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Normalized solutions of fractional mass supercritical NLS systems with linear couplings

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Abstract. In this paper, we consider normalized solutions for fractional NLS systems with Sobolev critical nonlinearities and linear couplings. Under mass supercritical condition, there are two cases discussed. First, we use the minimax method to get the existence of positive normalized solution when the exponent is Sobolev subcritical. Second, if it is Sobolev critical, we prove the nonexistence of positive normalized solution. Scaling transformation and concentration-compactness principle are used in this paper.

1. Introduction

In this paper, we investigate the following fractional coupled nonlinear Schrödinger equations with $N > 2s, s \in (0,1)$:

$$\begin{cases} (-\Delta)^s u - \lambda_1 u = \mu_1 |u|^{2_s^*-2} u + \eta\alpha |u|^{\alpha-2}|v|^{\beta} u + \kappa(x) v & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v - \lambda_2 v = \mu_2 |v|^{2_s^*-2} v + \eta\beta |u|^{\alpha}|v|^{\beta-2} v + \kappa(x) u & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

satisfying the conditions

$$\int_{\mathbb{R}^N} u^2 = a^2, \quad \int_{\mathbb{R}^N} v^2 = b^2, \tag{1.2}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ appear as Lagrange multipliers, $2_s^* = \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent, $\eta, a, b > 0$, $\alpha, \beta > 1$, $\alpha + \beta \in (2 + \frac{4s}{N}, 2_s^*]$ and $\kappa(x) \in L^\infty(\mathbb{R}^N)$.

The fractional Laplacian $(-\Delta)^s$ is expressed by

$$(-\Delta)^s u(x) = C(N,s)\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y,$$

where P.V. indicates the Cauchy principal value and $C(N,s)$ is a positive constant

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that depends on N, s. We can refer to [6, 9] for more information about the fractional Laplacian. Let the function space associated with $(-\Delta)^s$ be

$$H^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y < +\infty \right\},$$

equipped with the norm

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2 \right)^{\frac{1}{2}}.$$

Under the condition of mass supercritical, both fractional and integer order systems have been studied by many experts.

For example, for integer order systems with Sobolev critical nonlinearities and $p, q, \alpha + \beta > 2 + \frac{4}{N}$ as follows:

$$\begin{cases} -\Delta u - \lambda_1 u = |u|^{2^*-2} u + \mu_1 |u|^{p-2} u + \eta \alpha |u|^{\alpha-2} |v|^\beta u & \text{in } \mathbb{R}^N, \\ -\Delta v - \lambda_2 v = |v|^{2^*-2} v + \mu_2 |v|^{q-2} v + \eta \beta |u|^\alpha |v|^{\beta-2} v & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

Liu and Fang [14] investigated the existence of positive normalized solution of the systems (1.3) satisfying (1.2) in the case of $p, q, \alpha + \beta < 2^*$ and the nonexistence of positive normalized solution in the case of $p, q, \alpha + \beta = 2^*$.

In addition, for the following fractional systems with Sobolev critical terms:

$$\begin{cases} (-\Delta)^s u - \lambda_1 u = |u|^{2_s^*-2} u + \eta \alpha |u|^{\alpha-2} |v|^\beta u & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v - \lambda_2 v = |v|^{2_s^*-2} v + \eta \beta |u|^\alpha |v|^{\beta-2} v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

which were studied by Zou et al. [3], they proved the existence and nonexistence of normalized ground state in the case of $\eta > 0$ and $\eta < 0$. Furthermore, they also showed the asymptotic behavior of the minimizers for question (1.4) and (1.2). This fractional system with Sobolev critical terms models complex physical phenomena. The fractional Laplace operator $(-\Delta)^s$ captures non-local interactions, relevant to quantum tunneling or anomalous diffusion. Sobolev critical terms represent critical interaction strengths, like in nonlinear wave focusing or quantum many-body phase transitions. Coupling terms describe multi-field interactions, such as cross-modulation in nonlinear optics or electronic state coupling in quantum Hall effect. Overall, it mathematically abstracts non-local, critically nonlinear, multi-field coupled systems to analyze existence and asymptotic behavior.

Zhang and Yun [20] studied equations (1.5) with linear and nonlinear couplings

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 |u|^3 + \eta |v|^2 u + \kappa(x) v & \text{in } \mathbb{R}^3, \\ -\Delta v - \lambda_2 v = \mu_2 |v|^3 + \eta |u|^2 v + \kappa(x) u & \text{in } \mathbb{R}^3. \end{cases} \tag{1.5}$$

For the mass supercritical case, they proved that (1.5) has a radial positive normalized solution.

Of course, there are also numerous results for a single equation with Sobolev critical term. Li and Zou [12] studied the existence and nonexistence of normalized ground states in different cases for the Sobolev critical Schrödinger equation involving a Hardy term.

Based on the above works, we will consider equations with both linear couplings and Sobolev critical nonlinearities. The main features of this paper are the existence of Sobolev critical nonlinearities and the fractional Laplacian, which make the compactness conditions troublesome. The main methods we used are minimax theory, scaling transformation and concentration-compactness principle.

It is well known that to find the solutions of (1.1) means to look for the critical points of the energy functional

$$\begin{aligned} E(u,v) := {} & \frac{1}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2) - \frac{1}{2_s^*}\int_{\mathbb{R}^N}(\mu_1|u|^{2_s^*} + \mu_2|v|^{2_s^*}) \\ & - \eta\int_{\mathbb{R}^N}|u|^{\alpha}|v|^{\beta} - \int_{\mathbb{R}^N}\kappa(x)uv \end{aligned}$$

constrained on the L^2-spheres in $H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N)$. We define

$$S(a,b) := \left\{(u,v)\in H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N}u^2 = a^2,\ \int_{\mathbb{R}^N}v^2 = b^2\right\}.$$

We need to use the Palais's symmetric criticality principle [16] and compact embedding on the whole space to solve the problem (1.1)–(1.2). To achieve this purpose, we introduce such a symmetric radial subspace,

$$H_r^s := H_{\mathrm{rad}}^s(\mathbb{R}^N) = \{u\in H^s(\mathbb{R}^N) : u(x) = u(|x|)\}.$$

We know that the critical points of $E|_{S(a,b)}$ on $H_r^s\times H_r^s$ are the critical points of $E|_{S(a,b)}$ on $H^s(\mathbb{R}^N)\times H^s(\mathbb{R}^N)$ by Palais's symmetric criticality principle. So we just need to find the critical points of $E|_{S(a,b)}$ on $H_r^s\times H_r^s$.

Next, we talk about some notations. For the sake of simplicity, the norm of the space $L^p(\mathbb{R}^N)$ is represented by $|\cdot|_p$. Besides, we write

$$\begin{aligned} & \theta = \alpha+\beta,\ L^2\text{-critical exponent } \overline{p} = 2+\frac{4s}{N}, \\ & S(a) := \left\{u\in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N}u^2 = a^2\right\}, \\ & S_r(a,b) = S(a,b)\cap(H_r^s\times H_r^s). \end{aligned}$$

Now we state our main results.

Theorem 1.1. *Suppose that $N > 2s$, $\alpha + \beta \in (\overline{p}, 2_s^*)$ and $\eta, a, b > 0$, $\kappa(x) = \kappa(|x|) > 0$ and $\kappa(x) \in L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\frac{N}{2s} < p < \infty$. Moreover, if $\kappa(x)$ and $\nabla\kappa(x) \cdot x \geq 0$ are bounded and*

$$0 < |\kappa(x)|_\infty ab < \frac{\sigma K_2}{2}\left(1 - \frac{2}{\theta\gamma_\theta}\right),$$

where $\sigma \in (0,1)$ and K_2 is defined in Lemma 3.1, then there exists $\eta^* > 0$ such that (1.1)–(1.2) has a radial normalized solution $(\overline{u}, \overline{v})$ and $\overline{u} > 0$, $\overline{v} > 0$, $\lambda_1 < 0$, $\lambda_2 < 0$ for any $\eta \geq \eta^*$.

Remark 1.1. The difficulty of proving Theorem 1.1 lies in proving the Palais–Smale condition, because the existence of a Sobolev critical exponent makes compactness lack. Next, because the embeddings $H^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ and $H_r^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ are not compact, it is difficult to obtain strong convergence in $L^2(\mathbb{R}^N)$.

Theorem 1.2. *Suppose that $N > 2s$, $\alpha + \beta = 2_s^*$ and $\eta, a, b > 0$, $\kappa(x) = \kappa(|x|) < 0$ and $\kappa(x) \in L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\frac{N}{2s} < p < \infty$. Then, problem (1.1)–(1.2) has no positive normalized solution.*

This paper is organized as follows. In Section 2, we display some preliminary knowledge and known results. In Section 3, we show Theorem 1.1 by minimax theory, scaling transformation and concentration-compactness principle. Finally, Section 4 gives the proof of Theorem 1.2.

2. Preliminaries

In this section, we describe some fundamental properties of fractional Schrödinger systems. First, we review the following fractional Gagliardo–Nirenberg inequality (see [15, Lemma 1.1]) and the fractional Sobolev inequality (see [18]), which play a major role in later proofs.

Lemma 2.1. (i) *For $u \in H^s(\mathbb{R}^N)$, $p \in (2, 2_s^*)$, the following estimate holds:*

$$\int_{\mathbb{R}^N} |u|^p \leq C^p(N,p,s)\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\right)^{\frac{p\gamma_p}{2}}\left(\int_{\mathbb{R}^N} |u|^2\right)^{p(1-\gamma_p)}, \tag{2.1}$$

where $\gamma_p = \frac{N(p-2)}{2ps}$ and $C(N,p,s)$ is the best constant dependent on N, p, s.

