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Fine structures inside the pre-Lie operad revisited

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Abstract. We prove a conjecture of Chapoton from 2010 stating that the pre-Lie operad, as a Lie algebra in the symmetric monoidal category of linear species, is freely generated by the free operad on the species of cyclic Lie elements.

Introduction

Recall that a pre-Lie algebra is a vector space with a binary operation $a, b \mapsto a \lhd b$ satisfying the identity

$$(a_1 \lhd a_2) \lhd a_3 - a_1 \lhd (a_2 \lhd a_3) = (a_1 \lhd a_3) \lhd a_2 - a_1 \lhd (a_3 \lhd a_2).$$

This is a remarkable algebraic structure appearing in many different areas of mathematics, including category theory, combinatorics, deformation theory, differential geometry, and mathematical physics, to name a few. It seems to have first appeared independently in the work of Gerstenhaber [9] and Vinberg [15] in the 1960s. In particular, in each pre-Lie algebra the commutator $[a_1, a_2] = a_1 \lhd a_2 - a_2 \lhd a_1$ satisfies the Jacobi identity, and hence defines a Lie algebra structure on the same vector space.

A fact that could have been discovered by Cayley [3] in 1857, but had to wait till the work of Chapoton and Livernet [6] in 2000, states that free pre-Lie algebras can be described using rooted trees. For the case of the free pre-Lie algebra on one generator, the corresponding commutator algebra, the Lie algebra of rooted trees, plays an important role in the celebrated construction of Connes and Kreimer [12]. A result going back to the work of Foissy [8] in 2001 states that the Lie algebra of rooted trees is a free Lie algebra; in fact, it is easy to generalize that proof to the case of the commutator algebra of any free pre-Lie algebra. In 2007, Chapoton [4] and Bergeron and Livernet [1], established the same result in a more functorial way, proving that the pre-Lie operad is free as a Lie algebra in the symmetric monoidal category of linear species.

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In 2010, Bergeron and Loday proved [2] that in any pre-Lie algebra, the symmetrized product $a_1 \bullet a_2 = a_1 \lhd a_2 + a_2 \lhd a_1$ does not satisfy any identity other than commutativity, which is a striking contrast with the symmetrized product in associative algebras [11]. Soon after their work was circulated, Chapoton proposed in [5] a beautiful conjecture substantially generalizing this result. Informally, that conjecture, which we shall state precisely below after some recollections, is as follows. There exists a sequence $\mathcal{W} = \{\mathcal{W}(n)\}$ of representations of the symmetric groups S_n $(n \geq 2)$ with $\dim \mathcal{W}(n) = (n-2)!$, often referred to as the Whitehouse modules [14,16]. One may use that sequence to generate a free operad $\mathcal{T}(\mathcal{W})$. The conjecture of Chapoton states that the species that freely generates the pre-Lie operad as a Lie algebra is isomorphic to the underlying species of $\mathcal{T}(\mathcal{W})$. In this paper, a simple proof of the conjecture of Chapoton is given.

1. Recollections

All vector spaces and chain complexes are defined over a ground field $\Bbbk$ of zero characteristic. For necessary information about operads, we refer the reader to [13]. We emphasize that it is important for our purposes to think of operads as monoids in the category of linear species, since that viewpoint, in particular, greatly helps the intuition on operadic modules and bimodules upon which the below proof crucially relies. We shall repeatedly use the fact that a morphism of operads $\phi\colon \mathcal{P} \to \mathcal{Q}$ makes $\mathcal{Q}$ into a $\mathcal{P}$-bimodule. If one uses the monoidal definition of operads to define bimodules, that fact is a tautology: both the left action $\mathcal{P} \circ \mathcal{Q} \to \mathcal{Q}$ and the right action $\mathcal{Q} \circ \mathcal{P} \to \mathcal{Q}$ arise from the morphism ϕ and the operad structure of $\mathcal{Q}$ as composites $\mathcal{P} \circ \mathcal{Q} \to \mathcal{Q} \circ \mathcal{Q} \to \mathcal{Q}$ and $\mathcal{Q} \circ \mathcal{P} \to \mathcal{Q} \circ \mathcal{Q} \to \mathcal{Q}$, respectively.

We shall now introduce one of the protagonists of the conjecture we prove in this paper, the species *CycLie* of cyclic Lie elements. For our purposes, it is convenient to use the following definition that places it in the realm of modules over non-cyclic operads.

