E, violating that the diagram is reduced. (Though we do not use this example, we use its underlying idea. See the proofs of Propositions 2.1.7 and 2.1.8 in [6].) Coming back to the context of our work, notice that by Theorem C, every bigonal $\mathcal{R}$-diagram M, that is, $\partial M = \omega_1\omega_2^{-1}$ has a generalised Greendlinger region on one of the sides ω_i of the bigon. Since $\widetilde{\mathbb{M}}^t$ does satisfy C(4) and T(4), hence it has Greendlinger regions. The natural candidates for Greendlinger's regions in M are the images (by Ψ_t^{-1}) of the Greendlinger regions of $\widetilde{\mathbb{M}}^t$. The problem is that while Greendlinger regions are boundary regions, our candidates in M need not be. So we need a technique to "move" inner regions to the boundary. More precisely, to show that there is a diagram with the same boundary label as M in which the candidates are boundary regions. We do this by observing that since the generators of the second homotopy group of the defining complex are prisms, we can replace one half of the prism by its complement by a rotation of the prism. In M, this has the effect of moving a region which occurs both in M and in the prism. See [6, Section 3]. The involvement of the second homotopy group requires the introduction of extended presentations.

Finally, we have to count $|\mathrm{Reg}_2(M)|$. To this end, it is enough to show that two bands cannot intersect more than once (see Section 4). Essentially, we prove this simultaneously with the other main proposition (see Section 4).

The ideas developed here lead to further results for Artin groups A dealt with in the present work. Recall that a parabolic subgroup of an Artin group is a conjugate of a standard parabolic subgroup. In [7], we show that the intersection of parabolic subgroups is parabolic. Also, we describe fusion in A. In particular, we show that in even Artin groups, every standard parabolic subgroup P controls fusion in P (i.e., if two elements of P are conjugate in A, then they are already conjugate in P).

Together with further ideas, we show in [8] that locally reducible Artin groups (i.e., no (2.3.3), (2.3.4) and (2.3.5) type standard parabolics occur) have polynomial (n^6) isoperimetric functions.

The work is organised as follows:

- In Section 2, we introduce the diagrams $\mathbb{M}$ and $\mathbb{M}_t$ and show that they are adequate. Also, M is adequate.
- In Section 3, we consider extended presentations, Howie diagrams, $\mathcal{I}$-moves and banded diagrams.
- In Section 4, we consider the connection between $\mathbb{M}_t$ and $\widetilde{\mathbb{M}^t}$ and prove the main theorems.

2. The diagrams $\mathbb{M}$, $\mathbb{M}_t$ and $\widetilde{\mathbb{M}}^t$, $t \in T(M)$

2.1. The diagrams $\mathbb{M}$ and $\mathbb{M}_t$, $t \in T(M)$

Let M be a simply connected $\mathcal{R}$-diagram over F, $\mathcal{R}$ given by (1.3), and let $t \in T(M)$. Denote by $\mathrm{Reg}_{H_t}(M)$ the set of all the regions of M the boundaries of which are labelled only by letters from H_t ($H_t = \langle X - \{t\}\rangle$). Denote by $\mathrm{Reg}_t(M)$ the set of all the regions of M the boundaries of which have labels from $\langle t\rangle$ and also from H_t. Thus $\mathrm{Reg}(M) = \mathrm{Reg}_t(M) \cup \mathrm{Reg}_{H_t}(M)$. We subdivide $\mathrm{Reg}_t(M)$ further by

$$\mathrm{Reg}_t(M) = \mathrm{Reg}_{4^+,t}(M) \cup \mathrm{Reg}_{2,t}(M),$$

where $\mathrm{Reg}_{4^+,t}(M)$ is the set of regions in $\mathrm{Reg}_t(M)$ with boundary length at least 8 and $\mathrm{Reg}_{2,t}(M)$ is the set of regions in $\mathrm{Reg}_t(M)$ with boundary length 4. We have for $t \in T$

$$\mathrm{Reg}(M) = \mathrm{Reg}_{4^+,t}(M) \cup \mathrm{Reg}_{H_t}(M) \cup \mathrm{Reg}_{2,t}(M). \tag{2.1}$$

For the definition of the diagrams $\mathbb{M}$ and $\mathbb{M}_t$, we introduce equivalence relations "$\underset{t}{\sim}$", "$\underset{H_t}{\sim}$" and "$\underset{B_t}{\sim}$" on $\mathrm{Reg}_{4^+,t}(M)$, $\mathrm{Reg}_{H_t}(M)$ and $\mathrm{Reg}_{2,t}(M)$, respectively.

Definition 2.1.1 (of $\underset{t}{\sim}$). Let D_1 and D_2 be regions in $\mathrm{Reg}_{4^+,t}(M)$. Say that they are *t-friends* if the following hold:

(i) $\partial D_1 \cap \partial D_2$ contains an edge.

(ii) $\mathrm{Supp}(\partial D_1) = \mathrm{Supp}(\partial D_2)$.

Let $\underset{t}{\sim}$ be the transitive closure of t-friendness. It is easy to see that $\underset{t}{\sim}$ is an equivalence relation on $\mathrm{Reg}_{4^+,t}(M)$. Denote by $[D]_t$ the $\underset{t}{\sim}$ equivalence class of D.

Definition 2.1.2 (of $\underset{H_t}{\sim}$). Let D_1 and D_2 be regions in $\mathrm{Reg}_{H_t}(M)$. Say that they are H_t-*friends* if $\partial D_1 \cap \partial D_2$ contains an edge. Let $\underset{H_t}{\sim}$be the transitive closure of H_t-friendness. Then $\underset{H_t}{\sim}$ is an equivalence relation on $\mathrm{Reg}_{H_t}(M)$. Denote by $[D]_{H_t}$ the equivalence class of $D \in \mathrm{Reg}_{H_t}(M)$.

Definition 2.1.3 (of $\underset{B_t}{\sim}$). Let D_1 and D_2 be regions in $\mathrm{Reg}_{2,t}(M)$. Say that D_1 and D_2 are B_t-*friends* if $\partial D_1 \cap \partial D_2$ consists of an edge with label $t^{\pm 1}$. Let $\underset{B_t}{\sim}$be the transitive closure of B_t-friendness. Then $\underset{B_t}{\sim}$is an equivalence relation. For $D \in \mathrm{Reg}_{2,t}(M)$, denote by $[D]_{B_t}$ the equivalence class of D in $\mathrm{Reg}_{2,t}(M)$.

Definition 2.1.4 (The class $\mathcal{M}_3(W)$ of diagrams). The following statements hold:

(a) Let W be a cyclically reduced word which represents 1 in $A(\Gamma)$. Assume that $W \neq 1$ in $A(\Gamma_2)$. Here Γ_2 is the graph obtained from γ by removing all edges not labelled by 2. Denote by $\mathcal{M}(W)$ the set of all the van Kampen diagrams with boundary label W. By [10, Chapter V], $\mathcal{M}(W) \neq \emptyset$. Let $\mathcal{M}_1(W)$ denote the set of all the diagrams in $\mathcal{M}(W)$ for which $|\mathrm{Reg}_{4^+}(M)|$ is minimal possible. Clearly, $\mathcal{M}_1(W) \neq \emptyset$. Let $\mathcal{M}_2(W)$ be the set of the diagrams in $\mathcal{M}_1(W)$ which contain minimal number of equivalence classes $[D]_M$ of regions in $\mathrm{Reg}_{4^+}(M)$. Clearly, $\mathcal{M}_2(W) \neq \emptyset$. Finally, let $\mathcal{M}_3(W)$ be the set of the diagrams in $\mathcal{M}_2(W)$ with adequate bands. Say that M is *minimal* if $M \in \mathcal{M}_2(W)$ and $|\mathrm{Reg}_2(M)|$ is minimal possible. It follows that if M is minimal, then $M \in \mathcal{M}_3(W)$. Hence $\mathcal{M}_3(W) \neq \emptyset$.

(b) Definition 2.1.1 makes sense without the requirement that D_1 and D_2 are in $\mathrm{Reg}_{4^+,t}(M)$, namely, $D_1 \sim D_2$ if $\mathrm{Supp}(\partial D_1) = \mathrm{Supp}(\partial D_2)$ and $\partial D_1 \cap \partial D_2$ contains an edge. We shall call the corresponding equivalence relation "equivalence". For $D \in \mathrm{Reg}_Z(M)$, define $\kappa_0(D) = \mathrm{Int}(\cup\{\overline{E} \mid E \in [D]_Z\})$. Then $\kappa_0(D)$ is connected. If $Z = t^{4^+}$, then define $n(\kappa_0(D)) = n(D)$. Notice that $n(\kappa_0(D))$ is well defined. Also, denote $\kappa_0(D)$ by $\Delta(D)$ for this case. Denote the diagram obtained by $\mathbb{M}$.

Definition 2.1.5 (Adequate). Let M be a connected, simply connected $\mathcal{R}$-diagram over F, $M \in \mathcal{M}_3(W)$. Say that M is *adequate*, if for every $D \in \mathrm{Reg}(M)$, D is homeomorphic to the open unit disc, ∂D is simple such that the boundary label is not 1 and is cyclically reduced.

Definition 2.1.6 ($\mathbb{M}_t$). Let M be a van Kampen $\mathcal{R}$-diagram over F, and let $t \in T(M)$. If $\kappa_0(D)$ is homeomorphic to the open unit disc, for every $D \in \mathrm{Reg}(M)$, then we may consider $\{\kappa_0(D) \mid D \in \mathrm{Reg}(M)\}$ as a set of regions such that their totality generates a diagram which we denote by $\mathbb{M}_t$. Thus $\mathrm{Reg}(\mathbb{M}_t) = \{\kappa_0(D) \mid D \in \mathrm{Reg}(M)\}$. We have $\partial\mathbb{M}_t = \partial M$. Also,

$$\mathrm{Reg}(\mathbb{M}_t) = \mathrm{Reg}_{4^+,t}(\mathbb{M}_t) \cup \mathrm{Reg}_{B_t}(\mathbb{M}_t) \cup \mathrm{Reg}_{H_t}(\mathbb{M}_t) \quad \text{for every } t \in T(M). \tag{2.2}$$

The main results of this section are Propositions 2.1.7 and 2.1.8.

Proposition 2.1.7. *Let $M \in \mathcal{M}_3(W)$. Then $\mathbb{M}_t$ is adequate for every $t \in T(M)$. Moreover, if K_1 and K_2 are regions of $\mathbb{M}_t$, not both bands, then $\partial K_1 \cap \partial K_2 \neq \emptyset$ implies that $\partial K_1 \cap \partial K_2$ is connected.*

Proposition 2.1.8. *Let $M \in \mathcal{M}_3(W)$. Then M is adequate. Moreover, if $\partial D_1 \cap \partial D_2 \neq \emptyset$, then $\partial D_1 \cap \partial D_2$ is connected.*

Remark 2.1.9. Let $K \in \mathrm{Reg}_{4^+,t}(\mathbb{M}_t)$. When passing from $\mathbb{M}_t$ to $\widetilde{\mathbb{M}^t}$ by shrinking edges labelled by elements of H_t to a point and shrinking t-bands to edges, it may happen that the interior of the obtained region $\widetilde{K^t}$ in $\widetilde{\mathbb{M}^t}$ becomes disconnected. Since M is adequate, we can replace $\widetilde{K^t}$ by a disc (and we shall do so), the boundary of which is labelled by $\Phi(\partial(\widetilde{K^t}))$.

