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Evaluating a permutation of the power and complement operations of a graph with respect to its independence numbers

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Abstract. Let G be a simple graph and $k \in \mathbb{N}$. Two important graph operations are: the k-th graph power of G, denoted by G^k, where G^k is the graph obtained from G by adding an edge between every pair of vertices that have a distance at most k, and the complement graph of G, denoted by $\overline{G}$. In this paper, we studied the relation between the independence numbers of $\overline{G}^k$ and $\overline{G^k}$.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A path with $n \geq 2$ vertices, denoted by P_n, is a simple connected graph with 2 leaves (vertices of degree 1) and the other vertices having degree 2. The length of P_n is the number of its edges. Let $u, v \in V(G)$. If u, v belong to the same connected component of G, the path between u and v in G is a subgraph of G that is a path where u and v are its leaves. In this case, the distance between u and v, denoted by $\mathrm{dist}_G(u, v)$ or simply by $\mathrm{dist}(u, v)$, is the minimum length of all paths between u and v in G. If u, v belong to different connected components of G then $\mathrm{dist}_G(u, v) = \infty$. The diameter of G, denoted by $\mathrm{diam}(G)$, is the farthest distance between any two of its vertices, that is,

$$\mathrm{diam}(G) = \max_{u,v \in V(G)} \{\mathrm{dist}_G(u, v)\}.$$

Let $S \subseteq V(G)$. The subgraph of G formed with all vertices of S and all edges of G with both endpoints in S is called the subgraph in G induced by S.

Let k be a positive integer. The set consisting of all the vertices of G whose distance to u is positive and at most k is denoted by $N_k^G(u)$. Therefore, the degree of u in G, $d_G(u)$, is the cardinality of $N_1^G(u)$. The k-th graph power G^k of G is the graph whose vertex set is $V(G)$ in which two distinct vertices are adjacent if and only if their distance in G is at most k.

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A set $S \subseteq V(G)$ is a k-independent set of vertices of G if the distance between any two elements of S is greater than k in G. The k-independence number of G, denoted by $\alpha_k(G)$ or simply α_k, is the cardinality of the largest k-independent set of G. It is known that determining $\alpha_k(G)$ is NP-hard in general [8]. A k-independent set S of G is called an α_k-set of G if $|S| = \alpha_k(G)$. When $k = 1$ we write $\alpha(G)$ instead of $\alpha_1(G)$. It is well known that the unique graph G that satisfies $\alpha(G) = 1$ is the complete graph and the unique graph that satisfies $\alpha(G) = |V(G)|$ is the null graph.

Several researchers have noted the relationship $\alpha_1(G^k) = \alpha_k(G)$. However, this fact did not simplify the problem of deducing the independence number of G^k. Consequently, in the last few years, many papers have been dedicated to this matter. In 1997, Firby et al. described an upper bound for $\alpha_k(G)$ when G is a connected graph, [6]. In 2005, Beis et al. [2] proved some upper bounds for $\alpha_k(G)$ in random regular graphs. In 2019, the k-independence number was studied from an algebraic point of view by Abiad et al., [1]. In 2021, [9], Suil O et al. published a work dedicated to upper bounds for the k-independence number in connected graphs using their minimum and maximum degrees. Furthermore, in 2021, [7], Koerts wrote his bachelor's thesis on the k-independent sets. More recently, in 2026, [5], Fernandes et al. studied the k-independence number of trees.

As usual, we denote by $\overline{G}$ the complement graph of G, that is, the graph obtained from G by removing all edges and adding an edge between any two vertices not adjacent in G. For more basic definitions and notation of graphs, see [3, 4].

In all papers on the k-independence number of a graph, we have seen that the main problem in calculating this number is the existence of cycles in the graph. In some cases, this problem is solved by considering the complement graph. For example:

- if G has n vertices, with $n \geq 4$, and $\frac{n(n-1)}{2} - (n-1)$ edges, then $\overline{G}$ has $n-1$ edges. Furthermore, if $\overline{G}$ is a connected graph, then $\overline{G}$ is a tree, and it is possible to use known results in this matter for the k-independence number of $\overline{G}$;
- if G has more cycles than $\overline{G}$ and it is easier to calculate the k-independence number of $\overline{G}$.

Therefore, it is important to know the relation between the k-independent numbers of G and $\overline{G}$.

Mixing the two mentioned graph operations, the complement graph and the k-th graph power, we obtain the graphs $\overline{G}^k$ and $\overline{G^k}$. So, we get these three numbers $\alpha(\overline{G}^k)$, $\alpha(\overline{G^k})$, $\alpha_k(\overline{G})$. As we mentioned before, two of these numbers are equal, $\alpha(\overline{G}^k) = \alpha_k(\overline{G})$. Consequently, the focus of this paper is on the relationship between $\alpha(\overline{G}^k)$ and $\alpha(\overline{G^k})$. More specifically, whether one of them can be said to be an upper bound of the other. Since the complement graph of the complement graph is the initial graph, each graph is the complement graph of a graph. Therefore, we want to find an upper or lower bound for the k-independence number of a graph.

