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Sigma-convergence for thin heterogeneous domains and application to the upscaling of Darcy–Lapwood–Brinkmann flow

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Abstract. The sigma-convergence concept has been up to now used to derive macroscopic models in full space dimensions. In this work, we generalize it to thin heterogeneous domains given rise to phenomena in lower space dimensions. More precisely, we provide a new approach of the sigma-convergence method that is suitable for the study of phenomena occurring in thin heterogeneous media. This is made through a systematic study of the sigma-convergence method for thin heterogeneous domains. Assuming that the thin heterogeneous layer is made of microstructures that are distributed inside in a deterministic way including as special cases the periodic and the almost periodic distributions, we make use of the concept of algebras with mean value to state and prove the main compactness results. As an illustration, we upscale a Darcy–Lapwood–Brinkmann micro-model for thin flow. We prove that, according to the magnitude of the permeability of the porous domain, we obtain as effective models, the Darcy law in lower dimensions. The effective models are derived through the solvability of either the local Stokes–Brinkmann problems or the local Hele-Shaw problems.

1. Introduction and the main results

This work is concerned with two main challenges: 1) build a framework enabling the study of physical/natural phenomena occurring in thin heterogeneous media by taking into account the distribution of heterogeneities inside the thin layer; 2) apply the results obtained in the first part 1) to upscale a double porosity model, viz. a Darcy–Lapwood–Brinkmann flow occurring in thin heterogeneous domains.

The first main challenge is to prove some important compactness results in thin heterogeneous domains. To make it a little more precise, let $G_i \subset \mathbb{R}^{d_i}$ ($i = 1, 2$) be a bounded open set in $\mathbb{R}^{d_i}$ (integer $d_i \geq 1$) with $0 \in \overline{G}_2$ (the closure of G_2). We set

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$G_\varepsilon = G_1 \times \varepsilon G_2$ (where $\varepsilon > 0$ is a small parameter) and $G_0 = G_1 \times \{0\}$. The first main result states as follows.

Theorem 1.1. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(G_\varepsilon)$ satisfying*

$$\sup_{\varepsilon\in E} \varepsilon^{-\frac{d_2}{p}} \|u_\varepsilon\|_{L^p(G_\varepsilon)} \leq C,$$

where $C > 0$ is a constant independent of ε, $1 < p < \infty$ and E is an ordinary sequence of positive real numbers ε tending to zero. Then there exist a subsequence E' of E and $u_0 \in L^p(G_0; \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)))$ such that, as $E' \ni \varepsilon \to 0$,

$$\varepsilon^{-d_2} \int_{G_\varepsilon} u_\varepsilon(x) f\left(\overline{x}, \frac{x}{\varepsilon}\right) dx \to \int_{G_0}\int_{G_2} M(u_0(\overline{x}, \cdot, \zeta) f(\overline{x}, \cdot, \zeta)) d\zeta d\overline{x} \tag{1.1}$$

for all $f \in L^{p'}(G_0; \mathcal{A}(\mathbb{R}^{d_1}; L^{p'}(G_2)))$ $(1/p' = 1 - 1/p)$.

We denote the convergence property (1.1) by "$u_\varepsilon \to u_0$ in $L^p(G_\varepsilon)$-weak $\Sigma_{\mathcal{A}}$".

In Theorem 1.1, $\mathcal{A}$ is an algebra with mean value on $\mathbb{R}^{d_1}$ while $\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2))$ stands for the vector-valued generalized Besicovitch space associated to $\mathcal{A}$; M denotes the mean value on $\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2))$. Especially, we denote by $u_0(\overline{x}, \cdot, \zeta)$ the function $\overline{y} \mapsto u_0(\overline{x}, \overline{y}, \zeta)$, and the mean value operator acts on the $\overline{y}$ variable that belongs to $\mathbb{R}^{d_1}$. This shows that in the integral

$$\int_{G_0}\int_{G_2} M(u_0(\overline{x}, \cdot, \zeta) f(\overline{x}, \cdot, \zeta)) d\zeta d\overline{x},$$

the mean value is with respect to the dot "$\cdot$". In the special setting of periodic functions with periodicity 1 in each coordinate, one has

$$\int_{G_0}\int_{G_2} M(u_0(\overline{x}, \cdot, \zeta) f(\overline{x}, \cdot, \zeta)) d\zeta d\overline{x} = \int_{G_0}\int_{G_2}\int_Y u_0(\overline{x}, \overline{y}, \zeta) f(\overline{x}, \overline{y}, \zeta) d\overline{y} d\zeta d\overline{x},$$

where $Y = (-\frac{1}{2}, \frac{1}{2})^{d_1}$.

