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p-perfection and group completion of $\mathbb{E}_\infty$-monoids

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Abstract. We study $\mathbb{E}_\infty$-monoids on which a prime p acts invertibly, which we call p-perfect, in the non-group-complete situation. In particular, we prove that in many examples, they almost embed in their group-completion.

We further study the p-perfection functor, and describe it in terms of Quillen's $+$-construction, similarly to group-completion. This gives an alternative description of the p-inverted higher algebraic K-theory of a ring.

Introduction

In homotopy theory, many constructions which classically appear rather harmless, such as group completion of commutative monoids, become much more complicated. For example, the sphere spectrum is the (homotopical) group completion of the groupoid of finite sets, and while the latter is an ordinary 1-groupoid, the former has non-trivial homotopy groups in arbitrarily high positive degrees.

Thus, it is somewhat of a miracle that on connective spectra, also known as grouplike $\mathbb{E}_\infty$-monoids, or grouplike commutative monoids, the process of inverting a prime p or a set of primes S is rather harmless: it is given by a simple telescope construction

$$\operatorname{colim}(X \xrightarrow{p} X \xrightarrow{p} X \to \cdots),$$

see e.g. [4, Proposition C.3] for a published account. In particular, the homotopy groups of $X[\frac{1}{p}]$ are simply those of X where we formally invert p in the classical sense.

However, the proof of this fact truly uses the grouplike-ness of spectra, and more specifically, the fact that their underlying spaces are simple. This paper stems from the observation that this miracle fails away from the grouplike situation—it fails already for the groupoid of finite sets $\mathrm{Fin}^{\simeq}$, as we explain in detail in Proposition 1.12 and Remark 1.13.

Our goal in this paper is to study (not-necessarily-grouplike) commutative monoids in the ∞-category of spaces, aka $\mathbb{E}_\infty$-monoids, on which a prime p acts invertibly. We call such commutative monoids *p-perfect*. Note that by [9, Proposition 5.5.4.15], the inclusion of the ∞-category of p-perfect commutative monoids in the ∞-category of commutative

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monoids admits a left adjoint. However, as such, its existence is obtained purely abstractly and our second goal in this paper is therefore to study this left adjoint, which we call the p-perfection functor, and denote by $(-)[\frac{1}{p}]$.

The first thing we prove is some kind of elementary structure theorem that allows us to understand the failure of the telescope construction a bit better—the following is Theorem 1.2.

Theorem A. *Let M be a p-perfect commutative monoid. Then:*

(1) *M is a simple space, that is, $\pi_1(M,m)$ is abelian for every $m \in M$, and its action on $\pi_n(M,m)$ is trivial for all $n \geq 1$;*

(2) *$\pi_n(M,m)$ is uniquely p-divisible for all $n \geq 1$.*

Remark. It will be clear from the proof of this theorem that it holds more generally: it works for any homotopy commutative homotopy monoid in spaces for which the map $M \to M^{\times p} \to M$ is an equivalence (of spaces). In particular, it applies to $\mathbb{E}_k$-monoids, $k \geq 2$. We restrict to the setting of commutative or $\mathbb{E}_\infty$-monoids only because the rest of our methods seem to require this amount of structure. Specifically, for other aspects of this paper, it seems crucial that the map $p \cdot - : M \to M$ is itself a map of commutative monoids (while if M is $\mathbb{E}_{k+1}$, this map is a priori only $\mathbb{E}_k$). It would be interesting to try and see whether our other results can be generalized to $\mathbb{E}_k$-monoids, $k \geq 2$ (or maybe $k \geq 3$).

With this in hand, the similarity with group-complete commutative monoids becomes somewhat striking, and in fact we are able to compare p-perfection of a commutative monoid and the p-perfection of its group completion in an *a priori* rather surprising way. We refer to Definition 2.3 for the definition of "locally monogenic"—we simply point out here that for R a commutative ring with connected $\mathrm{Spec}(R)$, the groupoid of finitely generated projective R-modules $\mathrm{Proj}(R)^{\simeq}$ is locally monogenic, and so is $\mathrm{Fin}^{\simeq}$. With this definition, the following is Theorem 2.8.

Theorem B. *Let M be a locally monogenic commutative monoid. Consider the pullback square of commutative monoids, induced by the group completion map $M \to M^{\mathrm{gp}}$:*

$$\begin{array}{ccc} N & \longrightarrow & M^{\mathrm{gp}}[\frac{1}{p}] \\ \downarrow & & \downarrow \\ \pi_0(M)[\frac{1}{p}] & \longrightarrow & \pi_0(M)^{\mathrm{gp}}[\frac{1}{p}]. \end{array}$$

Then the natural map $M[\frac{1}{p}] \to N$ is an isomorphism on π_0 and an equivalence at all components except 0.

Finally, again by analogy with group-completion, rather than try to compute the p-perfection functor on the nose (which, by the above, may be as hard as group completion!), we give a way to describe its homology, analogously to the group-completion theorem. In

the following statement, the $+$-superscript refers to Quillen's $+$-construction (see [7] or [2, Section 3] for a modern account), and $\mathrm{tel}_p(M)$ refers to the telescope construction $\mathrm{colim}(M \xrightarrow{p} M \xrightarrow{p} M \to \cdots)$. With this notation, the following is Theorem 3.1.

Theorem C. *Let M be a commutative monoid. There are canonical maps of commutative monoids*

$$\mathrm{tel}_p(M)^+ \to \mathrm{tel}_p(M^+) \to M\left[\frac{1}{p}\right]$$

that are both equivalences.

Remark. We note that, while we use the symbol p throughout the introduction, p does not need to be a prime, it simply needs to be a natural number ≥ 2. Not only are the proofs the same, but the relevant results also follow from the case of a prime.

In particular, this theorem provides a homology isomorphism between $\mathrm{tel}_p(M)$ and $M[\frac{1}{p}]$. In Theorem 5.1, we give a second proof of this homology isomorphism with $\mathbb{F}_p$-coefficients using the "bialgebra Frobenius" on $H_*(M;\mathbb{F}_p)$, where now we assume that p is prime.

This theorem also suggests a comparison with a slightly different formal inversion of p, namely the one where we view M as a module (in spaces) over the groupoid of finite sets equipped with the *cartesian* product and invert p as a module map. One can also imagine remembering only the module structure over the full subgroupoid spanned by p. Informally speaking, this means that we only remember how to add "elements" in M to themselves a number of times, or a p-power number of times respectively, and invert p in that context. Surprisingly, all these three inversions give the same result, as we prove in Corollary 4.6.

We use this result to extend to higher homotopy groups, and generalize, a result due to Bass in degrees 0 and 1 of a multiplicative group completion of projective R-modules, see Example 4.7.

Corollary D. *Let k be a commutative ring and consider the groupoid $((\mathrm{Proj}_k^{\mathrm{ff}})^{\simeq}, \otimes_k)$ of faithful, finitely generated projective k-modules under tensor product. Its group completion has π_n isomorphic to $K_n(k) \otimes \mathbb{Q}$ for $n > 0$, and π_0 is the multiplicative submonoid of $K_0(k) \otimes \mathbb{Q}$ spanned by classes of everywhere positive rank.*

Linear overview

The early sections match the order in which we stated our main results: Section 1 deals with generalities about p-perfect commutative monoids and there we prove Theorem 1.2; Section 2 is about the proof of Theorem 2.8 and Section 3 deals with the proof of Theorem 3.1. Section 4 is concerned with the comparison of p-perfection with another formal inversion of p, and there we prove Corollary 4.6. We added an extra section, Section 5, which gives a different proof of a special case of Theorem 3.1, because we thought the proof technique worth recording.[1]

[1] It also happens that this proof was obtained earlier than Theorem 3.1.

Related work

This work arose partly from a discussion with Ben Antieau about his work in [1, Section 2.2]. In the meantime, he obtained a different proof of Theorem 5.1 (which also implies Corollary 1.8), cf. [1, Corollary 2.10].

After a first version of this paper appeared on the arXiv, it was pointed out to the authors by Mamuka Jibladze that our Theorem 2.8 had a precedent in the work of Bass [3]. He proves in [3, Chapter IX, Theorems 7.1 and 7.3] a version of this for the 0th and 1st homotopy groups in the case of $M = \mathrm{Proj}_R^{\simeq}$, and after inverting all primes. The method of proof in this case is similar, though in the case of π_0, π_1, there are explicit constructions of the group completion which simplify some of the details and thus remove the need for many of our preliminaries.

Notation

We freely use the language of ∞-categories as developed by Lurie in [9, 10].

