J. Eur. Math. Soc. 28, 4239–4320 (2026) DOI 10.4171/JEMS/1549

Claudia García · Javier Gómez-Serrano

Self-similar spirals for the generalized surface quasi-geostrophic equations

Received October 13, 2022; revised July 26, 2023

Abstract. We construct a large class of nontrivial (nonradial) self-similar solutions of the generalized surface quasi-geostrophic equations (gSQG). To the best of our knowledge, this is the first rigorous construction of any self-similar solution for these equations. The solutions are of spiral type, locally integrable, and may have mixed sign. Moreover, they bear some resemblance to the finite time singularity scenario numerically proposed by Scott–Dritschel [J. Fluid Mechanics 863, article no. R2 (2019)] in the SQG patch setting.

Keywords: self-similar, surface quasi-geostrophic, bifurcation.

Contents

Claudia García: Departament de Matemàtiques i Informàtica, Universitat de Barcelona, 08007 Barcelona; current affiliation: Departamento de Matemática Aplicada & Research Unit “Modeling Nature” (MNat), Universidad de Granada, 18071 Granada, Spain; claudiagarcia@ugr.es

Javier Gómez-Serrano: Departament de Matemàtiques i Informàtica, Universitat de Barcelona, 08007 Barcelona; Centre de Recerca Matemàtica, Edifici C, Campus Bellaterra, 08193 Bellaterra, Spain; current affiliation: Department of Mathematics, Brown University, Providence, RI 02912, USA; javier_gomez_serrano@brown.edu

Mathematics Subject Classification 2020: 35Q35 (primary); 76B03, 35B32 (secondary).

1. Introduction

The generalized surface quasi-geostrophic equations describing the evolution of the potential temperature θ read

$$\begin{cases} \theta_t + (u \cdot \nabla)\theta = 0 & \text{in } [0,\infty) \times \mathbb{R}^2, \\ u = -\nabla^{\perp}(-\Delta)^{-1+\gamma/2}\theta & \text{in } [0,\infty) \times \mathbb{R}^2, \\ \theta(0,x) = \theta_0(x) & \text{for } x \in \mathbb{R}^2. \end{cases} \tag{1.1}$$

In this system u refers to the velocity field, $\nabla^{\perp} = (-\partial_2, \partial_1)$ and $\gamma \in [0, 2)$. The velocity field is linked to the potential temperature θ via the operator $-(-\Delta)^{-1+\gamma/2}$ amounting to $u = \nabla^{\perp}\psi$ and

$$\psi(t,x) = -(-\Delta)^{-1+\gamma/2}\theta(x) = \int_{\mathbb{R}^2} K_\gamma(x-y)\theta(t,y)\,dy, \tag{1.2}$$

where

$$K_\gamma(x) = \begin{cases} \frac{1}{2\pi}\log|x|, & \gamma = 0, \\ -\frac{C_\gamma}{2\pi}\frac{1}{|x|^\gamma}, & \gamma \in (0,2), \end{cases}$$

and $C_\gamma = \frac{\Gamma(\gamma/2)}{2^{1-\gamma}\Gamma((1-\gamma)/2)}$. The case $\gamma = 0$ corresponds to the Euler equations and $\gamma = 1$ to the surface quasi-geostrophic (SQG) model. In this work we will be interested in the case $\gamma \in (0, 1)$.

The SQG equation ($\gamma = 1$) models the evolution of the temperature from a general quasi-geostrophic system for atmospheric and oceanic flows (see [21,56,68,75] for more details). Constantin–Majda–Tabak [21] pointed out the analogy between the SQG equation and the incompressible 3D Euler equations and carried out the first numerical and analytical study of the equation. The gSQG or $(\mathrm{SQG})_\gamma$ model (1.1) was proposed by

Córdoba–Fontelos–Mancho–Rodrigo [26] as an interpolation between Euler and surface quasi-geostrophic equations. Yet, almost 30 years after [21], the global existence vs finite time singularity question remains open, even in the case $\gamma > 0$. As a first step in the finite time singularity direction, in this paper we construct self-similar solutions which have infinite energy. We leave for future work the transformation of those solutions into finite energy ones that may develop a finite time singularity.

1.1. Main result

Our main task is to construct solutions with a spiraling vortex singularity at the origin that are perturbations of a stationary solution. Indeed, we will perturb appropriately the following radial solution:

$$\theta(t,x) = \theta_0(x) = |x|^{-1/\mu} \tag{1.3}$$

for $\mu > 0$. The singularity of the stationary solution (1.3) depends on the parameter μ. However, we need some condition on μ to ensure the existence of the associated stream function, that is, the integrability of the singular operator (1.2). In particular, assuming that θ is equal to (1.3), one needs

$$\frac{1}{2} < \mu < \frac{1}{2-\gamma} \tag{1.4}$$

to have $\psi(x) = C_0|x|^{2-\gamma-1/\mu}$, where C_0 is defined in (A.2).

In this work, we will be interested in self-similar solutions. That means

$$\theta(t,x) = \lambda^{1+\alpha-\gamma}\theta(\lambda^{1+\alpha}t, \lambda x), \quad \forall \lambda > 0. \tag{1.5}$$

By taking $\lambda = 1/t^{\frac{1}{1+\alpha}}$, the problem reduces to finding a self-similar profile $\hat{\theta}$ such that

$$\theta(t,x) = \frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}}}\hat{\theta}\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right). \tag{1.6}$$

Moreover, for the trivial singular radial solution (1.3) to be self-similar, one needs the following condition on μ and α:

$$1/\mu = 1 + \alpha - \gamma, \tag{1.7}$$

which, together with (1.4), implies that $\alpha \in (1, 1+\gamma)$.

Our main theorem reads as follows.

Theorem 1.1. *Let $\gamma \in (0,1)$ and $\alpha \in (1, 1+\gamma)$. Then there exists $\varepsilon > 0$ such that for any $\frac{2\pi}{m}$-periodic $\tilde{\Omega} \in B_{L^p([0,2\pi])}(1,\varepsilon)$ and $p > \frac{1}{1-\gamma}$, the initial condition*

$$\theta_0(re^{i\vartheta}) = r^{-(1+\alpha-\gamma)}\tilde{\Omega}(\vartheta)$$

determines a self-similar solution of the type (1.6) *for* $(\mathrm{SQG})_\gamma$*, for any $m \geq 1$ and where $B_{L^p([0,2\pi])}(1,\varepsilon)$ refers to the ball of real-valued periodic functions defined in $L^p([0,2\pi])$ which are less than ε away from the constant function* 1.

Let us remark that Theorem 1.1 states that any initial condition of that type leads to a self-similar solution, and thus it has an associated self-similar profile. However, uniqueness is not known in this class of solutions, and thus we cannot ensure that this is the only solution with such initial data. Let us refer to Figure 1 to illustrate the kind of self-similar profiles obtained in Theorem 1.1. Let us remark that compared to the Euler case studied in [5, 36, 37], condition (1.4) gives $\theta_0 \in L^1_{\text{loc}}$. More precisely, we can prove that the initial vorticity θ_0 belongs to L^q_{loc} with $q < \min\{p, 2\mu\}$ provided that $\tilde{\Omega} \in L^p$.

Fig. 1. Left: the trivial self-similar radial solution (1.3). Right: the perturbed solution of Theorem 1.1.

We now summarize the previous literature related to this problem. We will first discuss the 2D Euler equation, and then discuss the SQG and gSQG equations.

1.2. Spiral solutions for 2D Euler

The study of spiraling solutions for the 2D Euler equation has a long history due to its connections to turbulence and nonuniqueness, which starts from the seminal work of Kaden [64] and Pullin [76] in the context of vortex sheets. These vortex spiral solutions exhibit nonuniqueness phenomena. Elling [36] constructed spiral solutions close to constant in a high symmetry class, and later upgraded them to sign-changing solutions [37]. Lately, Cieślak–Kokocki–Ożański [18, 19] have proved global in time solutions in the form of logarithmic spiral vortex sheets. See also [40] where spirals are reported in the context of water waves.

Bressan, in a series of papers together with Shen and Murray [5, 6], has laid out a very ambitious program to prove nonuniqueness by constructing initial data that can be continued as either one spiral or two spirals. The idea developed in those works amounts to proving some a priori estimates and establishing the existence of some of the building blocks of the program, as well as providing compelling numerical evidence for it.

We remark that in our case, in contrast to the previous literature, we are able to achieve solutions in any symmetry class due to the study of the full linear operator, as opposed to (1) either a Neumann series in the symmetry class, where the contribution of one of the pieces of the linear operator dominates the other whenever the solution is sufficiently symmetric [36, 37], or (2) the case of an inner domain where the problem has fewer constraints [5]. Moreover, let us emphasize one fundamental difference between the Euler case and the gSQG equations: in [5, 36, 37] the authors substantially exploit the fact that the relation between the stream function and the vorticity is local, which does not hold for the gSQG equations, and hence new techniques must be developed.

In [35], Elgindi–Jeong obtained global well-posedness for a class of patches with boundary singular at the origin which includes spirals. In [33], they constructed a patch solution whose boundary forms filaments which spirals around the center of the patch, as time goes to infinity, which was suggested numerically by [32].

1.3. Self-similar solutions, nonuniqueness and finite time singularities for SQG and gSQG

Resnick, in his thesis [79], showed global existence of weak solutions in L^2 using the oddness of the Riesz transform to obtain an extra cancellation. Marchand [70] extended Resnick's result to the class of initial data belonging to L^p with $p > 4/3$.

The question of nonuniqueness for weak solutions of SQG is still a challenging problem which has received a lot of attention lately, starting from [2,63] up to the breakthrough by Buckmaster–Shkoller–Vicol [7], alternative new proofs by Isett and Ma [62] and the study of the stationary problem by Cheng–Kwon–Li [17]. It is likely that the scenario crafted by Bressan and his collaborators for 2D Euler also holds in the gSQG case.

The problem of whether the SQG and gSQG systems present finite time singularities or there is global existence is one of the most outstanding problems in mathematical fluid mechanics. Kiselev and Nazarov [66] constructed solutions for which there is norm inflation, and Friedlander and Shvydkoy [42] showed the existence of unstable eigenvalues of the spectrum. We also refer to the construction of Castro and Córdoba [10] of singular solutions with infinite energy.

Aside from the analytical work, there have been many numerical results in this framework. Constantin–Majda–Tabak [21] reported a possible singularity in the form of a hyperbolic saddle closing in finite time. By contrast, Ohkitani and Yamada [72] and Constantin–Nie–Schörghofer [22] suggested that the growth was double exponential instead. Córdoba in his thesis [24] bounded the growth by a quadruple exponential (see also later work by Córdoba and Fefferman [25] on a double exponential bound and numerical simulations by Deng–Hou–Li–Yu [31]), thus settling the question. 20 years later, using improved algorithms and more computational resources, Constantin et al. [20] recalculated the hyperbolic saddle scenario, finding no evidence of blowup. Scott [82], starting from elliptical configurations, proposed a candidate that develops filamentation and after a few cascades, blowup of $\nabla\theta$. This is the only potential scenario in the smooth setting known to date.

A simpler setting for the SQG and gSQG problem is the so-called *patch setting*, where the scalar takes the form of the characteristic function of a time-dependent domain with smooth boundary. Rodrigo [80] (in a C^∞ space) and Gancedo [43] and Constantin et al. [16] (in a Sobolev space) proved local existence for $0 < \gamma \leq 1$ and $\gamma > 1$ respectively. Some 20 years ago, Córdoba–Fontelos–Mancho–Rodrigo [26] found numerically strong evidence of finite time singularity formation in the form of a self-intersecting patch where at the point of intersection the curvature blows up (the curve has to lose regularity at the self-intersection point due to Gancedo–Strain [45]). They conducted experiments for different values of $\gamma \in (0, 1]$ and reported blowup for all of them; see also [69]. Moreover, they numerically checked that the singularity was of asymptotically self-similar type. Later, Scott–Dritschel performed a careful study using ellipses as initial condition: On the one hand, in [83], after taking an ellipse with aspect ratio $a = 0.16$, a self-similar cascade of filamentation is reported, where the filament width collapses to zero in a finite time. On the other hand, in [84], a more detailed analysis is done (for initial aspect ratios $a = 0.094, 0.132, 0.180$), and the boundary is split into three scales: an inner (faster) scale that develops a self-similar corner, a middle scale that develops a self-similar spiral, and the outermost region. We note that there is some resemblance between the middle scale of this analysis and the solutions constructed in this paper which are also self-similar spirals.

Finally, we mention the formation of singularities in other geometries (in the presence of a boundary) by Kiselev–Ryzhik–Yao–Zlatoš [67] (for $0 < \gamma < 1/12$) and Gancedo–Patel [44] (for $0 < \gamma < 1/3$). We also refer the reader to [1, 11, 14, 27, 30, 46, 47, 51, 52, 54, 55, 57, 59, 60, 71, 78] for other results concerning global solutions for the gSQG equations.

Regarding negative results on self-similar solutions (i.e. when the only possible solution is the trivial one) Cannone–Xue [9] and Chae [15] proved that there are no locally self-similar solutions under suitable technical assumptions, one of them being integrable. Although our solutions decay at infinity, they do not decay sufficiently fast to be in L^q with $q > 2$ as needed in [9].

1.4. Main ideas of the proof

Let us give the main ideas of the proof. By inserting the ansatz (1.6) in the gSQG equations, we find the equivalent stationary equation for $\hat{\theta}$:

$$\left\{\frac{1+\alpha-\gamma}{1+\alpha} - \frac{2}{1+\alpha}\right\}\hat{\theta}(z) - \nabla\cdot\left\{\left(\hat{u}(z) - \frac{z}{1+\alpha}\right)\hat{\theta}(z)\right\} = 0, \tag{1.8}$$

where the vector field

$$q(z) := \hat{u}(z) - \frac{z}{1+\alpha} = \nabla^{\perp}\hat{\psi}(z) - \frac{z}{1+\alpha} \tag{1.9}$$

is called the *pseudo-velocity*, and its integral curves are the *pseudo-streamlines*. Hence, in order to find nontrivial self-similar solutions one needs to solve (1.8).

Motivated by numerous works in the literature about the existence of nontrivial steady vortex patches in different active scalar equations [8, 29, 30, 46–51, 54, 57, 58, 78], we will solve (1.8) via perturbative arguments by means of the infinite-dimensional implicit function theorem. That means that we will look for nontrivial self-similar solutions which are *close* to the trivial stationary solution (1.3). However, perturbative arguments cannot be directly applied to (1.8) since the linearized operator at a trivial radial solution (1.3) is not Fredholm. That comes from the fact that the solution is singular at 0. However, the kind of singularity at the linear level of the terms with the radial and angular derivatives is different, which complicates the choice of the appropriate weighted space. That problem also appeared when perturbing radial functions in the Euler and gSQG equations [12, 13, 50]. The authors overcame such issues by either performing an appropriate change of variables by means of level sets or by working in a suitable class of solutions.

In this work, we will tackle the lack of Fredholmness of the linearized operator when working directly with (1.8) in the physical variables, by using the *adapted coordinates* introduced by Elling [36]. Throughout the construction we will consider variables $\beta \in [0, \infty)$ and $\phi \in \mathbb{T}$. This will enable us to better understand the vorticity equation and skip the Fredholmness issue. In particular, the linearized operator is Fredholm in the new variables and we can analyze its kernel using the Mellin transform and, as a result, we will be able to use the infinite-dimensional implicit function theorem to obtain a solution

$$\Theta(\beta, \phi) := \hat{\theta}(z(\beta, \phi)), \quad \Psi(\beta, \phi) := \hat{\psi}(z(\beta, \phi)).$$

We will define a mapping $(\beta, \phi) \mapsto (r, \vartheta)$ via

$$\vartheta = \beta + \phi, \quad r = (-(1+\alpha)\Psi_\beta)^{1/2}. \tag{1.10}$$

We will show that this is a bijection between suitable function spaces, and that the Jacobian of the transformation does not vanish (Section 7.1). This implies that the change of variables is well-defined, and we will show that we can come back to $(\hat{\theta}, \hat{\psi})$ and thus find the solution claimed by Theorem 1.1.

Let us remark that the above transformation is a nonlinear change of coordinates which is implicit in the sense that it depends on the solution itself. Here β refers to the angle along a particular pseudo-streamline. Moreover, $\beta = 0$ will correspond to spatial infinity, while $\beta = \infty$ approaches the spiral center. On the other hand, ϕ is the polar angle at $r \to \infty$. Indeed, we refer to our solutions as *spiral solutions* due to the fact that for each solution constructed in Theorem 1.1 there exist spiral variables (β, ϕ) (as described before) such that the solution is a function of β only. Moreover, the spiral variables (β, ϕ) resemble algebraic spirals in the sense that $\vartheta = \phi + \frac{1}{r^{1/\mu}} + g$, where g is a smooth perturbation. Hence, for ϕ fixed, we observe in the ϑ variable the algebraic spiraling vortex singularity at the origin. See Figure 2 for an illustration of the adapted coordinates.

The second equation above implies that the pseudo-velocity (1.9) has vanishing ϕ-component; see Proposition 2.2. Indeed, note that

$$\begin{aligned}\left(\nabla^\perp \hat{\psi} - \frac{z}{1+\alpha}\right) \cdot \nabla_z \hat{\theta} = \Bigg(&(\beta_{z_2}\phi_{z_1} - \beta_{z_1}\phi_{z_2})\Psi_\phi - \frac{1}{1+\alpha}(z_1\beta_{z_1} + z_2\beta_{z_2}), \\ &(\beta_{z_1}\phi_{z_2} - \beta_{z_2}\phi_{z_1})\Psi_\beta - \frac{1}{1+\alpha}(z_1\phi_{z_1} + z_2\phi_{z_2})\Bigg) \cdot \nabla_{\beta,\phi}\Theta.\end{aligned}$$

Hence (1.10) implies

$$(\beta_{z_1}\phi_{z_2} - \beta_{z_2}\phi_{z_1})\Psi_\beta - \frac{1}{1+\alpha}(z_1\phi_{z_1} + z_2\phi_{z_2}) = 0. \tag{1.11}$$

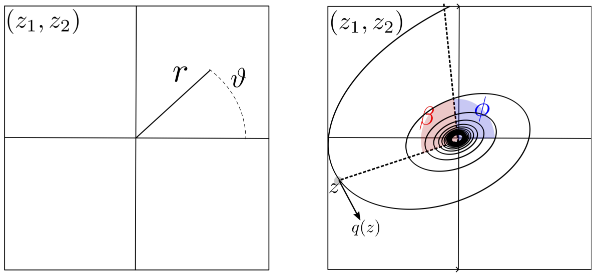

Fig. 2. Comparison between the polar coordinates in the (z_1, z_2) plane with the adapted (β, ϕ) coordinates. The pseudo-streamlines satisfy ϕ = constant and the point z corresponds to the intersection of the line $\vartheta = \beta + \phi$ and the pseudo-streamline satisfying $\beta = 0$ at $|z| = \infty$. We remark that values of β between $2k\pi$ and $2(k+1)\pi$ correspond to k rotations of the spiral.

It is convenient to note that when assuming (1.11) we have

$$\left(\nabla^{\perp}\hat{\psi} - \frac{z}{1+\alpha}\right)\cdot\nabla_z\hat{\theta} = \left\{(\beta_{z_2}\phi_{z_1} - \beta_{z_1}\phi_{z_2})\Psi_\phi - \frac{1}{1+\alpha}(z_1\beta_{z_1} + z_2\beta_{z_2})\right\}\Theta_\beta, \tag{1.12}$$

where only ∂_β survives. It follows that the vorticity (1.8) written in the adapted coordinates depends only on the β-derivative of Θ. Thanks to that, a trivial solution to (1.8) is found as

$$\Theta(\beta, \phi) = (\Psi_\varphi)^{-\frac{1+\alpha-\gamma}{2}}\Omega(\phi) = (\Psi_\varphi)^{-\frac{1/\mu}{2}}\Omega(\phi), \tag{1.13}$$

for any Ω, where $\partial_\varphi := \partial_\phi - \partial_\beta$. However, Θ depends on the stream function Ψ and it remains to check the elliptic relation (1.2) between these two functions. Written in the adapted coordinates and using the functions $(f, \tilde{\Omega})$ as in

$$\Psi(\beta, \phi) = -C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\beta^{\frac{\alpha-1}{1+\alpha}}(1 + f(\beta, \phi)), \quad \Omega(\phi) = (1+\alpha)^{-\frac{1}{2\mu}}\tilde{\Omega}(\phi),$$

the elliptic relation between the vorticity and the stream function amounts to

$$\tilde{F}(f, \tilde{\Omega})(\beta, \phi) = 0, \quad (\beta, \phi) \in [0, \infty) \times \mathbb{T},$$

where

$$\begin{aligned} &\tilde{F}(f, \tilde{\Omega})(\beta, \phi) \\ &= 1 + f(\beta, \phi) - (-1)^{\frac{-1}{2\mu}}\frac{C_\gamma(\alpha-1)^{\frac{\alpha-1}{2}}}{4\pi C_0(1+\alpha)^{\frac{\alpha-1}{2}}}\beta^{-\frac{\alpha-1}{1+\alpha}} \\ &\quad\times \iint \frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}\left\{-\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\varphi}}f(b, \Phi)\right\}^{-\frac{1}{2\mu}}\left\{\frac{2(\alpha-1)}{(1+\alpha)^2} + \partial_{\overline{\beta\varphi}}f(b, \Phi)\right\}\tilde{\Omega}(\Phi)\, db\, d\Phi}{D(f)(\beta, \phi, b, \Phi)^{\gamma/2}}, \end{aligned} \tag{1.14}$$

with (here and throughout)

$$\iint := \int_0^{2\pi}\int_0^{\infty}$$

and

$$\begin{aligned}
D(f)(\beta,\phi,b,\Phi) &:= \tilde{\Psi}_\beta + \tilde{\Psi}_b - 2\tilde{\Psi}_\beta^{1/2}\tilde{\Psi}_b^{1/2}\cos(\phi+\beta-\Phi-b) \\
&= \beta^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar{\beta}} f(\beta,\phi)\right\} + b^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar{b}} f(b,\Phi)\right\} \\
&\quad - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar{\beta}} f(\beta,\phi)\right\}^{1/2}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar{b}} f(b,\Phi)\right\}^{1/2} \\
&\qquad \times \cos(\phi+\beta-\Phi-b) \\
&= \left(\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar{\beta}} f(\beta,\phi)\right\}^{1/2} - b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar{b}} f(b,\Phi)\right\}^{1/2}\right)^2 \\
&\quad + 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar{\beta}} f(\beta,\phi)\right\}^{1/2}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar{b}} f(b,\Phi)\right\}^{1/2} \\
&\qquad \times (1-\cos(\phi+\beta-\Phi-b)).
\end{aligned} \tag{1.15}$$

Notice that the expression of $\tilde{F}$ (see (1.14)) does not depend on ∂_ϕ, which is crucial for our analysis. Indeed, the choice of the adapted coordinates is the key point of this work due to two facts: first, we can solve the vorticity equation with (1.13), and second, the nonlinear equation only depends on one derivative, as opposed to (1.8).

The proof of the main theorem relies on the application of the infinite-dimensional implicit function theorem to $\tilde{F}$ in appropriate spaces. The choice of the function spaces is important since one needs to find not only that the nonlinear operator is well-defined in those spaces but also that the linear one is an isomorphism. Therefore, the spaces are somehow *constrained* in order to have the invertibility of the linear operator (however, the nonlinear one also has to work there). A "simple" unweighted – Hölder, Sobolev – space will make the analysis extremely complicated (at best) or simply fail. Moreover, it is natural to use weighted Hölder spaces since we are working in an unbounded domain. We also remark that the papers [5,6,36,37], which were set up in the context of the Euler equations, also propose versions of weighted Hölder spaces. Further, since our trivial solution has a singularity at $\beta \to \infty$, one has to deal with that issue, as well as with the decay at $\beta = 0$. It is crucial for this decay to be well understood not just at the linear level, but by the action of the nonlinear operator, which is a much harder task. Let us also point out that the function $\tilde{F}$ differs from the one used in [5,6,36,37] when working with the Euler equations since there the relation between the vorticity and the streamfunction is local, as opposed to the gSQG equations.

In particular, using Propositions 4.3, 4.5 and 5.1, together with Section 5.1, we find that the linearized operator has the following expression as a Fourier series:

$$\partial_f \tilde{F}(0,1)h(\beta,\phi) = \sum_{k\in\mathbb{Z}} e^{ik\phi}e^{ik\beta}\tilde{\mathfrak{L}}_k(g_k)$$

with $g_k(\beta) := e^{-ik\beta}h_k(\beta)$ and

$$\tilde{\mathfrak{L}}_k(g_k) := \frac{1+\alpha}{2}g_k(\beta) + ik\frac{1+\alpha}{2}\beta g_k(\beta) + \frac{1+\alpha}{2}\beta g_k'(\beta) - ikJ_k(g_k)(\beta),$$

where

$$J_k(\beta) = \int_0^\infty g_k(b) K_k(\beta/b)\, db$$

for some kernel K_k. In Section 4.2 we will check that $\partial_f \tilde{F}(0,1)$ is Fredholm of index zero from $X_1^{\sigma,\delta}$ to $Y^{\sigma,\delta}$, where the function spaces are defined in (3.5)–(3.8).

In order to apply the implicit function theorem to $\tilde{F}$, we shall check that the linearized operator is an isomorphism from $X_1^{\sigma,\delta}$ to $Y^{\sigma,\delta}$. However, due to its Fredholmness, it is enough to check that its kernel in $X_1^{\sigma,\delta}$ is trivial. In general, there is no simple and successful method to perform the spectral analysis of a linear operator, and the analysis depends on the problem itself. Here the Fourier coefficients depend on β. Thus, to study the kernel one has to solve a nonlocal integrodifferential equation and there is no clear way to do it. For example, in the past, we have been successful in using computer-assisted methods to overcome this difficulty [14, 28, 38], yielding very long papers and tens of thousands of lines of code. Other alternatives have been to use asymptotic expansions in the modes where the solution is supported [36, 37, 48, 49] at the cost of a weaker theorem. See also [3, 53, 61] for a brief sample of alternative solutions in fluid mechanics to address the linearized operator problem. In our case we found a good compromise between difficulty and outcome by resorting to the Mellin transform

$$M[f](s) = \int_0^\infty x^{s-1} f(x)\, dx,$$

and exploiting the scaling of the operator. The main advantage is to use the following properties:

(1) If $(f \wedge g)(x) := \int_0^\infty f(y) g(\frac{x}{y}) \frac{dy}{y}$, then $M[f \wedge g](s) = M[f]M[g]$.

(2) $M[x^\alpha f](s) = M[f](s+\alpha)$.

Using these properties, one can aim to understand better $M[J_k](s)$ as a function of $M[g_k](s)$ and $M[K_k](s)$, possibly evaluated at different values of s, by solving a functional equation for $M[J_k]$ instead of trying to deal with J_k itself. Naturally, the class of potential J_k's that could be amenable to this analysis is not restricted to the previous example since we only use very soft scaling properties of the kernel.

Concretely, motivated by some works on coagulation and fragmentation models [4, 39] we will apply the Mellin transform to obtain a recurrence equation with singular coefficients. The usage of the Mellin transform is new in this context as far as we can tell, and we look forward to our method becoming more mainstream. See also [23, 41] for other instances of the Mellin transform being used to perform energy estimates.In our case, we will relate our recurrence equation to the moment problem [81] concluding that the function in the kernel is singular and indeed does not belong to our function spaces. Finally, after showing that the implicit function theorem can be applied to the formulation in the adapted coordinates, we shall prove the invertibility of the adapted coordinates when we are *close* to the trivial solution, in order to come back to the physical variables.

1.5. Organization of the paper

The paper is organized as follows: In Section 2 we set up the problem and rewrite the equations in the adapted coordinates. In Section 3 we define the function spaces we will work with and perform estimates at the nonlinear level. On the other hand, in Section 4 we perform regularity estimates of the linear operator. Section 5 is devoted to the spectral study and in particular the invertibility of the linear operator. Finally, Sections 6 and 7 comprise the reduction to m-fold symmetry classes and the translation of Theorem 6.2 into standard coordinates. We also include two appendices: Appendix A concerns basic results on radial solutions of the fractional Laplacian, whereas Appendix B deals with basic properties of some special functions needed throughout the text.

2. Self-similar solutions

In this section, we provide the definition of a self-similar solution to the gSQG equations together with a formulation that characterizes them. Note that solutions to the gSQG equations (1.1) are invariant under the following scaling: for $\alpha > -1$,

$$\begin{aligned} \theta(t,x) &\mapsto \theta_\lambda(t,x) = \lambda^{1+\alpha-\gamma}\theta(\lambda^{1+\alpha}t, \lambda x), \\ u(t,x) &\mapsto u_\lambda(t,x) = \lambda^{\alpha}u(\lambda^{1+\alpha}t, \lambda x), \quad \forall \lambda > 0. \end{aligned}$$

We will say that a solution θ of (1.1) is *self-similar* if $\theta = \theta_\lambda$ for any $\lambda > 0$. That is,

$$\theta(t,x) = \lambda^{1+\alpha-\gamma}\theta(\lambda^{1+\alpha}t, \lambda x), \quad \forall \lambda > 0. \tag{2.1}$$

Hence, for the trivial solution (1.3) to be self-similar in the above sense, we have the following condition on the singularity parameter μ depending on α and γ:

$$\frac{1}{\mu} = 1 + \alpha - \gamma. \tag{2.2}$$

By (1.4) and (2.2) we find that

$$1 < \alpha < 1 + \gamma. \tag{2.3}$$

Now, by taking $\lambda = 1/t^{\frac{1}{1+\alpha}}$ in (2.1), our aim is to look for self-similar solutions of the form

$$\theta(t,x) = \frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}}}\hat{\theta}\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right) \tag{2.4}$$

for some self-similar profile $\hat{\theta}$.

In the following proposition, we give the expression of u with the self-similar ansatz.

Proposition 2.1. *If θ takes the form (2.4), then*

$$u(t,x) = \frac{1}{t^{\frac{\alpha}{1+\alpha}}}\hat{u}\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right),$$

where $\hat{u} = -\nabla^{\perp}(-\Delta)^{-1+\gamma/2}\hat{\theta}$. *Moreover,* (1.1) *is equivalent to*

$$\frac{1+\alpha-\gamma}{1+\alpha}\hat{\theta}(z) + \frac{1}{1+\alpha}z\cdot\nabla\hat{\theta}(z) - \hat{u}\cdot\nabla\hat{\theta}(z) = 0, \tag{2.5}$$

where $z = \frac{x}{t^{\frac{1}{1+\alpha}}}$.

Proof. First, let us compute u assuming (2.4):

$$\begin{aligned}
u(t,x) &= -\nabla^{\perp}(-\Delta)^{-1+\gamma/2}\theta(t,x)\\
&= -\frac{C_\gamma \gamma}{4\pi}\int_{\mathbb{R}^2}\frac{(x-y)^{\perp}}{|x-y|^{\gamma+2}}\theta(t,y)\,dA(y)\\
&= \frac{C_\gamma \gamma}{4\pi}\frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}}}\int_{\mathbb{R}^2}\frac{(x-y)^{\perp}}{|x-y|^{\gamma+2}}\hat{\theta}\left(\frac{y}{t^{\frac{1}{1+\alpha}}}\right)dA(y)\\
&= \frac{C_\gamma \gamma}{4\pi}\frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}}}\frac{t^{\frac{1}{1+\alpha}}}{t^{\frac{\gamma+2}{1+\alpha}}}\int_{\mathbb{R}^2}\frac{\left(\frac{x}{t^{\frac{1}{1+\alpha}}}-\frac{y}{t^{\frac{1}{1+\alpha}}}\right)^{\perp}}{\left|\frac{x}{t^{\frac{1}{1+\alpha}}}-\frac{y}{t^{\frac{1}{1+\alpha}}}\right|^{\gamma+2}}\hat{\theta}\left(\frac{y}{t^{\frac{1}{1+\alpha}}}\right)dA(y)\\
&= \frac{C_\gamma \gamma}{4\pi}\frac{1}{t^{\frac{\alpha}{1+\alpha}}}\int_{\mathbb{R}^2}\frac{\left(\frac{x}{t^{\frac{1}{1+\alpha}}}-y\right)^{\perp}}{\left|\frac{x}{t^{\frac{1}{1+\alpha}}}-y\right|^{\gamma+2}}\hat{\theta}(y)\,dA(y) = \frac{1}{t^{\frac{\alpha}{1+\alpha}}}\hat{u}\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right),
\end{aligned}$$

where $\hat{u} = -\nabla^{\perp}(-\Delta)^{-1+\gamma/2}\hat{\theta}$.

