J. Comb. Algebra 10 (2026), 335–363
DOI 10.4171/JCA/127

Join irreducible pseudovarieties of dimension six

Edmond W. H. Lee and John Rhodes

Abstract. A nontrivial pseudovariety of semigroups is *join irreducible* if its inclusion in the join of a collection of pseudovarieties implies its inclusion in at least one member of that collection. The *dimension* of a finitely generated pseudovariety is defined as the order of its minimal generating semigroup. There are precisely 30 join irreducible pseudovarieties of dimension 5 or less. While those of dimension 6 have not been fully identified, 16 such examples are known so far.

In this article, several pseudovarieties are examined to determine their join irreducible subpseudovarieties, achieving complete identification in some cases. By applying these results alongside existing sufficient conditions, join irreducible pseudovarieties of dimension 6 are completely identified. Notably, no additional examples exist beyond the 16 previously known. This is a surprising result, considering that there are 28,634 semigroups of order 6 up to isomorphism.

Dedicated to Jorge Almeida on the occasion of his 70th birthday in 2026

1. Introduction

1.1. Join irreducibility

The class $\mathbf{SEM}$ of all finite semigroups is closed under the formation of homomorphic images, subsemigroups, and finitary direct products; such a class is called a *pseudovariety*. The subpseudovarieties of $\mathbf{SEM}$ form a complete lattice $\mathfrak{L}(\mathbf{SEM})$ under class inclusion. In the 1970s, Eilenberg [5] and Schützenberger established an isomorphism between $\mathfrak{L}(\mathbf{SEM})$ and the lattice of varieties of regular languages. Since then, the lattice $\mathfrak{L}(\mathbf{SEM})$ rose to prominence and much effort has been devoted to the study of pseudovarieties of semigroups. Refer to Almeida [2] and Rhodes and Steinberg [19] for more information.

A natural way to study the lattice $\mathfrak{L}(\mathbf{SEM})$ is to examine its elements with important lattice properties, such as those described in Rhodes and Steinberg [19, Definition 6.1.5]. One such property—join irreducibility—is the main focus of the present

---

Mathematics Subject Classification 2020: 20M07.
Keywords: semigroup, pseudovariety, join irreducible.

article. Recall that a nontrivial pseudovariety $\mathbf{V}$ is *join irreducible* if the following implication holds for any collection $\{\mathbf{V}_i \mid i \in I\}$ of pseudovarieties:

$$\mathbf{V} \subseteq \bigvee\{\mathbf{V}_i \mid i \in I\} \quad \Longrightarrow \quad \mathbf{V} \subseteq \mathbf{V}_i \text{ for some } i.$$

Each join irreducible pseudovariety is necessarily finitely generated. Define the *dimension* of a finitely generated pseudovariety to be the order of its minimal generator.

A finite semigroup $\mathcal{S}$ is *join irreducible* if the pseudovariety $\langle\!\langle \mathcal{S} \rangle\!\rangle$ it generates is join irreducible. The join irreducibility of a finite semigroup can be characterized in several ways, one of which involves the class

$$\mathsf{Excl}(\mathcal{S}) = \{\mathcal{X} \in \mathbf{SEM} \mid \mathcal{S} \notin \langle\!\langle \mathcal{X} \rangle\!\rangle\},$$

called the *exclusion class* of $\mathcal{S}$.

Theorem 1.1 (Rhodes and Steinberg [19, Chapter 7]). *A finite semigroup $\mathcal{S}$ is join irreducible if and only if any of the following conditions hold:*

(i) *if $\mathcal{S}$ divides a direct product $\mathcal{S}_1 \times \mathcal{S}_2$ for some $\mathcal{S}_1, \mathcal{S}_2 \in \mathbf{SEM}$, then there exists $i \in \{1, 2\}$ such that $\mathcal{S}$ divides some finite direct product $\mathcal{S}_i \times \mathcal{S}_i \times \cdots \times \mathcal{S}_i$;*

(ii) $\mathsf{Excl}(\mathcal{S})$ *is a pseudovariety.*

In Theorem 1.1, the pseudovariety $\mathsf{Excl}(\mathcal{S})$ is defined by a single pseudoidentity, called an *exclusion pseudoidentity* for $\mathcal{S}$, and the intersection $\langle\!\langle \mathcal{S} \rangle\!\rangle \cap \mathsf{Excl}(\mathcal{S})$ coincides with the unique maximal subpseudovariety of $\langle\!\langle \mathcal{S} \rangle\!\rangle$.

1.2. Overview of results

A systematic study of join irreducible pseudovarieties was first initiated by Rhodes and Steinberg [19, Chapter 7] and more recently continued by Lee et al. [11, 12]. An overview of relevant results is given in this section.

1.2.1. Some examples of join irreducible semigroups.

Example 1.2 (Rhodes and Steinberg [19, Section 7.1.1]). All atoms in the lattice $\mathfrak{L}(\mathbf{SEM})$ are join irreducible, and they are generated by

(i) the nilpotent semigroup $\mathcal{N}_2 = \langle \mathsf{a} \mid \mathsf{a}^2 = 0 \rangle$;

(ii) the semilattice $\mathcal{S}\ell_2 = \{0, 1\}$;

(iii) the left zero semigroup $\mathcal{L}_2 = \langle \mathsf{e}, \mathsf{f} \mid \mathsf{e}^2 = \mathsf{ef} = \mathsf{e}, \mathsf{f}^2 = \mathsf{fe} = \mathsf{f} \rangle$;

(iv) the right zero semigroup $\mathcal{R}_2 = \langle \mathsf{e}, \mathsf{f} \mid \mathsf{e}^2 = \mathsf{fe} = \mathsf{e}, \mathsf{f}^2 = \mathsf{ef} = \mathsf{f} \rangle$;

(v) the cyclic group $\mathcal{Z}_p = \langle \mathsf{g} \mid \mathsf{g}^p = 1 \rangle$ of prime order p.

Up to isomorphism, $\mathcal{N}_2$, $\mathcal{S}\ell_2$, $\mathcal{L}_2$, $\mathcal{R}_2$, and $\mathcal{Z}_2$ exhaust all semigroups of order 2; see Table 1.

$\mathcal{N}_2$	0	a
0	0	0
a	0	0

$\mathcal{S\ell}_2$	0	1
0	0	0
1	0	1

$\mathcal{L}_2$	e	f
e	e	e
f	f	f

$\mathcal{R}_2$	e	f
e	e	f
f	e	f

$\mathcal{Z}_2$	1	g
1	1	g
g	g	1

Table 1. Multiplication tables of semigroups of order 2.

Example 1.3 (Lee et al. [11, Theorems 4.20 and 5.3]). The following groups are join irreducible:

(i) cyclic group $\mathcal{Z}_{p^m}$ of prime power order p^m;

(ii) dihedral group $\mathcal{D}_p$ of order $2p$ with odd prime p.

Conversely, the groups $\mathcal{Z}_n$ and $\mathcal{D}_n$ are not join irreducible if n has two or more distinct prime divisors.

Example 1.4 (Lee et al. [11, Theorem 5.7]). For each $n \geq 2$, the nilpotent semigroup

$$\mathcal{N}_n = \langle \mathsf{a} \mid \mathsf{a}^n = 0 \rangle$$

of order n is join irreducible.

Example 1.5 (Lee [7, Propositions 2.3 and 3.3]). For each $n \geq 2$, the $\mathcal{J}$-trivial semigroup

$$\mathcal{J}_n = \langle \mathsf{e}, \mathsf{f} \mid \mathsf{e}^2 = \mathsf{e}, \mathsf{f}^2 = \mathsf{f}, \underbrace{\mathsf{efefe}\cdots}_{\text{length } n} = 0 \rangle$$

of order $2n$ is join irreducible.

Example 1.6 (Rhodes and Steinberg [19, Examples 7.3.4 and 7.3.6]). The 0-simple semigroups

$$\mathcal{A}_2 = \langle \mathsf{a}, \mathsf{e} \mid \mathsf{a}^2 = 0, \mathsf{aea} = \mathsf{a}, \mathsf{e}^2 = \mathsf{eae} = \mathsf{e} \rangle,$$
$$\mathcal{B}_2 = \langle \mathsf{a}, \mathsf{b} \mid \mathsf{a}^2 = \mathsf{b}^2 = 0, \mathsf{aba} = \mathsf{a}, \mathsf{bab} = \mathsf{b} \rangle$$

of order 5 are join irreducible; see Table 2.

Example 1.7 (Rhodes and Steinberg [19, Theorem 7.3.10]). For each prime p, define the following $p \times p$ matrices over $\{0, 1\}$:

- the zero matrix O;
- the standard basis matrix $\mathsf{E}_{i,j}$ with the (i, j)-entry being its only nonzero entry; and

- the cyclic permutation matrix

$$\mathsf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then under usual matrix multiplication, the set

$$\mathcal{BZ}_p = \left\{ \mathsf{O}, \mathsf{E}_{i,j}, \mathsf{C}^k \mid i, j, k \in \{1, 2, \ldots, p\} \right\}$$

is a semigroup of order $p^2 + p + 1$ that is join irreducible.

1.2.2. Some constructions of join irreducible semigroups. An important approach to the study of join irreducible semigroups is to construct new examples from known ones. For any semigroup $\mathcal{S}$, let $\mathcal{S}^{\mathsf{op}}$ denote the *opposite* semigroup—obtained by reversing the multiplication on $\mathcal{S}$—and $\mathcal{S}^1$ denote the smallest monoid containing $\mathcal{S}$. Then the operators

$$\mathcal{S} \mapsto \mathcal{S}^{\mathsf{op}} \quad \text{and} \quad \mathcal{S} \mapsto \mathcal{S}^1$$

on **Sem** preserve join irreducibility [11, Examples 1.3 and 1.4].

A more substantial operator involves transformation semigroups. For any set $\mathcal{X}$ and semigroup $\mathcal{S}$, let $(\mathcal{X}, \mathcal{S})$ denote the right regular representation of $\mathcal{S}$ acting on $\mathcal{X}$ by right multiplication and $\overline{\mathcal{X}}$ denote the set of all constant maps on $\mathcal{X}$. More specifically, if $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$, then for any $\mathsf{x} \in \mathcal{X}$ and $\mathsf{s} \in \mathcal{S}$, the maps $\rho_{\mathsf{s}} \in (\mathcal{X}, \mathcal{S})$ and $\overline{\mathsf{x}} \in \overline{\mathcal{X}}$ are given by

$$\rho_{\mathsf{s}} = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ x_1\mathsf{s} & x_2\mathsf{s} & \cdots & x_m\mathsf{s} \end{pmatrix} \quad \text{and} \quad \overline{\mathsf{x}} = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ \mathsf{x} & \mathsf{x} & \cdots & \mathsf{x} \end{pmatrix}.$$

Define the *augmentation* of $\mathcal{S}$ to be

$$\mathcal{S}^{\mathsf{bar}} = \begin{cases} (\mathcal{S}, \mathcal{S}) \cup \overline{\mathcal{S}} & \text{if } (\mathcal{S}, \mathcal{S}) \text{ is a faithful transformation semigroup,} \\ (\mathcal{S}^1, \mathcal{S}) \cup \overline{\mathcal{S}}^1 & \text{otherwise.} \end{cases}$$

Then $\mathcal{S}^{\mathsf{bar}}$ is a semigroup under composition with independent variable written on the left, and the operator $\mathcal{S} \mapsto \mathcal{S}^{\mathsf{bar}}$ on **Sem** preserves join irreducibility. With respect to augmentations of semigroups, only a few specific examples are required in the present article. For more information, refer to Lee et al. [11, Section 2].

$\mathcal{A}_2$	0	a	e	ae	ea
0	0	0	0	0	0
a	0	0	ae	0	a
e	0	ea	e	e	ea
ae	0	a	ae	ae	a
ea	0	0	e	0	ea

$\mathcal{B}_2$	0	a	b	ab	ba
0	0	0	0	0	0
a	0	0	ab	0	a
b	0	ba	0	b	0
ab	0	a	0	ab	0
ba	0	0	b	0	ba

Table 2. Multiplication tables of $\mathcal{A}_2$ and $\mathcal{B}_2$.

Example 1.8. Since the semigroups $\mathcal{Z}_2 = \{1, \mathsf{g}\}$, $\mathcal{L}_2 = \{\mathsf{e}, \mathsf{f}\}$, $\mathcal{R}_2 = \{\mathsf{e}, \mathsf{f}\}$, and $\mathcal{N}_3 = \{0, \mathsf{a}, \mathsf{a}^2\}$ are join irreducible (see Examples 1.2 and 1.4), their augmentations $\mathcal{Z}_2^{\mathsf{bar}}$, $\mathcal{L}_2^{\mathsf{bar}}$, $\mathcal{R}_2^{\mathsf{bar}}$, and $\mathcal{N}_3^{\mathsf{bar}}$ are also join irreducible.

(i) The semigroup $\mathcal{Z}_2^{\mathsf{bar}} = (\mathcal{Z}_2, \mathcal{Z}_2) \cup \overline{\mathcal{Z}}_2$ consists of the transformations

$$\rho_1 = \begin{pmatrix} 1 & \mathsf{g} \\ 1 & \mathsf{g} \end{pmatrix}, \quad \rho_{\mathsf{g}} = \begin{pmatrix} 1 & \mathsf{g} \\ \mathsf{g} & 1 \end{pmatrix}, \quad \overline{1} = \begin{pmatrix} 1 & \mathsf{g} \\ 1 & 1 \end{pmatrix}, \quad \overline{\mathsf{g}} = \begin{pmatrix} 1 & \mathsf{g} \\ \mathsf{g} & \mathsf{g} \end{pmatrix}.$$

(ii) The semigroup $\mathcal{L}_2^{\mathsf{bar}} = (\mathcal{L}_2^1, \mathcal{L}_2) \cup \overline{\mathcal{L}}_2^1$ consists of the transformations

$$\rho_{\mathsf{e}} = \begin{pmatrix} \mathsf{e} & \mathsf{f} & 1 \\ \mathsf{e} & \mathsf{f} & \mathsf{e} \end{pmatrix}, \quad \rho_{\mathsf{f}} = \begin{pmatrix} \mathsf{e} & \mathsf{f} & 1 \\ \mathsf{e} & \mathsf{f} & \mathsf{f} \end{pmatrix},$$
$$\overline{\mathsf{e}} = \begin{pmatrix} \mathsf{e} & \mathsf{f} & 1 \\ \mathsf{e} & \mathsf{e} & \mathsf{e} \end{pmatrix}, \quad \overline{\mathsf{f}} = \begin{pmatrix} \mathsf{e} & \mathsf{f} & 1 \\ \mathsf{f} & \mathsf{f} & \mathsf{f} \end{pmatrix}, \quad \overline{1} = \begin{pmatrix} \mathsf{e} & \mathsf{f} & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(iii) In the semigroup $\mathcal{R}_2^{\mathsf{bar}} = (\mathcal{R}_2, \mathcal{R}_2) \cup \overline{\mathcal{R}}_2$, since

$$\rho_{\mathsf{e}} = \begin{pmatrix} \mathsf{e} & \mathsf{f} \\ \mathsf{e} & \mathsf{e} \end{pmatrix} = \overline{\mathsf{e}} \quad \text{and} \quad \rho_{\mathsf{f}} = \begin{pmatrix} \mathsf{e} & \mathsf{f} \\ \mathsf{f} & \mathsf{f} \end{pmatrix} = \overline{\mathsf{f}},$$

one has $\mathcal{R}_2^{\mathsf{bar}} \cong \mathcal{R}_2$.