We prove Propositions 2.1.7 and 2.1.8 by simultaneous induction on the size of M. See [6, Sections 2.2–2.4].

3. Extended presentations, $\mathcal{I}$-moves and banded diagrams

3.1. Identities among relations

We recall identities among relations from [2, p. 157]. Let C be a 2-complex. Let c_0 be a basepoint for C (some designated vertex of C). A *sequence σ over C at v_0* is $\sigma = (q_1, \dots, q_n)$, where q_i is a path in C consisting of a concatenation $\gamma_i \partial\Delta_i \gamma_i^{-1}$, Δ_i being a region (face) of C and γ_i a path from v_0 to $v(\Delta_i)$, $1 \le i \le n$, where $v(\Delta_i)$ is the basepoint of Δ_i. We call σ an *identity sequence* if the concatenation $q_1 \cdots q_n$ is freely equivalent to the trivial path at v_0.

Given a presentation $\mathcal{P} = \langle X \mid \mathcal{R}\rangle$, following [10], we construct a bouquet of tailed circles C with basepoint v_0. Let $\sigma = (q_1, \dots, q_n)$ be a sequence over C at v_0, $q_i = \gamma_i \Delta_i \gamma_i^{-1}$, γ_i tails. Label $\partial\Delta_i$ by $R_i \in \mathcal{R}$ and label γ_i by a word in $F(X)$, via a labelling function $\Phi_0 : C \to F$. After carrying out all free cancellations in $\Phi_0(\sigma) = \Phi_0(q_1) \cdots \Phi_0(q_n)$, we get an $\mathcal{R}$-diagram M over F the boundary label of which is the cyclically reduced word for $\Phi_0(q_1 \cdots q_n)$ in F.

If $\Phi_0(q_1 \cdots q_n) \neq 1_F$, then M is a connected and simply connected planar diagram (van Kampen diagram), and if $\Phi_0(q_1 \cdots q_n) = 1$ in F, then M is the union of a finite number of spines and tessellated spheres ($\cong S^2 \subseteq \mathbb{R}^3$) by the Δ_i. In this case, we call M a *spherical diagram*. We call M a *singular sphere* or just a sphere if it consists of a single sphere. When the q_i are labelled with R_i and $\phi(q_1 \cdots q_n) = R_1 \cdots R_n = 1$ in F, then we call the sequence $S = (R_1, \ldots, R_n)$ an *identity sequence over* $\mathcal{R}$ and call it a *simple identity sequence*, if the corresponding spherical diagram is a simple sphere.

Definition 3.1.1 (Extended presentations). Let $\mathcal{P} = \langle X \mid \mathcal{R} \rangle$ be a finite presentation of a group G, and let $\mathcal{I}$ be a set of simple identity sequences over $\mathcal{R}$. The corresponding *extended presentation* is the triple $\langle X \mid \mathcal{R} \mid \mathcal{I} \rangle$. This notion was introduced in [3] by Roger Fenn. Now let $A = A(\Gamma)$ be an Artin group given by $\langle X \mid \mathcal{R} \rangle$ in (1.3). The identities among relations $\mathcal{I}$ we choose are obtained from the 3-generated standard parabolic subgroups $A(i, j, k)$ with defining subgraphs $\Gamma_{i,j,k}$ with the vertex set $V(\Gamma_{i,j,k}) = \{x_i, x_j, x_k\}$ and the edge set $E(\Gamma_{i,j,k}) = \{e_{ij}, e_{i,k}, e_{j,k}\}$, $1 \leq i < j < k \leq n$, with labels $\lambda(e_{ij}) = \lambda(e_{j,k}) = 2$ and $\lambda(e_{i,k}) = m$, $m \geq 4$. The corresponding tessellated simple spheres $\Sigma_{i,j,k}$ are prisms with two regions of degree $2m$ (upper and lower) labelled with R_{ik}, which we call the *big regions*, and a band of length $2n$ in which the regions are labelled by R_{ij} and R_{jk} in an alternating manner. See Figure 1 for $n = 4$.

We introduce one of our main tools, $\mathcal{I}$*-moves*.

Let $\Sigma = \Sigma_{i,j,k}$, and let ω be a simple closed curve on Σ. Then ω subdivides Σ into two simply connected subdiagrams, ω_R and ω_L, with connected interiors, where ω_R is the submap of all regions to the right of ω and ω_L is the submap of all the regions to the left of ω. Clearly, $\overline{\omega_R} \cap \overline{\omega_L} = \omega$ and $\omega_R \cup \omega \cup \omega_L = \Sigma$.

Let $S(\Sigma)$ be the set of all the connected, simply connected submaps of Σ with connected interior, and let $\beta_\omega : S(\Sigma) \to S(\Sigma)$ be the function which for every closed curve ω on Σ sends ω_R to ω_L and ω_L to ω_R and leaves ω unaltered.

Let M be a van Kampen $\mathcal{R}$-diagram over F, and let M_0 be a simply connected subdiagram of M with connected interior and simple boundary cycle μ. Suppose that $\overline{M_0}$ can be embedded into $\Sigma = \Sigma_{i,j,k}$ for some (uniquely defined) i, j and k as an $\mathcal{R}$-diagram with image N in Σ with ∂N simple closed such that μ is mapped onto ∂N. Then $N \in S(\Sigma)$.

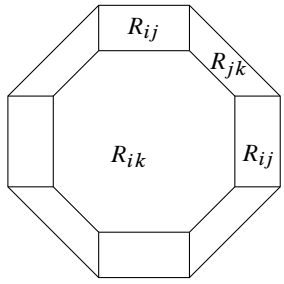

Figure 1. A big region realising R_{ik} in a half sphere for $n = 4$.

Definition 3.1.2 ($\mathcal{I}$-moves). Let $\mathcal{I}$ be a simple identity among relations realised by a uniquely defined sphere $\Sigma = \Sigma_{i,j,k}$. Let M, M_0, N, $S(\Sigma)$ and β_ω be as defined above for some ω. An *$\mathcal{I}$-move at M_0* is a surgery in M by which we cut out M_0 and fill in instead by $\beta_\omega(N)$. Since $\partial N = \partial(\beta_\omega N)$, this is well defined.

Below we consider some examples which illustrate the way $\mathcal{I}$-moves are used.

Example 3.1.3. Let $X = \{a, b, c\}$, $\mathcal{R} = \{R_1, R_2, R_3\}$, where $R_1 = aba^{-1}b^{-1}$, $R_2 = aca^{-1}c^{-1}$, $R_3 = bcb^{-1}c^{-1}$. Here $G \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Let $W = abca^{-1}b^{-1}c^{-1}$. Then W represents 1 in G. Figure 2 shows the corresponding van Kampen diagrams.

Example 3.1.4. Now let $\mathcal{E} = \langle a, b, c \mid \mathcal{R} \mid I \rangle$, $\mathcal{R} = \{R_1, R_2, R_3\}$, $I = R_1$, R_2^b, R_3, $(R_1^{-1})^c$, R_2^{-1}, $(R_3^{-1})^a$ be an extended presentation, where $R^x = xRx^{-1}$ and R_1, R_2, R_3 are as in Example 3.1.3. The effect of I on the diagrams of Figure 2 is the replacement of the diagram in Figure 2 (a) by the diagram in Figure 2 (b).

This changes the positions of the regions realising R_1, R_2 and R_3 and may reduce the number of regions. See Figure 3.

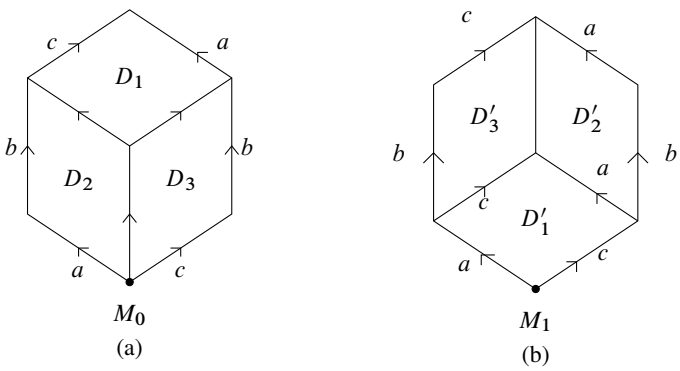

Figure 2. Application of an $\mathcal{I}$-move on $\langle D_1, D_2, D_3 \rangle$.

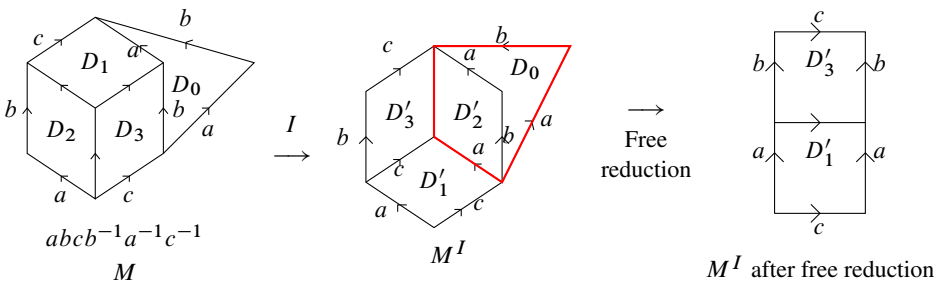

Figure 3. An $\mathcal{I}$-move followed by a free reduction which reduces the number of regions.

3.2. Banded subdiagrams and transfer of regions

In this subsection, we assume $M \in \mathcal{M}_3(W)$ and assumption $\mathcal{H}$.

Lemma 3.2.1. *Let $M \in \mathcal{M}_2(W)$, and let I be an I-move or extended I-move. Then $M^I \in \mathcal{M}_2(W)$.*

Proof. We have to show that I increases neither $|\mathrm{Reg}_{4^+}(M)|$ nor $|\mathrm{Reg}_{4^+}(\mathbb{M})|$. Suppose that I replaces the simply connected subdiagram N with connected interior by a diagram N'.

We use notation of Definition 3.1.1. If $n = 2$, then clearly I cannot alter $|\mathrm{Reg}_{4^+}(M)|$ and $|\mathrm{Reg}_{4^+}(\mathbb{M})|$. If $n \geq 4$, then N contains exactly one big region Δ and N' also contains exactly one big region Δ', which is a copy of Δ. Since $M \in \mathcal{M}_2(W)$, the result follows. ∎

Definition 3.2.2 (Banded subdiagrams). Let $B = \langle E_1, \dots, E_k \rangle$ be a band in M with sides σ_1 and σ_2, and let θ be a boundary path of a simply connected subdiagram M_0 with connected interior.

(a) Say that M_0 *is banded at* θ *by* B if $\|\theta\| \geq 1$ and $\theta \subseteq \sigma_1 \cup \sigma_2$. See Figure 4 (a).

(b) Let notation be as in part (a), and let $\theta = v_0\theta_1 v_1\theta_2 v_2 \cdots \theta_r v_r$ be a decomposition of θ, v_i vertices, $r \geq 1$. Say that M_0 is *banded at* $(\theta_1, \dots, \theta_r)$ *by bands* B_i, respectively, if M_0 is banded at θ_i by B_i for every i, $i = 1, \dots, r$.

