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On the difference between squares and integral S-units

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Abstract. Let $q_1, \dots, q_t$ be distinct prime numbers. Let $a_1, \dots, a_t$ be nonnegative integers and x a positive integer. We establish an effective lower bound for the greatest prime divisor of $|x^2 - q_1^{a_1} \cdots q_t^{a_t}|$, which tends to infinity with the maximum of $x, a_1, \dots, a_t$.

To Jan-Hendrik Evertse on his retirement

1. Introduction

In his seminal paper [20], Schinzel established fully explicit versions of two theorems of Gelfond [15, 16] on the Archimedean and the p-adic distances between integral powers of algebraic numbers (nowadays commonly referred to as linear forms in two logarithms, in the Archimedean and in the p-adic settings). He further gave many applications of his estimates, including the following result [20, Theorem 9].

Theorem 1.1 (Schinzel, 1967). *Let $m_1, \dots, m_k$ be positive integers. If x and $\ell_1, \dots, \ell_k$ are positive integers and $x^2 - m_1^{\ell_1} \cdots m_k^{\ell_k}$ is nonzero, then*

$$|x^2 - m_1^{\ell_1} \cdots m_k^{\ell_k}| > \exp\bigl(c(\log \max\{x^2, m_1^{\ell_1} \cdots m_k^{\ell_k}\})^{1/7}\bigr), \tag{1.1}$$

where c is a positive effectively computable number depending only on $m_1, \dots, m_k$.

By using more recent estimates for linear forms in two p-adic logarithms (established, e.g., in [12, 13]), the right hand side of (1.1) can be replaced by a small positive power of $\max\{x^2, m_1^{\ell_1} \cdots m_k^{\ell_k}\}$, see [1, 8]. Here, we go one step further, in the spirit of the papers [7, 10, 11, 17], where the authors bound from above the S-parts of various sequences of integers.

Let $S = \{p_1, \dots, p_s\}$ be a finite, non-empty set of distinct prime numbers. For a nonzero integer n, write $n = bp_1^{a_1} \cdots p_s^{a_s}$, where $a_1, \dots, a_s$ are nonnegative integers

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and b is an integer relatively prime to the product $p_1 \cdots p_s$. Then, we define the S-part $[n]_S$ of n by

$$[n]_S := p_1^{a_1} \cdots p_s^{a_s}.$$

In the sequel, we use the notation $\gg^{\mathrm{eff}}_{a,b,\ldots}$ and $\ll^{\mathrm{eff}}_{a,b,\ldots}$ (resp., $\gg^{\mathrm{ineff}}_{a,b,\ldots}$ and $\ll^{\mathrm{ineff}}_{a,b,\ldots}$) to indicate that the positive numerical constants implied by $\gg$ and $\ll$ can be given explicitly (resp., cannot be given explicitly from the known proofs) and depend at most on the parameters $a, b, \ldots$

Our main result is the following.

Theorem 1.2. *Let $S := \{p_1, \ldots, p_s\}$ and $T := \{q_1, \ldots, q_t\}$ be disjoint sets of distinct prime numbers. For every nonzero integer x coprime with $q_1 \cdots q_t$, every nonnegative integers $a_1, \ldots, a_t$, and every positive ε, we have*

$$[x^2 - q_1^{a_1} \cdots q_t^{a_t}]_S \ll^{\mathrm{ineff}}_{S,T,\varepsilon} |x^2 - q_1^{a_1} \cdots q_t^{a_t}|^{\frac{1}{2}+\varepsilon}.$$

Furthermore, for every nonzero integer x coprime with $q_1 \cdots q_t$ and every nonnegative integers $a_1, \ldots, a_t$, we have

$$[x^2 - q_1^{a_1} \cdots q_t^{a_t}]_S \ll^{\mathrm{eff}}_{S,T} |x^2 - q_1^{a_1} \cdots q_t^{a_t}|^{1-\kappa}, \tag{1.2}$$

where

$$\kappa = \left(c^s (\log\log P)((\log p_1) \cdots (\log p_s))^2\right)^{-1} \tag{1.3}$$

and c is an effectively computable positive number depending only on T, and P is the maximum of $p_1, \ldots, p_s$.