(ii) *There exists a fractional optimal Sobolev critical constant $S > 0$ such that*

$$S\left(\int_{\mathbb{R}^N} |u|^{2_s^*}\right)^{\frac{2}{2_s^*}} \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2, \quad \text{for all } u \in S(a). \tag{2.2}$$

The proof of two commonly used inequalities in Lemma 2.1 has been studied by many scholars, and we will not repeat it, interested readers can refer to [15, 18] and the references therein.

We now focus on the following scalar equations, i.e., $\kappa(x) = 0$ in (1.1),

$$\begin{cases} (-\Delta)^s u - \lambda_1 u = \mu_1 |u|^{2_s^*-2} u + \eta\alpha |u|^{\alpha-2}|v|^\beta u & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v - \lambda_2 v = \mu_2 |v|^{2_s^*-2} v + \eta\beta |u|^{\alpha}|v|^{\beta-2} v & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} u^2 = a^2, \quad \int_{\mathbb{R}^N} v^2 = b^2. \end{cases} \tag{2.3}$$

Bartsch et al. have already studied this system when $s = 1$, $\mu_1 = \mu_2 = 1$ in [3]. We know that solutions of (2.3) are critical points of the C^1 functional

$$\begin{aligned} I_\eta(u, v) := &\frac{1}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2) \\ &- \frac{1}{2_s^*}\int_{\mathbb{R}^N} (\mu_1 |u|^{2_s^*} + \mu_2 |v|^{2_s^*}) - \eta \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \end{aligned}$$

constrained on $S(a, b)$, and the parameters λ_1, λ_2 appear as Lagrange multipliers.

The Pohozaev identity is a crucial tool in studying elliptic equations and systems, relating solutions to the geometry of the domain and nonlinear terms. For fractional systems like the one here, it helps constrain critical points, as seen by restricting them to the manifold $\mathcal{P}(a, b)$ where $P(u, v) = 0$, enabling analysis of solution existence and properties. Regarding regularity analysis, solutions to such fractional elliptic systems, especially with Sobolev critical nonlinearities, are typically shown to be in suitable fractional Sobolev spaces initially. Through techniques like bootstrap arguments and estimates on fractional Laplace operators, one can further derive higher regularity, ensuring solutions are sufficiently smooth for rigorous mathematical analysis of the system. Because of the Pohozaev identity (see [7, Proposition 4.1]), we know that the critical points of $I_\eta(u, v)$ are all in the set $\mathcal{P}(a, b)$. We introduce the Pohozaev manifold

$$\mathcal{P}(a, b) = \{(u, v) \in S(a, b) : P(u, v) = 0\},$$

where

$$\begin{aligned} P(u, v) := &\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2) \\ &- \int_{\mathbb{R}^N} (\mu_1 |u|^{2_s^*} + \mu_2 |v|^{2_s^*}) - \eta\theta\gamma_\theta \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta. \end{aligned} \tag{2.4}$$

Next, we will consider the monotonicity of the problem

$$m_\eta(a, b) := \inf_{(u,v)\in\mathcal{P}(a,b)} I_\eta(u, v).$$

To prove Lemma 2.2 and later other lemmas, we introduce a dilation transformation

$$(t \star u)(x) := e^{\frac{Nst}{2}} u(e^{st} x),$$

where $(u, v) \in S(a, b)$ and $t \in \mathbb{R}$. The scaling map was inspired by [13]. Through some simple calculations, we can get some properties of this scaling transformation as below:

$$|(-\Delta)^{\frac{s}{2}}(t \star u)|_2^2 = e^{2s^2 t}|(-\Delta)^{\frac{s}{2}} u|_2^2,$$

$$|t \star u|_{2_s^*}^{2_s^*} = e^{\frac{2_s^*-2}{2} Nst}|u|_{2_s^*}^{2_s^*},$$

$$\int_{\mathbb{R}^N} |t \star u|^\alpha |t \star v|^\beta = e^{\frac{\theta-2}{2} Nst}\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta,$$

besides, we also have

$$|t \star u|_2^2 = |u|_2^2, \quad 0 \star u = u \quad \text{and} \quad -t \star (t \star u) = u.$$

The monotonicity of the scalar equations is given below in order to prove Lemma 3.7. The proof of Lemma 2.2 can refer to [3, Lemma 2.7].

Lemma 2.2. *Assume that $N > 2s$, $a, b, \eta > 0$ and $\theta \in (\bar{p}, 2_s^*)$. Then, $m_\eta(a, b)$ is non-increasing for $\eta \in (0, +\infty)$.*

Lemma 2.3 ([10, Theorem 3.2]). *Let $\widetilde{E}(\widetilde{\chi}(\tau))$ be a C^1-functional on a complete connected C^1-Finsler manifold $S(a, b)$ (without boundary) and consider a homotopy-stable family $\widetilde{\Gamma}$ of compact subsets of $S(a, b)$ with a closed boundary $|\widetilde{\chi}_n(\tau)|$. Set $c_\eta(a, b) = \inf_{\widetilde{\chi}\in\widetilde{\Gamma}} \max_{\tau\in[0,1]} \widetilde{E}(\widetilde{\chi}(\tau))$ and suppose that $\sup \widetilde{E}(|\widetilde{\chi}_n(\tau)|)$. Then, for any sequence of sets $(\chi_n)_n$ in $\widetilde{\Gamma}$ such that $\lim_{n\to\infty} \sup_{\tau\in[0,1]} \widetilde{E}(\widetilde{\chi}_n(\tau)) = c_\eta(a, b)$, there exists a sequence $(x_n)_n$ in $S(a, b)$ such that*

$$\lim_{n\to\infty} \operatorname{dist}(x_n, \chi_n) = 0.$$

The following Brézis–Lieb lemma (see [8, Lemma 2.3]) will be used, the proof of Lemma 2.4 is classical, so we drop it here.

Lemma 2.4. *Let $N > 2s$, $\alpha, \beta > 1$, $\theta \in [2, 2_s^*]$ and $(u_n, v_n) \rightharpoonup (u, v)$ in $H_r^s \times H_r^s$. Up to a subsequence, it holds that*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta - |u_n - u|^\alpha |v_n - v|^\beta - |u|^\alpha |v|^\beta = 0.$$

Lemma 2.5. *Suppose $N > 2s$, $\kappa(x) \geq 0$, $a, b, \eta > 0$, $\alpha, \beta > 1$, $\theta \in (2, 2_s^*)$ and $(u, v) \in H_r^s \times H_r^s$ is a positive solution of (1.1)–(1.2). Then, $\lambda_1 < 0$, $\lambda_2 < 0$.*

This lemma for the exclusion of trivial solutions has been explained in many articles, and the proof process is basically the same. That is, using proof by contradiction first and using [19, Proposition 2.17], we omit the process here.

3. Proof of Theorem 1.1

In this section, we will prove the existence of normalized solutions of (1.1). Since $E(u,v)$ is unbounded from below on $S(a,b)$, we cannot get $\inf_{(u,v)\in S(a,b)} E(u,v)$. In order to obtain the critical points of $E|_{S(a,b)}$, we need to find the minimax value of $E|_{S(a,b)}$ by constructing a mountain pass structure on $S(a,b)$.

Lemma 3.1. *Assume $K_1 > 0$ small enough and let $K_2 = (\theta\gamma_\theta M)^{\frac{2}{2-\theta\gamma_\theta}}$, where $M = \frac{\mu}{2_s^* S^{\frac{2_s^*}{2}}} + \eta\mathcal{C}$ and $\mu = \max\{\mu_1, \mu_2\}$, $\mathcal{C}$ is a constant. In addition, if there exists $\sigma \in (0,1)$ such that*

$$|\kappa(x)|_\infty ab < \frac{\sigma K_2}{2}\left(1 - \frac{2}{\theta\gamma_\theta}\right),$$

then we have

$$\sup_{(u,v)\in A} E(u,v) < \inf_{(u,v)\in B} E(u,v) \quad \text{and} \quad \inf_{(u,v)\in B} E(u,v) > 0,$$

$$E(u,v) > -C(\kappa(x),a,b) \text{ on } A \text{ where } C(\kappa(x),a,b) \text{ is a positive constant,}$$

with

$$A := \left\{(u,v) \in S(a,b) : \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 \le K_1\right\},$$
$$B := \left\{(u,v) \in S(a,b) : \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 = K_2\right\}.$$

Proof. First, let $w = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2$. For any $(u,v) \in A$, we see that $0 < w \le K_1$. It follows from the Hölder inequality and Lemma 2.1 that

$$\int_{\mathbb{R}^N} (|u|^{2_s^*} + |v|^{2_s^*}) \le S^{-\frac{2_s^*}{2}}\left(\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\right)^{\frac{2_s^*}{2}} + \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2\right)^{\frac{2_s^*}{2}}\right),$$

$$\eta\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \le \eta |u|_\theta^\alpha |v|_\theta^\beta \le \eta\mathcal{C}\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2\right)^{\frac{\theta\gamma_\theta}{2}},$$

where $\mathcal{C}$ is a constant related to $N, \alpha, \beta, s, a, b$ and S is mentioned in (2.2). Thus, we get

$$\begin{aligned}
E(u,v) &:= \frac{1}{2}w - \frac{1}{2_s^*}\int_{\mathbb{R}^N} (\mu_1|u|^{2_s^*} + \mu_2|v|^{2_s^*}) - \eta\int_{\mathbb{R}^N} |u|^\alpha|v|^\beta - \int_{\mathbb{R}^N} \kappa(x)uv \\
&\ge \frac{1}{2}w - \frac{\mu}{2_s^* S^{\frac{2_s^*}{2}}} w^{\frac{2_s^*}{2}} - \eta|u|_\theta^\alpha|v|_\theta^\beta - |\kappa(x)|_\infty ab \\
&\ge \frac{1}{2}w - \frac{\mu}{2_s^* S^{\frac{2_s^*}{2}}} w^{\frac{2_s^*}{2}} - \eta\mathcal{C} w^{\frac{\theta\gamma_\theta}{2}} - |\kappa(x)|_\infty ab
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{8}w - |\kappa(x)|_\infty ab \\
&> -|\kappa(x)|_\infty ab, \qquad (3.1)
\end{aligned}
$$

where $\mu := \max\{\mu_1, \mu_2\}$. Since $K_1 > 0$ small enough and $2_s^*, \theta\gamma_\theta > 2$, we can take $C(\kappa(x), a, b) = |\kappa(x)|_\infty ab > 0$ to make $E(u, v) > -C(\kappa(x), a, b)$ on A hold.