Definition 3.2.3 (Transfer of regions from M_0). Let M be a connected, simply connected $\mathcal{R}$-diagram over F with connected interior, and let M_0 be a connected, simply connected $\mathcal{R}$-subdiagram over F with connected interior. Let D be a boundary region of M_0 with $\mu := \partial D \cap \partial M_0$ connected, and let $B = B_1 \cup B_2 \cup B_3$ be such that $\partial D \cap \partial B = \partial D \cap \partial B_2$. Observe that $D \cup B_2$ embeds into the prism P_m, $m = 2n(D)$, $P_m = D \cup B_2 \cup D' \cup B_2'$, where $B_2 \cup B_2'$ is an annulus with m regions from $\mathrm{Reg}_2(M)$ (a closed band) and D is a region with $n(D') = n(D)$ and $\mathrm{Supp}(D) = \mathrm{Supp}(D')$. (We identify D and B_2 with their images in P_m.) Notice that B_2 and B_2' have the same poles α and β, hence $B' = B_1 \cup B_2' \cup B_3$ is a band. Replace $B \cup D$ with $B' \cup D'$ (which have the same boundaries) and denote the resulting diagram by M_0^J, where J is the $\mathcal{I}$-move which replaced $B \cup D$ with $B' \cup D'$. Thus $M_0 \cup B$ is replaced by $((M_0 \cup B) \setminus (D \cup B)) \cup (D' \cup B')$. Hence

$$M_0^J = M_0 \setminus D, \quad B^J = B' \quad \text{and} \quad M_0^J \text{ is banded by } B'. \tag{3.1}$$

In this situation, we say that J *transferred* D *from* M_0. See Figure 4 (b) and 4 (c). If every region can be transferred from M_0, we say that M_0 is *transferable*. The removal of $D \in \mathrm{Reg}(M_0)$ naturally extends to transfer of regions Δ from $\mathbb{M}$. (See Figure 4 (b′) and 4 (c′).)

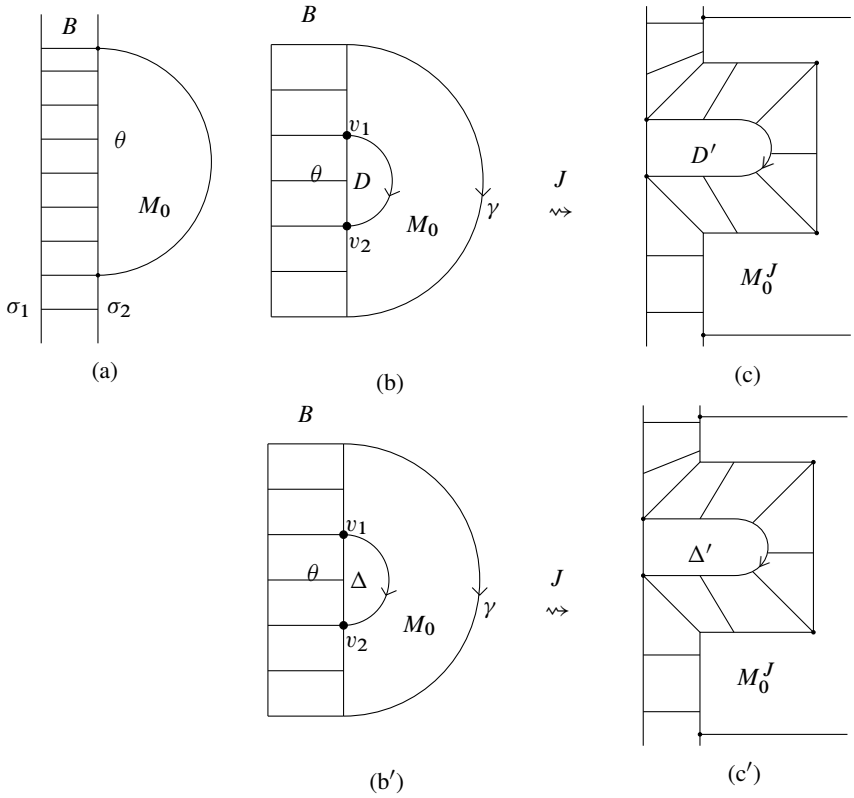

Figure 4. (a)–(c): Transfer of the region D via an adjacent band B by the application of the $\mathcal{I}$-move. (b′) and (c′): Transfer of the modified region Δ via an adjacent band B by the application of the $\mathcal{I}$-move.

Notice that if instead of B we have a band-bundle, then D can be transferred by a sequence of $\mathcal{I}$-moves beyond the band-bundle, without destroying the bundle structure. See Figure 5 (a) and 5 (b) for D and Figure 5 (a′) and 5 (b′) for Δ.

We are interested in a special type of banded subdiagrams (2-banded subdiagrams) which we consider in the next subsection.

Definition 3.2.4 (2-Banded subdiagram). Let $W \neq 1$ be a cyclically reduced word in F which represents 1 in A. Let $M \in \mathcal{M}_3(W)$, and let Q be a connected, simply connected subdiagram of M with connected interior. Say that Q is 2-*banded* if

(1) there is an a-band B_l and a b-band B_r in Q, $a, b \in X$ ($a = b$ is not excluded),

(2) there are connected and simply connected subdiagrams Q_r, Q_l and P in Q and paths θ_r and θ_l such that

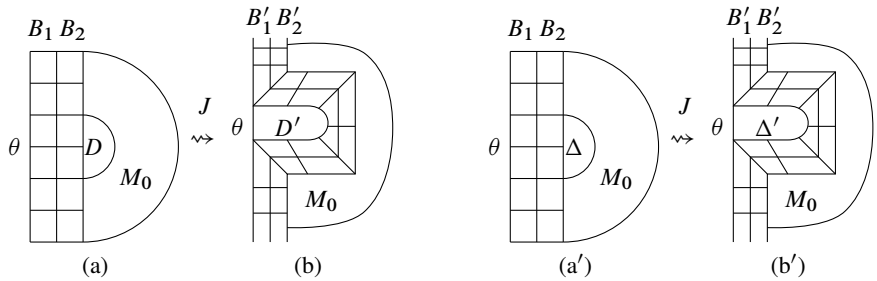

Figure 5. (a) and (b): Transfer of the region D via a band-bundle (β_1, β_2) by the application of the $\mathcal{I}$-move J. (a′) and (b′): Transfer of the modified region Δ via a band-bundle (β_1, β_2) by the application of the $\mathcal{I}$-move J.

(i) $Q = Q_l \cup B_l \cup_{\theta_l} P \cup_{\theta_r} B_r \cup Q_r$ and $\partial Q = \omega_1 \alpha_l^{-1} \gamma \alpha_r^{-1} \omega_3 \beta_r \beta_l$, where $\theta_l = \partial B_l \cap \partial P$, $\theta_r = \partial B_r \cap \partial P$, θ_l and θ_r are sides of B_l and B_r, respectively, $\omega_1 = \partial Q_l \cap \partial Q$, $\gamma = \partial P \cap \partial Q$, $\omega_3 = \partial Q_r \cap \partial Q$ are connected, α_l and β_l are poles of B_l and α_r and β_r are poles of B_r,

(ii) $o(\theta_l) = o(\theta_r)$,

(3) $\mathrm{Int}(P) \neq \emptyset$, then $\Phi(\partial P)$ is cyclically reduced.

Call Q *closed* 2*-banded* if $t(\theta_l) = t(\theta_r)$. Let U be a cyclically reduced word in F which represents 1 in A. Denote by $\mathcal{Q}_M(U)$ the collection of all the 2-banded subdiagrams of M with boundary label U.

We shall show that under certain conditions, there is a boundary region of M_0 which we can transfer from M_0. We wish to repeat the process of transferring until either no regions remain in M_0 or M_0 has a special structure. The next proposition shows that $\mathcal{I}$-moves preserve these "certain conditions", hence we can repeat transferring regions as long as there are regions in M_0^J, or M_0^J does not have the special structure. Here M_0^J is the result of applying J on M_0, J a sequence of $\mathcal{I}$-moves.

Definition 3.2.5 (Standard $\mathcal{I}$-moves). Say that an $\mathcal{I}$-move J is *standard* if θ is a maximal piece in the sense that θ in Figure 5 (b) cannot be extended as a common boundary of D and B.

Proposition 3.2.6. *Let W be a cyclically reduced word in F, $W \neq 1$, which represents* 1 *in A, and let $M \in \mathcal{M}_3(W)$. Let Q be a subdiagram of M with cyclically reduced boundary label U and suppose that $Q \in \mathcal{Q}_M(U)$. (Thus Q is* 2*-banded.) Let notation be as in* [6, Figure 33 (c)].

Assume that P contains a boundary region $\Delta \in \mathrm{Reg}_{4^+,t}(P)$ or $D \in \mathrm{Reg}_2(P)$ such that $\partial\Delta \cap \partial P$ or $\partial D \cap \partial P$, respectively, are connected and $\partial\Delta \cap \partial P = \partial\Delta \cap \theta$, or $\partial D \cap \partial P = \partial D \cap \theta$, $\theta = \theta_r$ or $\theta = \theta_l$ such that $\|\partial\Delta \cap \theta\| \geq 2$, or $\|\partial\theta \cap \partial D\| \geq 2$. Let I be the standard $\mathcal{I}$-move on $B \cup \Delta$, or $B \cup D$, respectively, where $B = B_r$ or $B = B_l$

according as $\theta = \theta_r$ or $\theta = \theta_l$, respectively, which transfers D or Δ, respectively, from Q. Then $Q^I \in \mathcal{Q}_{M^I}(U)$. See the proof of Proposition 3.2.6 *in* [6].

4. Lifting $\mathcal{D}(\widetilde{\mathbb{M}}^t)$ to $\mathbb{M}_t$, $t \in T(M)$, and the proofs of the main results

The idea behind showing that $\widetilde{\mathbb{M}}^t$ satisfies conditions C(4) and T(4) is to use it in M. Obviously, we cannot lift conditions C(4) and T(4) from $\widetilde{\mathbb{M}}^t$ to M; however, we can lift a version of Greendlinger's lemma. This is the content of Theorem C. The first step towards such a lifting is to establish the basic connection between $\widetilde{\mathbb{M}}^t$ and $\mathbb{M}_t$. We consider $\mathbb{M}_t$ as a map and consider $(\mathbb{M}_t)^1$, its 1-skeleton. Following the definition of $\widetilde{\mathbb{M}}^t$, we consider $(\widetilde{\mathbb{M}}^t)^1$ as a quotient of $(\mathbb{M}_t)^1$. Then we extend this from the 1-skeleton to the whole map and show that $|\mathrm{Reg}_{4^+,t}(\mathbb{M}_t)| = |\mathrm{Reg}(\widetilde{\mathbb{M}}^t)|$. Since $\widetilde{\mathbb{M}}^t$ satisfies conditions C(4) and T(4), this allows us to estimate $|\mathrm{Reg}_{4^+,t}(\mathbb{M})|$.

4.1. Preliminary results

4.1.1. The mappings Ψ_t and Ψ_t^{-1}, $t \in T(M)$. Let M be a connected, simply connected $\mathcal{R}$-diagram, $M \in \mathcal{M}_3(W)$ and $t \in T(M)$. Consider the diagram $\mathbb{M}_t$ (defined in Section 2.1). Assume first that it has connected interior. Denote by $(\mathbb{M}_t)^1$ the 1-skeleton of $\mathbb{M}_t$. Define an equivalence relation "$\underset{1}{\approx}$" on the vertices of $(\mathbb{M}_t)^1$ to be the transitive closure of the relation "$\underset{1}{\sim}$" defined as follows: vertices v_1 and v_2 satisfy $v_1 \underset{1}{\sim} v_2$ if either $v_1 = v_2$ or there is an edge e labelled by a letter from H_t (H_t-edge) with $\iota(e) = v_1$ and $\tau(e) = v_2$. Here $\iota(e)$ denotes the initial vertex of e and $\tau(e)$ denotes the terminal vertex of e. Edges e_1 and e_2 satisfy $e_1 \underset{1}{\sim} e_2$ if $e_1 = e_2$. Then by the definition of $\mathbb{M}^t$, $(\mathbb{M}_t)^1 / \underset{1}{\approx}$ is isomorphic to $(\mathbb{M}^t)^1$.