The first assertion of Theorem 1.2 is sharp. Indeed, let $T := \{q_1, \ldots, q_t\}$ be a set of distinct prime numbers. Then, there are finite sets of primes S with the smallest prime in S arbitrarily large, and, for each one of these sets S, infinitely many $(t+1)$-uples $(x, a_1, \ldots, a_t)$ of nonnegative integers such that

$$[x^2 - q_1^{a_1} \cdots q_t^{a_t}]_S \gg_{S,T} |x^2 - q_1^{a_1} \cdots q_t^{a_t}|^{\frac{1}{2}}.$$

This follows from Hensel's lemma, in a similar way as [11, Theorem 2.1, (ii)].

It would be interesting to solve completely Diophantine equations of the form $x^2 - q_1^{a_1} q_2^{a_2} = p^u$ (resp., $x^2 - q^a = p_1^{u_1} p_2^{u_2}$), for three given distinct prime numbers q_1, q_2, p (resp., q, p_1, p_2). The more general equation

$$x^2 - q_1^{a_1} \cdots q_t^{a_t} = y^n,$$

where $q_1, \ldots, q_t$ are fixed prime numbers, has been investigated in [2, 3, 5] when $t = 1$.

By applying Theorem 1.2 with the set S composed of the first s prime numbers, we derive a lower bound for the greatest prime factor of $|x^2 - q_1^{a_1} \cdots q_t^{a_t}|$. For a

nonzero integer n, let $P[n]$ denote the greatest prime factor of $|n|$ with the convention that $P[\pm 1] = 1$. For any positive real number x, we set $\log_* x = \max\{1, \log x\}$.

Corollary 1.3. *Let $T := \{q_1, \dots, q_t\}$ be a set of distinct prime numbers. Let x be a nonzero integer coprime with $q_1 \cdots q_t$ and set*

$$X := \max\{x^2, q_1^{a_1} \cdots q_t^{a_t}, 3\}.$$

Then, we have

$$P[x^2 - q_1^{a_1} \cdots q_t^{a_t}] \gg_T^{\text{eff}} \log_* \log X \frac{\log_* \log_* \log X}{\log_* \log_* \log_* \log X}.$$

To obtain this lower bound instead of the weaker estimate $\gg_T^{\text{eff}} \log_* \log X$, we argue as Győry and Yu in the proof of their [18, Theorem 2], see also [14] and the remark at the end of Section 3.

For completeness, we state the following result, whose proof follows the same lines as that of Theorem 1.2.

Theorem 1.4. *Let $T := \{q_1, \dots, q_t\}$ be a set of distinct prime numbers. Let m be a nonzero integer. There exists an effectively computable, positive real number c, depending only on T, such that, for every $(t+1)$-uple $(x, a_1, \dots, a_t)$ with*

$$x^2 - q_1^{a_1} \cdots q_t^{a_t} = m,$$

we have

$$|x| \le |2m|^c.$$

Let S, T, and P be as in the statement of Theorem 1.2. Let $a_1, \dots, a_t$ be nonnegative integers. For higher powers than squares, the analogue of the effective part of Theorem 1.2 is a particular case of [11, Theorem 2.5]. Indeed, for an integer $d \ge 3$, we write

$$x^d - q_1^{r_1} \cdots q_t^{r_t} \left(q_1^{\lfloor a_1/d \rfloor} \cdots q_t^{\lfloor a_t/d \rfloor}\right)^d,$$

where r_i is the remainder in the Euclidean division of a_i by d, for $i = 1, \dots, t$. Thus, we get t^d integer binary forms

$$F_{\underline{r}}(X, Y) = X^d - q_1^{r_1} \cdots q_t^{r_t} Y^d, \quad \underline{r} = (r_1, \dots, r_t),$$

of degree d, to which [11, Theorem 2.5] applies. Thus, for every nonzero integer x coprime with $q_1 \cdots q_t$ and any nonnegative integers $a_1, \dots, a_t$, we have

$$[x^d - q_1^{a_1} \cdots q_t^{a_t}]_S \ll_{S,T,d}^{\text{eff}} |x^d - q_1^{a_1} \cdots q_t^{a_t}|^{1-\kappa},$$

where

$$\kappa = \left(c^s (P(\log p_1) \cdots (\log p_s))^{d!}\right)^{-1} \tag{1.4}$$

and c is an effectively computable positive number depending only on T and d. The dependence on P is much better in (1.3) than in (1.4). The reason for this is that we do not use estimates for linear forms in p_i-adic logarithms in the proof of Theorem 1.2 (namely, we use estimates for linear forms in q_j-adic logarithms), unlike in the proof of [11, Theorem 2.5].