Second, it is not difficult to verify that $K_2 > 0$ is the maximum point of the function

$$
f(x) = \frac{1}{2}x - \left(\frac{\mu}{2_s^* S^{\frac{2_s^*}{2}}} + \eta\mathcal{C}\right)x^{\frac{\theta\gamma_\theta}{2}},
$$

and

$$
\begin{aligned}
f(K_2) &= \frac{1}{2}K_2 - \left(\frac{\mu}{2_s^* S^{\frac{2_s^*}{2}}} + \eta\mathcal{C}\right)K_2^{\frac{\theta\gamma_\theta}{2}} \\
&= \frac{K_2}{2}\left(1 - \frac{2}{\theta\gamma_\theta}\right).
\end{aligned}
$$

For any $(u_1, v_1) \in A, (u_2, v_2) \in B$, set

$$
w_1 := \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_1|^2 + |(-\Delta)^{\frac{s}{2}}v_1|^2 \leq K_1
$$

and

$$
w_2 := \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_2|^2 + |(-\Delta)^{\frac{s}{2}}v_2|^2 = K_2.
$$

By $|\kappa(x)|_\infty ab < \frac{\sigma K_2}{2}(1 - \frac{2}{\theta\gamma_\theta})$ and (3.1), for K_1 small enough we have

$$
\begin{aligned}
E(u_2, v_2) - E(u_1, v_1) &\geq \frac{1}{2}(w_2 - w_1) - \frac{\mu}{2_s^* S^{\frac{2_s^*}{2}}}w_2^{\frac{2_s^*}{2}} - \eta\mathcal{C}w_2^{\frac{\theta\gamma_\theta}{2}} - |\kappa(x)|_\infty ab \\
&\geq \frac{1}{2}K_2 - \frac{1}{2}K_1 - \frac{\mu}{2_s^* S^{\frac{2_s^*}{2}}}K_2^{\frac{2_s^*}{2}} - \eta\mathcal{C}K_2^{\frac{\theta\gamma_\theta}{2}} - |\kappa(x)|_\infty ab \\
&\geq \frac{1}{2}K_2 - \left(\frac{\mu}{2_s^* S^{\frac{2_s^*}{2}}} + \eta\mathcal{C}\right)K_2^{\frac{\theta\gamma_\theta}{2}} - \frac{1}{2}K_1 - |\kappa(x)|_\infty ab \\
&\geq \frac{K_2}{2}\left(1 - \frac{2}{\theta\gamma_\theta}\right) - \frac{1}{2}K_1 - \frac{\sigma K_2}{2}\left(1 - \frac{2}{\theta\gamma_\theta}\right) \\
&> 0,
\end{aligned}
$$

the inequality holds because $s \in (0, 1)$ and $K_2 > 0$, $\frac{\theta\gamma_\theta}{2} > 1$.

Finally, the third formula can be obtained in a similar way. For any $(u_2, v_2) \in B$, it follows from (3.1) that

$$\begin{aligned}
E(u_2, v_2) &\geq \frac{1}{2} w_2 - \frac{\mu}{2_s^* S^{\frac{2_s^*}{2}}} w_2^{\frac{2_s^*}{2}} - \eta \mathcal{C} w_2^{\frac{\theta\gamma_\theta}{2}} - |\kappa(x)|_\infty ab \\
&\geq \frac{1}{2} K_2 - \frac{\mu}{2_s^* S^{\frac{2_s^*}{2}}} K_2^{\frac{2_s^*}{2}} - \eta \mathcal{C} K_2^{\frac{\theta\gamma_\theta}{2}} - |\kappa(x)|_\infty ab \\
&\geq \frac{1}{2} K_2 - \left(\frac{\mu}{2_s^* S^{\frac{2_s^*}{2}}} + \eta \mathcal{C} \right) K_2^{\frac{\theta\gamma_\theta}{2}} - |\kappa(x)|_\infty ab \\
&\geq \frac{\sigma K_2}{2} \left(1 - \frac{2}{\theta\gamma_\theta} \right) \\
&> 0.
\end{aligned}$$

We complete the proof. ■

To find a functional that satisfies the mountain pass geometry on $S(a, b)$, we will use the scaling map mentioned earlier. It follows from [11] that $\widetilde{E}(t, u, v) := E(t \star u, t \star v)$ has the same mountain pass structure and mountain pass level as the functional $E(u, v)$.

Lemma 3.2. *For any $(u, v) \in S(a, b)$, the following results hold:*

(i) $|(-\Delta)^{\frac{s}{2}}(t \star u)|_2^2 + |(-\Delta)^{\frac{s}{2}}(t \star v)|_2^2 \to 0$ *as* $t \to -\infty$;

(ii) $|(-\Delta)^{\frac{s}{2}}(t \star u)|_2^2 + |(-\Delta)^{\frac{s}{2}}(t \star v)|_2^2 \to +\infty$ *and* $\widetilde{E}(t, u, v) \to -\infty$ *as* $t \to +\infty$.

Proof. Recalling the properties of the scaling mapping, we know that as $t \to -\infty$,

$$|(-\Delta)^{\frac{s}{2}}(t \star u)|_2^2 + |(-\Delta)^{\frac{s}{2}}(t \star v)|_2^2 = e^{2s^2 t}(|(-\Delta)^{\frac{s}{2}} u|_2^2 + |(-\Delta)^{\frac{s}{2}} v|_2^2) \to 0,$$

and as $t \to +\infty$,

$$|(-\Delta)^{\frac{s}{2}}(t \star u)|_2^2 + |(-\Delta)^{\frac{s}{2}}(t \star v)|_2^2 \to +\infty.$$

By simple calculation, we have

$$\begin{aligned}
\widetilde{E}(t, u, v) := &\frac{1}{2} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}(t \star u)|^2 + |(-\Delta)^{\frac{s}{2}}(t \star v)|^2) \\
&- \frac{1}{2_s^*} \int_{\mathbb{R}^N} (\mu_1 |t \star u|^{2_s^*} + \mu_2 |t \star v|^{2_s^*}) \\
&- \eta \int_{\mathbb{R}^N} |t \star u|^\alpha |t \star v|^\beta - \int_{\mathbb{R}^N} \kappa(x)(t \star u)(t \star v)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}(e^{2s^2t}|(-\Delta)^{\frac{s}{2}}u|_2^2 + e^{2s^2t}|(-\Delta)^{\frac{s}{2}}v|_2^2) \\
&\quad - \frac{1}{2_s^*}e^{\frac{2_s^*-2}{2}Nst}(\mu_1|u|_{2_s^*}^{2_s^*} + \mu_2|v|_{2_s^*}^{2_s^*}) \\
&\quad - \eta e^{\frac{\theta-2}{2}Nst}\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta - e^{Nst}\int_{\mathbb{R}^N}\kappa(x)u(e^{st}x)v(e^{st}x).
\end{aligned}$$

In light of $2_s^*, \theta > 2$, we obtain that

$$\frac{2_s^*-2}{2}Ns > \frac{\theta-2}{2}Ns > Ns \quad \text{and} \quad \frac{2_s^*-2}{2}Ns = \frac{N}{N-2s}2s^2 > 2s^2.$$

Then we can get $\widetilde{E}(t,u,v) \to -\infty$ as $t \to +\infty$. ■

Let us give a nonnegative fixed point (u_1, v_1) on A, then from Lemma 3.2, when $t \to +\infty$, we can get a point (u_0, v_0) such that $E(u_0, v_0)$ is negative enough and $\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}w_1|^2 + |(-\Delta)^{\frac{s}{2}}w_2|^2)$ is big enough, i.e.,

$$\begin{aligned}
|(-\Delta)^{\frac{s}{2}}u_1|_2^2 + |(-\Delta)^{\frac{s}{2}}v_1|_2^2 &\le K_1, & E(u_1, v_1) &> -C(\kappa(x), a, b), \\
e^{2s^2t}(|(-\Delta)^{\frac{s}{2}}u_1|_2^2 + |(-\Delta)^{\frac{s}{2}}v_1|_2^2) &> K_2, & E(t\star u_1, t\star v_1) &\le -C(\kappa(x), a, b),
\end{aligned}$$

where K_1, K_2 are defined in Lemma 3.1.

We take $(u_0, v_0) := (t\star u_1, t\star v_1)$, then any path from point (u_1, v_1) to (u_0, v_0) has to go through B. We have a mountain pass structure on the manifold $S_r(a,b)$.

Now, we have the structure of mountain pass, then we define the minimax class

$$\Gamma := \{\chi(\tau) \in C([0,1],\ S_r(a,b)) : \chi(0) = (u_1, v_1),\ \chi(1) = (u_0, v_0)\},$$

where $\chi(\tau) = (\chi_1(\tau), \chi_2(\tau))$ and the mountain pass value is

$$c_\eta(a,b) := \inf_{\chi\in\Gamma}\max_{\tau\in[0,1]} E(\chi(\tau)).$$

In fact, we set $\chi_0(\tau) := ((\tau t)\star u_1, (\tau t)\star v_1)$. Clearly, $\chi_0(\tau) \in S_r(a,b)$ and $\chi_0(0) = (u_1, v_1)$, $\chi_0(1) = (t\star u_1, t\star v_1)$, that is, Γ is not empty. Moreover, it is obvious that

$$c_\eta(a,b) \ge \inf_B E(u_2, v_2) > 0.$$

The next lemma gives the asymptotic behavior for the critical value $c_\eta(a,b)$ and uses the method in [23].