(iv) The semigroup $\mathcal{N}_3^{\mathsf{bar}} = (\mathcal{N}_3^1, \mathcal{N}_3) \cup \overline{\mathcal{N}}_3^1$ consists of the transformations

$$\rho_0 = \begin{pmatrix} 0 & \mathsf{a} & \mathsf{a}^2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{\mathsf{a}} = \begin{pmatrix} 0 & \mathsf{a} & \mathsf{a}^2 & 1 \\ 0 & \mathsf{a}^2 & 0 & \mathsf{a} \end{pmatrix}, \quad \rho_{\mathsf{a}^2} = \begin{pmatrix} 0 & \mathsf{a} & \mathsf{a}^2 & 1 \\ 0 & 0 & 0 & \mathsf{a}^2 \end{pmatrix},$$
$$\overline{\mathsf{a}} = \begin{pmatrix} 0 & \mathsf{a} & \mathsf{a}^2 & 1 \\ \mathsf{a} & \mathsf{a} & \mathsf{a} & \mathsf{a} \end{pmatrix}, \quad \overline{\mathsf{a}}^2 = \begin{pmatrix} 0 & \mathsf{a} & \mathsf{a}^2 & 1 \\ \mathsf{a}^2 & \mathsf{a}^2 & \mathsf{a}^2 & \mathsf{a}^2 \end{pmatrix}, \quad \overline{1} = \begin{pmatrix} 0 & \mathsf{a} & \mathsf{a}^2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

See Table 3.

$\mathcal{Z}_2^{\mathsf{bar}}$	1	g	$\overline{1}$	$\overline{g}$
1	1	g	$\overline{1}$	$\overline{g}$
g	g	1	$\overline{1}$	$\overline{g}$
$\overline{1}$	$\overline{1}$	$\overline{g}$	$\overline{1}$	$\overline{g}$
$\overline{g}$	$\overline{g}$	$\overline{1}$	$\overline{1}$	$\overline{g}$

$\mathcal{L}_2^{\mathsf{bar}}$	e	f	$\overline{e}$	$\overline{f}$	$\overline{1}$
e	e	e	$\overline{e}$	$\overline{f}$	$\overline{1}$
f	f	f	$\overline{e}$	$\overline{f}$	$\overline{1}$
$\overline{e}$	$\overline{e}$	$\overline{e}$	$\overline{e}$	$\overline{f}$	$\overline{1}$
$\overline{f}$	$\overline{f}$	$\overline{f}$	$\overline{e}$	$\overline{f}$	$\overline{1}$
$\overline{1}$	$\overline{e}$	$\overline{f}$	$\overline{e}$	$\overline{f}$	$\overline{1}$

$\mathcal{N}_3^{\mathsf{bar}}$	0	a	a^2	$\overline{a}$	$\overline{a}^2$	$\overline{1}$
0	0	0	0	$\overline{a}$	$\overline{a}^2$	$\overline{1}$
a	0	a^2	0	$\overline{a}$	$\overline{a}^2$	$\overline{1}$
a^2	0	0	0	$\overline{a}$	$\overline{a}^2$	$\overline{1}$
$\overline{a}$	0	$\overline{a}^2$	0	$\overline{a}$	$\overline{a}^2$	$\overline{1}$
$\overline{a}^2$	0	0	0	$\overline{a}$	$\overline{a}^2$	$\overline{1}$
$\overline{1}$	0	$\overline{a}$	$\overline{a}^2$	$\overline{a}$	$\overline{a}^2$	$\overline{1}$

Table 3. Multiplication tables of $\mathcal{Z}_2^{\mathsf{bar}}$, $\mathcal{L}_2^{\mathsf{bar}}$, and $\mathcal{N}_3^{\mathsf{bar}}$ with transformation ρ_{s} abbreviated to s.

Example 1.9. Consider the semigroup $\mathcal{J}_2 = \{0, \mathsf{e}, \mathsf{f}, \mathsf{fe}\}$ from Example 1.5.

(i) Since $\mathcal{J}_2$ is join irreducible, its augmentation $\mathcal{J}_2^{\mathsf{bar}}$ is also join irreducible. The transformations in $\mathcal{J}_2^{\mathsf{bar}} = (\mathcal{J}_2, \mathcal{J}_2) \cup \overline{\mathcal{J}_2}$ are

$$\rho_0 = \begin{pmatrix} 0 & \mathsf{e} & \mathsf{f} & \mathsf{fe} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{\mathsf{e}} = \begin{pmatrix} 0 & \mathsf{e} & \mathsf{f} & \mathsf{fe} \\ 0 & \mathsf{e} & \mathsf{fe} & \mathsf{fe} \end{pmatrix},$$
$$\rho_{\mathsf{f}} = \begin{pmatrix} 0 & \mathsf{e} & \mathsf{f} & \mathsf{fe} \\ 0 & 0 & \mathsf{f} & 0 \end{pmatrix}, \quad \rho_{\mathsf{fe}} = \begin{pmatrix} 0 & \mathsf{e} & \mathsf{f} & \mathsf{fe} \\ 0 & 0 & \mathsf{fe} & 0 \end{pmatrix},$$
$$\overline{\mathsf{e}} = \begin{pmatrix} 0 & \mathsf{e} & \mathsf{f} & \mathsf{fe} \\ \mathsf{e} & \mathsf{e} & \mathsf{e} & \mathsf{e} \end{pmatrix}, \quad \overline{\mathsf{f}} = \begin{pmatrix} 0 & \mathsf{e} & \mathsf{f} & \mathsf{fe} \\ \mathsf{f} & \mathsf{f} & \mathsf{f} & \mathsf{f} \end{pmatrix}, \quad \overline{\mathsf{fe}} = \begin{pmatrix} 0 & \mathsf{e} & \mathsf{f} & \mathsf{fe} \\ \mathsf{fe} & \mathsf{fe} & \mathsf{fe} & \mathsf{fe} \end{pmatrix}.$$

See Table 4.

(ii) Let $\widehat{\mathcal{J}}_2^{\mathsf{bar}}$ denote the subset $\mathcal{J}_2^{\mathsf{bar}} \backslash \{\overline{\mathsf{e}}\}$ of $\mathcal{J}_2^{\mathsf{bar}}$. Then it is clear from the multiplication table of $\mathcal{J}_2^{\mathsf{bar}}$ that $\widehat{\mathcal{J}}_2^{\mathsf{bar}}$ is its subsemigroup; see Table 4. The semigroups $\mathcal{J}_2^{\mathsf{bar}}$ and $\widehat{\mathcal{J}}_2^{\mathsf{bar}}$ generate the same pseudovariety because the mapping

$$\mathcal{J}_2^{\mathsf{bar}} \to \widehat{\mathcal{J}}_2^{\mathsf{bar}} \times \widehat{\mathcal{J}}_2^{\mathsf{bar}}$$

given by

$$\rho_0 \mapsto (\rho_0, \rho_0), \quad \rho_{\mathsf{e}} \mapsto (\rho_{\mathsf{f}}, \rho_{\mathsf{e}}), \quad \rho_{\mathsf{f}} \mapsto (\rho_0, \rho_{\mathsf{f}}), \quad \rho_{\mathsf{fe}} \mapsto (\rho_0, \rho_{\mathsf{fe}}),$$
$$\overline{\mathsf{e}} \mapsto (\overline{\mathsf{f}}, \rho_0), \quad \overline{\mathsf{f}} \mapsto (\rho_0, \overline{\mathsf{f}}), \quad \overline{\mathsf{fe}} \mapsto (\rho_0, \overline{\mathsf{fe}})$$

is an embedding. It is easily checked that $\langle\!\langle \mathcal{J}_2^{\mathsf{bar}} \rangle\!\rangle$ is not of dimension 5 or less [11], so that $\langle\!\langle \mathcal{J}_2^{\mathsf{bar}} \rangle\!\rangle = \langle\!\langle \widehat{\mathcal{J}_2^{\mathsf{bar}}} \rangle\!\rangle$ is of dimension 6.

$\mathcal{J}_2^{\mathsf{bar}}$	0	fe	f	e	$\overline{\mathsf{fe}}$	$\overline{\mathsf{f}}$	$\overline{\mathsf{e}}$
0	0	0	0	0	$\overline{\mathsf{fe}}$	$\overline{\mathsf{f}}$	$\overline{\mathsf{e}}$
fe	0	0	0	fe	$\overline{\mathsf{fe}}$	$\overline{\mathsf{f}}$	$\overline{\mathsf{e}}$
f	0	fe	f	fe	$\overline{\mathsf{fe}}$	$\overline{\mathsf{f}}$	$\overline{\mathsf{e}}$
e	0	0	0	e	$\overline{\mathsf{fe}}$	$\overline{\mathsf{f}}$	$\overline{\mathsf{e}}$
$\overline{\mathsf{fe}}$	0	0	0	$\overline{\mathsf{fe}}$	$\overline{\mathsf{fe}}$	$\overline{\mathsf{f}}$	$\overline{\mathsf{e}}$
$\overline{\mathsf{f}}$	0	$\overline{\mathsf{fe}}$	$\overline{\mathsf{f}}$	$\overline{\mathsf{fe}}$	$\overline{\mathsf{fe}}$	$\overline{\mathsf{f}}$	$\overline{\mathsf{e}}$
$\overline{\mathsf{e}}$	0	0	0	$\overline{\mathsf{e}}$	$\overline{\mathsf{fe}}$	$\overline{\mathsf{f}}$	$\overline{\mathsf{e}}$

Table 4. Multiplication table of $\mathcal{J}_2^{\mathsf{bar}}$ with transformation ρ_{s} abbreviated to s.

The operators $\mathcal{S} \mapsto \mathcal{S}^{\mathsf{op}}$ and $\mathcal{S} \mapsto \mathcal{S}^{\mathsf{bar}}$, when iterated appropriately, can further produce many join irreducible semigroups. For instance, by letting $\flat = \mathsf{op} \circ \mathsf{bar} \circ \mathsf{op}$, then starting with any join irreducible semigroup $\mathcal{S}$, the semigroups

$$\mathcal{S}^{\mathsf{bar}},\ (\mathcal{S}^{\mathsf{bar}})^{\flat}, ((\mathcal{S}^{\mathsf{bar}})^{\flat})^{\mathsf{bar}}, (((\mathcal{S}^{\mathsf{bar}})^{\flat})^{\mathsf{bar}})^{\flat}, \ldots$$

are all join irreducible and the pseudovarieties they generate form a strictly increasing chain in the lattice $\mathfrak{L}(\mathbf{SEM})$; see Lee et al. [11, Corollary 4.11].

On the other hand, not only do the operators $\mathcal{S} \mapsto \mathcal{S}^1$ and $\mathcal{S} \mapsto \mathcal{S}^{\mathsf{bar}}$ preserve join irreducibility, they can produce join irreducible semigroups from some non-join irreducible examples.

Example 1.10 (Lee et al. [11, 12]). For each $n \geq 2$,

(i) the semigroup $\mathcal{C}_n = \langle \mathsf{a}, \mathsf{b} \mid \mathsf{ab}^n = 0, \mathsf{ba} = \mathsf{a}, \mathsf{b}^{n+1} = \mathsf{b}^n \rangle$ of order $2n+1$ is not join irreducible but the monoid $\mathcal{C}_n^1$ is join irreducible;

(ii) the semigroup $\mathcal{O}_n = \langle \mathsf{a}, \mathsf{e} \mid \mathsf{a}^n = \mathsf{a}^{n-1}\mathsf{e} = 0, \mathsf{ea} = \mathsf{a}, \mathsf{e}^2 = \mathsf{e} \rangle$ of order $2n-1$ is not join irreducible but its augmentation $\mathcal{O}_n^{\mathsf{bar}}$ is join irreducible.

1.2.3. Determining join irreducibility. An ultimate goal in the study of join irreducible semigroups is to determine which finite semigroups satisfy the property. It is presently unknown if this problem is decidable. Until a full solution is found, examination of individual semigroups is unavoidable. It would therefore be convenient to be able to identify—and hence disregard from consideration—any semigroup $\mathcal{S}$ that is *redundant* in the loose sense that it does not generate a new join irreducible pseudovariety, that is, either $\mathcal{S}$ generates a known example of join irreducible pseudovariety or $\mathcal{S}$ is not join irreducible by virtue of belonging to the complete join of some pseudovarieties but not to any one of the pseudovarieties. If appropriate identities are known, then sufficient conditions would be available to determine redundancy of a finite input semigroup $\mathcal{S}$:

(C1) given any known join irreducible pseudovariety $\mathbf{V}$ defined by some identity system Σ and any identity σ defining its unique maximal subpseudovariety,

$$\mathcal{S} \models \Sigma \text{ and } \mathcal{S} \not\models \sigma \quad \Longrightarrow \quad \langle\!\langle \mathcal{S} \rangle\!\rangle = \mathbf{V} \text{ is join irreducible;}$$

(C2) given any known complete join $\mathbf{V} = \bigvee\{\mathbf{V}_i \mid i \in I\}$ defined by some identity system Σ and any identities $\{\sigma_i \mid i \in I\}$ such that $\mathbf{V}_i \models \sigma_i$ and $\mathbf{V} \not\models \sigma_i$,

$$\mathcal{S} \models \Sigma \text{ and } \mathcal{S} \not\models \sigma_i \text{ for all } i \in I \quad \Longrightarrow \quad \langle\!\langle \mathcal{S} \rangle\!\rangle \subseteq \mathbf{V} \text{ is not join irreducible.}$$

With the aid of a computer, these sufficient conditions can be applied to quickly check the redundancy of a large number of finite semigroups. A project that immensely benefited from this approach is the classification of join irreducible pseudovarieties generated by a small semigroup. Recall that the *dimension* of a finitely generated pseudovariety is the order of its minimal generator.

Theorem 1.11 (Lee et al. [11, Theorem 7.1]). *There are precisely* 30 *join irreducible pseudovarieties of dimension* 5 *or less*:

Dimension 1: $\langle\!\langle \mathcal{L}_2 \rangle\!\rangle$, $\langle\!\langle \mathcal{R}_2 \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_2 \rangle\!\rangle$, $\langle\!\langle \mathcal{S}\ell_2 \rangle\!\rangle$, $\langle\!\langle \mathcal{Z}_2 \rangle\!\rangle$;

Dimension 2: $\langle\!\langle \mathcal{L}_2^1 \rangle\!\rangle$, $\langle\!\langle \mathcal{R}_2^1 \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_3 \rangle\!\rangle$, $\langle\!\langle \mathcal{Z}_3 \rangle\!\rangle$;

Dimension 3: $\langle\!\langle \mathcal{J}_2 \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_2^{\mathsf{bar}} \rangle\!\rangle$, $\langle\!\langle (\mathcal{N}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_4 \rangle\!\rangle$, $\langle\!\langle \mathcal{Z}_2^{\mathsf{bar}} \rangle\!\rangle$, $\langle\!\langle (\mathcal{Z}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle$, $\langle\!\langle \mathcal{Z}_4 \rangle\!\rangle$;

Dimension 4: $\langle\!\langle \mathcal{A}_2 \rangle\!\rangle$, $\langle\!\langle \mathcal{B}_2 \rangle\!\rangle$, $\langle\!\langle \mathcal{J}_2^1 \rangle\!\rangle$, $\langle\!\langle \mathcal{L}_2^{\mathsf{bar}} \rangle\!\rangle$, $\langle\!\langle (\mathcal{L}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle$, $\langle\!\langle (\mathcal{N}_2^{\mathsf{bar}})^1 \rangle\!\rangle$, $\langle\!\langle ((\mathcal{N}_2^{\mathsf{bar}})^1)^{\mathsf{op}} \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_4^1 \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_5 \rangle\!\rangle$, $\langle\!\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\!\rangle$, $\langle\!\langle (\mathcal{O}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle$, $\langle\!\langle \mathcal{Z}_5 \rangle\!\rangle$.