Furthermore, by arguing as in the proof of [11, Theorem 2.4], we easily obtain that, for every integer x and every positive ε, we have

$$[x^d - q_1^{a_1} \cdots q_t^{a_t}]_S \ll_{S,T,\varepsilon}^{\text{ineff}} |x^d - q_1^{a_1} \cdots q_t^{a_t}|^{\frac{1}{d}+\varepsilon}.$$

Conversely, if $T := \{q_1, \ldots, q_t\}$ is a given set of distinct prime numbers, there are finite sets of primes S with the smallest prime in S arbitrarily large, and for each one of these sets S infinitely many $(t+1)$-uples $(x, a_1, \ldots, a_t)$ of nonnegative integers such that

$$[x^d - q_1^{a_1} \cdots q_t^{a_t}]_S \gg_{S,T} |x^d - q_1^{a_1} \cdots q_t^{a_t}|^{\frac{1}{d}}.$$

Finally, we point out that a result of similar strength to that of Theorem 1.2 has been proved in [8] for the difference (or the sum) of two integral T-units. Namely, [8, Theorem 3.8] asserts that if x and y are coprime integers whose prime divisors are in T and S is a finite set of prime numbers disjoint from T, then

$$[x + y]_S \ll_{S,T}^{\text{eff}} |x + y|^{1-\tau},$$

where τ is an effectively computable positive number depending only on S and T. By looking closely at the proof, we see that

$$\tau = \big(c^s (\log\log P)(\log p_1) \cdots (\log p_s)\big)^{-1},$$

where c is an effectively computable positive number depending only on T, and P is the maximum of $p_1, \ldots, p_s$.

2. Auxiliary results

In this section, we recall, in a simplified form, estimates from the theory of linear forms in logarithms. As usual, $h(\alpha)$ denotes the (logarithmic) Weil height of the algebraic number α and we set $h_*(\alpha) = \max\{h(\alpha), 1\}$.

We begin with an immediate consequence of an estimate of Matveev [19]; see also [6, Theorem 2.2].

Theorem 2.1. *Let $n \geq 2$ be an integer. Let $\alpha_1, \dots, \alpha_n$ be nonzero algebraic numbers. Let $b_1, \dots, b_n$ be integers with $b_n = \pm 1$. Let D be the degree over $\mathbb{Q}$ of the number field $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$. Set*

$$B = \max\{3, |b_1|, \dots, |b_n|\}.$$

If

$$\alpha_1^{b_1} \cdots \alpha_n^{b_n} \neq 1,$$

then there exists an effectively computable positive number c_1, depending only on D, such that

$$\log |\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1| \geq -c_1^n h_*(\alpha_1) \cdots h_*(\alpha_n) \log_* \frac{B \max\{h_*(\alpha_1), \dots, h_*(\alpha_{n-1})\}}{h_*(\alpha_n)}.$$

Let p be a prime number. For a nonzero rational number α, let $\mathrm{v}_p(\alpha)$ denote the exponent of p in the decomposition of α as a product of powers of prime numbers. More generally, if α is a nonzero algebraic number in a number field K, let $\mathrm{v}_{\mathfrak{p}}(\alpha)$ denote the exponent of $\mathfrak{p}$ in the decomposition of the fractional ideal αO_K in a product of prime ideals and set

$$\mathrm{v}_p(\alpha) = \frac{\mathrm{v}_{\mathfrak{p}}(\alpha)}{e_{\mathfrak{p}}}.$$

This defines a valuation v_p on K which extends the p-adic valuation v_p on $\mathbb{Q}$ normalized in such a way that $\mathrm{v}_p(p) = 1$. We reproduce, in a simplified form, an estimate obtained by Yu [21]; see also [6, Theorems 2.9 and 2.11].

Theorem 2.2. *Let $n \geq 2$ be an integer. Let p be a prime number and $\alpha_1, \dots, \alpha_n$ algebraic numbers in an algebraic number field of degree D. Let $b_1, \dots, b_n$ denote rational integers such that $b_n = \pm 1$ and $\alpha_1^{b_1} \cdots \alpha_n^{b_n}$ is not equal to 1. Set*

$$B = \max\{3, |b_1|, \dots, |b_n|\}.$$

There exists a positive effectively computable real number c_2, depending only on D, such that

$$\mathrm{v}_p(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) < c_2^n p^D h_*(\alpha_1) \cdots h_*(\alpha_n) \log_* \frac{B \max\{h_*(\alpha_1), \dots, h_*(\alpha_{n-1})\}}{h_*(\alpha_n)}.$$

We may also refer to [14, Theorem 3.2.8], which merges the statements of Theorems 2.1 and 2.2.