Lemma 3.3. *Assume that $c_\eta(a,b)$ is the critical value of $E(u,v)$ on $S_r(a,b)$. Then we get* $\lim_{\eta\to+\infty} c_\eta(a,b) = 0$.

Proof. For fixed $(u,v) \in S_r(a,b)$, set $\chi_0(\tau) := ((\tau t) \star u, (\tau t) \star v)$. By the definition of $c_\eta(a,b)$, we obtain

$$\begin{aligned} c_\eta(a,b) &\le \max_{\tau \in [0,1]} E(\chi_0(\tau)) \\ &\le \max_{l>0} \Big\{ \frac{1}{2} e^{2s^2\tau t} \Big(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2) \Big) \\ &\quad - \eta e^{\frac{(\theta-2)Ns\tau t}{2}} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \Big\} \\ &= \max_{l>0} \Big\{ \frac{1}{2} w l^2 - \eta z l^{\frac{N(\theta-2)}{2s}} \Big\}, \end{aligned}$$

where $l := e^{s^2\tau t}$ and $w = \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2)$, $z = \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta$. It is calculable that the maximum point $l_{\max}$ of the function

$$g(l) = \frac{1}{2} w l^2 - \eta z l^{\frac{N(\theta-2)}{2s}}, \qquad \text{for any } l \ge 0,$$

is

$$l_{\max} = \Big(\frac{2sw}{\eta z N(\theta-2)} \Big)^{\frac{2s}{N\theta - 2N - 4s}}.$$

Bring the maximum point $l_{\max}$ back to the function $g(l)$ yields,

$$\begin{aligned} g_{\max}(l) &= \max_{l>0} \Big\{ \frac{1}{2} l^2 \Big(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2) \Big) - \eta l^{\frac{N(\theta-2)}{2s}} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta \Big\} \\ &= \frac{1}{2} w \Big(\frac{2sw}{\eta z N(\theta-2)} \Big)^{\frac{4s}{N\theta-2N-4s}} - \eta z \Big(\frac{2sw}{\eta z N(\theta-2)} \Big)^{\frac{N\theta-2N}{N\theta-2N-4s}} \\ &\le \frac{1}{2} w \Big(\frac{2sw}{\eta z N(\theta-2)} \Big)^{\frac{4s}{N\theta-2N-4s}}. \end{aligned}$$

Again by $\theta > \overline{p}$, we obtain

$$c_\eta(a,b) \le \frac{1}{2} w \Big(\frac{2sw}{\eta z N(\theta-2)} \Big)^{\frac{4s}{N\theta-2N-4s}} \le C \Big(\frac{1}{\eta} \Big)^{\frac{4s}{N\theta-2N-4s}} \to 0 \quad \text{as } \eta \to +\infty,$$

where $C > 0$ depends on w, z and $N(\theta-2) > 4s$. This finishes the proof. ■

To prove the boundedness of the PS sequence of $E(u,v)$ at $c_\eta(a,b)$, we define the Pohozaev manifold $\widetilde{\mathcal{P}}$ of (1.1),

$$\widetilde{\mathcal{P}}(a,b) = \big\{ (u,v) \in S(a,b) : \widetilde{P}(u,v) = 0 \big\}.$$

In a similar way, to find the Pohozaev manifold of (2.3), we calculate that $\widetilde{P}(u,v) = P(u,v)$.

Through direct computation, we know that $\widetilde{E}(t,u,v) = \widetilde{E}(0, t \star u, t \star v)$, then the mountain pass structure of $\widetilde{E}$ is

$$\widetilde{\Gamma} := \left\{\widetilde{\chi} \in C([0,1], \mathbb{R} \times S_r(a,b)) : \widetilde{\chi}(0) = (0, u_1, v_1),\ \widetilde{\chi}(1) = (0, u_0, v_0)\right\},$$

where $\widetilde{\chi}(\tau) = (t(\tau), \chi_1(\tau), \chi_2(\tau))$ and the minimax value is

$$\widetilde{c}(a,b) := \inf_{\widetilde{\chi} \in \widetilde{\Gamma}} \max_{\tau \in [0,1]} \widetilde{E}(\widetilde{\chi}(\tau)).$$

We claim that $c_\eta(a,b) = \widetilde{c}(a,b)$, and the minimax principle can be applied. The similar proof of this conclusion can be found in [2], and we will sketch it here for the convenience of reading. On one hand, it is easy to know $\widetilde{c}(a,b) \le c_\eta(a,b)$ from $\Gamma \subset \widetilde{\Gamma}$. On the other hand, for any $\widetilde{\chi}(\tau) = (t(\tau), \chi_1(\tau), \chi_2(\tau))$, we infer from the definition of $\widetilde{E}$ that $\widetilde{E}(\widetilde{\chi}(\tau)) = E(t(\tau) \star \chi(\tau))$. Since $t(\tau) \star \chi(\tau) \in \Gamma$, we have

$$\max_{\tau \in [0,1]} \widetilde{E}(\widetilde{\chi}(\tau)) = \max_{\tau \in [0,1]} E(t(\tau) \star \chi(\tau)) \ge \inf_{\chi \in \Gamma} \max_{\tau \in [0,1]} E(\chi(\tau)),$$

i.e., $\widetilde{c}(a,b) \ge c_\eta(a,b)$. Hence, it holds that $\widetilde{c}(a,b) = c_\eta(a,b)$.

Next, we take a sequence $\widetilde{\chi}_n = (0, \chi_{1,n}, \chi_{2,n}) \in \widetilde{\Gamma}$ such that

$$c_\eta(a,b) = \lim_{n \to \infty} \sup_{\tau \in [0,1]} \widetilde{E}(\widetilde{\chi}_n(\tau)).$$

Moreover, thanks to $\kappa(x) > 0$ and the following inequations:

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} |u||^2 \le \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2, \quad \int_{\mathbb{R}^N} uv \le \int_{\mathbb{R}^N} |u||v|,$$

we get that $\widetilde{E}(t, |u|, |v|) \le \widetilde{E}(t,u,v)$. From this we can infer that

$$\lim_{n \to \infty} \sup_{\tau \in [0,1]} \widetilde{E}(0, |\chi_{1,n}(\tau)|, |\chi_{2,n}(\tau)|) \le \lim_{n \to \infty} \sup_{\tau \in [0,1]} \widetilde{E}(0, \chi_{1,n}(\tau), \chi_{2,n}(\tau)) = \widetilde{c}(a,b).$$

From (u_1, v_1) and (u_0, v_0) are both nonnegative points and $(0, \chi_{1,n}, \chi_{2,n}) \in \widetilde{\Gamma}$, we can infer that $(0, |\chi_{1,n}|, |\chi_{2,n}|) \in \widetilde{\Gamma}$. Again by

$$\sup_{\tau \in [0,1]} \widetilde{E}(0, |\chi_{1,n}(\tau)|, |\chi_{2,n}(\tau)|) \ge \widetilde{c}(a,b),$$

we can obtain that

$$\lim_{n \to \infty} \sup_{\tau \in [0,1]} \widetilde{E}(0, |\chi_{1,n}(\tau)|, |\chi_{2,n}(\tau)|) = \widetilde{c}(a,b) = c_\eta(a,b),$$

so we can suppose $\chi_{1,n}, \chi_{2,n} \ge 0$. By Lemma 2.3, we can choose a PS sequence $x_n := (t_n, u_n, v_n)$ of $\widetilde{E}$ at level $c_\eta(a,b)$ such that

$$\lim_{n \to \infty} |t_n| + \mathrm{dist}_{H_r^s}((u_n, v_n), (\chi_{1,n}, \chi_{2,n})) = 0,$$

so we have $t_n \to 0$ and $u_n^-, v_n^- \to 0$ in H_r^s. We set $(\widetilde{u}_n, \widetilde{v}_n) := (t_n \star u_n, t_n \star v_n)$, obviously, $(\widetilde{u}_n, \widetilde{v}_n) \in S(a,b)$. Then we have the lemma below.

Lemma 3.4. *$\{(\widetilde{u}_n, \widetilde{v}_n)\}$ is a PS sequence of $E(u,v)$ at level $c_\eta(a,b)$ on $S(a,b)$.*