In the proof of Theorem 1.11, all semigroups of order 5 or less are checked against 36 sufficient conditions of types (C1) and (C2) [11, Section 7]. In Table 5, the large number of semigroups [4, Table 3] involved in this proof is compared with the corresponding number of pseudovarieties [3, Table 4].

1.3. Main result and organization

In view of Theorem 1.11, a next step in the study of join irreducible pseudovarieties is to consider those of dimension 6. Achieving a complete classification would

Number of	$n = 2$	3	4	5	Total
non-isomorphic semigroups of order n	5	24	188	1,915	2,132
pseudovarieties of dimension n	5	14	53	147	219
join irreducible pseudovarieties of dimension n	5	5	8	12	30

Table 5. Number of non-isomorphic semigroups of order $n \leq 5$ and number of (join irreducible) pseudovarieties of dimension $n \leq 5$.

be nontrivial since the number of non-isomorphic semigroups of order 6 is 28,634 (seeDistler and Kelsey [4, Table 3]), while the number of pseudovarieties of dimension 6 has not yet been determined but is known to be at least 463 (see Araújo et al. [3, Table 4]). Presently, according to results from Section 1.2, there are 16 known join irreducible pseudovarieties of dimension 6:

$$\begin{gathered}\langle\!\langle \mathcal{A}_2^1 \rangle\!\rangle,\ \langle\!\langle \mathcal{B}_2^1 \rangle\!\rangle,\ \langle\!\langle \mathcal{C}_2^1 \rangle\!\rangle,\ \langle\!\langle (\mathcal{C}_2^1)^{\mathsf{op}} \rangle\!\rangle,\ \langle\!\langle \mathcal{D}_3 \rangle\!\rangle,\ \langle\!\langle \widehat{\mathcal{J}}_2^{\mathsf{bar}} \rangle\!\rangle,\\ \langle\!\langle (\widehat{\mathcal{J}}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle,\ \langle\!\langle \mathcal{J}_3 \rangle\!\rangle,\ \langle\!\langle (\mathcal{L}_2^{\mathsf{bar}})^1 \rangle\!\rangle,\ \langle\!\langle ((\mathcal{L}_2^{\mathsf{bar}})^1)^{\mathsf{op}} \rangle\!\rangle,\ \langle\!\langle \mathcal{N}_3^{\mathsf{bar}} \rangle\!\rangle,\\ \langle\!\langle (\mathcal{N}_3^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle,\ \langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle,\ \langle\!\langle \mathcal{N}_6 \rangle\!\rangle,\ \langle\!\langle \mathcal{Z}_3^{\mathsf{bar}} \rangle\!\rangle,\ \langle\!\langle (\mathcal{Z}_3^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle.\end{gathered} \tag{1.1}$$

Given the large number of semigroups of order 6, it seems inconceivable that there are only 16 join irreducible pseudovarieties of dimension 6: is there another example beyond those from (1.1)? The main goal of the present article is to show that surprisingly, the answer to this question is negative.

Theorem 1.12. *The* 16 *pseudovarieties from* (1.1) *are precisely all join irreducible pseudovarieties of dimension* 6.

Sufficient conditions stronger than those of types (C1) and (C2) will be developed for the proof of Theorem 1.12. For any nontrivial pseudovariety $\mathbf{V}$ and any $m \in \{2, 3, \ldots, \infty\}$, define

$$\mathsf{irr}_m \mathbf{V} = \{\langle\!\langle \mathcal{S} \rangle\!\rangle \mid \mathcal{S} \in \mathbf{V},\ \mathcal{S} \text{ is join irreducible, and } |\mathcal{S}| \leq m\}.$$

In other words, $\mathsf{irr}_m \mathbf{V}$ is the set of all join irreducible subpseudovarieties of $\mathbf{V}$ of dimension at most m; in particular, $\mathsf{irr}_\infty \mathbf{V}$ is the set of all join irreducible subpseudovarieties of $\mathbf{V}$. In general, computing the set $\mathsf{irr}_\infty \mathbf{V}$—or even $\mathsf{irr}_m \mathbf{V}$ for a moderate m—is highly nontrivial, but if it can be accomplished for some pseudovariety $\mathbf{V}$ with known basis of identities, then the resulting sufficient condition to determine redundancy of a finite input semigroup $\mathcal{S}$ would be simpler to apply than those of types (C1) and (C2):

(C3) given any known pseudovariety $\mathbf{V}$ defined by some identity system Σ,

$$\mathcal{S} \models \Sigma \text{ and } |\mathcal{S}| \leq m \quad \Longrightarrow \quad \langle\!\langle \mathcal{S} \rangle\!\rangle \in \mathsf{irr}_m \mathbf{V} \text{ or } \langle\!\langle \mathcal{S} \rangle\!\rangle \text{ is not join irreducible.}$$

In Section 2, the set $\mathsf{irr}_m\langle\!\langle \mathcal{S} \rangle\!\rangle$ is computed for several finite semigroups $\mathcal{S}$ and values $m \geq 6$. Sufficient conditions that determine redundancy, mostly of type (C3), are then established and collected in Section 3. Based on these sufficient conditions, the proof of Theorem 1.12 is given in Section 4.

The article ends with some open problems in Section 5.

2. Join irreducible subpseudovarieties of certain pseudovarieties

This section is concerned with computing the set $\mathsf{irr}_m\langle\!\langle \mathcal{S} \rangle\!\rangle$ for several finite semigroups $\mathcal{S}$ and values $m \geq 6$. Specifically,

- some basic and some known examples of $\mathsf{irr}_\infty\langle\!\langle \mathcal{S} \rangle\!\rangle$ are given in Section 2.1;
- $\mathsf{irr}_\infty\langle\!\langle \mathcal{Z}_p^{\mathsf{bar}} \rangle\!\rangle$ is given in Section 2.2;
- $\mathsf{irr}_\infty\langle\!\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\!\rangle$ and $\mathsf{irr}_\infty\langle\!\langle \mathcal{N}_5 \rangle\!\rangle$ are computed in Sections 2.3 and 2.4, respectively;
- $\mathsf{irr}_6\langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle$, $\mathsf{irr}_6\langle\!\langle \mathcal{N}_4^1 \rangle\!\rangle$, and $\mathsf{irr}_6\langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle$ are computed in Section 2.5.

Results in the present section are required in Section 3 to establish sufficient conditions to determine redundancy of finite semigroups.

2.1. Some basic and known examples

Lemma 2.1. *For any nontrivial pseudovarieties* $\mathbf{V}$ *and* $\mathbf{W}$ *with* $m \in \{2, 3, \ldots, \infty\}$,

(i) $\mathsf{irr}_m(\mathbf{V} \vee \mathbf{W}) = \mathsf{irr}_m\mathbf{V} \cup \mathsf{irr}_m\mathbf{W}$;

(ii) *if* $\mathbf{V} \subseteq \mathbf{W}$, *then* $\mathsf{irr}_m\mathbf{V} \subseteq \mathsf{irr}_m\mathbf{W}$;

(iii) *if* $\mathbf{W}$ *is join irreducible and* $\mathbf{V}$ *is its unique maximal subpseudovariety, then* $\mathsf{irr}_\infty\mathbf{W} = \{\mathbf{W}\} \cup \mathsf{irr}_\infty\mathbf{V}$.

Proof. This is routinely established. ■

Lemma 2.2. *The following equalities hold:*

(i) $\mathsf{irr}_\infty\langle\!\langle \mathcal{A}_2 \rangle\!\rangle = \{\langle\!\langle \mathcal{A}_2 \rangle\!\rangle, \langle\!\langle \mathcal{B}_2 \rangle\!\rangle, \langle\!\langle \mathcal{J}_2 \rangle\!\rangle, \langle\!\langle \mathcal{L}_2 \rangle\!\rangle, \langle\!\langle \mathcal{R}_2 \rangle\!\rangle, \langle\!\langle \mathcal{N}_2 \rangle\!\rangle, \langle\!\langle \mathcal{S}\ell_2 \rangle\!\rangle\}$;

(ii) $\mathsf{irr}_\infty\langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle = \{\langle\!\langle \mathcal{N}_2 \rangle\!\rangle, \langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle, \langle\!\langle \mathcal{S}\ell_2 \rangle\!\rangle\}$;

(iii) $\mathsf{irr}_\infty\langle\!\langle \mathcal{J}_2^1 \rangle\!\rangle = \{\langle\!\langle \mathcal{J}_2 \rangle\!\rangle, \langle\!\langle \mathcal{J}_2^1 \rangle\!\rangle, \langle\!\langle \mathcal{N}_2 \rangle\!\rangle, \langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle, \langle\!\langle \mathcal{S}\ell_2 \rangle\!\rangle\}$;

(iv) $\mathsf{irr}_\infty\langle\!\langle \mathcal{O}_2 \rangle\!\rangle = \{\langle\!\langle \mathcal{N}_2 \rangle\!\rangle, \langle\!\langle \mathcal{S}\ell_2 \rangle\!\rangle\}$.

Proof. Parts (i)–(iii) have been established in Lee [8, Propositions 7 and 9 (vi) and Theorem 10]. As for part (iv), since $\mathcal{O}_2$ is isomorphic to a subsemigroup of $\mathcal{J}_2$, the result easily follows from part (iii) and Lemma 2.1 (ii). ■

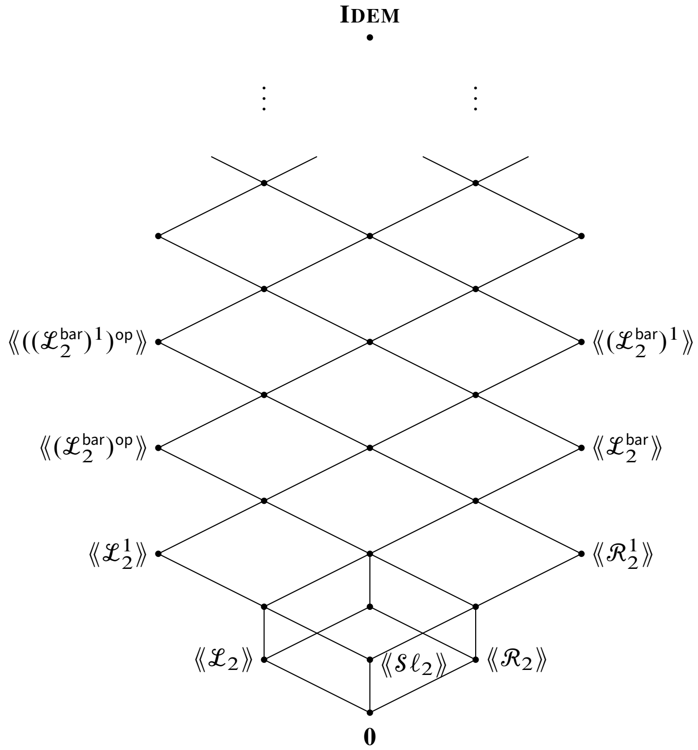

Figure 1. The lattice $\mathfrak{L}(\textbf{Idem})$.

Lemma 2.3. *The equality* $\mathsf{irr}_\infty \langle\!\langle \mathscr{N}_2^{\mathsf{bar}} \rangle\!\rangle = \{\langle\!\langle \mathscr{N}_2 \rangle\!\rangle, \langle\!\langle \mathscr{N}_2^{\mathsf{bar}} \rangle\!\rangle, \langle\!\langle \mathscr{R}_2 \rangle\!\rangle\}$ *holds.*

Proof. Since $\langle\!\langle \mathscr{N}_2^{\mathsf{bar}} \rangle\!\rangle = [\![xyz \approx yz]\!]$ by Lee et al. [11, Proposition 5.12 (i)], it follows from Tishchenko [21, Corollary 2.5 (c) and Proposition 3.4] that the subpseudovarieties of $\langle\!\langle \mathscr{N}_2^{\mathsf{bar}} \rangle\!\rangle$ form the following lattice:

$$\begin{array}{ccccc} \langle\!\langle \mathscr{N}_2 \rangle\!\rangle & \subset & \langle\!\langle \mathscr{N}_2 \rangle\!\rangle \vee \langle\!\langle \mathscr{R}_2 \rangle\!\rangle & \subset & \langle\!\langle \mathscr{N}_2^{\mathsf{bar}} \rangle\!\rangle \\ \cup & & \cup & & \\ \mathbf{0} & \subset & \langle\!\langle \mathscr{R}_2 \rangle\!\rangle & & \end{array}$$

It is clear that the join $\langle\!\langle \mathscr{N}_2 \rangle\!\rangle \vee \langle\!\langle \mathscr{R}_2 \rangle\!\rangle$ is not join irreducible, while the pseudovarieties $\langle\!\langle \mathscr{N}_2 \rangle\!\rangle$, $\langle\!\langle \mathscr{N}_2^{\mathsf{bar}} \rangle\!\rangle$, and $\langle\!\langle \mathscr{R}_2 \rangle\!\rangle$ are join irreducible by results in Section 1.2. ∎

Lemma 2.4. *The following equality holds:*

$$\mathsf{irr}_\infty \langle\!\langle (\mathscr{L}_2^{\mathsf{bar}})^1 \rangle\!\rangle = \{\langle\!\langle \mathscr{L}_2 \rangle\!\rangle, \langle\!\langle \mathscr{L}_2^1 \rangle\!\rangle, \langle\!\langle \mathscr{L}_2^{\mathsf{bar}} \rangle\!\rangle, \langle\!\langle (\mathscr{L}_2^{\mathsf{bar}})^1 \rangle\!\rangle, \langle\!\langle \mathscr{R}_2 \rangle\!\rangle, \langle\!\langle \mathscr{R}_2^1 \rangle\!\rangle, \langle\!\langle \mathscr{S}\ell_2 \rangle\!\rangle\}.$$

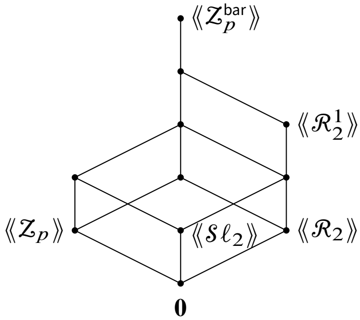

Figure 2. The lattice $\mathfrak{L}\langle\!\langle \mathcal{Z}_p^{\mathsf{bar}} \rangle\!\rangle$

Proof. The well-known lattice $\mathfrak{L}(\mathbf{IDEM})$ of all pseudovarieties of finite idempotent semigroups is shown in Figure 1; see, for example, Almeida [2, Section 5.5]. The unlabeled subpseudovarieties of $\langle\!\langle (\mathcal{L}_2^{\mathsf{bar}})^1 \rangle\!\rangle$ in this figure are clearly not join irreducible, while the other nontrivial subpseudovarieties are join irreducible by results in Section 1.2. ∎

2.2. Subpseudovarieties of $\langle\!\langle \mathcal{Z}_p^{\mathsf{bar}} \rangle\!\rangle$

Proposition 2.5. *The equality* $\mathsf{irr}_\infty \langle\!\langle \mathcal{Z}_p^{\mathsf{bar}} \rangle\!\rangle = \{\langle\!\langle \mathcal{R}_2 \rangle\!\rangle, \langle\!\langle \mathcal{R}_2^1 \rangle\!\rangle, \langle\!\langle \mathcal{S}\ell_2 \rangle\!\rangle, \langle\!\langle \mathcal{Z}_p \rangle\!\rangle, \langle\!\langle \mathcal{Z}_p^{\mathsf{bar}} \rangle\!\rangle\}$ *holds.*