$\mathbf{S}$ denotes the ∞-category of spaces. CMon denotes the ∞-category of commutative monoids in $\mathbf{S}$, also known as $\mathbb{E}_\infty$-monoids or $\mathbb{E}_\infty$-spaces – more precisely, this refers to commutative algebras formed in the (cartesian) symmetric monoidal ∞-category of spaces. Every monoid $M \in \mathrm{CMon}$ has canonical Frobenius map $p\colon M \to M$. $\mathrm{CMon}[\frac{1}{p}]$ denotes the full subcategory of monoids M such that Frobenius map $p\colon M \to M$ is an equivalence. The inclusion $\mathrm{CMon}[\frac{1}{p}] \hookrightarrow \mathrm{CMon}$ admits a left adjoint, which we denote $M \mapsto M[\frac{1}{p}]$. We denote mapping spaces by Map. We generally use additive notation for our commutative monoids, cf. Remark 2.2.

$\mathrm{Fin}^{\simeq}$ denotes the groupoid of finite sets, which we will consider as a commutative monoid under disjoint union. $\mathrm{Proj}(R)^{\simeq}$ denotes the groupoid of finitely generated projective R-modules, considered as a commutative monoid under direct sum.

1. Homotopy groups of p-perfect monoids

Definition 1.1. Let M be a commutative monoid. M is called *p-perfect* if it belongs to $\mathrm{CMon}[\frac{1}{p}]$, i.e., if the Frobenius map $p : M \to M$ is an equivalence.

In this section, we analyze the structure of homotopy groups of p-perfect commutative monoids. The main result of this section is the following theorem.

Theorem 1.2. *Let M be a p-perfect commutative monoid. Then:*

(1) *M is a simple space, that is, $\pi_1(M, m)$ is abelian for every $m \in M$, and its action on $\pi_n(M, m)$ is trivial for all $n \geq 1$;*

(2) *$\pi_n(M, m)$ is uniquely p-divisible for all $n \geq 1$.*

Before proving Theorem 1.2, we would like to point out two differences with the grouplike case that are at the heart of our difficulties. First, not all components of a com-

mutative monoid M are the same, and in particular we may not reduce to the component of $0 \in M$. Specifically, the Frobenius map typically goes from (M, m) to (M, pm) and is thus not an endomorphism of pointed spaces, which complicates π_*-considerations. Second, for essentially the same reason, a general commutative monoid need not be a simple space: its fundamental groups can be arbitrarily non-commutative (to wit, $\mathrm{Fin}^{\simeq}$ has Σ_n as the fundamental group at the basepoint n), and they can act non-trivially on higher homotopy groups.

The idea to prove Theorem 1.2 is to consider the following map of pointed spaces:

$$(M, m)^p \xrightarrow{\mu} (M, pm) \xrightarrow{\simeq} (M, m)$$

where the first map is the addition, and the second map is the inverse of the Frobenius map (which is an equivalence by the assumption that M is p-perfect).

This induces, for $n \geqslant 1$, a family of maps

$$\rho_n : \pi_n(M, m)^p \to \pi_n(M, m) \tag{1.1}$$

satisfying certain conditions coming from properties of M, thus providing an extra algebraic structure on these homotopy groups. The general claim is that this algebraic structure is enough to guarantee the conclusions of Theorem 1.2.

First, we will analyze the extra structure on the fundamental groups of M.

Proposition 1.3. *Let G be a group with a group morphism $\rho : G^p \to G$ such that:*

(1) *For all $g \in G$, $\rho(g, \dots, g) = g$;*

(2) *For any permutation $\sigma \in \Sigma_p$, there is a fixed $k_\sigma \in G$ such that $\rho \circ \sigma = k_\sigma \rho k_\sigma^{-1}$.*

In this case, G is abelian and uniquely p-divisible.

Proof. Specializing the second condition to diagonal elements, and using the first condition, we find that for any g and σ,

$$g = \rho(g, \dots, g) = \rho \circ \sigma(g, \dots, g) = k_\sigma \rho(g, \dots, g) k_\sigma^{-1} = k_\sigma g k_\sigma^{-1}.$$

Hence k_σ is central, and so the second condition simplifies to $\rho \circ \sigma = \rho$.

In particular, we find:

$$g = \rho(g, \dots, g) = \rho(g, 1, \dots, 1)\rho(1, g, 1, \dots, 1) \cdots \rho(1, \dots, 1, g) = \rho(g, 1, \dots, 1)^p$$

so that G is indeed p-divisible. Moreover, since $g \mapsto \rho(g, 1, \dots, 1)$ is a group homomorphism, G is uniquely p-divisible. We also find $gh = \rho(g, 1, \dots, 1)^p \rho(1, h, 1, \dots)^p$, and since $(g, 1, \dots, 1)$ and $(1, h, 1, \dots)$ commute in G^p, we get that $gh = hg$ and we deduce that G is abelian. ∎

Remark 1.4. Note that under the conditions of Proposition 1.3 we get that $\rho : G^p \to G$ is simply equal to $\frac{1}{p} \sum_{i=1}^{p}$.

We now verify that we have the relevant structure on $\pi_1(M, m)$.

Lemma 1.5. *Let M be a p-perfect commutative monoid, and let $m \in M$ be a point. The map ρ_1 defined in (1.1) satisfies the two assumptions of Proposition 1.3 with $G = \pi_1(M, m)$.*

Proof. Condition (1) is evident, as the composite

$$(M, m) \xrightarrow{\Delta} (M, m)^p \xrightarrow{\mu} (M, pm) \xrightarrow{\simeq} (M, m)$$

is by definition the composition of an equivalence and its inverse, i.e., the identity.

For condition (2), this comes from the fact that the addition on M is homotopy commutative. Note that the homotopy commutativity of the addition does not give pointed homotopies, it only gives homotopies, hence the need for conjugation by k_σ. In fact, for a general M, k_σ can be viewed as a power operation applied to m. ∎

We can then argue similarly for higher homotopy groups. The maps

$$\rho_n : \pi_n(M, m)^p \to \pi_n(M, m),$$

defined in (1.1), have the same first property as in Proposition 1.3, and in the second property the conjugation by k_σ is replaced by the *action* of $k_\sigma \in \pi_1(M, m)$ on $a \in \pi_n(M, m)$ which we denote as $k_\sigma \cdot a$. The relevant equation will then read: $\rho_n \circ \sigma = k_\sigma \cdot \rho_n$. To sum up, we want to prove the following statement.

Proposition 1.6. *Let G be a group acting on an abelian group A. Assume there is a morphism $\rho : A^p \to A$ of abelian groups such that:*

(1) *For all $a \in A$, $\rho(a, \dots, a) = a$;*

(2) *For any permutation $\sigma \in \Sigma_p$, there is a fixed $k_\sigma \in G$ such that $\rho \circ \sigma = k_\sigma \cdot \rho$.*

In this case, A is uniquely p-divisible. If, furthermore, G is abelian, uniquely p-divisible, and the map $\rho : A^p \to A$ is compatible with the action of G^p (where G^p acts on A via $G^p \xrightarrow{\frac{1}{p}\sum_{i=1}^p} G$), then the action of G on A is trivial.

Proof. We again start by plugging in a diagonal element, to find that

$$a = \rho(a, \dots, a) = \rho \circ \sigma(a, \dots, a) = k_\sigma \cdot \rho(a, \dots, a) = k_\sigma \cdot a$$

from which it follows that k_σ acts trivially on $\pi_n(M, m)$. So we have the same argument for unique p-divisibility as in Proposition 1.3.

Now for the trivial action, we note that $\rho : A^p \to A$ must also be $\frac{1}{p}\Sigma_{i=1}^p$ by the previous computation. So the equivariance condition for ρ reads as follows: for $(g_1, \dots, g_p) \in G^p$, $(a_1, \dots, a_p) \in A^p$,

$$\frac{1}{p}\sum_{i=1}^{p}(g_i \cdot a_i) = \left(\frac{1}{p}\sum_i g_i\right) \cdot \left(\frac{1}{p}\sum_i a_i\right).$$

Putting all of the g_i's except g_1 to be trivial, and all of the a_i's except for a_p to be trivial, we find

$$\frac{1}{p}a_p = \left(\frac{1}{p}g_1\right)\left(\frac{1}{p}a_p\right).$$

Multiplying this by p, we find that $\frac{1}{p}g_1$ acts trivially on a_p. However both g_1 and a_p were chosen to be arbitrary, so the action is indeed trivial. ∎

Similarly, we can verify that the maps ρ_n, $n \geq 2$ are examples of this algebraic structure.