Let E_t be the collection of all the edges of $\mathbb{M}_t$ which are labelled by $t^{\pm 1}$ and define E_{H_t} accordingly.

Denote the projection mapping of $(\mathbb{M}_t)^1$ which realises $\underset{1}{\approx}$ by $\phi_t : (\mathbb{M}_t)^1 \to (\mathbb{M}^t)^1$. Then ϕ_t sends a vertex v in $(\mathbb{M}_t)^1$ to the vertex $\phi_t(v)$ which we denote by v^t. If $e \in E_t((\mathbb{M}_t)^1)$, then $\phi_t(e)$ is an edge which we denote by e^t. If $e \in E_{H_t}((\mathbb{M}_t)^1)$, then $e^t := \phi_t(e)$ is a loop with $\iota(e) = \tau(e)$, which we remove. We can do this because there is no interrelation between a loop and an edge (or loop). So we define $\phi_t(e) = \emptyset$ and $\phi_t(\overline{e}) = v_1^t = v_2^t$, where $\iota(e) = v_1$ and $\tau(e) = v_2$. We define $\phi_t^{-1} : (\mathbb{M}^t)^1 \to (\mathbb{M}_t)^1$ by its set theoretical definition as follows.

Let v^t be a vertex in $\mathbb{M}^t$, $v^t = \phi_t(v)$, v a vertex of $\mathbb{M}_t$.

(a) Define $\phi_t^{-1}(v^t) = \{w \mid w \text{ is } H_t\text{-connected to } v\}$.

(b) If e^t is an edge in $(\mathbb{M}^t)^1$, then $\phi_t^{-1}(e^t)$ is the unique edge $e \in E_t((\mathbb{M}^t)^1)$ such that $\phi_t(e) = e^t$. (Notice that e is necessarily in $E_t((\mathbb{M}_t)^1)$.)

If $\mathrm{Int}(\mathbb{M}_t)$ is not connected, then consider each connected component individually.

Consider now $\widetilde{\mathbb{M}}^t$. Assume first that it has connected interior. Define an equivalence relation "$\underset{2}{\approx}$" on $(\mathbb{M}^t)^1$ to be the transitive closure of the relation "$\underset{2}{\sim}$" defined as follows:

(1) Vertices v_1^t and v_2^t of $(\mathbb{M}^t)^1$ satisfy $v_1 \underset{2}{\sim} v_2$ if and only if $v_1 = v_2$.

(2) Edges e_1^t and e_2^t satisfy $e_1^t \underset{2}{\sim} e_2^t$ if and only if $\tau(e_1^t) = \tau(e_2^t)$ and $\iota(e_1^t) = \iota(e_2^t)$.

It follows from the definition of $\widetilde{\mathbb{M}}^t$ that $(\mathbb{M}_t)^1$ / "$\underset{2}{\approx}$" is isomorphic to $(\widetilde{\mathbb{M}}^t)^1$.

Denote by ξ_t the projection from $(\mathbb{M}^t)^1$ onto $(\widetilde{\mathbb{M}}^t)^1$ which realises $\underset{2}{\approx}$. Then we have

(1) if $v^t \in V((\mathbb{M}^t)^1)$, then $\xi_t(v^t) = \widetilde{v}$, $\widetilde{v}$ a uniquely defined vertex in $\widetilde{\mathbb{M}^t}^1$,

(2) if $e^t \in E((\mathbb{M}^t)^1)$, then $\xi_t(e^t) = \widetilde{e}$, $\widetilde{e}$ a uniquely defined edge in $\widetilde{\mathbb{M}^t}^1$.

We define $\xi_t^{-1} : \widetilde{\mathbb{M}}^t \to \mathbb{M}^t$ as follows:

(1) If $\widetilde{v} \in V((\widetilde{\mathbb{M}^t})^1)$, then $\xi_t^{-1}(\widetilde{v}) = v^t$, where v^t is the unique vertex in $(\mathbb{M}^t)^1$ with $\xi_t(v^t) = \widetilde{v}$.

(2) If $\widetilde{e} \in E((\widetilde{\mathbb{M}^t})^1)$, then $\xi_t^{-1}(\widetilde{e}) = \{e^t \in \mathbb{M}^t \mid \iota(e^t) = \iota(e_0^t) \text{ and } \tau(e^t) = \tau(e_0^t)\}$, where $\xi_t(e_0^t) = \widetilde{e}$.

Finally, define $\Psi_t : (\mathbb{M}_t)^1 \to (\widetilde{\mathbb{M}}^t)^1$ by $\Psi_t = \xi_t \circ \phi_t$. Then $\Psi_t^{-1} = \phi_t^{-1} \circ \xi_t^{-1}$. It follows by checking Euler characteristics that $\widetilde{\mathbb{M}}^t$ is planar. We have the following.

Lemma 4.1.2. *The following statements hold:*

(1) *Let $e \in E_t((\mathbb{M}_t)^1)$, and let B_e^1 be the 1-skeleton of the t-band which contains e. Then $\Psi_t^{-1} \circ \Psi_t(\overline{e}) = B_e^1$. In particular, $\overline{e} \subseteq \Psi_t^{-1} \circ \Psi_t(\overline{e})$.*

(2) *Let $e \in E_t((\mathbb{M}_t)^1)$. Then $\Psi_t^{-1} \circ \Psi_t(e) = e$.*

(3) *Let $v \in V((\mathbb{M}_t)^1)$. Then $\Psi_t^{-1} \circ \Psi_t(v) = [v]_{H_t}$. In particular, $v \in \Psi_t^{-1} \circ \Psi_t(v)$.*

(4) *Let $\widetilde{v} \in V((\widetilde{\mathbb{M}}^t)^1)$. Then $\Psi_t \circ \Psi_t^{-1}(\widetilde{v}) = \widetilde{v}$.*

(5) *$\Psi_t(\iota(e)) = \iota(\Psi_t(e))$ and $\Psi_t(\tau(e)) = \tau(\Psi_t(e))$ for every edge $e \in V_t((\mathbb{M}_t)^1)$.*

Proof. Immediate by the definitions of Ψ_t and Ψ_t^{-1}. ∎

Proposition 4.1.3. *Let notation and assumptions be as above. Then* $|\mathrm{Reg}_{4^+,t}(\mathbb{M}_t)| = |\mathrm{Reg}(\widetilde{\mathbb{M}}^t)|$.

Proof. See [6, pp. 82–83]. ∎

4.2. The subdiagrams $\mathrm{Opt}(\Delta)$, $\mathrm{Opt}_r(\Delta)$ and $\mathrm{Opt}_l(\Delta)$, $\Delta = \Psi_t^{-1}(\widetilde{\Delta})$, $\widetilde{\Delta} \in \mathcal{D}(\widetilde{\omega}_i)$, $i = 1, 2$

Let $M \in \mathcal{M}_3(W)$ and assume $\mathrm{Int}(M)$ is connected. In general, this does not imply that $\widetilde{\mathbb{M}}^t$ has connected interior. In the rest of the work, we aim to prove Theorems A–C. It is enough to prove these theorems for every connected component of $\mathrm{Int}(\widetilde{\mathbb{M}}^t)$. Hence without loss of generality, we may work under the assumption that $\widetilde{\mathbb{M}}^t$ has connected interior, and we

shall do so. Thus *for the rest of the work, we shall assume that* $\widetilde{\mathbb{M}}^t$ *has connected interior, unless said otherwise.*

Definition 4.2.1 ($\mathrm{Opt}(\Delta)$, $\mathrm{Opt}_r(\Delta)$, $\mathrm{Opt}_l(\Delta)$, $\widehat{\mathrm{Opt}}(\Delta)$). Suppose $\widetilde{\Delta} \in \mathcal{D}(\widetilde{\mathbb{M}}^t)$, and let $\Delta = \Psi_t^{-1}(\widetilde{\Delta})$. Assume $\partial\widetilde{\Delta} \cap \partial\widetilde{\mathbb{M}}^t = \widetilde{v}_1\widetilde{e}_1\widetilde{v}_2\widetilde{e}_2\widetilde{v}_3\widetilde{e}_3 \cdots \widetilde{v}_k$, $k \geq 2$, $\widetilde{v}_i$ vertices, $\widetilde{e}_i$ edges. Define $\mathrm{Opt}(\Delta) = \Psi_t^{-1}(\widetilde{e}_1\widetilde{v}_2 \cdots \widetilde{e}_{k-1})$, define $\mathrm{Opt}_r(\Delta) = \Psi_t^{-1}(\widetilde{e}_1\widetilde{v}_2 \cdots \widetilde{e}_{k-1}\widetilde{v}_k)$ and define $\mathrm{Opt}_l(\Delta) = \Psi_t^{-1}(\widetilde{v}_1\widetilde{e}_1\widetilde{v}_2 \cdots \widetilde{e}_{k-1})$. Thus $\mathrm{Opt}_r(\Delta) = \mathrm{Opt}(\Delta) \cup \Psi_t^{-1}(\widetilde{v}_k)$ and $\mathrm{Opt}_l(\Delta) = \mathrm{Opt}(\Delta) \cup \Psi_t^{-1}(\widetilde{v}_1)$. Define $\widehat{\mathrm{Opt}}(\Delta) = \Psi_t^{-1}(\widetilde{v}_1\widetilde{e}_1 \cdots \widetilde{v}_k)$. It follows from the definition of ψ_t^{-1} that $\mathrm{Opt}(\Delta) = \Psi_t^{-1}(\widetilde{e}_1) \cup \Psi_t^{-1}(\widetilde{v}_2) \cup \Psi_t^{-1}(\widetilde{e}_2) \cup \cdots \cup \Psi_t^{-1}(\widetilde{e}_{k-1}) = B_1 \cup Z_2 \cup B_2 \cup Z_3 \cup \cdots \cup B_{k-1}$, where B_i are t-bands emanating from $\partial\Delta \cap \partial\,\mathrm{Opt}(\Delta)$ and Z_i are connected unions of regions from $\mathrm{Reg}_{H_t}(\mathbb{M}_t)$.

Thus $\Delta \cup \mathrm{Opt}(\Delta)$, $\Delta \cup \mathrm{Opt}_r(\Delta)$, $\Delta \cup \mathrm{Opt}_l(\Delta)$ and $\widehat{\mathrm{Opt}}(\Delta)$ are connected subdiagrams of $\mathbb{M}_t$. Due to Proposition 2.1.7, $\partial M \cap \partial B_i$, $\partial Z_i \cap \partial\Delta$ and $\partial B_i \cap \partial\Delta$ are connected, and it is easy to see that $\partial\Delta \cap \partial\,\mathrm{Opt}(\Delta)$, $\partial\Delta \cap \partial\,\mathrm{Opt}_r(\Delta)$, $\partial\Delta \cap \partial\,\mathrm{Opt}_l(\Delta)$ and $\partial\Delta \cap \partial\widehat{\mathrm{Opt}}(\Delta)$ are connected.

We are interested in $\mathrm{Opt}_r(\Delta)(\mathrm{Opt}_l(\Delta))$ for the special case when $\widetilde{v}_k(\widetilde{v}_1)$ has valency 3 in $\widetilde{\mathbb{M}}^t$.