Proof. The semigroup $\mathcal{Z}_p^{\mathsf{bar}} = (\mathcal{Z}_p, \mathcal{Z}_p) \cup \overline{\mathcal{Z}}_p$ consists of the transformations

$$\rho_{\mathsf{x}} = \begin{pmatrix} 1 & \mathsf{g} & \mathsf{g}^2 & \cdots & \mathsf{g}^{p-1} \\ \mathsf{x} & \mathsf{gx} & \mathsf{g}^2\mathsf{x} & \cdots & \mathsf{g}^{p-1}\mathsf{x} \end{pmatrix} \quad \text{and} \quad \overline{\mathsf{x}} = \begin{pmatrix} 1 & \mathsf{g} & \mathsf{g}^2 & \cdots & \mathsf{g}^{p-1} \\ \mathsf{x} & \mathsf{x} & \mathsf{x} & \cdots & \mathsf{x} \end{pmatrix},$$

where $\mathsf{x} \in \mathcal{Z}_p$. Since $\overline{\mathsf{x}} = \overline{1}\rho_{\mathsf{x}}$ for all $\mathsf{x} \in \mathcal{Z}_p$, the semigroup $\mathcal{Z}_p^{\mathsf{bar}}$ is generated by the elements $\overline{1}$ and ρ_{g}:

$$\mathcal{Z}_p^{\mathsf{bar}} = \langle \overline{1}, \rho_{\mathsf{g}} \mid \overline{1}^2 = \overline{1}, \rho_{\mathsf{g}}^p = 1, \rho_{\mathsf{g}}\overline{1} = \overline{1} \rangle.$$

It is then routinely checked that $\mathcal{Z}_p^{\mathsf{bar}}$ is isomorphic to the semigroup RG_p in Tishchenko [21, p. 111], so that the lattice $\mathfrak{L}\langle\!\langle \mathcal{Z}_p^{\mathsf{bar}} \rangle\!\rangle$ is as shown in Figure 2. The unlabeled pseudovarieties in this figure are clearly not join irreducible, while the other nontrivial pseudovarieties are join irreducible by results in Section 1.2. ∎

2.3. Subpseudovarieties of $\langle\!\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\!\rangle$

Proposition 2.6. *The following equality holds:*

$$\mathsf{irr}_\infty \langle\!\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\!\rangle = \{\langle\!\langle \mathcal{N}_2 \rangle\!\rangle, \langle\!\langle \mathcal{N}_2^{\mathsf{bar}} \rangle\!\rangle, \langle\!\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\!\rangle, \langle\!\langle \mathcal{R}_2 \rangle\!\rangle, \langle\!\langle \mathcal{R}_2^1 \rangle\!\rangle, \langle\!\langle \mathcal{S}\ell_2 \rangle\!\rangle\}.$$

Proof. Recall from Example 1.10 (ii) that the pseudovariety $\langle\!\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\!\rangle$ is join irreducible; let $\mathbf{M}$ denote its unique maximal subpseudovariety. In the remainder of this subsection, it is shown that $\mathbf{M} = \langle\!\langle \mathcal{N}_2^{\mathsf{bar}} \rangle\!\rangle \vee \langle\!\langle \mathcal{O}_2 \rangle\!\rangle \vee \langle\!\langle \mathcal{R}_2^1 \rangle\!\rangle$. Then by Lemma 2.1,

$$\begin{aligned} \mathsf{irr}_\infty \langle\!\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\!\rangle &= \{\langle\!\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\!\rangle\} \cup \mathsf{irr}_\infty \mathbf{M} \\ &= \{\langle\!\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\!\rangle\} \cup \mathsf{irr}_\infty \langle\!\langle \mathcal{N}_2^{\mathsf{bar}} \rangle\!\rangle \cup \mathsf{irr}_\infty \langle\!\langle \mathcal{O}_2 \rangle\!\rangle \cup \mathsf{irr}_\infty \langle\!\langle \mathcal{R}_2^1 \rangle\!\rangle. \end{aligned}$$

Therefore, the proposition follows from Lemmas 2.2 (iv), 2.3, and 2.4. ∎

A variable of a word is *simple* if it occurs precisely once. For any word $\mathbf{w}$,

- the *content* of $\mathbf{w}$, denoted by $\mathsf{con}(\mathbf{w})$, is the set of variables occurring in $\mathbf{w}$;
- the set of simple variables of $\mathbf{w}$ is denoted by $\mathsf{sim}(\mathbf{w})$;
- the suffix of $\mathbf{w}$ of length n is denoted by $\mathsf{suf}_n(\mathbf{w})$;
- the *final part* of $\mathbf{w}$, denoted by $\mathsf{fin}(\mathbf{w})$, is the word obtained by retaining the last occurrence of each variable in $\mathbf{w}$.

Lemma 2.7. *Let* $\mathbf{u} \approx \mathbf{v}$ *be any identity. Then*

(i) $\mathcal{N}_2^{\mathsf{bar}} \models \mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathsf{suf}_2(\mathbf{u}) = \mathsf{suf}_2(\mathbf{v})$*;*

(ii) $\mathcal{R}_2^1 \models \mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathsf{fin}(\mathbf{u}) = \mathsf{fin}(\mathbf{v})$*;*

(iii) $\mathcal{O}_2 \models \mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$ *and one of the following holds:*

  (a) $\mathsf{suf}_1(\mathbf{u}) \in \mathsf{sim}(\mathbf{u})$ *and* $\mathsf{suf}_1(\mathbf{v}) \in \mathsf{sim}(\mathbf{v})$ *with* $\mathsf{suf}_1(\mathbf{u}) = \mathsf{suf}_1(\mathbf{v})$*;*

  (b) $\mathsf{suf}_1(\mathbf{u}) \notin \mathsf{sim}(\mathbf{u})$ *and* $\mathsf{suf}_1(\mathbf{v}) \notin \mathsf{sim}(\mathbf{v})$.

Proof. Parts (i) and (ii) are well known and easily verified. As for part (iii), since $\langle\!\langle \mathcal{O}_2 \rangle\!\rangle = [\![x^2 y \approx xy, x^2 y^2 \approx y^2 x^2]\!]$ by Araújo et al. [3, Variety 8 in supplementary material], the result follows from Golubov and Sapir [6, Lemma 7]. ∎

Lemma 2.8. *The pseudovariety* $\mathbf{M}$ *is defined by the identities*

$$x^2 y \approx xy, \quad xyxz \approx yxz, \quad xyzx \approx yxzx. \tag{2.1}$$

Proof. This result follows from Lee et al. [11, Proposition 5.30], where the semigroup $\mathcal{O}_2$ also appears as ℓ_3. ∎

Lemma 2.9. *The identities satisfied by the direct product* $\mathcal{P} = \mathcal{N}_2^{\mathsf{bar}} \times \mathcal{O}_2 \times \mathcal{R}_2^1$ *are axiomatized by* (2.1)*. Consequently,* $\mathbf{M} = \langle\!\langle \mathcal{N}_2^{\mathsf{bar}} \rangle\!\rangle \vee \langle\!\langle \mathcal{O}_2 \rangle\!\rangle \vee \langle\!\langle \mathcal{R}_2^1 \rangle\!\rangle$.

Proof. It is routinely checked that the identities (2.1) are satisfied by the semigroups $\mathcal{N}_2^{\mathsf{bar}}$, $\mathcal{O}_2$, and $\mathcal{R}_2^1$, and so also $\mathcal{P}$. Therefore, it suffices to show that any identity $\mathbf{w}_1 \approx \mathbf{w}_2$ satisfied by $\mathcal{P}$ is deducible from (2.1). It is easily verified that the identities (2.1) can be used to convert each $\mathbf{w}_i$ into a word $\mathbf{w}_i'$ of one of the following types:

$$x_1x_2\cdots x_{m+1}, \quad x_1x_2\cdots x_m y^2, \quad x_1x_2\cdots x_m yzy, \quad m \geq 0.$$

In other words, the deduction (2.1) $\vdash \{\mathbf{w}_1 \approx \mathbf{w}_1', \mathbf{w}_2 \approx \mathbf{w}_2'\}$ holds and $\mathcal{P} \models \mathbf{w}_1' \approx \mathbf{w}_2'$. If the words $\mathbf{w}_1'$ and $\mathbf{w}_2'$ are of different types, then it easily follows from Lemma 2.7 (i), (iii) that the identity $\mathbf{w}_1' \approx \mathbf{w}_2'$ is contradictorily not satisfied by $\mathcal{P}$. Therefore, the words $\mathbf{w}_1'$ and $\mathbf{w}_2'$ are of the same type and so have to be identical by Lemma 2.7 (ii). Hence, (2.1) $\vdash \mathbf{w}_1 \approx \mathbf{w}_1' = \mathbf{w}_2' \approx \mathbf{w}_2$, as required. ∎

2.4. Subpseudovarieties of $\langle\!\langle \mathcal{N}_5 \rangle\!\rangle$

Proposition 2.10. *The equality* $\mathsf{irr}_\infty \langle\!\langle \mathcal{N}_5 \rangle\!\rangle = \{\langle\!\langle \mathcal{N}_2 \rangle\!\rangle, \langle\!\langle \mathcal{N}_3 \rangle\!\rangle, \langle\!\langle \mathcal{N}_4 \rangle\!\rangle, \langle\!\langle \mathcal{N}_5 \rangle\!\rangle\}$ *holds.*

Proof. The pseudovariety $\langle\!\langle \mathcal{N}_5 \rangle\!\rangle$ is defined by the identities

$$x^5 \approx y_1y_2y_3y_4y_5, \quad x^2yz \approx xy^2z, \quad xy \approx yx;$$

see Lee et al. [11, Proposition 5.8]. A description of the lattice $\mathfrak{L}\langle\!\langle \mathcal{N}_5 \rangle\!\rangle$ is given in Figure 3; see Mel'nik [17, Diagram 3], where the variety generated by $\mathcal{N}_5$ appears as the subvariety of $Q\mathfrak{N}_5$ defined by $B_5 = \{x^2yz \approx xy^2z\}$. The pseudovarieties $\langle\!\langle \mathcal{N}_2 \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_3 \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_4 \rangle\!\rangle$, and $\langle\!\langle \mathcal{N}_5 \rangle\!\rangle$ are join irreducible (see Example 1.4), while each unlabeled pseudovariety in Figure 3 is clearly not join irreducible. The remaining pseudovarieties in this figure are

$$\begin{aligned}
\mathbf{V}_1 &= [\![x^2 \approx y_1y_2y_3, xy \approx yx]\!],\\
\mathbf{V}_2 &= [\![x^2 \approx y_1y_2y_3y_4, xy \approx yx]\!],\\
\mathbf{V}_3 &= [\![x^2 \approx y_1y_2y_3y_4y_5, xy \approx yx]\!],\\
\mathbf{V}_4 &= [\![x^3 \approx y_1y_2y_3y_4, x^2y \approx xy^2, xy \approx yx]\!],\\
\mathbf{V}_5 &= [\![x^3 \approx y_1y_2y_3y_4, xy \approx yx]\!],\\
\mathbf{V}_6 &= [\![x^3 \approx y_1y_2y_3y_4y_5, x^2yz \approx xy^2z, xy \approx yx]\!],\\
\mathbf{V}_7 &= [\![x^4 \approx y_1y_2y_3y_4y_5, x^2yz \approx xy^2z, xy \approx yx]\!].
\end{aligned}$$

It follows from Almeida [2, Corollary 6.1.5] that $\langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle = [\![x^3 \approx x^2,\ xy \approx yx]\!]$; the inclusion $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3 \subseteq \langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle$ is routinely verified, so that $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ are not join irreducible by Lemma 2.2 (ii). The pseudovariety $\mathbf{V}_5$ is also not join irreducible [11, Corollary 6.15]. It is shown in Lemmas 2.12 and 2.13 below that the pseudovarieties $\mathbf{V}_4$, $\mathbf{V}_6$, and $\mathbf{V}_7$ are not join irreducible. ∎

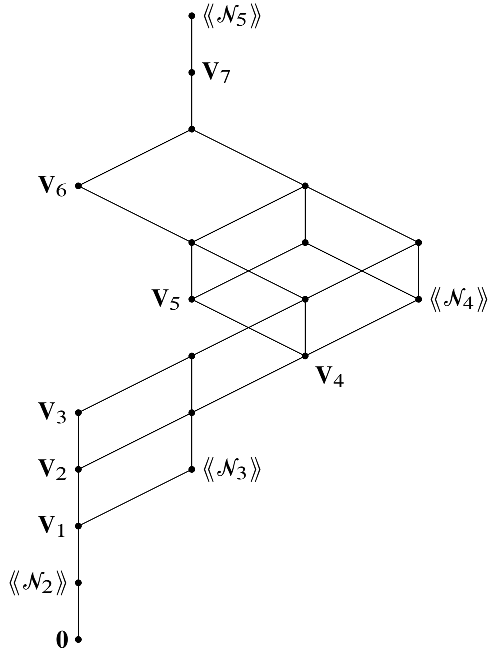

Figure 3. The lattice $\mathfrak{L}\langle\!\langle \mathcal{N}_5 \rangle\!\rangle$.

Lemma 2.11 (Aglianò and Nation [1, Lemma 1.4]). *Suppose that $\mathbb{V}$ is any locally finite variety, so that $\mathbb{V} \cap \mathbf{SEM}$ is a pseudovariety. Then the mapping $\mathbb{X} \mapsto \mathbb{X} \cap \mathbf{SEM}$ is an isomorphism from the lattice $\mathfrak{L}(\mathbb{V})$ onto the lattice $\mathfrak{L}(\mathbb{V} \cap \mathbf{SEM})$.*

The *length* of a word $\mathbf{w}$, denoted by $|\mathbf{w}|$, is the total number of variables in $\mathbf{w}$ counting multiplicity. A word $\mathbf{w}$ is an *isoterm* for a semigroup $\mathcal{S}$ if for any word $\mathbf{v}$,

$$\mathcal{S} \models \mathbf{w} \approx \mathbf{v} \quad \Longrightarrow \quad \mathbf{w} = \mathbf{v}.$$

Let $\mathsf{iso}(\mathcal{S})$ denote the set of isoterms for $\mathcal{S}$.

To show that the pseudovariety $\mathbf{V}_4$ is not join irreducible, the following pseudovarieties are required:

$$\mathbf{D} = [\![xyz_1z_2z_3 \approx yxz_1z_2z_3, x^3z \approx x^2z]\!],$$
$$\mathbf{D}^{\mathsf{op}} = [\![z_1z_2z_3xy \approx z_1z_2z_3yx, zx^3 \approx zx^2]\!].$$

Lemma 2.12. *The pseudovariety $\mathbf{V}_4$ is not join irreducible because*

(i) $\mathbf{V}_4 \subseteq \mathbf{D} \vee \mathbf{D}^{\mathsf{op}}$*;*

(ii) $\mathbf{V}_4 \nsubseteq \mathbf{D}$ *and* $\mathbf{V}_4 \nsubseteq \mathbf{D}^{\mathsf{op}}$.