Let us now check (2.5). Note that

$$\begin{aligned}
\partial_t\theta(t,x) &= -\frac{1+\alpha-\gamma}{1+\alpha}\frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}+1}}\hat{\theta}\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right) - \frac{1}{1+\alpha}\frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}+1}}\frac{x}{t^{\frac{1}{1+\alpha}}}\cdot(\nabla\hat{\theta})\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right)\\
&= -\frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}+1}}\left\{\frac{1+\alpha-\gamma}{1+\alpha}\hat{\theta}(z) + \frac{1}{1+\alpha}z\cdot\nabla\hat{\theta}(z)\right\},\\
\nabla\theta(t,x) &= \frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}}}\frac{1}{t^{\frac{1}{1+\alpha}}}(\nabla\hat{\theta})\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right) = \frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}}}\frac{1}{t^{\frac{1}{1+\alpha}}}\nabla\hat{\theta}(z).
\end{aligned}$$

Then

$$\begin{aligned}
&\partial_t\theta(t,x) + (u(t,x)\cdot\nabla)\theta(t,x)\\
&\qquad = -\frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}+1}}\left\{\frac{1+\alpha-\gamma}{1+\alpha}\hat{\theta}(z) + \frac{1}{1+\alpha}z\cdot\nabla\hat{\theta}(z) - \hat{u}(z)\cdot\nabla\hat{\theta}(z)\right\},
\end{aligned}$$

leading to the equivalent equation (2.5):

$$\frac{1+\alpha-\gamma}{1+\alpha}\hat{\theta}(z) + \frac{1}{1+\alpha}z\cdot\nabla\hat{\theta}(z) - \hat{u}(z)\cdot\nabla\hat{\theta}(z) = 0.$$

Note that since $\nabla\cdot\hat{u} = 0$, this equation can be written as

$$\left\{\frac{1+\alpha-\gamma}{1+\alpha} - \frac{2}{1+\alpha}\right\}\hat{\theta}(z) - \nabla\cdot\left\{\left(\hat{u}(z) - \frac{z}{1+\alpha}\right)\hat{\theta}(z)\right\} = 0. \qquad \blacksquare$$

Similarly, let us define

$$\hat{\psi}(z) = -(-\Delta)^{-1+\gamma/2}\hat{\theta}(z).$$

The vector field

$$q(z) := \hat{u}(z) - \frac{z}{1+\alpha} = \nabla^{\perp}\hat{\psi}(z) - \frac{z}{1+\alpha} \tag{2.6}$$

is called the *pseudo-velocity*, and its integral curves are the *pseudo-streamlines*.

2.1. Adapted coordinates

Our aim is to perturb the stationary radial solution (1.3) to find nontrivial self-similar solutions with a locally integrable singularity at the origin. However, perturbation techniques, for instance the implicit function theorem or the Crandall–Rabinowitz theorem, seem to be difficult to apply to radial solutions to gSQG or Euler equations due to the very special structure of the equation. Roughly speaking, the nonlinear operator and the linear one behave very differently in terms of the singularity at the origin. That makes the choice of the appropriate function spaces difficult and leads to problems in the Fredholm structure of the linearized operator. A similar issue appears when dealing with uniformly rotating solutions, which was solved in [12, 13, 50] by restricting the class of solutions or doing an appropriate change of variables by using level sets of the vorticity.

Although here we are not dealing with rotating solutions, our equation (2.5) has a very similar structure close to the radial solution (1.3). Moreover, we will have to deal with extra difficulties due to the singularity of (1.3) at 0 and its behavior at ∞. More precisely, we are now working in an unbounded domain and we need to use appropriate weights at 0 and ∞. In order to overcome the problem with the Fredholm structure of the linearized operator, we will work with some adapted coordinates instead of the usual polar coordinates.

Let (r, ϑ) be the polar coordinates with $z = re^{i\vartheta}$. We would like to find some adapted coordinates (β, ϕ) such that the pseudo-velocity (2.6) has vanishing ϕ-component. In order to define that change of coordinates one needs to give two equations relating (r, ϑ) and (β, ϕ). The first one will relate ϑ to β and ϕ, where we will think of ϕ as a new angle:

$$\vartheta = \beta + \phi, \quad \phi \in \mathbb{T}\,. \tag{2.7}$$

In the following proposition we find the second equation defining the change of coordinates, which will be a nonlinear expression of β depending on (r, ϑ) making the ϕ-component of the pseudo-velocity vanish (which will help us to solve later the vorticity equation in Proposition 2.4). We shall see that the nonlinear expression is indeed an implicit function that depends on the solution itself. Denote

$$\Theta(\beta, \phi) := \hat{\theta}(z(\beta, \phi)), \quad \Psi(\beta, \phi) := \hat{\psi}(z(\beta, \phi)).$$

Proposition 2.2. *If $r = (-(1+\alpha)\Psi_\beta)^{1/2}$ and $\vartheta = \beta + \phi$, then the ϕ-component of the pseudo-velocity $\nabla^{\perp}\hat{\psi} - \frac{z}{1+\alpha}$ vanishes.*

Proof. Let us compute the term $(\nabla^\perp\hat{\psi} - \frac{z}{1+\alpha})\cdot\nabla_z\hat{\theta}$ in terms of (β,ϕ):

$$\begin{aligned}\left(\nabla^\perp\hat{\psi} - \frac{z}{1+\alpha}\right)\cdot\nabla_z\hat{\theta} &= \left(-\hat{\psi}_{z_2} - \frac{z_1}{1+\alpha}\right)\hat{\theta}_{z_1} + \left(\hat{\psi}_{z_1} - \frac{z_2}{1+\alpha}\right)\hat{\theta}_{z_2}\\
&= \{-\Psi_\beta\beta_{z_2} - \Psi_\phi\phi_{z_2 - \frac{z_1}{1+\alpha}}\}\{\Theta_\beta\beta_{z_1} + \Theta_\phi\phi_{z_1}\}\\
&\quad + \{\Psi_\beta\beta_{z_1} + \Psi_\phi\phi_{z_1 - \frac{z_2}{1+\alpha}}\}\{\Theta_\beta\beta_{z_2} + \Theta_\phi\phi_{z_2}\}\\
&= \left\{(\beta_{z_1}\phi_{z_2} - \beta_{z_2}\phi_{z_1})\Psi_\beta - \frac{1}{1+\alpha}(z_1\phi_{z_1} + z_2\phi_{z_2})\right\}\Theta_\phi\\
&\quad + \left\{(\beta_{z_2}\phi_{z_1} - \beta_{z_1}\phi_{z_2})\Psi_\phi - \frac{1}{1+\alpha}(z_1\beta_{z_1} + z_2\beta_{z_2})\right\}\Theta_\beta\\
&= \Bigg((\beta_{z_2}\phi_{z_1} - \beta_{z_1}\phi_{z_2})\Psi_\phi - \frac{1}{1+\alpha}(z_1\beta_{z_1} + z_2\beta_{z_2}),\\
&\quad (\beta_{z_1}\phi_{z_2} - \beta_{z_2}\phi_{z_1})\Psi_\beta - \frac{1}{1+\alpha}(z_1\phi_{z_1} + z_2\phi_{z_2})\Bigg)\cdot\nabla_{\beta,\phi}\Theta.\end{aligned}$$

We claim that under the definition of (r,ϑ) in terms of the new variables (β,ϕ) we achieve

$$(\beta_{z_1}\phi_{z_2} - \beta_{z_2}\phi_{z_1})\Psi_\beta - \frac{1}{1+\alpha}(z_1\phi_{z_1} + z_2\phi_{z_2}) = 0. \tag{2.8}$$

Note that assuming (2.8), we have

$$\left(\nabla^\perp\hat{\psi} - \frac{z}{1+\alpha}\right)\cdot\nabla_z\hat{\theta} = \left\{(\beta_{z_2}\phi_{z_1} - \beta_{z_1}\phi_{z_2})\Psi_\phi - \frac{1}{1+\alpha}(z_1\beta_{z_1} + z_2\beta_{z_2})\right\}\Theta_\beta,$$

where the factor multiplying Θ_ϕ is vanishing and therefore we can conclude the proof of this proposition.

Let us check our claim. Note that by a change of variables one has

$$\Psi_\beta = -\frac{1}{1+\alpha}|z|^2.$$

Moreover, using $\vartheta(\beta,\phi) = \beta + \phi$ and $z = z(r,\vartheta)$ one gets

$$z\times z_\beta = |z|^2,$$

implying

$$\Psi_\beta = -\frac{1}{1+\alpha}(z\times z_\beta). \tag{2.9}$$

Now, let us write the expression of z_β. For that, let us compute the Jacobian of the transformation

$$\begin{bmatrix} z_{1,\beta} & z_{1,\phi}\\ z_{2,\beta} & z_{2,\phi}\end{bmatrix} = \begin{bmatrix}\beta_{z_1} & \beta_{z_2}\\ \phi_{z_1} & \phi_{z_2}\end{bmatrix}^{-1} = \frac{1}{\beta_{z_1}\phi_{z_2} - \beta_{z_2}\phi_{z_1}}\begin{bmatrix}\phi_{z_2} & -\beta_{z_2}\\ -\phi_{z_1} & \beta_{z_1}\end{bmatrix}. \tag{2.10}$$

Hence

$$z\times z_\beta = z_1 z_{2,\beta} - z_2 z_{1,\beta} = -\frac{1}{\beta_{z_1}\phi_{z_2} - \beta_{z_2}\phi_{z_1}}(z_1\phi_{z_1} + z_2\phi_{z_2}).$$

That implies that (2.9) agrees with

$$\Psi_\beta = \frac{1}{1+\alpha}\,\frac{1}{\beta_{z_1}\phi_{z_2} - \beta_{z_2}\phi_{z_1}}(z_1\phi_{z_1} + z_2\phi_{z_2}),$$

or equivalently

$$(\beta_{z_1}\phi_{z_2} - \beta_{z_2}\phi_{z_1})\Psi_\beta = \frac{1}{1+\alpha}(z_1\phi_{z_1} + z_2\phi_{z_2}),$$

which leads to (2.8). ∎

Let us remark again that the change of variables of the previous proposition is a nonlinear change of variables that depends on the solution; indeed, it depends on the stream function. Later, in Section 7.1 we will check that we can come back to the physical variables and thus the change of variables is invertible. Moreover, from now on define

$$\partial_\varphi = \partial_\phi - \partial_\beta. \tag{2.11}$$

In the following, let us compute the Jacobian of the transformation.

Proposition 2.3. *The Jacobian of the transformation* $(\beta,\phi) \mapsto (z_1, z_2)$ *is given by*

$$|J| = \frac{1+\alpha}{2}\Psi_{\beta\varphi},$$

where $\partial_\varphi = \partial_\phi - \partial_\beta$.

Proof. Recall that the matrix of the transformation is given by

$$J = \begin{bmatrix} z_{1,\beta} & z_{1,\phi} \\ z_{2,\beta} & z_{2,\phi} \end{bmatrix}.$$

From Proposition 2.2 we have

$$r = (-(1+\alpha)\Psi_\beta)^{1/2}, \quad \vartheta = \beta + \phi,$$

and using this expression, we can compute

$$\begin{aligned}
\partial_\phi z_1 &= \partial_r z_1 \partial_\phi r + \partial_\vartheta z_1 \partial_\phi \theta \\
&= -\tfrac{1}{2}\cos(\vartheta)(-(1+\alpha)\Psi_\beta)^{-1/2}(1+\alpha)\partial^2_{\phi\beta}\Psi - r\sin(\vartheta), \\
\partial_\beta z_1 &= -\tfrac{1}{2}\cos(\vartheta)(-(1+\alpha)\Psi_\beta)^{-1/2}(1+\alpha)\partial^2_{\beta\beta}\Psi - r\sin(\vartheta), \\
\partial_\phi z_2 &= -\tfrac{1}{2}\sin(\vartheta)(-(1+\alpha)\Psi_\beta)^{-1/2}(1+\alpha)\partial^2_{\phi\beta}\Psi + r\cos(\vartheta), \\
\partial_\beta z_2 &= -\tfrac{1}{2}\sin(\vartheta)(-(1+\alpha)\Psi_\beta)^{-1/2}(1+\alpha)\partial^2_{\beta\beta}\Psi + r\cos(\vartheta).
\end{aligned}$$

Then, after some straightforward computations, we finally find that

$$|J| = z_{1,\beta}z_{2,\phi} - z_{1,\phi}z_{2,\beta} = \frac{1+\alpha}{2}\Psi_{\beta\varphi}.$$ ∎

2.2. *The equation in the new coordinates*

In Section 2.1 we have defined the adapted coordinates that will be used throughout this work. Recall that we have chosen the coordinates in order to have the ϕ-component of the pseudo-velocity vanishing. In this way, we find

$$\left(\nabla^{\perp}\hat{\psi} - \frac{z}{1+\alpha}\right)\cdot\nabla\hat{\theta} = \left\{(\beta_{z_2}\phi_{z_1} - \beta_{z_1}\phi_{z_2})\Psi_{\phi} - \frac{1}{1+\alpha}(z_1\beta_{z_1} + z_2\beta_{z_2})\right\}\Theta_{\beta},$$

and then the vorticity equation (2.5) is equivalent to

$$\frac{1+\alpha-\gamma}{1+\alpha}\Theta(\beta,\phi) = \left\{(\beta_{z_2}\phi_{z_1} - \beta_{z_1}\phi_{z_2})\Psi_{\phi} - \frac{1}{1+\alpha}(z_1\beta_{z_1} + z_2\beta_{z_2})\right\}\Theta_{\beta}. \tag{2.12}$$

The very special structure of (2.12) involving only the β-derivative of Θ gives us a trivial solution. That will be discussed in the following result.

Proposition 2.4. *Equation* (2.12) *is equivalent to*

$$(1+\alpha-\gamma)\Theta(\beta,\phi)\Psi_{\beta\varphi}(\beta,\phi) + 2\Psi_{\varphi}(\beta,\phi)\Theta_{\beta}(\beta,\phi) = 0. \tag{2.13}$$

Proof. Let us start from (2.12) taking into account that

$$\beta_{z_2}\phi_{z_1} - \beta_{z_1}\phi_{z_2} = -|J^{-1}| = -\frac{1}{|J|} = -\frac{2}{(1+\alpha)\Psi_{\beta\varphi}},$$

that is, (2.12) amounts to

$$\frac{1+\alpha-\gamma}{1+\alpha}\Theta(\beta,\phi)\Psi_{\beta\varphi} + \left\{\frac{2}{1+\alpha}\Psi_{\phi} + \frac{\Psi_{\beta\varphi}}{1+\alpha}(z_1\beta_{z_1} + z_2\beta_{z_2})\right\}\Theta_{\beta} = 0. \tag{2.14}$$

Let us now compute $z_1\beta_{z_1} + z_2\beta_{z_2} = (z_1, z_2)\cdot\nabla_z\beta$. Note that

$$J^{-1} = \begin{bmatrix}\beta_{z_1} & \beta_{z_2}\\ \phi_{z_1} & \phi_{z_2}\end{bmatrix} = \frac{1}{|J|}\begin{bmatrix} z_{2,\phi} & -z_{1,\phi}\\ -z_{2,\beta} & z_{1,\beta}\end{bmatrix}. \tag{2.15}$$

Then

$$\beta_{z_1} = \frac{2}{(1+\alpha)\Psi_{\beta\varphi}}\left\{-\frac{\sin(\vartheta)(1+\alpha)}{2r}\partial^2_{\phi\beta}\Psi + r\cos(\vartheta)\right\},$$
$$\beta_{z_2} = -\frac{2}{(1+\alpha)\Psi_{\beta\varphi}}\left\{-\frac{\cos(\vartheta)(1+\alpha)}{2r}\partial^2_{\phi\beta}\Psi - r\sin(\vartheta)\right\}.$$

After some straightforward computations we find

$$z_1\beta_{z_1} + z_2\beta_{z_2} = \frac{2r^2}{(1+\alpha)\Psi_{\beta\varphi}} = -2\frac{\Psi_{\beta}}{\Psi_{\beta\varphi}}.$$

Putting everything together, we conclude that (2.14) is equivalent to (2.13). ∎

By taking into account the equivalence between (2.12) and (2.13) we find a trivial solution for the vorticity equation given in the following corollary.

Corollary 2.5. *Equations* (2.12) *and* (2.13) *have the following trivial solution:*

$$\Theta(\beta,\phi) = (\Psi_\varphi)^{-\frac{1+\alpha-\gamma}{2}}\Omega(\phi) = (\Psi_\varphi)^{-\frac{1/\mu}{2}}\Omega(\phi), \tag{2.16}$$

for any function Ω.

Via the adapted coordinates, we have simplified the nonlinear equation (2.5) to (2.13) where a trivial solution is found by (2.16). However, this does not take into account the relation between $\hat{\theta}$ and $\hat{\psi}$ via the singular integral operator (1.2), and this will be our aim from now on. Note that

$$\hat{\psi}(z) = -\frac{C_\gamma}{2\pi}\int_{\mathbb{R}^2}\frac{\hat{\theta}(y)}{|x-y|^\gamma}\,dA(y). \tag{2.17}$$

By writing (2.17) in the adapted coordinates with the help of the previous computation of the Jacobian of the transformation in Proposition 2.3 together with the trivial solution from Corollary 2.5, we find that it amounts to (recall that $\iint := \int_0^{2\pi}\int_0^\infty$)

$$\Psi(\beta,\phi) = \frac{C_\gamma(1+\alpha)^{1-\gamma/2}}{4\pi}\iint\frac{(\Psi_\varphi(b,\Phi))^{-\frac{1+\alpha-\gamma}{2}}\Omega(\Phi)\Psi_{b\varphi}(b,\Phi)\,db\,d\Phi}{\{-\Psi_\beta-\Psi_b+2\Psi_\beta^{1/2}\Psi_b^{1/2}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}. \tag{2.18}$$

In this expression we use the fact that $r=0$ corresponds to $\beta=\infty$, and $r=\infty$ means $\beta=0$ as we shall see later, which is the reason we have a change of sign in the integral (2.18). Hence, our aim is to find functions Ψ and Ω such that

$$\begin{aligned} F(\Psi,\Omega)(\beta,\phi) &:= \Psi(\beta,\phi)\\ &- \frac{C_\gamma(1+\alpha)^{1-\gamma/2}}{4\pi}\iint\frac{(\Psi_\varphi(b,\Phi))^{-\frac{1}{2\mu}}\Omega(\Phi)\Psi_{b\varphi}(b,\Phi)\,db\,d\Phi}{\{-\Psi_\beta-\Psi_b+2\Psi_\beta^{1/2}\Psi_b^{1/2}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}} = 0 \end{aligned} \tag{2.19}$$

for any $\beta\in[0,\infty)$ and $\phi\in\mathbb{T}$. In (2.19) we have used the fact that $1/\mu = 1+\alpha-\gamma$ coming from (2.2).

2.3. *Trivial solution*

The problem reduces to the study of the nonlinear equation (2.19) which is nothing but the relation between Ω and Ψ. In this section, we look for trivial solutions of that equation. The trivial solutions come from the fact that any radial function θ provides a stationary solution of (1.1), where the stream function is also radial. Here, we will use the computations of Appendix A for the fractional Laplacian.

Proposition 2.6. *If*

$$\Omega^0 = \frac{1}{(1+\alpha)^{\frac{1}{2\mu}}}$$

and

$$\Psi^0(\beta) = -C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\beta^{-\frac{1-\alpha}{1+\alpha}}, \tag{2.20}$$

then $F(\Psi^0,\Omega^0)(\beta,\phi) = 0$ *for any* $\beta \in [0,\infty)$ *and* $\phi \in \mathbb{T}$, *where* C_0 *is defined in* (A.2). *Moreover, the adapted coordinates* $(r,\vartheta) \mapsto (\beta,\phi)$ *at the trivial solution read*

$$\beta = C_0(\alpha-1)r^{-1-\alpha}, \quad \phi = \vartheta - C_0(\alpha-1)r^{-1-\alpha}.$$

Proof. The trivial solution that we will construct comes from the stationary solution

$$\theta(x) = |x|^{-1/\mu}$$

for $1/2 < \mu < \frac{1}{2-\gamma}$. From Corollary A.2 we know that ψ is also radial and is given by

$$\psi(x) = -C_0|x|^{2-\gamma-1/\mu} = -C_0|x|^{1-\alpha},$$

by using (2.2) and where C_0 is defined in (A.2). Using (2.4), we have $\hat{\theta}(z) = |z|^{-1/\mu}$ and $\hat{\psi}(z) = -C_0|z|^{1-\alpha}$. Since θ and ψ are a solution to the original elliptic equation, it suffices to write it in terms of (β,ϕ) to find the associated solution of (2.19). Hence, let us now express the solution in terms of (β,ϕ).

First, let us find an explicit relation between β and r via the nonlinear change of variables in Proposition 2.2. Indeed, we have $\Psi_\beta = -\frac{r^2}{1+\alpha}$ and

$$\partial_r\hat{\psi}(z) = C_0(\alpha-1)r^{-\alpha} = \partial_r\beta\partial_\beta\Psi = -\partial_r\beta\frac{r^2}{1+\alpha}.$$

Hence

$$\partial_r\beta = -C_0(1+\alpha)(\alpha-1)r^{-2-\alpha},$$

implying

$$\beta = C_0(\alpha-1)r^{-1-\alpha}, \tag{2.21}$$

which amounts to

$$r = (C_0(\alpha-1))^{\frac{1}{1+\alpha}}\beta^{-\frac{1}{1+\alpha}}.$$

Since $\alpha > 1$ and $r \in [0,\infty)$ we see that $\beta \in [0,\infty)$. In this way, we can compute Ψ depending on β as

$$\Psi^0(\beta) = -C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\beta^{-\frac{1-\alpha}{1+\alpha}}.$$

Let us now compute Ω. Note that

$$\Theta(\beta) = \left(\frac{1}{C_0(\alpha-1)}\right)^{\frac{1/\mu}{1+\alpha}}\beta^{\frac{1/\mu}{1+\alpha}},$$

and using the relation (2.16), we find

$$\Omega^0 = \frac{1}{(1+\alpha)^{\frac{1}{2\mu}}}.$$

That concludes the proof. ∎

Remark 2.7. The solutions that we aim to construct are perturbations of the singular radial stationary solution $\theta(x) = |x|^{-1/\mu}$. The associated initial stream function is also radial and can be computed via Corollary A.2 as $\psi(x) = -C_0|x|^{2-\gamma-1/\mu} = -C_0|x|^{1-\alpha}$. Note that both the potential temperature θ and the associated stream function have a singularity when $|x| = 0$ since $\alpha > 1$ via (2.3). However, the change of variables $(r, \vartheta) \mapsto (\beta, \phi)$ at the trivial solution gives us $\beta = C_0(1-\alpha)r^{-1-\alpha}$. Hence, θ and ψ written in terms of β are not singular at $\beta = 0$ any more, but they are singular as $\beta \to \infty$.

2.4. *Formulation with appropriate scaling*

Our aim is to construct solutions which are perturbations of the trivial one (2.20). Since $\alpha > 1$ we see that Ψ^0 in (2.20) vanishes at 0 and diverges at ∞. We shall use similar scaling for the perturbations. First, we shall simplify the constant in Ψ and Ω by working with $\tilde{\Psi}$ and $\tilde{\Omega}$ instead, where

$$\Psi(\beta,\phi) = -C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\tilde{\Psi}(\beta,\phi), \quad \Omega(\phi) = (1+\alpha)^{-\frac{1}{2\mu}}\tilde{\Omega}(\phi).$$

In that way, (2.19) reads

$$\frac{F(\Psi,\Omega)(\beta,\phi)}{-C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}} = \tilde{\Psi}(\beta,\phi) - (-1)^{-\frac{1}{2\mu}+\frac{\gamma}{2}}\frac{C_\gamma(1+\alpha)^{-\frac{\alpha-1}{2}}(\alpha-1)^{\frac{\alpha-1}{2}}}{4\pi C_0}$$
$$\times \iint \frac{(\tilde{\Psi}_\varphi(b,\Phi))^{-\frac{1}{2\mu}}\tilde{\Omega}(\Phi)\tilde{\Psi}_{b\varphi}(b,\Phi)\,db\,d\Phi}{\{-\tilde{\Psi}_\beta - \tilde{\Psi}_b + 2\tilde{\Psi}_\beta^{1/2}\tilde{\Psi}_b^{1/2}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}} = 0.$$

Hence, we take f satisfying

$$\tilde{\Psi}(\beta,\phi) = \beta^{\frac{\alpha-1}{1+\alpha}}(1+f(\beta,\phi)),$$

and we divide the equation by $\beta^{\frac{\alpha-1}{1+\alpha}}$ getting the equivalent formulation

$$\begin{aligned}
\tilde{F}(f,\tilde{\Omega})(\beta,\phi) &:= \beta^{-\frac{\alpha-1}{1+\alpha}}\frac{F(\Psi,\Omega)(\beta,\phi)}{-C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}} \\
&= 1 + f(\beta,\phi) \\
&\quad - (-1)^{-\frac{1}{2\mu}+\frac{\gamma}{2}}\frac{C_\gamma(1+\alpha)^{-\frac{\alpha-1}{2}}(\alpha-1)^{\frac{\alpha-1}{2}}}{4\pi C_0}\beta^{-\frac{\alpha-1}{1+\alpha}} \\
&\quad \times \iint \frac{(\tilde{\Psi}_\varphi(b,\Phi))^{-\frac{1}{2\mu}}\tilde{\Omega}(\Phi)\tilde{\Psi}_{b\varphi}(b,\Phi)\,db\,d\Phi}{\{-\tilde{\Psi}_\beta - \tilde{\Psi}_b + 2\tilde{\Psi}_\beta^{1/2}\tilde{\Psi}_b^{1/2}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}} = 0.
\end{aligned}$$

We can compute $\partial_\varphi\tilde{\Psi}$, $\partial_\beta\tilde{\Psi}$ and $\partial_{\beta\varphi}\tilde{\Psi}$ in terms of f:

$$\begin{aligned}
\partial_\beta\tilde{\Psi}(\beta,\phi) &= \beta^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\beta}} f(\beta,\phi)\right\}, \\
\partial_\varphi\tilde{\Psi}(\beta,\phi) &= \beta^{\frac{-2}{1+\alpha}}\left\{-\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\varphi}} f(\beta,\phi)\right\}, \qquad (2.22) \\
\partial_{\beta\varphi}\tilde{\Psi}(\beta,\phi) &= \beta^{-1}\beta^{\frac{-2}{1+\alpha}}\left\{\frac{2(\alpha-1)}{(1+\alpha)^2} + \partial_{\overline{\beta\varphi}} f\right\},
\end{aligned}$$

where the linear operators $\partial_{\overline{\beta}}$, $\partial_{\overline{\varphi}}$ and $\partial_{\overline{\beta\varphi}}$ are

$$\partial_{\overline{\beta}} f(\beta,\phi) := \frac{\alpha-1}{1+\alpha} f(\beta,\phi) + \beta f_\beta(\beta,\phi), \tag{2.23}$$

$$\partial_{\overline{\varphi}} f(\beta,\phi) := -\frac{\alpha-1}{1+\alpha} f(\beta,\phi) + \beta f_\varphi(\beta,\phi), \tag{2.24}$$

$$\begin{aligned} \partial_{\overline{\beta\varphi}} f &:= \frac{\alpha-1}{1+\alpha} f + \partial_{\overline{\beta}}\partial_{\overline{\varphi}} f \\ &= \frac{2(\alpha-1)}{(1+\alpha)^2} f + \frac{\alpha-1}{1+\alpha}\beta(f_\varphi - f_\beta) + \beta^2 f_{\beta\varphi}. \end{aligned} \tag{2.25}$$

Hence, $\tilde{F}$ reads

$$\begin{aligned} &\tilde{F}(f,\tilde{\Omega})(\beta,\phi) \\ &= 1 + f(\beta,\phi) - (-1)^{\frac{-1}{2\mu}} \frac{C_\gamma(\alpha-1)^{\frac{\alpha-1}{2}}}{4\pi C_0(1+\alpha)^{\frac{\alpha-1}{2}}} \beta^{-\frac{\alpha-1}{1+\alpha}} \\ &\quad\times \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} \left\{-\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\varphi}} f(b,\Phi)\right\}^{-\frac{1}{2\mu}} \left\{\frac{2(\alpha-1)}{(1+\alpha)^2} + \partial_{\overline{\beta\varphi}} f(b,\Phi)\right\} \tilde{\Omega}(\Phi)\, db\, d\Phi}{D(f)(\beta,\phi,b,\Phi)^{\gamma/2}}, \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} D(f)(\beta,\phi,b,\Phi) &:= \tilde{\Psi}_\beta + \tilde{\Psi}_b - 2\tilde{\Psi}_\beta^{1/2}\tilde{\Psi}_b^{1/2}\cos(\phi+\beta-\Phi-b) \\ &= \beta^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\beta}} f(\beta,\phi)\right\} + b^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{b}} f(b,\Phi)\right\} \\ &\quad - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\beta}} f(\beta,\phi)\right\}^{1/2} \left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{b}} f(b,\Phi)\right\}^{1/2} \\ &\qquad \times \cos(\phi+\beta-\Phi-b) \\ &= \left(\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\beta}} f(\beta,\phi)\right\}^{1/2} - b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{b}} f(b,\Phi)\right\}^{1/2}\right)^2 \\ &\quad + 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\beta}} f(\beta,\phi)\right\}^{1/2} \left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{b}} f(b,\Phi)\right\}^{1/2} \\ &\qquad \times (1 - \cos(\phi+\beta-\Phi-b)). \end{aligned} \tag{2.27}$$

Let us remark that in the expression of $\tilde{F}$ no ϕ-derivatives appear. Indeed, it only depends on ∂_β and ∂_φ. In that way, we will not assume any boundedness of ∂_ϕ in our function spaces. Moreover, since we are working in the unbounded space $[0,\infty)$ and the trivial solution is singular at $\beta = 0$, we shall insert some weights in the function spaces to be defined in the next section.

3. Functional regularity

3.1. Function spaces

The choice of function spaces is a crucial point in this work. In particular, application of the infinite-dimensional implicit function theorem to $\tilde{F}$ relies on those spaces: they need to be preserved by the nonlinear operator, and also the linearized operator has to be invertible in those spaces. Since we are dealing with an unbounded domain ($\beta \in [0,\infty)$), one needs to use weights in the function spaces that control the decay in β. Moreover, the fact that ∂_ϕ does not appear in the expression of $\tilde{F}$ is important to later check that the decay is also accounted for by the function spaces. That is why we will not assume any differentiability of f in ϕ. Let us now introduce the function spaces we will use throughout this work.

First, let us describe the function spaces for the scaled perturbation f. Indeed, we will assume that f is bounded and has appropriate decay at 0 and ∞. We need the decay in order to tackle the singularity at $\beta = 0$ and $\beta \to \infty$ in the nonlinear equation.

For $\sigma > 0$ (to be fixed later), let us define the weighted norm

$$\|f\|_{L^{\infty,\sigma}} := \sup_{\beta\in[0,\infty),\,\phi\in[0,2\pi]} \frac{(1+\beta)^{2\sigma}}{\beta^\sigma}|f(\beta,\phi)|. \tag{3.1}$$

We will assume that f is Hölder continuous in β and define

$$\|f\|_{C^\delta_\beta} := \sup_{\phi\in[0,2\pi]}\sup_{|\beta-b|<1} |\beta+b|^\delta \frac{|f(\beta,\phi)-f(b,\phi)|}{|\beta-b|^\delta}, \tag{3.2}$$

and Lipschitz continuous in the periodic variable ϕ, with

$$\|f\|_{\mathrm{Lip}_\phi} := \sup_{\beta\in[0,\infty)}\sup_{|\phi-\Phi|<1} \frac{|f(\beta,\phi)-f(\beta,\Phi)|}{\sin((\Phi-\phi)/2)}. \tag{3.3}$$

Hence, let us define

$$\mathcal{X}^{\sigma,\delta} := \{f : [0,\infty)\times[0,2\pi] \to \mathbb{C} \mid \|f\|_{\mathcal{X}^{\sigma,\delta}} := \|f\|_{L^{\infty,\sigma}} + \|f\|_{C^\delta_\beta} < \infty\}. \tag{3.4}$$

Note that the functions f lying in $\mathcal{X}^{\sigma,\delta}$ satisfy

$$f \sim_0 \beta^\sigma, \quad f \sim_\infty \beta^{-\sigma},$$

which will be crucial to work with the singularity at 0 and at ∞.

Now, we can define the function spaces for f:

$$X_1^{\sigma,\delta} := \Big\{ f \;\Big|\; f, \beta f_\beta, \beta f_\varphi, \beta^2 f_{\beta\varphi} \in \mathcal{X}^{\sigma,\delta},\ \|f\|_{\mathrm{Lip}_\phi} + \|\beta f_\beta\|_{\mathrm{Lip}_\phi} < \infty, \\ f(\beta,\phi) = \sum_{k\in\mathbb{Z}} f_k(\beta) e^{ik\phi} \Big\}, \tag{3.5}$$

with the norm

$$\|f\|_{X_1^{\sigma,\delta}} := \|f\|_{\mathcal{X}^{\sigma,\delta}} + \|\beta f_\beta\|_{\mathcal{X}^{\sigma,\delta}} + \|\beta f_\varphi\|_{\mathcal{X}^{\sigma,\delta}} + \|\beta^2 f_{\varphi\beta}\|_{\mathcal{X}^{\sigma,\delta}} + \|f\|_{\mathrm{Lip}_\phi} + \|\beta f_\beta\|_{\mathrm{Lip}_\phi}. \tag{3.6}$$

Notice that the expression of the nonlinear operator (2.26) only involves derivatives of the type ∂_φ and ∂_β, but not ∂_ϕ. For that reason, we only consider such derivatives in the function space (3.5) and we will take into account that $\partial_\phi = \partial_\varphi + \partial_\beta$. However, we shall require Lipschitz continuity in ϕ in order to tackle the singularity of the denominator in (2.26).

Moreover, let us denote by $B_{X_1^{\sigma,\delta}}(\epsilon)$ the ball in $X_1^{\sigma,\delta}$ centered at 0 and of radius ϵ. Let us remark that $1 \notin X_1^{\sigma,\delta}$ (this is the associated trivial solution after the scaling in (2.26)) because it does not have the same decay at 0 and ∞. Hence, we will have to subtract the contribution from the trivial solution from the nonlinear operator to obtain its well-posedness.

Remark 3.1. For $f \in X_1^{\sigma,\delta}$, one has $\partial_{\overline{\beta}} f, \partial_{\overline{\varphi}} f, \partial_{\overline{\beta\varphi}} f \in \mathcal{X}^{\sigma,\delta}$, where those linear operators are defined in (2.23)–(2.25).