We define the ρ_k-gap of G, denoted by $\rho_k(G)$, the difference between $\alpha(\overline{G^k})$ and $\alpha(\overline{G}^k)$, that is,

$$\rho_k(G) = \alpha(\overline{G^k}) - \alpha(\overline{G}^k).$$

It is well known that if G has n vertices then $1 \leq \alpha(G) \leq n$. This implies that

$$1 - n \leq \rho_k(G) \leq n - 1.$$

Let r, k, n be fixed as integers with $n \geq 1$, $1 - n \leq r \leq n - 1$ and $k \geq 2$. Two questions that naturally arise in this context are the following.

Is there a graph G with n vertices such that $\rho_k(G) = r$?

Which graphs G, with n vertices, satisfy $\rho_k(G) = r$?

The next tables show us the answers to these two questions when G is a graph with 4 vertices.

Let K_n, N_n, C_n, $K_{t,s}$ denote, respectively, the complete graph with n vertices, the null graph with n vertices, the cycle graph with n vertices, the complete bipartite graph having t vertices in one of its vertex classes, and s in the other, see [3,4].

In Table 1 we list all graphs with 4 vertices (first column of this table).

Note that in the third column of this table all graphs of the first appear (this is because each graph is the complement graph of a graph). In the sixth column, show us the 2-independence number of each graph with 4 vertices. In this case, $\rho_2(G) \geq 0$, for all graphs G with 4 vertices. Moreover, there are graphs G with $\rho_2(G) = 0, 1, 2$ and there is no graph G such that $\rho_2(G) = 3$.

G	G^2	$\overline{G}$	$\overline{G}^2$	$\overline{G^2}$	$\alpha(\overline{G}^2)$	$\alpha(\overline{G^2})$	$\rho_2(G)$
N_4	N_4	K_4	K_4	K_4	1	1	0
$K_2 \cup N_2$	$K_2 \cup N_2$	$\overline{K_2 \cup N_2}$	K_4	$\overline{K_2 \cup N_2}$	1	2	1
$K_2 \cup K_2$	$K_2 \cup K_2$	C_4	K_4	C_4	1	2	1
$P_3 \cup N_1$	$K_3 \cup N_1$	$\overline{P_3 \cup N_1}$	K_4	$K_{1,3}$	1	3	2
$K_{1,3}$	K_4	$K_3 \cup N_1$	$K_3 \cup N_1$	N_4	2	4	2
P_4	$\overline{K_2 \cup N_2}$	P_4	$\overline{K_2 \cup N_2}$	$K_2 \cup N_2$	2	3	1
$K_3 \cup N_1$	$K_3 \cup N_1$	$K_{1,3}$	K_4	$K_{1,3}$	1	3	2
$\overline{P_3 \cup N_1}$	K_4	$P_3 \cup N_1$	$K_3 \cup N_1$	N_4	2	4	2
C_4	K_4	$K_2 \cup K_2$	$K_2 \cup K_2$	N_4	2	4	2
$\overline{K_2 \cup N_2}$	K_4	$K_2 \cup N_2$	$K_2 \cup N_2$	N_4	3	4	1
K_4	K_4	N_4	N_4	N_4	4	4	0

Table 1. $\rho_2(G)$ when G is a graph with 4 vertices and $k = 2$.

G	G^k	$\overline{G}$	$\overline{G}^k$	$\overline{G^k}$	$\alpha(\overline{G}^k)$	$\alpha(\overline{G^k})$	$\rho_k(G)$
N_4	N_4	K_4	K_4	K_4	1	1	0
$K_2 \cup N_2$	$K_2 \cup N_2$	$\overline{K_2 \cup N_2}$	K_4	$\overline{K_2 \cup N_2}$	1	2	1
$K_2 \cup K_2$	$K_2 \cup K_2$	C_4	K_4	C_4	1	2	1
$P_3 \cup N_1$	$K_3 \cup N_1$	$\overline{P_3 \cup N_1}$	K_4	$K_{1,3}$	1	3	2
$K_{1,3}$	K_4	$K_3 \cup N_1$	$K_3 \cup N_1$	N_4	2	4	2
P_4	K_4	P_4	K_4	N_4	1	4	3
$K_3 \cup N_1$	$K_3 \cup N_1$	$K_{1,3}$	K_4	$K_{1,3}$	1	3	2
$\overline{P_3 \cup N_1}$	K_4	$P_3 \cup N_1$	$K_3 \cup N_1$	N_4	2	4	2
C_4	K_4	$K_2 \cup K_2$	$K_2 \cup K_2$	N_4	2	4	2
$\overline{K_2 \cup N_2}$	K_4	$K_2 \cup N_2$	$K_2 \cup N_2$	N_4	3	4	1
K_4	K_4	N_4	N_4	N_4	4	4	0

Table 2. $\rho_k(G)$ when G is a graph with 4 vertices and $k > 2$.

In Table 2 we did the same study as in Table 1 but with $k > 2$.

So, when G is a graph with 4 vertices, $\rho_k(G) \geq 0$, for all $k \geq 2$. Moreover, there are graphs G with $\rho_k(G) = 0, 1, 2$ and there is no graph G such that $\rho_2(G) = 3$. However, if $k \geq 3$ then $\rho_k(P_4) = 3$.