Proof. Recall that a PS sequence for a functional satisfies key compactness like properties, crucial for proving existence of critical points. Here, we aim to show $(\widetilde{u}_n, \widetilde{v}_n)$ is a PS sequence for $E(u,v)$ on $S(a,b)$. The proof relies on analyzing the functional $\widetilde{E}$ and leveraging transformations and properties of fractional Sobolev spaces, along with symmetry and convergence arguments related to the variables t_n, u_n and v_n. From previous discussion, we know that $\{(t_n, u_n, v_n)\}$ is a PS sequence of $\widetilde{E}$. Taking any $(\varphi, \psi) \in H_r^s \times H_r^s$, then

$$
\begin{aligned}
&\widetilde{E}'_{\mathbf{w}}(t_n, u_n, v_n)[(\varphi, \psi)] \\
&\quad = E'(t_n \star u_n, t_n \star v_n)[(\varphi, \psi)] \\
&\quad = e^{2s^2 t_n} \iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \\
&\qquad + \frac{(v_n(x) - v_n(y))(\psi(x) - \psi(y))}{|x-y|^{N+2s}} \\
&\qquad - e^{\frac{2_s^*-2}{2} N s t_n} \int_{\mathbb{R}^N} \mu_1 |u_n(x)|^{2_s^*-1} \varphi + \mu_2 |v_n(x)|^{2_s^*-1} \psi \\
&\qquad - \int_{\mathbb{R}^N} \kappa(e^{-st_n} x)(u_n(x)\psi + v_n(x)\varphi) \\
&\qquad - \eta e^{\frac{\theta-2}{2} N s t_n} \int_{\mathbb{R}^N} \big(\alpha |u_n(x)|^{\alpha-1} \varphi |v_n(x)|^{\beta} + \beta |u_n(x)|^{\alpha} |v_n(x)|^{\beta-1} \psi\big) \\
&\quad = \iint_{\mathbb{R}^{2N}} \frac{(\widetilde{u}_n(x) - \widetilde{u}_n(y))((t_n \star \varphi)(x) - (t_n \star \varphi)(y))}{|x-y|^{N+2s}} \\
&\qquad - \int_{\mathbb{R}^N} \mu_1 |\widetilde{u}_n|^{2_s^*-1} (t_n \star \varphi) \\
&\qquad + \iint_{\mathbb{R}^{2N}} \frac{(\widetilde{v}_n(x) - \widetilde{v}_n(y))((t_n \star \psi)(x) - (t_n \star \psi)(y))}{|x-y|^{N+2s}} \\
&\qquad - \int_{\mathbb{R}^N} \mu_2 |\widetilde{v}_n|^{2_s^*-1} (t_n \star \psi) \\
&\qquad - \eta \int_{\mathbb{R}^N} \big(\alpha |\widetilde{u}_n|^{\alpha-1} (t_n \star \varphi) |\widetilde{v}_n|^{\beta} + \beta |\widetilde{u}_n|^{\alpha} |\widetilde{v}_n|^{\beta-1} (t_n \star \psi)\big) \\
&\qquad - \int_{\mathbb{R}^N} \kappa(x)(\widetilde{u}_n(t_n \star \psi) + \widetilde{v}_n(t_n \star \varphi)) \\
&\quad = E'(\widetilde{u}_n, \widetilde{v}_n)[(t_n \star \varphi, t_n \star \psi)],
\end{aligned}
$$

where $\mathbf{w} = (u_n, v_n)$. According to $\forall t \in \mathbb{R}$, $-t \star (t \star u) = u$, then from the above conclusion we have

$$
\widetilde{E}'_{\mathbf{w}}(t_n, u_n, v_n)[(-t_n \star \varphi, -t_n \star \psi)] = E'(\widetilde{u}_n, \widetilde{v}_n)[(\varphi, \psi)].
$$

By [4, Lemma 3.6], for $(\varphi, \psi) \in T_{(\widetilde{u}_n,\widetilde{v}_n)}S(a,b)$, we get that

$$\begin{aligned}\int_{\mathbb{R}^N} (-t_n \star \widetilde{u}_n)(-t_n \star \varphi)\mathrm{d}x &= \int_{\mathbb{R}^N} e^{-Nst_n}\widetilde{u}_n(e^{st_n}x)\varphi(e^{st_n}x)\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \widetilde{u}_n(x)\varphi(x)\mathrm{d}x = 0.\end{aligned}$$

It can be seen that

$$(\varphi, \psi) \in T_{(\widetilde{u}_n,\widetilde{v}_n)}S(a,b) \quad \text{if and only if} \quad (-t_n \star \varphi, -t_n \star \psi) \in T_{(u_n,v_n)}S(a,b).$$

Since $t_n \to 0$ and $0 \star u = u$, we have $(-t_n \star \varphi, -t_n \star \psi) \to (\varphi, \psi)$ as $n \to \infty$ in $H_r^s \times H_r^s$. Hence, for n large enough and $(\varphi, \psi) \neq (0, 0)$, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 < \frac{\|(\varphi, \psi)\|_{H_r^s}}{\|(-t_n \star \varphi, -t_n \star \psi)\|_{H_r^s}} < C_2. \tag{3.2}$$

The norm of the cotangent space $T_{(u,v)}S(a,b)'$ is represented by $\|\cdot\|_*$, the norm of $T_{(u,v)}S(a,b)'$ is defined as

$$\|E'|_{S(a,b)}(u,v)\|_* := \sup_{\substack{(\varphi,\psi)\in T_{(u,v)}S(a,b),\\ \|(\varphi,\psi)\|_{H_r^s}=1}} \big|E'(u,v)[(\varphi,\psi)]\big|.$$

Then by

$$\begin{aligned}\big|E'|_{S(a,b)}(\widetilde{u}_n,\widetilde{v}_n)[(\varphi,\psi)]\big| &= \big|\widetilde{E}'_{\mathrm{w}}|_{S(a,b)}(t_n,u_n,v_n)[(-t_n \star \varphi, -t_n \star \psi)]\big| \\ &\leq \big|\|\widetilde{E}'_{\mathrm{w}}|_{S(a,b)}(t_n,u_n,v_n)\| \cdot \|(-t_n \star \varphi, -t_n \star \psi)\|_{H_r^s}\big|,\end{aligned}$$

we have

$$\left|\frac{E'|_{S(a,b)}(\widetilde{u}_n,\widetilde{v}_n)[(\varphi,\psi)]}{\|(-t_n \star \varphi, -t_n \star \psi)\|_{H_r^s}}\right| \leq \|\widetilde{E}'_{\mathrm{w}}|_{S(a,b)}(t_n,u_n,v_n)\|_* \to 0,$$

as $n \to \infty$. Using (3.2) and taking the supremum on both sides of the above formula, we obtain

$$C_1\|E'|_{S(a,b)}(\widetilde{u}_n,\widetilde{v}_n)\|_* \leq \|\widetilde{E}'_{\mathrm{w}}|_{S(a,b)}(t_n,u_n,v_n)\|_* \to 0, \qquad \text{as } n \to \infty.$$

Finally, we get

$$\|E'|_{S(a,b)}(\widetilde{u}_n,\widetilde{v}_n)\|_* \to 0, \qquad \text{as } n \to \infty,$$

on the other hand, due to $\{(t_n, u_n, v_n)\}$ is a PS sequence of $\widetilde{E}$ at level $c_\eta(a,b)$, we also have

$$E(\widetilde{u}_n,\widetilde{v}_n) = \widetilde{E}(t_n,u_n,v_n) \to c_\eta(a,b), \qquad \text{as } n \to \infty.$$

According to the definition of Palais–Smale sequence, we finish the proof. ∎

Lemma 3.5. *If $\kappa(x)$ and $\nabla\kappa(x)\cdot x$ are bounded in $\mathbb{R}^N$, then the PS sequence $\{(\widetilde{u}_n, \widetilde{v}_n)\}$ in Lemma 3.4 is bounded in $H^s_r \times H^s_r$. Moreover, there is also*

$$\lim_{n\to\infty} P(\widetilde{u}_n, \widetilde{v}_n) = 0$$

when $\nabla\kappa(x)\cdot x \geq 0$.

Proof. To further elaborate on the boundedness and convergence properties, recall that in fractional Sobolev spaces, the embeddings and compactness results play a pivotal role. Now, consider the nonlinear terms in the functional $\widetilde{E}$. Due to the strong convergence in L^q spaces and the growth conditions of the nonlinearities, the nonlinear functionals associated with $\widetilde{E}$ will converge continuously. In virtue of the fact that $\{(t_n, u_n, v_n)\}$ is a PS sequence of $\widetilde{E}$, it holds that

$$\partial_{t_n}\widetilde{E}(t_n, u_n, v_n) \to 0,$$

that is,

$$\begin{aligned}
&\partial_{t_n}\widetilde{E}(t_n, u_n, v_n)\\
&= \partial_{t_n}\Bigg(\frac{e^{2s^2 t_n}}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u_n|^2 + |(-\Delta)^{\frac{s}{2}}v_n|^2)\\
&\quad - \frac{1}{2^*_s}e^{\frac{2^*_s-2}{2}Nst_n}\int_{\mathbb{R}^N}\mu_1|u_n|^{2^*_s} + \mu_2|v_n|^{2^*_s}\Bigg)\\
&\quad - \partial_{t_n}\left(\eta e^{\frac{\theta-2}{2}Nst_n}\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta - \int_{\mathbb{R}^N}\kappa(e^{-st_n}x)u_n v_n\right)\\
&= s^2 e^{2s^2 t_n}(|(-\Delta)^{\frac{s}{2}}u_n|_2^2 + |(-\Delta)^{\frac{s}{2}}v_n|_2^2)\\
&\quad - \left(\frac{1}{2} - \frac{1}{2^*_s}\right)Nse^{\frac{2^*_s-2}{2}Nst_n}(\mu_1|u_n|_{2^*_s}^{2^*_s} + \mu_2|v_n|_{2^*_s}^{2^*_s})\\
&\quad - \left(\frac{\theta}{2} - 1\right)\eta Nse^{\frac{\theta-2}{2}Nst_n}\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\\
&\quad + s\int_{\mathbb{R}^N}\nabla\kappa(e^{-st_n}x)\cdot(e^{-st_n}x)u_n v_n\\
&= s^2(|(-\Delta)^{\frac{s}{2}}\widetilde{u}_n|_2^2 + |(-\Delta)^{\frac{s}{2}}\widetilde{v}_n|_2^2) - s^2(\mu_1|\widetilde{u}_n|_{2^*_s}^{2^*_s} + \mu_2|\widetilde{v}_n|_{2^*_s}^{2^*_s})\\
&\quad - \eta\theta\gamma_\theta s^2\int_{\mathbb{R}^N}|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta\\
&\quad + s\int_{\mathbb{R}^N}\nabla\kappa(e^{-st_n}x)\cdot(e^{-st_n}x)u_n v_n \to 0. \qquad (3.3)
\end{aligned}$$

On the other hand, we also get

$$
\begin{aligned}
\widetilde{E}(t_n, u_n, v_n) &= E(\widetilde{u}_n, \widetilde{v}_n) \\
&= \frac{1}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}\widetilde{u}_n|^2 + |(-\Delta)^{\frac{s}{2}}\widetilde{v}_n|^2) \\
&\quad - \frac{1}{2_s^*}\int_{\mathbb{R}^N} (\mu_1|\widetilde{u}_n|^{2_s^*} + \mu_2|\widetilde{v}_n|^{2_s^*}) \\
&\quad - \eta\int_{\mathbb{R}^N} |\widetilde{u}_n|^{\alpha}|\widetilde{v}_n|^{\beta} - \int_{\mathbb{R}^N} \kappa(e^{-st_n}x)u_n v_n \\
&\to c_\eta(a, b).
\end{aligned} \tag{3.4}
$$

Thus, by (3.3)–(3.4), $(u_n, v_n) \in S(a, b)$ and the boundedness of $\kappa(x)$ and $\nabla\kappa(x)\cdot x$, we can infer that $|(-\Delta)^{\frac{s}{2}}\widetilde{u}_n|_2^2 + |(-\Delta)^{\frac{s}{2}}\widetilde{v}_n|_2^2$ is bounded, again by $(\widetilde{u}_n, \widetilde{v}_n) \in S(a, b)$, we obtain $(\widetilde{u}_n, \widetilde{v}_n)$ is bounded in $H_r^s \times H_r^s$.