Lemma 1.7. *For any $n \geq 2$, the map ρ_n defined in (1.1) provides the group $A = \pi_n(M, m)$ with this structure, where $G = \pi_1(M, m)$ with the standard action of G on A.*

Proof. The proof is exactly the same as in Lemma 1.5. ∎

Proof of Theorem 1.2. For a commutative monoid M, combine Proposition 1.3 and Lemma 1.5 as well as Proposition 1.6 and Lemma 1.7 to find that all the actions of $\pi_1(M, m)$ on $\pi_n(M, m)$ are trivial for all basepoints $m \in M$, and all $\pi_n(M, m)$ are uniquely p-divisible. ∎

Corollary 1.8. *Let M be a p-perfect commutative monoid. The map $M \to \pi_0(M)$ is an $\mathbb{F}_p$-homology isomorphism.*

Proof. Indeed, from Theorem 1.2 and (a slight extension of) Serre's version of Hurewicz theorem (see e.g. the main theorem of [6, p. 55]) it follows that for a p-perfect monoid M, the groups $H_*(M; \mathbb{F}_p)$ are p-divisible for $* \geqslant 1$. Given that these groups are $\mathbb{F}_p$-vector spaces, it follows that they all vanish except $H_0(M; \mathbb{F}_p) \simeq H_0(\pi_0(M); \mathbb{F}_p)$. ∎

Remark 1.9. It is not hard to prove that $\pi_0(M[\frac{1}{p}]) \cong \pi_0(M)[\frac{1}{p}]$ where the latter is meant in the classical sense (in particular, it equals $\pi_0(\mathrm{tel}_p(M))$), so we can completely compute $H_*(M[\frac{1}{p}]; \mathbb{F}_p)$. We will improve on this calculation in the rest of the paper.

Next, we consider the following "approximation" to p-perfection.

Construction 1.10. Let M be a commutative monoid. The *p-telescope* of M is

$$\mathrm{tel}_p(M) := \mathrm{colim}(M \xrightarrow{p} M \xrightarrow{p} \ldots),$$

the colimit being taken in CMon.

Question 1.11. For which commutative monoids M is the canonical map $\mathrm{tel}_p(M) \to M[\frac{1}{p}]$ an equivalence?

For discrete commutative monoids, as well as grouplike commutative monoids (and more generally spectra) the answer to Question 1.11 is indeed positive. However, using Theorem 1.2 we can see that in general it can be negative, as in the following example.

Proposition 1.12. *Let $M = \mathrm{Fin}^{\simeq}$. In this case, the canonical map $\mathrm{tel}_p(M) \to M[\frac{1}{p}]$ is* not *an equivalence, i.e., the monoid $\mathrm{tel}_p(M)$ is* not *p-perfect.*

Proof. Fix $1 \in \mathrm{Fin}^{\simeq}$ and let $\underline{1} \in \mathrm{tel}_p(\mathrm{Fin}^{\simeq})$ be its image under the canonical map. Since the fundamental group functor commutes with filtered colimits, we get:

$$\pi_1\big(\mathrm{tel}_p(\mathrm{Fin}^{\simeq}), \underline{1}\big) \cong \mathrm{colim}_k\, \pi_1(\mathrm{Fin}^{\simeq}, p^k) \cong \mathrm{colim}_k\, \Sigma_{p^k}$$

and each of the maps $\Sigma_{p^k} \to \Sigma_{p^{k+1}}$ sends σ to $(\sigma, \dots, \sigma)$ and in particular is injective.

Thus Σ_p embeds in $\pi_1(\mathrm{tel}_p(\mathrm{Fin}^{\simeq}), \underline{1})$, which is therefore not abelian. It follows from Theorem 1.2 that $\mathrm{tel}_p(\mathrm{Fin}^{\simeq})$ is not p-perfect, and thus not equivalent to $\mathrm{Fin}^{\simeq}[\frac{1}{p}]$. ∎

Remark 1.13. With a similar argument, one may prove that the image of the group Σ_p in $\pi_1(\mathrm{tel}_p(\mathrm{Fin}^{\simeq}), \underline{1})$ is not p-divisible. The moral here is that $\sigma \in \Sigma_p$ is indeed sent to $(\sigma, \dots, \sigma) = (\sigma, 1, \dots, 1)(1, \sigma, 1, \dots, 1) \dots (1, \dots, 1, \sigma) \in \Sigma_{p^2}$ *but* each of the $(1, \dots, 1, \sigma, 1, \dots)$ differs from the others by a conjugation. We explain this conjugation below.

The failure of $\mathrm{tel}_p(\mathrm{Fin}^{\simeq})$ to be p-perfect is closely related to a similar phenomenon in the context of group completion, cf. [11]. The idea here is that the Frobenius map on the space $\mathrm{colim}(M \to M \to \cdots)$ is induced by a map of diagrams, namely a map which looks like:

$$\begin{array}{ccccccc} M & \xrightarrow{p} & M & \xrightarrow{p} & M & \longrightarrow & \dots \\ {\scriptstyle p}\downarrow & h & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow \\ M & \xrightarrow[p]{} & M & \xrightarrow[p]{} & M & \longrightarrow & \dots \end{array}$$

The homotopy h that is here is an *interesting* homotopy, it is the one that witnesses that $p : M \to M$ is a map of commutative monoids, and hence commutes with the "scalar" p. It is a *different* one than the homotopy $f \circ f = f \circ f$ that exists for any endomorphism f. If it were the latter, then the induced map on colimits would in fact be an equivalence, but because it is the former, there is no reason for it to be the case.

This difference in homotopies is related to the fact that $(\sigma, \dots, \sigma)$ is p-divisible *up to conjugacy* in Σ_{p^2}—note that conjugacies in π_1 arise precisely from *unpointed* homotopies, and they influence the induced map on colimits.

In fact, we can make that precise with the following result—this is analogous to [11, Proposition 6] (see also Corollary 4.6 for a formal relation between the two statements—note that the latter does not depend on this statement, so there is no circularity). Recall that a group G is called *hypoabelian* if it contains no non-trivial perfect subgroups.

Proposition 1.14. *Let M be a commutative monoid.*

(1) *Suppose $\mathrm{tel}_p(M)$ is p-perfect. Then it is the p-perfection of M.*

(2) *Suppose $\pi_1(M, x)$ is hypoabelian for all x. Then $\mathrm{tel}_p(M)$ is p-perfect. More generally, $\mathrm{tel}_p(M)$ is p-perfect if for every $m \in M$, the image of $p^{3\text{-cycle}} \in \Sigma_{p^3}$ in $\pi_1(M, p^3 m)$ is trivial, or the same statement with 3 replaced by some $n \geq 3$.*

(3) *More generally, suppose that for some $n \geq 3$ the image of $p^{n\text{-cycle}}$ in $\pi_1(\mathrm{tel}_p(M), m)$ is 0 for every $m \in M$. Then $\mathrm{tel}_p(M)$ is p-perfect.*

Here, we abused notation, and for a permutation σ of n, denoted p^σ the induced permutation of $p^n = p^{\times n}$. So for example, $p^{3-\text{cycle}}$ is p^σ where $\sigma = (123)$ or any 3-cycle in Σ_3.

Proof. Let N be a commutative monoid. Since $p : X \to X$ is a natural transformation of the identity on commutative monoids, the following two maps

$$\mathrm{Map}_{\mathrm{CMon}}(p \cdot -, N) : \mathrm{Map}_{\mathrm{CMon}}(M, N) \to \mathrm{Map}_{\mathrm{CMon}}(M, N)$$

and

$$\mathrm{Map}_{\mathrm{CMon}}(M, p \cdot -) : \mathrm{Map}_{\mathrm{CMon}}(M, N) \to \mathrm{Map}_{\mathrm{CMon}}(M, N)$$

are homotopic. Therefore, if N is p-perfect, the first one is an equivalence because the second one is. It follows that in this case, restriction along $M \to \mathrm{tel}_p(M)$ induces an equivalence

$$\mathrm{Map}_{\mathrm{CMon}}\left(\mathrm{tel}_p(M), N\right) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{CMon}}(M, N).$$

The first point then follows immediately by the universal property of $M[\frac{1}{p}]$.

Since the $p^{3\text{-cycle}}$ is a product of disjoint 3-cycles, it belongs to $A_{p^3} \subset \Sigma_{p^3}$, which is perfect for all p. The images of perfect subgroups are perfect, hence the hypoabelian case is indeed a special case of the "More generally" statement.