In comparison with other lower bounds for linear forms in logarithms, the crucial point in Theorems 2.1 and 2.2 is the replacement of the factor $\log_* B$ by the factor

$$\log_* \frac{B \max\{h_*(\alpha_1), \dots, h_*(\alpha_{n-1})\}}{h_*(\alpha_n)},$$

which is much smaller than $\log B$ when $h_*(\alpha_n)$ is large. This is precisely the situation occurring in Section 3; see also [9] for more explanations and further examples of Diophantine questions where this refined estimate appears to be crucial.

We state now a particular case of a classical lemma (see, e.g., [14, Proposition 4.3.12]).

Lemma 2.3. *Let K be a real quadratic number field. Let η denote its fundamental unit. For every nonzero algebraic integer α in K, there exists an integer m such that*

$$h(\eta^m \alpha) \le \frac{1}{2} \log |\alpha\sigma(\alpha)| + \frac{\log \eta}{2},$$

where σ denotes the Galois embedding of K different from the identity.

Proof. Set $N = |\alpha\sigma(\alpha)|$. Observe that $\sigma(\eta) = \pm\eta^{-1}$. There exist an algebraic integer δ in K and an integer m such that

$$\alpha = \delta\eta^m, \quad \sigma(\alpha) = \pm\sigma(\delta)\eta^{-m},$$

with

$$N^{1/2}\eta^{-1/2} < |\delta| \le N^{1/2}\eta^{1/2}.$$

It suffices to observe that we get

$$|\sigma(\delta)| = \frac{N}{|\delta|} \le \eta^{1/2} N^{1/2},$$

and the lemma follows from the definition of the height. ∎

The proof of the ineffective part of Theorem 1.2 rests on the p-adic Thue–Siegel–Roth theorem, formulated as in [4, Theorem 6.2.3]. In the next theorem, it is understood that, for a rational number x/y and a prime number p, the expression $|x/y - \infty|_p$ means $|y/x|_p$ (this can be seen by applying a Möbius transformation, see [4, Section 6.2.5]).

Theorem 2.4. *Let S_1 be a finite set of finite places of $\mathbb{Q}$. For each p in S_1, let β_p be algebraic in $\mathbb{Q}_p$ or put $\beta_p = \infty$. Let β_∞ be a real number. Let $\varepsilon > 0$. Then, there are only finitely many rational numbers x/y such that*

$$\min\left\{1, \left|\frac{x}{y} - \beta_\infty\right|\right\} \prod_{p \in S_1} \min\left\{1, \left|\frac{x}{y} - \beta_p\right|_p\right\} \le \max\{|x|, |y|\}^{-2-\varepsilon}.$$

In particular, if S_2 is a subset of S_1, then, for any $\varepsilon > 0$, there are only finitely many reduced rational numbers x/y such that all the prime divisors of y belong to S_2 and

$$\min\left\{1, \left|\frac{x}{y} - \beta_\infty\right|\right\} \prod_{p \in S_1 \setminus S_2} \min\left\{1, \left|\frac{x}{y} - \beta_p\right|_p\right\} \le \max\{|x|, |y|\}^{-1-\varepsilon}.$$

Proof. The second assertion follows from the first one by taking $\beta_p = \infty$ for every prime number p in S_2. ∎

3. Proof of Theorem 1.2

3.1. Proof of the ineffective estimate

We argue as in the proof of [11, Proposition 3.1]. We denote by $|\cdot|_\infty$ the ordinary absolute value, and by $|\cdot|_p$ the p-adic absolute value normalized in such a way that $|p|_p = p^{-1}$ for a prime number p. Further, we set $\mathbb{Q}_\infty := \mathbb{R}$ and $\overline{\mathbb{Q}}_\infty := \mathbb{C}$. The ineffective part of Theorem 1.2 follows straightforwardly from the next result.