Next, we want to show that $\lim_{n\to\infty} P(\widetilde{u}_n, \widetilde{v}_n) = 0$. According to (3.3) and $u_n^-, v_n^- \to 0$ in H_r^s, we can deduce that

$$
\begin{aligned}
&s^2 P(\widetilde{u}_n, \widetilde{v}_n) + s\int_{\mathbb{R}^N} \nabla\kappa(e^{-st_n}x)\cdot(e^{-st_n}x)u_n^+ v_n^+ \\
&= s^2\Big(|(-\Delta)^{\frac{s}{2}}\widetilde{u}_n|_2^2 + |(-\Delta)^{\frac{s}{2}}\widetilde{v}_n|_2^2 - (\mu_1|\widetilde{u}_n|_{2_s^*}^{2_s^*} + \mu_2|\widetilde{v}_n|_{2_s^*}^{2_s^*}) \\
&\quad - \eta\theta\gamma_\theta\int_{\mathbb{R}^N} |\widetilde{u}_n|^{\alpha}|\widetilde{v}_n|^{\beta}\Big) \\
&\quad + s\int_{\mathbb{R}^N} \nabla\kappa(e^{-st_n}x)\cdot(e^{-st_n}x)u_n^+ v_n^+ \\
&\to 0,
\end{aligned}
$$

namely,

$$
\lim_{n\to\infty}\left(s^2 P(\widetilde{u}_n, \widetilde{v}_n) + s\int_{\mathbb{R}^N} \nabla\kappa(e^{-st_n}x)\cdot(e^{-st_n}x)u_n^+ v_n^+\right) = 0.
$$

Let $\widetilde{w} = |(-\Delta)^{\frac{s}{2}}\widetilde{u}_n|_2^2 + |(-\Delta)^{\frac{s}{2}}\widetilde{v}_n|_2^2$ be small enough. Indeed, by Lemma 2.1 and the Hölder inequality it holds

$$
\begin{aligned}
P(\widetilde{u}_n, \widetilde{v}_n) &= |(-\Delta)^{\frac{s}{2}}\widetilde{u}_n|_2^2 + |(-\Delta)^{\frac{s}{2}}\widetilde{v}_n|_2^2 - (\mu_1|\widetilde{u}_n|_{2_s^*}^{2_s^*} + \mu_2|\widetilde{v}_n|_{2_s^*}^{2_s^*}) \\
&\quad - \eta\theta\gamma_\theta\int_{\mathbb{R}^N} |\widetilde{u}_n|^{\alpha}|\widetilde{v}_n|^{\beta} \\
&\geq \widetilde{w} - S^{-\frac{2_s^*}{2}}\mu_1(|(-\Delta)^{\frac{s}{2}}\widetilde{u}_n|_2^2)^{\frac{2_s^*}{2}} - S^{-\frac{2_s^*}{2}}\mu_2(|(-\Delta)^{\frac{s}{2}}\widetilde{v}_n|_2^2)^{\frac{2_s^*}{2}} \\
&\quad - \eta\theta\gamma_\theta|\widetilde{u}_n|_\theta^{\alpha}|\widetilde{v}_n|_\theta^{\beta}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \widetilde{w} - \mu S^{-\frac{2_s^*}{2}} \left(|(-\Delta)^{\frac{s}{2}} \widetilde{u}_n|_2^2 + |(-\Delta)^{\frac{s}{2}} \widetilde{v}_n|_2^2\right)^{\frac{2_s^*}{2}} \\
&\quad - \eta\theta\gamma_\theta C^\theta(N,\theta,s)\left(|(-\Delta)^{\frac{s}{2}} \widetilde{u}_n|_2^2\right)^{\frac{\alpha\gamma_\theta}{2}} |\widetilde{u}_n|_2^{\alpha(1-\gamma_\theta)} \\
&\quad \cdot \left(|(-\Delta)^{\frac{s}{2}} \widetilde{v}_n|_2^2\right)^{\frac{\beta\gamma_\theta}{2}} |\widetilde{v}_n|_2^{\beta(1-\gamma_\theta)} \\
&\geq \widetilde{w} - \mu S^{-\frac{2_s^*}{2}} \widetilde{w}^{\frac{2_s^*}{2}} - \eta C_3 \left(|(-\Delta)^{\frac{s}{2}} \widetilde{u}_n|_2^2 + |(-\Delta)^{\frac{s}{2}} \widetilde{v}_n|_2^2\right)^{\frac{\theta\gamma_\theta}{2}} \\
&= \widetilde{w} - \mu S^{-\frac{2_s^*}{2}} \widetilde{w}^{\frac{2_s^*}{2}} - \eta C_3 \widetilde{w}^{\frac{\theta\gamma_\theta}{2}} \\
&\geq 0,
\end{aligned}
$$

where C_3 is related to N, θ, s, a, b. In view of $\nabla\kappa(x) \cdot x \geq 0$, we can obtain that $\lim_{n\to\infty} P(\widetilde{u}_n, \widetilde{v}_n) = 0$. Besides, it also holds that

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla\kappa(e^{-st_n}x) \cdot (e^{-st_n}x) u_n^+ v_n^+ = 0.
$$

Hence, the proof is finished. ∎

Since $\{(\widetilde{u}_n, \widetilde{v}_n)\}$ is bounded in $H_r^s \times H_r^s$, then up to a subsequence, there exists $(\overline{u}, \overline{v}) \in H_r^s \times H_r^s$ such that

$$
\begin{aligned}
&(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (\overline{u}, \overline{v}) && \text{in } H_r^s(\mathbb{R}^N) \times H_r^s(\mathbb{R}^N); \\
&(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (\overline{u}, \overline{v}) && \text{in } L^{2_s^*}(\mathbb{R}^N) \times L^{2_s^*}(\mathbb{R}^N); \\
&(\widetilde{u}_n, \widetilde{v}_n) \to (\overline{u}, \overline{v}) && \text{in } L^\theta(\mathbb{R}^N) \times L^\theta(\mathbb{R}^N); \\
&(\widetilde{u}_n, \widetilde{v}_n) \to (\overline{u}, \overline{v}) && \text{a.e. in } \mathbb{R}^N
\end{aligned}
$$

as $n \to \infty$.

According to the fact that $\{(\widetilde{u}_n, \widetilde{v}_n)\}$ is a PS sequence of E, for any $(\varphi, \psi) \in H_r^s \times H_r^s$, there exists a sequence $\{(\lambda_{1,n}, \lambda_{2,n})\} \subset \mathbb{R} \times \mathbb{R}$ by the Lagrange multiplier method such that

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \left(\frac{(\widetilde{u}_n(x) - \widetilde{u}_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} + \frac{(\widetilde{v}_n(x) - \widetilde{v}_n(y))(\psi(x) - \psi(y))}{|x-y|^{N+2s}} \right) \\
&\qquad - \int_{\mathbb{R}^N} \mu_1 |\widetilde{u}_n|^{2_s^*-1}\varphi + \mu_2 |\widetilde{v}_n|^{2_s^*-1}\psi \\
&\qquad\qquad - \eta(\alpha|\widetilde{u}_n|^{\alpha-1}\varphi|\widetilde{v}_n|^\beta + \beta|\widetilde{u}_n|^\alpha|\widetilde{v}_n|^{\beta-1}\psi) \\
&\qquad - \int_{\mathbb{R}^N} \kappa(x)(\widetilde{u}_n\psi + \widetilde{v}_n\varphi) - \lambda_{1,n}\int_{\mathbb{R}^N} \widetilde{u}_n\varphi - \lambda_{2,n}\int_{\mathbb{R}^N} \widetilde{v}_n\psi \\
&\quad = o(1)\|(\varphi, \psi)\|_{H_s^r}
\end{aligned}
\tag{3.5}
$$

with $o(1) \to 0$ as $n \to \infty$. For more details we refer to [1, Lemma 3.2]. Taking $(\widetilde{u}_n, 0)$ and $(0, \widetilde{v}_n)$ as test functions in (3.5), we can obtain that

$$\begin{aligned}
\lambda_{1,n} a^2 &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \widetilde{u}_n|^2 - \int_{\mathbb{R}^N} \mu_1 |\widetilde{u}_n|^{2_s^*} \\
&\quad - \eta\alpha \int_{\mathbb{R}^N} |\widetilde{u}_n|^{\alpha} |\widetilde{v}_n|^{\beta} - \int_{\mathbb{R}^N} \kappa(x) \widetilde{u}_n \widetilde{v}_n, \\
\lambda_{2,n} b^2 &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \widetilde{v}_n|^2 - \int_{\mathbb{R}^N} \mu_2 |\widetilde{v}_n|^{2_s^*} \\
&\quad - \eta\beta \int_{\mathbb{R}^N} |\widetilde{u}_n|^{\alpha} |\widetilde{v}_n|^{\beta} - \int_{\mathbb{R}^N} \kappa(x) \widetilde{u}_n \widetilde{v}_n.
\end{aligned} \tag{3.6}$$

It is not difficult to deduce that $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ are bounded. Up to a subsequence, we may set $(\lambda_{1,n}, \lambda_{2,n}) \to (\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}$.