We will assume that Ω lies in the Lebesgue space L^p for some p that will be fixed later. Then we define

$$X_2^p := \left\{ f \in L^p(0,2\pi) \;\middle|\; f(\phi) = 1 + \sum_{0\neq k\in\mathbb{Z}} f_k e^{ik\phi} \right\}. \tag{3.7}$$

Remark 3.2. As we can observe in the statement of the main result of this work, Theorem 1.1, the generality of the choice of Ω depends on which space it belongs to. We will assume that Ω belongs to an L^p space for some appropriate p. Indeed, the parameter p will be defined in such a way that the nonlinear operator is well-defined; see Propositions 3.9 and 3.10. In particular, p has to satisfy $p > \frac{1}{1-\gamma}$.

Finally, we set the range space:

$$Y^{\sigma,\delta} := \{ f \mid f, \beta f_\varphi \in \mathcal{X}^{\sigma,\delta},\ \|f\|_{\mathrm{Lip}_\phi} < \infty \}, \tag{3.8}$$

with the norm

$$\|f\|_{Y^{\sigma,\delta}} := \|f\|_{\mathcal{X}^{\sigma,\delta}} + \|\beta f_\varphi\|_{\mathcal{X}^{\sigma,\delta}} + \|f\|_{\mathrm{Lip}_\phi}.$$

From now on, denote by $B_X(c,r)$ the ball in X centered at c and of radius r, and we write just $B_X(r)$ when $c = 0$.

Remark 3.3. Note that any function $f \in X_1^{\sigma,\delta}$ can be decomposed into a Fourier series in ϕ, that is, there exist $f_k(\beta)$ such that

$$f(\beta,\phi) = \sum_{k\in\mathbb{Z}} f_k(\beta) e^{ik\phi}.$$

Although we are looking for a real function f, which will be the perturbation of the initial stream function, we will consider it initially as a complex function. Later, we may just include in the function spaces the assumption that f is real.

Moreover, we can translate the decay properties of f into the coefficients f_k. In that way, we find that $f_k \in \mathcal{X}_0$ where

$$\mathcal{X}_0 := \{f : [0,\infty) \to \mathbb{R} \mid f, \beta f' \in \mathcal{X}^{\sigma,\delta}\}. \tag{3.9}$$

Although the space $\mathcal{X}^{\sigma,\delta}$ was defined in (3.4) for functions depending on (β,ϕ), we will also use it for functions only depending on β. Note that functions with Fourier coefficients in $\mathcal{X}_0$ are not necessarily in $X_1^{\sigma,\delta}$. We can do the same for $f \in Y^{\sigma,\delta}$ since the associated Fourier coefficients lie in the space

$$\mathcal{Y}_0 := \{f : [0,\infty) \to \mathbb{R} \mid f \in \mathcal{X}^{\sigma,\delta}\}. \tag{3.10}$$

These spaces will be used in order to check that the kernel of the linearized operator is trivial.

Remark 3.4. The function spaces used in [5, 36, 37] for the Euler equations cannot be used here. This is due to the different formulation of the nonlinear function F used in those papers. Note that the relation between the stream function ψ and the vorticity ω in the Euler equations is given via a local relation. Therefore, there is no need to use the nonlocal expression via a singular integral, as we have to do for gSQG equations.

Remark 3.5. We assume that Ω is of the form

$$\Omega(\phi) = 1 + \sum_{0\neq k\in\mathbb{Z}} g_k e^{ik\phi},$$

where 1 is the trivial solution and then no perturbation of the mode $k = 0$ is done. This ensures that we do not obtain more trivial solutions and that the solution is nonradial.

Remark 3.6. Function spaces with weights are commonly used when working in an unbounded space. Moreover, a weight must be considered when taking any derivative. That happens also in [34], where the authors have to include some weight even for the Hölder norm. Note that if f only depends on the angle ϕ (and not on β), then these weighted function spaces are the usual spaces C^1, L^∞, etc.

3.2. Deformation of the Euclidean norm

In order to study the regularity of the nonlinear function (2.26), we shall work with the denominator given in (2.27). The denominator is nothing other than the Euclidean norm deformed by the adapted coordinates and the scaling. That occurred also in [48,49], where the authors introduced some cylindrical coordinates amounting to a deformation of the

Euclidean norm. Hence, estimating from below the denominator

$$
\begin{aligned}
D(f)(\beta,\phi,b,\Phi) &:= \tilde{\Psi}_\beta + \tilde{\Psi}_b - 2\tilde{\Psi}_\beta^{1/2}\tilde{\Psi}_b^{1/2}\cos(\phi+\beta-\Phi-b) \\
&= \left(\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}} f(\beta,\phi)\right\}^{1/2} - b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}} f(b,\Phi)\right\}^{1/2}\right)^2 \\
&\quad + 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}} f(\beta,\phi)\right\}^{1/2}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}} f(b,\Phi)\right\}^{1/2} \\
&\quad \times (1-\cos(\phi+\beta-\Phi-b)) \qquad (3.11)
\end{aligned}
$$

is crucial to study the regularity of (2.26). That is the goal of the following proposition.

Proposition 3.7. *There exists* $\epsilon > 0$ *such that if* $f \in B_{X_1^{\sigma,\delta}}(0,\epsilon)$ *then*

$$
D(f)(\beta,\phi,b,\Phi) \geq C(\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}})\left\{\frac{|\beta-b|^2}{|\beta+b|^2} + \sin^2((\beta-b+\phi-\Phi)/2)\right\}. \qquad (3.12)
$$

Remark 3.8. In particular, we find that $D(f)(\beta,\phi,b,\Phi) \geq CD(0)(\beta,\phi,b,\Phi)$. Moreover, we also get $D(f)(\beta,\phi,b,\Phi) \leq CD(0)(\beta,\phi,b,\Phi)$.

Proof of Proposition 3.7. Take f such that $f \in B_{X_1^{\sigma,\delta}}(\epsilon)$. In particular, $\partial_{\overline{\beta}} f \in \mathcal{X}^{\sigma,\delta}$ and $\|\partial_{\overline{\beta}} f\|_{\mathrm{Lip}_\phi} < \infty$. Hence,

$$
\begin{aligned}
C^{-1} &< \frac{\alpha-1}{1+\alpha} + \partial_{\overline{\beta}} f(\beta,\phi) < C, \\
C^{-1} &< \frac{\alpha-1}{1+\alpha} + \partial_{\overline{b}} f(b,\Phi) < C.
\end{aligned}
$$

By symmetry, that implies

$$
\begin{aligned}
& D(f)(\beta,\phi,b,\Phi) \\
&\geq \left(\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}} f(\beta,\phi)\right\}^{1/2} - b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}} f(b,\Phi)\right\}^{1/2}\right)^2 \\
&\quad + C^{-1}(\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}})(1-\cos(\phi+\beta-\Phi-b))
\end{aligned}
$$

for some C uniformly in b, β, ϕ, Φ. Let us work with the first part. Note that

$$
\begin{aligned}
& \left|\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}} f(\beta,\phi)\right\}^{1/2} - b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}} f(b,\Phi)\right\}^{1/2}\right| \\
&= \left|\frac{\beta^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}} f(\beta,\phi)\right\} - b^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}} f(b,\Phi)\right\}}{\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}} f(\beta,\phi)\right\}^{1/2} + b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}} f(b,\Phi)\right\}^{1/2}}\right| \\
&\gtrsim \frac{\left|\beta^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}} f(\beta,\phi)\right\} - b^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}} f(b,\Phi)\right\}\right|}{\beta^{\frac{-1}{1+\alpha}} + b^{\frac{-1}{1+\alpha}}}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\left|\beta^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}} f(\beta,\phi)\right\}-b^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}} f(b,\Phi)\right\}\right| \\
&\qquad\gtrsim |\beta^{\frac{-2}{1+\alpha}}-b^{\frac{-2}{1+\alpha}}|-(\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}})|\partial_{\overline{\beta}} f(\beta,\phi)-\partial_{\overline{b}} f(b,\Phi)| \\
&\qquad\gtrsim |\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}|\left\{\frac{|\beta-b|}{\beta+b}-|\partial_{\overline{\beta}} f(\beta,\phi)-\partial_{\overline{b}} f(b,\Phi)|\right\}.
\end{aligned}
$$

Then we find

$$
\begin{aligned}
&\left|\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}} f(\beta,\phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}} f(b,\Phi)\right\}^{1/2}\right| \\
&\qquad\gtrsim |\beta^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}}|\left\{\frac{|\beta-b|}{\beta+b}-|\partial_{\overline{\beta}} f(\beta,\phi)-\partial_{\overline{b}} f(b,\Phi)|\right\},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left|\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}} f(\beta,\phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}} f(b,\Phi)\right\}^{1/2}\right|^2 \\
&\qquad\gtrsim |\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}|\left\{\frac{|\beta-b|^2}{|\beta+b|^2}-|\partial_{\overline{\beta}} f(\beta,\phi)-\partial_{\overline{b}} f(b,\Phi)|^2\right\}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
&D(f)(\beta,\phi,b,\Phi)\gtrsim(\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}) \\
&\quad\times\left\{\frac{|\beta-b|^2}{|\beta+b|^2}+(1-\cos(\phi+\beta-\Phi-b))-|\partial_{\overline{\beta}} f(\beta,\phi)-\partial_{\overline{b}} f(b,\Phi)|^2\right\}. \qquad (3.13)
\end{aligned}
$$

It remains to work with

$$|\partial_{\overline{\beta}} f(\beta,\phi)-\partial_{\overline{b}} f(b,\Phi)|.$$

Since we should bound this quantity by $\frac{|\beta-b|}{|\beta+b|}$ or $1-\cos(\phi+\beta-\Phi-b)$, we need to define the auxiliary function

$$g(\beta,x)=\partial_{\overline{\beta}} f(\beta,x-\beta).$$

In that way, taking $x=\beta+\phi$ and $y=b+\Phi$ we have

$$
\begin{aligned}
|\partial_{\overline{\beta}} f(\beta,\phi)-\partial_{\overline{b}} f(b,\Phi)|&=|g(\beta,x)-g(b,y)| \\
&\le|g(\beta,x)-g(b,x)|+|g(b,x)-g(b,y)|. \qquad (3.14)
\end{aligned}
$$

For the first term we use the mean value theorem. Note that $\partial_\beta g$ behaves like $\partial_\varphi f$ and $\beta\partial_{\beta\varphi} f$, by the definition of $\partial_{\overline{\beta}}$. Then, from the definition of $X_1^{\sigma,\delta}$ in (3.5) we infer that $\beta\partial_\varphi g$ is bounded. In that way, we find that

$$|g(\beta,x)-g(b,x)|\le C\|f\|_{X_1^{\sigma,\delta}}\frac{|\beta-b|}{\beta+b}.$$

For the second term, we use the Lipschitz continuity of f and βf_β (and hence $\partial_{\overline{\beta}} f$) with respect to ϕ, obtaining

$$|g(b,x)-g(b,y)| \leq C\|f\|_{X_1^{\sigma,\delta}}|\sin((x-y)/2)|.$$

By using $x=\beta+\phi$ and $y=b+\Phi$ we find

$$|\partial_{\overline{\beta}} f(\beta,\phi)-\partial_{\overline{b}} f(b,\Phi)| \leq C\|f\|_{X_1^{\sigma,\delta}}\left\{\frac{|\beta-b|}{\beta+b}+\sin^2((\beta-b+\phi-\Phi)/2)\right\}.$$

Coming back to (3.13) and using the fact that $\|f\|_{X_1^{\sigma,\delta}}$ is small enough we get (3.12). That concludes the proof. ∎

3.3. *Regularity of the nonlinear function*

In the following proposition, we deal with singular operators defined as

$$\mathcal{H}_f(h_1,h_2)(\beta,\phi) := \beta^{-\frac{\alpha-1}{1+\alpha}} \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} h_1(b,\Phi)h_2(\Phi)\, db\, d\Phi}{D(f)(\beta,\phi,b,\Phi)^{\gamma/2}}. \tag{3.15}$$

This will be crucial in order to prove later the regularity of the nonlinear functional (2.19).

Proposition 3.9. *Let $\gamma,\sigma\in(0,1)$ and $\alpha\in(1,2)$. Then there exists $\epsilon<1$ such that if $f\in B_{X_1^{\sigma,\delta}}(0,\epsilon)$ and*

$$0<\sigma<\min\left\{\frac{\alpha-1}{1+\alpha},\frac{\gamma-\alpha+1}{1+\alpha}\right\}, \quad 1<\alpha<1+\gamma, \tag{3.16}$$

then

$$\|\mathcal{H}_f(h_1,h_2)\|_{Y^{\sigma,\delta}} \leq C\|h_1\|_{X^{\sigma,\delta}}\|h_2\|_{X_2^p}, \tag{3.17}$$

and $f\mapsto\mathcal{H}_f$ is C^1, for any $p>\frac{1}{1-\gamma}$ and $\delta<\min\{1-\gamma,\sigma\}$.

Proof. The proof is very technical and will be divided into several steps. We will provide the key points and the general ideas.

- $\mathcal{H}_f\in L^{\infty,\sigma}$. Using Proposition 3.7, we can bound $\mathcal{H}_f$ as

$$\begin{aligned}
&|\mathcal{H}_f(h_1,h_2)(\beta,\phi)| \\
&\quad\leq C\beta^{-\frac{\alpha-1}{1+\alpha}}\beta^{\frac{\gamma}{1+\alpha}}\iint b^{\frac{-2}{1+\alpha}}\frac{|h_1(b,\phi+\beta-\eta-b)|\,|h_2(\phi+\beta-\eta-b)|\,db\,d\eta}{(\beta^{\frac{\gamma}{1+\alpha}}+b^{\frac{\gamma}{1+\alpha}})\left\{\frac{|\beta-b|^2}{|\beta+b|^2}+\sin^2\left(\frac{\eta}{2}\right)\right\}^{\gamma/2}}
\end{aligned}$$

$$\leq C\beta^{-\frac{\alpha-1}{1+\alpha}}\beta^{\frac{\gamma}{1+\alpha}}\iint \frac{b^{\frac{-2}{1+\alpha}}b^{\sigma}(1+b)^{-2\sigma}|h_2(\phi+\beta-\eta-b)|}{(\beta^{\frac{\gamma}{1+\alpha}}+b^{\frac{\gamma}{1+\alpha}})\{\frac{|\beta-b|^2}{|\beta+b|^2}+\sin^2(\frac{\eta}{2})\}^{\gamma/2}} \times \frac{(1+b)^{2\sigma}}{b^{\sigma}}|h_1(b,\phi+\beta-\eta-b)|\,db\,d\eta$$

$$= C\beta^{\sigma}\iint \frac{z^{\frac{-2}{1+\alpha}}z^{\sigma}(1+\beta z)^{-2\sigma}|h_2(\phi+\beta-\eta-\beta z)|}{(1+z^{\frac{\gamma}{1+\alpha}})\{\frac{|1-z|^2}{|1+z|^2}+\sin^2(\frac{\eta}{2})\}^{\gamma/2}} \times \frac{(1+\beta z)^{2\sigma}}{(\beta z)^{\sigma}}|h_1(\beta z,\phi+\beta-\eta-\beta z)|\,dz\,d\eta,$$

where we have made the change of variables $b=\beta z$ in the last equality. Then, we find

$$\|\mathcal{H}_f(h_1,h_2)\|_{L^{\infty,\sigma}} \leq C\|h_1\|_{L^{\infty,\sigma}}\|h_2\|_{L^1}\sup_{\beta\in[0,\infty)}\int_0^\infty \frac{(1+\beta)^{2\sigma}}{(1+\beta z)^{2\sigma}}\frac{z^{\frac{-2}{1+\alpha}}z^{\sigma}|1+z|^{\gamma}}{|1-z|^{\gamma}|1+z|^{\frac{\gamma}{1+\alpha}}}\,dz.$$

Recall that $\alpha>1$ by (1.7). In order to estimate the above integral, we split into several cases. Take $\varepsilon>0$, and let us work with $\beta\in[0,\varepsilon)$ and $\beta\in[\varepsilon,\infty)$ separately. First, if $\beta\in[0,\varepsilon)$ we find that

$$\sup_{\beta\in[0,\varepsilon)}\int_0^\infty \frac{(1+\beta)^{2\sigma}}{(1+\beta z)^{2\sigma}}\frac{z^{\frac{-2}{1+\alpha}}z^{\sigma}|1+z|^{\gamma}}{|1-z|^{\gamma}|1+z|^{\frac{\gamma}{1+\alpha}}}\,dz \leq C\sup_{\beta\in[0,\varepsilon)}\int_0^\infty \frac{z^{\frac{-2}{1+\alpha}}z^{\sigma}|1+z|^{\gamma}}{|1-z|^{\gamma}|1+z|^{\frac{\gamma}{1+\alpha}}}\,dz \leq C$$

if

$$0<\sigma<\frac{\gamma-\alpha+1}{1+\alpha},\quad 1<\alpha<1+\gamma. \tag{3.18}$$

In the case $\beta\in[\varepsilon,\infty)$ we have

$$\sup_{\beta\in[\varepsilon,\infty)}\int_0^\infty \frac{(1+\beta)^{2\sigma}}{(1+\beta z)^{2\sigma}}\frac{z^{\frac{-2}{1+\alpha}}z^{\sigma}|1+z|^{\gamma}}{|1-z|^{\gamma}|1+z|^{\frac{\gamma}{1+\alpha}}}\,dz \leq C\sup_{\beta\in[\varepsilon,\infty)}\int_0^\infty \frac{1}{z^{\sigma}}\frac{z^{\frac{-2}{1+\alpha}}|1+z|^{\gamma}}{|1-z|^{\gamma}|1+z|^{\frac{\gamma}{1+\alpha}}}\,dz \leq C$$

if

$$0<\sigma<\frac{\alpha-1}{1+\alpha}. \tag{3.19}$$

Hence, we find that

$$0<\sigma<\min\left\{\frac{\alpha-1}{1+\alpha},\frac{\gamma-\alpha+1}{1+\alpha}\right\},\quad 1<\alpha<1+\gamma. \tag{3.20}$$

• $\mathcal{H}_f \in C^{\delta}_{\beta}$. Let us now check the Hölder seminorm. Take $\beta_1 < \beta_2$ and so

$$\beta_2^{-\frac{\alpha-1}{1+\alpha}} < \beta_1^{-\frac{\alpha-1}{1+\alpha}}. \tag{3.21}$$

Taking the difference of $\mathcal{H}_f$ at two points we arrive at

$$\begin{aligned}
&|\mathcal{H}_f(h_1,h_2)(\beta_1,\phi) - \mathcal{H}_f(h_1,h_2)(\beta_2,\phi)| \\
&\quad = \Bigg|\beta_1^{-\frac{\alpha-1}{1+\alpha}} \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} h_1(b,\Phi)h_2(\Phi)\,db\,d\Phi}{D(f)(\beta_1,\phi,b,\Phi)^{\gamma/2}} \\
&\qquad\qquad - \beta_2^{-\frac{\alpha-1}{1+\alpha}} \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} h_1(b,\Phi)h_2(\Phi)\,db\,d\Phi}{D(f)(\beta_2,\phi,b,\Phi)^{\gamma/2}}\Bigg| \\
&\quad \le C\frac{|\beta_1-\beta_2|^{\delta}}{|\beta_1+\beta_2|^{\delta}}\left(\beta_1^{-\frac{\alpha-1}{1+\alpha}} + \beta_2^{-\frac{\alpha-1}{1+\alpha}}\right)\left|\iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} h_1(b,\Phi)h_2(\Phi)\,db\,d\Phi}{D(f)(\beta_1,\phi,b,\Phi)^{\gamma/2}}\right| \\
&\qquad + \beta_2^{-\frac{\alpha-1}{1+\alpha}}\Bigg|\iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} h_1(b,\Phi)h_2(\Phi)\,db\,d\Phi}{D(f)(\beta_1,\phi,b,\Phi)^{\gamma/2}} \\
&\qquad\qquad - \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} h_1(b,\Phi)h_2(\Phi)\,db\,d\Phi}{D(f)(\beta_2,\phi,b,\Phi)^{\gamma/2}}\Bigg|.
\end{aligned} \tag{3.22}$$

Due to (3.21) and using the previous analysis to bound $\mathcal{H}_f$ in $L^{\infty,\sigma}$ we get

$$\left(\beta_1^{-\frac{\alpha-1}{1+\alpha}} + \beta_2^{-\frac{\alpha-1}{1+\alpha}}\right)\left|\iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} h_1(b,\Phi)h_2(\Phi)\,db\,d\Phi}{D(f)(\beta_1,\phi,b,\Phi)^{\gamma/2}}\right| < C.$$

Hence, we need to estimate the second term in (3.22). Due to its expression, we will need to work with the difference of the denominator at two points. Thus

$$\begin{aligned}
&D(f)(\beta_1,\phi,b,\Phi) - D(f)(\beta_2,\phi,b,\Phi) \\
&= \left(\beta_1^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\beta}} f(\beta_1,\phi)\right\}^{1/2} - b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{b}} f(b,\Phi)\right\}^{1/2}\right)^2 \\
&\quad - \left(\beta_2^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\beta}} f(\beta_2,\phi)\right\}^{1/2} - b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{b}} f(b,\Phi)\right\}^{1/2}\right)^2 \\
&\quad + 2\beta_1^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\beta}} f(\beta_1,\phi)\right\}^{1/2}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{b}} f(b,\Phi)\right\}^{1/2} \\
&\qquad\qquad \times (1-\cos(\phi+\beta_1-\Phi-b)) \\
&\quad - 2\beta_2^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\beta}} f(\beta_2,\phi)\right\}^{1/2}\left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{b}} f(b,\Phi)\right\}^{1/2} \\
&\qquad\qquad \times (1-\cos(\phi+\beta_2-\Phi-b)).
\end{aligned} \tag{3.23}$$

Let us estimate it in several steps. First,

$$
\begin{aligned}
&\Bigg|\left(\beta_1^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_1,\phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar b}f(b,\Phi)\right\}^{1/2}\right)^2\\
&\qquad-\left(\beta_2^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_2,\phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar b}f(b,\Phi)\right\}^{1/2}\right)^2\Bigg|\\
&\qquad\le\left|\beta_1^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_1,\phi)\right\}^{1/2}-\beta_2^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_2,\Phi)\right\}^{1/2}\right|\\
&\qquad\quad\times\Bigg|\left(\beta_1^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_1,\phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar b}f(b,\Phi)\right\}^{1/2}\right)\\
&\qquad\quad+\left(\beta_2^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_2,\phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar b}f(b,\Phi)\right\}^{1/2}\right)\Bigg|,
\end{aligned}
$$

where

$$
\begin{aligned}
&\left|\beta_1^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_1,\phi)\right\}^{1/2}-\beta_2^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_2,\Phi)\right\}^{1/2}\right|\\
&\quad\le C\left|\frac{\beta_1^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_1,\phi)\right\}-\beta_2^{\frac{-2}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_2,\Phi)\right\}}{\beta_1^{\frac{-1}{1+\alpha}}+\beta_2^{\frac{-1}{1+\alpha}}}\right|\\
&\quad\le C\left|\frac{(\beta_1^{\frac{-2}{1+\alpha}}-\beta_2^{\frac{-2}{1+\alpha}})\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_1,\phi)\right\}+\beta_2^{\frac{-2}{1+\alpha}}\left\{\partial_{\bar\beta}f(\beta_1,\Phi)-\partial_{\bar\beta}f(\beta_2,\Phi)\right\}}{\beta_1^{\frac{-1}{1+\alpha}}+\beta_2^{\frac{-1}{1+\alpha}}}\right|\\
&\quad\le C\frac{\beta_1^{\frac{-2}{1+\alpha}}+\beta_2^{\frac{-2}{1+\alpha}}}{\beta_1^{\frac{-1}{1+\alpha}}+\beta_2^{\frac{-1}{1+\alpha}}}\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}\le C\{\beta_1^{\frac{-1}{1+\alpha}}+\beta_2^{\frac{-1}{1+\alpha}}\}\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\Bigg|\left(\beta_1^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_1,\phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar b}f(b,\Phi)\right\}^{1/2}\right)^2\\
&\qquad-\left(\beta_2^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_2,\phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar b}f(b,\Phi)\right\}^{1/2}\right)^2\Bigg|\\
&\qquad\le C\Bigg|\left(\beta_1^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_1,\phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar b}f(b,\Phi)\right\}^{1/2}\right)\\
&\qquad\quad+\left(\beta_2^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar\beta}f(\beta_2,\phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\bar b}f(b,\Phi)\right\}^{1/2}\right)\Bigg|\\
&\qquad\qquad\times\{\beta_1^{\frac{-1}{1+\alpha}}+\beta_2^{\frac{-1}{1+\alpha}}\}\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}\\
&\quad\le C(D(f)(\beta_1,\phi,b,\Phi)^{1/2}+D(f)(\beta_2,\phi,b,\Phi)^{1/2})\{\beta_1^{\frac{-1}{1+\alpha}}+\beta_2^{\frac{-1}{1+\alpha}}\}\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}.
\end{aligned}
$$

Now let us estimate the second part of (3.23) using similar ideas:

$$
\begin{aligned}
&\Bigg|\beta_1^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\beta}} f(\beta_1,\phi)\right\}^{1/2} \left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{b}} f(b,\Phi)\right\}^{1/2} \\
&\qquad\qquad \times (1-\cos(\phi+\beta_1-\Phi-b)) \\
&\qquad - \beta_2^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\beta}} f(\beta_2,\phi)\right\}^{1/2} \left\{\frac{\alpha-1}{1+\alpha} + \partial_{\overline{b}} f(b,\Phi)\right\}^{1/2} \\
&\qquad\qquad \times (1-\cos(\phi+\beta_2-\Phi-b))\Bigg| \\
&\leq C\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}(\beta_1^{\frac{-1}{1+\alpha}} + \beta_2^{\frac{-1}{1+\alpha}}) b^{\frac{-1}{1+\alpha}} \\
&\qquad \times \{\sin^2((\phi+\beta_1-\Phi-b)/2) + \sin^2((\phi+\beta_2-\Phi-b)/2)\} \\
&\quad + C|\sin^\delta((\beta_1-\beta_2)/2)|(\beta_1^{\frac{-1}{1+\alpha}} + \beta_2^{\frac{-1}{1+\alpha}}) b^{\frac{-1}{1+\alpha}} \\
&\qquad \times \{\sin^{2-\delta}((\phi+\beta_1-\Phi-b)/2) + \sin^{2-\delta}((\phi+\beta_2-\Phi-b)/2)\} \\
&\leq C\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}(\beta_1^{\frac{-1}{1+\alpha}} + \beta_2^{\frac{-1}{1+\alpha}}) b^{\frac{-1}{1+\alpha}} \\
&\qquad \times \{\sin^2((\phi+\beta_1-\Phi-b)/2) + \sin^2((\phi+\beta_2-\Phi-b)/2)\} \\
&\quad + C\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}(\beta_1^{\frac{-1}{1+\alpha}+\delta} + \beta_2^{\frac{-1}{1+\alpha}+\delta}) b^{\frac{-1}{1+\alpha}} \\
&\qquad \times \{\sin^{2-\delta}((\phi+\beta_1-\Phi-b)/2) + \sin^{2-\delta}((\phi+\beta_2-\Phi-b)/2)\} \\
&\leq C\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}(\beta_1^{\frac{-1}{1+\alpha}} + \beta_2^{\frac{-1}{1+\alpha}})(D(f)(\beta_1,\phi,b,\Phi)^{1/2} + D(f)(\beta_2,\phi,b,\Phi)^{1/2}) \\
&\quad + C\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}(\beta_1^{\frac{-1}{1+\alpha}+\delta} + \beta_2^{\frac{-1}{1+\alpha}+\delta}) \\
&\qquad \times \{\sin^{1-\delta}((\phi+\beta_1-\Phi-b)/2) + \sin^{1-\delta}((\phi+\beta_2-\Phi-b)/2)\} \\
&\qquad \times (D(f)(\beta_1,\phi,b,\Phi)^{1/2} + D(f)(\beta_2,\phi,b,\Phi)^{1/2}).
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
&|D(f)(\beta_2,\phi,b,\Phi) - D(f)(\beta_1,\phi,b,\Phi)| \\
&\qquad \leq C\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}(\beta_1^{\frac{-1}{1+\alpha}} + \beta_2^{\frac{-1}{1+\alpha}} + \beta_1^{\frac{-1}{1+\alpha}+\delta} + \beta_2^{\frac{-1}{1+\alpha}+\delta}) \\
&\qquad\quad \times (D(f)(\beta_1,\phi,b,\Phi)^{1/2} + D(f)(\beta_2,\phi,b,\Phi)^{1/2}).
\end{aligned} \tag{3.24}
$$

Hence

$$
\begin{aligned}
&\left|\frac{1}{D(f)(\beta_1,\phi,b,\Phi)^{\gamma/2}} - \frac{1}{D(f)(\beta_2,\phi,b,\Phi)^{\gamma/2}}\right| \\
&\qquad \leq \frac{D(f)(\beta_2,\phi,b,\Phi)^{\gamma/2} - D(f)(\beta_1,\phi,b,\Phi)^{\gamma/2}}{D(f)(\beta_1,\phi,b,\Phi)^{\gamma/2} D(f)(\beta_2,\phi,b,\Phi)^{\gamma/2}}
\end{aligned}
$$

$$\lesssim \frac{|D(f)(\beta_2,\phi,b,\Phi)-D(f)(\beta_1,\phi,b,\Phi)|}{D(f)(\beta_1,\phi,b,\Phi)^{\gamma/2}D(f)(\beta_2,\phi,b,\Phi)^{\gamma/2}} \times \frac{1}{(D(f)(\beta_2,\phi,b,\Phi)^{\frac{2-\gamma}{2}}+D(f)(\beta_1,\phi,b,\Phi)^{\frac{2-\gamma}{2}})},$$

and using (3.24) we find

$$\left|\frac{1}{D(f)(\beta_1,\phi,b,\Phi)^{\gamma/2}}-\frac{1}{D(f)(\beta_2,\phi,b,\Phi)^{\gamma/2}}\right| \leq C\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}\times \frac{\beta_1^{\frac{-1}{1+\alpha}}+\beta_2^{\frac{-1}{1+\alpha}}+\beta_1^{\frac{-1}{1+\alpha}+\delta}+\beta_2^{\frac{-1}{1+\alpha}+\delta}}{D(f)(\beta_1,\phi,b,\Phi)^{1/2}D(f)(\beta_2,\phi,b,\Phi)^{\gamma/2}+D(f)(\beta_2,\phi,b,\Phi)^{1/2}D(f)(\beta_1,\phi,b,\Phi)^{\gamma/2}}. \tag{3.25}$$

Let us come back to (3.22) to obtain

$$\begin{aligned}&|\mathcal{H}_f(h_1,h_2)(\beta_1,\phi)-\mathcal{H}_f(h_1,h_2)(\beta_2,\phi)|\\ &\leq C\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}+C\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}\beta_2^{-\frac{\alpha-1}{1+\alpha}}\iint b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}|h_1(b,\Phi)|\,|h_2(\Phi)|\times\\ &\frac{\beta_1^{\frac{-1}{1+\alpha}}+\beta_2^{\frac{-1}{1+\alpha}}+\beta_1^{\frac{-1}{1+\alpha}+\delta}+\beta_2^{\frac{-1}{1+\alpha}+\delta}}{D(f)(\beta_1,\phi,b,\Phi)^{1/2}D(f)(\beta_2,\phi,b,\Phi)^{\gamma/2}+D(f)(\beta_2,\phi,b,\Phi)^{1/2}D(f)(\beta_1,\phi,b,\Phi)^{\gamma/2}}\\ &\times\, db\, d\Phi.\end{aligned}$$

By virtue of Proposition 3.7 we can bound the denominator as

$$\begin{aligned}&|\mathcal{H}_f(h_1,h_2)(\beta_1,\phi)-\mathcal{H}_f(h_1,h_2)(\beta_2,\phi)|\\ &\leq C\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}+C\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}\iint b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}|h_1(b,\Phi)|\,|h_2(\Phi)|\\ &\times\Bigg(\frac{\beta_2^{-\frac{\alpha-1}{1+\alpha}}\beta_1^{\frac{-1}{1+\alpha}}}{(\beta_1^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})(\beta_2^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}})\{\frac{|\beta_1-b|^2}{|\beta_1+b|^2}+\sin^2((\beta_1+\phi-b-\Phi)/2)\}^{1/2}}\\ &\qquad\times\frac{1}{\{\frac{|\beta_2-b|^2}{|\beta_2+b|^2}+\sin^2((\beta_2+\phi-b-\Phi)/2)\}^{\gamma/2}}\\ &+\frac{\beta_2^{-\frac{\alpha-1}{1+\alpha}}\beta_2^{\frac{-1}{1+\alpha}}}{(\beta_2^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})(\beta_1^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}})\{\frac{|\beta_2-b|^2}{|\beta_2+b|^2}+\sin^2((\beta_2+\phi-b-\Phi)/2)\}^{1/2}}\\ &\qquad\times\frac{1}{\{\frac{|\beta_1-b|^2}{|\beta_1+b|^2}+\sin^2((\beta_1+\phi-b-\Phi)/2)\}^{\gamma/2}}\end{aligned}$$

$$+\frac{\beta_2^{-\frac{\alpha-1}{1+\alpha}}\beta_1^{\frac{-1}{1+\alpha}}\beta_1^{\delta}}{(\beta_1^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})(\beta_2^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}})\left\{\frac{|\beta_1-b|^2}{|\beta_1+b|^2}+\sin^2((\beta_1+\phi-b-\Phi)/2)\right\}^{1/2}}$$
$$\times\frac{1}{\left\{\frac{|\beta_2-b|^2}{|\beta_2+b|^2}+\sin^2((\beta_2+\phi-b-\Phi)/2)\right\}^{\gamma/2}}$$
$$+\frac{\beta_2^{-\frac{\alpha-1}{1+\alpha}}\beta_2^{\frac{-1}{1+\alpha}}\beta_2^{\delta}}{(\beta_2^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})(\beta_1^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}})\left\{\frac{|\beta_2-b|^2}{|\beta_2+b|^2}+\sin^2((\beta_2+\phi-b-\Phi)/2)\right\}^{1/2}}$$
$$\left.\times\frac{1}{\left\{\frac{|\beta_1-b|^2}{|\beta_1+b|^2}+\sin^2((\beta_1+\phi-b-\Phi)/2)\right\}^{\gamma/2}}\right)db\,d\Phi.$$