We refer the reader to Section 2 for general concepts about the algebras with mean value and the subsequent spaces. Theorem 1.1 is at the heart of some other important compactness results stated and proved in Section 3 of this work. It generalizes its homologue stated in [22], which is concerned with the periodic version. It is worth noticing that the generalization is not straightforward as it heavily relies on Lemma 3.5 stated and proved in [24, Proposition 3.2]. We use Lemma 3.5 to get rid of separability issues, as some algebras with mean value are not separable, in contrast with the algebra of continuous periodic functions on $\mathbb{R}^{d_1}$. To the best of our knowledge, there is no result available so far in the literature dealing with such kind of compactness results in thin domain beyond the periodic setting. So our result is new and most likely, of great applicability.

The second main result deals with the sigma-convergence in thin heterogeneous domains with oscillating lateral boundaries. Let Ω be a bounded open Lipschitz domain in $\mathbb{R}^{d-1}$ (integer $d \geq 2$), and let h_1 and h_2 be two Lipschitz continuous functions defined on $\mathbb{R}^{d-1}$ and satisfying $h_1, h_2 \in W^{1,\infty}(\mathbb{R}^{d-1}) \cap \mathcal{A}$ ($\mathcal{A}$ an algebra with mean value on $\mathbb{R}^{d-1}$). We define the thin domain here as follows:

$$\Omega^\varepsilon = \left\{x = (\overline{x}, x_d) \in \mathbb{R}^d : \overline{x} \in \Omega \text{ and } \varepsilon h_1\left(\frac{\overline{x}}{\varepsilon}\right) < x_3 < \varepsilon h_2\left(\frac{\overline{x}}{\varepsilon}\right)\right\}. \tag{1.2}$$

Set $I = (\min_{\mathbb{R}^{d-1}} h_1, \max_{\mathbb{R}^{d-1}} h_2)$ and assume that $0 \in [\min_{\mathbb{R}^{d-1}} h_1, \max_{\mathbb{R}^{d-1}} h_2]$, and define $G_\varepsilon = \Omega \times \varepsilon I$. Finally, define the set

$$\mathbb{J} = \left\{y = (\overline{y}, y_d) \in \mathbb{R}^d : \overline{y} \in \mathbb{R}^{d-1} \text{ and } h_1(\overline{y}) < y_3 < h_2(\overline{y})\right\},$$

and we denote by $\chi_{\mathbb{J}}$ its characteristic function in $\mathbb{R}^d$. We are in a position to state the second main compactness result.

Theorem 1.2. *Let $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(\Omega^\varepsilon)$ ($1 < p < \infty$) be a sequence satisfying*

$$\sup_{\varepsilon\in E} \varepsilon^{-\frac{1}{p}} \|u_\varepsilon\|_{L^p(\Omega^\varepsilon)} \leq C,$$

where $C > 0$ is independent of ε and E is as in Theorem 1.1. For any function $v \in L^p(\Omega^\varepsilon)$, we denote by $P_\varepsilon v$ its extension by 0 on G_ε: $P_\varepsilon v = \chi_{\Omega^\varepsilon} v$ (where $\chi_{\Omega^\varepsilon}$ stands for the characteristic function of Ω^ε in G_ε). Let E' and $u_0 \in L^p(G_0; \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d-1}; L^p(I)))$ be defined by Theorem 1.1 and such that $P_\varepsilon u_\varepsilon \to u_0$ in $L^p(G_\varepsilon)$-weak $\Sigma_{\mathcal{A}}$. Then, as $E' \ni \varepsilon \to 0$,

$$\frac{1}{\varepsilon}\int_{\Omega^\varepsilon} u_\varepsilon(x) f\left(\overline{x}, \frac{x}{\varepsilon}\right) dx \to \int_{G_0}\int_I M(\chi_{\mathbb{J}} u_0(\overline{x}, \cdot, y_d) f(\overline{x}, \cdot, y_d)) dy_d d\overline{x}$$

for all $f \in L^{p'}(G_0; \mathcal{A}(\mathbb{R}^{d-1}; \mathcal{C}(\overline{I})))$, $1/p' = 1 - 1/p$.