Suppose then that the $p^{3\text{-cycle}}$ is trivial in each $\pi_1(M, p^3 m)$, and consider the diagram of commutative monoids $M \xrightarrow{p^2} M \xrightarrow{p^2} M \to \cdots$.

By *naturality* of multiplication by p, this induces a commutative diagram of spaces—we added basepoints but it is not a priori a diagram of pointed spaces:

$$\begin{array}{ccccccc}
(M,m) & \xrightarrow{p^2} & (M,p^2m) & \xrightarrow{p^2} & (M,p^4m) & \cdots & (M,p^{2k}m) \\
{\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\
(M,pm) & \xrightarrow[p^2]{} & (M,p^3m) & \xrightarrow[p^2]{} & (M,p^5m) & \cdots & (M,p^{2k+1}m)
\end{array}$$

Here the 2-cells are obtained using the fact that p^2 is a map of commutative monoids and p is natural with respect to those.

Unwinding this, we find that the homotopy $p \circ p^2 \simeq p^2 \circ p$ sends the point m to the image of $p^{3\text{-cycle}} \in \Sigma_{p^3}$ in $\pi_1(M, p^3 m)$.

This element is trivial in each $\pi_1(M, p^{2k+3}m)$ by assumption, and so the 2-cells are homotopies of pointed maps, and so this map of diagrams lifts to a map of diagrams of pointed spaces. Then we may take π_n, and we find that the vertical map induces an isomorphism on π_n on colimits. Since π_n commutes with colimits, this implies that $p : \mathrm{tel}_p(M) \to \mathrm{tel}_p(M)$ induces an isomorphism on π_n at every point and for all n, and is thus an equivalence, which is what we wanted to prove.

More generally, if the image of $p^{3\text{-cycle}}$ only vanishes in $\pi_1(\mathrm{tel}_p(M), m)$, we know it must vanish at some finite stage of the telescope, and we can run the same argument simply by cofinality. ∎

Remark 1.15. In particular, if $\pi_1(M)$ is hypoabelian at all points, then $\mathrm{tel}_p(M)$ is simple by Theorem 1.2 and hence hypoabelian as well. This is not *a priori* clear, in the sense that in general, a filtered colimit of hypoabelian groups need not be hypoabelian. To give a concrete example, note that subgroups of free groups are free, so that if F is a free group on countably many generators, there is an abstract isomorphism $f : F \cong [F, F]$. In particular, we may follow it up with the inclusion $[F, F] \subset F$ to build a sequence $F \to F \to F \to \cdots$ of inclusions whose filtered colimit is a perfect group, while each F is hypoabelian.

As in [2, Example 5.2], the above also works in the context of motivic spaces, i.e., $\mathbb{A}^1$-invariant sheaves on the site of smooth S-schemes Sm_S for some scheme S. Considering, for example, $M = \mathrm{Vect} \simeq \coprod_n \mathrm{BGL}_n$, we find that for any ring R, the relevant matrix corresponding to $p^{3\text{-cycle}}$ is given by permutation matrix.

Since this matrix belongs to $\mathrm{SL}_{p^3}(\mathbb{Z})$ (determinants of permutation matrices agree with the signature of the permutation), it is $\mathbb{A}^1$-homotopic to the identity. Thus we find the following.

Corollary 1.16. *The canonical map of presheaves of spaces on* Sm_S

$$\mathrm{tel}_p(\mathrm{Vect}) \to \mathrm{Vect}\left[\frac{1}{p}\right]$$

is an $\mathbb{A}^1$*-homotopy equivalence on affines. In particular, it is a motivic equivalence.*

2. Comparison of p-perfection and group completion

In this section, we prove our main result for many commutative monoids, including $\mathrm{Fin}^{\simeq}$ and $\mathrm{Proj}(R)^{\simeq}$ with connected $\mathrm{Spec}\, R$: after p-perfection, connected components of such commutative monoid embed as "positive" components of its group completion, except for the component of 0.

First, we introduce some notation.

Notation 2.1. Let $x \in M$ be a point in a commutative monoid.

- The *telescope of M at x* is

$$\mathrm{tel}^x(M) := \mathrm{colim}(M \xrightarrow{+x} M \xrightarrow{+x} \cdots),$$

  the colimit being taken in $\mathbf{S}$.
- The formal inversion of x on M is $M[-x]$, which is typically not the same as the telescope $\mathrm{tel}^x(M)$.

Remark 2.2. Here, the formal inversion of x on M is the initial commutative monoid $M[-x]$ equipped with a map $M \to M[-x]$ such that x is sent to an invertible element in $\pi_0(M[-x])$. The existence of such an $M[-x]$ is guaranteed by the adjoint functor theorem, see also the discussion in [2, Section 3], before Proposition 3.2 in *loc. cit.*

Please note that we write $M[-x]$ since we are using additive notation for our commutative monoids: while this is purely conventional, we want to advise care here to not confuse this operation with the *a priori* completely unrelated[2] operation of p-perfection (where we invert p "in a multiplicative way"), which we denote throughout by $M[\frac{1}{p}]$.

Definition 2.3. Let M be a commutative monoid. We say that M is *locally monogenic* if for any non-zero $x, y \in M$ there exists $n \in \mathbb{N}$ and $z \in M$ such that $nx \simeq y + z$. Note that this condition only depends on $\pi_0(M)$.

We introduce locally monogenic monoids because of the following nice property, which follows immediately from the definition.

Proposition 2.4. *Let M be a locally monogenic monoid. Then for any non-zero $x \in M$ the map $M[-x] \to M^{\mathrm{gp}}$ is an equivalence.*

Proof. Since for a fixed non-zero x and for any y we have a presentation of the form $nx = y + z$, we get that any y becomes invertible in $M[-x]$, so the two share the same universal property. ∎

Example 2.5. The monoid $\mathrm{Fin}^{\simeq}$ is locally monogenic.

Example 2.6. For a ring R with connected $\mathrm{Spec}\, R$, the monoid $\mathrm{Proj}(R)^{\simeq}$ is locally monogenic by Proposition 2.7 below.

Proposition 2.7. *Let R be a ring and P a finitely generated projective R-module. Assume that the rank of P is a non-vanishing function on* $\mathrm{Spec}\, R$ *(for example, P is non-zero and* $\mathrm{Spec}\, R$ *is connected). Then there exists $n \in \mathbb{N}$ such that P^n splits off a trivial summand R.*

Proof. P is a projective module hence locally free. Since $\mathrm{Spec}\, R$ is quasicompact, there is a finite cover by distinguished open sets $\mathrm{Spec}\, R = \bigcup_{i=1}^{n} D(f_i)$ such that $P[\frac{1}{f_i}]$ is a free $R[\frac{1}{f_i}]$-module for every i.

For each i, pick a surjection $\bar{r}_i \colon P[\frac{1}{f_i}] \to R[\frac{1}{f_i}]$.

Since P is finitely generated projective, up to a power of f_i we can lift $\bar{r}_i$ to a map $r_i \colon P \to R$. Consider the map of R-modules $r = \bigoplus_{i=1}^{n} r_i \colon P^n \to R$. The map r induces a map of R-modules which is a surjection locally on $\mathrm{Spec}\, R$, hence it is surjective. Since its target is R, it automatically splits. ∎

Theorem 2.8. *Let M be a locally monogenic commutative monoid. Consider the pullback square of monoids, induced by the group completion map $M \to M^{\mathrm{gp}}$:*

$$\begin{array}{ccc} N & \longrightarrow & M^{\mathrm{gp}}[\frac{1}{p}] \\ \downarrow & & \downarrow \\ \pi_0(M)[\frac{1}{p}] & \longrightarrow & \pi_0(M)^{\mathrm{gp}}[\frac{1}{p}]. \end{array}$$

[2] For example, if $M = (\mathrm{Fin}^{\simeq}, \coprod)$, and therefore contains an element called "p", $M[\frac{1}{p}]$ and $M[-p]$ are defined completely differently.

Then the natural map $M[\frac{1}{p}] \to N$ is an isomorphism on π_0 and an equivalence at all components except 0.

The statement of Theorem 2.8 has this "except 0" condition for a concrete reason. To describe it, we introduce the following.

Definition 2.9. Let M be a commutative monoid. We say that 0 *is isolated in* M if the following two conditions hold:

- The inclusion $0 \to M$ is an inclusion of components, i.e., $\Omega(M, 0) \simeq \mathrm{pt}$.
- For $x, y \in M$, $(x + y \simeq 0) \implies (x \simeq y \simeq 0)$.