Let us show the bound for the third integral, and the others will follow similarly. Since $h_1\in\mathcal{X}^{\sigma,\delta}$, we have

$$\iint\frac{\beta_2^{-\frac{\alpha-1}{1+\alpha}}\beta_1^{\frac{-1}{1+\alpha}+\delta}b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}|h_1(b,\Phi)|\,|h_2(\Phi)|\,db\,d\Phi}{(\beta_1^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})(\beta_2^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}})\left\{\frac{|\beta_1-b|^2}{|\beta_1+b|^2}+\sin^2((\beta_1+\phi-b-\Phi)/2)\right\}^{1/2}}$$
$$\times\frac{db\,d\Phi}{\left\{\frac{|\beta_2-b|^2}{|\beta_2+b|^2}+\sin^2((\beta_2+\phi-b-\Phi)/2)\right\}^{\gamma/2}}$$
$$\leq C\|h_1\|_{L^{\infty,\sigma}}\beta_1^{\delta}\iint\frac{\beta_2^{-\frac{\alpha-1}{1+\alpha}}b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b^{\sigma}(1+b)^{-2\sigma}|h_2(\Phi)|}{(\beta_2^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}})\left\{\frac{|\beta_1-b|^2}{|\beta_1+b|^2}+\sin^2((\beta_1+\phi-b-\Phi)/2)\right\}^{1/2}}$$
$$\times\frac{db\,d\Phi}{\left\{\frac{|\beta_2-b|^2}{|\beta_2+b|^2}+\sin^2((\beta_2+\phi-b-\Phi)/2)\right\}^{\gamma/2}}$$
$$\leq C\|h_1\|_{L^{\infty,\sigma}}\beta_1^{\delta}\iint\frac{\beta_2^{\frac{\gamma}{1+\alpha}}\beta_2^{-\frac{\alpha-1}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b^{\sigma}(1+b)^{-2\sigma}|h_2(\Phi)|\,db\,d\Phi}{(\beta_2^{\frac{\gamma}{1+\alpha}}+b^{\frac{\gamma}{1+\alpha}})\frac{|\beta_1-b|^{1-\nu}}{|\beta_1+b|^{1-\nu}}|\sin((\beta_1+\phi-b-\Phi)/2)|^{\nu}\frac{|\beta_2-b|^{\gamma}}{|\beta_2+b|^{\gamma}}}$$

for $\nu\in(0,1)$ with

$$1-\nu+\gamma\in(0,1). \tag{3.26}$$

Hence

$$\iint\frac{\beta_2^{-\frac{\alpha-1}{1+\alpha}}\beta_1^{\frac{-1}{1+\alpha}+\delta}b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}|h_1(b,\Phi)|\,|h_2(\Phi)|}{(\beta_1^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})(\beta_2^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}})\left\{\frac{|\beta_1-b|^2}{|\beta_1+b|^2}+\sin^2((\beta_1+\phi-b-\Phi)/2)\right\}^{1/2}}$$
$$\times\frac{db\,d\Phi}{\left\{\frac{|\beta_2-b|^2}{|\beta_2+b|^2}+\sin^2((\beta_2+\phi-b-\Phi)/2)\right\}^{\gamma/2}}$$

$$
\begin{aligned}
&\le C\|h_1\|_{L^{\infty,\sigma}}\beta_1^{\delta}\int_0^\infty \frac{\beta_2^{\frac{\gamma}{1+\alpha}}\beta_2^{-\frac{\alpha-1}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b^{\sigma}(1+b)^{-2\sigma}}{(\beta_2^{\frac{\gamma}{1+\alpha}}+b^{\frac{\gamma}{1+\alpha}})\frac{|\beta_1-b|^{1-\nu}}{|\beta_1+b|^{1-\nu}}\frac{|\beta_2-b|^{\gamma}}{|\beta_2+b|^{\gamma}}}\\
&\quad\times\int_0^{2\pi}|\sin^{-\nu}((\beta_2+\phi-b-\Phi)/2)|\,|h_2(\Phi)|\,d\Phi\,db.
\end{aligned}
$$

Note that by interpolation we get

$$
\begin{aligned}
&\int_0^{2\pi}|\sin^{-\nu}((\beta_2+\phi-b-\Phi)/2)|\,|h_2(\Phi)|\,d\Phi\\
&\qquad\le C\left(\int_0^{2\pi}|h_2(\Phi)|^p\,d\Phi\right)^{1/p}\left(\int_0^{2\pi}|\sin^{-\nu}((\Phi)/2)|^q\,d\Phi\right)^{1/q}\\
&\qquad\le C\|h_2\|_{X_2^p}
\end{aligned}
\tag{3.27}
$$

provided that $\nu q<1$ and $1/p+1/q=1$. By using (3.26), we can choose any p such that

$$
\frac{1}{1-\gamma}<p. \tag{3.28}
$$

Finally, using the change of variables $b=\beta_2 z$ we get

$$
\begin{aligned}
&\iint \frac{\beta_2^{-\frac{\alpha-1}{1+\alpha}}\beta_1^{\frac{-1}{1+\alpha}+\delta}b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}|h_1(b,\Phi)|\,|h_2(\Phi)|}{(\beta_1^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})(\beta_2^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}})\{\frac{|\beta_1-b|^2}{|\beta_1+b|^2}+\sin^2((\beta_1+\phi-b-\Phi)/2)\}^{1/2}}\\
&\qquad\qquad\times\frac{db\,d\Phi}{\{\frac{|\beta_2-b|^2}{|\beta_2+b|^2}+\sin^2((\beta_2+\phi-b-\Phi)/2)\}^{\gamma/2}}\\
&\le C\|h_1\|_{L^{\infty,\sigma}}\|h_2\|_{X_2^p}\beta_1^{\delta}\int_0^\infty\frac{\beta_2^{\frac{\gamma}{1+\alpha}}\beta_2^{-\frac{\alpha-1}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b^{\sigma}(1+b)^{-2\sigma}}{(\beta_2^{\frac{\gamma}{1+\alpha}}+b^{\frac{\gamma}{1+\alpha}})\frac{|\beta_1-b|^{1-\nu}}{|\beta_1+b|^{1-\nu}}\frac{|\beta_2-b|^{\gamma}}{|\beta_2+b|^{\gamma}}}\,db\\
&\le C\|h_1\|_{L^{\infty,\sigma}}\|h_2\|_{X_2^p}\beta_2^{\delta}\int_0^\infty\frac{z^{\frac{-2}{1+\alpha}}(z\beta_2)^{\sigma}(1+z\beta_2)^{-2\sigma}}{(1+z^{\frac{\gamma}{1+\alpha}})\frac{|\beta_1-z\beta_2|^{1-\nu}}{|\beta_1+z\beta_2|^{1-\nu}}\frac{|1-z|^{\gamma}}{|1+z|^{\gamma}}}\,dz\\
&\le C\|h_1\|_{L^{\infty,\sigma}}\|h_2\|_{X_2^p},
\end{aligned}
\tag{3.29}
$$

where we have used $\beta_1<\beta_2$, and which is bounded provided that σ satisfies (3.18) and (3.19) and

$$
\delta<\sigma,
$$

in order to get the boundedness as β_2 goes to ∞. Hence, putting everything together we find that

$$
|\mathcal{H}_f(\beta_1,\phi,b,\Phi)-\mathcal{H}_f(\beta_2,\phi,b,\Phi)|\le C\|h_1\|_{L^{\infty,\sigma}}\|h_2\|_{X_2^p}\frac{|\beta_1-\beta_2|^{\delta}}{|\beta_1+\beta_2|^{\delta}},
$$

implying that $\mathcal{H}_f\in C^{\delta}_{\beta}$.

• $\mathcal{H}_f \in \mathrm{Lip}_\phi$. We shall prove that

$$|\mathcal{H}_f(\beta,\phi_1,b,\Phi)-\mathcal{H}_f(\beta,\phi_2,b,\Phi)|\leq C|\sin((\phi_1-\phi_2)/2)|. \tag{3.30}$$

For that we need to estimate the difference of the denominator at two points:

$$\begin{aligned}
&D(f)(\beta,\phi_1,b,\Phi)-D(f)(\beta,\phi_2,b,\Phi)\\
&=\left(\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi_1)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\Phi)\right\}^{1/2}\right)^2\\
&\quad-\left(\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi_2)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\Phi)\right\}^{1/2}\right)^2\\
&\quad+2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi_1)\right\}^{1/2}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\Phi)\right\}^{1/2}\\
&\qquad\qquad\times(1-\cos(\phi_1+\beta-\Phi-b))\\
&\quad-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi_2)\right\}^{1/2}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\Phi)\right\}^{1/2}\\
&\qquad\qquad\times(1-\cos(\phi_2+\beta-\Phi-b)).
\end{aligned}$$

Hence

$$\begin{aligned}
&|D(f)(\beta,\phi_1,b,\Phi)-D(f)(\beta,\phi_2,b,\Phi)|\\
&\leq C|\sin((\phi_1-\phi_2)/2)|(\beta^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})(D(f)(\beta,\phi_1,b,\Phi)^{1/2}+D(f)(\beta,\phi_2,b,\Phi)^{1/2}).
\end{aligned}\tag{3.31}$$

Therefore

$$\begin{aligned}
&\left|\frac{1}{D(f)(\beta,\phi_1,b,\Phi)^{\gamma/2}}-\frac{1}{D(f)(\beta,\phi_2,b,\Phi)^{\gamma/2}}\right|\\
&\qquad\leq\frac{D(f)(\beta,\phi_1,b,\Phi)^{\gamma/2}-D(f)(\beta,\phi_2,b,\Phi)^{\gamma/2}}{D(f)(\beta,\phi_1,b,\Phi)^{\gamma/2}D(f)(\beta,\phi_2,b,\Phi)^{\gamma/2}}\\
&\qquad\leq\frac{|D(f)(\beta,\phi_1,b,\Phi)-D(f)(\beta,\phi_2,b,\Phi)|}{D(f)(\beta,\phi_1,b,\Phi)^{\gamma/2}D(f)(\beta,\phi_2,b,\Phi)^{\gamma/2}}\\
&\qquad\quad\times\frac{1}{D(f)(\beta,\phi_2,b,\Phi)^{\frac{2-\gamma}{2}}+D(f)(\beta,\phi_1,b,\Phi)^{\frac{2-\gamma}{2}}},
\end{aligned}$$

and using (3.31) we find

$$\begin{aligned}
&\left|\frac{1}{D(f)(\beta,\phi_1,b,\Phi)^{\gamma/2}}-\frac{1}{D(f)(\beta,\phi_2,b,\Phi)^{\gamma/2}}\right|\leq\\
&\frac{C|\sin((\phi_1-\phi_2)/2)|(\beta^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})}{D(f)(\beta,\phi_1,b,\Phi)^{1/2}D(f)(\beta,\phi_2,b,\Phi)^{\gamma/2}+D(f)(\beta,\phi_2,b,\Phi)^{1/2}D(f)(\beta,\phi_1,b,\Phi)^{\gamma/2}}.
\end{aligned}$$

In that way, we obtain

$$
\begin{aligned}
&|\mathcal{H}_f(\beta,\phi_1,b,\Phi)-\mathcal{H}_f(\beta,\phi_2,b,\Phi)|\\
&\leq C|\sin((\phi_1-\phi_2)/2)|\beta^{-\frac{\alpha-1}{1+\alpha}}\\
&\times\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}(\beta^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})|h_1(b,\Phi)|\,|h_2(\Phi)|\,db\,d\Phi}{D(f)(\beta,\phi_1,b,\Phi)^{1/2}D(f)(\beta,\phi_2,b,\Phi)^{\gamma/2}+D(f)(\beta,\phi_2,b,\Phi)^{1/2}D(f)(\beta,\phi_1,b,\Phi)^{\gamma/2}}\\
&\leq C|\sin((\phi_1-\phi_2)/2)|\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}(\beta^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})|h_1(b,\Phi)|\,|h_2(\Phi)|\,db\,d\Phi}{D(f)(\beta,\phi_1,b,\Phi)^{1/2}D(f)(\beta,\phi_2,b,\Phi)^{\gamma/2}}\\
&\leq |\sin((\phi_1-\phi_2)/2)|\beta^{-\frac{\alpha-1}{1+\alpha}}\\
&\times\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}(\beta^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})|h_1(b,\Phi)|\,|h_2(\Phi)|\,db\,d\Phi}{\{\beta^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}}\}\{\beta^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}}\}\frac{|\beta-b|^\gamma}{|\beta+b|^\gamma}\left\{\frac{|\beta-b|^2}{|\beta+b|^2}+\sin^2((\beta-b+\phi_1-\Phi)/2)\right\}^{1/2}}\\
&\leq C|\sin((\phi_1-\phi_2)/2)|\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}|h_1(b,\Phi)|\,|h_2(\Phi)|\,db\,d\Phi}{\{\beta^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}}\}\frac{|\beta-b|^{\gamma+1-\nu}}{|\beta+b|^{\gamma+1-\nu}}|\sin((\beta-b+\phi_1-\Phi)/2)|^\nu}.
\end{aligned}
$$

Using now (3.27) we get

$$
\begin{aligned}
&|\mathcal{H}_f(\beta,\phi_1,b,\Phi)-\mathcal{H}_f(\beta,\phi_2,b,\Phi)|\\
&\qquad\leq C\|h_1\|_{X_1^{\sigma,\delta}}\|h_2\|_{X_2^p}|\sin((\phi_1-\phi_2)/2)|\beta^{-\frac{\alpha-1}{1+\alpha}}\int_0^\infty\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b^\sigma(1+b)^{-2\sigma}\,db}{\{\beta^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}}\}\frac{|\beta-b|^{\gamma+1-\nu}}{|\beta+b|^{\gamma+1-\nu}}}
\end{aligned}
$$

for p with $1/p+1/q=1$ and $q\nu<1$ as in (3.27). The change of variable $b=z\beta$ leads to

$$
\begin{aligned}
&|\mathcal{H}_f(\beta,\phi_1,b,\Phi)-\mathcal{H}_f(\beta,\phi_2,b,\Phi)|\\
&\qquad\qquad\leq C\|h_1\|_{\mathcal{X}^{\sigma,\delta}}\|h_2\|_{X_2^p}|\sin((\phi_1-\phi_2)/2)|\beta^\sigma\int_0^\infty\frac{z^{\frac{-2}{1+\alpha}}z^\sigma(1+z\beta)^{-2\sigma}\,dz}{\{1+z^{\frac{\gamma}{1+\alpha}}\}\frac{|1-z|^{\gamma+1-\nu}}{|1+z|^{\gamma+1-\nu}}}\\
&\qquad\qquad\leq C\|h_1\|_{\mathcal{X}^{\sigma,\delta}}\|h_2\|_{X_2^p}
\end{aligned}
$$

provided that $\gamma+1-\nu<1$, p satisfies (3.28) and σ satisfies (3.18) and (3.19).

- $\beta\partial_\varphi\mathcal{H}_f\in L^{\infty,\sigma}$. By using the expression of $\mathcal{H}_f$ we find

$$
\begin{aligned}
\partial_\varphi\mathcal{H}_f(h_1,h_2)(\beta,\phi)=&\,\frac{\alpha-1}{1+\alpha}\,\frac{1}{\beta}\mathcal{H}_f(h_1,h_2)(\beta,\phi)\\
&-\frac{\gamma}{2}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{\frac{-\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}h_1(b,\Phi)h_2(\Phi)\partial_\varphi D(f)(\beta,\phi,b,\Phi)}{D(f)(\beta,\phi,b,\Phi)^{\frac{2+\gamma}{2}}}\,db\,d\Phi.
\end{aligned}
$$

Note that

$$\begin{aligned}\beta\partial_\varphi \mathcal{H}_f(h_1,h_2)(\beta,\phi) &= \frac{\alpha-1}{1+\alpha}\mathcal{H}_f(h_1,h_2)(\beta,\phi)\\ &\quad -\frac{\gamma}{2}\beta\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{\frac{-\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}h_1(b,\Phi)h_2(\Phi)\partial_\varphi D(f)(\beta,\phi,b,\Phi)}{D(f)(\beta,\phi,b,\Phi)^{\frac{2+\gamma}{2}}}\,db\,d\Phi\\ &=: \mathcal{H}_1+\mathcal{H}_2, \qquad (3.32)\end{aligned}$$

where the first term was already studied in the previous steps. For the second term, let us compute the derivative of the denominator:

$$\begin{aligned}&\partial_\varphi D(f)(\beta,\phi,b,\Phi)\\ &= 2\left(\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\Phi)\right\}^{1/2}\right)\\ &\quad\times\beta^{\frac{-1}{1+\alpha}}\frac{\partial_\varphi\partial_{\overline{\beta}}f(\beta,\phi)}{2\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi)\}^{1/2}}\\ &\quad+\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi)\right\}^{\frac{-1}{2}}\partial_\varphi\partial_{\overline{\beta}}f(\beta,\phi)\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\Phi)\right\}^{1/2}\\ &\qquad\times(1-\cos(\phi+\beta-\Phi-b))\\ &\quad-\frac{2}{1+\alpha}\frac{1}{\beta}\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi)\right\}^{1/2}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\Phi)\right\}^{1/2}\\ &\qquad\times(1-\cos(\phi+\beta-\Phi-b)), \qquad (3.33)\end{aligned}$$

implying that

$$|\partial_\varphi D(f)(\beta,\phi,b,\Phi)|\le CD(f)(\beta,\phi,b,\Phi)^{1/2}\beta^{\frac{-1}{1+\alpha}}\frac{1}{\beta}. \qquad (3.34)$$

In this way, we find that

$$\begin{aligned}&|\beta\partial_\varphi\mathcal{H}(h_1,h_2)(\beta,\phi)|\le C|\mathcal{H}_f(h_1,h_2)(\beta,\phi)|\\ &\quad+C\beta^{\frac{-1}{1+\alpha}}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}|h_1(b,\phi+\beta-\eta-b)|\,|h_2(\phi+\beta-\eta-b)|}{D(f)(\beta,\phi,b,\phi+\beta-\eta-b)^{\frac{1+\gamma}{2}}}\,db\,d\eta\\ &\le C|\mathcal{H}_f(h_1,h_2)(\beta,\phi)|\\ &\quad+C\|h_1\|_{X_1^{\sigma,\delta}}\beta^\sigma\beta^{\frac{-1}{1+\alpha}}\iint\frac{z^{-\frac{\gamma}{1+\alpha}}z^{\frac{-2}{1+\alpha}}b^\sigma(1+\beta z)^{-2\sigma}|h_2(\phi+\beta-\eta-b)|\,db\,d\eta}{(\beta^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}})(\beta^{\frac{-1}{1+\alpha}}+b^{\frac{-1}{1+\alpha}})\{\frac{|\beta-b|^2}{|\beta+b|^2}+\sin^2(\eta/2)\}^{\frac{1+\gamma}{2}}}.\end{aligned}$$

Using (3.27) we get

$$\begin{aligned}&|\beta\partial_\varphi\mathcal{H}(h_1,h_2)(\beta,\phi)|\le C|\mathcal{H}_f(h_1,h_2)(\beta,\phi)|\\ &\qquad+C\|h_1\|_{\mathcal{X}^{\sigma,\delta}}\|h_2\|_{X_2^p}\beta^\sigma\int_0^\infty\frac{z^{-\frac{\gamma}{1+\alpha}}z^{\frac{-2}{1+\alpha}}b^\sigma(1+\beta z)^{-2\sigma}}{(\beta^{\frac{-\gamma}{1+\alpha}}+b^{\frac{-\gamma}{1+\alpha}})\frac{|\beta-b|^{1+\gamma-\nu}}{|\beta+b|^{1+\gamma-\nu}}}\,db.\end{aligned}$$

Then

$$\|\beta\partial_\varphi \mathcal{H}(h_1,h_2)\|_\sigma \le C\|h_1\|_{\mathcal{X}^{\sigma,\delta}}\|h_2\|_{X_2^p} \tag{3.35}$$

provided that (3.18) and (3.19) hold, as well as (3.28) and $1+\gamma-\nu<1$.

• $\beta\partial_\varphi\mathcal{H}_f(h_1,h_2)\in C^\delta_\beta$. Since $\mathcal{H}_f(h_1,h_2)\in C^\delta_\beta$, we have $\mathcal{H}_1\in C^\delta_\beta$. Hence, it remains to check the term $\mathcal{H}_2$ in (3.32). Take $\beta_1<\beta_2$; then they satisfy (3.21). Let us write the difference of $\mathcal{H}_2$ for these two values as follows:

$$\begin{aligned}
&\mathcal{H}_2(\beta_1)-\mathcal{H}_2(\beta_2)\\
&= C\beta_1^{\frac{2}{1+\alpha}}\iint \frac{b^{\frac{-\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}h_1(b,\Phi)h_2(\Phi)\partial_\varphi D(f)(\beta_1,\phi,b,\Phi)}{D(f)(\beta_1,\phi,b,\Phi)^{1/2}}\\
&\qquad\times\left\{\frac{1}{D(f)(\beta_1,\phi,b,\Phi)^{\frac{1+\gamma}{2}}}-\frac{1}{D(f)(\beta_2,\phi,b,\Phi)^{\frac{1+\gamma}{2}}}\right\}db\,d\Phi\\
&\quad + C\beta_1^{\frac{2}{1+\alpha}}\iint \frac{b^{\frac{-\gamma-2}{1+\alpha}}h_1(b,\Phi)h_2(\Phi)(\partial_\varphi D(f)(\beta_1,\phi,b,\Phi)-\partial_\varphi D(f)(\beta_2,\phi,b,\Phi))}{D(f)(\beta_1,\phi,b,\Phi)^{1/2}D(f)(\beta_2,\phi,b,\Phi)^{\frac{1+\gamma}{2}}}\\
&\qquad\times db\,d\Phi\\
&\quad + C\beta_1^{\frac{2}{1+\alpha}}\iint \frac{b^{\frac{-\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}h_1(b,\Phi)h_2(\Phi)\partial_\varphi D(f)(\beta_2,\phi,b,\Phi)}{D(f)(\beta_2,\phi,b,\Phi)^{\frac{1+\gamma}{2}}}\\
&\qquad\times\left\{\frac{1}{D(f)(\beta_1,\phi,b,\Phi)^{1/2}}-\frac{1}{D(f)(\beta_2,\phi,b,\Phi)^{1/2}}\right\}db\,d\Phi\\
&\quad + C(\beta_1^{\frac{2}{1+\alpha}}-\beta_2^{\frac{2}{1+\alpha}})\iint \frac{b^{\frac{-\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}h_1(b,\Phi)h_2(\Phi)\partial_\varphi D(f)(\beta_2,\phi,b,\Phi)}{D(f)(\beta_2,\phi,b,\Phi)^{\frac{2+\gamma}{2}}}\,db\,d\Phi\\
&=: \mathcal{H}_{2,1}+\mathcal{H}_{2,2}+\mathcal{H}_{2,3}+\mathcal{H}_{2,4}.
\end{aligned}$$

By using (3.25) and (3.34) we can bound $\mathcal{H}_{2,1}$ and $\mathcal{H}_{2,3}$. Let us estimate $\mathcal{H}_{2,1}$. From (3.24) we find for $\delta=1$ that

$$\begin{aligned}
&|D(f)(\beta_2,\phi,b,\Phi)-D(f)(\beta_1,\phi,b,\Phi)|\\
&\qquad\le C\frac{|\beta_1-\beta_2|}{|\beta_1+\beta_2|}(\beta_1^{\frac{-1}{1+\alpha}}+\beta_2^{\frac{-1}{1+\alpha}}+\beta_1^{\frac{-1}{1+\alpha}+1}+\beta_2^{\frac{-1}{1+\alpha}+1})\\
&\qquad\quad\times(D(f)(\beta_1,\phi,b,\Phi)^{1/2}+D(f)(\beta_2,\phi,b,\Phi)^{1/2}).
\end{aligned}$$

By interpolating with

$$\begin{aligned}
&|D(f)(\beta_2,\phi,b,\Phi)-D(f)(\beta_1,\phi,b,\Phi)|\\
&\qquad\le |D(f)(\beta_2,\phi,b,\Phi)+D(f)(\beta_1,\phi,b,\Phi)|,
\end{aligned}$$

we find

$$
\begin{aligned}
&|D(f)(\beta_2,\phi,b,\Phi)-D(f)(\beta_1,\phi,b,\Phi)| \\
&\qquad\qquad \le C\frac{|\beta_1-\beta_2|^{\delta}}{|\beta_1+\beta_2|^{\delta}}\left(\beta_1^{\frac{-\delta}{1+\alpha}}+\beta_2^{\frac{-\delta}{1+\alpha}}+\beta_1^{\frac{-\delta}{1+\alpha}+\delta}+\beta_2^{\frac{-\delta}{1+\alpha}+\delta}\right)\\
&\qquad\qquad\quad \times\left(D(f)(\beta_1,\phi,b,\Phi)^{\frac{2-\delta}{2}}+D(f)(\beta_2,\phi,b,\Phi)^{\frac{2-\delta}{2}}\right).
\end{aligned}
\tag{3.36}
$$

Hence using the ideas in (3.25) together with (3.36) we find

$$
\begin{aligned}
&\left|\frac{1}{D(f)(\beta_1,\phi,b,\Phi)^{\frac{1+\gamma}{2}}}-\frac{1}{D(f)(\beta_2,\phi,b,\Phi)^{\frac{1+\gamma}{2}}}\right|\le C\frac{|\beta_1-\beta_2|^{\delta}}{|\beta_1+\beta_2|^{\delta}}\times\\
&\frac{\beta_1^{\frac{-\delta}{1+\alpha}}+\beta_2^{\frac{-\delta}{1+\alpha}}+\beta_1^{\frac{-\delta}{1+\alpha}+\delta}+\beta_2^{\frac{-\delta}{1+\alpha}+\delta}}{D(f)(\beta_1,\phi,b,\Phi)^{\delta/2}D(f)(\beta_2,\phi,b,\Phi)^{\frac{1+\gamma}{2}}+D(f)(\beta_2,\phi,b,\Phi)^{\delta/2}D(f)(\beta_1,\phi,b,\Phi)^{\frac{1+\gamma}{2}}}.
\end{aligned}
\tag{3.37}
$$

Hence, (3.37) together with (3.34) amounts to

$$
\begin{aligned}
&|\mathcal{H}_{2,1}|\\
&\le \frac{C|\beta_1-\beta_2|^{\delta}}{|\beta_1+\beta_2|^{\delta}}\beta_1^{\frac{2}{1+\alpha}-\frac{1}{1+\alpha}-1}\iint\frac{b^{\frac{-\gamma-2}{1+\alpha}}|h_1(b,\Phi)||h_2(\Phi)|(\beta_1^{\frac{-\delta}{1+\alpha}}+\beta_1^{\frac{-\delta}{1+\alpha}+\delta})\,db\,d\Phi}{D(f)(\beta_2,\phi,b,\Phi)^{\delta/2}D(f)(\beta_1,\phi,b,\Phi)^{\frac{1+\gamma}{2}}}\\
&\le C\frac{|\beta_1-\beta_2|^{\delta}}{|\beta_1+\beta_2|^{\delta}}\beta_1^{\frac{2}{1+\alpha}}\beta_1^{\frac{-1}{1+\alpha}}\beta_1^{-1}\\
&\quad\times\iint\frac{b^{\frac{-\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}|h_1(b,\Phi)||h_2(\Phi)|(\beta_1^{\frac{-\delta}{1+\alpha}}+\beta_1^{\frac{-\delta}{1+\alpha}+\delta})}{|\beta_2^{\frac{-\delta}{1+\alpha}}+b^{\frac{-\delta}{1+\alpha}}|\{\frac{|\beta_2-b|^2}{|\beta_2+b|^2}+\sin^2((\beta_2-b+\phi-\Phi)/2)\}^{\delta/2}|\beta_1^{\frac{-1-\gamma}{1+\alpha}}+b^{\frac{-1-\gamma}{1+\alpha}}|}\\
&\qquad\qquad\qquad\qquad\times\frac{db\,d\Phi}{\{\frac{|\beta_1-b|^2}{|\beta_1+b|^2}+\sin^2((\beta_1-b+\phi-\Phi)/2)\}^{\frac{1+\gamma}{2}}}.
\end{aligned}
$$

Using the ideas for (3.29) we finally get

$$
|\mathcal{H}_{2,1}|\le C\frac{|\beta_1-\beta_2|^{\delta}}{|\beta_1+\beta_2|^{\delta}}\|h_1\|_{L^{\infty,\sigma}}\|h_2\|_{L^p}
$$

for some constant C provided that $\delta<1-\gamma$, $\delta<\sigma$ and $p>\frac{1}{1-\gamma}$.

Let us turn to $\mathcal{H}_{2,3}$ and show the main idea. In order to work with that term, we should estimate the difference of $\partial_\varphi D(f)$ at two points. Note that

$$
\begin{aligned}
&\partial_\varphi D(f)(\beta,\phi,b,\Phi)\\
&=2\left(\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\Phi)\right\}^{1/2}\right)\\
&\qquad\qquad\qquad\qquad\times\partial_\varphi\left[\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi)\right\}^{1/2}\right]\\
&\quad+2\partial_\varphi\left[\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi)\right\}^{1/2}\right]b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\Phi)\right\}^{1/2}\\
&\qquad\qquad\qquad\qquad\times(1-\cos(\phi+\beta-\Phi-b)).
\end{aligned}
$$

By working with that term as for (3.24) we obtain

$$
\begin{aligned}
&|\partial_\varphi D(f)(\beta_1,\phi,b,\Phi)-\partial_\varphi D(f)(\beta_2,\phi,b,\Phi)|\\
&\quad\le C\frac{|\beta_1-\beta_2|}{|\beta_1+\beta_2|}(\beta_1^{-1}\beta_1^{\frac{-2}{1+\alpha}}+\beta_2^{-1}\beta_2^{\frac{-2}{1+\alpha}}+\beta_1^{-1}\beta_1^{\frac{-2}{1+\alpha}+1}+\beta_2^{-1}\beta_2^{\frac{-2}{1+\alpha}+1}),
\end{aligned}
\tag{3.38}
$$

which, interpolating with (3.34), is equivalent to

$$
\begin{aligned}
&|\partial_\varphi D(f)(\beta_1,\phi,b,\Phi)-\partial_\varphi D(f)(\beta_2,\phi,b,\Phi)|\\
&\le C\frac{|\beta_1-\beta_2|^\delta}{|\beta_1+\beta_2|^\delta}(\beta_1^{-\delta}\beta_1^{\frac{-2\delta}{1+\alpha}}+\beta_2^{-\delta}\beta_2^{\frac{-2\delta}{1+\alpha}}+\beta_1^{-\delta}\beta_1^{\frac{-\delta}{1+\alpha}+\delta}+\beta_2^{-\delta}\beta_2^{\frac{-\delta}{1+\alpha}+\delta})\\
&\quad\times(\beta_1^{-\frac{1-\delta}{1+\alpha}}\beta_1^{-(1-\delta)}D(f)(\beta_1,\phi,b,\Phi)^{\frac{1-\delta}{2}}+\beta_2^{-\frac{1-\delta}{1+\alpha}}\beta_2^{-(1-\delta)}D(f)(\beta_2,\phi,b,\Phi)^{\frac{1-\delta}{2}}).
\end{aligned}
\tag{3.39}
$$

Then, the key point to bound $\mathcal{H}_{2,2}$ is the use of (3.39) instead of just (3.38).

Finally, the estimate for the last term $\mathcal{H}_{2,4}$ follows identically as for $\beta\partial_\varphi\mathcal{H}_f\in L^{\infty,\sigma}$ taking into account that

$$
|\beta_1^{\frac{2}{1+\alpha}}-\beta_2^{\frac{2}{1+\alpha}}|\le C\frac{|\beta_1-\beta_2|}{|\beta_1+\beta_2|}(\beta_1^{\frac{2}{1+\alpha}}+\beta_2^{\frac{2}{1+\alpha}})\le C\frac{|\beta_1-\beta_2|}{|\beta_1+\beta_2|}\beta_2^{\frac{2}{1+\alpha}}.
$$

The C^1 regularity of $\mathcal{H}_f$ in f follows similarly by estimating the associated Fréchet derivatives. ∎

In the next proposition, we state the well-definedness of the nonlinear function together with its C^1 regularity.