To illustrate the previous results, we consider the upscaling of a double porosity model in thin heterogeneous layers. The model problem is stated as follows. In the thin heterogeneous layer Ω^ε, we consider the flow of a fluid described by the Darcy–Lapwood–Brinkmann system

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla \boldsymbol{u}_\varepsilon\right) + \frac{\mu}{K_\varepsilon}\boldsymbol{u}_\varepsilon + \frac{\rho}{\phi^2}(\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon = \boldsymbol{f} & \text{in } \Omega^\varepsilon, \\ \operatorname{div} \boldsymbol{u}_\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ \boldsymbol{u}_\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{1.3}$$

where the assumptions on A and $\boldsymbol{f}$ ensure the existence (for each fixed $\varepsilon > 0$) of a solution $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in H^1_0(\Omega^\varepsilon)^3 \times L^2_0(\Omega^\varepsilon)$. It is worth noting that in (1.3), the parameters μ, ρ and ϕ are positive constants independent of ε. We consider two different kind of thin layers: a thin layer with flat parallel boundaries and a thin layer with highly oscillating boundaries. In each case, we obtain the following results, which are respectively, the third and the fourth main results of this work.

Theorem 1.3. *For each $\varepsilon > 0$, let Ω^ε be given by (1.2) with $h_1 = -1$ and $h_2 = 1$ (i.e., $\Omega^\varepsilon = \Omega \times (-\varepsilon, \varepsilon)$, where Ω is a bounded open connected Lipschitz subset in $\mathbb{R}^2$), and let $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ be a solution of (1.3). Assume that $A \in (B^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I)))^{3\times 3}$. Then:*

(i) *If $K_\varepsilon = O(\varepsilon^2)$ with $K_\varepsilon/\varepsilon^2 \to K$ when $\varepsilon \to 0$, $0 < K < \infty$, then $(\boldsymbol{u}_\varepsilon/\varepsilon^2, p_\varepsilon)$ weakly $\Sigma_{\mathcal{A}}$-converges (as $\varepsilon \to 0$) in $L^2(\Omega^\varepsilon)^3 \times L^2(\Omega^\varepsilon)$ to $(\boldsymbol{u}_0, p_0)$ belonging to $[L^2(\Omega; \mathcal{B}^{1,2}_{\mathcal{A}}(\mathbb{R}^2; H^1_0(I)))]^3 \times L^2(\Omega)$. Moreover, $p_0 \in H^1(\Omega)$ and, defining $\boldsymbol{u}(\overline{x}) = \int_{-1}^1 M(\boldsymbol{u}_0(\overline{x}, \cdot, \zeta))d\zeta \equiv (\boldsymbol{u}', u_3)$, one has $u_3 = 0$ and $(\boldsymbol{u}', p_0)$ is the unique solution to the effective problem*

$$\begin{cases} \boldsymbol{u}' = \widehat{A}(\boldsymbol{f}_1 - \nabla_{\overline{x}} p_0) & \text{in } \Omega, \\ \operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 & \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}' \cdot \nu = 0 \text{ on } \partial\Omega \end{cases}$$

where $\widehat{A} = (\widehat{a}_{ij})_{1\le i,j\le 2}$ is a symmetric, positive definite 2×2 matrix defined by its entries

$$\widehat{a}_{ij} = \int_{-1}^1 M(A\overline{\nabla}_y \boldsymbol{w}_i : \overline{\nabla}_y \boldsymbol{w}_j)d\zeta + \frac{\mu}{K}\int_{-1}^1 M(\boldsymbol{w}_i\boldsymbol{w}_j)d\zeta.$$

Here $\boldsymbol{w}_i$ $(1 \le i \le 2)$ is the unique solution in $[\mathcal{B}^{1,2}_{\mathcal{A}}(\mathbb{R}^2; H^1_0(I))]^3$ of the Stokes–Brinkmann system

$$\begin{cases} -\overline{\operatorname{div}}_y(A(y)\overline{\nabla}_y \boldsymbol{w}_i) + \dfrac{\mu}{K}\boldsymbol{w}_i + \overline{\nabla}_y \pi_i = e_i & \text{in } \mathbb{R}^2 \times I, \\ \overline{\operatorname{div}}_y \boldsymbol{w}_i = 0 & \text{in } \mathbb{R}^2 \times I, \end{cases}$$

e_i being the ith vector of the canonical basis in $\mathbb{R}^3$.