Proposition 3.1. *Let $T = \{q_1, \dots, q_t\}$ be a finite, non-empty set of prime numbers. Let S be a finite set of prime numbers, disjoint from T. Then*

$$\frac{|x^2 - q_1^{a_1} \cdots q_t^{a_t}|}{[x^2 - q_1^{a_1} \cdots q_t^{a_t}]_S} \gg^{\mathrm{ineff}}_{S,T,\varepsilon} \max\{x^2, q_1^{a_1} \cdots q_t^{a_t}\}^{\frac{1}{2}-\varepsilon}$$

for every $\varepsilon > 0$ and every positive integer x coprime with $q_1 \cdots q_t$.

Proof. Set $y = q_1^{\lfloor a_1/2 \rfloor} \cdots q_t^{\lfloor a_t/2 \rfloor}$. Our assumption implies that, for each p in $S \cup \{\infty\}$, there exists β_p in $\overline{\mathbb{Q}}_p$ algebraic over $\mathbb{Q}$ such that

$$x^2 - q_1^{a_1} \cdots q_t^{a_t} = (x - \beta_p y)(x + \beta_p y).$$

Then we have

$$\begin{aligned}\frac{|x^2 - q_1^{a_1} \cdots q_t^{a_t}|}{[x^2 - q_1^{a_1} \cdots q_t^{a_t}]_S \cdot \max\{x, y\}^2} &= \frac{\prod_{p \in S \cup \{\infty\}} |x^2 - q_1^{a_1} \cdots q_t^{a_t}|_p}{\max\{x, y\}^2} \\ &\gg_{S,T} \prod_{p \in S \cup \{\infty\}} \min\left\{1, \left|\frac{x}{y} - \beta_p\right|_p, \left|\frac{x}{y} + \beta_p\right|_p\right\} \\ &\gg_{S,T} \prod_{\substack{p \in S \cup \{\infty\}, \\ \beta_p \in \mathbb{Q}_p}} \min\left\{1, \left|\frac{x}{y} - \beta_p\right|_p, \left|\frac{x}{y} + \beta_p\right|_p\right\}.\end{aligned}$$

Since the prime factors of y are in a finite set, the latter quantity is $\gg^{\mathrm{ineff}}_{S,T,\varepsilon} \max\{x, y\}^{-1-\varepsilon}$ for every $\varepsilon > 0$, by Theorem 2.4. ∎

3.2. Proof of the effective estimate

We adapt Schinzel's argument [20] and the proof of [1, Theorem 1.2]; see also [6, Theorem 6.3]. Throughout this subsection, all the numerical constants $c_1, c_2, \dots$ and those implicit in $\ll$ and $\gg$ are effective and depend at most on $q_1, \dots, q_t$.

Define

$$Q_1 = \prod_{\substack{1\le i\le t,\\ a_i\equiv 1 \pmod 2}} q_i, \quad Q_2 = \sqrt{\frac{q_1^{a_1}\cdots q_t^{a_t}}{Q_1}},$$

thus, $q_1^{a_1}\cdots q_t^{a_t} = Q_1Q_2^2$. Set

$$b := \frac{|x^2 - q_1^{a_1}\cdots q_t^{a_t}|}{[x^2 - q_1^{a_1}\cdots q_t^{a_t}]_S}$$

and write

$$[x^2 - q_1^{a_1}\cdots q_t^{a_t}]_S = p_1^{2u_1+v_1}\cdots p_s^{2u_s+v_s}, \quad v_1,\dots,v_s \in \{0,1\}.$$

Set

$$A := \max\{|a_1|,\dots,|a_t|,2\}, \quad U := \max\{|u_1|,\dots,|u_s|,2\}, \quad X := \max\{x, Q_2, 2\}.$$

For later use, observe that

$$A \ll \log X, \quad U \ll \log X, \quad |x^2 - Q_1Q_2^2| \ll X^2.$$

Assume first that $Q_1 = 1$. Then, $a_1,\dots,a_t$ are all even and we get

$$|x - Q_2|\cdot|x + Q_2| = bp_1^{2u_1+v_1}\cdots p_s^{2u_s+v_s}.$$

Since the greatest common divisor of $x - Q_2$ and $x + Q_2$ is equal to 1 or 2, there exist an integer h in $\{0,1,\dots,s\}$ (below, an empty product is always equal to 1) and integers b_1, b_2 with $b_1b_2 = 4b$ such that (we reorder $p_1,\dots,p_s$ if necessary)

$$x - Q_2 = \frac{b_1}{2}p_1^{2u_1+v_1}\cdots p_h^{2u_h+v_h}, \quad x + Q_2 = \frac{b_2}{2}p_{h+1}^{2u_{h+1}+v_{h+1}}\cdots p_s^{2u_s+v_s}.$$