Example 2.10. Let C be a category with coproducts such that $x \coprod y \simeq 0$ implies $x = 0$ or $y = 0$, e.g. if C is semi-additive, or $C = \mathrm{Fin}$. In this case, 0 is isolated in $(C^{\simeq}, \coprod)$.

Lemma 2.11. *Let M be a commutative monoid.* 0 *is isolated in M if and only if $M = N_+$ for some non-unital commutative monoid N, where $(-)_+$ is the left adjoint to the forgetful functor from commutative monoids to non-unital commutative monoids.*

Proof. "If": By [10, Proposition 5.4.4.8], the underlying space of N_+ is $N \coprod \{0\}$, proving the first condition of 0 being isolated; the map $N \to N_+$ being a map of non-unital commutative monoids shows that the second condition is also satisfied.

"Only if": Suppose 0 is isolated in M. By the second condition in the definition of isolated, the full sub-space of M spanned by non-zero components is closed under products and therefore defines a non-unital commutative monoid N with a non-unital commutative monoid map $N \to M$, which induces a map $N_+ \to M$. Again by [10, Proposition 5.4.4.8], and now using the first condition in the definition, we see that $N_+ \to M$ is an equivalence. ∎

Proposition 2.12. *If M is a commutative monoid and* 0 *is an isolated point, then* 0 *is also isolated in $M[\frac{1}{p}]$.*

Proof. If $M = N_+$ where N is a non-unital commutative monoid then $M[\frac{1}{p}] \simeq N[\frac{1}{p}]_+$, where now we consider $(-)[\frac{1}{p}]$ as functor on non-unital commutative monoids. Indeed, $(-)[\frac{1}{p}]$ and $(-)_+$ commute as they are left adjoints of commuting forgetful functors. ∎

Remark 2.13. Theorem 2.8 is not true for an arbitrary commutative monoid, for example, it does not hold for $M = \mathrm{Proj}(R)^{\simeq}$ when $R = \mathbb{Z} \times \mathbb{Z}$. Indeed, consider the point $(\mathbb{Z}, 0)$ in $M[\frac{1}{p}] \simeq \mathrm{Proj}(\mathbb{Z})^{\simeq}[\frac{1}{p}] \times \mathrm{Proj}(\mathbb{Z})^{\simeq}[\frac{1}{p}]$. By Proposition 2.12, 0 is isolated in the second summand and so the component of $(\mathbb{Z}, 0)$ in this product is equivalent to a component of $K(\mathbb{Z})[\frac{1}{p}] \times \{0\}$. However, the point 0 is not isolated in the second summand of

$$M^{\mathrm{gp}}\left[\frac{1}{p}\right] \simeq K(\mathbb{Z})\left[\frac{1}{p}\right] \times K(\mathbb{Z})\left[\frac{1}{p}\right]$$

and the component of $(\mathbb{Z}, 0)$ therein is equivalent to a product of a component of $K(\mathbb{Z})[\frac{1}{p}]$ with itself.

On the other hand, Theorem 2.8 can also be true for a monoid that is not locally monogenic: for example, it holds for formal reasons for the monoid $M = K(\mathbb{Z})_{\geqslant 0} \times K(\mathbb{Z})_{\geqslant 0}$, where $K(\mathbb{Z})_{\geqslant 0} = K(\mathbb{Z}) \times_{\mathbb{Z}} \mathbb{N}$.

We first prove the following partial case of Theorem 2.8.

Proposition 2.14. *Let M be a p-perfect commutative monoid such that for some $x \in M$ the group completion map $M[-x] \to M^{\mathrm{gp}}$ is an equivalence. Denote $T = \mathrm{tel}^x(M)$. Then the map*

$$(M, x) \to (T, x) \to (M^{\mathrm{gp}}, x)$$

is an equivalence between the component of x in M and the component of (the image of) x in M^{gp}.

Proof. Since M is a simple space by Theorem 1.2, we have $M[-x] \simeq M^{\mathrm{gp}} \simeq T$ by [2, Corollary 3.4].[3] We want to show that the map $(M, x) \to (T, x)$ is an equivalence of corresponding connected components of M and T.

Since M is p-perfect, the composition $(M, x) \xrightarrow{\Delta} (M, x)^p \xrightarrow{\mu} (M, px)$ is an equivalence. On $G = \pi_n(M, x)$ and $H = \pi_n(M, px)$ this composition induces $G \to G \times \cdots \times G \to H$ which was computed in [1, Proof of Proposition 2.9]: this composition sends every element g to $p \cdot \mathrm{trans}(g)$ where trans: $(M, x) \to (M, px)$ is the translation by $(p-1)x$. Since the composition is an isomorphism of groups and G is uniquely p-divisible for $n \geqslant 1$ by Theorem 1.2, the map trans: $(M, x) \xrightarrow{+(p-1)x} (M, px)$ must be an equivalence as well. Observe that $T = \mathrm{tel}^x(M) \simeq \mathrm{tel}^{(p-1)x}(M)$. Hence the map $(M, x) \to (T, x)$ is an equivalence of components, as we wanted to show. ∎

Proof of Theorem 2.8. The theorem follows immediately from Proposition 2.14 applied to the p-perfect locally monogenic monoid $M[\frac{1}{p}]$, thanks to Proposition 2.4.

Note that $M[\frac{1}{p}]$ is indeed locally monogenic because of Remark 1.9 (and the fact that being locally monogenic is a π_0-property): any non-zero $x, y \in \pi_0(M)[\frac{1}{p}] \cong \pi_0(M[\frac{1}{p}])$ are of the form $\frac{x_0}{p^n}, \frac{y_0}{p^n}$ for some non-zero $x_0, y_0 \in \pi_0(M)$, so we can find $n \in \mathbb{N}, z \in \pi_0(M)$ such that $nx_0 = y_0 + z$ in $\pi_0(M)$. Moving to $\pi_0(M)[\frac{1}{p}]$, we find $nx = y + \frac{z}{p^n}$. ∎

Example 2.15. Let $M = \mathrm{Fin}^{\simeq}$. In this case, $M^{\mathrm{gp}} \simeq \mathbb{S}$ is the sphere spectrum, so Theorem 2.8 tells us that

$$\mathrm{Fin}^{\simeq}\left[\frac{1}{p}\right] \setminus \{0\} \simeq \mathbb{S}\left[\frac{1}{p}\right] \times_{\mathbb{Z}[\frac{1}{p}]} \left(\mathbb{N}\left[\frac{1}{p}\right] \setminus \{0\}\right),$$

i.e., the union of positive components of $\mathbb{S}[\frac{1}{p}]$.

We already know that p-perfection of M cannot be computed as $\mathrm{tel}_p(M)$ for general M, see Proposition 1.12. Now we can deduce a more general statement, similar to the group-completion situation.[4]

[3] See also [11, Corollary 7] for a more elementary but unpublished account.

[4] But in contrast to the group-complet*e* situation.

Corollary 2.16. *For a general commutative monoid M its p-perfection cannot be expressed as a filtered colimit of maps on M.*

Proof. If it could, then for 1-truncated M, $M[\frac{1}{p}]$ would also be 1-truncated. However we know that $M^{\mathrm{gp}}[\frac{1}{p}]$ need not be n-truncated for any n: $M = \mathrm{Fin}^{\simeq}$ serves as an example. ∎

We will see how to compute the p-perfection functor in the next section.

3. p-perfection via $+$-construction

The goal of this section is to give a $+$-construction model for p-perfection, analogously to the McDuff–Segal group completion theorem which describes group completion in terms of $+$-constructions. More precisely, we prove the following theorem.

Theorem 3.1. *Let M be a commutative monoid. There are canonical maps of commutative monoids*

$$\mathrm{tel}_p(M)^+ \to \mathrm{tel}_p(M^+) \to M\left[\frac{1}{p}\right]$$

that are both equivalences.

Our reference for the $+$-construction is Hoyois' note [7], see also [2, Section 3]. As a consequence of this description of the $+$-construction, we find that it defines a functor $(-)^+ : \mathbf{S} \to \mathbf{S}$ which preserves products, and thus also induces a functor $(-)^+ : \mathrm{CMon} \to \mathrm{CMon}$.

The localization introduced there is at acyclic maps, and hence we call the local objects (equivalently the essential image of $(-)^+$) "acyclic-local" spaces.

Lemma 3.2. *Let $M \in \mathrm{CMon}$ be a commutative monoid. The map $M \to M^+$ induces an equivalence $M[\frac{1}{p}] \to M^+[\frac{1}{p}]$.*

In particular, the canonical map $\mathrm{tel}_p(M^+) \to M[\frac{1}{p}]$ obtained from $M[\frac{1}{p}]$ being simple and hence acyclic-local is an equivalence.