Definition 1. The species *CycLie* of cyclic Lie elements is the underlying species of the right *Lie*-module generated by the species $\mathcal{X}$ with

$$\mathcal{X}(I) = \begin{cases} \Bbbk e_I, & |I| = 2, \\ 0, & |I| \neq 2, \end{cases}$$

modulo the relations

$$e_{\{a,\star\}} \circ_\star [b,c] = e_{\{c,\star\}} \circ_\star [a,b].$$

Consequently, a cyclic Lie element on a finite set I is an element of the vector space $\mathit{CycLie}(I)$.

In plain words, for a finite set I, the vector $CycLie(I)$ is spanned by expressions of the form (a, b), where $(-, -)$ is symmetric and bilinear, a and b are Lie monomials in the variables x_i $(i \in I)$, each variable x_i appears in (a, b) exactly once, and the relation

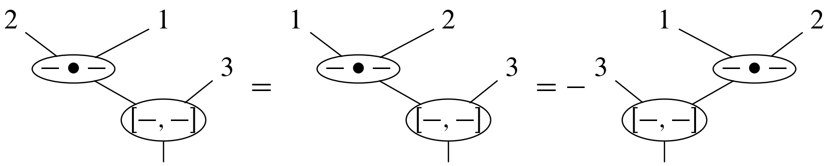
$$\big([a, b], c\big) = \big(a, [b, c]\big)$$

is satisfied for all a, b, c. Thus, in the same way as the Lie operad is made of universal Lie expressions, the species $CycLie$ is made of universal values of invariant bilinear forms on Lie algebras, see [10, Section 4].

Another crucial ingredient of the conjecture of Chapoton is the free operad functor. By contrast with bimodules, for working with free operads it is more convenient to think of an operad as an algebra over the monad of trees, see [13, Section 5.6]. In particular, for a linear species $\mathcal{X}$ with $\mathcal{X}(\varnothing) = 0$, the free operad $\mathcal{T}(\mathcal{X})$ generated by $\mathcal{X}$ has components

$$\mathcal{T}(\mathcal{X})(I) = \bigoplus_{t \in RT_I} \bigotimes_{v \text{ an internal vertex of } T} \mathcal{X}\big(\mathrm{in}(v)\big).$$

Here RT_I is the set of rooted trees with leaves bijectively labelled by I, and $\mathrm{in}(v)$ is the set of incoming edges of v. In other words, we consider trees whose internal vertices are labelled by elements of the species $\mathcal{X}$, and use tensor products to ensure that these are *tree tensors*, that is, labelled trees that are linear in each individual vertex label. Usually, we draw our labelled trees on the plane, and changing a planar drawing of a tree tensor by imposing a different planar order of subtrees of a given vertex v gives the same result as keeping the drawing but instead acting on the label of the vertex v by the bijection of the finite set $\mathrm{in}(v)$ that corresponds to changing the order of subtrees. Let us illustrate this by an example. We shall consider the free operad $\mathcal{T}([-, -], - \bullet -)$ generated by one anti-symmetric binary operation $[-, -]$ and one symmetric binary operation $- \bullet -$, that is, by a species $\mathcal{X}$ supported on two-element sets and such that for $I = \{a, b\}$, the vector space $\mathcal{X}(I)$ has a basis made of the symbol $[a, b]$ that is anti-symmetric in a, b and the symbol $a \bullet b$ that is symmetric in a, b. Then the above construction of the free operad amounts to saying that each component $\mathcal{T}([-, -], - \bullet -)(I)$ has a basis of equivalence classes of planar rooted binary trees whose leaves are in bijection with I, and whose internal vertices are labelled $[-, -]$ or $- \bullet -$ each. The equivalence relation we impose is generated by elementary equivalences coming from symmetry of generators, so that, for example,



in $\mathcal{T}([-,-],-\bullet-)(\{1,2,3\})$. For a tree tensor S with leaves labelled $1,\dots,p$ and tree tensors $T_1,\dots,T_p$ with disjoint sets of leaf labels $I_1,\dots,I_p$, the operadic composition map $\gamma(S;T_1,\dots,T_p)$ grafts the trees $T_1,\dots,T_p$ at the respective leaves of the tree S.

In terms of the notions we recalled, the precise version of the conjecture of Chapoton that we prove below states that the species that freely generates the pre-Lie operad as a Lie algebra is isomorphic to the underlying species of $\mathcal{T}(CycLie)$.

2. The main theorem

We are now ready to state and prove the main result of the paper.