Proposition 3.10. *Let $\gamma,\sigma\in(0,1)$ and $\alpha\in(1,2)$ satisfying (3.16). There exists $\epsilon<1$ such that $\tilde F: B_{X_1^{\sigma,\delta}}(\epsilon)\times X_2^p\to Y^{\sigma,\delta}$ is well-defined and C^1 for any $p>\frac{1}{1-\gamma}$ and $\delta<\min\{1-\gamma,\sigma\}$.*

Proof. Let us prove that $\tilde F$ is well-defined and C^1 in f, where we shall use the expression of $\tilde F(f,\Omega)$ in (2.26). Note that the first term in that expression is just the constant 1, which is not in $Y^{\sigma,\delta}$ since it does not decay appropriately at 0 and ∞. For that reason, we need to add and subtract the contribution of the trivial solution in $\tilde F$, obtaining

$$
\begin{aligned}
&\tilde F(f,\Omega)(\beta,\phi)=\\
&f(\beta,\phi)-(-1)^{\frac{-1}{2\mu}}\frac{C_\gamma(\alpha-1)^{\frac{\alpha-1}{2}}}{4\pi C_0(1+\alpha)^{\frac{\alpha-1}{2}}}(I_1(f,\Omega)+I_2(f,\Omega)+I_3(f,\Omega)+I_4(f,\Omega))(\beta,\phi),
\end{aligned}
$$

where

$$
I_1(f,\Omega):=\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}\left\{-\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\varphi}}f(b,\Phi)\right\}^{-\frac{1}{2\mu}}\partial_{\overline{\beta\varphi}}f(b,\Phi)\Omega(\Phi)\,db\,d\Phi}{D(f)(\beta,\phi,b,\Phi)^{\gamma/2}},
$$

$$I_2(f,\Omega) := \frac{2(\alpha-1)}{(1+\alpha)^2}\beta^{-\frac{\alpha-1}{1+\alpha}}$$
$$\times \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} \left[\left\{-\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\varphi}} f(b,\Phi)\right\}^{-\frac{1}{2\mu}} - \left\{-\frac{\alpha-1}{1+\alpha}\right\}^{-\frac{1}{2\mu}}\right]\Omega(\Phi)\, db\, d\Phi}{D(f)(\beta,\phi,b,\Phi)^{\gamma/2}},$$
$$I_3(f,\Omega) := \frac{2(\alpha-1)}{(1+\alpha)^2}\left\{-\frac{\alpha-1}{1+\alpha}\right\}^{-\frac{1}{2\mu}}\beta^{-\frac{\alpha-1}{1+\alpha}} \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}}\Omega(\Phi)}{D(f)(\beta,\phi,b,\Phi)^{\gamma/2} D(0)(\beta,\phi,b,\Phi)^{\gamma/2}}$$
$$\times \{D(0)(\beta,\phi,b,\Phi)^{\gamma/2} - D(f)(\beta,\phi,b,\Phi)^{\gamma/2}\}\, db\, d\Phi,$$
$$I_4(f,\Omega) := \frac{2(\alpha-1)}{(1+\alpha)^2}\left\{-\frac{\alpha-1}{1+\alpha}\right\}^{-\frac{1}{2\mu}}\beta^{-\frac{\alpha-1}{1+\alpha}} \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}}(\Omega(\Phi)-1)\, db\, d\Phi}{D(0)(\beta,\phi,b,\Phi)^{\gamma/2}}.$$

Note that now $f \in Y^{\sigma,\delta}$ and it remains to prove that $I_i \in Y^{\sigma,\delta}$ for $i = 1,2,3,4$.

By using the operator $\mathcal{H}$ defined in Proposition 3.9 we can write

$$I_1(f,\Omega) = \alpha_1 \mathcal{H}\left(\left\{-\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\varphi}} f\right\}^{-\frac{1}{2\mu}} \partial_{\overline{\beta\varphi}} f, \Omega\right),$$
$$I_2(f,\Omega) = \alpha_2 \mathcal{H}\left(\left[\left\{-\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\varphi}} f\right\}^{-\frac{1}{2\mu}} - \left\{-\frac{\alpha-1}{1+\alpha}\right\}^{-\frac{1}{2\mu}}\right], \Omega\right),$$

for some constants α_1, α_2. Then, by Proposition 3.9,

$$\begin{aligned}\|I_1\|_{Y^{\sigma,\delta}} + \|I_2\|_{Y^{\sigma,\delta}} &\le C\|\Omega\|_{X_2^p}\left\{\left\|\left\{-\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\varphi}} f\right\}^{-\frac{1}{2\mu}} \partial_{\overline{\beta\varphi}} f\right\|_{\mathcal{X}_{\sigma,\alpha}}\right.\\ &\quad \left. + \left\|\left[\left\{-\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\varphi}} f\right\}^{-\frac{1}{2\mu}} - \left\{-\frac{\alpha-1}{1+\alpha}\right\}^{-\frac{1}{2\mu}}\right]\right\|_{\mathcal{X}_{\sigma,\alpha}}\right\}\\ &\le C\|\Omega\|_{X_2^p}\|f\|_{X_1^{\sigma,\alpha}}.\end{aligned}$$

Moreover, Proposition 3.9 shows that I_1 and I_2 are C^1 in f. It remains to check the well-posedness and C^1 regularity of I_3 and I_4. Let us start with I_4.

Note that

$$\begin{aligned}I_4(f,\Omega)(\beta,\phi) &= C\beta^{-\frac{\alpha-1}{1+\alpha}} \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}}(\Omega(\Phi)-1)\, db\, d\Phi}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\\ &= C\beta^{-\frac{\alpha-1}{1+\alpha}}\beta^{\frac{\gamma}{1+\alpha}} \iint \frac{b^{\frac{-2}{1+\alpha}}(\Omega(\Phi)-1)\, db\, d\Phi}{\{\beta^{\frac{2}{1+\alpha}} + b^{\frac{2}{1+\alpha}} - 2\beta^{\frac{1}{1+\alpha}} b^{\frac{1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\\ &=: C\beta^{-\frac{\alpha-1}{1+\alpha}}\beta^{\frac{\gamma}{1+\alpha}} I_{4,1}(f,\Omega)(\beta,\phi),\end{aligned}$$

where C is some constant which is not important in this proof. Let us first check the $L^{\infty,\sigma}$ norm. Take first $\beta \in [0,\beta_0]$. Since we have $\Omega - 1$ in the numerator, we get an extra cancellation at $\beta = 0$. This comes from the expression of $\Omega - 1$:

$$\Omega(\phi) - 1 = \sum_{0 \ne k \in \mathbb{Z}} g_k e^{ik\phi},$$

where the mode $k = 0$ vanishes, implying that

$$\int_0^{2\pi} (\Omega(\Phi) - 1)\, d\Phi = 0.$$

In this way, we obtain

$$\begin{aligned} I_{4,1}(f,\Omega)(0,\phi) &= \lim_{\varepsilon\to 0} \int_0^{2\pi} \int_\varepsilon^{1/\varepsilon} \frac{b^{\frac{-2}{1+\alpha}}(\Omega(\Phi)-1)\, db\, d\Phi}{\{b^{\frac{2}{1+\alpha}}\}^{\gamma/2}} \\ &= \lim_{\varepsilon\to 0} \int_\varepsilon^{1/\varepsilon} \frac{b^{\frac{-2}{1+\alpha}}\, db}{\{b^{\frac{2}{1+\alpha}}\}^{\gamma/2}} \int_0^{2\pi} (\Omega(\Phi)-1)\, d\Phi = 0. \end{aligned}$$

Then

$$\sup_{\beta\in[0,\beta_0]} \frac{(1+\beta)^{2\sigma}}{\beta^\sigma} I_4(f,\Omega)(\beta,\phi) \le \sup_{\beta\in[0,\beta_0]} \frac{\beta^{-\frac{\alpha-1}{1+\alpha}}\beta^{\frac{\gamma}{1+\alpha}}}{\beta^\sigma} < C,$$

since $\sigma < \frac{\gamma+1-\alpha}{1+\alpha}$ due to (3.16). Now take $\beta \in [\beta_0, \infty)$ and note that I_4 can also be written as

$$\begin{aligned} I_4(f,\Omega)(\beta,\phi) &= C\beta^{-\frac{\alpha-1}{1+\alpha}} \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}}(\Omega(\Phi)-1)\, db\, d\Phi}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \cos(\phi+\beta-\Phi-b)\}^{\gamma/2}} \\ &=: C\beta^{-\frac{\alpha-1}{1+\alpha}} I_{4,2}(f,\Omega)(\beta,\phi), \end{aligned} \tag{3.40}$$

where

$$\lim_{\beta\to\infty} I_{4,2}(f,\Omega)(\beta,\phi) = 0,$$

since $\Omega - 1$ has a vanishing 0 Fourier mode as before. Since $\sigma < \frac{\alpha-1}{1+\alpha}$ by (3.16), we get

$$\sup_{\beta\in[\beta_0,\infty)} I_4(f,\Omega)(\beta,\phi) < C.$$

Now make the change of variables $b = \beta z$ in the expression of I_4 to get

$$I_4(f,\Omega)(\beta,\phi) = C \iint \frac{z^{\frac{-2}{1+\alpha}}(\Omega(\Phi)-1)\, db\, d\Phi}{\{1 + z^{\frac{2}{1+\alpha}} - 2z^{\frac{1}{1+\alpha}} \cos(\phi+\beta-\Phi-z\beta)\}^{\gamma/2}}, \tag{3.41}$$

where we can prove easily the C^δ regularity in β and the Lip regularity in ϕ using that expression and the ideas developed in Proposition 3.9. Applying ∂_φ to (3.40) we find

$$\begin{aligned} \partial_\varphi I_4(f,\Omega)(\beta,\phi) &= C\beta^{-1} I_4(f,\Omega)(\beta,\phi) + C\beta^{-1}\beta^{-\frac{\alpha-1}{1+\alpha}} \\ &\times \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}}(\Omega(\Phi)-1)\{\beta^{\frac{-2}{1+\alpha}} - \beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \cos(\phi+\beta-\Phi-b)\}\, db\, d\Phi}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \cos(\phi+\beta-\Phi-b)\}^{\frac{2+\gamma}{2}}}. \end{aligned}$$

Let us check the $L^{\infty,\sigma}$ norm of $\beta\partial_\varphi I_4$. The first term is clearly in $L^{\infty,\sigma}$ and the second term can be written as

$$\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}(\Omega(\Phi)-1)\{\beta^{\frac{-2}{1+\alpha}}-\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\frac{2+\gamma}{2}}}$$
$$=: \beta^{-\frac{\alpha-1}{1+\alpha}}I_{4,3}(\beta,\phi) =: \beta^{-\frac{\alpha-1}{1+\alpha}}\beta^{\frac{\gamma}{1+\alpha}}I_{4,4}(\beta,\phi)$$

with $I_{4,4}(0,\phi)=0$ due to the structure of $\Omega-1$, and

$$\lim_{\beta\to\infty} I_{4,3}(\beta,\phi)=0.$$

Arguing as before for I_4 working with $I_{4,4}$ for $\beta\in[0,\beta_0]$ and with $I_{4,3}$ for $\beta\in[\beta_0,\infty)$ we find that $\beta\partial_\varphi I_4\in L^{\infty,\sigma}$. In the same way we can prove the Hölder continuity of that term and the C^1 regularity in f.

Let us consider I_3. A priori, it cannot be written in terms of $\mathcal{H}$, but it has the same scaling and the estimation goes similarly. Let us just give the first ideas. Recall

$$I_3(f,\Omega)=C\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{\frac{-\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}\Omega(\Phi)}{D(f)(\beta,\phi,b,\Phi)^{\gamma/2}D(0)(\beta,\phi,b,\Phi)^{\gamma/2}}$$
$$\times\{D(0)(\beta,\phi,b,\Phi)^{\gamma/2}-D(f)(\beta,\phi,b,\Phi)^{\gamma/2}\}\,db\,d\Phi.$$

Notice that

$$|D(0)(\beta,\phi,b,\Phi)^{\gamma/2}-D(f)(\beta,\phi,b,\Phi)^{\gamma/2}|$$
$$\leq C\frac{|D(0)(\beta,\phi,b,\Phi)-D(f)(\beta,\phi,b,\Phi)|}{|D(0)(\beta,\phi,b,\Phi)^{1-\gamma/2}+D(f)(\beta,\phi,b,\Phi)^{1-\gamma/2}|}.$$

Via Proposition 3.7 and Remark 3.8, we know that $D(f)$ is bounded below by $D(0)$:

$$D(f)(\beta,\phi,b,\Phi)\geq D(0)(\beta,\phi,b,\Phi)$$

and so

$$|I_3(f,\Omega)|\leq C\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{\frac{-\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}|\Omega(\Phi)|}{D(f)(\beta,\phi,b,\Phi)^{\gamma/2}}$$
$$\times\frac{|D(0)(\beta,\phi,b,\Phi)-D(f)(\beta,\phi,b,\Phi)|}{D(0)(\beta,\phi,b,\Phi)}\,db\,d\Phi. \quad (3.42)$$

Moreover, using the expression of $D(f)$ in (3.11) we find

$$
\begin{aligned}
&|D(0)(\beta,\phi,b,\Phi)-D(f)(\beta,\phi,b,\Phi)| \leq C|D(0)(\beta,\phi,b,\Phi)^{1/2}+D(f)(\beta,\phi,b,\Phi)^{1/2}| \\
&\times\Bigg[\left|\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi)\right\}^{1/2}-\beta^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}\right\}^{1/2}\right| \\
&+\left|b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\Phi)\right\}^{1/2}-b^{\frac{-1}{1+\alpha}}\left\{\frac{\alpha-1}{1+\alpha}\right\}^{1/2}\right| \\
&+\beta^{\frac{-1}{1+\alpha}}\left|\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi)\right\}^{1/2}-\left\{\frac{\alpha-1}{1+\alpha}\right\}^{1/2}\right|\left|\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\phi)\right|^{1/2} \\
&\hspace{20em}\times|\sin((\beta+\phi-b-\Phi)/2)| \\
&+b^{\frac{-1}{1+\alpha}}\left|\frac{\alpha-1}{1+\alpha}\right|^{1/2}\left|\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\Phi)\right\}^{1/2}-\left\{\frac{\alpha-1}{1+\alpha}\right\}^{1/2}\right||\sin((\beta+\phi-b-\Phi)/2)|\Bigg].
\end{aligned}
$$

Since $\partial_b f(\beta,\Phi)$ is bounded in L^∞ we get

$$
\begin{aligned}
&|D(0)(\beta,\phi,b,\Phi)-D(f)(\beta,\phi,b,\Phi)| \\
&\qquad\leq C|D(0)(\beta,\phi,b,\Phi)^{1/2}+D(f)(\beta,\phi,b,\Phi)^{1/2}| \\
&\qquad\quad\times\Bigg[\beta^{\frac{-1}{1+\alpha}}\left|\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{\beta}}f(\beta,\phi)\right\}^{1/2}-\left\{\frac{\alpha-1}{1+\alpha}\right\}^{1/2}\right| \\
&\qquad\qquad+b^{\frac{-1}{1+\alpha}}\left|\left\{\frac{\alpha-1}{1+\alpha}+\partial_{\overline{b}}f(b,\Phi)\right\}^{1/2}-\left\{\frac{\alpha-1}{1+\alpha}\right\}^{1/2}\right|\Bigg].
\end{aligned}
$$

Now, we wish to take into account the bound for f and using the fact that the norm of f is small, we can write

$$
\begin{aligned}
|D(0)(\beta,\phi,b,\Phi)-D(f)(\beta,\phi,b,\Phi)| &\leq C|D(0)(\beta,\phi,b,\Phi)^{1/2}+D(f)(\beta,\phi,b,\Phi)^{1/2}| \\
&\quad\times\left[\beta^{\frac{-1}{1+\alpha}}|\partial_{\overline{\beta}}f(\beta,\phi)|+b^{\frac{-1}{1+\alpha}}|\partial_{\overline{b}}f(b,\Phi)|\right].
\end{aligned}
$$

In this way, we decompose I_3 into two integrals and use again Remark 3.8 to compare $D(f)$ with $D(0)$:

$$
\begin{aligned}
&|I_3(f,\Omega)| \\
&\quad\leq C\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{\frac{-\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}|\Omega(\Phi)|}{D(f)(\beta,\phi,b,\Phi)^{\gamma/2}}\,\frac{\beta^{\frac{-1}{1+\alpha}}|\partial_{\overline{\beta}}f(\beta,\phi)|+b^{\frac{-1}{1+\alpha}}|\partial_{\overline{b}}f(b,\Phi)|}{D(0)(\beta,\phi,b,\Phi)^{1/2}}\,db\,d\Phi.
\end{aligned}
$$

By using similar arguments to those in Proposition 3.9, we finally get

$$
\|I_3\|_{L^{\infty,\sigma}}\leq C\|\Omega\|_{L^p}\|\partial_{\overline{\beta}}f\|_{L^{\infty,\sigma}}
$$

for any $p>1$.

By working with $D(f)$ and using the ideas of Proposition 3.9, we can check that the other norms in $X_1^{\sigma,\delta}$ are also bounded. ∎

4. Linearized operator

This section aims to give a different expression of the linearized operator of $\tilde{F}$ at the trivial solution $(0, 1)$. First, we will show that there is an extra cancellation that helps us to prove that the linearized operator is a compact perturbation of an isomorphism. Hence, we deduce that the linearized operator is Fredholm of zero index, which will be useful in checking that it is an isomorphism in the following section. Moreover, at the end of this section, we give its expression in Fourier series.

4.1. Linearization at the equilibrium

First, let us give the expression for the linearized operator of $\tilde{F}$ at $(0, 1)$.

Proposition 4.1. *The linearized operator of $\tilde{F}$ at the trivial solution $(0, 1)$ reads*

$$\begin{aligned}
\partial_f \tilde{F}(0,1)h(\beta,\phi) &= \frac{1+\alpha}{2}\partial_\beta(\beta h(\beta,\phi)) \\
&\quad - \frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\left(\frac{\alpha-1}{1+\alpha}\right)^{\gamma/2}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{\beta^{-\frac{\gamma}{1+\alpha}}\beta^{\frac{-2}{1+\alpha}}\partial_{\overline{b\Psi}}h(b,\Phi)\,db\,d\Phi}{D(0)(\beta,\phi,b,\Phi)^{\gamma/2}} \\
&\quad - \frac{1}{\mu}\frac{C_\gamma}{4\pi C_0}\frac{1}{\alpha-1}\left(\frac{\alpha-1}{1+\alpha}\right)^{\gamma/2}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{\beta^{-\frac{\gamma}{1+\alpha}}\beta^{\frac{-2}{1+\alpha}}\partial_{\overline{\Psi}}h(b,\Phi)\,db\,d\Phi}{D(0)(\beta,\phi,b,\Phi)^{\gamma/2}} \\
&\quad + \frac{\gamma}{2}\frac{C_\gamma}{2\pi C_0}\frac{1}{1+\alpha}\left(\frac{\alpha-1}{1+\alpha}\right)^{\gamma/2}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{\beta^{-\frac{\gamma}{1+\alpha}}\beta^{\frac{-2}{1+\alpha}}}{D(0)(\beta,\phi,b,\Phi)^{\gamma/2+1}} \\
&\qquad\qquad \times \{b^{\frac{-2}{1+\alpha}} - \beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}\partial_{\overline{b}}h(b,\Phi)\,db\,d\Phi \\
&=: \frac{1+\alpha}{2}\partial_\beta(\beta h(\beta,\phi)) + I_1 + I_2 + I_3. \qquad (4.1)
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
&\partial_f \tilde{F}(0,1)h(\beta,\phi) = h(\beta,\phi) - L(h)(\beta,\phi) \\
&- \frac{C_\gamma(\alpha-1)^{\frac{\alpha-1}{2}}}{4\pi C_0(1+\alpha)^{\frac{\alpha-1}{2}}}\left\{\frac{\alpha-1}{1+\alpha}\right\}^{-\frac{1}{2\mu}}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{\beta^{-\frac{\gamma}{1+\alpha}}\beta^{\frac{-2}{1+\alpha}}\partial_{\overline{b\Psi}}h(b,\Phi)\,db\,d\Phi}{D(0)(\beta,\phi,b,\Phi)^{\gamma/2}} \\
&- \frac{1}{2\mu}\frac{C_\gamma(\alpha-1)^{\frac{\alpha-1}{2}}}{4\pi C_0(1+\alpha)^{\frac{\alpha-1}{2}}}\left\{\frac{\alpha-1}{1+\alpha}\right\}^{-\frac{1}{2\mu}-1}\left\{\frac{2(\alpha-1)}{(1+\alpha)^2}\right\}\beta^{-\frac{\alpha-1}{1+\alpha}} \\
&\qquad\qquad \times \iint \frac{\beta^{-\frac{\gamma}{1+\alpha}}\beta^{\frac{-2}{1+\alpha}}\partial_{\overline{\Psi}}h(b,\Phi)\,db\,d\Phi}{D(0)(\beta,\phi,b,\Phi)^{\gamma/2}}, \\
&+ \frac{\gamma}{2}\frac{C_\gamma(\alpha-1)^{\frac{\alpha-1}{2}}}{4\pi C_0(1+\alpha)^{\frac{\alpha-1}{2}}}\left\{\frac{\alpha-1}{1+\alpha}\right\}^{-\frac{1}{2\mu}}\left\{\frac{2(\alpha-1)}{(1+\alpha)^2}\right\}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{\beta^{-\frac{\gamma}{1+\alpha}}\beta^{\frac{-2}{1+\alpha}}}{D(0)(\beta,\phi,b,\Phi)^{\gamma/2+1}} \\
&\qquad\qquad \times \{b^{\frac{-2}{1+\alpha}} - \beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}\partial_{\overline{b}}h(b,\Phi)\,db\,d\Phi.
\end{aligned}$$

Let us explain the meaning of L. The function L is the linearization of the local part associated to $-(-\Delta)^{-1+\gamma/2}\hat{\theta}$ after passing to the adapted coordinates and the appropriate scaling from Ψ to f. That is, we take the initial stream function (which is a radial function)

$$\hat{\psi}^0(re^{i\theta}) = -C_0 r^{1-\alpha},$$

and we use the fact that $r = (1+\alpha)^{1/2}(-\Psi_\beta)^{1/2}$, where we recall

$$\Psi = -C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\beta^{\frac{\alpha-1}{1+\alpha}}(1+f).$$

Hence, after the scaling done from F to $\tilde{F}$ in Section 2.4, we obtain

$$\begin{aligned}
L(h) &= \beta^{-\frac{\alpha-1}{1+\alpha}}\{-C_0^{\frac{-2}{1+\alpha}}(\alpha-1)^{\frac{\alpha-1}{1+\alpha}}\}\partial_f[-C_0(1+\alpha)^{\frac{1-\alpha}{2}}(-\Psi_\beta)^{\frac{1-\alpha}{2}}]h \\
&= -\beta^{-\frac{\alpha-1}{1+\alpha}}\{-C_0^{\frac{-2}{1+\alpha}}(\alpha-1)^{\frac{\alpha-1}{1+\alpha}}\}C_0(1+\alpha)^{\frac{1-\alpha}{2}}\frac{1-\alpha}{2}[(-\Psi_\beta^0)^{\frac{1-\alpha}{2}-1}] \\
&\quad\times C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\partial_\beta(\beta^{\frac{\alpha-1}{1+\alpha}}h).
\end{aligned}$$

Using $\partial_\beta(\beta^{\frac{\alpha-1}{1+\alpha}}h) = \beta^{\frac{-2}{1+\alpha}}\partial_{\overline{\beta}}h$ together with

$$(-\Psi_\beta^0)(\beta) = C_0^{\frac{2}{1+\alpha}}\frac{(\alpha-1)^{\frac{2}{1+\alpha}}}{1+\alpha}\beta^{\frac{-2}{1+\alpha}},$$

we find

$$\begin{aligned}
L(h) = &-\beta^{-\frac{\alpha-1}{1+\alpha}}\{-C_0^{\frac{-2}{1+\alpha}}(\alpha-1)^{\frac{\alpha-1}{1+\alpha}}\}C_0(1+\alpha)^{\frac{1-\alpha}{2}}\frac{1-\alpha}{2}C_0^{\frac{1-\alpha}{1+\alpha}}\frac{(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}}{(1+\alpha)^{\frac{1-\alpha}{2}}}\beta^{\frac{\alpha-1}{1+\alpha}} \\
&\times C_0^{\frac{-2}{1+\alpha}}\frac{(\alpha-1)^{\frac{-2}{1+\alpha}}}{(1+\alpha)^{-1}}\beta^{\frac{2}{1+\alpha}}C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\beta^{\frac{-2}{1+\alpha}}\partial_{\overline{\beta}}h \\
= &-\frac{1+\alpha}{2}\partial_{\overline{\beta}}h.
\end{aligned}$$

Using now the expression of $\partial_{\overline{\beta}}h$ we finally get

$$L(h) = -\frac{\alpha-1}{2}h - \frac{1+\alpha}{2}\beta h_\beta.$$

Hence, we deduce (4.1). ∎

Remark 4.2. The denominator $D(0)$ equals

$$\begin{aligned}
&D(0)(\beta,\phi,b,\Phi) \\
&\quad = \frac{\alpha-1}{1+\alpha}\{(\beta^{\frac{-1}{1+\alpha}} - b^{\frac{-1}{1+\alpha}})^2 + 2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}(1-\cos(\phi+\beta-\Phi-b))\}.
\end{aligned}$$

In the following proposition, we show an extra cancellation of the linearized operator leading to a more compact expression.

Proposition 4.3. *The linearized operator in Proposition 4.1 can be written as*

$$\begin{aligned}\partial_f \tilde{F}(0,1)h(\beta,\phi) &= \frac{1+\alpha}{2}\partial_\beta(\beta h(\beta,\phi))\\ &-\frac{1}{\mu}\frac{C_\gamma}{4\pi C_0}\frac{1}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}(\partial_{\overline{\Psi}}h(b,\Phi)+\partial_{\overline{b}}h(b,\Phi))\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}.\end{aligned} \tag{4.2}$$

Proof. To obtain the above expression, we will integrate by parts in I_3 in (4.1).

First, using $\partial_\Psi = \partial_\Phi - \partial_b$ we get

$$\partial_\Psi \frac{1}{D(0)(\beta,\phi,b,\Phi)^{\gamma/2}} = -\frac{\gamma}{2}\frac{2}{1+\alpha}\frac{\alpha-1}{1+\alpha}b^{-1}\frac{b^{\frac{-2}{1+\alpha}}-b^{\frac{-1}{1+\alpha}}\beta^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)}{D(0)(\beta,\phi,b,\Phi)^{\gamma/2+1}}.$$

Then

$$\begin{aligned}I_3 &= -\frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\left(\frac{\alpha-1}{1+\alpha}\right)^{\gamma/2}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b\partial_{\overline{b}}h(b,\Phi)\partial_\Psi\frac{1}{D(0)(\beta,\phi,b,\Phi)^{\gamma/2}}\\ &\qquad\times db\,d\Phi\\ &= -\frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\left(\frac{\alpha-1}{1+\alpha}\right)^{\gamma/2}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b\partial_{\overline{b}}h(b,\Phi)\partial_\Phi\frac{1}{D(0)(\beta,\phi,b,\Phi)^{\gamma/2}}\\ &\qquad\times db\,d\Phi\\ &+\frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\left(\frac{\alpha-1}{1+\alpha}\right)^{\gamma/2}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b\partial_{\overline{b}}h(b,\Phi)\partial_b\frac{1}{D(0)(\beta,\phi,b,\Phi)^{\gamma/2}}\\ &\qquad\times db\,d\Phi.\end{aligned}$$

Taking into account the definition of Ψ, we can integrate by parts.

Let us start with ∂_Φ. Note that

$$\begin{aligned}&\int_0^{2\pi} b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b\partial_{\overline{b}}h(b,\Phi)\\ &\qquad\times\partial_\Phi\frac{1}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}d\Phi\\ &\qquad= -\int_0^{2\pi}\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b\partial_\Phi\partial_{\overline{b}}h(b,\Phi)}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}d\Phi,\end{aligned}$$

since the boundary terms vanish due to the periodicity of the function. However, the boundary terms with respect to the integration in b will depend on the choice of the parameters:

$$\begin{aligned}
&\int_0^\infty b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} b\partial_{\bar{b}} h(b,\Phi) \\
&\qquad\qquad \times \partial_b \frac{1}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\, db \\
&= \lim_{b\to\infty} \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} b\partial_{\bar{b}} h(b,\Phi)}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \cos(\phi+\beta-\Phi-b)\}^{\gamma/2}} \\
&\quad - \lim_{b\to 0} \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} b\partial_{\bar{b}} h(b,\Phi)}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \cos(\phi+\beta-\Phi-b)\}^{\gamma/2}} \\
&\quad - \int_0^\infty \frac{\frac{-\gamma-1+\alpha}{1+\alpha} b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} \partial_{\bar{b}} h(b,\Phi) + b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} b\partial_b\partial_{\bar{b}} h(b,\Phi)}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\, db.
\end{aligned}$$

Since $\partial_{\bar{b}} h \in \mathcal{X}^{\sigma,\delta}$, we have

$$\begin{aligned}
&\lim_{b\to\infty} \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} b\partial_{\bar{b}} h(b,\Phi)}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \cos(\phi+\beta-\Phi-b)\}^{\gamma/2}} \\
&\qquad\qquad = \lim_{b\to\infty} \beta^{\frac{\gamma}{1+\alpha}} b^{\frac{-\gamma}{1+\alpha}} b^{\frac{\alpha-1}{1+\alpha}} b^{-\sigma} = 0,
\end{aligned}$$

because $\gamma > \alpha - 1$ by (3.16). On the other hand,

$$\begin{aligned}
&\lim_{b\to 0} \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} b\partial_{\bar{b}} h(b,\Phi)}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \cos(\phi+\beta-\Phi-b)\}^{\gamma/2}} \\
&\qquad\qquad = \lim_{b\to 0} b^{\frac{\gamma}{1+\alpha}} b^{\frac{-\gamma}{1+\alpha}} b^{\frac{\alpha-1}{1+\alpha}} b^{\sigma} = 0,
\end{aligned}$$

since $\alpha > 1$. Hence, in this term the boundary terms also vanish, yielding

$$\begin{aligned}
&\int_0^\infty b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} b\partial_{\bar{b}} h(b,\Phi) \\
&\qquad\qquad \times \partial_b \frac{1}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\, db \\
&= -\int_0^\infty \frac{\frac{-\gamma-2+\alpha}{1+\alpha} b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} \partial_{\bar{b}} h(b,\Phi) + b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} b\partial_b\partial_{\bar{b}} h(b,\Phi)}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}} \cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\, db.
\end{aligned}$$

By putting everything together we find

$$
\begin{aligned}
\mathcal{I}_3 &= \frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b\partial_\Phi\partial_{\overline{b}}h(b,\Phi)\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\\
&\quad-\frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\left(\frac{-\gamma-2+\alpha}{1+\alpha}\right)\\
&\qquad\times\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}\partial_{\overline{b}}h(b,\Phi)+b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b\partial_b\partial_{\overline{b}}h(b,\Phi)\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\\
&=\frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b\partial_\Psi\partial_{\overline{b}}h(b,\Phi)}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\,db\,d\Phi\\
&\quad-\frac{1/\mu-2}{1+\alpha}\frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}\partial_{\overline{b}}h(b,\Phi)\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}.
\end{aligned}
$$

Note now that

$$
b\partial_\Psi\partial_{\overline{b}}h=\partial_{\overline{b\Psi}}h-\frac{2}{1+\alpha}\partial_{\overline{b}}h,
$$

implying

$$
\begin{aligned}
\mathcal{I}_3 &= \frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}\partial_{\overline{b\Psi}}h(b,\Phi)\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\\
&\quad-\frac{2}{1+\alpha}\frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}\partial_{\overline{b}}h(b,\Phi)\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\\
&\quad-\frac{1/\mu-2}{1+\alpha}\frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}\partial_{\overline{b}}h(b,\Phi)\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\\
&=\frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}\partial_{\overline{b\Psi}}h(b,\Phi)}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\,db\,d\Phi\\
&\quad-\frac{1/\mu}{1+\alpha}\frac{C_\gamma}{4\pi C_0}\frac{1+\alpha}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}\partial_{\overline{b}}h(b,\Phi)\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}.
\end{aligned}
$$

Combining this with I_1 and I_2 we get (4.2). ∎

4.2. *Fredholmness at the trivial solution*

By using the expression from Proposition 4.3 we can prove that the linear operator is Fredholm of index zero in the following proposition. Since our goal is the use of the implicit function theorem, and then proving that $\partial_f F(0,1)$ is an isomorphism, we need to study its kernel and range. However, by the general theory of Fredholm operators [65] we can restrict ourselves to checking the kernel.

Proposition 4.4. *Let $\gamma, \sigma \in (0,1)$ and $\alpha \in (1,2)$ satisfy (3.16). Then the linear operator $\partial_f \tilde{F}(0,1) : X_1^{\sigma,\delta} \to Y^{\sigma,\delta}$ is Fredholm of index zero for any $\delta < \min\{1-\gamma, \sigma\}$.*

Proof. We can decompose the linear operator as follows:

$$\begin{aligned}
\partial_f \tilde{F}(0,1)(\beta,\phi) &= \frac{1+\alpha}{2}\partial_\beta(\beta h(\beta,\phi)) \\
&- \frac{1}{\mu}\frac{C_\gamma}{4\pi C_0}\frac{1}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}}(\partial_{\overline{\varphi}} h(b,\Phi) + \partial_{\overline{b}} h(b,\Phi))\, db\, d\Phi}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}} \\
&= \frac{1+\alpha}{2}\mathcal{L}_1 - \frac{1}{\mu}\frac{C_\gamma}{4\pi C_0}\frac{1}{\alpha-1}\mathcal{L}_2, \qquad (4.3)
\end{aligned}$$

where

$$\mathcal{L}_1(h) := \partial_\beta(\beta h(\beta,\phi)),$$

and

$$\mathcal{L}_2(h) := \beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}}(\partial_{\overline{\varphi}} h(b,\Phi) + \partial_{\overline{b}} h(b,\Phi))\, db\, d\Phi}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}.$$

The idea of the proof is to show that $\mathcal{L}_1 : X_1^{\sigma,\delta} \to Y^{\sigma,\delta}$ is an isomorphism, whereas $\mathcal{L}_2 : X_1^{\sigma,\delta} \to Y^{\sigma,\delta}$ is compact. Since compact perturbations of Fredholm operators remain Fredholm of index zero, we can conclude the proof.