This paper is organized as follows. In Section 2, we provide that $\alpha(\overline{G^k})$ is an upper bound to $\alpha(\overline{G}^k)$ and characterize all graphs that achieve the equality. In Section 3, we explore the possible values of $\rho_k(G)$. The focus of Section 4 is the case $\rho_k(G) = |V(G)| - 1$. Furthermore, when $k = 2$ we describe necessary conditions for a graph G to have $\rho_2(G) = |V(G)| - 1$ without imposing any conditions on $\overline{G}$. We finish this paper with an open problem.

2. The permutation of the two operations

As we saw in Tables 1 and 2, there are graphs G, with 4 vertices, such that $\rho_k(G) \neq 0$, for $k \geq 2$. The goal of this section is to find necessary and sufficient conditions for a graph to satisfy $\alpha(\overline{G^k}) = \alpha(\overline{G}^k)$, that is, $\rho_k(G) = 0$. The following lemma shows us that every independent set of $\overline{G}^k$ is also an independent set of $\overline{G^k}$.

Lemma 2.1. *Let G be a graph and $k \in \mathbb{N}$. If $S \subseteq X$ is an independent set of $\overline{G}^k$, then*

(1) *S is an independent set of $\overline{G^k}$;*

(2) *the subgraph in G induced by S is a complete graph.*

Proof. If $|S| = 1$ then S is an independent set of $\overline{G}^k$ and $\overline{G^k}$.

Suppose that $|S| \geq 2$. Let $x, y \in S$, $x \neq y$. By definition of an independent set of $\overline{G}^k$, $\text{dist}_{\overline{G}^k}(x, y) > 1$. Then $\text{dist}_{\overline{G}}(x, y) > k \geq 1$, implying that x and y are adjacent in G and G^k. Consequently, x and y are not adjacent in $\overline{G^k}$. Due to the arbitrariness of the choice of the vertices, we conclude that S is an independent set of $\overline{G^k}$ and its elements are the vertices of a complete subgraph of G. ■

Lemma 2.2. *Let G be a graph with n vertices and let $k \geq 1$. Then,*

(1) $\alpha(\overline{G^k}) = 1$ *if and only if G is the null graph;*

(2) $\alpha(\overline{G}^k) = n$ *if and only if G is the complete graph.*

Proof. (1) As mentioned in the introduction, $\alpha(\overline{G^k}) = 1$ if and only if $\overline{G^k}$ is the complete graph. Since the complement of the complete graph is the null graph, $\overline{G^k}$ is the complete graph if and only if G^k is the null graph. Because G is a subgraph of G^k formed by all vertices of G^k, G^k is the null graph if and only if G is the null graph.

(2) As mentioned in the introduction, $\alpha(\overline{G}^k) = n$ if and only if $\overline{G}^k$ is the null graph. Because $\overline{G}$ is a subgraph of $\overline{G}^k$ formed by all vertices of $\overline{G}^k$, $\overline{G}^k$ is the null graph if and only if $\overline{G}$ is the null graph. Since the complement of the null graph is the complete graph, $\overline{G}$ is the null graph if and only if G is the complete graph. ■

Lemma 2.3. *Let G be a non-null and non-complete graph, with $\alpha(\overline{G}^k) \geq 2$ and let $k \in \mathbb{N}$, with $k \geq 2$. Let S be an α_1-set of $\overline{G}^k$. Then, for all $y \in V(G) \setminus S$, the set $S \cup \{y\}$ is an independent set of $\overline{G^k}$.*

Proof. Let $n = |V(G)|$. Since G is a non-null and non-complete graph, $n \geq 3$. By Lemma 2.2, $|S| \leq n - 1$. Let $y \in V(G) \setminus S$. Suppose that for all $s \in S$, $\{y, s\} \notin E(G)$. Then, for all $s \in S$, $\{y, s\} \in E(\overline{G})$. As $k \geq 2$, we get that $S \cup \{y\}$ is the set of vertices of a complete subgraph of $\overline{G}^k$. This implies that S, with $|S| \geq 2$, is the set of vertices of a complete subgraph of $\overline{G}^k$, contradicting the fact that S is an independent set of $\overline{G}^k$. Therefore, there is $s \in S$ such that $\{y, s\} \in E(G)$. By Lemma 2.1, S is the set of vertices of a complete subgraph of G. Again, as $k \geq 2$, we get that $S \cup \{y\}$ is the set of vertices of a complete subgraph of G^k. Thus, $S \cup \{y\}$ is an independent set of $\overline{G^k}$. ■

The next result follows from Lemma 2.1 and gives us the first relation between $\alpha(\overline{G}^k)$ and $\alpha(\overline{G^k})$.

Theorem 2.4. *Let G be a simple graph and $k \in \mathbb{N}$ with $k \geq 2$. Then,*

$$\alpha(\overline{G}^k) \leq \alpha(\overline{G^k}).$$

The equality holds if and only if G is the complete graph or the null graph.

Proof. Let S be an α_1-set of $\overline{G}^k$. By Lemma 2.1, S is an independent set of $\overline{G^k}$. Therefore, $\alpha(\overline{G}^k) = |S| \leq \alpha(\overline{G^k})$.

If G is the complete graph with n vertices, then $\alpha(\overline{G}^k) = n = \alpha(\overline{G^k})$. If G is the null graph, then $\alpha(\overline{G}^k) = 1 = \alpha(\overline{G^k})$. Therefore, in these two situations we get $\alpha(\overline{G}^k) = \alpha(\overline{G^k})$.