Theorem 1. *We have an isomorphism of Lie algebras*

$$PreLie \cong Lie \circ \mathcal{T}(CycLie).$$

More precisely, if we denote by $\mathcal{Y}$ the subspecies of PreLie spanned by all elements of the form $\ell \triangleleft \ell' + \ell' \triangleleft \ell$, where ℓ, ℓ' are Lie monomials, the following statements hold:

(1) *we have an isomorphism $\mathcal{Y} \cong CycLie$;*

(2) *the suboperad $\mathcal{P}$ generated by $\mathcal{Y}$ is free;*

(3) *the Lie-subalgebra of PreLie generated by $\mathcal{P}$ is free and coincides with PreLie.*

Proof. While the operad *PreLie* is defined in terms of one operation $\triangleleft$, namely as the quotient of the free operad by the operadic ideal generated by the pre-Lie identity, we shall use a different presentation, using the operations $[a_1,a_2] = a_1 \triangleleft a_2 - a_2 \triangleleft a_1$ and $a_1 \bullet a_2 = a_1 \triangleleft a_2 + a_2 \triangleleft a_1$ mentioned in the introduction. It is known [7] that the S_3-module of identities satisfied by these operations is generated by the Jacobi identity for $[a_1,a_2]$ and the identity

$$\begin{aligned}(a_1 \bullet a_2) \bullet a_3 - a_1 \bullet (a_2 \bullet a_3) - a_1 \bullet [a_2,a_3] - [a_1,a_2] \bullet a_3 \\ - 2[a_1,a_3] \bullet a_2 + [a_1, a_2 \bullet a_3] + [a_1 \bullet a_2, a_3] + [[a_1,a_3],a_2] = 0, \end{aligned} \tag{2.1}$$

Let us start with considering the free operad $\mathcal{T}([-,-],-\bullet-)$ generated by the anti-symmetric operation $[-,-]$ and the symmetric operation $-\bullet-$ (before imposing either the Jacobi identity or Identity (2.1)). Note that the operad $\mathcal{T}([-,-],-\bullet-)$ has a suboperad $\mathcal{T}([-,-])$ generated by the anti-symmetric operation $[-,-]$ only; therefore, as indicated above, we may consider $\mathcal{T}([-,-],-\bullet-)$ as a bimodule over that suboperad. The argument that follows is a bit intricate, and so we split it into six relatively simple steps.

Step one: Factorizing elements of the free operad. Let us consider a typical tree tensor T in the free operad $\mathcal{T}([-,-],-\bullet-)$. We may consider the maximal subtree of the underlying tree that contains the root of the tree and all whose internal vertices are labelled $[-,-]$; that includes the possibility that the corresponding subtree is the trivial tree (corresponding to the unit element of the free operad). If we denote by S the tree tensor corresponding to this subtree, and by $T_1, \dots, T_p$ the tree tensors corresponding to the subtrees of the underlying tree of T grafted at the leaves of S, in the free operad we have thus the unique factorization

$$T = \gamma(S; T_1, \dots, T_p),$$

where each of the underlying trees of each tree tensor T_i is either a trivial tree or a tree whose root vertex is labelled $- \bullet -$.

Step two: Bimodule filtration of the free operad. Using the obtained factorization, one can define a sort of grading of the free operad. Concretely, if T is a tree tensor factorized in the form $T = \gamma(S; T_1, \dots, T_p)$ as above, we define the weight of T as the sum of p (arity of S) and the number of vertices labelled $- \bullet -$ in T. Note that this weight does not make our free operad a weight graded operad: it is not additive with respect to operadic compositions.

Let us show that the weight grading we just defined has some compatibility with the $\mathcal{T}([-,-])$-bimodule structure on $\mathcal{T}([-,-], - \bullet -)$. First, we see that the left action of $[-,-]$ respects the weight in a very strong sense: for α of weight ℓ and α' of weight ℓ', the element $[\alpha, \alpha']$ has weight exactly $\ell + \ell'$. Indeed, both the parameter p and the number of vertices labelled $- \bullet -$ are additive in this case. Next, we observe that the right action of $[-,-]$ exhibits certain compatibility with the weight: for α of weight ℓ, we have $\alpha \circ_i [-,-]$ is at least ℓ. Indeed, our substitution may increase the parameter p, and does not change the number of vertices labelled $- \bullet -$. This means that if we define the decreasing filtration $F^\bullet \mathcal{T}$ of the free operad with the filtered component $F^\ell \mathcal{T}$ being the linear span of all tree tensors of weight at least ℓ, this filtration is a $\mathcal{T}([-,-])$-bimodule filtration.