Lemma 4.2.2. *Let $M \in \mathcal{M}_3(W)$. Assume that ∂M decomposes by $\partial M = u\omega_1 w\omega_2^{-1}$, u, w vertices and $t \in T(M)$. Let $\widetilde{\Delta}_1^t$ and $\widetilde{\Delta}_2^t$ be boundary regions of $\widetilde{\mathbb{M}}^t$, $\widetilde{\Delta}_1^t$ and $\widetilde{\Delta}_2^t$ in $N(\widetilde{\omega}_1)$, $\Delta_i = \Psi_t^{-1}(\widetilde{\Delta}_i^t)$, $\mathrm{Supp}(\Delta_i) = \{a, t\}$, $\widetilde{\omega}_1 = \Psi_t(\omega_1)$. Assume that $\partial\widetilde{\Delta}_1 \cap \partial\widetilde{\Delta}_2 \cap \partial\widetilde{\omega}_1$ is a vertex $\widetilde{v}^t$ with valency 3 in $\widetilde{\mathbb{M}}^t$. Let $Q = \Psi_t^{-1}(\widetilde{v})$, and let $P = \Psi_t^{-1}(N(\widetilde{v}))$. Thus $P = \Psi_t^{-1}(\widetilde{\Delta^t}_1 \cup \widetilde{\Delta_2^t} \cup (\partial\widetilde{\Delta^t}_1 \cap \partial\widetilde{\Delta^t}_2) \cup \widetilde{e}_k \cup \widetilde{f}_1 \cup \widetilde{v})$, where $\widetilde{e}_k$ and $\widetilde{f}_1$ are the boundary paths of $\widetilde{\Delta}_1^t$ and $\widetilde{\Delta}_2^t$ with common endpoint $\widetilde{v}$. Then each of the following holds:*

(a) *Q is connected and simply connected with cyclically reduced boundary label such that each connected component Q_i, $i = 1, \ldots, k$ of Q has cyclically reduced boundary labels U_i. Moreover, $Q_i \in \mathcal{M}_3(U_i)$, $i = 1, \ldots, k$.*

(b) *P is connected and simply connected with connected interior and cyclically reduced boundary label V, $P \in \mathcal{M}_3(V)$.*

(c) *$\partial Q = \theta_1 p_1 \alpha_1 p_2 \theta_2 p_3 \alpha_2 p_4 \theta_3 p_5 \gamma^{-1}$, where p_i are vertices, θ_i are sides of t-bands $i = 1, 2, 3$ and $\alpha_j = \partial\Delta_j \cap \partial Q$, $j = 1, 2$, $\Phi(\alpha_j) = a^{c_j}$, $c_j \neq 0$ and $\gamma = \partial Q \cap \omega_1$.*

Proof of Lemma 4.2.2. See the proof of Lemma 5.2.2 in [6]. ∎

4.3. Definitions and assumptions

Generalised Greendlinger regions. In our earlier work [6, Section 1.3], we introduced (classical) Greendlinger regions. In the rest of the work, we shall need more general notions of Greendlinger regions. Recall from [6, Section 1.3] $\mathcal{D}_i(M)$, $i = 1, 2$, and $\mathcal{D}(M)$ that M is a connected, simply connected map with connected interior which contains at least two regions.

If $\partial M = u\mu v\nu^{-1}$, μ and ν boundary paths, we shall call M *bigonal with sides μ and ν*. We define $\mathcal{D}_i(\mu)$, $\mathcal{D}_i(\nu)$, $i = 1, 2$ and $\mathcal{D}(\mu)$ and $\mathcal{D}(\nu)$ as follows.

Definitions 4.3.1. Let M be a bigonal map with sides μ and ν. Define

(i) $\mathcal{D}_j(\mu) = \{D \in \mathrm{Reg}(M), \partial D \cap \mu$ is connected and $i(D) = j\}$ $j = 1, 2$. $\mathcal{D}_j(\nu)$ is defined accordingly. Furthermore, define $\mathcal{D}(\mu) = \mathcal{D}_1(\mu) \cup \mathcal{D}_2(\mu) \cup \mathcal{D}_0(\mu)$, where $\mathcal{D}_0(\mu)$ is defined only if $M = \{D\}$. In this case, $\mathcal{D}_0(\mu) = \{D\}$ if $|\mu| \geq |\nu|$ and $\mathcal{D}_0(\nu) = \{D\}$ if $|\nu| \geq |\mu|$. In case $|\mu| = |\nu|$, we choose $\mathcal{D}_0(\mu) = \{D\}$.

(ii) Let M be an $\mathcal{R}$-diagram, $\mathcal{R}$ given by (1.1). Assume $M \in \mathcal{M}_3(W)$ and $\partial M = \mu\nu^{-1}$.
Define

$$\mathbb{D}(\mu) = \{\Delta \in \mathrm{Reg}_{4+}(\mathbb{M}) \mid \partial\Delta \cap \mu \text{ is connected}, \|\partial\Delta \cap \mu\| \geq n(\Delta) \text{ and } |\Phi(\xi)| \leq |\Phi(\partial\Delta \cap \mu)|\}.$$

Here, ξ is the complement of $\partial\Delta \cap \mu$ on $\partial\Delta$. Define $\mathbb{D}(\nu)$ accordingly.

(iii) Define

$$\mathcal{D}^v(\mu) = \{\Delta \in \mathrm{Reg}_{4+}(\mathbb{M}) \mid \Delta \text{ contains a subdiagram } S \text{ with } S \in \mathrm{O.L.}(\theta_1, \theta_2), S \subseteq N(\mu), \partial S \cap \mu = \theta_1, \|\theta_1\| \geq n(\Delta) \geq 4 \text{ and } |\Phi(\theta_2)| \leq |\Phi(\theta_1)|\}.$$

Define $\mathcal{D}^v(\nu)$ accordingly.

(iv) Define
$\mathcal{D}_2^1(\mu) = \{D \in \mathcal{D}_1(\mu), n(D) = 2\}$, $\mathcal{D}_2^2(\mu) = \{D \in \mathcal{D}_2(\mu), n(D) = 2\}$.

(v) Define σ : boundary paths $\to \{0, 1\}$ as follows:
$\sigma(\mu) = 1$ if one of the following holds:

(1) $\mathcal{D}_2^1(\mu) \neq \emptyset$.

(2) $|\mathcal{D}_2^2(\mu)| \geq 2$.

(3) $|\mathcal{D}_2^2(\mu)| = 1$, and there is a region Δ in $\mathbb{M}_{4+}$ and a sequence J of $\mathcal{I}$-moves such that $\partial\Delta^J \cap \mu$ is connected and $|\partial\Delta^J \cap \mu| \geq 2$.

(4) $\mathcal{D}_2^2(\mu) = \emptyset$, and $\mathbb{D}(\mu) \cup \mathcal{D}^v(\mu) \neq \emptyset$.

In all other cases, $\sigma(\mu) = 0$.

(vi) Let $\widetilde{\mu} = \Psi_t(\mu)$. Define

$$\mathcal{D}'(\widetilde{\mu}) = \{\widetilde{\Delta} \in \mathrm{Reg}(\widetilde{\mathbb{M}}^t) \mid \Delta \in \mathcal{D}^v(\mu), \Delta = \Phi_t^{-1}(\widetilde{\Delta})\}.$$

Assumptions 4.3.2. The proofs of Theorems A–C rely on other results as well (below) which are proved by simultaneous induction. They are

(I) Theorems A–C,

(II) Theorems S, L and T.

Theorem 4.3.3 (Theorem S). *Let $M \in \mathcal{M}_3(W)$ and assume that ∂M decomposes by $\omega_1 v_1 \omega_2^{-1} v_2$, where ω_1 and ω_2 are boundary paths and v_1 and v_2 are boundary vertices. Let $t \in T(M)$, and let $\widetilde{\Delta} \in \mathcal{D}(\widetilde{\omega}_1)$, $\widetilde{\omega}_1 = \Psi_t(\omega_1)$, $\Delta = \Psi_t^{-1}(\widetilde{\Delta})$, $\mathrm{Supp}(\Delta) = \{a, t\}$.*

Assume $\sigma(\omega_1) = 0$ in parts (a), (b) and (c). Then there is a sequence J of $\mathcal{I}$-moves such that $\Delta^J = \Delta$, $M^J \in \mathcal{M}_3(W)$, and each of the following holds:

(a) *If β is an edge of $(\partial\Delta \cap \partial\widehat{\mathrm{Opt}}(\Delta))^J$, then there is a band which has a pole on β and the other pole on ω_1. In particular, $(\partial\Delta \cap_B \omega_1)^J$ is connected.*

(b) *If B_1 and B_2 are bands emanating from $(\partial\Delta \cap_B \omega_1)^J$ with adjacent poles on $\partial\Delta$, then $\partial B_1 \cap \partial B_2$ is connected. Moreover, let $L_1, \dots, L_k$ be the bands that emanate from $(\partial\Delta \cap_B \omega_1)^J$. Then $\partial L_i \cap \partial L_{i+1}$ is connected, $i = 1, \dots, k-1$, where L_i and L_{i+1} have adjacent poles on $\partial\Delta$.*

(c) *Let $Z := \widehat{\mathrm{Opt}}(\Delta)^J \cup \omega_1 \setminus \{L_1, \dots, L_k\} = Q_1 \dot\cup \cdots \dot\cup Q_l$, where Q_i are the connected components.*

  (i) *∂Q_i are simple closed curves and ∂Q_i decompose by $\partial Q_i = \rho_i \xi_i \gamma_i^{-1}$, where $\rho_i = \partial Q_i \cap \partial L_j$, $\xi_i = \partial Q_i \cap \partial L_{j+1}$ and $\gamma_i = \partial Q_i \cap \omega_1$, $j \in \{1, \dots, k\}, i = 1, \dots, l$.*

  (ii) *Up to diamond moves, $U_i := \Phi(\partial Q_i)$ is cyclically reduced and $Q_i \in \mathcal{M}_3(U_i), i = 1, \dots, l$, and either $Q_i = \{D_i\}, n(D_i) = 2$ or Q_i is transferable via $L_j \cup L_{j+1}$. See Figure 6.*

  (iii) *For every $D \in \mathrm{Reg}(Q_i), i = 1, \dots, l, \mathrm{Supp}(\Phi(\partial D)) \subseteq X \setminus \{a, t\}$.*

(d) *Let $\eta = (\partial\Delta \cap_B \omega_1)^J$. Then the following hold:*

  (i) *If $\Delta \in \mathbb{D}(\eta)$, then $\sigma(\omega_1) = 1$.*

  (ii) *If $\widetilde{\mathbb{M}}^t \in$ O.L., $\widetilde{\mathbb{M}}^t = \langle \widetilde{\Delta}_1, \dots, \widetilde{\Delta}_m \rangle$, $m \geq 2$ and $\Delta_j \in \mathcal{D}^v(\eta)$, for $j = 1$ and $j = m$, then $\sigma(\omega_1) = 1$.*

Remarks (Pushing up Δ). (1) It follows from part (c) that if $\mathcal{D}_2^2(\omega_1) = \emptyset$, then $\mathrm{Opt}(\Delta)$ is a homogeneous band-bundle, hence we can push up Δ by a sequence J_1 of $\mathcal{I}$-moves to ω_1, such that $\Delta^j \in \mathbb{D}(\mu)$, obtaining the classical result on Greendlinger regions.