• *$\mathcal{L}_1 : X_1^{\sigma,\delta} \to Y^{\sigma,\delta}$ is an isomorphism.* Take $g \in Y^{\sigma,\delta}$; we wish to solve $\mathcal{L}_1(h) = g$. We can explicitly solve this differential equation as

$$h(\beta,\phi) = \frac{K}{\beta} + \frac{1}{\beta}\int_0^\beta g(s,\phi)\, ds.$$

for any K. However, h must lie in $X_1^{\sigma,\delta}$. In particular, h vanishes at 0 and $K = 0$:

$$h(\beta,\phi) = \frac{1}{\beta}\int_0^\beta g(s,\phi)\, ds = \int_0^1 g(s\beta,\phi)\, ds. \qquad (4.4)$$

Let us now check that if $g \in Y^{\sigma,\delta}$, then $h \in X_1^{\sigma,\delta}$, where the norm of $X_1^{\sigma,\delta}$ is given in (3.6). Let us start by showing that

$$\|h\|_{\mathcal{X}^{\sigma,\delta}} = \|h\|_{L^{\infty,\sigma}} + \|h\|_{C^\delta_\beta} < C.$$

We find that

$$\begin{aligned}
\|h\|_{L^{\infty,\sigma}} &= \sup_{\beta\in[0,\infty)} \frac{(1+\beta)^{2\sigma}}{\beta^\sigma}\int_0^1 g(s\beta,\phi)\, ds \\
&\le \|g\|_{L^{\infty,\sigma}} \sup_{\beta\in[0,\infty)} (1+\beta)^{2\sigma}\int_0^1 \frac{s^\sigma}{(1+s\beta)^{2\sigma}}\, ds < C,
\end{aligned}$$

since $\sigma < 1$. Let us now show that $\|h\|_{C^\delta_\beta} < C$:

$$\begin{aligned}|h(\beta,\phi)-h(b,\phi)| &\leq \int_0^1 |g(s\beta,\phi)-g(sb,\phi)|\,ds\\ &\leq C\|g\|_{C^\delta_\beta}\int_0^1 \frac{|s\beta-sb|^\alpha}{|s\beta+sb|^\alpha}\,ds \leq C\|g\|_{C^\delta_\beta}\frac{|\beta-b|^\delta}{|\beta+b|^\delta},\end{aligned}$$

implying that $\|h\|_{C^\delta_\beta} < C\|g\|_{C^\delta_\beta}$. Let us show that $\|h\|_{\mathrm{Lip}_\phi} < C$:

$$|h(\beta,\phi)-h(\beta,\Phi)| \leq \int_0^1 |g(s\beta,\phi)-g(s\beta,\Phi)|\,ds \leq \|g\|_{\mathrm{Lip}_\phi}|\sin((\Phi-\phi)/2)|,$$

implying that

$$\|h\|_{\mathrm{Lip}_\phi} < \|g\|_{\mathrm{Lip}_\phi}.$$

Let us turn to βh_β and note that

$$\beta h_\beta(\beta,\phi) = -h(\beta,\phi)+g(\beta,\phi),$$

and so

$$\|\beta h_\beta\|_{\mathcal{X}^{\sigma,\delta}} + \|\beta h_\beta\|_{\mathrm{Lip}_\phi} \leq \|h\|_{\mathcal{X}^{\sigma,\delta}} + \|h\|_{\mathrm{Lip}_\phi} + \|g\|_{\mathcal{X}^{\sigma,\delta}} + \|g\|_{\mathrm{Lip}_\phi} \leq C.$$

Next, we will show that $\beta h_\varphi \in \mathcal{X}^{\sigma,\delta}$. We will consider separately the cases $\beta \in [0,\beta_0)$ and $\beta \in [\beta_0,\infty)$. For the first case, we can write

$$\begin{aligned}h_\varphi &= \frac{1}{\beta}\int_0^\beta g_\phi(s,\phi)\,ds + \frac{1}{\beta}h(\beta,\phi) - \frac{1}{\beta}g(\beta,\phi)\\ &= \frac{1}{\beta}\int_0^\beta g_\varphi(s,\phi)\,ds + \frac{1}{\beta}\int_0^\beta g_s(s,\phi)\,ds + \frac{1}{\beta}h(\beta,\phi) - \frac{1}{\beta}g(\beta,\phi)\\ &= \frac{1}{\beta}\int_0^\beta g_\varphi(s,\phi)\,ds + \frac{1}{\beta}g(\beta,\phi)\,ds + \frac{1}{\beta}h(\beta,\phi) - \frac{1}{\beta}g(\beta,\phi),\end{aligned}$$

and so

$$\beta h_\varphi = \int_0^\beta g_\varphi(s,\phi)\,ds + h(\beta,\phi). \tag{4.5}$$

Since we have shown that $h \in \mathcal{X}^{\sigma,\delta}$, it remains to check the first term:

$$\begin{aligned}\sup_{\beta\in[0,\beta_0)} \frac{(1+\beta)^{2\sigma}}{\beta^\sigma}\int_0^\beta g_\varphi(s,\phi)\,ds &\leq \|\beta g_\varphi\|_{L^{\infty,\sigma}} \sup_{\beta\in[0,\beta_0)} \frac{1}{\beta^\sigma}\int_0^\beta \frac{1}{s}\frac{s^\sigma}{(1+s)^{2\sigma}}\,ds\\ &\leq \|\beta g_\varphi\|_{L^{\infty,\sigma}} \sup_{\beta\in[0,\beta_0)}\int_0^1 s^{\sigma-1}\,ds \leq \|\beta g_\varphi\|_{L^{\infty,\sigma}}.\end{aligned}$$

In the case $\beta \in [\beta_0,\infty)$ we write

$$h(\beta,\phi) = \frac{1}{\beta}\int_\infty^\beta g(s,\phi)\,ds$$

instead of (4.4) by using the fact that h must vanish at 0 and ∞. We then have

$$\beta h_\varphi = \int_\infty^\beta g_\varphi(s,\phi)\,ds + h(\beta,\phi), \tag{4.6}$$

and hence it remains to check the first term:

$$\sup_{\beta\in[\beta_0,\infty)} \frac{(1+\beta)^{2\sigma}}{\beta^\sigma}\int_\infty^\beta g_\varphi(s,\phi)\,ds \le \|\beta g_\varphi\|_{L^{\infty,\sigma}} \sup_{\beta\in[\beta_0,\infty)} \beta^\sigma \int_\infty^\beta \frac{1}{s^{1+\sigma}}\,ds$$
$$\lesssim \|\beta g_\varphi\|_{L^{\infty,\sigma}},$$

since $\sigma\in(0,1)$. In that way, we have shown that

$$\|\beta h_\varphi\|_{L^{\infty,\sigma}} \le C\|\beta g_\varphi\|.$$

Let us now check that $\|\beta h_\varphi\|_{C^\delta_\beta} < C$, which amounts to checking this property for the first term in (4.5), that is,

$$\left|\int_\beta^b g_\varphi(s,\phi)\,ds\right| \le C\frac{|\beta-b|^\delta}{|\beta+b|^\delta}.$$

Take first $|\beta+b|<\beta_0<1$; then

$$\left|\int_\beta^b g_\varphi(s,\phi)\,ds\right| \le C\|g_\varphi\|_{\mathcal{X}^{\sigma,\delta}}\int_\beta^b s^{\sigma-1}\,ds \le C|\beta-b|^\sigma \le C\frac{|\beta-b|^\delta}{|\beta+b|^\delta}$$

if $\delta<\sigma<1$. If $|\beta+b|>\beta_0$ we use the expression of βh_φ to find that

$$\begin{aligned}\left|\int_\beta^b g_\varphi(s,\phi)\,ds\right| &\le C\|g_\varphi\|_{\mathcal{X}^{\sigma,\delta}}\int_\beta^b s^{-\sigma-1}\,ds\\ &\le C\left|\frac{1}{\beta^\sigma}-\frac{1}{b^\sigma}\right| \le C\frac{|\beta-b|^\sigma}{|\beta+b|^{2\sigma}}\\ &\le C\frac{|\beta-b|^\sigma}{|\beta+b|^\sigma} \le C\frac{|\beta-b|^\delta}{|\beta+b|^\delta}.\end{aligned}$$

Let us finish with $\beta^2 h_{\beta,\varphi}$, noting that

$$\begin{aligned}h_{\beta,\varphi}(\beta,\phi) &= -\frac{1}{\beta}\frac{1}{\beta}\int_0^\beta g_\varphi(s,\phi)\,ds + \frac{1}{\beta}g_\varphi(\beta,\phi) + \frac{1}{\beta}h_\beta(\beta,\phi) - \frac{1}{\beta^2}h(\beta,\phi)\\ &= -\frac{1}{\beta}\left(h_\varphi(\beta,\phi) - \frac{1}{\beta}h(\beta,\phi)\right) + \frac{1}{\beta}g_\varphi(\beta,\phi) + \frac{1}{\beta}h_\beta(\beta,\phi) - \frac{1}{\beta^2}h(\beta,\phi),\end{aligned}$$

and so

$$\beta^2 h_{\beta,\varphi}(\beta,\phi) = -\beta h_\varphi(\beta,\phi) + \beta g_\varphi(\beta,\phi) + \beta h_\beta(\beta,\phi),$$

implying that $\beta^2 h_{\beta,\varphi}\in\mathcal{X}_{\sigma,\alpha}$.

• $\mathcal{L}_2 : X_1^{\sigma,\delta} \to Y^{\sigma,\delta}$ *is compact.* From Proposition 3.10 we know that $\mathrm{Range}(\mathcal{L}_2)\subset Y^{\sigma,\delta}$. To prove that $\mathcal{L}_2$ is compact, we will show that $\mathrm{Range}(\mathcal{L}_2)\subset Y^{\sigma,\delta'}$ for some $\delta'>\delta$. Then, since the embedding $Y^{\sigma,\delta'} \subset Y^{\sigma,\delta}$ is compact (we refer to [73, Lemma 2.2]) we conclude that $\mathcal{L}_2 : X_1^{\sigma,\delta} \to Y^{\sigma,\delta}$ is a compact operator.

Let us prove that $\beta\partial_\varphi\mathcal{L}_2 \in C_\beta^{\delta'}$ for some $\delta' > \delta$. For that, we compute $\partial_\varphi\mathcal{L}_2$:

$$
\begin{aligned}
\partial_\varphi\mathcal{L}_2(h) &= C\beta^{-1}\mathcal{L}_2(h)\\
&\quad + C\beta^{-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}(\partial_{\overline{\varphi}}h(b,\Phi)+\partial_{\overline{b}}h(b,\Phi))}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\frac{2+\gamma}{2}}}\\
&\qquad \times\{\beta^{\frac{-2}{1+\alpha}}-\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}\,db\,d\Phi
\end{aligned}
$$

for some constant C. Then

$$
\begin{aligned}
\beta\partial_\varphi\mathcal{L}_2(h) &= C\mathcal{L}_2(h)\\
&\quad + C\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}(\partial_{\overline{\varphi}}h(b,\Phi)+\partial_{\overline{b}}h(b,\Phi))}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\frac{2+\gamma}{2}}}\\
&\qquad \times\{\beta^{\frac{-1}{1+\alpha}}(\beta^{\frac{-1}{1+\alpha}}-b^{\frac{-1}{1+\alpha}})+\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}(1-\cos(\phi+\beta-\Phi-b))\}\,db\,d\Phi.
\end{aligned}
$$

Note that in this expression the singularity is critical. To solve this problem we should integrate by parts somehow but we cannot differentiate $\partial_{\overline{\varphi}}h$ with respect to ∂_φ and so we have to make the following decomposition:

$$
\begin{aligned}
\beta\partial_\varphi\mathcal{L}_2(h) &= C\mathcal{L}_2(h)\\
&\quad + C\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}(\partial_{\overline{\varphi}}h(b,\Phi)-\partial_{\overline{\varphi}}h(\beta,\phi))}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\frac{2+\gamma}{2}}}\\
&\qquad \times\{\beta^{\frac{-1}{1+\alpha}}(\beta^{\frac{-1}{1+\alpha}}-b^{\frac{-1}{1+\alpha}})+\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}(1-\cos(\phi+\beta-\Phi-b))\}\,db\,d\Phi\\
&\quad + C\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}(\partial_{\overline{b}}h(b,\Phi))}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\frac{2+\gamma}{2}}}\\
&\qquad \times\{\beta^{\frac{-1}{1+\alpha}}(\beta^{\frac{-1}{1+\alpha}}-b^{\frac{-1}{1+\alpha}})+\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}(1-\cos(\phi+\beta-\Phi-b))\}\,db\,d\Phi\\
&\quad + C\partial_{\overline{\varphi}}h(\beta,\phi)\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\frac{2+\gamma}{2}}}\\
&\qquad \times\{\beta^{\frac{-1}{1+\alpha}}(\beta^{\frac{-1}{1+\alpha}}-b^{\frac{-1}{1+\alpha}})+\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}(1-\cos(\phi+\beta-\Phi-b))\}\,db\,d\Phi\\
&=: C\mathcal{L}_2(h)+L_{2,1}(h)+L_{2,2}(h)+L_{2,3}(h).
\end{aligned}
$$

Let us show the details for one of the terms, say $L_{2,2}$ due to its difficulty. Note that the singularity coming from the denominator is not a priori integrable and we need to use some extra cancellation. To achieve that, we need to integrate by parts and so we have to add and subtract an appropriate function in the denominator. Indeed, denote

$$
J(\beta,\phi,b,\Phi) := \{\beta_1^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta_1^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta_1-\Phi-b)\}.
$$

Notice that J is indeed the denominator at the trivial solution and so $J = CD(0)$ for some constant C, where $D(f)$ is defined in (3.11). In particular, following Proposition 3.7 and the expression for J one finds that

$$|J(\beta,\phi,b,\Phi)| \geq C(\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}})\left\{\frac{|\beta-b|^2}{|\beta+b|^2} + \sin^2((\beta-b+\phi-\Phi)/2)\right\}. \quad (4.7)$$

By using the expression of J we also get

$$\begin{aligned}&\partial_\varphi J(\beta,\phi,b,\Phi)\\ &\quad= \frac{2}{1+\alpha}\beta^{-1}\{\beta^{\frac{-1}{1+\alpha}}(\beta^{\frac{-1}{1+\alpha}} - b^{\frac{-1}{1+\alpha}}) + \beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}(1-\cos(\beta+\phi-b-\Phi)\},\end{aligned}$$

and so

$$\begin{aligned}&\beta\partial_\varphi J(\beta,\phi,b,\Phi) + b\partial_\Psi J(\beta,\phi,b,\Phi)\\ &\qquad= \frac{2}{1+\alpha}\{(\beta^{\frac{-1}{1+\alpha}} - b^{\frac{-1}{1+\alpha}})^2 + 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}(1-\cos(\beta+\phi-b-\Phi)\}\\ &\qquad= \frac{2}{1+\alpha}J(\beta,\phi,b,\Phi).\end{aligned}$$

Motivated by the previous computation, we decompose $L_{2,2}$ as follows:

$$\begin{aligned}L_{2,2}(h)(\beta) &= C\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}}\partial_{\bar b} h(b,\Phi)\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\\ &\quad+ \frac{2}{\gamma}C\beta^{-\frac{\alpha-1}{1+\alpha}}\iint bb^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}}\partial_{\bar b} h(b,\Phi)\\ &\qquad\times\partial_\Psi \frac{1}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\,db\,d\Phi.\end{aligned}$$

If we integrate by parts in the last term, the boundary terms vanish since $h \in X_1^{\sigma,\delta}$ (we refer to the proof of Proposition 4.3) and so

$$\begin{aligned}L_{2,2}(h)(\beta) &= C\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}}\partial_{\bar b} h(b,\Phi)\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\\ &\quad- \frac{2}{\gamma}C\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{bb^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}}\partial_{\bar b,\Psi} h(b,\Phi)\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\\ &\quad- \frac{2}{\gamma}C\frac{1+\gamma-\alpha}{1+\alpha}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}}\partial_{\bar b} h(b,\Phi)\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\\ &=: L_{2,2,1} + L_{2,2,2} + L_{2,2,3}.\end{aligned}$$

Take now $\beta_1 < \beta_2$ implying

$$\beta_2^{-\frac{\alpha-1}{1+\alpha}} < \beta_1^{-\frac{\alpha-1}{1+\alpha}}, \quad (4.8)$$

and let us prove the Hölder regularity of $L_{2,2}$. By taking the difference of $L_{2,2}$ at the two points we obtain

$$
\begin{aligned}
&|L_{2,2}(h)(\beta_1) - L_{2,2}(h)(\beta_2)| \\
&\leq C \frac{|\beta_1-\beta_2|^{\delta'}}{|\beta_1+\beta_2|^{\delta'}} (\beta_1^{-\frac{\alpha-1}{1+\alpha}} + \beta_2^{-\frac{\alpha-1}{1+\alpha}}) \left\{ \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} \partial_{\overline{b}} h(b,\Phi)}{J(\beta_1,\phi,b,\Phi)^{\gamma/2}} \, db \, d\Phi \right. \\
&\qquad\qquad \left. + \iint \frac{b b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} \partial_{\overline{b},\Psi} h(b,\Phi) \, db \, d\Phi}{J(\beta_1,\phi,b,\Phi)^{\gamma/2}} \right\} \\
&\quad + \beta_2^{-\frac{\alpha-1}{1+\alpha}} \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} \partial_{\overline{b}} h(b,\Phi)}{J(\beta_1,\phi,b,\Phi)^{\gamma/2} J(\beta_2,\phi,b,\Phi)^{\gamma/2}} \\
&\qquad\qquad \times \frac{|J(\beta_1,\phi,b,\Phi) - J(\beta_2,\phi,b,\Phi)|}{J(\beta_1,\phi,b,\Phi)^{\frac{2-\gamma}{2}} + J(\beta_2,\phi,b,\Phi)^{\frac{2-\gamma}{2}}} \, db \, d\Phi \\
&\quad + \beta_2^{-\frac{\alpha-1}{1+\alpha}} \iint \frac{b b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} \partial_{\overline{b},\Psi} h(b,\Phi)}{J(\beta_1,\phi,b,\Phi)^{\gamma/2} J(\beta_2,\phi,b,\Phi)^{\gamma/2}} \\
&\qquad\qquad \times \frac{|J(\beta_1,\phi,b,\Phi) - J(\beta_2,\phi,b,\Phi)|}{J(\beta_1,\phi,b,\Phi)^{\frac{2-\gamma}{2}} + J(\beta_2,\phi,b,\Phi)^{\frac{2-\gamma}{2}}} \, db \, d\Phi,
\end{aligned}
$$

where δ' is any number in $(0,1)$. Taking into account (4.8), the first three terms can be related to the operator $\mathcal{H}$ defined in (3.15), and for the last three we use the fact that $h \in X_1^{\sigma,\delta}$, obtaining

$$
\begin{aligned}
&|L_{2,2}(h)(\beta_1) - L_{2,2}(h)(\beta_2)| \\
&\qquad \leq C \frac{|\beta_1-\beta_2|^{\delta'}}{|\beta_1+\beta_2|^{\delta'}} \{ \|\mathcal{H}_0(\partial_{\overline{b}} h(b,\Phi),1)\|_{L^\infty} + \|\mathcal{H}_0(b(\partial_{\overline{b},\Psi} h(b,\Phi)),1)\|_{L^\infty} \} \\
&\qquad\quad + C \|h\|_{X_1^{\sigma,\delta}} \beta_2^{-\frac{\alpha-1}{1+\alpha}} \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} b^{\sigma} (1+b)^{-2\sigma}}{J(\beta_1,\phi,b,\Phi)^{\gamma/2} J(\beta_2,\phi,b,\Phi)^{\gamma/2}} \\
&\qquad\qquad \times \frac{|J(\beta_1,\phi,b,\Phi) - J(\beta_2,\phi,b,\Phi)| \, db \, d\Phi}{J(\beta_1,\phi,b,\Phi)^{\frac{2-\gamma}{2}} + J(\beta_2,\phi,b,\Phi)^{\frac{2-\gamma}{2}}}.
\end{aligned}
$$

By Proposition 3.9 we have

$$
\|\mathcal{H}_0(\partial_{\overline{b}} h(b,\Phi),1)\|_{L^\infty} + \|\mathcal{H}_0(b(\partial_{\overline{b},\Psi} h(b,\Phi)),1)\|_{L^\infty} \leq C \|h\|_{X_1^{\sigma,\delta}}.
$$

Next, we need to estimate the difference of J at these two points:

$$
\begin{aligned}
&|J(\beta_1,\phi,b,\Phi) - J(\beta_2,\phi,b,\Phi)| \\
&\leq C \frac{|\beta_1-\beta_2|}{|\beta_1+\beta_2|} (\beta_1^{\frac{-1}{1+\alpha}} + \beta_2^{\frac{-1}{1+\alpha}}) (J(\beta_1,\phi,b,\Phi)^{1/2} + J(\beta_2,\phi,b,\Phi)^{1/2}) \\
&\quad + C \frac{|\beta_1-\beta_2|}{|\beta_1+\beta_2|} \beta_2^{\frac{-1}{1+\alpha}} |\beta_1+\beta_2| (J(\beta_1,\phi,b,\Phi)^{1/2} + J(\beta_2,\phi,b,\Phi)^{1/2}) \\
&\leq C \frac{|\beta_1-\beta_2|}{|\beta_1+\beta_2|} (1+|\beta_1+\beta_2|) (\beta_1^{\frac{-1}{1+\alpha}} + \beta_2^{\frac{-1}{1+\alpha}}) (J(\beta_1,\phi,b,\Phi)^{1/2} + J(\beta_2,\phi,b,\Phi)^{1/2})
\end{aligned} \tag{4.9}
$$

and interpolating with

$$|J(\beta_1,\phi,b,\Phi)-J(\beta_2,\phi,b,\Phi)|\leq C|J(\beta_1,\phi,b,\Phi)+J(\beta_2,\phi,b,\Phi)|,$$

we get

$$\begin{aligned}|J(\beta_1,\phi,b,\Phi)-J(\beta_2,\phi,b,\Phi)|\leq C\frac{|\beta_1-\beta_2|^{\delta'}}{|\beta_1+\beta_2|^{\delta'}}(1+|\beta_1+\beta_2|^{\delta'})(\beta_1^{\frac{-1}{1+\alpha}}+\beta_2^{\frac{-1}{1+\alpha}})^{\delta'}\\ \times(J(\beta_1,\phi,b,\Phi)^{1-\delta'/2}+J(\beta_2,\phi,b,\Phi)^{1-\delta'/2})\end{aligned} \tag{4.10}$$

for all $\delta'\in(0,1)$. Finally,

$$\begin{aligned}&|L_{2,2}(h)(\beta_1)-L_{2,2}(h)(\beta_2)|\leq C\|h\|_{X_1^{\sigma,\delta}}\frac{|\beta_1-\beta_2|^{\delta'}}{|\beta_1+\beta_2|^{\delta'}}\\ &\times\Bigg[1+(1+|\beta_1+\beta_2|^{\delta'})(\beta_1^{\frac{-1}{1+\alpha}}+\beta_2^{\frac{-1}{1+\alpha}})^{\delta'}\beta_2^{-\frac{\alpha-1}{1+\alpha}}\\ &\times\iint\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b^{\sigma}(1+b)^{-2\sigma}}{J(\beta_1,\phi,b,\Phi)^{\gamma/2}J(\beta_2,\phi,b,\Phi)^{\gamma/2}}\frac{J(\beta_1,\phi,b,\Phi)^{-\frac{\delta'}{2}}+J(\beta_2,\phi,b,\Phi)^{-\frac{\delta'}{2}}}{J(\beta_1,\phi,b,\Phi)^{\frac{-\gamma}{2}}+J(\beta_2,\phi,b,\Phi)^{\frac{-\gamma}{2}}}\,db\,d\Phi\Bigg]\\ &\leq C\|h\|_{X_1^{\sigma,\delta}}\frac{|\beta_1-\beta_2|^{\delta'}}{|\beta_1+\beta_2|^{\delta'}}\\ &\quad\times\Bigg[1+(1+|\beta_1+\beta_2|^{\delta'})(\beta_1^{\frac{-1}{1+\alpha}}+\beta_2^{\frac{-1}{1+\alpha}})^{\delta'}\beta_2^{-\frac{\alpha-1}{1+\alpha}}\iint b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}b^{\sigma}(1+b)^{-2\sigma}\\ &\times\left(\frac{1}{J(\beta_1,\phi,b,\Phi)^{\delta'/2}J(\beta_2,\phi,b,\Phi)^{\gamma/2}}+\frac{1}{J(\beta_1,\phi,b,\Phi)^{\delta'/2}J(\beta_2,\phi,b,\Phi)^{\gamma/2}}\right)db\,d\Phi\Bigg].\end{aligned}$$

The last integral is similar to the one in (3.29) and so it is bounded if $\delta'<\sigma$. Furthermore, since δ' is arbitrary we can choose $\delta'>\delta$ to have compactness. ■

4.3. *Expression of the linearized operator in Fourier series*

In the next proposition, we show that the linearized operator can be computed in Fourier modes depending on the Fourier coefficients of h. Assume that h takes the form

$$h(\beta,\phi)=\sum_{k\in\mathbb{Z}}h_k(\beta)e^{ik\phi}. \tag{4.11}$$

Proposition 4.5. *Let h be expressed as in (4.11). Then*

$$\begin{aligned}\partial_f\tilde{F}(0,1)h(\beta,\phi)=\sum_{k\in\mathbb{Z}}e^{ik\phi}\Bigg\{\frac{1+\alpha}{2}\partial_\beta(\beta h_k(\beta))\\ -ik\frac{1}{\mu}\frac{C_\gamma}{4\pi C_0}\frac{1}{\alpha-1}e^{ik\beta}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint\frac{e^{-ikb}b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}bh_k(b)\cos(k\eta)\,db\,d\eta}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\eta)\}^{\gamma/2}}\Bigg\}.\end{aligned}$$

Proof. We will insert (4.11) in the expression of Proposition 4.3. First, using the definition of $\partial_{\overline{\beta}}$ and $\partial_{\overline{\varphi}}$ we find

$$\partial_{\overline{\beta}} h(\beta,\phi) = \frac{\alpha-1}{1+\alpha} h(\beta,\phi) + \beta\partial_\beta h(\beta,\phi),$$
$$\partial_{\overline{\varphi}} h(\beta,\phi) = -\frac{\alpha-1}{1+\alpha} h(\beta,\phi) + \beta\partial_\varphi h(\beta,\phi),$$

obtaining

$$\partial_{\overline{\varphi}} h(\beta,\phi) + \partial_{\overline{\beta}} h(\beta,\phi) = \beta(\partial_\varphi h(\beta,\phi) + \partial_\beta h(\beta,\phi)) = \beta\partial_\phi h(\beta,\phi),$$

taking into account that $\partial_\varphi = \partial_\phi - \partial_\beta$. Hence, we can compute the Fourier series of the above term as

$$\partial_{\overline{\varphi}} h(\beta,\phi) + \partial_{\overline{\beta}} h(\beta,\phi) = \sum_{k\in\mathbb{Z}} ik\beta h_k(\beta) e^{ik\phi}. \tag{4.12}$$

Moreover,

$$\partial_\beta(\beta h(\beta,\phi)) = \sum_{k\in\mathbb{Z}} \partial_\beta(\beta h_k(\beta)) e^{ik\phi}. \tag{4.13}$$

Inserting (4.12) and (4.13) into (4.2) we get

$$\begin{aligned}\partial_f \tilde{F}(0,1)(\beta,\phi) = \sum_{k\in\mathbb{Z}} e^{ik\phi}\Bigg\{&\frac{1+\alpha}{2}\partial_\beta(\beta h_k(\beta))\\ &- ike^{-ik\phi}\frac{1}{\mu}\frac{C_\gamma}{4\pi C_0}\frac{1}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\\ &\times \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} b h_k(b) e^{ik\Phi}\, db\, d\Phi}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\Bigg\}. \end{aligned} \tag{4.14}$$

Note that we can write the last integral as

$$\begin{aligned} e^{ik\phi}e^{ik\beta} &\iint \frac{e^{-ikb} b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} b h_k(b) e^{ik(\Phi+b-\phi-\beta)}\, db\, d\Phi}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\cos(\phi+\beta-\Phi-b)\}^{\gamma/2}}\\ &= e^{ik\phi}e^{ik\beta}\iint \frac{e^{-ikb} b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} b h_k(b)\cos(k\eta)\, db\, d\eta}{\{\beta^{\frac{-2}{1+\alpha}} + b^{\frac{-2}{1+\alpha}} - 2\beta^{\frac{-1}{1+\alpha}} b^{\frac{-1}{1+\alpha}}\cos(\eta)\}^{\gamma/2}}. \end{aligned}$$

Hence, coming back to (4.14) we find the desired formula. ■

5. Invertibility of the linearized operator

In this section, we shall study the invertibility of the linearized operator at the trivial solution in order to later apply the infinite-dimensional implicit function theorem. From Proposition 4.4 we know that the operator is a compact perturbation of an isomorphism and thus it is a Fredholm operator of index zero. In that way, it remains to check that its kernel is trivial. In order to study the kernel equation, we relate this problem to the moment problem by using the Mellin and Laplace transforms. More specifically, once we apply the Mellin transform to the kernel equation, we arrive at a recurrence equation with singular coefficients. However, we can overcome this by studying the special structure of the recurrence equation. Later, we shall relate the solution of the recurrence equation to the moment problem by using the Laplace transform, concluding that the solution is singular and thus it does not belong to our function space. That will mean a trivial kernel.

5.1. Kernel issue

Note that the kernel issue reduces to the following problem:

$$\partial_f F(0,1)h = 0, h \in X_1^{\sigma,\delta} \implies h = 0. \tag{5.1}$$

Since we can restrict ourselves to work with the linearized operator as a Fourier series, we can directly check that the operator at the mode k is trivial, for any $k \in \mathbb{Z}$. The operator is given by Proposition 4.5 and can be written as

$$\begin{aligned}\partial_f \tilde{F}(0,1)(\beta,\phi) &= \sum_{k\in\mathbb{Z}} e^{ik\phi}\Bigg\{\frac{1+\alpha}{2}\partial_\beta(\beta h_k(\beta)) \\ &\quad - ik\frac{1}{\mu}\frac{C_\gamma}{4\pi C_0}\frac{1}{\alpha-1}e^{ik\beta}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{e^{-ikb}b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}bh_k(b)\cos(k\eta)\,db\,d\eta}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\eta)\}^{\gamma/2}}\Bigg\} \\ &= \sum_{k\in\mathbb{Z}} e^{ik\phi}e^{ik\beta}\left\{\frac{1+\alpha}{2}e^{-ik\beta}\partial_\beta(\beta h_k(\beta)) - ikJ_k(e^{-ikb}h_k)(\beta)\right\},\end{aligned}$$

where

$$J_k(g_k)(\beta) := \frac{1}{\mu}\frac{C_\gamma}{4\pi C_0}\frac{1}{\alpha-1}\beta^{-\frac{\alpha-1}{1+\alpha}}\iint \frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}bg_k(b)\cos(k\eta)\,db\,d\Phi}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\eta)\}^{\gamma/2}},$$

and

$$g_k(\beta) := e^{-ik\beta}h_k(\beta).$$

We also define

$$\mathcal{L}_k(h)(\beta) := \frac{1+\alpha}{2}e^{-ik\beta}\partial_\beta(\beta h_k(\beta)) - ikJ_k(e^{-ikb}h_k)(\beta), \tag{5.2}$$

and we wish to prove that $\mathcal{L}_k$ has a trivial kernel for each $k \in \mathbb{Z}$. If $h \in X_1^{\sigma,\delta}$ one sees that h_k is in $\mathcal{X}_0$ defined in (3.9) where

$$\mathcal{X}_0 = \{f : \beta \in [0,\infty) \to \mathbb{R} \mid f, \beta f' \in \mathcal{X}^{\sigma,\delta}\}.$$

As stated in Remark 3.3, knowing that the Fourier coefficients belong to $\mathcal{X}_0$ does not imply that the full function lies in $X_1^{\sigma,\delta}$. However, since our goal is to check that $\mathcal{L}_k$ has a trivial kernel for any $k \in \mathbb{Z}$, it is enough to check it in the spaces of functions whose Fourier coefficients lie in $\mathcal{X}_0$.

The case $k = 0$ is straightforward since

$$\mathcal{L}_0(h)(\beta) := \frac{1+\alpha}{2} e^{-ik\beta} \partial_\beta (\beta h_k(\beta)),$$

which has a trivial kernel.

In the following proposition we use an auxiliary function g_k to work with (5.2).

Proposition 5.1. *Let $g_k(\beta) = e^{-ik\beta} h_k(\beta)$. Then $\mathcal{L}_k(h) = 0$, defined in (5.2), amounts to*

$$\tilde{\mathcal{L}}_k(g_k) := \frac{1+\alpha}{2} g_k(\beta) + ik\frac{1+\alpha}{2} \beta g_k(\beta) + \frac{1+\alpha}{2} \beta g_k'(\beta) - ikJ_k(g_k)(\beta) = 0,$$

where

$$J_k(\beta) = \int_0^\infty g_k(b) K_k(\beta/b)\, db, \tag{5.3}$$

and

$$K_k(z) := \frac{C_\gamma}{\mu 4\pi C_0(\alpha-1)} \int_0^{2\pi} \frac{z^{-\frac{\alpha-1}{1+\alpha}} \cos(k\eta)\, d\eta}{\{z^{\frac{-2}{1+\alpha}} + 1 - 2z^{\frac{-1}{1+\alpha}} \cos(\eta)\}^{\gamma/2}}.$$

Proof. Just note that

$$e^{ik\beta} \beta h_k'(\beta) = \beta g_k'(\beta) + ik\beta g_k(\beta),$$

and use the expression of J_k. ∎

Note that $K_{-k} = K_k$ and although we will work with $k \in \mathbb{N}^\star$, we can always change k to $-k$. In the following, we state the monotonicity of K_k with respect to k.