Step three: Further factorization in the free operad. For an element T of the free operad factorized as above, each of the tree tensors T_i above can be uniquely written as an iterated operadic composition of tree tensors for which *only* the root is labelled $- \bullet -$, and all other internal vertices are labelled $[-,-]$; this includes the possibility of the empty composition, for which the result is the trivial tree. Indeed, to find such composition, one should locate all internal vertices of T_i labelled $- \bullet -$, and cut the underlying tree of T_i into maximal subtrees rooted at such vertices with all other internal vertices labelled $[-,-]$. Overall, our factorization can be summarized by the formula

$$\mathcal{T}\big([-,-], - \bullet -\big) \cong \mathcal{T}\big([-,-]\big) \circ \mathcal{T}(\overline{\mathcal{Y}}),$$

where $\overline{\mathcal{Y}}$ is the subspecies of $\mathcal{T}([-,-], - \bullet -)$ spanned by all elements of the form $\alpha \bullet \alpha'$, where $\alpha, \alpha' \in \mathcal{T}([-,-])$. Indeed, $\overline{\mathcal{Y}}$ is precisely the species spanned by tree tensors for which only the root is labelled $-\bullet-$, and all other internal vertices are labelled $[-,-]$, and taking the free operad on $\overline{\mathcal{Y}}$ amounts to taking all possible iterated operadic compositions.

Step four: Passing to the quotient. Let us now record what all this means for the quotient operad *PreLie*. Each element of *PreLie* is a linear combination of cosets of tree tensors, each of which can be written in the way described above, that is as a composition of a coset of a tree tensor all whose internal vertices are labelled $[-,-]$ with iterated operadic compositions of cosets of tree tensors whose root is labelled $-\bullet-$ and all other internal vertices are labelled $[-,-]$. The former cosets clearly span the suboperad $\mathit{Lie}\subset\mathit{PreLie}$, and the latter cosets span the subspecies $\mathcal{Y}\subset\mathit{PreLie}$. Thus, the isomorphism

$$\mathcal{T}\big([-,-],-\bullet-\big)\cong\mathcal{T}\big([-,-]\big)\circ\mathcal{T}(\overline{\mathcal{Y}}),$$

induces a surjective map of left *Lie*-modules

$$\pi\colon \mathit{Lie}\circ\mathcal{T}(\mathcal{Y})\twoheadrightarrow \mathit{PreLie}\,.$$

Let us also note that one can define a filtration $F^\bullet\,\mathit{PreLie}$ of the underlying species of the operad *PreLie* analogously to the filtration of the underlying species of the free operad defined above; namely, the filtered component $F^\ell\,\mathit{PreLie}$ is defined as the species of elements that can be written as linear combinations of tree tensors of weight at least ℓ. The same argument as above (at the end of the "step two" part) shows that this filtration is compatible with the *Lie*-bimodule structure on *PreLie* obtained from the morphism of operads $\mathit{Lie}\to\mathit{PreLie}$ (at this point, just existence of a linear combination of tree tensors of appropriate weight is used, so non-uniqueness coming from the fact that we imposed some relations does not create problems). Note that this filtration is *not* compatible with the full operad structure; however, as the remaining part of the proof will show, compatibility with the *Lie*-bimodule structure is sufficient to furnish a proof of our main result. This is perhaps the most subtle aspect of our argument.

Step five: Passing to the associated graded bimodule. Let us now consider the associated graded *Lie*-bimodule $\mathrm{gr}_F\,\mathit{PreLie}$. To make a key advance in the proof, we shall do a small calculation. Denoting (2.1) by r, and computing $\frac{1}{3}(r-2r.(23))$, we obtain the relation

$$\begin{aligned}[][a_1,a_2]\bullet a_3-a_1\bullet[a_2,a_3]=\frac{1}{3}\big(&-(a_1\bullet a_2)\bullet a_3-a_1\bullet(a_2\bullet a_3)+2(a_1\bullet a_3)\bullet a_2\\&-[a_1\bullet a_2,a_3]+[a_1,a_2\bullet a_3]+2[a_1\bullet a_3,a_2]\\&+[a_1,[a_2,a_3]]+[[a_1,a_2],a_3]\big).\end{aligned}$$

Let us understand precisely the filtration levels of the elements in this formula.