(2) The same results hold if we use ν in place of μ.

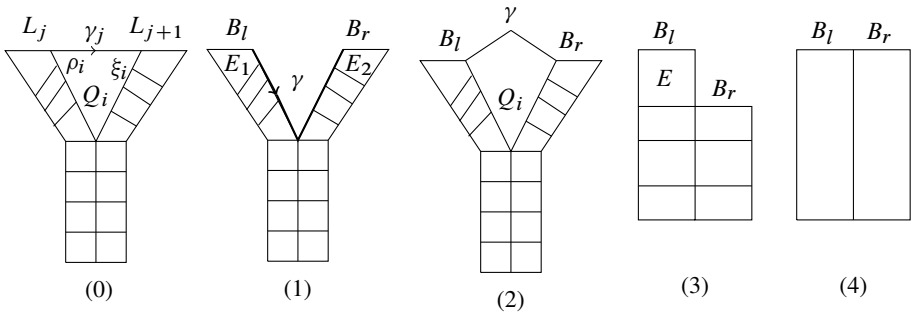

Figure 6. The possible configurations of the adjacent bands in $\mathrm{Opt}(\Delta)$ emanating from Δ.

Theorem 4.3.4 (Theorem L). *Let $M \in \mathcal{M}_3(W)$, $\partial M = u\omega_1 v\omega_2^{-1}$, u and v vertices, ω_1 and ω_2 boundary paths of M. Assume $T(M) \neq \emptyset$, and let $t \in T(\mathbb{M})$. Let $\widetilde{\omega}_1 = \Psi_t(\omega_1)$, and let $\widetilde{\Delta} \in \mathrm{Reg}(\widetilde{\mathbb{M}}^t)$. Let $\Delta = \Psi_t^{-1}(\widetilde{\Delta})$, and let $\eta = \partial\Delta \cap_B \omega_1$. Let* $\mathrm{Supp}(\Delta) = \{a, t\}$.

(a) *If $\widetilde{\Delta} \in \mathcal{D}(\widetilde{\omega}_1)$, then $\sigma(\omega_1) = 1$, $\|\eta\|_a \geq 2$ and $\|\eta\|_t \geq 2$.*

(b) *If $\widetilde{\Delta} \in \mathcal{D}'(\widetilde{\omega}_1)$, then $\sigma(\omega_1) = 1$, $\|\eta\|_a \geq 2$ and $\|\eta\|_t \geq 2$.*

Theorem 4.3.5 (Theorem T). *Let $M \in \mathcal{M}_3(W)$, and let P be a simply connected, connected subdiagram of M with at least one region. Assume that P is closed 2-banded in M by bands B_r and B_l. If ∂P has cyclically reduced label U and $P \in \mathcal{M}_3(U)$, then P is transferable. In particular, there is a sequence J of $\mathcal{I}$-moves such that $B_r^J \cap B_l^J$ is connected and $B_r^J \cup B_l^J$ is a band-bundle. Moreover, $|B_r^J| = |B_r|$ and $|B_l^J| = |B_l|$.*

In each of these theorems, $M \in \mathcal{M}_3(W)$ and $\partial M = u\mu v\nu^{-1}$, u and v vertices. The induction hypothesis claims that if N is a proper subdiagram of M with cyclically reduced boundary label U_N such that $N \in \mathcal{M}_3(U_N)$, then the corresponding result holds for N. We shall denote these hypotheses by $\mathcal{H}(A)$, $\mathcal{H}(B)$, $\mathcal{H}(C)$, $\mathcal{H}(S)$, $\mathcal{H}(L)$ and $\mathcal{H}(T)$, respectively. So we assume in the sequel that

$$\mathcal{H}(A), \mathcal{H}(B), \mathcal{H}(C), \mathcal{H}(S), \mathcal{H}(L) \text{ and } \mathcal{H}(T) \text{ hold.} \tag{$\mathcal{H}_1$}$$

4.4. The proof of Theorem A

Lemma 4.4.1. *Let W be a cyclically reduced word in F, $W = 1$ in A and $W \neq 1$ in $A(\Gamma_2)$. Let M be a van Kampen $\mathcal{R}$-diagram over F, $\mathcal{R}$ given by (1.1). Assume that $M \in \mathcal{M}_1(W)$ and all bands are adequate. Let Δ_1 and Δ_2 be regions in $\mathrm{Reg}_{4+}(\mathbb{M})$, with $\mathrm{Supp}(\Delta_1) = \mathrm{Supp}(\Delta_2) = \{a, t\}$, $\Delta_1 \neq \Delta_2$. Assume that there are bands B_1 and B_2 connecting adjacent edges $e_1 v_1 e_2$ of $\partial\Delta_1$ with adjacent edges $f_1 w_1 f_2$ of $\partial\Delta_2$, v_1 and w_1 are vertices such that $\Phi(e_1) = a$ and $\Phi(e_2) = t$. Then $M \notin \mathcal{M}_2(W)$. In particular, if $M \in \mathcal{M}_3(W)$, then the poles of B_1 and B_2 cannot be adjacent both on $\partial\Delta_1$ and on $\partial\Delta_2$.*

Proof. See the proof of Lemma 5.4.1 in [6]. ∎

Lemma 4.4.2. *Let $M \in \mathcal{M}_3(W)$ and assume $T(M) \neq \emptyset$. Let $t \in T(M)$. Then*

(a) *$\widetilde{\mathbb{M}}^t$ has no inner vertices with valency 2,*

(b) *$\widetilde{\mathbb{M}}^t$ satisfies condition* C(4).

Proof. See the proof of Lemma 5.4.2 in [6]. ∎

4.4.3. Proof of Theorem A. If $\widetilde{\mathbb{M}}^t$ does not satisfy condition T(4), then due to Lemma 4.4.2, $\widetilde{\mathbb{M}}^t$ contains an inner vertex $\widetilde{v}$ with valency 3. Let $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$ and $\widetilde{\Delta}_3$ be the regions of $\widetilde{\mathbb{M}}^t$ which contain $\widetilde{v}$ on their boundary. Let $\Delta_i = \Psi_t^{-1}(\widetilde{\Delta}_i)$, $i = 1, 2, 3$, and let $Q = \Psi_t^{-1}(\widetilde{v})$. It is not difficult to show by arguments similar to those used in the proof of Lemma 4.2.2 that $\psi_t^{-1}\left(\bigcup_{i=1}^{i=3} \widetilde{\Delta}_i\right)$ is connected and simply connected, and $\partial Q = \alpha_1\theta_1\alpha_2\theta_2\alpha_3\theta_3$, $\alpha_i = \partial\Delta_i \cap \partial Q$ and θ_i are sides of bands B_i, $\theta_i = \partial B_i \cap \partial Q$, $i = 1, 2, 3$,

and α_i is either a point or a path with label $a_i^{k_i}$, $k_i \neq 0, \mathrm{Supp}(\Delta_i) = \{t, a_i\}$, $i = 1, 2, 3$. Consider $U := \Phi(\partial Q)$. Since $\mathrm{Supp}(\Phi(\theta_i)) \in C_\Gamma(t)$ while if α_i is not a point, then α_i is labelled with $a_i^{k_i} \notin C_\Gamma(t)$, the only occurrences of letters in U which do not commute with t are the non-trivial labels of α_i. In particular,

$$\left.\begin{array}{l}\text{there are at most three disjoint occurrences of letters in } U\\ \text{which do not commute with } t\end{array}\right\}. \qquad (*)$$

We claim that if α_i and α_j are not points, then $a_i = a_j$. Suppose not. By [6, Lemma 1.1.c], there are at least two disjoint occurrences of a_i and two disjoint occurrences of a_j in U, hence U has at least four such occurrences violating (*). Hence $a_i = a_j$. Denote $a = a_i = a_j$. It is easy to see that U is cyclically reduced and $Q \in \mathcal{M}_3(U)$. In particular, Theorem B applies to Q ($|Q| < |M|$). Hence if D is a region of Q with $a \in \mathrm{Supp}(D)$, then $n(D) \leq 3$, due to (*). Since $n(D) \neq 3$, hence $n(D) = 2$. Hence Q is a-abelian. Consequently, if α_i is not a point, then from each edge of α_i, there emanates a band L in Q which has one of its poles on α_i and the other on α_j. Observe that L and B_i have adjacent poles on Δ_i and on Δ_j. But this violates Lemma 4.4.1. Hence all α_i are points. But then since three is odd, the three changes of orientations of the poles of $\beta_i, i = 1, 2, 3$, force $t = t^{-1}$. This violates [6, Lemma 1.1 (a)]. Hence $d(\widetilde{v}) \neq 3$. It follows from Lemma 4.4.2 (b) that $\widetilde{\mathbb{M}}^t$ satisfies conditions C(4) and T(4). ∎

4.5. Preparatory results for the proofs of Theorems S, L, T and B

Lemma 4.5.1. *Let $M \in \mathcal{M}_3(W)$ be a connected, simply connected $\mathcal{R}$-diagram with connected interior. Suppose $\partial M = u_1\omega_1 u_2 \omega_2^{-1}$ and $T(M) \neq \emptyset$, where u_1 and u_2 are vertices.*

Let $t \in T(M)$ and consider $\widetilde{\mathbb{M}}^t$. Assume $\mathrm{Reg}(\widetilde{M}^t) = \{\widetilde{\Delta}\}$. Let $\Delta = \Psi_t^{-1}(\widetilde{\Delta})$, $\mathrm{Supp}(\Delta) = \{t, a\}$ and define $\mathrm{Opt}_{\omega_i}(\Delta) := \Psi_t^{-1}(\widetilde{\omega}_i), i = 1, 2$. Assume Theorem S holds for M. Then $\sigma(\omega_i) = 1$ for $i = 1$ or $i = 2$, and $\|\omega_i\|_t \geq 2$.

Proof. See the proof of Lemma 5.5.1 in [6]. ∎

Lemma 4.5.2. *Let $M \in \mathcal{M}_3(W)$ with boundary cycle $\mu u v^{-1} v$, where u and v are vertices. Assume that $T(M) \neq \emptyset$, and let $t \in T(M)$. Let $\widetilde{\Delta} \in \mathrm{Reg}(\widetilde{\mathbb{M}}^t)$, and let $\Delta = \Psi_t^{-1}(\widetilde{\Delta})$. Let $\eta = (\partial\Delta \cap \partial\widetilde{\mathrm{Opt}}(\Delta)) \cap_B \mu$, and let $\eta_0 = (\partial\Delta \cap \partial\mathrm{Opt}(\Delta)) \cap_B \mu$. Assume η is connected. Then in each of the following cases, $\Delta \in \mathbb{D}(\eta)$ and $\|\Phi(\mu)\|_t \geq 2$:*

(a) $\widetilde{\Delta} \in \mathcal{D}_1(\widetilde{\mu})$.

(b) $\widetilde{\Delta} \in \mathcal{D}_2(\widetilde{\mu})$, *and there is an H_t-band L emanating from $\eta \setminus \eta_0$ and ending on μ.*

Moreover, in case (a), $\|\Phi(\mu)\|_t \geq 3$,

Proof. See the proof of Lemma 5.5.2 in [6]. ∎

Remark. Lemma 4.5.2 remains true if we replace μ by ν.