Conversely, suppose that G has n vertices and $\alpha(\overline{G}^k) = \alpha(\overline{G^k})$. If $\alpha(\overline{G}^k) = \alpha(\overline{G^k}) = 1$ or $\alpha(\overline{G}^k) = \alpha(\overline{G^k}) = n$, by Lemma 2.2 we have G is the null graph or the complete graph. If $2 \leq \alpha(\overline{G}^k) = \alpha(\overline{G^k}) \leq n-1$, using Lemma 2.2, G is a non-null and non-complete graph and $n \geq 3$. By Lemma 2.3 we obtain $\alpha(\overline{G}^k) < \alpha(\overline{G^k})$. Contradiction. Therefore, the result follows. ■

3. Different values of $\rho_k(G)$

First, we prove a result regarding the power graphs of the same graph and their independence numbers.

Proposition 3.1. *Let G be a graph and $k \in \mathbb{N}$. Then,*

$$\alpha(G^{k+1}) \leq \alpha(G^k).$$

Proof. Since G^k is a subgraph of G^{k+1} with the same vertex set, an α-set of G^{k+1} is an α-set of G^k. Thus, $\alpha(G^{k+1}) \leq \alpha(G^k)$. ■

Theorem 3.2. *Let G be a graph, and $k, j \in \mathbb{N}$ with $k \leq j$. Then,*

$$\rho_k(G) \leq \rho_j(G).$$

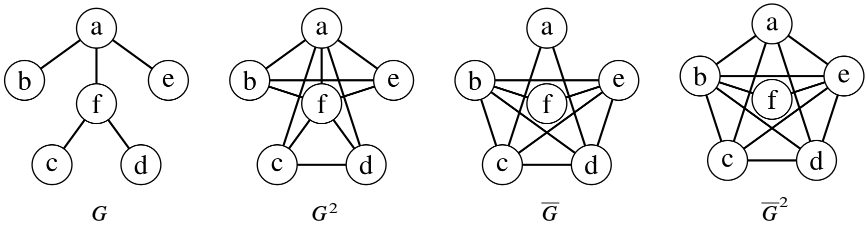
Proof. If $k = j$ then $\rho_k(G) \leq \rho_j(G)$. Suppose that $k < j$. From Proposition 3.1, we get $\alpha(\overline{G}^j) \leq \alpha(\overline{G}^k)$. On the other hand, since $k < j$, G^k is a subgraph of G^j, with the same vertex set. Then $\overline{G^j}$ is a subgraph of $\overline{G^k}$, with the same vertex set. Consequently, $\alpha(\overline{G^k}) \leq \alpha(\overline{G^j})$. Thus, $\rho_k(G) \leq \rho_j(G)$. ■

By the last result, we know that for any graph G and $k \in \mathbb{N}$,

$$\rho_{k+1}(G) - \rho_k(G) \geq 0.$$

It is important to emphasize that if $\rho_{k+1}(G) - \rho_k(G) \neq 0$, it does not mean that it is equal to 1. There are graphs that verify $\rho_{k+1}(G) - \rho_k(G) \geq 2$, for some integer k, with $k \geq 2$. The following picture shows a graph G with 6 vertices such that $\rho_2(G) = 2$ and $\rho_3(G) = 5$. In this case,

$$\rho_3(G) - \rho_2(G) = 5 - 2 = 3.$$



Note that G^2 has a clique with 4 vertices and does not have a clique with 5 vertices. So, $\alpha(\overline{G^2}) = 4$ and $\alpha(\overline{G}^2) = 2$. On the other hand, $\overline{G}^3$ and G^3 are the complete graph.

Proposition 3.3. *Let G be a connected graph and p be the number of connected components of $\overline{G}$. Then, for all $k \in \mathbb{N}$,*

$$0 \leq \rho_k(G) \leq |V(G)| - p.$$

Note that if G is a graph and $\overline{G}$ has $p \geq 2$ connected components, then G is a connected graph. However, if $p = 1$ then G may or may not be connected. Therefore, condition G be a connected graph in Proposition 3.3 is important.

Proof. As G is a connected graph, there is an integer r such that, for $\ell \geq r$, G^ℓ is the complete graph. Consequently, $\alpha(\overline{G^\ell}) = |V(G)|$. On the other hand, since $\overline{G}$ has p connected components, there is an integer s such that, for $j \geq s$, $\overline{G}^j$ is the disjoint union of p complete graphs and $\alpha(\overline{G}^j) = p$. Let $m = \max\{r, s\}$. Then, for all integers d, $d \geq m$, we get $\rho_d(G) = \alpha(\overline{G^d}) - \alpha(\overline{G}^d) = |V(G)| - p$. Consequently, if $s, t \in \mathbb{N}$, with $s \leq m \leq t$ then, by Theorems 2.4 and 3.2, we get $0 \leq \rho_s(G) \leq \rho_m(G) = |V(G)| - p = \rho_t(G)$. Thus, the result follows. ∎

Let p be a positive integer and $p \geq 2$. A complete p-partite graph is a graph whose vertices are partitioned into p disjoint sets, called parts, such that no two vertices within the same part are adjacent, but every vertex in one part is adjacent to every vertex in all other parts.

Corollary 3.4. *Let G be a complete p-partite graph. Then,*

$$\rho_k(G) = |V(G)| - p,$$

for all $k \in \mathbb{N}$, $k \geq 2$.