(ii) *If $K_\varepsilon \ll \varepsilon^2$, then $(\boldsymbol{u}_\varepsilon/(\varepsilon K_\varepsilon^{1/2}), (K_\varepsilon^{1/2}/\varepsilon)p_\varepsilon)_{\varepsilon>0}$ weakly $\Sigma_{\mathcal{A}}$-converges (as $\varepsilon \to 0$) in $L^2(\Omega^\varepsilon)^3 \times L^2(\Omega^\varepsilon)$ toward $(\boldsymbol{u}_0, p_0) \in [L^2(\Omega; \mathcal{B}^{1,2}_{\mathcal{A}}(\mathbb{R}^2; H^1_0(I)))]^3 \times L^2(\Omega)$. Furthermore, defining $\boldsymbol{u}$ as in* (i) *above, we have $u_3 = 0$ and $(\boldsymbol{u}', p_0)$ is a solution of*

$$\boldsymbol{u}' = -\widehat{A}\nabla_{\overline{x}} p_0 \quad \text{in } \Omega, \quad \operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}' \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where $\widehat{A}$ is a symmetric matrix defined by

$$\widehat{A} = (\widehat{a}_{ij})_{1\le i,j\le 2} \quad \text{with } \widehat{a}_{ij} = \mu\int_{-1}^1 M(\boldsymbol{w}_i\boldsymbol{w}_j)d\zeta,$$

$\boldsymbol{w}_i$ $(1 \le i \le 2)$ being the unique solution in $\mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I))^3$ of

$$\mu\boldsymbol{w}_i + \overline{\nabla}_y \pi_i = e_i \quad \text{in } \mathbb{R}^2 \times I \quad \text{and} \quad \overline{\operatorname{div}}_y \boldsymbol{w}_i = 0 \quad \text{in } \mathbb{R}^2 \times I.$$

(iii) *If $K_\varepsilon \gg \varepsilon^2$, then $(\boldsymbol{u}_\varepsilon/\varepsilon^2, p_\varepsilon)_{\varepsilon>0}$ weakly $\Sigma_{\mathcal{A}}$-converges (as $\varepsilon \to 0$) in $L^2(\Omega^\varepsilon)^{3\times 3} \times L^2(\Omega^\varepsilon)$ to $(\boldsymbol{u}_0, p_0)$ where $\boldsymbol{u}_0 \in [L^2(\Omega; B_{\mathcal{A}}^{1,2}(\mathbb{R}^2; H_0^1(I)))]^3$ and $p_0 \in L^2(\Omega)$. Furthermore, $p_0 \in H^1(\Omega)$ and, defining $\boldsymbol{u} = (\boldsymbol{u}', u_3)$ as in* (i) *above, one has $u_3 = 0$ and $(\boldsymbol{u}', p_0)$ is the unique solution of the effective problem*

$$\begin{cases} \boldsymbol{u}' = \widehat{A}(\boldsymbol{f}_1 - \nabla_{\overline{x}} p_0) & \text{in } \Omega, \\ \operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 & \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}' \cdot \nu = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\widehat{A} = (\widehat{a}_{ij})_{1\le i,j\le 2}$ is given by

$$\widehat{a}_{ij} = \int_{-1}^{1} M(A\overline{\nabla}_y \boldsymbol{w}_i : \overline{\nabla}_y \boldsymbol{w}_j) dy_3, \quad 1 \le i, j \le 2,$$

where here, $\boldsymbol{w}_i$ ($1 \le i \le 2$) is the unique solution in $(\mathcal{B}_{\mathcal{A}}^{1,2}(\mathbb{R}^2; H_0^1(I)))^3$ of the Stokes system

$$\begin{cases} -\overline{\operatorname{div}}_y (A(y)\overline{\nabla}_y \boldsymbol{w}_i) + \overline{\nabla}_y \pi_i = e_i & \text{in } \mathbb{R}^2 \times I, \\ \overline{\operatorname{div}}_y \boldsymbol{w}_i = 0 & \text{in } \mathbb{R}^2 \times I. \end{cases}$$

Remark 1.4. Let us recall the following concept that has been used in the statement of Theorem 1.3 above. Let U and V be two positive functions of a small positive variable ε, such that $U, V \to 0$ when $\varepsilon \to 0$. We say that

$$U = O(V) \quad \text{provided that } \frac{U}{V} \to K \text{ as } \varepsilon \to 0, \text{ where } 0 < K < \infty;$$
$$U \ll V \quad \text{iff } \frac{U}{V} \to 0 \text{ as } \varepsilon \to 0, \quad \text{and} \quad U \gg V \quad \text{iff } V \ll U.$$

In the next result, we assume that Ω^ε is given by (1.2) where h_1 and h_2 satisfy $\max_{\mathbb{R}^2} h_1 < \min_{\mathbb{R}^2} h_2$, and $h_1, h_2 \in \mathcal{A}$ with $M(h_2 - h_1) \neq 0$, $\mathcal{A}$ being an ergodic algebra with mean value on $\mathbb{R}^2$. The result reads as follows.