Proof. The varieties $\mathbb{V}_4$, $\mathbb{D}$, and $\mathbb{D}^{\mathrm{op}}$ defined by the identities that define $\mathbf{V}_4$, $\mathbf{D}$, and $\mathbf{D}^{\mathrm{op}}$, respectively, are locally finite varieties [20, Proposition 3.1] such that

$$\mathbf{V}_4 = \mathbb{V}_4 \cap \mathbf{SEM}, \quad \mathbf{D} = \mathbb{D} \cap \mathbf{SEM}, \quad \mathbf{D}^{\mathrm{op}} = \mathbb{D}^{\mathrm{op}} \cap \mathbf{SEM}.$$

(i) Let $\mathbf{u} \approx \mathbf{v}$ be any nontrivial identity of $\mathbb{D} \vee \mathbb{D}^{\mathrm{op}}$, so that $\mathbf{u}, \mathbf{v} \notin \mathsf{iso}(\mathbb{D}) \cup \mathsf{iso}(\mathbb{D}^{\mathrm{op}})$. If $|\mathbf{u}|, |\mathbf{v}| \geq 4$, then $\mathbf{u} \approx \mathbf{v}$ is easily seen to be an identity of $\mathbb{V}_4$. Therefore, it remains to assume that either $|\mathbf{u}| \leq 3$ or $|\mathbf{v}| \leq 3$. It is routinely verified that up to renaming of variables, x^3 is the only word of length 3 or less that is not in $\mathsf{iso}(\mathbb{D}) \cup \mathsf{iso}(\mathbb{D}^{\mathrm{op}})$. Therefore, generality is not lost by assuming that $\mathbf{u} = x^3$. Since $\mathbf{u} \approx \mathbf{v}$ is a nontrivial identity of $\mathbb{D}$, it is easily seen that $\mathbf{v} = x^{3+k}$ for some $k \geq 1$, whence $\mathbf{u} \approx \mathbf{v}$ is an identity of $\mathbb{V}_4$. Consequently, $\mathbb{V}_4 \subseteq \mathbb{D} \vee \mathbb{D}^{\mathrm{op}}$, so that $\mathbf{V}_4 \subseteq \mathbf{D} \vee \mathbf{D}^{\mathrm{op}}$ by Lemma 2.11.

(ii) It is easily shown that the variety $\mathbb{V}_4$ violates the identity $x^3 y \approx x^2 y$ of $\mathbb{D}$, so that $\mathbb{V}_4 \nsubseteq \mathbb{D}$. The exclusion $\mathbb{V}_4 \nsubseteq \mathbb{D}^{\mathrm{op}}$ holds by symmetry. Therefore, $\mathbf{V}_4 \nsubseteq \mathbf{D}$ and $\mathbf{V}_4 \nsubseteq \mathbf{D}^{\mathrm{op}}$ by Lemma 2.11. ∎

To show that the pseudovarieties $\mathbf{V}_6$ and $\mathbf{V}_7$ are not join irreducible, the following pseudovarieties are required:

$$\mathbf{E} = [\![xyz_1z_2z_3z_4 \approx yxz_1z_2z_3z_4,\, x^3z_1z_2 \approx x^2z_1z_2]\!],$$
$$\mathbf{E}^{\mathrm{op}} = [\![z_1z_2z_3z_4xy \approx z_1z_2z_3z_4yx,\, z_1z_2x^3 \approx z_1z_2x^2]\!].$$

Lemma 2.13. *The pseudovarieties* $\mathbf{V}_6$ *and* $\mathbf{V}_7$ *are not join irreducible because*

(i) $\mathbf{V}_6, \mathbf{V}_7 \subseteq \mathbf{E} \vee \mathbf{E}^{\mathrm{op}}$;

(ii) $\mathbf{V}_6, \mathbf{V}_7 \nsubseteq \mathbf{E}$ *and* $\mathbf{V}_6, \mathbf{V}_7 \nsubseteq \mathbf{E}^{\mathrm{op}}$.

Proof. The varieties $\mathbb{V}_6$, $\mathbb{V}_7$, $\mathbb{E}$, and $\mathbb{E}^{\mathrm{op}}$ defined by the identities that define $\mathbf{V}_6$, $\mathbf{V}_7$, $\mathbf{E}$, and $\mathbf{E}^{\mathrm{op}}$, respectively, are locally finite varieties [20, Proposition 3.1] such that

$$\mathbf{V}_6 = \mathbb{V}_6 \cap \mathbf{SEM}, \quad \mathbf{V}_7 = \mathbb{V}_7 \cap \mathbf{SEM}, \quad \mathbf{E} = \mathbb{E} \cap \mathbf{SEM}, \quad \text{and} \quad \mathbf{E}^{\mathrm{op}} = \mathbb{E}^{\mathrm{op}} \cap \mathbf{SEM}.$$

(i) Let $\mathbf{u} \approx \mathbf{v}$ be any nontrivial identity of $\mathbb{E} \vee \mathbb{E}^{\mathrm{op}}$, so that $\mathbf{u}, \mathbf{v} \notin \mathsf{iso}(\mathbb{E}) \cup \mathsf{iso}(\mathbb{E}^{\mathrm{op}})$. If $|\mathbf{u}|, |\mathbf{v}| \geq 5$, then $\mathbf{u} \approx \mathbf{v}$ is easily seen to be an identity of $\mathbb{V}_6$ and $\mathbb{V}_7$. Therefore, it remains to assume that either $|\mathbf{u}| \leq 4$ or $|\mathbf{v}| \leq 4$. It is routinely verified that up to renaming of variables, x^2y^2 and x^4 are the only words of length 4 or less that are not in $\mathsf{iso}(\mathbb{E}) \cup \mathsf{iso}(\mathbb{E}^{\mathrm{op}})$. Therefore, generality is not lost by assuming that $\mathbf{u} \in \{x^2y^2, x^4\}$. It is easily checked that

(a) if $x^2y^2 \approx \mathbf{v}$ is a nontrivial identity of $\mathbb{E}$, then $\mathbf{v} \in \{x^{2+k}y^2 \mid k \geq 1\}$;

(b) if $x^2y^2 \approx \mathbf{v}$ is a nontrivial identity of $\mathbb{E}^{\mathrm{op}}$, then $\mathbf{v} \in \{x^2y^{2+k} \mid k \geq 1\}$.

Since $\mathbf{u} \approx \mathbf{v}$ is a nontrivial identity of $\mathbb{E} \vee \mathbb{E}^{\mathrm{op}}$, it follows from (a) and (b) that the case $\mathbf{u} = x^2y^2$ is impossible. Therefore, $\mathbf{u} = x^4$ is the only possibility; in this case, it is easily seen that $\mathbf{v} = x^{4+k}$ for some $k \geq 1$, whence $\mathbf{u} \approx \mathbf{v}$ is an identity of $\mathbb{V}_6$ and $\mathbb{V}_7$. Consequently, $\mathbb{V}_6, \mathbb{V}_7 \subseteq \mathbb{E} \vee \mathbb{E}^{\mathrm{op}}$, so that $\mathbf{V}_6, \mathbf{V}_7 \subseteq \mathbf{E} \vee \mathbf{E}^{\mathrm{op}}$ by Lemma 2.11.

(ii) It is easily shown that $\mathbb{V}_6$ and $\mathbb{V}_7$ violate both the identity $x^3z_1z_2 \approx x^2z_1z_2$ of $\mathbb{E}$ and the identity $z_1z_2x^3 \approx z_1z_2x^2$ of $\mathbb{E}^{\mathrm{op}}$. Therefore, $\mathbb{V}_6, \mathbb{V}_7 \notin \mathbb{E}$ and $\mathbb{V}_6, \mathbb{V}_7 \notin \mathbb{E}^{\mathrm{op}}$, whence $\mathbf{V}_6, \mathbf{V}_7 \nsubseteq \mathbf{E}$ and $\mathbf{V}_6, \mathbf{V}_7 \nsubseteq \mathbf{E}^{\mathrm{op}}$ by Lemma 2.11. ∎

2.5. Subpseudovarieties of $\langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle$

For $n \geq 2$, define the set

$$\mathfrak{N}_n = \{\langle\!\langle \mathcal{N}_2 \rangle\!\rangle, \langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle, \langle\!\langle \mathcal{N}_3 \rangle\!\rangle, \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle, \ldots, \langle\!\langle \mathcal{N}_n \rangle\!\rangle, \langle\!\langle \mathcal{N}_n^1 \rangle\!\rangle, \langle\!\langle \mathcal{S}\ell_2 \rangle\!\rangle\}.$$

Proposition 2.14. *The equality* $\mathsf{irr}_6 \langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle = \mathfrak{N}_5$ *holds.*

The proof of this result is given at the end of the subsection after a few intermediate results are first established.

Lemma 2.15. *Let $\mathcal{S}$ be any finite commutative semigroup.*

(i) *If $\mathcal{S} \models \{x^4 \approx x^3, x^3y^2 \approx x^2y^3\}$ and $\mathcal{S} \not\models x^3z_1z_2z_3 \approx x^2z_1z_2z_3$, then $|\mathcal{S}| \geq 7$.*

(ii) *If $\mathcal{S} \models \{x^5 \approx x^4, x^4y^3 \approx x^3y^4\}$ and $\mathcal{S} \not\models x^4z_1z_2 \approx x^3z_1z_2$, then $|\mathcal{S}| \geq 9$.*

(iii) *If $\mathcal{S} \models \{x^6 \approx x^5, x^5y^4 \approx x^4y^5\}$ and $\mathcal{S} \not\models x^5z \approx x^4z$, then $|\mathcal{S}| \geq 10$.*

Proof. These results can be confirmed by `Mace4` [16]. For instance, for part (i), consider the following identities given in Mace4 syntax:

$$(\mathtt{x} * \mathtt{y}) * \mathtt{z} = \mathtt{x} * (\mathtt{y} * \mathtt{z}), \tag{2.2}$$

$$((\mathtt{x} * \mathtt{x}) * \mathtt{x}) * \mathtt{x} = (\mathtt{x} * \mathtt{x}) * \mathtt{x}, \tag{2.3}$$

$$(((\mathtt{x} * \mathtt{x}) * \mathtt{x}) * \mathtt{y}) * \mathtt{y} = (((\mathtt{x} * \mathtt{x}) * \mathtt{y}) * \mathtt{y}) * \mathtt{y}, \tag{2.4}$$

$$((((\mathtt{x} * \mathtt{x}) * \mathtt{x}) * \mathtt{y}) * \mathtt{z}) * \mathtt{u} = (((\mathtt{x} * \mathtt{x}) * \mathtt{y}) * \mathtt{z}) * \mathtt{u}. \tag{2.5}$$

When `Mace4` is used to show that the assumptions {(2.2), (2.3), (2.4)} do not imply the goal (2.5), a minimal counterexample of order 7 is returned. ∎

Lemma 2.16 (Lee et al. [11, Propositions 5.10 and 6.14]). *The following equalities hold:*

(i) $\langle\!\langle \mathcal{N}_n^1 \rangle\!\rangle = [\![x^{n+1} \approx x^n, xy \approx yx]\!]$;

(ii) $\langle\!\langle \mathcal{N}_n^1 \rangle\!\rangle \cap \mathsf{Excl}(\mathcal{N}_n^1) = \langle\!\langle \mathcal{N}_n^1 \rangle\!\rangle \cap [\![x^n y^{n-1} \approx x^{n-1}y^n]\!]$;

(iii) $\langle\!\langle \mathcal{N}_5 \rangle\!\rangle \vee \langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle = [\![x^3z_1z_2z_3 \approx x^2z_1z_2z_3, xy \approx yx]\!]$;

(iv) $\langle\!\langle \mathcal{N}_5 \rangle\!\rangle \vee \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle = [\![x^4 z_1 z_2 \approx x^3 z_1 z_2, xy \approx yx]\!]$;

(v) $\langle\!\langle \mathcal{N}_5 \rangle\!\rangle \vee \langle\!\langle \mathcal{N}_4^1 \rangle\!\rangle = [\![x^5 z \approx x^4 z, xy \approx yx]\!]$.

Lemma 2.17. *The following equalities hold:*

(i) $\mathsf{irr}_6 \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle = \mathfrak{N}_3$;

(ii) $\mathsf{irr}_6 \langle\!\langle \mathcal{N}_4^1 \rangle\!\rangle = \mathfrak{N}_4$.

Proof. (i) Results in Section 1.2 imply that $\mathfrak{N}_3 \subseteq \mathsf{irr}_6 \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle$. Conversely, let $\mathcal{S}$ be any join irreducible semigroup in $\langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle$ with $|\mathcal{S}| \leq 6$, so that $\langle\!\langle \mathcal{S} \rangle\!\rangle \in \mathsf{irr}_6 \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle$. Clearly, if $\langle\!\langle \mathcal{S} \rangle\!\rangle = \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle$, then $\langle\!\langle \mathcal{S} \rangle\!\rangle \in \mathfrak{N}_3$. If $\langle\!\langle \mathcal{S} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{N}_5 \rangle\!\rangle \vee \langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle$, then either

$$\langle\!\langle \mathcal{S} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{N}_5 \rangle\!\rangle \quad \text{or} \quad \langle\!\langle \mathcal{S} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle,$$

so that $\langle\!\langle \mathcal{S} \rangle\!\rangle \in \mathfrak{N}_3$ by Lemma 2.2 (ii) and Proposition 2.10. Hence, assume that

$$\langle\!\langle \mathcal{S} \rangle\!\rangle \neq \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle \quad \text{and} \quad \langle\!\langle \mathcal{S} \rangle\!\rangle \nsubseteq \langle\!\langle \mathcal{N}_5 \rangle\!\rangle \vee \langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle.$$

Then it follows from Lemma 2.16 (i), (ii), (iii) that

$$\mathcal{S} \models \{x^4 \approx x^3, x^3 y^2 \approx x^2 y^3, xy \approx yx\} \quad \text{and} \quad \mathcal{S} \not\models x^3 z_1 z_2 z_3 \approx x^2 z_1 z_2 z_3.$$

However, this is impossible by Lemma 2.15 (i).