Proof. If G is a complete p-partite graph then G is a connected graph and $\overline{G}$ is the disjoint union of p complete graphs. Moreover, if u, v are two vertices of G then $\text{dist}_G(u, v) \leq 2$. Consequently, G^2 is the complete graph. The result follows from the proof of Proposition 3.3. ■

Corollary 3.5. *Let G be the graph obtained from the complete graph by removing an edge and $|V(G)| \geq 3$. Then,*

$$\rho_k(G) = 1,$$

for all $k \in \mathbb{N}$, $k \geq 2$.

Proof. Since $|V(G)| \geq 3$, we have $\text{dist}_G(u, v) \leq 2$, for all $u, v \in V(G)$. Consequently, G^2 is the complete graph. Moreover, $\overline{G}$ is the disjoint union of $|V(G)| - 1$ complete graphs. The result follows from the proof of Proposition 3.3. ■

Proposition 3.6. *Let G be a graph with $p \geq 2$ connected components. Let i be the number of vertices of the connected component of G with the most vertices. Then, for all $k \in \mathbb{N}$,*

$$0 \leq \rho_k(G) \leq i - 1.$$

Proof. As G is not a connected graph, $\overline{G}$ is a connected graph. So, there is an integer r such that, for $\ell \geq r$, $\overline{G}^\ell$ is the complete graph. Consequently, $\alpha(\overline{G}^\ell) = 1$. On the other hand, since G has p connected components, there is an integer s such that, for $j \geq s$, G^j is the disjoint union of p complete graphs and $\alpha(\overline{G^j}) = i$. Let $m = \max\{r, s\}$. Consequently, if $s, t \in \mathbb{N}$, with $s \leq m \leq t$ then, by Theorem 3.2, $0 \leq \rho_s(G) \leq \rho_m(G) = i - 1 = \rho_t(G)$. Thus, the result follows. ■

Corollary 3.7. *Let G be a graph that is the disjoint union of p complete graphs, with $p \geq 2$. Let i be the number of vertices of the connected component of G with the most vertices. Then,*

$$\rho_k(G) = i - 1,$$

for all $k \in \mathbb{N}$, $k \geq 2$.

4. The case $\rho_k(G) = n - 1$

As we proved in Section 2, for a given graph G and $k \geq 2$ integer, $\rho_k(G) \geq 0$. It is known that the independence number of any graph is at most the number of its vertices and at least 1. This means that $0 \leq \rho_k(G) \leq n - 1$, where n is the number of vertices of G. In Section 2, we characterized the graphs G that verify $\rho_k(G) = 0$, with $k \geq 2$. In this section, we will explore the other extreme of the inequality, that is, $\rho_k(G) = n - 1$, trying to characterize the graphs that make $\rho_k(G)$ reach that value.

Note that if G has only one vertex, then G is the null graph and by Theorem 2.4, $\rho_k(G) = 0 = 1 - 1$. If G has only two vertices, then G is either the null graph or the complete graph. In these two cases, we have $\rho_k(G) = 0 \neq 2 - 1 = 1$. Consequently, in this section we will only consider graphs with at least three vertices.

Theorem 4.1. *Let G be a graph with $n \geq 3$ vertices and let $k \in \mathbb{N}$ with $k \geq 2$. Then, $\rho_k(G) = n - 1$ if and only if* $\max\{\mathrm{diam}(G), \mathrm{diam}(\overline{G})\} \leq k$.

Proof. We have $\rho_k(G) = n - 1$ if and only if $\alpha(\overline{G^k}) = n$ and $\alpha(\overline{G}^k) = 1$. This is equivalent to $\overline{G^k}$ being the null graph and $\overline{G}^k$ the complete graph. And this is equivalent to G^k and $\overline{G}^k$ verifying $\max\{\mathrm{diam}(G), \mathrm{diam}(\overline{G})\} \leq k$. ∎

A question that naturally arises from this theorem is whether there are conditions under which a graph G will verify $\rho_k(G) = n - 1$, without imposing any conditions on $\overline{G}$, rather solely on G. The graphs G and $\overline{G}$ will need to be connected because

$$\max\{\mathrm{diam}(G), \mathrm{diam}(\overline{G})\} \leq k$$

is a finite number. Consequently, if a given graph G and its complement graph are connected graphs, then taking $k = \max\{\mathrm{diam}(G), \mathrm{diam}(\overline{G})\}$, we get $\rho_k(G) = |V(G)| - 1$. Moreover, by Theorem 3.2, for every $j > k$, $\rho_j(G) = |V(G)| - 1$. Then, there is an infinite number of values $j \in \mathbb{N}$ satisfying $\rho_j(G) = |V(G)| - 1$ for every graph, as long as the graph and its complement graph satisfy the condition of being connected.

One might also ask if the conditions of $G, \overline{G}$ being connected graphs and

$$\mathrm{diam}(G) \leq k$$

are sufficient to say that $\mathrm{diam}(\overline{G}) \leq k$, for a given natural $k \geq 2$. The answer is no. For example, if we consider $k = 3$ and $G = \overline{P_5}$ (the house graph), we have $\mathrm{diam}(G) = 2$, $\overline{G} = P_5$ but $\mathrm{diam}(P_5) = 4 > 3$.