Lemma 4.5.3. *Let $M \in \mathcal{M}_3(W)$ be a connected, simply connected $\mathcal{R}$-diagram over F with connected interior. Let $\partial M = v_1\omega_1 v_2\omega_2^{-1}$, v_1 and v_2 vertices and ω_1 and ω_2 boundary paths. Let $t \in T(M)$ and assume that $|\widetilde{\mathbb{M}}^t| \geq 2$ and $\widetilde{\mathbb{M}}^t \in \mathrm{O.L.}(\widetilde{\omega_1}, \widetilde{\omega_2})$, $\widetilde{\omega_i} = \Psi_t(\omega_i)$, $\widetilde{\mathbb{M}}^t = \langle \widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_k \rangle$, $k \geq 2$. If $\sigma(\omega_1) = 0$ and Theorem S holds for M, then each of the following holds:*

(a) $\sigma(\partial \operatorname{Opt}(\Delta_i) \cap_B \omega_2) = 1$, $i = 1, k$.

(b) $\|\omega_2\|_t \geq 3$.

Proof. See [6, pp. 98–99]. ∎

Lemma 4.5.4. *Let notation and assumptions be as in Lemma 4.2.2. Let $\eta_i = \partial\Delta_i \cap_B \omega_i$, $i = 1, i = 2$. Let $\omega_1 = \gamma$ and $\omega_2 = \theta_1\alpha_1\theta_3\alpha_2\theta_2$. (Thus $|Q| < |M|$, and $\omega_1 u \omega_2^{-1}$ is a boundary cycle of Q.) If $\sigma(\omega_1) = 0$, then Q is a-abelian.*

Proof. See the proof of Lemma 5.5.4 in [6]. ∎

Corollary. *Let notation and assumptions be as in the lemma. If $\widetilde{\Delta} \in \mathcal{D}_2(\widetilde{\mu})$, then $\Delta \in \mathbb{D}(\mu)$.*

Proof. Due to the lemma, condition (b) of Lemma 4.5.2 is satisfied. Hence the result follows from part (b) of Lemma 4.5.2. ∎

4.6. Proof of Theorem S

(a) Let $\tau_1 w_1 \alpha w_2 \tau_2$ be a subpath of $\partial\Delta \cap \operatorname{Opt}(\Delta)$, w_1 and w_2 vertices, $\Phi(\tau_1) = t^{\pm 1}$, $\Phi(\tau_2) = t^{\pm 1}$, $\Phi(\alpha) = a^f$, $f \neq 0$. It follows from the definition of $\operatorname{Opt}(\Delta)$ that there are t-bands L_1 and L_2 emanating from τ_1 and τ_2, respectively, and ending on ∂M. Let θ_1 be the side of L_1 which has w_1 as an endpoint, and let θ_2 be the side of L_2 which has w_2 as an endpoint. Then $\operatorname{Supp}(\Phi(\theta_i)) \subseteq C_\Gamma(t)$, $i = 1$ and $i = 2$. Let z_1 and z_2 be the endpoints of θ_1 and θ_2, respectively, on ∂M, and let γ be the subpath of $\partial M \cap \partial \operatorname{Opt}(\Delta)$ which starts at z_1 and ends at z_2. Then $\sigma(\gamma) = 0$ as $\gamma \subseteq \omega_1$ and $\sigma(\omega_1) = 0$. Also, $U := \Phi(\theta_1\alpha\theta_2\gamma^{-1})$ is cyclically reduced, since $a \notin C_\Gamma(t)$. Let Q be the bounded component of $\mathbb{E}^2 \setminus (\theta_1\alpha\theta_2\gamma^{-1})$. Assume first that $\partial Q (= \theta_1\alpha\theta_2\gamma^{-1})$ is a simple closed path. Hence $\operatorname{Int}(Q)$ is connected. It follows from [6, Lemma 2.2.1] that $Q \in \mathcal{M}_3(U)$. Hence Theorem A applies to Q, and it follows from the proof of Lemma 4.5.4 that if Q is not a-abelian, then one of the four cases mentioned in the proof of Lemma 5.5.4 in [6] occurs. From this point on, we follow the proof of Lemma 4.5.4, with $\|\omega_2\|_a = 1$ (here) instead of $\|\omega_2\|_a = 2$, showing that each case leads to a contradiction, hence Q is a-abelian, more precisely, in the first case, by Lemma 4.5.1, in the second case, by Lemma 4.5.3, in the third case, by Lemma 4.5.2 and in the fourth case, by Theorem L. Therefore, if $\beta = \beta_1 \cdots \beta_k$, $\Phi(\beta_i) = a^{\pm 1}$, then from every β_i, there emanates an a-band which ends on γ, since $a \notin C_\Gamma(t)$. (Notice that it cannot

end on β.) If $\mathrm{Int}(Q)$ is not connected, then a similar argument applies to each component.

(b) Let B_1 and B_2 be bands emanating from $\partial\Delta \cap \partial\,\mathrm{Opt}(\Delta)$ and ending on ω_1. Assume that their poles on $\partial\Delta$ are adjacent. Let ξ_1 and ξ_2 be the sides of B_1 and B_2, respectively, with $w := \xi_1 \cap \xi_2 \cap \partial\Delta \neq \emptyset$ a point. We claim that $\xi_1 \cap \xi_2$ is connected. Suppose not and consider $\mathbb{E}^2 \setminus (\xi_1 \cup \xi_2)$. Let P be a bounded connected component of it. Then $\partial P = (\partial P \cap \xi_1) \cup (\partial P \cap \xi_2)$. Let $V := \Phi(\partial P)$. Then applying diamond moves at the endpoints of $\partial P \cap \xi_1$ if needed, we may assume that V is cyclically reduced. It follows from [6, Lemma 2.2.1] that $P \in \mathcal{M}_3(V)$. Hence Theorem T applies to P ($|P| < |M|$). Hence by a sequence J_0 of $\mathcal{I}$-moves, P is transferable beyond B_1 and B_2 such that P^J contains no regions. Doing so independently for every connected component P_i, by $J_i, i = 1, \dots, r$, we get that $(B_1 \cap B_2)^J$ is connected, where $J = J_1 \circ J_2 \circ \cdots \circ J_r$.
Consider now $\mathbb{E}^2 \setminus \{L_1, \dots, L_k\}$. Let $Q_1, \dots, Q_m$ be its bounded connected components.
Assume $k = 3$ and $\partial L_1 \cap \partial L_2$ is connected. Assume that for some $i, 1 \le i \le m$, Q_i occurs between L_2 and L_3 such that $\partial Q_i = (\partial Q_i \cap \partial L_2) \cup (\partial Q_i \cap \partial L_3)$. Suppose that we would like to transfer a region D from Q_i via L_2. In principle, it is possible that a previous step of transferring regions D occurred between L_1 and L_2 and was transferred to Q_1. Hence now we transfer it back, and by this we create a loop in the process. The point is that because $\partial L_1 \cap \partial L_2$ is connected, we may regard $L_1 \cup L_2$ as a single band, and this enables us to transfer D further beyond L_1, out of $\widehat{\mathrm{Opt}}(\Delta)$, and, by this, to "empty" (by $\mathcal{I}$-moves J) each Q_i, so to get $L_1^J \cap L_2^J$ and $L_2^J \cap L_3^J$ simultaneously connected. From this, it is clear how to prove for general k.

(c) (i) This is clear from the construction of Z and the definition of Q_i.
(ii) and (iii) It follows from [6, Lemma 2.2.1] that $Q_i \in \mathcal{M}_3(U_i)$. It remains to show that either $Q_i = \{D\}$ with $n(D) = 2$ or Q_i is transferable. Thus Theorem C is applicable to Q_i, implying that $\sigma(\rho_i \xi_i) = 1$. (Recall that $\sigma(\gamma) = 0$.) Consequently, if Q_i is not abelian, then $N(\rho_i \xi_i)$ contains a transferable region. Hence by applying a finite number of $\mathcal{I}$-moves, we can transfer all the regions D with $n(D) \ge 4$ beyond L_i and/or L_{i+1} and remaining either with an abelian diagram Q_i' or with a diagram with $\mathcal{D}_2^2(\omega_1') = \emptyset$, where ω_1' is obtained from ω_1 by the appropriate I-move. Applying [6, Lemma 4.1.2], the result follows. Finally, since either every region D in Q is transferable via $L_j \cup L_{j+1}$ or $Q = \{D\}$, $\mathrm{Supp}(\partial D) \subseteq C_\Gamma(a) \cup C_\Gamma(t)$. But $\{a, t\} \cap (C_\Gamma(a) \cup C_\Gamma(t)) = \emptyset$. Hence $\{a, t\} \cap \mathrm{Supp}(D) = \emptyset$.

(d) We follow notation of part (c).
(i) By assumption, $\Delta \in \mathbb{D}(\eta)$. Hence $\|\partial\Delta \cap \eta\| \ge \|n(\Delta)\| \ge 4$. Consider $\widehat{\mathrm{Opt}}(\Delta)$. Let $\{L_1, \dots, L_k\}$ be the bands which emanate from η and end on ω_1. Consider $L_1 \cup Q_1 \cup L_2$. If $Q_1 \neq \{D\}$ with $n(D) = 2$, then we can transfer Q_1

beyond L_1 and L_2 via a suitable sequence J of $\mathcal{I}$-moves such that in M^J, configurations (1), (2), (3) and (4) as in [6, Figure 41] occur. In configuration (1), $\mathcal{D}_2^2(\omega_1) \geq 2$, in configurations (2) and (3), $\mathcal{D}_2^2(\omega_1) \geq 1$, and in configuration (4), $\mathcal{D}_2^2(\omega_1) \geq 0$. Similar arguments apply to $L_{k-1} \cup Q_{k-1} \cup L_k$. Notice that if $L_1 \cup Q_1 \cup L_2$ and $L_{k-1} \cup Q_{k-1} \cup L_k$ are in configurations (1), (2) or (3), then $|\mathcal{D}_2^2(\omega_1)| \geq 2$. This is part (2) in the definition of $\sigma(\omega_1) = 1$. Hence we have to show that if $\mathcal{D}_2^2(\omega_1) \leq 1$, then $\sigma(\omega_1) = 1$, which contradicts $\sigma(\omega_1) = 0$.
Let $L_1 \cup L_2$ be in configuration i and $L_{k-1} \cup L_k$ in configuration j, $1 \leq i \leq 4$ and $1 \leq j \leq 4$. If $|\mathcal{D}_2^2(\omega_1)| \leq 1$ then either $i = 4$ or $j = 4$ or both. Assume $L_1 \cup L_2$ is in configuration (4) and $L_{k-1} \cup L_k$ in configuration $j \leq 3$. Then $L_1 \cup L_2$ is a homogeneous band-bundle. See [6] for the definition. Let $\mathbb{B}_0$ be the maximal band-bundle that contains $L_1 \cup L_2$. Then we claim that $\mathbb{B}_0 = L_1 \cup \cdots \cup L_{k-1}$. Thus suppose $\mathbb{B}_0 = L_1 \cup \cdots \cup L_l$, $l < k-1$ and consider L_{l+1}. If $\mathrm{Reg}(Q_l) \neq \emptyset$, then we transfer Q_l via $L_l \cup L_{l+2}$ by a sequence J of $\mathcal{I}$-moves such that $\mathrm{Reg}(Q_l^J) = \emptyset$. Hence $L_l \cup L_{L+1}$ is in configurations (1), (2) or (3), since $|\mathcal{D}_2^2(\omega_1)| = 1$ implies that $l = k-1$. Observe that we can push up Δ by a sequence J_1 of $\mathcal{I}$-moves along $\mathbb{B}_0$ and $\|\partial\Delta^J \cap \omega_1\| \geq n(\Delta) - 2 \geq 2$. This is part (3) in the definition of $\sigma(\omega_1) = 1$. So it remains to deal with the case when both $L_1 \cup L_2$ and $L_{k-1} \cup L_k$ are in configuration (4). Let $\mathbb{B}_1$ be the maximal band-bundle which contains $L_{k-1} \cup L_k$. Then $\mathbb{B}_0 \cup \mathbb{B}_1 = \mathrm{Opt}(\Delta)$. If $\mathbb{B}_0 = \mathbb{B}_1$, then $\mathrm{Opt}(\Delta)$ is a band-bundle, and by pushing up Δ via $\mathrm{Opt}(\Delta)$, we get $\Delta^{J_2} \in \mathcal{D}(\omega_1)$. But then by part (4) of the definition of $\sigma(\omega_1)$, $\sigma(\omega_1) = 1$. Finally, if $\mathbb{B}_0 = \mathbb{B}_1$, then either $\|\eta \cap \partial\mathbb{B}_0\| \geq 2$ or $\|\eta \cap \partial\mathbb{B}_1\| \geq 2$, and we fulfil part (3) of the definition of $\sigma(\omega_1)$, showing that $\sigma(\omega_1) = 1$. If one of the Q_i consists of a single region D with $n(D) = 2$, then $|\mathcal{D}_2^2(\omega_1)| \geq 1$, and the above arguments apply to this case.
(ii) The case when $\widetilde{\mathbb{M}}^t \in \mathrm{O.L.}(\omega_1, \omega_2)$ is dealt with similarly. ■