Theorem 1.5. *Assume that Ω^ε is given by* (1.2). *Let $(\boldsymbol{u}_\varepsilon, p_\varepsilon = p_\varepsilon^0 + \varepsilon p_\varepsilon^1) \in H_0^1(\Omega^\varepsilon)^3 \times L_0^2(\Omega^\varepsilon)$ be a solution of* (1.3). *Assume that $A \in (B_{\mathcal{A}}^2(\mathbb{R}^2; L^\infty(I)))^{3\times 3}$. Then:*

(i) *If $K_\varepsilon = O(\varepsilon^2)$ with $K_\varepsilon/\varepsilon^2 \to K$ when $\varepsilon \to 0$, $0 < K < \infty$, then, still denoting by $\boldsymbol{u}_\varepsilon$ the extension by zero of $\boldsymbol{u}_\varepsilon$ on $G_\varepsilon = \Omega \times (\varepsilon h_1^-, \varepsilon h_2^+)$, one has*

$$\frac{\boldsymbol{u}_\varepsilon}{\varepsilon^2} \to \boldsymbol{u}_0 \quad \text{in } L^2(G_\varepsilon)^3\text{-weak } \Sigma_{\mathcal{A}},$$

and

$$p_\varepsilon^0 \to p_0 \quad \text{in } H^1(\Omega)\text{-weak and in } L^2(\Omega)\text{-strong.}$$

Defining $\boldsymbol{u} = (\boldsymbol{u}', u_3)$ *by* $\boldsymbol{u}(\overline{x}) = M(\int_{h_1}^{h_2} \boldsymbol{u}_0(\overline{x}, \cdot, y_3) dy_3)$*, we have* $u_3 = 0$ *and* $(\boldsymbol{u}', p_0)$ *is the unique solution of the homogenized problem*

$$\begin{cases} \boldsymbol{u}' = \widehat{A}(\boldsymbol{f}_1 - \nabla_{\overline{x}} p_0) & \text{in } \Omega, \\ \operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 & \text{in } \Omega, \quad \text{and} \quad \boldsymbol{u}' \cdot \nu = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\widehat{A} = (\widehat{a}_{ij})_{1 \le i,j \le 2}$ *is a symmetric, positive definite* 2×2 *matrix defined by its entries*

$$\widehat{a}_{ij} = M\left(\int_{h_1}^{h_2} A \overline{\nabla}_y \boldsymbol{w}_i \cdot \overline{\nabla}_y \boldsymbol{w}_j \, dy_3 \right) + \frac{\mu}{K} M\left(\int_{h_1}^{h_2} \boldsymbol{w}_i \cdot \boldsymbol{w}_j \, dy_3 \right).$$

Here $\boldsymbol{w}_i$ $(1 \le i \le 2)$ *is the unique solution in* $B^{1,2}_{\#,\mathrm{div}}(\mathbb{J})$ *of the Stokes–Brinkmann system*

$$\begin{cases} -\overline{\operatorname{div}}_y (A(y) \overline{\nabla}_y \boldsymbol{w}_i) + \dfrac{\mu}{K} \boldsymbol{w}_i + \overline{\nabla}_y \pi_i = e_i & \text{in } \mathbb{J}, \\ \overline{\operatorname{div}}_y \boldsymbol{w}_i = 0 & \text{in } \mathbb{J}, \\ \boldsymbol{w}_i = 0 & \text{on } \partial\mathbb{J}, \end{cases}$$

e_i *being the* i*th vector of the canonical basis in* $\mathbb{R}^3$ *and*

$$\mathbb{J} = \left\{ y = (\overline{y}, y_3) \in \mathbb{R}^3 : \overline{y} \in \mathbb{R}^2 \text{ and } h_1(\overline{y}) < y_3 < h_2(\overline{y}) \right\}.$$