Two immediate results follow from Theorem 4.1.

Corollary 4.2. *Let G be a graph with $n \geq 3$ vertices and $\rho_2(G) = n - 1$. Then*

$$\mathrm{diam}(G) = 2.$$

Corollary 4.3. *Let G be a graph with $n \geq 3$ vertices and let $k \in \mathbb{N}$ with $k \geq 2$. If* $\mathrm{diam}(G) \leq k$ *and G is isomorphic to $\overline{G}$, then*

$$\rho_k(G) = n - 1.$$

Using Corollary 4.3, we conclude that the graph C_5, whose diameter is 2, will always satisfy $\rho_k(C_5) = |V(C_5)| - 1 = 4$, for every $k \geq 2$. On the other hand, as we have seen in Tables 1 and 2, the graph P_4, whose diameter is 3, will always satisfy $\rho_k(P_4) = |V(P_4)| - 1 = 3$, for every $k \geq 3$, but not for $k = 2$.

Proposition 4.4. *Let G be a graph with $n \geq 3$ vertices and $k \in \mathbb{N}$, $k \geq 2$. Then,*

$$\rho_k(G) = n - 1 \quad \textit{if and only if} \quad \rho_k(\overline{G}) = n - 1.$$

Proof. By Theorem 4.1, $\rho_k(G) = n - 1$ if and only if both G^k and $\overline{G}^k$ are complete graphs. That happens if and only if $\overline{\overline{G}^k}$ is the null graph. So, $\alpha(\overline{\overline{G}^k}) = n$, meaning $\rho_k(\overline{G}) = \alpha(\overline{\overline{G}^k}) - \alpha(G^k) = n - 1$. ■

We will now focus on $k = 2$.

Proposition 4.5. *Let G be a graph with $n \geq 3$ vertices. If $\rho_2(G) = n - 1$, then $d_G(v) \geq 2$, for all $v \in V(G)$.*

Proof. Suppose that there is a vertex v in G of degree 1. In order to $\rho_2(G) = n - 1 \geq 2$, G^2 should be the complete graph, which means that the distance between any two vertices of G should be at most 2. But, if v has degree one, there is w such that $\{v, w\}$ is an edge of G which implies that w is adjacent to every vertex of G. Consequently, w is an isolated vertex of $\overline{G}$, which is impossible by Theorem 4.1. ■

An immediate consequence of Propositions 4.4 and 4.5 is the one that follows.

Corollary 4.6. *Let G be a graph with $n \geq 3$ vertices. If $\rho_2(G) = n - 1$ then*

$$\min_{v \in V(G)} \{d_G(v), d_{\overline{G}}(v)\} \geq 2.$$

This last corollary implies that if $\rho_2(G) = |V(G)| - 1$ then $|V(G)| \geq 5$, because $4 \leq d_G(v) + d_{\overline{G}}(v) = |V(G)| - 1$, for all $v \in V(G)$. The following proposition is a characterization of the graphs G, with 5 vertices, such that $\rho_2(G) = |V(G)| - 1$.

Proposition 4.7. *Let G be a graph with 5 vertices. Then, $\rho_2(G) = |V(G)| - 1$ if and only if G is the graph C_5.*

Proof. From Theorem 4.1, the graphs G and $\overline{G}$ should be connected graphs of diameter 2. By Corollary 4.6, we get $d_G(v) \geq 2$ and $d_{\overline{G}}(v) \geq 2$, for all $v \in V(G)$. Then,

$$d_G(v) = 2 = d_{\overline{G}}(v) \geq 2,$$

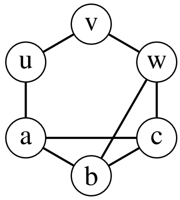
for all $v \in V(G)$ and G is the graph C_5. ■

We already know that, given a connected graph G such that $\overline{G}$ is a connected graph, there is an integer $k \geq 2$ such that $\rho_k(G) = |V(G)| - 1$. The question is, for $k = 2$, will it always be possible to find a graph with $n \geq 5$ vertices such that $\rho_2(G) = n - 1$? The answer is yes.

Theorem 4.8. *Let $s \geq 2$ be a positive integer. Let (A, B) be a partition of $V(K_s)$. Let G be the graph obtained from K_s by adding three distinct vertices, u, v, w and making u adjacent to all elements of A, w adjacent to all elements of B and v adjacent to both u and w. Then,*

$$\rho_2(G) = n - 1.$$

The following graph G is an example of the graphs described in Theorem 4.8. G has $n = 6$ vertices and is obtained from K_3 (the subgraph in G induced by the vertices a, b, c) by adding the vertices, u, v, w and making u adjacent to a, w adjacent to b, c and v adjacent to u, w. In this case, $A = \{a\}$ and $B = \{b, c\}$.



G

Note that the graphs described in Theorem 4.8 are a kind of generalization of the graph C_5.

Proof. In $\overline{G}$, u is adjacent to all elements of B, w is adjacent to all elements of A, v is adjacent to all elements of $A \cup B$ and u is adjacent to w. Besides these, there are no other edges in $\overline{G}$.

So, $\max\{\text{diam}(G), \text{diam}(\overline{G})\} = 2$. By Theorem 4.1 we get $\rho_2(G) = n - 1$. ■

We now prove conditions that are necessary for every graph G that satisfies $\rho_2(G) = |V(G)| - 1$.