4.7. Proof of Theorem L

(a) If $|\widetilde{\mathbb{M}}^t| = 1$, then by Lemma 4.5.1, $\Delta \in \mathbb{D}(\eta)$. Assume $|\widetilde{\mathbb{M}}^t| \geq 2$. By Theorem S part (d), it is enough to show that $\Delta \in \mathbb{D}(\eta)$. By [6, Lemma 1.3.2], $i(\widetilde{\Delta}) \leq 2$. If $i(\widetilde{\Delta}) = 1$, then by Lemma 4.5.2 (a), $\Delta \in \mathbb{D}(\eta)$. If $i(\widetilde{\Delta}) = 2$, then by [6, Lemma 1.3.4], $\partial\widetilde{\Delta} \cap \widetilde{\omega}_1$ has an endpoint $\widetilde{v}$ with valency 3. Let $Q = \Psi_t^{-1}(\widetilde{v})$. If $\sigma(\omega_1) = 0$, then Q is an a-abelian, by Lemma 4.5.4, hence there is an a-band-bundle emanating from α_1 and ending at ω_1. (We follow the notation of Lemma 4.5.4.) Consequently, $\Delta \in \mathbb{D}(\eta)$, by Lemma 4.5.2 (b), as required. Since $n(\Delta) \geq 4$, the rest follow.

(b) Let $\widetilde{\mathbb{M}}^t = \langle \widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_k \rangle$, $k \geq 2$ and consider Δ_1. By [6, Proposition 1.5.2], $\partial S \cap \partial\Delta_1$ is connected, $\|\partial S \cap \partial\Delta_1\| \geq n(\Delta_1)$ and $|\xi| \leq |\partial S \cap \partial\Delta_1|$, where ξ is the complement of $\partial S \cap \partial\Delta_1$ on $\partial\Delta_1$. Consequently, $S \in \mathbb{D}^v(\eta)$. See Definition 4.3.1 (iii). Therefore, the result follows from Theorem S part (d). The rest

follows from the fact that $\|\partial S \cap \partial\Delta_1\| \geq n(\Delta_1) \geq 4$, hence $\|\partial S \cap \partial\Delta_1\|_a \geq 2$ and $\|\partial S \cap \partial\Delta_1\|_t \geq 2$, due to the alternating nature (in a and t) of the boundary label of Δ_1. ■

4.8. Proof of Theorem T

By induction on $|P|$. Let $\omega_r = \partial P \cap \partial B_r$, and let $\omega_l = \partial P \cap \partial B_l$. Then $u\omega_r v\omega_l^{-1} = \partial P$, u and v boundary vertices. We start with all the transferable regions from $\mathrm{Reg}_2(P)$ by applying successively a sequence J_1 of $\mathcal{I}$-moves. Let $P_1 = P^{J_1}$ be the resulting diagram. Assume $\mathrm{Reg}(P_1) \neq \emptyset$. If $P_1 \notin \mathcal{M}_3(U_1)$, then we can replace P_1 by $P_1' \in \mathcal{M}_3(U_1)$ and make the same argument. Hence we may assume that $P_1 \in \mathcal{M}_3(U_1)$, where U_1 is a boundary label of P_1, which we may assume to be cyclically reduced. Since we removed only regions of $\mathrm{Reg}_2(M)$ and this cannot increase $|\mathrm{Reg}_{4+}(M)|$ and $|\mathrm{Reg}_{4+}(\mathbb{M})|$, Theorem C applies to P_1 and implies that $\sigma(\omega_1^*) = 1$ or $\sigma(\omega_2^*) = 1$, where $\omega_i^* = \omega_i^{J_1}$, $i = 1, 2$. Consequently, there is a region Δ with $n(\Delta) \geq 4$ or a submap $S \in \mathbb{D}^v(\omega_i^*)$ which can be transferred from P_1. See Remarks following Theorem S. Let the resulting diagram obtained from P_1 be denoted by P_2. Notice that $|P_2| < |P|$. Then we may continue transferring from P all the regions Δ with $n(\Delta) \geq 4$, ending up either with a diagram without regions proving the theorem or with an abelian diagram. In this case, [6, Lemma 4.1.2] completes the proof. ■

4.9. Proofs of Theorems B and C

Proof of Theorem B. We start with the proof of Theorem B.

Case 1. $T(M) = \emptyset$.

Then $\mathrm{Reg}_{4+}(M) = \emptyset$, hence $\mathrm{Reg}(M) = \mathrm{Reg}_2(M)$. Hence if $\mathrm{Supp}(D) = \{a, b\}$, $D \in \mathrm{Reg}(M)$, then $\|\partial D\|_a = \|\partial D\|_b = 2$. Now, by [6, Lemma 1.1.1 (c)], $\|\partial M\|_a \geq 2$. Hence $\|\partial M\|_a \geq \|\partial D\|_a$ and $\|\partial M\|_b \geq \|\partial D\|_b$, as required.

Case 2. $T(M) \neq \emptyset$.

Let $t \in T$ and consider $\widetilde{\mathbb{M}}^t$. Recall that for every region Δ in $\mathbb{M}_t$, $\|\partial\Delta\|_t = \|\partial\widetilde{\Delta}\|$ and $n(\Delta) = n(\widetilde{\Delta})$, where $\widetilde{\Delta} = \Psi_t(\Delta)$. Without loss of generality, we may assume that $\widetilde{\mathbb{M}}^t$ has connected interior. Hence due to Theorem A and [6, Lemma 1.3.2], either $\widetilde{\mathbb{M}}^t = \{\widetilde{\Delta}\}$ or $|\mathcal{D}(\widetilde{\mathbb{M}}^t)| \geq 2$. In the first case, the result is clear. Assume, therefore, that $\mathcal{D}(\widetilde{\mathbb{M}}^t) = \{\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_k\}$, $k \geq 2$. We show that

$$\|\partial\widetilde{\mathbb{M}}^t\| \geq n(\Delta_j), \quad j = 1, \ldots, k. \tag{4.1}$$

From this, the result follows. We show this for $j = 1$. The proof is the same for $j = 2, \ldots, k$. Recall the notation of Lemma 4.5.2. By (3) in the proof of Lemma 4.5.2, $\|\eta_0\|_t \geq n(\Delta_j) - i(\widetilde{\Delta}_j)$, $j = 1, \ldots, k$. Hence by Theorem A and due to [6, Lemma 1.3.2], the individual contribution of Δ_j to $\|\partial M\|_t$ is at least $n(\Delta_j) - 2$. However, due to possible consolidation of labels of adjacent Δ_j, the actual contribution of Δ_j to $\|\partial M\|_t$ is at least $n(\Delta_j) - 3$. Observe that

$$\|\partial\mathbb{M}\|_t \geq \|\eta_0\|_t. \tag{4.2}$$

Consequently, due to (4.2),

$$\|\partial\mathbb{M}\|_t \geq \sum_{j=1}^{k}[n(\Delta_j)-3]. \tag{4.3}$$

Now, $\sum_{j=1}^{k}[n(\Delta_j)-3] = n(\Delta_1) + \left(\sum_{j=2}^{k}[n(\Delta_j)-3]\right) - 3 \geq n(\Delta_1) + \left(\sum_{j=2}^{k}[4-3]\right) - 3 = n(\Delta_1) + (k-1) - 3 = n(\Delta_1) + k - 4$, that is,

$$\|\partial\mathbb{M}\|_t \geq n(\Delta_1) + k - 4 \quad \text{(recall that } n(\Delta_j) \geq 4). \tag{4.4}$$

Hence if $k \geq 4$, then $\|\partial\mathbb{M}\|_t \geq n(\Delta_1)$, as required. If $k = 3$, then due to [6, Lemma 1.3.2], $i(\widetilde{\Delta}_j) = 1$, $j = 1, 2, 3$, hence $\|\partial\mathbb{M}\|_t \geq (n(\Delta_1)-2) + (n(\Delta_2)-2) + (n(\Delta_3)-2) \geq n(\Delta_1) + (4-2) + (4-2) - 2 > n(\Delta_1)$. Finally, if $k = 2$, then by [6, Lemma 1.3.2], $i(\Delta_j) = 1$, $j = 1, 2$, hence $(n(\Delta_1)-2) + (n(\Delta_2)-2) = n(\Delta_1) + [(n(\Delta_2)-2)-2] \geq n(\Delta_1)$, that is, due to (4.3), $\|\partial\mathbb{M}\|_t \geq n(\Delta_1)$, as required. ∎

Proof of Theorem C. We have to show that either $\sigma(\omega_1) = 1$ or $\sigma(\omega_2) = 1$. Let $t \in T(M)$ and consider $\widetilde{\mathbb{M}}^t$. By [6, Lemma 1.3.2], either $\widetilde{\mathbb{M}}^t = \{\widetilde{\Delta}\}$ or $|\mathcal{D}(\widetilde{\mathbb{M}}^t)| \geq 2$. If $|\mathcal{D}(\widetilde{\mathbb{M}}^t)| \geq 3$, then by the Pigeon Hole Principle (P.H.P), either $|\mathcal{D}(\widetilde{\omega}_1)| \neq \emptyset$ or $|\mathcal{D}(\widetilde{\omega}_2)| \neq \emptyset$. Assume $|\mathcal{D}(\widetilde{\omega}_2)| = \emptyset$. Then $|\mathcal{D}(\widetilde{\omega}_1)| \neq \emptyset$, hence $\sigma(\omega_1) = 1$, by Theorem L, as required. If $|\mathcal{D}(\widetilde{\mathbb{M}}^t)| = 2$, then by [6, Lemma 1.3.2], $\widetilde{\mathbb{M}}^t \in$ O.L. Hence, again, by Theorem L, $\sigma(\omega_1) = 1$. Finally, if $\widetilde{\mathbb{M}}^t = \{\widetilde{\Delta}\}$, then by Lemma 4.5.3, $\sigma(\omega_1) = 1$. ∎

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Arye Juhász
Department of Mathematics, Technion – Israel Institute of Technology, 32000 Haifa, Israel;
arju@technion.ac.il