Lemma 3.6. *There exists $\eta^* > 0$ dependent on a, b large enough, such that for any $\eta \ge \eta^*$, $(\widetilde{u}_n, \widetilde{v}_n) \to (\overline{u}, \overline{v})$ in $L^{2_s^*}(\mathbb{R}^N) \times L^{2_s^*}(\mathbb{R}^N)$ and $(\overline{u}, \overline{v}) \neq (0, 0)$.*

Proof. First, we claim that $\widetilde{u}_n \to \overline{u}$ in $L^{2_s^*}(\mathbb{R}^N)$. By the concentration-compactness principle (see [17, Theorem 5] and [21, Lemma 3.3–3.4]), there exist two nonnegative measures ω, υ and an index set J such that $|(-\Delta)^{\frac{s}{2}} \widetilde{u}_n|^2 \rightharpoonup \omega$ in $\mathcal{M}(\mathbb{R}^N)$, $\mu_1 |\widetilde{u}_n|^{2_s^*} \rightharpoonup \upsilon$ in $\mathcal{M}(\mathbb{R}^N)$ as $n \to \infty$. In addition,

$$\begin{aligned}
&\omega \ge |(-\Delta)^{\frac{s}{2}} \overline{u}|^2 + \sum_{j \in J} \omega_j \delta_{x_j}, && \omega_j \ge 0, \\
&\upsilon = \mu_1 |\overline{u}|^{2_s^*} + \sum_{j \in J} \upsilon_j \delta_{x_j}, && \upsilon_j \ge 0, \\
&\upsilon_j \le S^{\frac{-2_s^*}{2}} \omega_j^{\frac{2_s^*}{2}}, && \forall j \in J,
\end{aligned} \tag{3.7}$$

where x_j is a point in $\mathbb{R}^N$, δ_{x_j} represents the Dirac measure at x_j and S is the Sobolev optimal constant in (2.2). Besides, set

$$\begin{aligned}
\omega_\infty &:= \lim_{r \to +\infty} \limsup_{n \to \infty} \int_{|x|>r} |(-\Delta)^{\frac{s}{2}} \widetilde{u}_n|^2 \mathrm{d}x, \\
\upsilon_\infty &:= \lim_{r \to +\infty} \limsup_{n \to \infty} \int_{|x|>r} \mu_1 |\widetilde{u}_n|^{2_s^*} \mathrm{d}x.
\end{aligned}$$

And by [21, Lemma 3.3], we know that $\omega_\infty, \upsilon_\infty$ satisfy

$$\begin{aligned}
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \widetilde{u}_n|^2 \mathrm{d}x &= \int_{\mathbb{R}^N} \mathrm{d}\omega + \omega_\infty, \\
\limsup_{n \to \infty} \int_{\mathbb{R}^N} \mu_1 |\widetilde{u}_n|^{2_s^*} \mathrm{d}x &= \int_{\mathbb{R}^N} \mathrm{d}\upsilon + \upsilon_\infty, \\
\upsilon_\infty &\le S^{\frac{-2_s^*}{2}} \omega_\infty^{\frac{2_s^*}{2}}.
\end{aligned} \tag{3.8}$$

We take the cut-off function $\Psi_\varepsilon(x) \in C_0^\infty(\mathbb{R}^N)$ as follows:

$$\begin{cases} \quad\Psi_\varepsilon(x) \equiv 1 & \text{in } B_\varepsilon(x_j), \\ \quad\Psi_\varepsilon(x) \equiv 0 & \text{in } B_{2\varepsilon}^c(x_j), \\ 0 \le \Psi_\varepsilon(x) \le 1, \end{cases}$$

where $B_\varepsilon(x_j)$ denotes a ball with center x_j and radius ε. It follows from Lemma 3.5 that $\{\Psi_\varepsilon(x)\widetilde{u}_n\}$ is bounded in H_r^s. Now we take $(\Psi_\varepsilon(x)\widetilde{u}_n, 0)$ as a test function in (3.5) and let $\varepsilon \to 0$, then from the absolute continuity of the Lebesgue integral, we have

$$\begin{aligned} \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} \lambda_{1,n}\widetilde{u}_n^2\Psi_\varepsilon \mathrm{d}x &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \lambda_1\overline{u}^2\Psi_\varepsilon \mathrm{d}x = 0, \\ \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} |\widetilde{u}_n|^\alpha|\widetilde{v}_n|^\beta\Psi_\varepsilon \mathrm{d}x &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} |\overline{u}|^\alpha|\overline{v}|^\beta\Psi_\varepsilon \mathrm{d}x = 0, \\ \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} \kappa(x)\widetilde{u}_n\widetilde{v}_n\Psi_\varepsilon \mathrm{d}x &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \kappa(x)\overline{u}\overline{v}\Psi_\varepsilon \mathrm{d}x = 0. \end{aligned} \tag{3.9}$$

We infer from (3.5) and (3.9) that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\widetilde{u}_n|^2\Psi_\varepsilon \mathrm{d}x = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N} \mu_1|\widetilde{u}_n|^{2_s^*}\Psi_\varepsilon \mathrm{d}x,$$

which implies that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \Psi_\varepsilon \mathrm{d}\omega = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} \Psi_\varepsilon \mathrm{d}\upsilon. \tag{3.10}$$

Combining with (3.7) and (3.10), we get $\upsilon_j \ge \omega_j$, thereby

$$\text{either} \quad \omega_j = 0 \quad \text{or} \quad \omega_j \ge S^{\frac{N}{2s}} \quad \text{for } j \in J. \tag{3.11}$$

For any $j \in J$, if $\omega_j = 0$, again by (3.7) we have $\upsilon_j = 0$, thus $\mu_1|\widetilde{u}_n|_{2_s^*}^{2_s^*} \to \mu_1|\overline{u}|_{2_s^*}^{2_s^*}$. We show that $\widetilde{u}_n \to \overline{u}$ in $L^{2_s^*}$ by the Brézis–Lieb lemma (see [5]).

If $\omega_j \ge S^{\frac{N}{2s}}$ for some $j \in J$, according to Lemma 3.3, there exists $\eta^* > 0$ large enough such that $c_\eta(a,b) < (\frac{1}{2} - \frac{1}{\theta\gamma_\theta})S^{\frac{N}{2s}}$ for any $\eta \ge \eta^*$. By (3.7), for a big positive constant C_4 we have

$$\begin{aligned} &\left(\frac{1}{2} - \frac{1}{\theta\gamma_\theta}\right)S^{\frac{N}{2s}} > c_\eta(a,b) \\ &\quad= \lim_{n\to\infty}\left(E(\widetilde{u}_n,\widetilde{v}_n) - \frac{1}{\theta\gamma_\theta}\widetilde{P}(\widetilde{u}_n,\widetilde{v}_n) + C_4\int_{\mathbb{R}^N} \nabla\kappa(e^{-st_n}x)\cdot(e^{-st_n}x)u_n^+v_n^+\right) \\ &\quad\ge \left(\frac{1}{2} - \frac{1}{\theta\gamma_\theta}\right)\limsup_{n\to\infty}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\widetilde{u}_n|^2\Psi_\varepsilon \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2} - \frac{1}{\theta\gamma_\theta}\right) \limsup_{n\to\infty} \int_{\mathbb{R}^N} \Psi_\varepsilon \mathrm{d}\omega \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta\gamma_\theta}\right) \omega_j \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta\gamma_\theta}\right) S^{\frac{N}{2s}},
\end{aligned}
$$

which is impossible. Therefore, the strong convergence of $\widetilde{u}_n$ in $L^{2_s^*}(\mathbb{R}^N)$ is obtained.

Similarly, we can also get $\widetilde{v}_n \to \overline{v}$ in $L^{2_s^*}(\mathbb{R}^N)$. Let $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$ be another cut-off function,

$$
\begin{cases}
\varphi(x) \equiv 1 & \text{in } \mathbb{R}^N \backslash B_r(0), \\
\varphi(x) \equiv 0 & \text{in } B_{\frac{r}{2}}(0), \\
0 \leq \varphi(x) \leq 1,
\end{cases}
$$

and denote $\varphi_r(x)$ by $\varphi(\frac{x}{r})$ for any r. We infer from Lemma 3.5 that $\{\varphi_r(x)\widetilde{u}_n\}$ is bounded in H_r^s. Now we take $(\varphi_r(x)\widetilde{u}_n, 0)$ as a test function in (3.5) and let $r \to +\infty$, then we get

$$
\begin{aligned}
\lim_{r\to+\infty}\lim_{n\to\infty} \int_{\mathbb{R}^N} \lambda_{1,n}\widetilde{u}_n^2\varphi_r \mathrm{d}x &= \lim_{r\to+\infty} \int_{\mathbb{R}^N} \lambda_1 \overline{u}^2 \varphi_r \mathrm{d}x = 0, \\
\lim_{r\to+\infty}\lim_{n\to\infty} \int_{\mathbb{R}^N} |\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta \varphi_r \mathrm{d}x &= \lim_{r\to+\infty} \int_{\mathbb{R}^N} |\overline{u}|^\alpha |\overline{v}|^\beta \varphi_r \mathrm{d}x = 0, \\
\lim_{r\to+\infty}\lim_{n\to\infty} \int_{\mathbb{R}^N} \kappa(x)\widetilde{u}_n\widetilde{v}_n\varphi_r \mathrm{d}x &= \lim_{r\to+\infty} \int_{\mathbb{R}^N} \kappa(x)\overline{u}\overline{v}\varphi_r \mathrm{d}x = 0.
\end{aligned}
\tag{3.12}
$$

It follows from (3.12) that

$$
\lim_{r\to+\infty}\lim_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\widetilde{u}_n|^2 \varphi_r \mathrm{d}x = \lim_{r\to+\infty}\lim_{n\to\infty} \int_{\mathbb{R}^N} \mu_1 |\widetilde{u}_n|^{2_s^*} \varphi_r \mathrm{d}x,
$$

that is, $\omega_\infty = \upsilon_\infty$. Again by (3.8), we obtain

$$
\text{either} \quad \omega_\infty = 0 \quad \text{or} \quad \omega_\infty \geq S^{\frac{N}{2s}}.
$$

The rest of the proof is the same as before, and we will not repeat it.