(ii) Results in Section 1.2 imply that $\mathfrak{N}_4 \subseteq \mathsf{irr}_6 \langle\!\langle \mathcal{N}_4^1 \rangle\!\rangle$. Conversely, let $\mathcal{S}$ be any join irreducible semigroup in $\langle\!\langle \mathcal{N}_4^1 \rangle\!\rangle$ with $|\mathcal{S}| \leq 6$, so that $\langle\!\langle \mathcal{S} \rangle\!\rangle \in \mathsf{irr}_6 \langle\!\langle \mathcal{N}_4^1 \rangle\!\rangle$. Clearly, if $\langle\!\langle \mathcal{S} \rangle\!\rangle = \langle\!\langle \mathcal{N}_4^1 \rangle\!\rangle$, then $\langle\!\langle \mathcal{S} \rangle\!\rangle \in \mathfrak{N}_4$. If $\langle\!\langle \mathcal{S} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{N}_5 \rangle\!\rangle \vee \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle$, then either

$$\langle\!\langle \mathcal{S} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{N}_5 \rangle\!\rangle \quad \text{or} \quad \langle\!\langle \mathcal{S} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle,$$

so that $\langle\!\langle \mathcal{S} \rangle\!\rangle \in \mathfrak{N}_4$ by Proposition 2.10 and part (i). Hence, assume that

$$\langle\!\langle \mathcal{S} \rangle\!\rangle \neq \langle\!\langle \mathcal{N}_4^1 \rangle\!\rangle \quad \text{and} \quad \langle\!\langle \mathcal{S} \rangle\!\rangle \nsubseteq \langle\!\langle \mathcal{N}_5 \rangle\!\rangle \vee \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle.$$

Then it follows from Lemma 2.16 (i), (ii), (iv) that

$$\mathcal{S} \models \{x^5 \approx x^4, x^4 y^3 \approx x^3 y^4, xy \approx yx\} \quad \text{and} \quad \mathcal{S} \not\models x^4 z_1 z_2 \approx x^3 z_1 z_2.$$

However, this is impossible by Lemma 2.15 (ii). ∎

Proof of Proposition 2.14. Results in Section 1.2 imply that $\mathfrak{N}_5 \subseteq \mathsf{irr}_6 \langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle$. Conversely, let $\mathcal{S}$ be any join irreducible semigroup in $\langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle$ with $|\mathcal{S}| \leq 6$, so that $\langle\!\langle \mathcal{S} \rangle\!\rangle \in \mathsf{irr}_6 \langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle$. If $\langle\!\langle \mathcal{S} \rangle\!\rangle = \langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle$, then $\langle\!\langle \mathcal{S} \rangle\!\rangle \in \mathfrak{N}_5$. If $\langle\!\langle \mathcal{S} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{N}_5 \rangle\!\rangle \vee \langle\!\langle \mathcal{N}_4^1 \rangle\!\rangle$, then either

$$\langle\!\langle \mathcal{S} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{N}_5 \rangle\!\rangle \quad \text{or} \quad \langle\!\langle \mathcal{S} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{N}_4^1 \rangle\!\rangle,$$

so that $\langle\!\langle \mathcal{S} \rangle\!\rangle \in \mathfrak{N}_5$ by Proposition 2.10 and Lemma 2.17 (ii). Hence, assume that

$$\langle\!\langle \mathcal{S} \rangle\!\rangle \neq \langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle \quad \text{and} \quad \langle\!\langle \mathcal{S} \rangle\!\rangle \not\subseteq \langle\!\langle \mathcal{N}_5 \rangle\!\rangle \vee \langle\!\langle \mathcal{N}_4^1 \rangle\!\rangle.$$

Then it follows from Lemma 2.16 (i), (ii), (v) that

$$\mathcal{S} \models \{x^6 \approx x^5, x^5y^4 \approx x^4y^5, xy \approx yx\} \quad \text{and} \quad \mathcal{S} \not\models x^5z \approx x^4z.$$

However, this is impossible by Lemma 2.15 (iii). ∎

3. Sufficient conditions to determine join irreducibility

Recall that a finite semigroup $\mathcal{S}$ is informally said to be *redundant* if either $\langle\!\langle \mathcal{S} \rangle\!\rangle$ is a known example of join irreducible pseudovariety or $\langle\!\langle \mathcal{S} \rangle\!\rangle$ is not join irreducible. In particular, any finite semigroup that generates one the 46 join irreducible pseudovarieties in Theorem 1.11 and (1.1) is redundant.

Results from Section 2 will figure prominently in the present section to establish sufficient conditions that determine redundancy. Hence, it is convenient to give a summary of results from Section 2:

(S1) semigroups in any of the following pseudovarieties are redundant:

$$\langle\!\langle \mathcal{A}_2 \rangle\!\rangle,\ \langle\!\langle \mathcal{J}_2^1 \rangle\!\rangle,\ \langle\!\langle (\mathcal{L}_2^{\mathsf{bar}})^1 \rangle\!\rangle,\ \langle\!\langle ((\mathcal{L}_2^{\mathsf{bar}})^1)^{\mathsf{op}} \rangle\!\rangle,\ \langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle,\ \langle\!\langle \mathcal{N}_2^{\mathsf{bar}} \rangle\!\rangle,$$
$$\langle\!\langle (\mathcal{N}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle,\ \langle\!\langle \mathcal{N}_5 \rangle\!\rangle,\ \langle\!\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\!\rangle,\ \langle\!\langle (\mathcal{O}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle,\ \langle\!\langle \mathcal{Z}_p^{\mathsf{bar}} \rangle\!\rangle,\ \langle\!\langle (\mathcal{Z}_p^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle,$$

where p is any prime;

(S2) finite cyclic groups are redundant;

(S3) semigroups in $\langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle$ of order 6 or less are redundant.

Recall that **SEM** denotes the pseudovariety of all finite semigroups. Let $\mathbf{SEM}_n$ denote the class of all semigroups of order n or less.

Condition 3.1. *Suppose that $\mathcal{S} \in \mathbf{SEM}$ satisfies the identity $xyz \approx xyhyz$. Then $\mathcal{S} \in \langle\!\langle \mathcal{N}_2^{\mathsf{bar}}, (\mathcal{N}_2^{\mathsf{bar}})^{\mathsf{op}}, \mathcal{N}_3 \rangle\!\rangle$ is redundant.*

Proof. By Monzo [18, Corollary 6 and corrigendum], the decomposition

$$[\![xyz \approx xyhyz]\!] = [\![xyz \approx xy]\!] \vee [\![xyz \approx yz]\!] \vee [\![x^3 \approx y_1y_2y_3, xy \approx yx]\!]$$

holds; since the three pseudovarieties on the right coincide with $\langle\!\langle (\mathcal{N}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_2^{\mathsf{bar}} \rangle\!\rangle$, and $\langle\!\langle \mathcal{N}_3 \rangle\!\rangle$, respectively [11, Propositions 5.8 (i) and 5.12 (i)], it follows that $\mathcal{S}$ belongs to the join $\langle\!\langle (\mathcal{N}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle \vee \langle\!\langle \mathcal{N}_2^{\mathsf{bar}} \rangle\!\rangle \vee \langle\!\langle \mathcal{N}_3 \rangle\!\rangle$. If $\mathcal{S}$ is join irreducible, then it belongs to one of $\langle\!\langle (\mathcal{N}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_2^{\mathsf{bar}} \rangle\!\rangle$, and $\langle\!\langle \mathcal{N}_3 \rangle\!\rangle$, and so is redundant by (S1). ∎

Condition 3.2. *Suppose that $\mathcal{S} \in \mathbf{SEM}_6$ satisfies the identities*

$$x^{15} \approx x^3, \quad x^{14}hx \approx x^2hx, \quad x^{13}hx^2 \approx xhx^2, \quad x^{13}hxtx \approx xhxtx,$$
$$x^3hx \approx xhx^3, \quad xhx^2tx \approx x^3htx, \quad xhx^2y^2ty \approx xhy^2x^2ty,$$
$$xhykxytxdy \approx xhykyxtxdy, \quad xhykxytydx \approx xhykyxtydx.$$

Then $\mathcal{S} \in \langle\!\langle \mathcal{N}_3^1, \mathcal{Z}_2^{\mathsf{bar}}, (\mathcal{Z}_2^{\mathsf{bar}})^{\mathsf{op}}, \mathcal{Z}_3, \mathcal{Z}_4 \rangle\!\rangle$ is redundant.

Proof. The identities define the join $\langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle \vee \langle\!\langle \mathcal{Z}_2^{\mathsf{bar}} \rangle\!\rangle \vee \langle\!\langle (\mathcal{Z}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle \vee \langle\!\langle \mathcal{Z}_3 \rangle\!\rangle \vee \langle\!\langle \mathcal{Z}_4 \rangle\!\rangle$ (see Lee and Li [10, Theorem 1.1 and Proposition 3.1]), so that $\mathcal{S}$ belongs to this join. If $\mathcal{S}$ is join irreducible, then it belongs to one of $\langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle$, $\langle\!\langle \mathcal{Z}_2^{\mathsf{bar}} \rangle\!\rangle$, $\langle\!\langle (\mathcal{Z}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\!\rangle$, $\langle\!\langle \mathcal{Z}_3 \rangle\!\rangle$, and $\langle\!\langle \mathcal{Z}_4 \rangle\!\rangle$, and so is redundant by (S1)–(S3). ∎

Condition 3.3. *Suppose that $\mathcal{S} \in \mathbf{SEM}_7$ is an idempotent semigroup. Then either $\mathcal{S} \in \langle\!\langle (\mathcal{L}_2^{\mathsf{bar}})^1 \rangle\!\rangle$ or $\mathcal{S} \in \langle\!\langle ((\mathcal{L}_2^{\mathsf{bar}})^1)^{\mathsf{op}} \rangle\!\rangle$, so that $\mathcal{S}$ is redundant.*

Proof. It is long known that $\langle\!\langle (\mathcal{L}_2^{\mathsf{bar}})^1 \rangle\!\rangle = [\![x^2 \approx x, xyxzxyz \approx xyz]\!]$; see Almeida [2, Figure 5.1], where $\langle\!\langle (\mathcal{L}_2^{\mathsf{bar}})^1 \rangle\!\rangle$ is denoted by $[\![R_3^{\rho} = S_3^{\rho}]\!]_{\mathbf{B}}$. Consider the following identities given in Mace4 syntax:

$$(\mathtt{x} * \mathtt{y}) * \mathtt{z} = \mathtt{x} * (\mathtt{y} * \mathtt{z}), \tag{3.1}$$
$$\mathtt{x} * \mathtt{x} = \mathtt{x}, \tag{3.2}$$
$$(((((\mathtt{x} * \mathtt{y}) * \mathtt{x}) * \mathtt{z}) * \mathtt{x}) * \mathtt{y}) * \mathtt{z} = (\mathtt{x} * \mathtt{y}) * \mathtt{z}, \tag{3.3}$$
$$(((((\mathtt{x} * \mathtt{y}) * \mathtt{z}) * \mathtt{x}) * \mathtt{z}) * \mathtt{y}) * \mathtt{z} = (\mathtt{x} * \mathtt{y}) * \mathtt{z}. \tag{3.4}$$

When `Mace4` is used to show that the assumptions {(3.1), (3.2)} do not imply the goals {(3.3), (3.4)}, a minimal counterexample of order 8 is returned. Hence any idempotent semigroup that is excluded from both $\langle\!\langle (\mathcal{L}_2^{\mathsf{bar}})^1 \rangle\!\rangle$ and $\langle\!\langle ((\mathcal{L}_2^{\mathsf{bar}})^1)^{\mathsf{op}} \rangle\!\rangle$ is of order at least 8. Consequently, if $\mathcal{S} \in \mathbf{SEM}_7$ is an idempotent semigroup, then $\mathcal{S}$ belongs to either $\langle\!\langle (\mathcal{L}_2^{\mathsf{bar}})^1 \rangle\!\rangle$ or $\langle\!\langle ((\mathcal{L}_2^{\mathsf{bar}})^1)^{\mathsf{op}} \rangle\!\rangle$, and so is redundant by (S1). ∎

Condition 3.4. *Suppose that $\mathcal{S} \in \mathbf{SEM}$ satisfies the identities*

$$x^3 \approx x^2, \quad xyxyx \approx xyx, \quad xyxzx \approx xzxyx.$$

Then $\mathcal{S} \in \langle\!\langle \mathcal{A}_2 \rangle\!\rangle$ is redundant.

Proof. The identities define the pseudovariety $\langle\!\langle \mathcal{A}_2 \rangle\!\rangle$ (see Trahtman [22]), so that $\mathcal{S} \in \langle\!\langle \mathcal{A}_2 \rangle\!\rangle$. Therefore, $\mathcal{S}$ is redundant by (S1). ∎

Condition 3.5. *Suppose that $\mathcal{S} \in \mathbf{SEM}$ satisfies the identities*

$$x^{p+1} \approx x, \quad xyxy \approx yx^2y \tag{3.5}$$

for some prime p. Then $\mathcal{S} \in \langle\!\langle \mathcal{Z}_p^{\mathsf{bar}} \rangle\!\rangle$ is redundant.

Proof. As observed in the proof of Proposition 2.5, the semigroup $\mathcal{Z}_p^{\mathsf{bar}}$ is isomorphic to the semigroup RG_p in Tishchenko [21, p. 111]. Let $\mathcal{Z}_p \wr \mathcal{S}\ell_2$ denote the wreath product of $\mathcal{Z}_p$ and $\mathcal{S}\ell_2$. Then the pseudovariety $\langle\!\langle (\mathcal{Z}_p \wr \mathcal{S}\ell_2)^{\mathsf{op}} \rangle\!\rangle$ is generated by the semigroup RG_p and is defined by the identities (3.5); see Tishchenko [21, Proposition 2.25 and Lemma 3.17]. It follows that $\langle\!\langle \mathcal{Z}_p^{\mathsf{bar}} \rangle\!\rangle$ is also defined by (3.5). Therefore, $\mathcal{S}$ belongs to $\langle\!\langle \mathcal{Z}_p^{\mathsf{bar}} \rangle\!\rangle$ and so is redundant by (S1). ∎

Condition 3.6. *Suppose that $\mathcal{S} \in \mathbf{SEM}_6$ satisfies the identities*

$$x^{11} \approx x^5, \quad x^{10}yx \approx x^4yx, \quad x^2yx \approx xyx^2, \quad xyxzx \approx x^2yzx.$$

Then $\mathcal{S} \in \langle\!\langle \mathcal{J}_2^1, \mathcal{L}_2^1, \mathcal{R}_2^1, \mathcal{N}_5^1, \mathcal{Z}_6 \rangle\!\rangle$ is redundant.

Proof. The identities define the join $\langle\!\langle \mathcal{J}_2^1 \rangle\!\rangle \vee \langle\!\langle \mathcal{L}_2^1 \rangle\!\rangle \vee \langle\!\langle \mathcal{R}_2^1 \rangle\!\rangle \vee \langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle \vee \langle\!\langle \mathcal{Z}_6 \rangle\!\rangle$ (see Lee et al. [11, Proposition 6.5]), so that $\mathcal{S}$ belongs to this join. If $\mathcal{S}$ is join irreducible, then it belongs to one of $\langle\!\langle \mathcal{J}_2^1 \rangle\!\rangle$, $\langle\!\langle \mathcal{L}_2^1 \rangle\!\rangle$, $\langle\!\langle \mathcal{R}_2^1 \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle$, and $\langle\!\langle \mathcal{Z}_6 \rangle\!\rangle$, and so is redundant by (S1)–(S3). ∎

Condition 3.7. *Suppose that $\mathcal{S} \in \mathbf{SEM}_6$ satisfies the pseudoidentities*

$$x^{\omega+1} \approx x, \quad x^{\omega}y \approx yx^{\omega}.$$

Then $\mathcal{S}$ is redundant.

Proof. The pseudoidentities define the join $\mathbf{G} \vee \langle\!\langle \mathcal{S}\ell_2 \rangle\!\rangle$, where $\mathbf{G}$ is the pseudovariety of finite groups [2, Figure 9.1]. Hence, $\mathcal{S} \in \mathbf{G} \vee \langle\!\langle \mathcal{S}\ell_2 \rangle\!\rangle$. If $\mathcal{S}$ is join irreducible, then it belongs to either $\mathbf{G}$ or $\langle\!\langle \mathcal{S}\ell_2 \rangle\!\rangle$, and so is redundant by (S1) and Example 1.3. ∎

Condition 3.8. *Suppose that $\mathcal{S} \in \mathbf{SEM}_6$ satisfies the pseudoidentities*

$$x^{\omega+1} \approx x^{\omega}, \quad (xy)^{\omega} \approx (yx)^{\omega}.$$

Then $\mathcal{S}$ is redundant.