Proof. Observe that $+$-construction commutes with p-perfection as left adjoints of commuting functors. The claim then follows directly from the universal properties using that p-perfect commutative monoids are simple and hence acyclic-local.

The "In particular" part follows from the first part and from Proposition 1.14. ∎

We can now prove the theorem.

Proof of Theorem 3.1. Lemma 3.2 deals with the second map. In particular, $\mathrm{tel}_p(M^+)$ is hypoabelian and so the map $M \to M^+$ induces a map $\mathrm{tel}_p(M)^+ \to \mathrm{tel}_p(M^+)$.

Since $M \to M^+$ is a homology isomorphism, and homology commutes with filtered colimits, we find that this map is a homology isomorphism, and hence (both spaces are acyclic-local) an equivalence, as was to be proved. ∎

As a straightforward corollary of Theorem 3.1, we get an alternative approach to computing the 1-component of the p-inverted K-theory space $\Omega^\infty_1(K(R)[\frac{1}{p}])$.

Corollary 3.3. *Let R be a ring. Then there is a canonical isomorphism*

$$\operatorname{colim}_k H_*\big(\mathrm{BGL}_{p^k}(R);\mathbb{Z}\big) \xrightarrow{\simeq} H_*\left(\Omega^\infty_1\left(K(R)\left[\frac{1}{p}\right]\right);\mathbb{Z}\right)$$

where the colimit is induced by the map

$$\begin{aligned} \mathrm{GL}_{p^k}(R) &\to \mathrm{GL}_{p^{k+1}}(R) \\ A &\mapsto \begin{pmatrix} A & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & A \end{pmatrix}. \end{aligned}$$

Remark 3.4. Corollary 3.3 could be useful, for example, if the direct system

$$\big\{\cdots \to \mathrm{GL}_{p^k}(R) \to \mathrm{GL}_{p^{k+1}}(R) \to \cdots\big\}$$

is simpler for computations than the standard direct system

$$\big\{\cdots \to \mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R) \to \cdots\big\}.$$

In the case when $R = \mathbb{F}_q$, Quillen computed all homology groups $H_*(\mathrm{BGL}_n(R);\mathbb{Z}[\frac{1}{p}])$ unstably [12, Theorem 3], so taking a different direct system does not simplify the computation of $K(\mathbb{F}_q)$. However, for other rings R it could possibly be useful for computing p-inverted K-theory of R.

4. Comparison with an alternative construction

In this section we show that the following three informally described ways of inverting p on a commutative monoid agree: inverting p among commutative monoids, inverting it among spaces where one can add points to themselves any number of times, and inverting it among spaces where we can add points any p-power number of times.

To do that, we observe that a result similar to Theorem 3.1 appears in [2, Proposition 3.2]. We describe here the relationship between the two and how we could in principle deduce the results from the previous section from *loc. cit*. Before doing so, let us recall a special case of *loc. cit.* for the convenience of the reader.

Proposition 4.1 ([2, Proposition 3.2, Remark 3.3]). *Let Q be an $\mathbb{E}_2$-monoid in spaces and $q \in Q$. For every Q-module E, the canonical map*

$$\mathrm{tel}_q(E) \to E[q^{-1}]$$

is acyclic.

Here, $\mathrm{tel}_q(E)$ is the colimit of the diagram $E \xrightarrow{q\cdot -} E \xrightarrow{q\cdot -} E \to \cdots$ which can canonically be made a diagram of Q-modules since Q is $\mathbb{E}_2$, and $E[q^{-1}]$ is the initial Q-module below E on which q acts invertibly.

The relevant connection here is as follows: let $Q = \mathrm{End}(\mathrm{id}_{\mathrm{CMon}}) \simeq (\mathrm{Fin}^{\times})^{\simeq}$ be the endomorphism monoid of the identity on CMon (or the full submonoid thereof spanned by powers of p).

Remark 4.2. That $\mathrm{End}(\mathrm{id}_{\mathrm{CMon}}) \simeq (\mathrm{Fin}^{\times})^{\simeq}$ can be proved by combining Corollary 2.5 (iii) in [5] with the Yoneda lemma: letting U denote the forgetful functor, *loc. cit.* implies $\mathrm{End}(\mathrm{id}_{\mathrm{CMon}}) \simeq \mathrm{End}(U)$ and since U is representable by $\mathrm{Fin}^{\simeq}$, the Yoneda lemma provides the result, where one simply needs to identify the composition of endomorphisms with cartesian product of finite sets.

There is a forgetful functor

$$U : \mathrm{CMon} \to \mathbf{Mod}_Q(\mathbf{S}).$$

This forgetful functor is what we described in the introduction informally as "remembering only how to add elements of a commutative monoid to themselves (a p-power number of times)".

For $M \in \mathrm{CMon}$, $\mathrm{tel}_p(M)$ as defined in Construction 1.10 agrees with $\mathrm{tel}_{p_Q}(M)$ where M is considered as a Q-module and $p_Q \in Q$ is the element corresponding to multiplication by p.

In particular, this forgetful functor sends p-perfect commutative monoids to modules on which $p_Q \in Q$ acts invertibly. Thus we get a commutative square of categories which induces a Beck–Chevalley square:

$$\begin{array}{ccc} \mathrm{CMon} & \longrightarrow & \mathbf{Mod}_Q \\ \downarrow & \swarrow & \downarrow \\ \mathrm{CMon}[\frac{1}{p}] & \longrightarrow & \mathbf{Mod}_{Q[p^{-1}]} \end{array}$$

which is, concretely, for a commutative monoid E, the map $E[p_Q^{-1}] \to E[\frac{1}{p}]$ obtained from considering the map $E \to E[\frac{1}{p}]$ as a map of Q-modules. Furthermore, $\mathrm{tel}_p(E)$ is the same here as in our earlier considerations and we get a commutative triangle:

$$\begin{array}{ccc} & & \mathrm{tel}_p(E) \\ & \swarrow & \downarrow \\ E[p_Q^{-1}] & \longrightarrow & E[\frac{1}{p}] \end{array}$$

As a consequence of our work, the right hand map is acyclic, and [2, Proposition 3.2] states that the left hand map also is, therefore the lower horizontal map also is.

The following question is then natural: *is the bottom horizontal map an equivalence?* We prove below that the answer is yes, which provides an alternative proof of Theorem 3.1

(as well as Proposition 1.14). This also justifies the intuition that we explained in the beginning of the section, by taking $Q = (\mathrm{Fin}^\times)^\simeq$ and the full sub-monoid spanned by powers of p.

We will need the following lemmas.

Lemma 4.3. *Let Q be an $\mathbb{E}_2$-monoid and $q \in Q$. The functor $[q^{-1}] : \mathbf{Mod}_Q \to \mathbf{Mod}_{Q[q^{-1}]}$ preserves finite products, and in particular the same is true for the composite functor*

$$\mathbf{Mod}_Q \to \mathbf{Mod}_{Q[q^{-1}]} \to \mathbf{Mod}_Q.$$

Proof. For the duration of the proof, let hom denote the cartesian internal hom on $\mathbf{Mod}_Q$. Since Q is the free Q-module on a point, we find that the underlying space of $\hom(X, M)$ is $\mathrm{Map}_Q(Q \times X, M)$.

Since Q is $\mathbb{E}_2$, we have a natural transformation $q : \mathrm{id} \to \mathrm{id}$ on $\mathbf{Mod}_Q$ which is a Q-linear lift of the natural map $q : \mathrm{forget} \to \mathrm{forget}$ which simply multiplies by q (more precisely, it is given by the composite

$$X \xrightarrow{(q, \mathrm{id}_X)} Q \times X \to X,$$

where the second map is the action map). By naturality, we get a homotopy between $q_X^* : \mathrm{Map}_Q(X, M) \to \mathrm{Map}_Q(X, M)$ and $(q_M)_* : \mathrm{Map}_Q(X, M) \to \mathrm{Map}_Q(X, M)$, which shows that q acts invertibly on a Q-module M if and only if M is q_X-local for every $X \in \mathbf{Mod}_Q$.

We furthermore have, for $X, Y \in \mathbf{Mod}_Q$, that $q_X \times \mathrm{id}_Y$ commutes with $\mathrm{id}_X \times q_Y$ and that their composite is $q_{X\times Y}$. In particular, if M is $q_{X\times Y}$-local, it is also $q_X \times \mathrm{id}_Y$-local. It follows that if M is q_X-local for all $X \in \mathbf{Mod}_Q$, then for any $Y \in \mathbf{Mod}_Q$, $\hom(Y, M)$ is also q_X-local for all X: indeed $\mathrm{Map}_Q(X, \hom(Y, M)) \simeq \mathrm{Map}_Q(X \times Y, M)$ and q_X on the left corresponds to $q_X \times \mathrm{id}_Y$ on the right.