Finally, we show that $(\overline{u}, \overline{v}) \neq (0, 0)$. Suppose on the contrary that $(\overline{u}, \overline{v}) = (0, 0)$, by Lemma 3.5 and the above conclusions, we have

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}\widetilde{u}_n|^2 + |(-\Delta)^{\frac{s}{2}}\widetilde{v}_n|^2) = 0.
$$

It means that $c_\eta(a, b) = \lim_{n\to\infty} E(\widetilde{u}_n, \widetilde{v}_n) = 0$, which is a contradiction. ∎

Lemma 3.7. *Suppose that $c_\eta(a, b) < \min\{m_\eta(a, 0), m_\eta(0, b)\}$, then $(\overline{u}, \overline{v})$ is a positive normalized solution for problem (1.1)–(1.2) and $\lambda_1 < 0$, $\lambda_2 < 0$.*

Proof. We infer from Lemma 3.6 that $(\overline{u}, \overline{v}) \neq (0,0)$. Now we suppose by contradiction and divide into two cases.

Case 1: Set $\overline{u} > 0$, $\overline{v} = 0$. In view of Lemma 2.2 and [22, Theorem 1.3], we get that $\overline{u}$ is a positive radial symmetric solution of problem (2.3) with parameters a_1, b_1, where $a_1^2 = |\overline{u}|_2^2 \leq a^2$ and $b_1^2 = |\overline{v}|_2^2 \leq b^2$. Thus, there is $m_\eta(a,b) \leq m_\eta(a_1,b_1) \leq I_\eta(\overline{u},\overline{v})$. Combining with Lemma 2.4, Lemmas 3.5–3.6 and Brézis–Lieb lemmas, we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} P(\widetilde{u}_n, \widetilde{v}_n) \\
&= \lim_{n\to\infty} P(\widetilde{u}_n - \overline{u}, \widetilde{v}_n) + P(\overline{u}, 0) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\widetilde{u}_n - \overline{u})|^2 + |(-\Delta)^{\frac{s}{2}}\widetilde{v}_n|^2,
\end{aligned}$$

and

$$\begin{aligned}
c_\eta(a,b) &= \lim_{n\to\infty} E(\widetilde{u}_n, \widetilde{v}_n) \\
&= \lim_{n\to\infty} E(\widetilde{u}_n - \overline{u}, \widetilde{v}_n) + E(\overline{u}, 0) \\
&\geq \frac{1}{2}\lim_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\widetilde{u}_n - \overline{u})|^2 + |(-\Delta)^{\frac{s}{2}}\widetilde{v}_n|^2 + m_\eta(a,0) \\
&= m_\eta(a,0).
\end{aligned}$$

This contradicts that $c_\eta(a,b) < m_\eta(a,0)$.

Case 2: Set $\overline{u} = 0$, $\overline{v} > 0$. The same proof gives us $c_\eta(a,b) \geq m_\eta(0,b)$, which is still a contradiction. Hence, we obtain that $\overline{u} > 0$, $\overline{v} > 0$ and $(\overline{u}, \overline{v})$ is a positive normalized solution for problem (1.1)–(1.2). It follows from Lemma 2.5 that $\lambda_1 < 0$, $\lambda_2 < 0$. ∎

Proof of Theorem 1.1. Due to Lemma 3.3, for any $a, b > 0$, we can choose an η^* big enough such that $c_\eta(a,b) < \min\{m_\eta(a,0), m_\eta(0,b)\}$ for any $\eta \geq \eta^*$. By [3, Lemma 2.7], we know that $m_\eta(a,b) > 0$. Again by Lemma 3.7, we can deduce that $(\lambda_1, \lambda_2, \overline{u}, \overline{v})$ is a solution of problem (1.1)–(1.2), where $\overline{u}, \overline{v}$ are nonnegative thanks to $u_n^-, v_n^- \to 0$ in H_r^s. Next, we just need to verify $(\widetilde{u}_n, \widetilde{v}_n) \to (\overline{u}, \overline{v})$ in $H_r^s \times H_r^s$. We have the following four cases.

(i) If $(\overline{u}, \overline{v}) = (0,0)$, then by Lemma 3.6 we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}}\widetilde{u}_n|^2 + |(-\Delta)^{\frac{s}{2}}\widetilde{v}_n|^2) = 0,$$

this contradicts that $c_\eta(a,b) > 0$;

(ii) if $\overline{u} \neq 0$, $\overline{v} = 0$, because $(\lambda_1, \lambda_2, \overline{u}, \overline{v})$ is a solution of problem (1.1)–(1.2) and $\kappa(x) > 0$, we must have $\overline{u} = 0$, which is impossible;

(iii) if $\overline{u} = 0$, $\overline{v} \neq 0$, it is the same as (ii), which is impossible;

(iv) if $\overline{u} > 0$ and $\overline{v} > 0$, by Lemma 3.5 and (3.5)–(3.6), we obtain

$$\begin{aligned}
&\lambda_1 a^2 + \lambda_2 b^2 \\
&\quad = \lim_{n\to\infty} (\lambda_{1,n} a^2 + \lambda_{2,n} b^2) \\
&\quad = \lim_{n\to\infty} \bigg(\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} \widetilde{u}_n|^2 + |(-\Delta)^{\frac{s}{2}} \widetilde{v}_n|^2) \\
&\qquad - \int_{\mathbb{R}^N} (\mu_1 |\widetilde{u}_n|^{2_s^*} + \mu_2 |\widetilde{v}_n|^{2_s^*}) \\
&\qquad - \eta\theta \int_{\mathbb{R}^N} |\widetilde{u}_n|^{\alpha} |\widetilde{v}_n|^{\beta} - 2\int_{\mathbb{R}^N} \kappa(x) \widetilde{u}_n \widetilde{v}_n \bigg) \\
&\quad = \lim_{n\to\infty} \bigg((\gamma_\theta - 1)\eta\theta \int_{\mathbb{R}^N} |\widetilde{u}_n|^{\alpha} |\widetilde{v}_n|^{\beta} - \int_{\mathbb{R}^N} 2\kappa(x) \widetilde{u}_n \widetilde{v}_n \bigg) \\
&\quad = \lambda_1 |\overline{u}|_2^2 + \lambda_2 |\overline{v}|_2^2,
\end{aligned}$$

which implies that

$$\lambda_1(|\overline{u}|_2^2 - a^2) + \lambda_2(|\overline{v}|_2^2 - b^2) = 0.$$

Thus, we get $|\overline{u}|_2^2 = a^2$, $|\overline{v}|_2^2 = b^2$, i.e., $(\widetilde{u}_n, \widetilde{v}_n) \to (\overline{u}, \overline{v})$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Combining with (3.5) and compact embedding, we also have

$$\begin{aligned}
o(1) &= (E'(\widetilde{u}_n, \widetilde{v}_n) - E'(\overline{u}_n, \overline{v}_n))[(\widetilde{u}_n - \overline{u}, 0)] - \lambda_1 \int_{\mathbb{R}^N} (\widetilde{u}_n - \overline{u})^2 \\
&= \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} (\widetilde{u}_n - \overline{u})|^2 - \lambda_1 |\widetilde{u}_n - \overline{u}|^2) + o(1).
\end{aligned}$$

Since $\lambda_1 < 0$, we can get $\widetilde{u}_n \to \overline{u}$ in H_r^s. Similarly, it holds that $\widetilde{v}_n \to \overline{v}$ in H_r^s. From maximum principle, we have $\overline{u}, \overline{v} > 0$.

We complete the proof of Theorem 1.1. ■

4. Proof of Theorem 1.2

In this part, we want to prove that problem (1.1)–(1.2) has no positive normalized solution when $\theta = 2_s^*$ and $\kappa(x) < 0$.

Proof of Theorem 1.2. Now we are going to prove this result by contradiction. Assume that (u, v) is a positive normalized solution for equations (1.1)–(1.2) when $\theta = 2_s^*$. It follows from Lemma 2.5 that $\lambda_1 < 0$, $\lambda_2 < 0$. Recalling (2.4), we can infer the Pohozaev identity of the new system

$$\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2) - \int_{\mathbb{R}^N} (\mu_1 |u|^{2_s^*} + \mu_2 |v|^{2_s^*}) - 2_s^* \eta \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta} = 0.$$

Since (u, v) is a positive normalized solution of (1.1)–(1.2), we deduce from the definition of weak solutions that

$$\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2) - \int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2)$$
$$= \int_{\mathbb{R}^N} (\mu_1 |u|^{2_s^*} + \mu_2 |v|^{2_s^*}) + 2_s^* \eta \int_{\mathbb{R}^N} |u|^{\alpha} |v|^{\beta} + \int_{\mathbb{R}^N} \kappa(x) uv.$$

Combining with the above two formulas, we have

$$\int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2) + \int_{\mathbb{R}^N} \kappa(x) uv = 0,$$

this is a contradiction. Hence, there is no positive normalized solution to the equations under these conditions. We finish the proof of Theorem 1.2. ∎

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