Proposition 5.2. *For any $z \in (0, \infty)$, the sequence $\mathbb{N}^\star \ni k \mapsto K_k(z)$ is strictly decreasing.*

Proof. Let us define the auxiliary function

$$T_k(z) := \int_0^{2\pi} \frac{\cos(k\eta)\, d\eta}{\{z^2 + 1 - 2z\cos(\eta)\}^{\gamma/2}}. \tag{5.4}$$

Then K_k can be written as

$$K_k(z) = \frac{C_\gamma}{\mu 4\pi C_0(\alpha-1)} z^{-\frac{\alpha-1}{1+\alpha}} T_k(z^{\frac{-1}{1+\alpha}}). \tag{5.5}$$

Hence, the monotonicity of $k \mapsto K_k(z)$ with $z \in (0, \infty)$ reduces to the monotonicity of $k \mapsto T_k(z)$ with $z \in (0, \infty)$, which is true by [14, Lemma 4.11]. ∎

5.2. *Mellin transform*

Mellin transforms are useful when dealing with integral differential equations where the integral operator is of the type (5.3) due to the product property (B.4):

$$M[J_k](s) = M[K](s) M[g_k](s+1). \tag{5.6}$$

Hence, applying the Mellin transform to the expression given in Proposition 5.1 we obtain the following equivalent equation.

Proposition 5.3. *The equation* $\tilde{\mathcal{L}}_k(g_k) = 0$ *amounts to*

$$M[\tilde{\mathcal{L}}_k(g_k)](s) = \frac{1+\alpha}{2}(1-s)G_k(s) - ik\left(\tilde{K}_k(s) - \frac{1+\alpha}{2}\right)G_k(s+1) = 0, \quad (5.7)$$

where $G_k = M[g_k]$ *and* $\tilde{K}_k = M[K_k]$.

Proof. The proof is based on the properties (B.4)–(B.6) together with (5.6). ∎

Remark 5.4. If $h_k \in \mathcal{X}_0$, then $M[g_k]$ is well-defined in

$$\{s \in \mathbb{C} \mid -\sigma < \mathrm{Re}(s) < \sigma\}.$$

Remark 5.5. Let us again remark that the kernel K_k is symmetric with respect to k: $K_{-k} = K_k$. Hence, if $k < 0$ then (5.7) reads

$$M[\tilde{\mathcal{L}}_k(g_k)](s) = \frac{1+\alpha}{2}(1-s)G_k(s) + ik\left(\tilde{K}_{|k|}(s) - \frac{1+\alpha}{2}\right)G_k(s+1) = 0.$$

In that way, (5.7) can be written as

$$G_k(s+1) = \frac{1}{k}F_k(s)G_k(s), \quad (5.8)$$

where

$$F_k(s) := i\frac{1-s}{1 - \frac{2}{1+\alpha}\tilde{K}_k(s)}.$$

In the following, let us give a better expression for F_k together with some properties. As a consequence, we shall look for the values of s such that (5.8) is well-defined.

Proposition 5.6. *The Mellin transform of* K_k *can be written as follows:*

$$\tilde{K}_k(s) = \frac{C_\gamma(1+\alpha)}{\mu 4\pi C_0(\alpha-1)}M[T_k(z)](-(1+\alpha)s + \alpha - 1)$$

for

$$\frac{\alpha - 1 - k - \gamma}{1+\alpha} < s < \frac{\alpha - 1 + k}{1+\alpha},$$

where T_k *is defined in* (5.4). *Moreover,*

$$\lim_{s\to\frac{\alpha-1-k-\gamma}{1+\alpha}} \tilde{K}_k(s) = \lim_{s\to\frac{\alpha-1+k}{1+\alpha}} \tilde{K}_k(s) = \infty$$

for $k \in \mathbb{N}^\star$.

Proof. We will use some properties of the Mellin transform described in Appendix B. Recalling (5.5), by (B.5) we get

$$M[K_k](s) = \tilde{K}_k(s) = \frac{C_\gamma}{\mu 4\pi C_0(\alpha-1)}M[T_k(z^{\frac{-1}{1+\alpha}})]\left(s - \frac{\alpha-1}{1+\alpha}\right).$$

Moreover, by (B.8),

$$M[T_k(z^{\frac{-1}{1+\alpha}})](s) = (1+\alpha)M[T_k(z)](-(1+\alpha)s),$$

and hence

$$M[T_k(z^{\frac{-1}{1+\alpha}})]\left(s - \frac{\alpha-1}{1+\alpha}\right) = (1+\alpha)M[T_k(z)](-(1+\alpha)s + \alpha - 1),$$

which implies

$$\tilde{K}_k(s) = \frac{C_\gamma(1+\alpha)}{\mu 4\pi C_0(\alpha-1)} M[T_k(z)](-(1+\alpha)s + \alpha - 1). \tag{5.9}$$

Let us check where $M[T_k(z)](s)$ is well-defined. Since T_k can be related to the Gauss hypergeometric function by using Lemma B.1, we get

$$\begin{aligned} M[T_k](s) &= \int_0^\infty \int_0^{2\pi} \frac{z^{s-1}\cos(k\eta)\,d\eta\,dz}{\{z^2+1-2z\cos(\eta)\}^{\gamma/2}} \\ &= \frac{2\pi 2^{2k}\left(\frac{\gamma}{2}\right)_k\left(\frac{1}{2}\right)_k}{(2k)!}\int_0^\infty \frac{z^{s-1+k}}{(1+z)^{\gamma+2k}}\,{}_2F_1\left(k+\frac{\gamma}{2}, k+\frac{1}{2}, 2k+1, \frac{4z}{(1+z)^2}\right)dz. \end{aligned} \tag{5.10}$$

Since the parameters of the Gauss hypergeometric function are positive, this function increases and so for any $k \in \mathbb{N}^\star$, $\gamma \in (0,1)$ and $z \in (0,\infty)$ one has

$$1 < F\left(k+\frac{\gamma}{2}, k+\frac{1}{2}, 2k+1, \frac{4z}{(1+z)^2}\right) < {}_2F_1\left(k+\frac{\gamma}{2}, k+\frac{1}{2}, 2k+1, 1\right) < \infty.$$

Hence $M[T_k](s)$ is well-defined iff $s \in (-k, k+\gamma)$, a condition coming from the integrability of the integral. Moreover

$$M[T_k](s) \geq \frac{2\pi 2^{2k}\left(\frac{\gamma}{2}\right)_k\left(\frac{1}{2}\right)_k}{(2k)!}\int_0^\infty \frac{z^{s-1+k}}{(1+z)^{\gamma+2k}}\,dz, \tag{5.11}$$

implying

$$\lim_{s\to -k} T_k(s) = \lim_{s\to k+\gamma} T_k(s) = \infty.$$

Coming back to $\tilde{K}$ through (5.9), we find that $\tilde{K}$ is well-defined iff

$$\frac{\alpha-1-k-\gamma}{1+\alpha} < s < \frac{\alpha-1+k}{1+\alpha},$$

and

$$\lim_{s\to\frac{\alpha-1-k-\gamma}{1+\alpha}} \tilde{K}(s) = \lim_{s\to\frac{\alpha-1+k}{1+\alpha}} \tilde{K}(s) = \infty. \qquad \blacksquare$$

As a consequence of Proposition 5.2 we get the monotonicity of the Mellin transform of K, that is, $\tilde{K}$:

Corollary 5.7. *For any $s \in [0, \frac{\alpha-1+k}{1+\alpha})$, the sequence $\mathbb{N}^\star \ni k \mapsto \tilde{K}_k(s)$ is strictly decreasing.*

In the following proposition, we state the monotonicity of $\tilde{K}_k$ with respect to s.

Proposition 5.8. *The function $[0, \frac{\alpha-1+k}{1+\alpha}) \ni s \mapsto \tilde{K}_k(s)$ increases, for any $k \in \mathbb{N}^\star$. Moreover, $\frac{2}{1+\alpha}\tilde{K}_k(0) < 1$.*

Proof. By the expression of $\tilde{K}$ in Proposition 5.6, we can relate $\tilde{K}_k$ to $M[T_k]$. Hence if $(-k, \alpha-1] \ni s \mapsto M[T_k](s)$ decreases, we deduce that $[0, \frac{\alpha-1+k}{1+\alpha}) \ni s \mapsto \tilde{K}_k(s)$ increases. Moreover, using the explicit expression of T_k we obtain $T_k(z^{-1}) = z^\gamma T_k(z)$. In that way, $M[T_k]$ can be written as follows:

$$\begin{aligned} M[T_k](s) &= \int_0^\infty z^{s-1} T_k(z)\, dz \\ &= \int_0^1 z^{s-1} T_k(z)\, dz + \int_1^\infty z^{s-1} T_k(z)\, dz \\ &= \int_0^1 T_k(z)(z^{s-1} + z^{1-s} z^\gamma z^{-2})\, dz. \end{aligned}$$

Computing the derivative we observe

$$\begin{aligned} M[T_k]'(s) &= \int_0^1 \ln(z) T_k(z)(z^{s-1} - z^{1-s} z^\gamma z^{-2})\, dz \\ &= \int_0^1 \ln(z) T_k(z) z^{s-1}(1 - z^{\gamma-2s})\, dz. \end{aligned}$$

Since $s \in (-k, \alpha-1)$ and α satisfies (3.16), we conclude that $\gamma - 2s > 0$ and hence $M[T_k]'(s) < 0$. As a consequence, $M[T_k]$ is strictly decreasing.

For the "moreover" statement, by Corollary 5.7, it is enough to prove $\frac{2}{1+\alpha}\tilde{K}_1(0) < 1$. Moreover, let us check that $\alpha \mapsto \frac{2}{1+\alpha}\tilde{K}_1(0)$ strictly decreases and hence we can restrict ourselves to $\alpha = 1$. From Proposition 5.1 we get

$$\frac{2}{1+\alpha}\tilde{K}_1(0) = \frac{C_\gamma}{\mu 2\pi C_0(\alpha-1)} M[T_1](\alpha-1).$$

From Proposition A.3 and the expression of μ in (2.2) we get

$$\frac{2}{1+\alpha}\tilde{K}_1(0) = \frac{1+\alpha-\gamma}{(\alpha-1)M[T_0](1-\alpha+\gamma)} M[T_1](\alpha-1).$$

By the previous estimates concerning the monotonicity of $s \mapsto M[T_k](s)$, we know that $(1, 1+\gamma) \ni \alpha \mapsto M[T_1](\alpha-1)$ is strictly decreasing. Moreover, $M[T_0]'(1-\alpha+\gamma) < 0$ by similar arguments. Hence, we claim that

$$g(\alpha) := \frac{1+\alpha-\gamma}{(\alpha-1)M[T_0](1-\alpha+\gamma)}$$

is positive and strictly decreasing. The positivity comes from the fact that $M[T_0]$ is positive. In order to prove the monotonicity, let us differentiate

$$\begin{aligned} g'(\alpha) &= \frac{1}{(\alpha-1)M[T_0](1-\alpha+\gamma)^2} \\ &\quad\times \big((\alpha-1)M[T_0](1-\alpha+\gamma) - (1+\alpha-\gamma)M[T_0](1-\alpha+\gamma) \\ &\qquad + (\alpha-1)(1+\alpha-\gamma)M[T_0]'(1-\alpha+\gamma)\big) \\ &= \frac{-(2-\gamma)M[T_0](1-\alpha+\gamma) + (\alpha-1)(1+\alpha-\gamma)M[T_0]'(1-\alpha+\gamma)}{(\alpha-1)M[T_0](1-\alpha+\gamma)^2}, \end{aligned}$$

which is negative since $\gamma < 1$. Hence, we have proved our claim. Then

$$\begin{aligned} \frac{2}{1+\alpha}\tilde{K}_1(0) &\le \lim_{\alpha\to 1^+} \frac{1+\alpha-\gamma}{(\alpha-1)M[T_0](1-\alpha+\gamma)} M[T_1](\alpha-1) \\ &= \lim_{\alpha\to 1^+} \frac{2-\gamma}{(\alpha-1)M[T_0](1-\alpha+\gamma)} M[T_1](0). \end{aligned}$$

Note that

$$\lim_{\alpha\to 1^+} M[T_0](1-\alpha+\gamma) = \infty,$$

and hence there is a competition in the denominator that we will have to handle. Note that we can again relate $M[T_0]$ to the Gauss hypergeometric function:

$$\begin{aligned} M[T_0](1-\alpha+\gamma) &= 2\pi \int_0^\infty \frac{z^{\gamma-\alpha}}{(1+z)^\gamma}\, {}_2F_1\left(\frac{\gamma}{2}, \frac{1}{2}, 1, \frac{4z}{(1+z)^2}\right) dz \\ &> 2\pi \int_0^\infty \frac{z^{\gamma-\alpha}}{(1+z)^\gamma}\, dz = 2\pi \frac{\Gamma(1+\gamma-\alpha)\Gamma(\alpha-1)}{\Gamma(\gamma)}. \end{aligned}$$

Since $x\Gamma(x) = \Gamma(x+1)$ we get

$$\lim_{\alpha\to 1} (\alpha-1)\Gamma(\alpha-1) = 1,$$

and hence

$$\lim_{\alpha\to 1^+} \frac{2-\gamma}{(\alpha-1)M[T_0](1-\alpha+\gamma)} M[T_1](0) < \frac{2-\gamma}{2\pi} M[T_1](0).$$

Using now the expression (5.10) in terms of the Gauss hypergeometric function, we find

$$M[T_1](0) = \pi\gamma \int_0^\infty \frac{1}{(1+z)^{2+\gamma}}\, {}_2F_1\left(1+\frac{\gamma}{2}, \frac{3}{2}, 3, \frac{4z}{(1+z)^2}\right) dz.$$

By writing the Gauss hypergeometric function in a series, integrating and summing again we find that

$$M[T_1](0) = \pi\gamma \frac{\Gamma\left(-\frac{\gamma}{2}\right)\Gamma\left(\frac{\gamma-1}{2}\right)}{\Gamma\left(\frac{1-\gamma}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)}.$$

Using

$$\Gamma(z) = -\frac{\Gamma(-z)\Gamma(1+z)}{\Gamma(1-z)},$$

we obtain

$$M[T_1](0) = \pi\gamma \frac{\Gamma\left(1-\frac{\gamma}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(1+\frac{\gamma}{2}\right)\Gamma\left(\frac{3}{2}-\frac{\gamma}{2}\right)},$$

and hence

$$\lim_{\alpha\to 1^+} \frac{2-\gamma}{(\alpha-1)M[T_0](1-\alpha+\gamma)} M[T_1](0) < \frac{(2-\gamma)\gamma}{2} \frac{\Gamma\left(1-\frac{\gamma}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(1+\frac{\gamma}{2}\right)\Gamma\left(\frac{3}{2}-\frac{\gamma}{2}\right)} =: p(\gamma).$$

Note that $p(1) = 1$ and if we check that p is strictly increasing, we can arrive at the announced result given that

$$\lim_{\alpha\to 1^+} \frac{2-\gamma}{(\alpha-1)M[T_0](1-\alpha+\gamma)} M[T_1](0) < \frac{(2-\gamma)\gamma}{2} \frac{\Gamma\left(1-\frac{\gamma}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(1+\frac{\gamma}{2}\right)\Gamma\left(\frac{3}{2}-\frac{\gamma}{2}\right)} < p(1) = 1$$

for any $\gamma \in (0, 1)$.

Hence, it remains to check that p is strictly increasing. Note that p can be written as

$$p(\gamma) = 2\frac{\Gamma\left(2-\frac{\gamma}{2}\right)\Gamma\left(\frac{1+\gamma}{2}\right)}{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)},$$

and its derivative reads

$$p'(\gamma) = \frac{\Gamma\left(2-\frac{\gamma}{2}\right)\Gamma\left(\frac{1+\gamma}{2}\right)}{\Gamma\left(\frac{3-\gamma}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)} \left\{\psi^0\left(\frac{3-\gamma}{2}\right) - \psi^0\left(2-\frac{\gamma}{2}\right) - \psi^0\left(\frac{\gamma}{2}\right) + \psi^0\left(\frac{1+\gamma}{2}\right)\right\},$$

where ψ^0 is the digamma function. Now, using the integral representation (B.3) of the digamma function we find

$$\begin{aligned}\psi^0\left(\frac{3-\gamma}{2}\right) - \psi^0\left(2-\frac{\gamma}{2}\right) - \psi^0\left(\frac{\gamma}{2}\right) + \psi^0\left(\frac{1+\gamma}{2}\right)& \\ = \int_0^1 \frac{-t^{\frac{1-\gamma}{2}} - t^{-\frac{1-\gamma}{2}} + t^{\frac{2-\gamma}{2}} + t^{-\frac{2-\gamma}{2}}}{1-t}\, dt&,\end{aligned}$$

where we can prove that the integrand is positive for any $t \in (0, 1)$ since $\gamma \in (0, 1)$. Hence $p'(\gamma) > 0$ for any $\gamma \in (0, 1)$, and the claim holds true. That concludes the proof. ∎

As a consequence of Propositions 5.6 and 5.8, and using the intermediate value theorem, we get the following corollary.

Corollary 5.9. *For any $k \in \mathbb{N}^\star$, there is a unique $s_0 \in [0, \frac{\alpha-1+k}{1+\alpha})$ such that $\frac{2}{1+\alpha}\tilde{K}_k(s_0) = 1$.*

Hence, using the expression of $\tilde{K}$ in Proposition 5.6 we find

$$(1-s)G_k(s) + ik\left(1 - \frac{C_\gamma}{\mu 2\pi C_0(\alpha-1)} M[T_k(z)](-(1+\alpha)s+\alpha-1)\right)G_k(s+1) = 0,$$

amounting to

$$G_k(s+1) = \frac{i}{k}\frac{1-s}{1-\frac{C_\gamma}{\mu 2\pi C_0(\alpha-1)}M[T_k(z)](-(1+\alpha)s+\alpha-1)}G_k(s). \tag{5.12}$$

Note that by Proposition 5.6, $M[T_k(z)](-(1+\alpha)s+\alpha-1)$ is well-defined for

$$\frac{\alpha-1-k-\gamma}{1+\alpha} < s < \frac{\alpha-1+k}{1+\alpha}.$$

Using F_k one has

$$F_k(s) = i\frac{1-s}{1-\frac{C_\gamma}{\mu 2\pi C_0(\alpha-1)}M[T_k(z)](-(1+\alpha)s+\alpha-1)}.$$

Moreover, due to the limits of $\tilde{K}$ in Proposition 5.6, we get

$$\lim_{s\to\frac{\alpha-1-k-\gamma}{1+\alpha}} F_k(s) = \lim_{s\to\frac{\alpha-1+k}{1+\alpha}} F_k(s) = 0,$$

and hence we can extend it to zero outside $(\frac{\alpha-1-k-\gamma}{1+\alpha}, \frac{\alpha-1+k}{1+\alpha})$ due to the bound (5.11). Moreover, in view of (3.16) we have

$$\frac{\alpha-1-\gamma-k}{1+\alpha} < \frac{\alpha-1-\gamma}{1+\alpha} < -\sigma < s < \sigma < \frac{\alpha-1}{1+\alpha} < \frac{\alpha-1+k}{1+\alpha}.$$

Since a priori G_k is well-defined for $s \in (-\sigma,\sigma)$, one finds that (5.12) is well-defined for $s \in (-\sigma,\sigma)$, and so G_k is well-defined for $s \in (-\sigma+1,\sigma+1)$.

Finally, in the following proposition we show that the kernel is trivial. For this, we view (5.12) as a recurrence equation and we relate the solutions to the moment problem. In that way, using the Laplace transform we show that the solution h_k is singular and thus it does not belong to our function spaces, proving that the kernel is trivial.

Proposition 5.10. *If G_k solves (5.12) with $g_k(\beta) = e^{-ik\beta}h_k(\beta)$ and $h_k \in \mathcal{X}_0^{\sigma,\delta}$, then $h_k = 0$.*

Proof. Taking $s = M \in \mathbb{N}^\star$, we can solve the recurrence equation (5.12) as

$$G_k(M) = \frac{1}{k}G_k(0)\prod_{n=1}^{M-1} F_k(n).$$

Note that $G_k(0)$ is well-defined and so

$$G_k(1) = \frac{1}{k}F_k(0)G_k(0),$$

which is finite due to Proposition 5.8.

First, assume that $\frac{2}{1+\alpha}\tilde{K}_k(N) \neq 1$ for any $N \in \mathbb{N}$ with $N \leq \frac{\alpha-1+k}{1+\alpha}$. In that case, since $F_k(1) = 0$ we get $G_k(M) = 0$ for any $M \geq 2$. Let $\mathcal{L}$ be the Laplace transform operator. Thus

$$\begin{aligned}\mathcal{L}[g_k](t) &= \int_0^\infty e^{-tz} g_k(z)\, dz = \sum_{n=0}^\infty \frac{(-t)^n}{n!} \int_0^\infty z^n g_k(z)\, dz \\ &= \sum_{n=0}^\infty \frac{(-t)^n}{n!} G_k(n+1) = G_k(1).\end{aligned}$$

Since the inverse Laplace transform of a constant is a Dirac mass and g_k is continuous, we see that necessarily $g_k = 0$.

Second, assume that there exists $N_0 \in \mathbb{N}$ with $N_0 \in [1, \frac{\alpha-1+k}{1+\alpha})$ such that

$$\frac{2}{1+\alpha}\tilde{K}_k(N_0) = 1.$$

In that case $\frac{2}{1+\alpha}\tilde{K}_k$ has a simple root at N_0. On the one hand, if $N_0 = 1$ we have

$$\lim_{s\to 1} \frac{1}{i} F_k(s) = \lim_{s\to 1} \frac{s-1}{\frac{2}{1+\alpha}\tilde{K}_k(s) - 1} \in \mathbb{R},$$

since the root is simple. In that case $G_k(2)$ is well-defined and we can write

$$G_k(M) = \frac{1}{k} G_k(0) \prod_{n=1}^{M-1} F_k(M).$$

Moreover, note that for $M \geq \frac{\alpha-1+k}{1+\alpha}$ we have $F_k(M) = 0$, implying that $G_k(M) = 0$ for such M. Hence the Laplace transform of g_k is a polynomial. On the other hand, if $N_0 \geq 2$, we have

$$\lim_{s\to N_0} F_k(1) F_k(s) \in \mathbb{R},$$

since $F_k(1) = 0$ and the pole of F_k at N_0 is simple. In that case, we find again that the Laplace transform of g_k is a polynomial. Now, by using the expression of the Laplace transform we find

$$\mathcal{L}[g_k](t) = \int_0^\infty e^{-tz} g_k(z)\, dz.$$

Since $g_k \in L^\infty([0,\infty))$ we get

$$\lim_{t\to\infty} |\mathcal{L}[g_k](t)| = 0,$$

which contradicts the fact that the Laplace transform is a polynomial. Hence $g_k = 0$. ■

6. *m*-fold symmetry and main theorem in the variables (β, ϕ)

In this section, we introduce the m-fold symmetry to the function spaces, and then we will work with the extra parameter m. Moreover, we will give the main result in the adapted coordinates which is a consequence of the infinite-dimensional implicit function theorem applied to $\tilde{F}$.

6.1. m-fold symmetry in function spaces

In the following, we introduce the m-fold symmetry in the function spaces (3.5), (3.7) and (3.8):

$$\begin{aligned}
X^{\sigma,\delta}_{1,m} &:= \{f \in X^{\sigma,\delta}_1 \mid f(\beta,\phi+2\pi/m) = f(\beta,\phi),\ (\beta,\phi) \in [0,\infty)\times[0,2\pi]\},\\
X^{p}_{2,m} &:= \{f \in X^{p}_2 \mid f(\phi+2\pi/m) = f(\phi),\ \phi \in [0,2\pi]\},\\
Y^{\sigma,\delta}_{m} &:= \{f \in Y^{\sigma,\delta} \mid f(\beta,\phi+2\pi/m) = f(\beta,\phi),\ (\beta,\phi) \in [0,\infty)\times[0,2\pi]\}.
\end{aligned}$$

Note that $f \in X^{\sigma,\delta}_{1,m}$ can be written as a Fourier series

$$f(\beta,\phi) = \sum_{k\in\mathbb{Z}} f_k(\beta)e^{ikm\phi}.$$

Since the only variation that we have introduced in the function spaces is m-fold symmetry, all the previous analysis also works here. We only need to check the symmetry persistence of the nonlinear functional.

Proposition 6.1. *If $f(\beta,\phi+2\pi/m) = f(\beta,\phi)$ and $\Omega(\phi+2\pi/m) = \Omega(\phi)$ then*

$$\tilde{F}(f,\Omega)(\beta,\phi+2\pi/m) = \tilde{F}(f,\Omega)(\beta,\phi).$$

Proof. Let us write $\tilde{F}(f,\Omega)(\beta,\phi+2\pi/m)$ using (2.26):

$$\tilde{F}(f,\Omega)(\beta,\phi+2\pi/m) = 1 + f(\beta,\phi+2\pi/m) - (-1)^{\frac{-1}{2\mu}} \frac{C_\gamma(\alpha-1)^{\frac{\alpha-1}{2}}}{4\pi C_0(1+\alpha)^{\frac{\alpha-1}{2}}}\beta^{-\frac{\alpha-1}{1+\alpha}}$$
$$\times \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} \{-\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\varphi}} f(b,\Phi)\}^{-\frac{1}{2\mu}} \{\frac{2(\alpha-1)}{(1+\alpha)^2} + \partial_{\overline{\beta\varphi}} f(b,\Phi)\}\tilde{\Omega}(\Phi)\, db\, d\Phi}{D(f)\big(\beta,\phi+\frac{2\pi}{m},b,\Phi\big)^{\gamma/2}}.$$

The change of variables $\Phi \mapsto \Phi + 2\pi/m$ yields

$$\tilde{F}(f,\Omega)(\beta,\phi+2\pi/m) = 1 + f(\beta,\phi+2\pi/m) - (-1)^{\frac{-1}{2\mu}} \frac{C_\gamma(\alpha-1)^{\frac{\alpha-1}{2}}}{4\pi C_0(1+\alpha)^{\frac{\alpha-1}{2}}}\beta^{-\frac{\alpha-1}{1+\alpha}}$$
$$\times \iint \frac{b^{-\frac{\gamma}{1+\alpha}} b^{\frac{-2}{1+\alpha}} \{-\frac{\alpha-1}{1+\alpha} + \partial_{\overline{\varphi}} f(b,\Phi+2\pi/m)\}^{-\frac{1}{2\mu}} \{\frac{2(\alpha-1)}{(1+\alpha)^2} + \partial_{\overline{\beta\varphi}} f(b,\Phi+2\pi/m)\}}{D(f)\big(\beta,\phi+2\pi/m,b,\Phi+2\pi/m\big)^{\gamma/2}}$$
$$\times \tilde{\Omega}(\Phi+2\pi/m)\, db\, d\Phi.$$

Using $f(\beta,\phi+2\pi/m) = f(\beta,\phi)$ and $\Omega(\phi+2\pi/m) = \Omega(\phi)$, and the expression of $D(f)$ in (2.27), we deduce that $\tilde{F}(f,\Omega)(\beta,\phi+2\pi/m) = \tilde{F}(f,\Omega)(\beta,\phi)$, concluding the proof. ∎

Let us remark that the spectral study developed in Section 5 follows similarly – one just has to change $k \mapsto km$.

6.2. *Main theorem in the adapted coordinates*

Finally, let us state the main theorem in the adapted coordinates, which is a consequence of the infinite-dimensional implicit function theorem.

Theorem 6.2. *Let $\gamma, \sigma \in (0,1)$ and $\alpha \in (1, 1+\gamma)$ satisfy (3.16). Then, for each $m \geq 1$, there exists $\varepsilon > 0$ and a C^1 function $B_{X^p_{2,m}}(1,\varepsilon) \ni \tilde{\Omega} \mapsto \tilde{f}(\tilde{\Omega}) \in X^{\sigma,\delta}_{1,m}$ such that*

$$\tilde{F}(f, \tilde{\Omega}) = 0,\ (f, \tilde{\Omega}) \in B_{X^{\sigma,\delta}_{1,m}}(0,\varepsilon) \times B_{X^p_{2,m}}(1,\varepsilon) \iff f = \tilde{f}(\Omega),$$

for $\delta < \min\{1-\gamma, \sigma\}$ and $p > \frac{1}{1-\gamma}$.

Proof. From Propositions 3.10 and 6.1 we know that $\tilde{F} : B_{X^{\sigma,\delta}_{1,m}}(0,\varepsilon) \times B_{X^p_{2,m}}(1,\varepsilon) \to Y^{\sigma,\delta}_m$ is well-defined and C^1. Moreover, by Proposition 2.6 we have $\tilde{F}(0,1) = 0$. Furthermore, $\partial_f \tilde{F}(0,1)$ is Fredholm of index zero by Proposition 4.4 and has a trivial kernel by (5.10). Hence, $\partial_f \tilde{F}(0,1)$ is an isomorphism. An application of the infinite-dimensional implicit function theorem to $\tilde{F}$ concludes the proof. ∎

Remark 6.3. A priori Ψ and Ω are complex functions due to the definitions of the function spaces, but we are interested in real functions. That is not a problem since assuming in the function space that f and $\tilde{\Omega}$ are real functions we find that the nonlinear operator is also a real function.

Remark 6.4. Taking $\tilde{f}$ constructed in Theorem 6.2, we find

$$\begin{aligned}\Psi(\beta,\phi) &= -C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\beta^{\frac{\alpha-1}{1+\alpha}}(1+\tilde{f}(\tilde{\Omega})(\beta,\phi)),\\ \Omega(\phi) &= (1+\alpha)^{-\frac{1}{2\mu}}\tilde{\Omega}(\phi),\end{aligned}$$

satisfying $F(\Psi,\Omega) = 0$. Moreover, we can come back to Θ and find

$$\Theta(\beta,\phi) = (\Psi_\varphi)^{-\frac{1+\alpha-\gamma}{2}}\Omega(\phi),$$

which together with Ψ satisfies (2.12), which is the vorticity equation in the adapted coordinates (β,ϕ). Note that Θ is just a perturbation of the trivial solution. Indeed, we can compute Ψ_φ as

$$\begin{aligned}\Psi_\varphi(\beta,\phi) &= -C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\beta^{\frac{-2}{1+\alpha}}\left(-\frac{\alpha-1}{1+\alpha} - \frac{\alpha-1}{1+\alpha}\tilde{f}(\tilde{\Omega})(\beta,\phi) + \beta\tilde{f}_\varphi(\tilde{\Omega})(\beta,\phi)\right)\\ &= -C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\beta^{\frac{-2}{1+\alpha}}\left(-\frac{\alpha-1}{1+\alpha} + \tilde{f_{\bar{\varphi}}}(\tilde{\Omega})(\beta,\phi)\right),\end{aligned} \tag{6.1}$$

where recall that $\tilde{f_\varphi}(\tilde{\Omega})$ decays at 0 and ∞ in β. Hence Ψ_φ is singular at $\beta = 0$ as the trivial solution, and decays at $\beta \to \infty$ as the trivial solution. Coming back to Θ we get

$$\begin{aligned}&\Theta(\beta,\phi)\\ &= (1+\alpha)^{-\frac{1}{2\mu}}\left[C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\right]^{-\frac{1+\alpha-\gamma}{2}}\beta^{\frac{1+\alpha-\gamma}{1+\alpha}}\left(\frac{\alpha-1}{1+\alpha} - \tilde{f_{\bar{\varphi}}}(\tilde{\Omega})(\beta,\phi)\right)^{-\frac{1+\alpha-\gamma}{2}}\tilde{\Omega}(\phi).\end{aligned}$$

Note that $1+\alpha-\gamma>0$ and hence $\Theta(0,\phi)=0$ for any $\phi\in[0,2\pi]$. In particular,

$$\Theta(\beta,\phi)\sim\beta^{\frac{1+\alpha-\gamma}{1+\alpha}},\quad \beta\sim 0,\infty.$$

7. Recovering the solution in the original coordinates

This section is devoted to recovering the solution in the physical variables $x\in\mathbb{R}^2$ and hence giving the associated Theorem 6.2 in those variables. First, we study the invertibility of the change of coordinates, which is a nonlinear implicit change of variables, and later we state the main result.