Lemma 4.9. *Let G be a graph with $n \geq 3$ vertices such that $\rho_2(G) = n - 1$. Let $v \in V(G)$. Then, $|N_1^G(v)| \geq 2$ and $|N_2^G(v) \setminus N_1^G(v)| \geq 2$.*

Proof. As mentioned after Corollary 4.6, $n \geq 5$. By Theorem 4.1, we have $\text{diam}(G) \leq 2$. By Corollary 4.6, $|N_1^G(v)| \geq 2$. If $|N_2^G(v) \setminus N_1^G(v)| = 0$ then v is adjacent to all vertices of $V(G) \setminus \{v\}$. Consequently, $d_{\overline{G}}(v) = 0$ which is a contradiction with Corollary 4.6. Suppose that $|N_2^G(v) \setminus N_1^G(v)| = 1$. Let $\{u\} = N_2^G(v) \setminus N_1^G(v)$. Consequently, $d_{\overline{G}}(v) = 1$ (in $\overline{G}$, the unique vertex adjacent to v is u) which is a contradiction with Corollary 4.6. So, $|N_2^G(v) \setminus N_1^G(v)| \geq 2$. ■

Lemma 4.10. *Let G be a graph with $n \geq 2$ vertices such that $\rho_2(G) = n - 1$. Let $v \in V(G)$ such that the subgraph in G induced by $N_2^G(v) \setminus N_1^G(v)$ is a complete graph. Then, the subgraph in G induced by $N_1^G(v)$ is not a complete graph.*

Proof. Suppose that the subgraph in G induced by $N_1^G(v)$ is a complete graph. Since $N_1^{\overline{G}}(v) = N_2^G(v) \setminus N_1^G(v)$, $N_2^{\overline{G}}(v) \setminus N_1^{\overline{G}}(v) = N_1^G(v)$ (note that by Theorem 4.1, the diameter of G is 2), we conclude that the subgraphs in $\overline{G}$ induced by $N_1^{\overline{G}}(v)$ and by $N_2^{\overline{G}}(v) \setminus N_1^{\overline{G}}(v)$ are null graphs. Using Lemma 4.9, we get $|N_1^G(v)| \geq 2$ and $|N_2^G(v) \setminus N_1^G(v)| \geq 2$. Suppose there are $a \in N_1^G(v)$ and $b \in N_2^G(v) \setminus N_1^G(v)$ such that $\{a, b\} \notin E(\overline{G})$. Then $\mathrm{dist}_{\overline{G}}(a, b) \geq 3$, contradicting Theorem 4.1. Consequently, $\{a, b\} \in E(\overline{G})$, for all $a \in N_1^G(v)$ and for all $b \in N_2^G(v) \setminus N_1^G(v)$. This implies that $\{a, b\} \notin E(G)$, for all $a \in N_1^G(v)$ and for all $b \in N_2^G(v) \setminus N_1^G(v)$, which is a contradiction with Corollary 4.2. Therefore, we get the result. ■

With a similar proof, we conclude the next lemma.

Lemma 4.11. *Let G be a graph with $n \geq 3$ vertices such that $\rho_2(G) = n - 1$. Let $v \in V(G)$ such that the subgraph in G induced by $N_1^G(v)$ is a complete graph. Then, the subgraph in G induced by $N_2^G(v) \setminus N_1^G(v)$ is not a complete graph.*

Lemma 4.12. *Let G be a graph with $n \geq 3$ vertices such that $\rho_2(G) = n - 1$. Let $v \in V(G)$ such that the subgraph in G induced by $N_2^G(v) \setminus N_1^G(v)$ is a complete graph. Then, there is no $a \in N_2^G(v) \setminus N_1^G(v)$ such that $\{a, b\} \in E(G)$, for all $b \in N_1^G(v)$.*

Proof. Suppose there is $a \in N_2^G(v) \setminus N_1^G(v)$ such that $\{a, b\} \in E(G)$, for all $b \in N_1^G(v)$. Since the subgraph in $\overline{G}$ induced by $N_1^{\overline{G}}(v)$ is the null graph, $d_{\overline{G}}(a) = 1$ (the vertex a is only adjacent to v). This is impossible by Corollary 4.6. ■

Lemma 4.13. *Let G be a graph with $n \geq 3$ vertices such that $\rho_2(G) = n - 1$. Let $v \in V(G)$. Then, there is no $b \in N_1^G(v)$ such that $\{a, b\} \in E(G)$, for all $a \in N_2^G(v) \setminus N_1^G(v)$.*

Proof. Suppose there is $b \in N_1^G(v)$ such that $\{a, b\} \in E(G)$, for all $a \in N_2^G(v) \setminus N_1^G(v)$. Then, $b \in N_2^{\overline{G}}(v) \setminus N_1^{\overline{G}}(v)$ and $\{a, b\} \notin E(\overline{G})$, for all $a \in N_1^{\overline{G}}(v)$, Consequently,

$$\mathrm{dist}_{\overline{G}}(b, v) \geq 3.$$

This is impossible by Theorem 4.1. ■

Theorem 4.14. *Let G be a graph with $n \geq 3$ vertices such that $\rho_2(G) = n - 1$. Let $v \in V(G)$ such that the subgraph in G induced by $N_2^G(v) \setminus N_1^G(v)$ is a complete graph and the subgraph in G induced by $N_1^G(v)$ is a null graph. Then, there exists a subset S of $V(G)$ such that the subgraph induced in G by S is C_5.*