(ii) *If* $K_\varepsilon \ll \varepsilon^2$*, then, up to a subsequence, one has*

$$\frac{\boldsymbol{u}_\varepsilon}{\varepsilon K_\varepsilon^{\frac{1}{2}}} \to \boldsymbol{u}_0 \quad \text{in } L^2(G_\varepsilon)^3\text{-weak } \Sigma_{\mathcal{A}},$$
$$\frac{K_\varepsilon^{\frac{1}{2}}}{\varepsilon} p_\varepsilon^0 \to p_0 \quad \text{in } H^1(\Omega)\text{-weak and in } L^2(\Omega)\text{-strong.}$$

Furthermore, defining $\boldsymbol{u}$ *as in* (i) *above, we have* $u_3 = 0$ *and* $(\boldsymbol{u}', p_0)$ *is a solution of*

$$\boldsymbol{u}' = -\widehat{A} \nabla_{\overline{x}} p_0 \quad \text{in } \Omega, \quad \operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}' \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where $\widehat{A}$ *is a symmetric matrix defined by*

$$\widehat{A} = (\widehat{a}_{ij})_{1 \le i,j \le 2} \quad \text{with } \widehat{a}_{ij} = \mu M\left(\int_{h_1}^{h_2} \boldsymbol{w}_i \boldsymbol{w}_j \, dy_3 \right),$$

$\boldsymbol{w}_i$ $(1 \le i \le 2)$ *being the unique solution in* $\mathcal{B}^2_{\mathcal{A}}(\mathbb{J})^3$ *of*

$$\mu \boldsymbol{w}_i + \overline{\nabla}_y \pi_i = e_i \quad \text{in } \mathbb{J} \quad \text{and} \quad \overline{\operatorname{div}}_y \boldsymbol{w}_i = 0 \quad \text{in } \mathbb{J}.$$

(iii) *If $K_\varepsilon \gg \varepsilon^2$, then, still denoting by $\boldsymbol{u}_\varepsilon$ and p_ε^1 the extension of $\boldsymbol{u}_\varepsilon$ and p_ε^1 by zero on G_ε, we have,*

$$\begin{aligned} \frac{\boldsymbol{u}_\varepsilon}{\varepsilon^2} &\to \boldsymbol{u}_0 && \text{in } L^2(G_\varepsilon)^3\text{-weak } \Sigma_{\mathcal{A}}, \\ \frac{1}{\varepsilon}\nabla \boldsymbol{u}_\varepsilon &\to \overline{\nabla}_y \boldsymbol{u}_0 && \text{in } L^2(G_\varepsilon)^{3\times 3}\text{-weak } \Sigma_{\mathcal{A}}, \\ p_\varepsilon^0 &\to p_0 && \text{in } H^1(\Omega)\text{-weak and in } L^2(\Omega)\text{-strong,} \end{aligned}$$

still defining $\boldsymbol{u}$ as in (i) *above, it holds that*

$$\begin{cases} \boldsymbol{u}' = \widehat{A}(\boldsymbol{f}_1 - \nabla_{\overline{x}} p_0) & \text{in } \Omega, \\ \operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 & \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}' \cdot \nu = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\widehat{A} = (\widehat{a}_{ij})_{1\le i,j\le 2}$ is given by

$$\widehat{a}_{ij} = M\left(\int_{h_1}^{h_2} A\overline{\nabla}_y \boldsymbol{w}_i : \overline{\nabla}_y \boldsymbol{w}_j \, dy_3 \right), \quad 1 \le i, j \le 2,$$

with $\boldsymbol{w}_i$ ($1 \le i \le 2$) being the unique solution in $B^{1,2}_{\#,\mathrm{div}}(\mathbb{J})$ of the Stokes system

$$\begin{cases} -\overline{\operatorname{div}}_y(A(y)\overline{\nabla}_y \boldsymbol{w}_i) + \overline{\nabla}_y \pi_i = e_i & \text{in } \mathbb{J}, \\ \overline{\operatorname{div}}_y \boldsymbol{w}_i = 0 & \text{in } \mathbb{J}. \end{cases}$$