It follows that the localization $[q^{-1}] : \mathbf{Mod}_Q \to \mathbf{Mod}_{Q[q^{-1}]}$ is compatible with the symmetric monoidal structure in the sense of [10, Definition 2.2.1.6]. There is a unique induced symmetric monoidal structure on $\mathbf{Mod}_{Q[q^{-1}]}$ for which $[q^{-1}]$ is strong symmetric monoidal. To conclude, we are left with proving that it is cartesian, or, equivalently, that the forgetful functor $\mathbf{Mod}_{Q[q^{-1}]} \to \mathbf{Mod}_Q$ is strong symmetric monoidal (as opposed to lax). This follows from the fact that a product of q-local Q-modules is still q-local. ∎

Remark 4.4. For convenience, let us spell out a proof of the above without resorting to symmetric monoidal localizations: for $X, Y \in \mathbf{Mod}_Q$ and $M \in \mathbf{Mod}_{Q[q^{-1}]}$, we have a chain of equivalences, where the second and second to last equivalences follow from the observation that $\hom(Z, M)$ is q-local for every Z:

$$\begin{aligned}\mathrm{Map}_Q(X\times Y, M) &\simeq \mathrm{Map}_Q\left(X, \hom(Y, M)\right) \simeq \mathrm{Map}_Q\left(X[q^{-1}], \hom(Y, M)\right)\\ &\simeq \mathrm{Map}_Q\left(X[q^{-1}] \times Y, M\right) \simeq \mathrm{Map}_Q\left(Y, \hom\left(X[q^{-1}], M\right)\right)\\ &\simeq \mathrm{Map}_Q\left(Y[q^{-1}], \hom\left(X[q^{-1}], M\right)\right) \simeq \mathrm{Map}_Q\left(X[q^{-1}]\times Y[q^{-1}], M\right).\end{aligned}$$

To conclude from there, we crucially use the fact that $X[q^{-1}] \times Y[q^{-1}]$ is q-local.

It follows that $[q^{-1}]$ induces a functor $\mathrm{CMon}(\mathbf{Mod}_Q) \to \mathrm{CMon}(\mathbf{Mod}_{Q[q^{-1}]})$. To conclude, we will need a further lemma. To state it, note that Q acts on commutative monoids in any category with finite products, that is, for every such C we have a forgetful functor $\mathrm{CMon}(C) \to \mathbf{Mod}_Q(C)$.

Lemma 4.5. *The forgetful functor* $\mathrm{CMon} \to \mathbf{Mod}_Q$ *preserves products, so it also lifts to* $\mathrm{CMon}(\mathbf{Mod}_Q)$. *The latter has a forgetful functor to* $\mathbf{Mod}_Q(\mathbf{Mod}_Q) \simeq \mathbf{Mod}_{Q\times Q}$, *and this has* two *forgetful functors to* $\mathbf{Mod}_Q$. *The following composites agree:*

$$\mathrm{CMon} \to \mathrm{CMon}(\mathbf{Mod}_Q) \to \mathbf{Mod}_{Q\times Q} \rightrightarrows \mathbf{Mod}_Q.$$

Proof. This follows from the fact that the two forgetful functors $\mathrm{CMon}(\mathrm{CMon}) \rightrightarrows \mathrm{CMon}$ agree, as follows from [5, Corollary 4.4, Theorem 4.6] (and the fact that for an idempotent algebra A in a symmetric monoidal category, the two unit maps $A \to A \otimes A$ are inverse to the multiplication map $A \otimes A \to A$, and hence are equivalent). ∎

Corollary 4.6. *Let* E *be a commutative monoid. The map* $E[p_Q^{-1}] \to E[\frac{1}{p}]$ *described above is an equivalence.*

Proof. Since both $U : \mathrm{CMon} \to \mathbf{Mod}_Q$ and $[q^{-1}]$ preserve products, we find that $E[p_Q^{-1}]$ has a canonical commutative monoid structure, and the unit map $E \to E[p_Q^{-1}]$ a canonical commutative monoid structure as well.

We claim that $E[p_Q^{-1}]$ is p-perfect, from which the statement will follow. For this, we need to identify p and p_Q on it. We have the following commutative diagram in $\mathbf{Mod}_Q$:

$$\begin{array}{ccc} E & \longrightarrow & E[p_Q^{-1}] \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ E & \longrightarrow & E[p_Q^{-1}]. \end{array}$$

By the universal property of $E[p_{Q^{-1}}]$, to prove that p and p_Q are equivalent on it, it suffices to prove so after restricting to E, and since we also have a commutative diagram like the above with p_Q, it suffices to actually prove that in $\mathbf{Mod}_Q$, $p : E \to E$ and $p_Q : E \to E$ are homotopic, which is the content of Lemma 4.5. ∎

Theorem 2.8 tells us that we can often describe p-perfection in terms of group completion. In general, p-perfection cannot really be seen as inverting an element in the same sense as group completion, but we point out in the following example that this can happen, and that together with Corollary 4.6, this can be used to compute some "unusual" group completions:

Example 4.7. Let R be a commutative semi-ring space, that is, a commutative algebra in $\mathrm{CMon}(\mathbf{S})$ where the latter is equipped with the unique symmetric monoidal structure making the free functor $\mathbf{S} \to \mathrm{CMon}(\mathbf{S})$ symmetric monoidal, cf. [5, Theorem 5.1], see also Corollary 8.11 in *loc. cit.* for examples of such gadgets.

In particular, there is a map from the initial commutative semi-ring, $\mathrm{Fin}^{\simeq}$ to R.

Remembering only the multiplication on R, we get a commutative monoid $(R,\times)$, and we can consider $(R,\times)[-p]$.[5]

Comparing universal properties, one sees easily that this is equivalent to the pushout of commutative monoids

$$(\mathrm{Fin}^{\simeq},\times)[-p] \coprod_{(\mathrm{Fin}^{\simeq},\times)} (R,\times).$$

In general, a pushout of commutative monoids can be understood as a base-change of modules, so that this pushout is itself equivalent, as a space, to $R[p_Q^{-1}]$ in the sense described above (since $(\mathrm{Fin}^{\simeq},\times)\simeq Q$). Finally, by Corollary 4.6, the latter is equivalent to $R[\frac{1}{p}]$. Thus we find that as spaces, we have an equivalence $(R,\times)[-p]\simeq R[\frac{1}{p}]$. Note that while completely trivial in classical algebra, this fact does compare inverting p in a sense which only knows about the multiplication on R, and inverting p in a sense which only knows about the addition on R, and so is potentially surprising in higher degrees.

As an example, let $R=(\mathrm{Proj}_k^{\mathrm{ff}})^{\simeq}$ be the commutative semi-ring of finitely generated faithful-or-0 projective k-modules[6] for k some commutative ring (cf. again [5, Corollary 8.11] for a construction of the associated commutative semi-ring), and for simplicity invert all primes.[7] Combining the above discussion with Theorem 2.8, and paying attention to the 0 component, we find an equivalence of spaces

$$\left((\mathrm{Proj}_k^{\mathrm{ff}})^{\simeq},\otimes_k\right)^{\mathrm{gp}} \simeq \{0\}\coprod K(k)\otimes\mathbb{Q}\times_{K_0(k)\otimes\mathbb{Q}}\left(\pi_0\left((\mathrm{Proj}_k^{\mathrm{ff}})^{\simeq}\right),\otimes_k\right)\otimes\mathbb{Q}_{\geq 0}.$$

As in [3, Chapter IX, Theorem 7.1], we find that π_0 is exactly the subset of $K_0(k)\otimes\mathbb{Q}$ spanned by projective modules of everywhere strictly positive rank, and unwinding the above, that π_1 at the element $k\in(\mathrm{Proj}_k^{\mathrm{ff}})^{\simeq}$ (or any non-zero element) is exactly $K_1(k)\otimes\mathbb{Q}$. This recovers precisely Bass' [3, Chapter IX, Theorem 7.3] and extends it to the higher homotopy groups.

5. The bialgebra Frobenius and colimit perfections

Our goal in this section is to provide an alternative proof of the following theorem.