Proof. The pseudoidentities define the pseudovariety $\mathbf{J}$ of all finite $\mathcal{J}$-trivial semigroups; see Almeida [2, Exercise 5.1.5 (b)]. By assumption, the dimension of $\langle\!\langle S \rangle\!\rangle$ is at most 6. If $\langle\!\langle S \rangle\!\rangle$ is join irreducible and of dimension 5 or less, then it is one of the pseudovarieties in Theorem 1.11. If $\langle\!\langle S \rangle\!\rangle$ is join irreducible and of dimension 6, then it is one of $\langle\!\langle \mathcal{C}_2^1 \rangle\!\rangle$, $\langle\!\langle (\mathcal{C}_2^1)^{\mathsf{op}} \rangle\!\rangle$, $\langle\!\langle \mathcal{J}_3 \rangle\!\rangle$, $\langle\!\langle \mathcal{N}_5^1 \rangle\!\rangle$, and $\langle\!\langle \mathcal{N}_6 \rangle\!\rangle$; see Lee et al. [12, Theorem 5.1]. ∎

Condition 3.9. *Suppose that $\mathcal{S} \in \mathbf{SEM}$ satisfies the identities*

$$x^2y \approx xy, \quad xyxz \approx yxz.$$

Then $\mathcal{S} \in \langle\!\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\!\rangle$ is redundant.

Proof. The identities define the pseudovariety $\langle\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\rangle$ (see Lee et al. [11, Proposition 5.30]), so that $\mathcal{S} \in \langle\langle \mathcal{O}_2^{\mathsf{bar}} \rangle\rangle$. Therefore, $\mathcal{S}$ is redundant by (S1). ■

Condition 3.10. *Suppose that $\mathcal{S} \in \mathbf{SEM}_6$ satisfies the identities*

$$x^6 \approx x^5, \quad x^5yx \approx x^4yx, \quad xyzt \approx yxzt.$$

Then $\mathcal{S} \in \langle\langle \mathcal{N}_2^{\mathsf{bar}}, \mathcal{N}_5^1 \rangle\rangle$ is redundant.

Proof. The identities define the join $\langle\langle \mathcal{N}_2^{\mathsf{bar}} \rangle\rangle \vee \langle\langle \mathcal{N}_5^1 \rangle\rangle$ (see Lee et al. [11, Proposition 6.17]), so that $\mathcal{S}$ belongs to this join. If $\mathcal{S}$ is join irreducible, then it belongs to either $\langle\langle \mathcal{N}_2^{\mathsf{bar}} \rangle\rangle$ or $\langle\langle \mathcal{N}_5^1 \rangle\rangle$, and so is redundant by (S1) and (S3). ■

Condition 3.11. *Suppose that $\mathcal{S} \in \mathbf{SEM}$ satisfies the identities*

$$x^3 \approx x^2, \quad x^2yx^2 \approx xyx, \quad xyhxty \approx xhxty^2, \quad xyhytx \approx xhy^2tx.$$

Then $\mathcal{S} \in \langle\langle \mathcal{L}_2^{\mathsf{bar}}, \mathcal{N}_2^1 \rangle\rangle$ is redundant.

Proof. The identities define the join $\langle\langle \mathcal{L}_2^{\mathsf{bar}} \rangle\rangle \vee \langle\langle \mathcal{N}_2^1 \rangle\rangle$ (see Lee et al. [11, Proposition 6.21]), so that $\mathcal{S}$ belongs to this join. If $\mathcal{S}$ is join irreducible, then it belongs to either $\langle\langle \mathcal{L}_2^{\mathsf{bar}} \rangle\rangle$ or $\langle\langle \mathcal{N}_2^1 \rangle\rangle$, and so is redundant by (S1). ■

Condition 3.12. *Suppose that $\mathcal{S} \in \mathbf{SEM}_6$ satisfies the identities*

$$\begin{gathered} x^2yx \approx xyx^2, \quad x^2\mathtt{H}_1x\mathtt{H}_2x\mathtt{H}_3x\mathtt{H}_4x \approx x\mathtt{H}_1x\mathtt{H}_2x\mathtt{H}_3x\mathtt{H}_4x, \\ xy\mathtt{H}_1x\mathtt{H}_2y \approx yx\mathtt{H}_1x\mathtt{H}_2y, \quad x\mathtt{H}_1xy\mathtt{H}_2y \approx x\mathtt{H}_1yx\mathtt{H}_2y, \\ x\mathtt{H}_1y\mathtt{H}_2xy \approx x\mathtt{H}_1y\mathtt{H}_2yx, \quad \mathtt{H}_i \in \{1, h_i\}. \end{gathered}$$

Then $\mathcal{S}$ is redundant.

Proof. The above identities define certain subpseudovariety of $\langle\langle (\mathcal{L}_2^{\mathsf{bar}})^1 \rangle\rangle \vee \langle\langle \mathcal{N}_5^1 \rangle\rangle$ (see Lee et al. [12, Proposition 4.4 and Lemma 4.5]), so that $\mathcal{S}$ belongs this join. If $\mathcal{S}$ is join irreducible, then it belongs to either $\langle\langle (\mathcal{L}_2^{\mathsf{bar}})^1 \rangle\rangle$ or $\langle\langle \mathcal{N}_5^1 \rangle\rangle$, and so is redundant by (S1) and (S3). ■

Condition 3.13. *Suppose that $\mathcal{S} \in \mathbf{SEM}$ satisfies the identities*

$$x\mathtt{H}_1x\mathtt{H}_2x \approx \mathtt{H}_1x\mathtt{H}_2x, \quad xy\mathtt{H}_1x\mathtt{H}_2y \approx yx\mathtt{H}_1x\mathtt{H}_2y, \quad \mathtt{H}_i \in \{1, h_i\},$$

but violates the identity $xyxyh^2 \approx x^2y^2h^2$. Then $\mathcal{S}$ generates $\langle\langle (\mathcal{N}_2^{\mathsf{bar}})^1 \rangle\rangle$ and so is redundant.

Proof. This result follows from Lee and Li [9, Corollary 6.6 and Lemma 6.7], where the opposite of the semigroup $\mathcal{N}_2^{\mathsf{bar}}$ appears as P_2. ■

$\mathcal{W}$	a	b	c	d	e
a	a	a	a	d	e
b	a	a	b	d	e
c	a	a	c	d	e
d	a	a	d	d	e
e	a	d	a	d	e

Table 6. Multiplication table of $\mathcal{W}$.

Condition 3.14 (Lee et al. [12, Corollary 4.3]). *Suppose that $\mathcal{S} \in \mathbf{SEM}$ satisfies the identities*

$$x\mathtt{H}_1x\mathtt{H}_2x \approx \mathtt{H}_1x\mathtt{H}_2x, \quad xy\mathtt{H}_1x\mathtt{H}_2y \approx yx\mathtt{H}_1x\mathtt{H}_2y,$$
$$x\mathtt{H}_1xy\mathtt{H}_2y \approx x\mathtt{H}_1yx\mathtt{H}_2y, \quad \mathtt{H}_i \in \{1, h_i\},$$

but violates both the identities $x^2 \approx x$ and $xyhxy \approx xyhyx$. Then $\mathcal{S} \in \langle\!\langle \mathcal{C}_2^1, \mathcal{R}_2^1 \rangle\!\rangle$ is not join irreducible and so is redundant.

Condition 3.15 (Lee et al. [12, Corollary 4.9]). *Suppose that $\mathcal{S} \in \mathbf{SEM}$ satisfies the identities*

$$xy^2x^2 \approx y^2x^2, \quad x^2\mathtt{H}_1x \approx x\mathtt{H}_1x, \quad xy\mathtt{H}_1x\mathtt{H}_2y \approx yx\mathtt{H}_1x\mathtt{H}_2y, \quad \mathtt{H}_i \in \{1, h_i\},$$

but violates both the identities $x^2 \approx x$ and $xyx^2 \approx yx^2$. Then $\mathcal{S} \in \langle\!\langle \mathcal{W}^1 \rangle\!\rangle$ is not join irreducible and so is redundant, where $\mathcal{W} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}\}$ is the semigroup in Table 6.

4. Proof of Theorem 1.12

A finite semigroup is *subdirectly indecomposible* if it has a unique minimal congruence.

Lemma 4.1. *Suppose that $\mathcal{S}$ is any join irreducible semigroup that is not subdirectly indecomposible. Then the pseudovariety $\langle\!\langle \mathcal{S} \rangle\!\rangle$ is of dimension less than $|\mathcal{S}|$.*

Proof. By assumption, there exist distinct minimal congruences μ_1 and μ_2 on $\mathcal{S}$. Then $\mathcal{S}$ is embeddable in $\mathcal{S}/\mu_1 \times \mathcal{S}/\mu_2$, so that $\langle\!\langle \mathcal{S} \rangle\!\rangle \subseteq \langle\!\langle \mathcal{S}/\mu_1 \rangle\!\rangle \vee \langle\!\langle \mathcal{S}/\mu_2 \rangle\!\rangle$. Since the pseudovariety $\langle\!\langle \mathcal{S} \rangle\!\rangle$ is join irreducible, $\langle\!\langle \mathcal{S} \rangle\!\rangle = \langle\!\langle \mathcal{S}/\mu_i \rangle\!\rangle$ for some $i \in \{1, 2\}$. Consequently, $\langle\!\langle \mathcal{S} \rangle\!\rangle$ is of dimension at most $|\mathcal{S}/\mu_i| < |\mathcal{S}|$. ∎

Hence, to identify all new join irreducible pseudovarieties of dimension 6, by Theorem 1.11 and Lemma 4.1, it suffices to consider only those generated by subdirectly indecomposible semigroups of order 6. Up to isomorphism and anti-isomorphism, there exist 15,973 semigroups of order 6, among which 1,823 are subdirectly indecomposible. With the aid of a computer, it can be routinely checked that by Conditions 3.1–3.15 or their dual conditions, 1,816 of these subdirectly indecomposible semigroups are redundant, while the remaining 7 semigroups to which the conditions do not apply are the semigroups $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_7$ in Table 7.

By referring to Tables 2–4, it is easily checked that the semigroups $\mathcal{Y}_2$, $\mathcal{Y}_5$, $\mathcal{Y}_6$, and $\mathcal{Y}_7$ are isomorphic to $(\mathcal{N}_3^{\mathsf{bar}})^{\mathsf{op}}$, $(\widehat{\mathcal{J}}_2^{\mathsf{bar}})^{\mathsf{op}}$, $\mathcal{B}_2^1$, and $\mathcal{A}_2^1$, respectively, and so are redundant. In the remainder of this section, the pseudovarieties $\langle\!\langle \mathcal{Y}_1 \rangle\!\rangle$, $\langle\!\langle \mathcal{Y}_3 \rangle\!\rangle$, and $\langle\!\langle \mathcal{Y}_4 \rangle\!\rangle$ are shown to be non-join irreducible, so that the semigroups $\mathcal{Y}_1$, $\mathcal{Y}_3$, and $\mathcal{Y}_4$ are also redundant. Consequently, there are no join irreducible pseudovarieties of dimension 6 beyond those from (1.1), and the proof of Theorem 1.12 is complete.

4.1. The pseudovariety $\langle\!\langle \mathcal{Y}_1 \rangle\!\rangle$

The identities satisfied by the semigroup $\mathcal{Y}_1$ are axiomatized by

$$\begin{gathered} x\mathtt{H}_1x\mathtt{H}_2x^2 \approx x\mathtt{H}_1x\mathtt{H}_2x, \quad x\mathtt{H}_1x\mathtt{H}_2x \approx x\mathtt{H}_2x\mathtt{H}_1x, \quad x\mathtt{H}_1x\mathtt{H}_2y\mathtt{H}_3y \approx x\mathtt{H}_1x\mathtt{H}_3y\mathtt{H}_2y, \\ x\mathtt{H}_1x\mathtt{H}_2y\mathtt{H}_3y \approx x\mathtt{H}_2y\mathtt{H}_1x\mathtt{H}_3y, \quad x\mathtt{H}_1y\mathtt{H}_2y\mathtt{H}_3x \approx x\mathtt{H}_3x\mathtt{H}_1y\mathtt{H}_2y, \\ x\mathtt{H}_1x\mathtt{H}_2y\mathtt{H}_3y\mathtt{H}_4z\mathtt{H}_5z \approx x\mathtt{H}_1x\mathtt{H}_4z\mathtt{H}_5z\mathtt{H}_2y\mathtt{H}_3y, \quad \mathtt{H}_i \in \{1, h_i\}; \end{gathered} \tag{4.1}$$

see Lee and Zhang [13, Proposition 20.7]. By the proof of this result, any identity violated by $\mathcal{Y}_1$ is also violated by one of the following subsemigroups:

$$\langle \mathsf{d}, \mathsf{e} \rangle = \{\mathsf{a}, \mathsf{c}, \mathsf{d}, \mathsf{e}\} \cong \mathcal{N}_3^1 \quad \text{and} \quad \mathcal{X}_1 = \langle \mathsf{b}, \mathsf{e}, \mathsf{f} \rangle = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{e}, \mathsf{f}\}.$$

It follows that the subpseudovariety $\langle\!\langle \mathcal{N}_3^1, \mathcal{X}_1 \rangle\!\rangle$ of $\langle\!\langle \mathcal{Y}_1 \rangle\!\rangle$ is not proper, whence

$$\langle\!\langle \mathcal{Y}_1 \rangle\!\rangle = \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle \vee \langle\!\langle \mathcal{X}_1 \rangle\!\rangle.$$

Since the identity $xy \approx yx$ of $\mathcal{N}_3^1$ and the identity $x^3 \approx x^2$ of $\mathcal{X}_1$ are both violated by $\mathcal{Y}_1$, one has

$$\langle\!\langle \mathcal{Y}_1 \rangle\!\rangle \nsubseteq \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle \quad \text{and} \quad \langle\!\langle \mathcal{Y}_1 \rangle\!\rangle \nsubseteq \langle\!\langle \mathcal{X}_1 \rangle\!\rangle.$$

Therefore, the pseudovariety $\langle\!\langle \mathcal{Y}_1 \rangle\!\rangle$ is not join irreducible.

Corollary 4.2. *Suppose that $\mathcal{S} \in$ **Sem** satisfies the identities (4.1) but violates both the identities $x^3 \approx x^2$ and $xy \approx yx$. Then $\langle\!\langle \mathcal{S} \rangle\!\rangle$ is a subpseudovariety of $\langle\!\langle \mathcal{Y}_1 \rangle\!\rangle = \langle\!\langle \mathcal{N}_3^1, \mathcal{X}_1 \rangle\!\rangle$ that is not join irreducible.*

$\mathcal{Y}_1$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	a	a	c
c	a	a	a	a	c	c
d	a	a	a	c	d	d
e	a	b	c	d	e	e
f	a	b	c	d	f	f

$\mathcal{Y}_2$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	a	a	c
c	c	c	c	c	c	c
d	a	a	a	b	c	e
e	e	e	e	e	e	e
f	f	f	f	f	f	f

$\mathcal{Y}_3$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	a	a	c
c	c	c	c	c	c	c
d	a	b	c	d	e	a
e	a	b	c	d	e	c
f	f	f	f	f	f	f

$\mathcal{Y}_4$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	a	b	c
a	a	a	a	c	c	c
d	a	b	c	d	d	d
e	a	b	c	d	e	f
f	a	b	c	f	f	f

$\mathcal{Y}_5$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	a	b	c
c	c	c	c	c	c	c
d	a	b	c	d	b	c
e	a	a	a	a	e	f
f	f	f	f	f	f	f

$\mathcal{Y}_6$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	b	b	c
c	a	b	c	a	c	a
d	a	a	a	d	d	f
e	a	b	c	d	e	f
f	a	d	f	a	f	a

$\mathcal{Y}_7$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	b	b	c
c	a	b	c	b	c	c
d	a	a	a	d	d	f
e	a	b	c	d	e	f
f	a	d	f	d	f	f

Table 7. Multiplication tables of $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_7$.