Let us first and foremost compare our result in Theorem 1.5 with the existing ones in the literature. In [27], problem (1.3) has been considered in a thin domain with periodic oscillatory boundary. To be more precise, the authors of [27] considered the problem

$$\begin{cases} -\Delta \boldsymbol{u}_\varepsilon + \dfrac{\mu}{K_\varepsilon}\boldsymbol{u}_\varepsilon + \dfrac{\rho}{\phi^2}(\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon = \boldsymbol{f} & \text{in } \Lambda^\varepsilon, \\ \operatorname{div} \boldsymbol{u}_\varepsilon = 0 & \text{in } \Lambda^\varepsilon, \\ \boldsymbol{u}_\varepsilon = 0 & \text{on } \partial\Lambda^\varepsilon, \end{cases}$$

where $\Lambda_\varepsilon = \{(\overline{x}, x_3) \in \mathbb{R}^3 : \overline{x} \in \Omega \text{ and } \varepsilon h_1(\frac{\overline{x}}{\varepsilon}) < x_3 < \varepsilon h_2(\frac{\overline{x}}{\varepsilon})\}$ with h_1, h_2 being smooth periodic functions. They proved the analog of Theorem 1.5. However, with respect to the results in [27], a few remarks are in order: 1) In Theorem 1.5, the functions h_1 and h_2 assume several behaviours such as the periodicity, the almost periodicity, the asymptotic periodicity, the asymptotic almost periodicity, and many more besides. So the main result in [27] (Theorem 3.1 for instance) is a special case of Theorem 1.5. It is also worth noting that, instead of considering only a periodic framework like in [27], we consider the case where the thin domain is highly heterogeneous, the heterogeneities being distributed inside the domain in a general deterministic fashion.

This imposes the use of a more general concept of convergence taking into account the properties of the domain, in order to pass to the limit. 2) We also notice that our estimates leading to the homogenization process are not the same compared to those obtained in [27, Sections 2 and 3]. Indeed, in [27, Sections 2 and 3], from the estimates

$$\|\boldsymbol{u}\|_{L^2(\Omega^\varepsilon)^3} \leq C\varepsilon^{\frac{5}{2}} \quad \text{and} \quad \|\boldsymbol{u}\|_{L^2(\Omega^\varepsilon)^3} \leq C\varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}},$$

the authors used the following estimate:

$$\|\boldsymbol{u}\|_{L^2(\Omega^\varepsilon)^3} \leq C(\varepsilon^{\frac{5}{2}} + \varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}}),$$

while we rather inferred the following one (which seems to be more realistic):

$$\|\boldsymbol{u}\|_{L^2(\Omega^\varepsilon)^3} \leq C \min(\varepsilon^{\frac{5}{2}}, \varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}}).$$

This gives rise in our work, to uniform estimates different from the ones they obtained, especially in the cases when $K_\varepsilon \ll \varepsilon^2$ and when $K_\varepsilon \gg \varepsilon^2$. 3) Instead of using the unfolding periodic method, we use a direct approach: the sigma-convergence method generalizing the two-scale convergence method. The use of this approach is also justified by the fact that there is no variant up to now, of the unfolding operator beyond the periodic setting. 4) In the proof of Theorem 1.3, we derive the estimates of the pressure by using a classical tool relying on the solution of a divergence equation; see Lemma 4.4 and Proposition 4.5. However, in the proof of Theorem 1.5, the estimates of the pressure are obtained from a crucial trick stemming from [13, Theorem 3.1]: the decomposition of the pressure as $L_0^2(\Omega^\varepsilon) \ni p_\varepsilon = p_\varepsilon^0 + \varepsilon p_\varepsilon^1$ with $p_\varepsilon^0 \in H^1(\Omega)$ and $p_\varepsilon^1 \in L^2(\Omega^\varepsilon)$ (see, e.g., (5.5) and (5.6)). This very important result allows us to work in the same domain Ω^ε instead of adding further assumptions on the geometry of Ω as it is the case in [27] (see hypotheses **H1**, **H2** and **H3** therein).

Very few works address rigorously the asymptotic analysis of (1.3) or related problems, in the literature. To the best of our knowledge the only ones available to date are [3, 4, 8, 13, 27, 32]. For other works in thin heterogeneous domains, we may refer to, e.g., [1, 2, 5, 9, 12, 14–16, 20–22, 26, 29], to cite a few. In [15, 16, 29] the authors consider the formal asymptotic developments to upscale a Stokes–Brinkmann model.