7.1. Invertibility of the change of coordinates

In order to come back to the physical variable $z\in\mathbb{R}^2$ we should invert the adapted coordinates. From Proposition 2.2 we have

$$r(\beta,\phi)=(-(1+\alpha)\Psi_\beta)^{1/2},\quad \vartheta(\beta,\phi)=\beta+\phi. \tag{7.1}$$

In the following proposition, we study the regularity of $r(\beta,\phi)$ and $\vartheta(\beta,\phi)$. For that, we define the following function space:

$$L_0^{\infty,\sigma}:=\{g=g(0)+\tilde{g}\mid \tilde{g}\in L^{\infty,\sigma}\}.$$

Proposition 7.1. *Let* $(r,\vartheta):[0,\infty)\times[0,2\pi]\to[0,\infty)\times[0,2\pi]$ *be defined by* (7.1). *There exists* $\varepsilon<1$ *such that if* $f\in B_{X_1^{\sigma,\delta}}(0,\varepsilon)$ *then* $\vartheta\in C^\infty([0,\infty)\times[0,2\pi])$ *and*

$$\beta^{\frac{-1}{1+\alpha}}r\in L_0^{\infty,\sigma}\cap C_\beta^\delta\cap \mathrm{Lip}_\phi,\quad \beta^{\frac{-1}{1+\alpha}}\beta r_\varphi\in L^{\infty,\sigma}\cap C_\beta^\delta.$$

Proof. First, note that $\vartheta\in C^\infty([0,\infty)\times[0,2\pi])$. Next, from (6.1) we find

$$(-\Psi_\beta)^{1/2}(\beta,\phi)=C_0^{\frac{1}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{2(1+\alpha)}}\beta^{\frac{-1}{1+\alpha}}\left(\frac{\alpha-1}{1+\alpha}+f_{\overline{\beta}}(\beta,\phi)\right)^{1/2}. \tag{7.2}$$

Let us check the regularity of this function, which will give us the regularity of r due to (7.1). Denote

$$g(\beta,\phi):=\left(\frac{\alpha-1}{1+\alpha}+f_{\overline{\beta}}(\beta,\phi)\right)^{1/2}=\left(\frac{\alpha-1}{1+\alpha}+\frac{\alpha-1}{1+\alpha}f(\beta,\phi)+\beta f_\beta(\beta,\phi)\right)^{1/2}.$$

Since $f\in X_1^{\sigma,\delta}$, we have $f,\beta f_\beta\in L^{\infty,\sigma}\cap C_\beta^\delta\cap\mathrm{Lip}_\phi$. Hence, denoting

$$g_0=g(0,\phi)=\left(\frac{\alpha-1}{1+\alpha}\right)^{1/2},$$

we get

$$g-g_0\in L^{\infty,\sigma},\quad g\in C_\beta^\delta\cap\mathrm{Lip}_\phi,$$

and so

$$\beta^{\frac{-1}{1+\alpha}} r \in L_0^{\infty,\sigma} \cap C_\beta^\delta \cap \mathrm{Lip}_\phi.$$

Hence, we find the following asymptotics for g:

$$g \sim_{0,\infty} g_0 + \frac{\beta^\sigma}{(1+\beta)^{2\sigma}}.$$

Coming back to $(\Psi_\beta)^{1/2}$ we get

$$\beta^{\frac{1}{1+\alpha}} (\Psi_\beta)^{1/2} \in L_0^{\infty,\sigma} \cap C_\beta^\delta \cap \mathrm{Lip}_\phi,$$

with the following asymptotics:

$$(\Psi_\beta)^{1/2} \sim_{0,\infty} \beta^{\frac{-1}{1+\alpha}} \left(1 + \frac{\beta^\sigma}{(1+\beta)^{2\sigma}}\right),$$

and

$$(\Psi_\beta)^{1/2} \sim_{0,\infty} \beta^{\frac{-1}{1+\alpha}}.$$

We can also compute $\partial_\varphi g$:

$$\partial_\varphi g(\beta,\phi) = \frac{\frac{\alpha-1}{1+\alpha} f_\varphi(\beta,\phi) + \beta f_{\beta\varphi}(\beta,\phi)}{2\big(\frac{\alpha-1}{1+\alpha} + \frac{\alpha-1}{1+\alpha} f(\beta,\phi) + \beta f_\beta(\beta,\phi)\big)^{1/2}}.$$

Since $\beta f_\varphi, \beta^2 f_{\beta,\varphi} \in L^{\infty,\sigma} \cap C_\beta^\delta$ we obtain

$$\beta \partial_\varphi g \in L^{\infty,\sigma} \cap C_\beta^\delta,$$

implying

$$\beta^{\frac{-1}{1+\alpha}} \beta \partial_\varphi (\Psi_\beta)^{1/2} \in L^{\infty,\sigma} \cap C_\beta^\delta.$$

This yields the regularity as in the statement. ∎

In the next proposition, we show that if $f \in B_{X_1^{\sigma,\delta}}(0,\varepsilon)$ then the Jacobian of the transformation $(\beta,\phi) \mapsto (z_1,z_2)$ is nonvanishing except at $\beta = \infty$.

Proposition 7.2. *There exists $\varepsilon < 1$ such that if $f \in B_{X_1^{\sigma,\delta}}(0,\varepsilon)$ then the Jacobian $|J|$ of the transformation $(\beta,\phi) \mapsto (z_1,z_2)$ does not vanish for $(\beta,\phi) \in (0,\infty) \times [0,2\pi]$.*

Proof. From Proposition 2.3 we have

$$|J| = \frac{1+\alpha}{2} \Psi_{\beta\varphi}.$$

Writing Ψ in terms of f and using (6.1) we find

$$\Psi_{\beta\varphi} = -C_0^{\frac{2}{1+\alpha}} (\alpha-1)^{\frac{1-\alpha}{1+\alpha}} \beta^{-1} \beta^{\frac{-2}{1+\alpha}} \left\{ \frac{2(\alpha-1)}{(1+\alpha)^2} + \partial_{\overline{\beta\varphi}} f(\beta,\phi) \right\}.$$

Note that

$$\frac{2(\alpha-1)}{(1+\alpha)^2} + \partial_{\overline{\beta\varphi}} f(\beta,\phi) \geq \frac{2(\alpha-1)}{(1+\alpha)^2} - \|f\|_{X_1^{\sigma,\delta}} \geq \varepsilon_0 > 0,$$

where we have used that $f \in B_{X_1^{\sigma,\delta}}(0,\varepsilon)$. This, together with $\beta \in (0,\infty)$, yields $|J| > 0$. Moreover, notice that

$$\lim_{\beta\to 0} |J| = \infty, \qquad \lim_{\beta\to\infty} |J| = 0. \qquad \blacksquare$$

In view of Proposition 7.2 we can invert the change of coordinates and find $(\beta(z_1,z_2),\phi(z_1,z_2))$, for $(\beta,\phi) \in (0,\infty)\times[0,2\pi]$. Since the Jacobian of $(z_1,z_2) \mapsto (\beta,\phi)$ is $|J|^{-1}$, which vanishes at $\beta = 0$, we will identify that point with $(z_1,z_2) = (0,0)$.

Indeed, we shall first use the polar coordinates and then the main task is to invert

$$(r(\beta,\phi),\vartheta(\beta,\phi)) = (\tilde{r},\tilde{\vartheta}),$$

finding $\beta = \beta(\tilde{r},\tilde{\vartheta})$ and $\phi = \phi(\tilde{r},\tilde{\vartheta})$, which is possible due to Proposition 7.2 by using the inverse function theorem. More precisely, in the next proposition we understand better the functions $\beta(r,\vartheta)$ and $\phi(r,\vartheta)$. For that, we define

$$Z^{\sigma,\alpha} := \{\tilde{\beta} \in C_b([0,\infty)\times[0,2\pi]) \mid \tilde{\beta}, r\partial_r\tilde{\beta} \in L^{\infty,\sigma(1+\alpha)}\}.$$

Proposition 7.3. *Let $(r,\vartheta) : [0,\infty)\times[0,2\pi] \to [0,\infty)\times[0,2\pi]$ be as in (7.1). There exists $\varepsilon > 0$ such that for $f \in B_{X_1^{\sigma,\delta}}(0,\varepsilon)$ the following is satisfied. There exists a function $\tilde{\beta} \in Z^{\sigma,\alpha}$ such that defining*

$$\beta(\tilde{r},\tilde{\vartheta}) = r^{-1-\alpha}(1+\tilde{\beta}(\tilde{r},\tilde{\vartheta})), \quad \phi(\tilde{r},\tilde{\vartheta}) = \tilde{\vartheta} - \beta(\tilde{r},\tilde{\vartheta}),$$

we find

$$(r(\beta(\tilde{r},\tilde{\vartheta}),\phi(\tilde{r},\tilde{\vartheta})),\vartheta(\beta(\tilde{r},\tilde{\vartheta}),\phi(\tilde{r},\tilde{\vartheta}))) = (\tilde{r},\tilde{\vartheta})$$

for any $(\tilde{r},\tilde{\vartheta}) \in [0,\infty)\times[0,2\pi]$.

Proof. Using (7.1), our main task is to invert

$$\left((-(1+\alpha)\Psi_\beta)^{1/2}, \beta+\phi\right) = (r,\vartheta),$$

for any $(r,\vartheta) \in [0,\infty)\times[0,2\pi]$. Hence, we want to find two functions $\beta,\phi : [0,\infty)\times[0,2\pi] \to [0,\infty)\times[0,2\pi]$ such that

$$(-(1+\alpha)\Psi_\beta(\beta(r,\vartheta),\phi(r,\vartheta))^{1/2} = r, \quad \beta(r,\vartheta)+\phi(r,\vartheta) = \vartheta.$$

From the second equation we find

$$\phi(r,\vartheta) = \vartheta - \beta(r,\vartheta), \tag{7.3}$$

and inserting it in the first equation we shall solve

$$(-(1+\alpha)\Psi_\beta(\beta(r,\vartheta),\vartheta-\beta(r,\vartheta)))^{1/2} - r = 0 \tag{7.4}$$

for any $(r,\vartheta) \in [0,\infty)\times[0,2\pi]$. Let us recall that Ψ is indeed a perturbation of the trivial solution, and by (7.2) we find

$$(-\Psi_\beta)^{1/2}(\beta,\phi) = C_0^{\frac{1}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{2(1+\alpha)}}\beta^{\frac{-1}{1+\alpha}}\left(\frac{\alpha-1}{1+\alpha} + f_{\overline{\beta}}(\beta,\phi)\right)^{1/2}$$

for $f \in B_{X_1^{\sigma,\delta}}(0,\varepsilon)$. Hence, (7.4) amounts to

$$(1+\alpha)^{1/2} C_0^{\frac{1}{1+\alpha}} (\alpha-1)^{\frac{1-\alpha}{2(1+\alpha)}} (\beta(r,\vartheta))^{\frac{-1}{1+\alpha}} \\ \times \left(\frac{\alpha-1}{1+\alpha} + f_{\tilde{\beta}}(\beta(r,\vartheta), \vartheta-\beta(r,\vartheta))\right)^{1/2} - r = 0. \quad (7.5)$$

Define now

$$\beta_0(r) = C_0(\alpha-1) r^{-1-\alpha},$$

agreeing with the change of coordinates for the trivial solution, which was given in Proposition 2.6. Hence, decomposing β as

$$\beta = \beta_0(1+\tilde{\beta}),$$

we can write (7.5) as $G(\tilde{\beta}, f)(r,\vartheta) = 0$ for any $(r,\vartheta) \in (0,\infty) \times [0,2\pi]$, where

$$G(\tilde{\beta}, f)(r,\vartheta) \\ := \frac{(1+\alpha)^{1/2}}{(\alpha-1)^{1/2}} (1+\tilde{\beta}(r,\vartheta))^{\frac{-1}{1+\alpha}} \left(\frac{\alpha-1}{1+\alpha} + f_{\tilde{\beta}}(\beta(r,\vartheta), \vartheta-\beta(r,\vartheta))\right)^{1/2} - 1. \quad (7.6)$$

We claim that there exists $\varepsilon < 1$ such that $G : B_{Z^{\sigma,\alpha}}(0,\varepsilon) \times B_{X_1^{\sigma,\delta}(0,\varepsilon)} \to Z^{\sigma,\alpha}$ is well-defined and C^1. Hence, we would like to find nontrivial $(\tilde{\beta}, f)$ such that $G(\tilde{\beta}, f)(r) = 0$ for any $r \in [0,\infty)$. We have the trivial solution given by

$$G(0,0)(r,\vartheta) = 0, \quad \forall (r,\vartheta) \in [0,\infty) \times [0,2\pi].$$

We wish to apply the implicit function theorem and thus we need to compute the linearized operator:

$$\partial_{\tilde{\beta}} G(0,0) h(r,\vartheta) = -\frac{1}{1+\alpha} h(r,\vartheta),$$

which is clearly an isomorphism from $Z^{\sigma,\alpha}$ into $Z^{\sigma,\alpha}$.

Finally, it remains to show that G is well-defined. For that, let us write

$$G(\tilde{\beta}, f)(r) \\ = \frac{(1+\alpha)^{1/2}}{(\alpha-1)^{1/2}} \frac{(1+\tilde{\beta}(r,\vartheta))^{\frac{-2}{1+\alpha}} \left(\frac{\alpha-1}{1+\alpha} + f_{\tilde{\beta}}(\beta(r,\vartheta), \vartheta-\beta(r,\vartheta))\right) - \frac{\alpha-1}{1+\alpha}}{(1+\tilde{\beta}(r,\vartheta))^{\frac{-1}{1+\alpha}} \left(\frac{\alpha-1}{1+\alpha} + f_{\tilde{\beta}}(\beta(r,\vartheta), \vartheta-\beta(r,\vartheta))\right)^{1/2} + \frac{(\alpha-1)^{1/2}}{(1+\alpha)^{1/2}}} \\ = \frac{(1+\alpha)^{1/2}}{(\alpha-1)^{1/2}} \frac{[(1+\tilde{\beta}(r,\vartheta))^{\frac{-2}{1+\alpha}} - 1]\left(\frac{\alpha-1}{1+\alpha} + f_{\tilde{\beta}}(\beta(r,\vartheta), \vartheta-\beta(r,\vartheta))\right)}{(1+\tilde{\beta}(r,\vartheta))^{\frac{-1}{1+\alpha}} \left(\frac{\alpha-1}{1+\alpha} + f_{\tilde{\beta}}(\beta(r,\vartheta), \vartheta-\beta(r,\vartheta))\right)^{1/2} + \frac{(\alpha-1)^{1/2}}{(1+\alpha)^{1/2}}} \\ + \frac{(1+\alpha)^{1/2}}{(\alpha-1)^{1/2}} \frac{f_{\tilde{\beta}}(\beta(r,\vartheta), \vartheta-\beta(r,\vartheta))}{(1+\tilde{\beta}(r,\vartheta))^{\frac{-1}{1+\alpha}} \left(\frac{\alpha-1}{1+\alpha} + f_{\tilde{\beta}}(\beta(r,\vartheta), \vartheta-\beta(r,\vartheta))\right)^{1/2} + \frac{(\alpha-1)^{1/2}}{(1+\alpha)^{1/2}}}.$$

First, note that $f_{\bar{\beta}} \in L^{\infty,\sigma}$. Assuming that $\tilde{\beta} \in L^{\infty}$, we find that

$$\beta(r,\vartheta) \sim_{0,\infty} r^{-1-\alpha},$$

and therefore

$$f_{\bar{\beta}}(\beta(r,\vartheta),\vartheta-\beta(r,\vartheta)) \sim_{0,\infty} \frac{r^{\sigma(1+\alpha)}}{(1+r)^{2\sigma(1+\alpha)}}. \tag{7.7}$$

Moreover, for ε small and $\tilde{\beta} \in B_{L^{\infty,\sigma(1+\alpha)}}(0,\varepsilon)$, all the denominators in the expression of G are bounded. Using (7.7) we find that $G(\tilde{\beta}, f) \in L^{\infty,\sigma(1+\alpha)}$.

Since in the expression of G we have $\partial_{\bar{\beta}} f(\beta(r,\vartheta),\vartheta-\beta(r,\vartheta))$, we can only take one derivative in r. Indeed,

$$\partial_r(\partial_{\bar{\beta}} f(\beta(r,\vartheta),\vartheta-\beta(r,\vartheta))) = (\partial_\varphi \partial_{\bar{\beta}} f)(\beta(r,\vartheta),\vartheta-\beta(r,\vartheta))\partial_r\beta(r,\vartheta).$$

Since $\beta(\partial_\varphi \partial_{\bar{\beta}} f) \in L^{\infty,\sigma}$ we find that

$$r\partial_r(\partial_{\bar{\beta}} f(\beta(r,\vartheta),\vartheta-\beta(r,\vartheta))) \sim_{0,\infty} \frac{r^{\sigma(1+\alpha)}}{(1+r)^{2\sigma(1+\alpha)}}.$$

That motivates us to include $r\partial_r\tilde{\beta} \in L^{\infty,\sigma(1+\alpha)}$ in the space $Z^{\sigma,\alpha}$. In such a space, G is well-defined and we conclude the proof by applying the implicit function theorem.

By (7.3) we find that $\phi \in Z_0^{\sigma,\alpha}$, where

$$Z_0^{\sigma,\alpha} := \{\tilde{\beta} \in C_b([0,\infty)\times[0,2\pi]) \mid \tilde{\beta}, r\partial_r\tilde{\beta} \in L_0^{\infty,\sigma(1+\alpha)}\}. \qquad \blacksquare$$

7.2. *Solution in the physical variables*

From Theorem 6.2 together with Remark 6.4 we have constructed solutions (Ψ,Θ) to (2.12), which is the vorticity equation in the adapted coordinates, and (2.19) which refers to the elliptic relation between Ψ and Θ. In particular, we find (Ψ,Θ) as

$$\begin{aligned}
\Psi(\beta,\phi) &= -C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\beta^{\frac{\alpha-1}{1+\alpha}}(1+\tilde{f}(\tilde{\Omega})(\beta,\phi)),\\
\Theta(\beta,\phi) &= (1+\alpha)^{-\frac{1}{2\mu}}\left[C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\right]^{-\frac{1+\alpha-\gamma}{2}}\beta^{\frac{1+\alpha-\gamma}{1+\alpha}}\\
&\quad\times\left(\frac{\alpha-1}{1+\alpha}-\tilde{f}_{\bar{\varphi}}(\tilde{\Omega})(\beta,\phi)\right)^{-\frac{1+\alpha-\gamma}{2}}\tilde{\Omega}(\phi),
\end{aligned}$$

where $\tilde{\Omega} \in B_{X_2^p}(1,\varepsilon)$ and $f(\tilde{\Omega}) \in B_{X_1^{\sigma,\delta}}(0,\varepsilon)$.

Let us now come back to $\hat{\theta}$ by inverting the adapted coordinates using Proposition 7.3. Denote

$$\tilde{\theta}(r,\vartheta) = \hat{\theta}(re^{i\vartheta})$$

and so

$$
\begin{aligned}
&\tilde{\theta}(r,\vartheta) = \Theta(\beta(r,\vartheta),\phi(r,\vartheta)) \\
&= \Theta\big(C_0(\alpha-1)r^{-1-\alpha}(1+\tilde{\beta}(r,\vartheta)),\vartheta - C_0(\alpha-1)r^{-1-\alpha}(1+\tilde{\beta}(r,\vartheta))\big) \\
&= (1+\alpha)^{-\frac{1}{2\mu}}(\alpha-1)^{\frac{1+\alpha-\gamma}{2}} r^{-(1+\alpha-\gamma)} \\
&\times\left(\frac{\alpha-1}{1+\alpha} - \tilde{f}_{\tilde{\varphi}}(\tilde{\Omega})\big(C_0(\alpha-1)r^{-1-\alpha}(1+\tilde{\beta}(r,\vartheta)),\vartheta - C_0(\alpha-1)r^{-1-\alpha}(1+\tilde{\beta}(r,\vartheta))\big)\right)^{-\frac{1+\alpha-\gamma}{2}} \\
&\times \tilde{\Omega}\big(\vartheta - C_0(\alpha-1)r^{-1-\alpha}(1+\tilde{\beta}(r,\vartheta))\big),
\end{aligned}
$$

with $\tilde{\beta} \in Z^{\sigma,\alpha}$. From (2.4) we have

$$
\theta(t,x) = \frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}}}\hat{\theta}\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right) = \frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}}}\tilde{\theta}\left(\frac{r}{t^{\frac{1}{1+\alpha}}},\vartheta\right),
$$

where $x = re^{i\vartheta}$. Using $f \in X_1^{\sigma,\delta}$ we find that the initial condition amounts to

$$
\theta_0(x) = r^{-(1+\alpha-\gamma)}\tilde{\Omega}(\vartheta),
$$

where $\tilde{\Omega} \in B_{X_2^p}(1,\varepsilon)$.

Moreover, θ_0 is locally integrable:

$$
\|\theta_0\|_{L^1(B_R)} = \|\tilde{\Omega}\|_{L^1([0,2\pi])}\int_0^R r^{\gamma-\alpha}\,dr = \frac{1}{1-\alpha+\gamma}\|\tilde{\Omega}\|_{L^1([0,2\pi])}R^{1-\alpha+\gamma}
$$

for any $R < \infty$, where $1-\alpha+\gamma > 0$. Note that $\tilde{\Omega} \in L^1$ since $\tilde{\Omega} \in L^p$ for some $p > \frac{1}{1-\gamma}$.

7.3. *Main result*

Our main result reads as follows.

Theorem 7.4. *Let $\gamma \in (0,1)$ and $\alpha \in (1, 1+\gamma)$. Then there exists $\varepsilon > 0$ such that for any $\frac{2\pi}{m}$-periodic $\tilde{\Omega} \in B_{L^p([0,2\pi])}(1,\varepsilon)$ and $p > \frac{1}{1-\gamma}$, the initial condition*

$$
\theta_0(re^{i\vartheta}) = r^{-(1+\alpha-\gamma)}\tilde{\Omega}(\vartheta)
$$

determines a self-similar solution of the type (2.4) *for* $(\mathrm{SQG})_\gamma$, *for any* $m \geq 1$.

Appendix A. Radial solutions

In this section, we review the expression of the stream function ψ associated to a radial vorticity θ.

Proposition A.1. *If $f = (-\Delta)^\beta g$ with $\beta \in (-1,0)$ and $g(x) = g(|x|)$ radial, then*

$$
f(x) = 2\pi C_\beta \int_0^\infty G_0(r,|x|)g(r)r\,dr,
$$

where

$$G_0(r,|x|) = \begin{cases} \frac{1}{\Gamma(-\beta)}\left(\frac{2}{r}\right)^{2+2\beta}\Gamma(1+\beta)_2F_1\left(1+\beta,1+\beta,1,\frac{|x|^2}{r^2}\right), & |x|<r,\\ \frac{1}{\Gamma(-\beta)}\left(\frac{2}{|x|}\right)^{2+2\beta}\Gamma(1+\beta)_2F_1\left(1+\beta,1+\beta,1,\frac{r^2}{|x|^2}\right), & |x|>r.\end{cases}$$

Proof. We want to calculate the explicit kernels of

$$f = (-\Delta)^\beta g$$

whenever $-1 < \beta < 0$ and for general g of the type

$$g(r,\theta) = g_k(r)\cos(k\theta), \quad g(r,\theta) = g_k(r)\sin(k\theta).$$

Note that we are indeed interested in radial functions g corresponding to $k = 0$, but let us compute it in general. To do so, we will calculate the Fourier transform:

$$f(x) = \int \hat{g}(\xi)|\xi|^{2\beta}e^{ix\cdot\xi}\,d\xi.$$

We start by calculating $\hat{g}(\xi)$. In the first case,

$$\hat{g}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi}g(x)\,dx = \int_0^\infty g_k(r)r\int_0^{2\pi}\cos(k\theta)e^{-ir(\cos(\theta)\xi_1+\sin(\theta)\xi_2)}\,d\theta\,dr.$$

Writing

$$\cos(\psi) = \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}}, \quad \sin(\psi) = \frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}}, \quad z = \sqrt{\xi_1^2+\xi_2^2}\,r, \tag{A.1}$$

the inner integral becomes

$$\begin{aligned}\int_0^{2\pi} e^{-iz\cos(\theta-\psi)}\cos(k\theta)\,d\theta &= \cos(k\psi)\int_0^{2\pi} e^{-iz\cos(u)}\cos(ku)\,du\\ &= 2\pi T_k\left(\frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}}\right)(-i)^k J_k\left(\sqrt{\xi_1^2+\xi_2^2}\,r\right),\end{aligned}$$

where J_k is the Bessel function of order k, and T_k is the k-th Chebyshev polynomial of the first kind.

In the second case

$$\hat{g}(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi}g(x)\,dx = \int_0^\infty g_k(r)r\int_0^{2\pi}\sin(k\theta)e^{-ir(\cos(\theta)\xi_1+\sin(\theta)\xi_2)}\,d\theta\,dr$$

Using (A.1), the inner integral becomes

$$\begin{aligned}\int_0^{2\pi} e^{-iz\cos(\theta-\psi)}\sin(k\theta)\,d\theta &= \sin(k\psi)\int_0^{2\pi} e^{-iz\cos(u)}\sin(ku)\,du\\ &= 2\pi\frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}}U_{k-1}\left(\frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}}\right)(-i)^k J_k\left(\sqrt{\xi_1^2+\xi_2^2}\,r\right),\end{aligned}$$

where U_k is the k-th Chebyshev polynomial of the second kind.

We now move on to the computation of the outer integral. Switching the integrals in r and in ξ, and integrating first in ξ, in the first case we obtain

$$\begin{aligned}
&\int_0^\infty (-i)^k 2\pi g_k(r) r \int_{\mathbb{R}^2} T_k\Big(\frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}}\Big) J_k\big(\sqrt{\xi_1^2+\xi_2^2}\, r\big)|\xi|^{2\beta} e^{ix\cdot\xi}\, d\xi\, dr \\
&= \int_0^\infty (-i)^k 2\pi g_k(r) r \int_0^\infty J_k(|\xi| r)|\xi|^{2\beta+1} \\
&\qquad\qquad \times \int_0^{2\pi} \cos(k\psi) e^{i|\xi|(x_1\cos(\psi)+x_2\sin(\psi))}\, d\psi\, d|\xi|\, dr \\
&= (-i)^{2k}(2\pi)^2\cos(k\theta)\int_0^\infty g_k(r) r \int_0^\infty J_k(|\xi| r)|\xi|^{2\beta+1} J_k(|x|\,|\xi|)\, d|\xi|\, dr.
\end{aligned}$$

Since $-1<\beta<0$, we get

$$\begin{aligned}
&\int_0^\infty J_k(|\xi| r)|\xi|^{2\beta+1} J_k(|x|\,|\xi|)\, d|\xi| \equiv G_k(r,|x|) \\
&= \begin{cases} \frac{1}{2\Gamma(-\beta)}\big(\frac{2}{r}\big)^{2+2\beta}\big(\frac{|x|}{r}\big)^k \Gamma(1+\beta+k)\Gamma(k+1)\, {}_2F_1(1+\beta, 1+\beta+k, 1+k, \frac{|x|^2}{r^2}) \\ \hfill \text{if } |x|<r, \\ \frac{1}{2\Gamma(-\beta)}\big(\frac{2}{x}\big)^{2+2\beta}\big(\frac{r}{|x|}\big)^k \Gamma(1+\beta+k)\Gamma(k+1)\, {}_2F_1(1+\beta, 1+\beta+k, 1+k, \frac{r^2}{|x|^2}) \\ \hfill \text{if } |x|>r, \end{cases}
\end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function. This yields

$$f(x) = (-i)^{2k}(2\pi)^2\cos(k\theta)\int g_k(r) G_k(r,|x|) r\, dr.$$

The computation in the second case is similar, and we obtain

$$f(x) = (-i)^{2k}(2\pi)^2\sin(k\theta)\int g_k(r) G_k(r,|x|) r\, dr.$$

Now, taking $k=0$ we arrive at the announced result. ■

In the following, we apply the previous proposition to our trivial solution.

Corollary A.2. *If $\theta(x) = |x|^{-1/\mu}$, then*

$$\psi(x) = -(-\Delta)^{-1+\gamma/2}\theta(x) = -C_0|x|^{2-\gamma-1/\mu},$$

where

$$\begin{aligned}
C_0 = \frac{\pi 2^\gamma \Gamma\big(\frac{\gamma}{2}\big)\mu}{\Gamma(1-\gamma/2)}\Big\{ &\frac{1}{2\mu-1}\, {}_3F_2\Big(\Big\{\frac{\gamma}{2}, \frac{\gamma}{2}, 1-\frac{1}{2\mu}\Big\}, \Big\{1, 2-\frac{1}{2\mu}\Big\}, 1\Big) \\
&+ \frac{1}{1+\mu(-2+\gamma)}\, {}_3F_2\Big(\Big\{\frac{\gamma}{2}, \frac{\gamma}{2}, -1+\frac{\gamma}{2}+\frac{1}{2\mu}\Big\}, \Big\{1, \frac{\gamma}{2}+\frac{1}{2\mu}\Big\}, 1\Big)\Big\}, \quad \text{(A.2)}
\end{aligned}$$

where

$$ {}_3F_2(a_1, a_2, a_3; b_1, b_2; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n} \frac{z^n}{n!} $$

is the $(3, 2)$ *hypergeometric function.*

Proof. Use the previous proposition with $\beta = -1 + \gamma/2$ and $g(x) = |x|^{-1/\mu}$. ∎

The constant C_0 is important in studying the kernel of the linearized operator. For that reason, let us give its expression in terms of the integral operator T_k defined in (5.4).

Proposition A.3. *The constant C_0 can also be written as*

$$ C_0 = \frac{C_\gamma}{2\pi} \int_0^\infty \int_0^{2\pi} \frac{z^{\gamma-\alpha}}{\{1 + z^2 - 2z\cos(\eta)\}^{\gamma/2}}\, d\eta\, dz = \frac{C_\gamma}{2\pi} M[T_0](1 - \alpha + \gamma). $$

Proof. From the definition of the $(-\Delta)^{-1+\gamma/2}$ operator one has

$$ C_0 = r^{-2+\gamma+1/\mu} \frac{C_\gamma}{2\pi} \int_0^\infty \int_0^{2\pi} \frac{z^{\gamma-\alpha}}{\{r^2 + z^2 - 2rz\cos(\eta)\}^{\gamma/2}}\, d\eta\, dz. $$

By performing the change of variable $z \mapsto rz$ and using the expression of μ in (2.2) and T_k in (5.4) we arrive at the announced identity. ∎

Appendix B. Special functions

In this section we overview some special functions such as the Gauss hypergeometric function, the gamma function and the digamma function. At the end, we also introduce the Mellin transform and review some of its properties.

For any $a, b \in \mathbb{R}$ and $c \in \mathbb{R} \setminus (-\mathbb{N})$ we define the Gauss hypergeometric function $z \mapsto {}_2F_1(a, b; c; z)$ on the open unit disc $\mathbb{D}$ by the power series

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad \forall z \in \mathbb{D}, \tag{B.1} $$

where the Pochhammer symbol $(x)_n$ is defined by

$$ (x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)\cdots(x+n-1), & n \geq 1, \end{cases} $$

and satisfies

$$ (x)_n = x\,(1+x)_{n-1}, \quad (x)_{n+1} = (x+n)\,(x)_n. $$

Let us recall the integral representation of the hypergeometric function (see for instance [77, p. 47]). If $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, then

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}\, dx, \quad \forall z \in \mathbb{C} \setminus [1, \infty). \tag{B.2} $$

The function $\Gamma : \mathbb{C} \setminus \{-\mathbb{N}\} \to \mathbb{C}$ is the analytic continuation to the negative half-plane of the usual gamma function defined on the positive half-plane $\{\mathrm{Re}(z) > 0\}$ by

$$\Gamma(z) = \int_0^\infty \tau^{z-1} e^{-\tau}\, d\tau,$$

and satisfies the relation $\Gamma(z+1) = z\,\Gamma(z)$ for all $z \in \mathbb{C} \setminus (-\mathbb{N})$. Moreover, the digamma function is defined through the logarithmic derivative of the gamma function:

$$\psi^0(z) = \frac{\Gamma'(z)}{\Gamma(z)},$$

and has the following integral representation:

$$\psi^0(z) = \int_0^1 \frac{1 - t^{z-1}}{1-t} dt - \gamma_0 \tag{B.3}$$

for $\mathrm{Re}(z) > 0$, where γ_0 stands for the Euler constant.

In the following lemma, we give a representation of the Gauss hypergeometric function in terms of some special trigonometric integrals; the proof can be found in [48].

Lemma B.1. *Let $k \in \mathbb{N}$, $\gamma \geq 0$ and $A > 1$. Then*

$$\int_0^{2\pi} \frac{\cos(k\eta)}{(A - \cos(\eta))^{\gamma/2}}\, d\eta = \frac{2\pi}{(1+A)^{\gamma/2+k}} \frac{\left(\frac{\gamma}{2}\right)_k 2^k \left(\frac{1}{2}\right)_k}{(2k)!}\, {}_2F_1\left(k + \frac{\gamma}{2}, k + \frac{1}{2}; 2k+1; \frac{2}{1+A}\right).$$

Finally, let us define the Mellin transform of f by

$$M[f](s) = \int_0^\infty x^{s-1} f(x)\, dx.$$

In the following lemma, we give some properties of the Mellin transform; we refer to [74] for the proof and for more information about this transform.

Lemma B.2. *The Mellin transform of f has the following properties:*

- *If*

$$(f \wedge g)(x) = \int_0^\infty f(y) g\left(\frac{x}{y}\right) \frac{dy}{y},$$

  then

$$M[f \wedge g](s) = M[f]M[g]. \tag{B.4}$$

- *Moreover,*

$$M[x^\alpha f](s) = M[f](s+\alpha), \tag{B.5}$$

$$M[f'](s) = -(s-1)M[f](s-1), \tag{B.6}$$

$$M[x^\alpha f](s) = M[f](s+\alpha), \tag{B.7}$$

$$M[f(x^a)] = |a|^{-1} M[f](s/a). \tag{B.8}$$

Funding. C.G. and J.G.-S. have been partially supported by the European Research Council ERC-StG-852741 (CAPA) and the MICINN (Spain) research grant number PID2021-125021NA-I00. C.G. has been supported by RYC2022-035967-I (MCIU/AEI/10.13039/501100011033 and FSE+), and partially by the AEI grants PID2022-137228OB-I00, PID2022-140494NA-I00, project C-EXP-265-UGR23 by Programa Operativo FEDER Andalucía, and project QUAL21-011 by Modeling Nature Research Unit. J.G.-S. was partially supported by the AGAUR project 2021-SGR-0087 (Catalunya), and by NSF under Grants DMS-2245017 and DMS-2247537. This material is based upon work supported by the National Science Foundation under Grant No. DMS–1929284 while C.G. and J.G.-S. were in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the program "Hamiltonian Methods in Dispersive and Wave Evolution Equations". This work is supported by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M).

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