Theorem 5.1. *Let M be a commutative monoid. Then the map $\mathrm{tel}_p(M)\to\pi_0(\mathrm{tel}_p(M))$ is an $\mathbb{F}_p$-homology isomorphism.*

[5] We have fixed throughout the additive convention for our commutative monoids. While the notation "$-p$" for what feels more like a "p^{-1}" may be confusing, this is the only way to be consistent with our conventions and not get confused about why some statements are not obvious.

[6] These are exactly the finitely generated projective modules P for which there exists a Q such that $P\otimes_k Q$ is free, cf. [3, Chapter IX, Proposition 4.6].

[7] This inverts all elements, by the characterization of faithful projective modules given in [3, Chapter IX, Proposition 4.6].

Combining Theorem 5.1 with Theorem 3.1, we get the following result.

Corollary 5.2. *For any commutative monoid M we have*

$$H_*\left(M\left[\frac{1}{p}\right];\mathbb{F}_p\right) \simeq H_*\big(\mathrm{tel}_p(M);\mathbb{F}_p\big) = 0 \quad \textit{when } * > 0;$$

$$H_0\left(M\left[\frac{1}{p}\right];\mathbb{F}_p\right) \simeq H_0\big(\mathrm{tel}_p(M);\mathbb{F}_p\big) \simeq \mathrm{colim}\big(H_0(M;\mathbb{F}_p) \to H_0(M;\mathbb{F}_p) \to \cdots\big)$$

where the colimit is taken along Frobenius maps $p : M \to M$.

Remark 5.3. Theorem 5.1 can be deduced from the combination of Theorem 3.1 with Corollary 1.8, and also from [1, Corollary 2.10]. However, we wanted to provide another proof because it has a different flavor: in this section, we make use of the structure of bialgebra on $H_*(M;\mathbb{F}_p)$, and the fact that $H_*(\mathrm{tel}_p(M);\mathbb{F}_p)$ can be interpreted in terms of this structure.

Remark 5.4. We note that Tyler Lawson gave in [8] a proof that the monoid $Q[p_Q^{-1}]$ from Section 4 is discrete (and therefore isomorphic to $\mathbb{Z}$). That proof is closely related to our proof of Theorem 5.1.

Construction 5.5. There is a refinement of $\mathbb{F}_p$-homology to a symmetric monoidal functor

$$H_*(-;\mathbb{F}_p) : \mathbf{S} \to \mathrm{coCAlg}^{\mathrm{gr}}(\mathbb{F}_p)$$

because $\mathbb{F}_p$ is a field. Here, $\mathrm{coCAlg}^{\mathrm{gr}}(\mathbb{F}_p)$ is the (ordinary) category of cocommutative[8] coalgebras in the category of ($\mathbb{N}$-)graded vector spaces over $\mathbb{F}_p$. In particular, it sends commutative monoids in $\mathbf{S}$ to commutative, cocommutative bialgebras in graded $\mathbb{F}_p$-vector spaces.

Definition 5.6. Let H be a commutative, cocommutative bialgebra in some symmetric monoidal category. It has a *bialgebra Frobenius map* Φ_p given by $H \xrightarrow{\Delta^{(p)}} H^{\otimes p} \xrightarrow{\mu_p} H$, where $\Delta^{(p)}$, resp. μ_p, is the iterated comultiplication, resp. multiplication.

Observation 5.7. Let M be a commutative monoid. The Frobenius map $p : M \to M$ induces the bialgebra Frobenius map on $H_*(M;\mathbb{F}_p)$.

Therefore, we can describe $H_*(\mathrm{tel}_p(M);\mathbb{F}_p)$ entirely in terms of the bialgebra structure on $H_*(M;\mathbb{F}_p)$, as a colimit along bialgebra Frobenii.

Notation 5.8. From now on, all our homology will be with coefficients in $\mathbb{F}_p$.

To prove the theorem, we will need a simple observation about the comultiplication of H.

Observation 5.9. Let X be a space. $H_*(X)$ is a graded cocommutative coalgebra. Its degree 0 part is generated by grouplike elements, that is, elements x such that $\Delta(x) = x \otimes x$, namely, they are the generators corresponding to connected components of X.

[8] Cocommutativity is to be understood in the graded sense, using the Koszul sign rule.

Furthermore, one can prove relatively easily using the Künneth formula that in the decomposition

$$H_n(X \times X) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(X), \quad \text{for } x \in H_n(C) \subset \bigoplus_{D \in \pi_0(X)} H_n(D),$$

we have

$$\Delta(x) = x \otimes [C] + [C] \otimes x + \text{terms in } H_p(X) \otimes H_q(X), \quad p, q < n,$$

where $[C] \in H_0(X)$ is the corresponding generator.

We now axiomatize this in the following definition.

Definition 5.10. Let H_* be a graded commutative and cocommutative bialgebra. An element $x \in H_n$ is called *weakly primitive* if $\Delta(x) = x \otimes \alpha + \alpha \otimes x +$ terms in $H_p \otimes H_q$, $p, q < n$ where α is a grouplike element in H_0, i.e., $\Delta(\alpha) = \alpha \otimes \alpha$.

Example 5.11. We have essentially argued that the homogeneous elements of $H_*(M)$ that are "$\pi_0(M)$-homogeneous" are weakly primitive whenever M is a commutative monoid.

Our key ingredient is the following proposition.

Proposition 5.12. *Let H_* be a graded commutative and cocommutative bialgebra, and let $x \in H_n$, $n \geq 1$ be weakly primitive. In this case, for any $m \geq 1$, $\Phi_m(x) = m\alpha^{m-1}x +$ terms in the ideal generated by the H_p's, $0 < p < n$.*

Proof. We prove a stronger statement on $\Delta^{(m)}(x) \in H^{\otimes m}$ by induction. Namely, let $t_i^m(\alpha, x)$ denote the tensor $\alpha \otimes \cdots \otimes x \otimes \cdots \otimes \alpha$ with m terms and a single x in position i. We have

$$\Delta^{(m)}(x) = \sum_i t_i^m(\alpha, x) + \text{ terms where at least one tensor factor is in degree } 0 < p < n.$$

Given how $\Delta^{(m)}$ is defined, we can simply prove this by induction. We note that for $i > 1$, Δ applied to the first slot of $t_i^m(\alpha, x)$ is simply $t_{i+1}^{m+1}(\alpha, x)$ because α is grouplike, while for $i = 1$, Δ applied to the first slot of $t_1^m(\alpha, x) = x \otimes \alpha \cdots \otimes \alpha$ is $t_1^{m+1}(\alpha, x) + t_2^{m+1}(\alpha, x) +$some terms containing at least one tensor factor of degree $< n$, and in total this proves the induction step, the case $m = 1$ being our assumption.

Multiplying terms together proves the proposition, since α is in degree 0 and therefore strictly commutes with x so that $\mu_m(t_i^m(\alpha, x)) = \alpha^{m-1}x$. ∎

Corollary 5.13. *Let H_* be a non-negatively graded commutative and cocommutative bialgebra over $\mathbb{F}_p$, and suppose that each H_n, $n \geq 1$ is generated under sums by weakly primitive elements. In this case, for every $x \in H_n$, $\Phi_p^{\circ n}(x) = \Phi_{p^n}(x) = 0$.*

As a consequence, the colimit of $H \xrightarrow{\Phi_p} H \xrightarrow{\Phi_p} H \to \cdots$ is concentrated in degree 0.

Proof. We prove this by induction on $n \geq 1$. For $x \in H_1$ weakly primitive, we find that the previous proposition specializes to $\Phi_p(x) = p\alpha^{p-1}x = 0$, as $p = 0$.

Now assume the statement has been proved for $m < n$, and let $x \in H_n$ be weakly primitive. It suffices to prove that $\Phi_p^{\circ n}(x) = 0$, since such x generate H_n under sums. By Proposition 5.12, $\Phi_p(x) = p\alpha^{p-1}x +$ terms in the ideal generated by $H_p, 0 < p < n$. Since $p = 0$ in $\mathbb{F}_p$, we may apply the induction hypothesis and the fact that $\Phi_p^{\circ n-1}$ is an algebra morphism to conclude that $\Phi_p^{\circ n}(x) = 0$, as was to be shown.

The claim about the colimit follows at once. ∎

Proof of Theorem 5.1. By Example 5.11, if M is a commutative monoid, $H_*(M; \mathbb{F}_p)$ fits into the hypotheses of Corollary 5.13, and $H_*(\mathrm{tel}_p(M))$ is the colimit of $H_*(M)$ along its bialgebra Frobenius, and so Corollary 5.13 guarantees that it is concentrated in degree 0, from which the claim follows. ∎

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