Consequently,

$$\frac{-2Q_2}{x+Q_2} = \frac{b_1}{b_2}p_1^{2u_1+v_1}\cdots p_h^{2u_h+v_h}p_{h+1}^{-2u_{h+1}-v_{h+1}}\cdots p_s^{-2u_s-v_s} - 1. \tag{3.1}$$

Observe that

$$h\Big(\frac{b_1}{b_2}\Big) \ll \log_* b.$$

If $x > Q_2^2$, then it follows from Theorem 2.1 applied to (3.1) that

$$\begin{aligned}\log X &\le c_1^s(\log_* b)\prod_{i=1}^{s}(\log p_i)\cdot\Big(\log\frac{U}{\log_* b}\Big)\\ &\le c_2^s(\log_* b)\prod_{i=1}^{s}(\log p_i)\cdot\Big(\log\frac{\log X}{\log_* b}\Big),\end{aligned}$$

and, setting

$$\mathcal{P} := (\log p_1)\cdots(\log p_s), \quad P := \max\{p_1,\dots,p_s\},$$

we get

$$\frac{\log X}{\log_* b} \le c_2^s \mathcal{P}\Big(\log \frac{\log X}{\log_* b}\Big),$$

thus, assuming that $\log X \ge 10 \log_* b$, we get

$$\frac{\log X}{\log_* b} \le 2c_2^s \mathcal{P} \log(c_2^s \mathcal{P}) \le 2c_2^s \mathcal{P} s(\log\log P + \log c_2).$$

Consequently, we have established the lower bound

$$\max\{b,3\} \ge X^{c_3^{-s}(\mathcal{P}\log\log P)^{-1}} \ge |x^2 - Q_2^2|^{c_4^{-s}(\mathcal{P}\log\log P)^{-1}}.$$

If $x \le Q_2^2$, then $A \gg U$ and we apply Theorem 2.2 to (3.1) to bound $\lfloor a_i/2 \rfloor = \mathrm{v}_{q_i}(Q_2)$ and we get

$$\begin{aligned} A &\le c_5^s (\log_* b) q_i \prod_{i=1}^{s} (\log p_i) \cdot \Big(\log \frac{U}{\log_* b}\Big) \\ &\le c_6^s (\log_* b) q_i \prod_{i=1}^{s} (\log p_i) \cdot \Big(\log \frac{A}{\log_* b}\Big). \end{aligned}$$

This gives

$$\max\{b,3\} \ge 2^{Ac_7^{-s}(\mathcal{P}\log\log P)^{-1}} \ge X^{c_8^{-s}(\mathcal{P}\log\log P)^{-1}} \ge |x^2 - Q_2^2|^{c_9^{-s}(\mathcal{P}\log\log P)^{-1}}.$$

Assume now that $Q_1 \ge 2$. Let K denote the real quadratic field $\mathbb{Q}(\sqrt{Q_1})$ and h its class number. The quadratic number

$$\alpha = x - \sqrt{Q_1}\, Q_2$$

is an algebraic integer in K. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_{s'}$ denote the prime ideals in K that divide $p_1,\dots,p_s$. Since at most two prime ideals in K divide p_i for $i = 1,\dots,s$, we have $s' \le 2s$. There is an ideal $\mathfrak{b}$ in K, coprime with $\mathfrak{p}_1\cdots\mathfrak{p}_{s'}$, and nonnegative integers $w_1,\dots,w_{s'}$ such that

$$(\alpha) = \mathfrak{b}\mathfrak{p}_1^{w_1}\cdots\mathfrak{p}_{s'}^{w_{s'}}.$$

For $i = 1,\dots,s'$, let z_i and r_i denote, respectively, the quotient and the remainder in the Euclidean division of w_i by h. Then, write

$$(\alpha) = (\mathfrak{b}\mathfrak{p}_1^{r_1}\cdots\mathfrak{p}_{s'}^{r_{s'}})(\mathfrak{p}_1^h)^{z_1}\cdots(\mathfrak{p}_{s'}^h)^{z_{s'}}$$

and observe that all the ideals in this equality are principal. Let η denote the fundamental unit in K. By Lemma 2.3, there is an integer m, an algebraic integer γ in K and, for $i = 1, \dots, s'$, an algebraic integer π_i generating the ideal $(\mathfrak{p}_i^h)$ such that

$$\alpha = \pm \eta^m \gamma \pi_1^{z_1} \cdots \pi_{s'}^{z_{s'}} \tag{3.2}$$

and

$$h(\gamma) \le c_{10} \log_* b, \quad h(\pi_i) \le c_{11} \log N\mathfrak{p}_i, \quad i = 1, \dots, s',$$

where $N\mathfrak{p}$ is the norm of the ideal $\mathfrak{p}$. We need now to bound $|m|, z_1, \dots, z_{s'}$ from above.