Note that for the two elements on the left-hand side, their unique factorization of step one in the free operad writes them as a composition of the unit element (of arity one) with an element corresponding to a tree tensor with the root vertex labelled $-\bullet-$, and the other vertex labelled $[-,-]$ (that is, of total weight $1+1=2$). Moreover,

$$[a_1,a_2]\bullet a_3 \in F^2\,\mathit{PreLie}\setminus F^3\,\mathit{PreLie}\,.$$

Indeed, if we assume the contrary, that is, if we could write the coset of the tree tensor $[a_1,a_2]\bullet a_3$ in the operad *PreLie* as a linear combination of tree tensors of weight at least 3, the S_3-action shows that the same would be true for $[a_1,a_3]\bullet a_2$ and $[a_2,a_3]\bullet a_1$. However, the S_3-module generated by the relation (2.1) is of dimension two, so we do not have enough relations to rewrite three different cosets of tree tensors.

If we now consider the elements on the right-hand side, their unique factorizations of step one in the free operad are as follows:

- for the first three of them, the factorization writes them as a composition of the unit element (of arity one) with an element with two vertices labelled $-\bullet-$ (that is, of total weight $1+2=3$),
- for the second three of them, the factorization writes them as a composition of the Lie bracket (of arity two) with elements that have in total one vertex labelled $-\bullet-$ (that is, of total weight $2+1=3$),
- for the last two of them, the factorization writes them as a composition of a Lie monomial of arity three with the trivial trees (that is, of total weight $3+0=3$).

Therefore, the right-hand side belongs to $F^3\,\mathit{PreLie}$. This implies that in the associated graded *Lie*-bimodule, we have a non-trivial relation

$$[a_1,a_2]\bullet a_3 = a_1\bullet[a_2,a_3].$$

This relation is precisely the defining relation of the cyclic Lie module. Since $\mathcal{Y}$ is clearly a right *Lie*-submodule of *PreLie*, and since $F^\bullet\,\mathit{PreLie}$ is a filtration by *Lie*-bimodules, we conclude that there is a surjective map of right *Lie*-modules

$$\widetilde{\phi}\colon \mathit{CycLie} \twoheadrightarrow \widetilde{\mathcal{Y}}$$

where $\widetilde{\mathcal{Y}}$ is the right submodule of $\mathrm{gr}_F\,\mathit{PreLie}$ generated by the coset of the element $-\bullet-$.

Step six: Dimension counting and bijectivity. Since components of these species are $\Bbbk S_n$-modules and $\Bbbk$ is of characteristic zero, passing to the associated graded does not change the module structure, our previous argument implies that there is a surjective map of right *Lie*-modules

$$\phi\colon \mathit{CycLie} \twoheadrightarrow \mathcal{Y},$$

and hence a surjective map of linear species

$$Lie \circ \mathcal{T}(CycLie) \twoheadrightarrow Lie \circ \mathcal{T}(\mathcal{Y}) \twoheadrightarrow PreLie\,.$$

However, a calculation already performed in [5, Section 9] shows that the species $Lie \circ \mathcal{T}(CycLie)$ and $PreLie$ are of the same dimension in each arity, so a surjective map between the species $Lie \circ \mathcal{T}(CycLie)$ and $PreLie$ must be an isomorphism. Our surjection is constructed using two different surjections, and since the final result is an isomorphism, each of those surjections is an isomorphism. In particular, the surjection

$$\phi\colon CycLie \twoheadrightarrow \mathcal{Y},$$

is an isomorphism, so we have a species isomorphism $CycLie \cong \mathcal{Y}$, proving the first claim of the theorem. Furthermore, the surjective map of left *Lie*-modules

$$Lie \circ \mathcal{T}(\mathcal{Y}) \twoheadrightarrow PreLie$$

is an isomorphism, which implies both that the suboperad of *PreLie* generated by $\mathcal{Y}$ is free and the left *Lie*-module generated by that suboperad is free and coincides with *PreLie*, proving the last two claims of the theorem. ■

Remark 1. In our previous work [7, Theorem 3], an alternative proof of the result of Bergeron and Loday [2] on freeness of the suboperad of *PreLie* generated by $- \bullet -$ was suggested. However, that proof contains a gap that cannot be fixed by any moderate adjustment: it is possible to show that there is no quadratic Gröbner basis or convergent rewriting system of the operad *PreLie* with the indicated leading terms. The proof above uses the same presentation of *PreLie* in a conceptually different way, and, in particular, furnishes a new proof of the result of Bergeron and Loday.

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