Proof. By Lemma 4.9, $|N_1^G(v)| \geq 2$ and $|N_2^G(v) \setminus N_1^G(v)| \geq 2$. By Corollary 4.6,

$$\forall a \in N_1^G(v),\ \exists b \in N_2^G(v) \setminus N_1^G(v) :\ \{a, b\} \in E(G).$$

Furthermore, according to Theorem 4.1,

$$\forall c \in N_2^G(v) \setminus N_1^G(v),\ \exists d \in N_1^G(v) :\ \{c, d\} \in E(G).$$

By Lemma 4.12, let $u, w \in N_1^G(v)$ be such that $N_1^G(u) \setminus N_1^G(w) \neq \emptyset$. Then $u \neq w$. Let $b \in N_1^G(u) \setminus N_1^G(w)$. If $N_1^G(w) \setminus N_1^G(u) \neq \emptyset$, let $a \in N_1^G(w) \setminus N_1^G(u)$. Then, the subgraph induced in G by $\{v, u, b, a, w\}$ is C_5.

Suppose that $N_1^G(w) \setminus N_1^G(u) = \emptyset$. Since $N_2^G(v) \setminus (N_1^G(v) \cup N_1^G(u)) \neq \emptyset$, let $c \in N_2^G(v) \setminus (N_1^G(v) \cup N_1^G(u))$ and $r \in N_1^G(v)$ such that $\{r, c\} \in E(G)$. Then $r \neq w$, $r \neq u$ and $c \in N_1^G(r) \setminus N_1^G(u)$. If $N_1^G(u) \setminus N_1^G(r) \neq \emptyset$, let $p \in N_1^G(u) \setminus N_1^G(r)$. Then, the subgraph induced in G by $\{v, u, p, c, r\}$ is C_5.

Suppose that $N_1^G(u) \setminus N_1^G(r) = \emptyset$, we repeat the process. Since

$$N_2^G(v) \setminus (N_1^G(v) \cup N_1^G(r)) \neq \emptyset,$$

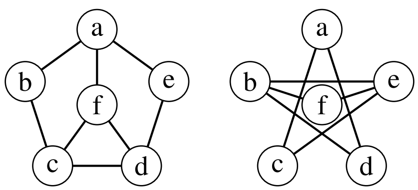
let $d \in N_2^G(v) \setminus (N_1^G(v) \cup N_1^G(r))$ and $x \in N_1^G(v)$ such that $\{x, d\} \in E(G)$. Then $x \neq w$, $x \neq u$, $x \neq r$ and $d \in N_1^G(x) \setminus N_1^G(r)$. If $N_1^G(r) \setminus N_1^G(x) \neq \emptyset$, let $j \in N_1^G(r) \setminus N_1^G(x)$. Then, the subgraph induced in G by $\{v, r, j, d, x\}$ is C_5.

Suppose that $N_1^G(r) \setminus N_1^G(x) = \emptyset$, we repeat the process. By induction, this process ends because G has a finite number of vertices. Therefore, we get the result. ■

Using Proposition 4.4 and the last theorem, we get the following result.

Theorem 4.15. *Let G be a graph with $n \geq 3$ vertices such that $\rho_2(G) = n - 1$. Let $v \in V(G)$ such that the subgraph in G induced by $N_1^G(v)$ is a complete graph and the subgraph in G induced by $N_2^G(v) \setminus N_1^G(v)$ is a null graph. Then, there exists a subset S of $V(G)$ such that the subgraph induced in G by S is C_5.*

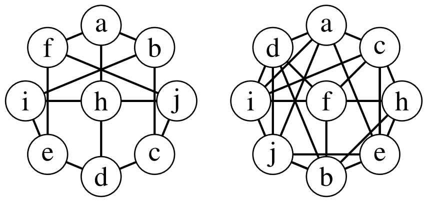
We should note that the necessary condition of Theorem 4.14 is not sufficient for a graph G, with n vertices, to verify $\rho_2(G) = n - 1$. For example, the following graph G has $n = 6$ vertices and the subgraph induced in G by $\{a, b, c, d, e\}$ is C_5. However, $\text{dist}_{\overline{G}}(a, f) = 3$ and by Theorem 4.1, $\rho_2(G) \neq 5 = n - 1$.



G $\overline{G}$

Furthermore, it is not true that if a graph G verifies $\rho_2(G) = |V(G)| - 1$, then there is no $R \subseteq V(G)$ such that the subgraph induced in G by R is C_p, with $|V(G)| \geq p \geq 6$.

The next graph G is a graph that verifies $\rho_2(G) = 8$ (the distance between any two vertices in G and in $\overline{G}$ is at most 2) and the subgraph induced in G by $\{a, f, e, d, c, b\}$ is C_6.



G $\overline{G}$

However, looking at Theorems 4.14 and 4.15, we finish this section with an open problem.

Open Problem 4.16. *Characterize all graphs G, with $n \geq 3$ vertices such that $\rho_2(G) = n - 1$ and having a subset S of $V(G)$ such that the subgraph induced in G by S is C_5.*

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