4.2. The pseudovariety $\langle\langle \mathcal{Y}_3 \rangle\rangle$

The identities satisfied by the semigroup $\mathcal{Y}_3$ are axiomatized by

$$x\mathtt{H}_1x^2 \approx x\mathtt{H}_1x, \quad x\mathtt{H}_1y\mathtt{H}_2xy \approx x\mathtt{H}_1y\mathtt{H}_2y,$$
$$x\mathtt{H}_1y\mathtt{H}_2yx \approx x\mathtt{H}_1y\mathtt{H}_2x, \quad \mathtt{H}_i \in \{1, h_i\}; \tag{4.2}$$

see Lee and Zhang [13, Proposition 22.9]. By the proof of this result, any identity violated by $\mathcal{Y}_3$ is also violated by one of the following subsemigroups:

$$\langle \mathsf{d}, \mathsf{e}, \mathsf{f} \rangle = \{\mathsf{a}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}\} \cong (\mathcal{L}_2^{\mathsf{bar}})^{\mathsf{op}} \quad \text{and} \quad \langle \mathsf{b}, \mathsf{d}, \mathsf{f} \rangle = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{f}\} \cong \mathcal{W}^{\mathsf{op}}.$$

It follows that the subpseudovariety $\langle\langle (\mathcal{L}_2^{\mathsf{bar}})^{\mathsf{op}}, \mathcal{W}^{\mathsf{op}} \rangle\rangle$ of $\langle\langle \mathcal{Y}_3 \rangle\rangle$ is not proper, whence

$$\langle\langle \mathcal{Y}_3 \rangle\rangle = \langle\langle (\mathcal{L}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\rangle \vee \langle\langle \mathcal{W}^{\mathsf{op}} \rangle\rangle.$$

Since the identity $x^2 \approx x$ of $(\mathcal{L}_2^{\mathsf{bar}})^{\mathsf{op}}$ and the identity $xyx \approx xy^2$ of $\mathcal{W}^{\mathsf{op}}$ are both violated by $\mathcal{Y}_3$, one has $\langle\langle \mathcal{Y}_3 \rangle\rangle \nsubseteq \langle\langle (\mathcal{L}_2^{\mathsf{bar}})^{\mathsf{op}} \rangle\rangle$ and $\langle\langle \mathcal{Y}_3 \rangle\rangle \nsubseteq \langle\langle \mathcal{W}^{\mathsf{op}} \rangle\rangle$. Therefore, the pseudovariety $\langle\langle \mathcal{Y}_3 \rangle\rangle$ is not join irreducible.

Corollary 4.3. *Suppose $\mathcal{S} \in \mathbf{SEM}$ satisfies the identities (4.2), but violates both the identities $x^2 \approx x$ and $xyx \approx xy^2$. Then $\langle\langle \mathcal{S} \rangle\rangle$ is a subpseudovariety of $\langle\langle \mathcal{Y}_3 \rangle\rangle = \langle\langle (\mathcal{L}_2^{\mathsf{bar}})^{\mathsf{op}}, \mathcal{W}^{\mathsf{op}} \rangle\rangle$ that is not join irreducible.*

4.3. The pseudovariety $\langle\langle \mathcal{Y}_4 \rangle\rangle$

Let $\mathcal{X}_4$ be the subsemigroup of $\mathcal{Y}_4$ obtained by removing the identity element e, so that $\mathcal{X}_4^1 \cong \mathcal{Y}_4$. It is routinely checked that $\mathcal{X}_4$ satisfies the identities

$$x^3 \approx x^2, \quad x^2yx^2 \approx xyx, \quad xyhxty \approx xhxty^2, \quad xyhytx \approx xhy^2tx.$$

These identities define the pseudovariety $\langle\langle \mathcal{L}_2^{\mathsf{bar}}, \mathcal{N}_2^1 \rangle\rangle$ (see Lee et al. [11, Proposition 6.21]), so that $\mathcal{X}_4 \in \langle\langle \mathcal{L}_2^{\mathsf{bar}}, \mathcal{N}_2^1 \rangle\rangle$. It follows that

$$\langle\langle \mathcal{Y}_4 \rangle\rangle = \langle\langle \mathcal{X}_4^1 \rangle\rangle \subseteq \langle\langle (\mathcal{L}_2^{\mathsf{bar}})^1 \rangle\rangle \vee \langle\langle \mathcal{N}_2^1 \rangle\rangle.$$

However, the identity $x^2 \approx x$ of $(\mathcal{L}_2^{\mathsf{bar}})^1$ and the identity $xy \approx yx$ of $\mathcal{N}_2^1$ are both violated by $\mathcal{Y}_4$, so that $\langle\langle \mathcal{Y}_4 \rangle\rangle \nsubseteq \langle\langle (\mathcal{L}_2^{\mathsf{bar}})^1 \rangle\rangle$ and $\langle\langle \mathcal{Y}_4 \rangle\rangle \nsubseteq \langle\langle \mathcal{N}_2^1 \rangle\rangle$. Therefore, the pseudovariety $\langle\langle \mathcal{Y}_4 \rangle\rangle$ is not join irreducible.

5. Open problems

Naturally, the first open problem on the list is to improve the main result.

Problem 5.1. Identify all join irreducible pseudovarieties of dimension 7.

Much perseverance is required to solve this problem since up to isomorphism, the number of semigroups of order 7 is 1,627,672; see Distler and Kelsey [4, Table 3]. Presently, the only known join irreducible pseudovarieties of dimension 7 are

$$\langle\!\langle \mathcal{BZ}_2 \rangle\!\rangle,\ \langle\!\langle \mathcal{J}_3^1 \rangle\!\rangle,\ \langle\!\langle (\mathcal{N}_3^{\mathsf{bar}})^1 \rangle\!\rangle,\ \langle\!\langle ((\mathcal{N}_3^{\mathsf{bar}})^{\mathsf{op}})^1 \rangle\!\rangle,\ \langle\!\langle \mathcal{N}_6^1 \rangle\!\rangle,\ \langle\!\langle \mathcal{N}_7 \rangle\!\rangle,\ \langle\!\langle \mathcal{Z}_7 \rangle\!\rangle.$$

It would also be helpful to compute, for as many pseudovarieties $\mathbf{V}$ as possible, the set $\mathsf{irr}_m\mathbf{V}$ with some $m \geq 7$.

Computation of the sets $\mathsf{irr}_\infty\langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle$ and $\mathsf{irr}_\infty\langle\!\langle \mathcal{N}_5 \rangle\!\rangle$ relied heavily on the descriptions of the lattices $\mathfrak{L}\langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle$ and $\mathfrak{L}\langle\!\langle \mathcal{N}_5 \rangle\!\rangle$; see Lee [8, Proposition 7] and Proposition 2.10. Since the descriptions of the lattices $\mathfrak{L}\langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle$ and $\mathfrak{L}\langle\!\langle \mathcal{N}_6 \rangle\!\rangle$ have not yet been found, the problem of identifying all their join irreducible pseudovarieties is probably very challenging.

Problem 5.2. Compute $\mathsf{irr}_\infty\langle\!\langle \mathcal{N}_n^1 \rangle\!\rangle$ for $n \geq 3$ and $\mathsf{irr}_\infty\langle\!\langle \mathcal{N}_n \rangle\!\rangle$ for $n \geq 6$.

Presently, among all pseudovarieties of nilpotent semigroups, the only known join irreducible examples are $\{\langle\!\langle \mathcal{N}_n \rangle\!\rangle \mid n \geq 2\}$. Therefore, in view of Proposition 2.10, it seems plausible that

$$\mathsf{irr}_\infty\langle\!\langle \mathcal{N}_n \rangle\!\rangle = \{\langle\!\langle \mathcal{N}_2 \rangle\!\rangle, \langle\!\langle \mathcal{N}_3 \rangle\!\rangle, \ldots, \langle\!\langle \mathcal{N}_n \rangle\!\rangle\}.$$

Similarly, by referring to Lemma 2.2 (ii), it is reasonable to conjecture that

$$\mathsf{irr}_\infty\langle\!\langle \mathcal{N}_n^1 \rangle\!\rangle = \{\langle\!\langle \mathcal{S}\ell_2 \rangle\!\rangle, \langle\!\langle \mathcal{N}_2 \rangle\!\rangle, \langle\!\langle \mathcal{N}_2^1 \rangle\!\rangle, \langle\!\langle \mathcal{N}_3 \rangle\!\rangle, \langle\!\langle \mathcal{N}_3^1 \rangle\!\rangle, \ldots, \langle\!\langle \mathcal{N}_n \rangle\!\rangle, \langle\!\langle \mathcal{N}_n^1 \rangle\!\rangle\}.$$

As mentioned in Section 1, an ultimate goal in the present study is to determine which finite semigroups are join irreducible, but it is unknown if this problem is even decidable [11, Question 1.2].

Problem 5.3. Is join irreducibility decidable for finite semigroups?

There has been some pessimism about solving this problem [11], but it turns out to be a little easier than first thought because the formation of joins in the lattice $\mathfrak{L}(\mathbf{SEM})$ may be "dual" to the wreath product. First, one evidence for this "dual" is that the Presentation lemma (of Krohn–Rhodes complexity) produces surprising join decompositions; see Rhodes and Steinberg [19, Theorem 4.14.19] and Margolis et al. [14]. Second, regarding the application of sufficient conditions of type (C2), techniques from Krohn–Rhodes complexity [19, Chapter 4] can be used to find a finite semigroup $\mathcal{S}$ that divides the wreath product $\mathcal{A} \wr \mathcal{B}$ of two given finite semigroups $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A}$ is not join irreducible, say it divides $\mathcal{A}' \times \mathcal{A}''$, then $\mathcal{S}$ divides $(\mathcal{A}' \wr \mathcal{B}) \times (\mathcal{A}'' \wr \mathcal{B})$, resulting in test cases for (C2).

The recent result of Margolis et al. [15] further demonstrates the strength of these techniques.

Acknowledgments. The authors are indebted to: the anonymous reviewer, for a very detailed review and for correcting some errors in Lemma 2.15; Wei Li, for providing computing hardware; João Araújo and Rui Barradas Pereira, for providing multiplication tables of the 1,823 subdirectly indecomposible semigroups of order 6; and Wendy Wong, for checking the sufficient conditions in Section 3, with a computer, against these semigroups.

References

[1] P. Aglianò and J. B. Nation, Lattices of pseudovarieties. *J. Austral. Math. Soc. Ser. A* **46** (1989), no. 2, 177–183 Zbl 0671.08007 MR 0973311

[2] J. Almeida, *Finite semigroups and universal algebra*. Ser. Algebra 3, World Scientific, River Edge, NJ, 1994 Zbl 0844.20039 MR 1331143

[3] J. Araújo, J. P. Araújo, P. J. Cameron, E. W. H. Lee, and J. Raminhos, A survey on varieties generated by small semigroups and a companion website. *J. Algebra* **635** (2023), 698–735 Zbl 1522.20218 MR 4642193

[4] A. Distler and T. Kelsey, The semigroups of order 9 and their automorphism groups. *Semigroup Forum* **88** (2014), no. 1, 93–112 Zbl 1291.20054 MR 3164153

[5] S. Eilenberg, *Automata, languages, and machines. Vol. B*. Pure Appl. Math., Academic Press 59, Academic Press, New York, 1976 Zbl 0359.94067 MR 0530383

[6] È. A. Golubov and M. V. Sapir, Varieties of finitely approximable semigroups (Russian). *Izv. Vyssh. Uchebn. Zaved. Mat.* (1982), no. 11, 21–29. English translation. *Soviet Math.* **26** (1982), no. 11, 25–36 Zbl 0521.20047 MR 0687309

[7] E. W. H. Lee, On a class of completely join prime J-trivial semigroups with unique involution. *Algebra Universalis* **78** (2017), no. 2, 131–145 Zbl 1373.20071 MR 3697185

[8] E. W. H. Lee, Join irreducible 2-testable semigroups. *Discuss. Math. Gen. Algebra Appl.* **41** (2021), no. 1, 103–112 Zbl 1491.20108 MR 4227480

[9] E. W. H. Lee and J. R. Li, Minimal non-finitely based monoids. *Dissertationes Math.* **475** (2011), 1–65 Zbl 1238.20067 MR 2663812

[10] E. W. H. Lee and J. R. Li, The variety generated by all monoids of order four is finitely based. *Glas. Mat. Ser. III* **50(70)** (2015), no. 2, 373–396 Zbl 1341.20058 MR 3437496

[11] E. W. H. Lee, J. Rhodes, and B. Steinberg, Join irreducible semigroups. *Internat. J. Algebra Comput.* **29** (2019), no. 7, 1249–1310 Zbl 1483.20089 MR 4022706

[12] E. W. H. Lee, J. Rhodes, and B. Steinberg, On join irreducible J-trivial semigroups. *Rend. Semin. Mat. Univ. Padova* **147** (2022), 43–78 Zbl 1506.20089 MR 4450783

[13] E. W. H. Lee and W. T. Zhang, Finite basis problem for semigroups of order six. *LMS J. Comput. Math.* **18** (2015), no. 1, 1–129 Zbl 1325.20051 MR 3303600

[14] S. Margolis, J. Rhodes, and A. Schilling, Decidability of Krohn–Rhodes complexity $c = 1$ of finite semigroups and automata. [v1] 2021, [v4] 2026, arXiv:2110.10373v4

[15] S. Margolis, J. Rhodes, and A. Schilling, Decidability of Krohn–Rhodes complexity for all finite semigroups and automata. 2024, arXiv:2406.18477

[16] W. McCune, Prover9 and Mace4, https://www.cs.unm.edu/~mccune/mace4, visited on 18 May 2026

[17] I. I. Mel'nik, A description of certain lattices of varieties of semigroups (Russian). *Izv. Vysš. Učebn. Zaved. Matematika* (1972), no. 7(122), 65–74 Zbl 0245.20065 MR 0373973

[18] R. A. R. Monzo, Pre-compatible almost endomorphisms and semigroups whose cube is a band. *Semigroup Forum* **67** (2003), no. 3, 355–372 Zbl 1059.20055 MR 2002940. Correction. *Semigroup Forum* **97** (2018), no. 1, 181–182 Zbl 1406.20058 MR 3818358

[19] J. Rhodes and B. Steinberg, *The q-theory of finite semigroups*. Springer Monogr. Math., Springer, New York, 2009 Zbl 1186.20043 MR 2472427

[20] M. V. Sapir, Problems of Burnside type and the finite basis property in varieties of semigroups (Russian). *Izv. Akad. Nauk SSSR Ser. Mat.* **51** (1987), no. 2, 319–340. English translation. *Math. USSR-Izv.* **30** (1988), no. 2, 295–314 Zbl 0646.20047 MR 0897000

[21] A. V. Tishchenko, The wreath product of atoms of the lattice of semigroup varieties (Russian). *Tr. Mosk. Mat. Obs.* **68** (2007), 107–132. English translation. *Trans. Moscow Math. Soc.* (2007), 93–118 Zbl 1160.20058 MR 2429268

[22] A. N. Trahtman, Identities of a five-element 0-simple semigroup. *Semigroup Forum* **48** (1994), no. 3, 385–387 Zbl 0813.20062 MR 1263003

Received 12 June 2023; revised 28 January 2026.

Edmond W. H. Lee
Department of Mathematics, Nova Southeastern University, 3300 S. University Drive, Fort Lauderdale, FL 33328, USA; edmond.lee@nova.edu
Author IDs: zbMATH lee.edmond-w-h MR 733616 ORCID 0000-0002-1662-3734

John Rhodes
Department of Mathematics, University of California, Berkeley, 970 Evans Hall #3840, Berkeley, CA 94720, USA; rhodes@math.berkeley.edu
Author IDs: zbMATH rhodes.john-l MR 147630 ORCID 0000-0001-7879-4021