Since the algebraic integer α is a root of the quadratic polynomial

$$Z^2 - 2xZ + x^2 - q_1^{a_1} \cdots q_t^{a_t},$$

its height $h(\alpha)$ satisfies

$$h(\alpha) \ll \log X.$$

For $i = 1, \dots, s'$, the $\mathfrak{p}_i$-adic valuation of γ is bounded by the class number h and the $\mathfrak{p}_i$-adic valuations of the π_j's are zero for $j \ne i$. It then follows from the definition of the height that

$$U \ll \max\{z_1, \dots, z_{s'}\} \ll h(\alpha).$$

Then, we deduce from (3.2) that

$$|m| \ll h(\eta^m) = h(\alpha\gamma^{-1}\pi_1^{-z_1} \cdots \pi_{s'}^{-z_{s'}}) \ll (h_*(\gamma) + \log X).$$

Let σ denote the complex embedding sending $\sqrt{Q_1}$ to $-\sqrt{Q_1}$. Then,

$$\Lambda := \frac{\alpha - \sigma(\alpha)}{\sigma(\alpha)} = \frac{-2\sqrt{Q_1}Q_2}{x + \sqrt{Q_1}Q_2} = \frac{\gamma}{\sigma(\gamma)} \Big(\frac{\eta}{\sigma(\eta)}\Big)^m \prod_{i=1}^{s'} \Big(\frac{\pi_i}{\sigma(\pi_i)}\Big)^{z_i} - 1. \tag{3.3}$$

If $x > Q_2^2$, then we apply Theorem 2.1 to (3.3) to get

$$\log X \ll -\log \Lambda \le c_{12}^s h(\eta) h_*(\gamma) \Big(\prod_{i=1}^{s'} h_*(\pi_i) \Big) \log \frac{h_*(\gamma) + \log X}{h_*(\gamma)}.$$

This gives the lower bound

$$\max\{b, 3\} \ge X^{c_{13}^{-s} \mathcal{P}^{-2} (\log\log P)^{-1}} \ge |x^2 - Q_1 Q_2^2|^{c_{14}^{-s} \mathcal{P}^{-2} (\log\log P)^{-1}}.$$

If $x \le Q_2^2$, then we apply Theorem 2.2 to (3.3) to bound $\lfloor a_i/2 \rfloor = \mathrm{v}_{q_i}(Q_2)$ and we get

$$a_i \le c_{15}^s q_i h(\eta) h_*(\gamma) \Big(\prod_{i=1}^{s'} h_*(\pi_i) \Big) \log \frac{h_*(\gamma) + \log X}{h_*(\gamma)}.$$

Since $\log X \ll A$, this gives the lower bound

$$\max\{b,3\} \geq X^{c_{16}^{-s}\mathcal{P}^{-2}(\log\log P)^{-1}} \geq |x^2 - Q_1 Q_2^2|^{c_{17}^{-s}\mathcal{P}^{-2}(\log\log P)^{-1}},$$

and the proof of the effective part of Theorem 1.2 is complete.

We have established that, if $b = 1$, then

$$X \leq 3^{c_{18}^{s}\mathcal{P}^{2}(\log\log P)}.$$

By using the Prime Number Theorem and taking for p_j the j-th prime number for $j = 1, \ldots, s$, we obtain Corollary 1.3.

We conclude with a remark. Instead of several applications of Lemma 2.3, we could have applied only once [14, Proposition 4.3.12] to write α as a product of an algebraic number in K of controlled height times an S-unit (the finite places in S being $\mathfrak{p}_1, \ldots, \mathfrak{p}_{s'}$). We would then have obtained (1.2) with a different expression for κ and the estimate

$$P[x^2 - q_1^{a_1} \cdots q_t^{a_t}] \gg_T^{\text{eff}} \log_* \log X,$$

which is weaker than our Corollary 1.3.

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