The rest of the work is organized as follows. Section 2 deals with some fundamental tools around the concept of algebra with mean value. In Section 3, we are concerned with the concept of sigma-convergence in thin heterogeneous domains. We prove therein some compactness results related to that concept such as Theorems 1.1 and 1.2. We apply in Sections 4 and 5 the results developed in the previous section to upscale a Darcy–Lapwood–Brinkmann flow occurring in thin heterogeneous layers. Finally, in Section 6, we provide some concrete applications of Theorems 1.3 and 1.5.

Unless otherwise specified, the vector spaces throughout are assumed to be real vector spaces, and the scalar functions are assumed to take real values. We shall always assume that the numerical space $\mathbb{R}^m$ (integer $m \geq 1$) and its open sets are each provided with the Lebesgue measure denoted by $dx = dx_1 \cdots dx_m$.

Note. This work inspired the works [10, 28] where the evolutionary counterpart of results contained in Section 3.1 have been considered.

2. Algebras with mean value and related Sobolev-type spaces

In this section, we gather the reader with some basic concepts about the algebras with mean value [18, 35] and the associated Sobolev-type spaces [17, 24, 30].

Let $\mathcal{A}$ be an algebra with mean value (algebra wmv in short) on $\mathbb{R}^m$ (integer $m \geq 1$) [18, 35], that is, a closed subalgebra of the $\mathcal{C}^*$-algebra of bounded uniformly continuous real-valued functions on $\mathbb{R}^m$, $\mathrm{BUC}(\mathbb{R}^m)$, which contains the constants, is translation invariant and is such that any of its elements possesses a mean value in the following sense: for every $u \in \mathcal{A}$, the sequence $(u^\varepsilon)_{\varepsilon>0}$ ($u^\varepsilon(x) = u(x/\varepsilon)$) weakly $*$-converges in $L^\infty(\mathbb{R}^m)$ to some real number $M(u)$ (called the mean value of u) as $\varepsilon \to 0$. The mean value expresses as

$$M(u) = \lim_{R\to\infty} \fint_{B_R} u(y)dy \quad \text{for } u \in \mathcal{A}$$

where we have set $\fint_{B_R} = |B_R|^{-1} \int_{B_R}$.

To an algebra with mean value $\mathcal{A}$ are associated its regular subalgebras $\mathcal{A}^k = \{\psi \in \mathcal{C}^k(\mathbb{R}^m) : D_y^\alpha \psi \in \mathcal{A} \forall \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m \text{ with } |\alpha| \leq k\}$ ($k \geq 0$ an integer with $\mathcal{A}^0 = \mathcal{A}$, and $D_y^\alpha \psi = \frac{\partial^{|\alpha|}\psi}{\partial y_1^{\alpha_1} \ldots \partial y_m^{\alpha_m}}$). Under the norm $\||u|\|_k = \sup_{|\alpha|\leq k} \|D_y^\alpha \psi\|_\infty$, $\mathcal{A}^k$ is a Banach space. We also define the space $\mathcal{A}^\infty = \{\psi \in \mathcal{C}^\infty(\mathbb{R}^m) : D_y^\alpha \psi \in \mathcal{A} \forall \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m\}$, a Fréchet space when endowed with the locally convex topology defined by the family of norms $\||\cdot|\|_m$. The space $\mathcal{A}^\infty$ is dense in any $\mathcal{A}^m$ (integer $m \geq 0$).

The notion of a vector-valued algebra with mean value will be very useful in this study.

Let $\mathbb{F}$ be a Banach space. We denote by $\mathrm{BUC}(\mathbb{R}^m; \mathbb{F})$ the Banach space of bounded uniformly continuous functions $u : \mathbb{R}^m \to \mathbb{F}$, endowed with the norm

$$\|u\|_\infty = \sup_{y\in\mathbb{R}^m} \|u(y)\|_{\mathbb{F}}$$

where $\|\cdot\|_{\mathbb{F}}$ stands for the norm in $\mathbb{F}$. Let $\mathcal{A}$ be an algebra with mean value on $\mathbb{R}^m$. We denote by $\mathcal{A} \otimes \mathbb{F}$ the usual space of functions of the form

$$\sum_{\text{finite}} u_i \otimes v_i \quad \text{with } u_i \in \mathcal{A} \text{ and } v_i \in \mathbb{F}$$

where $(u_i \otimes v_i)(y) = u_i(y)v_i$ for $y \in \mathbb{R}^m$. With this in mind, we define the vector-valued algebra with mean value $\mathcal{A}(\mathbb{R}^m;\mathbb{F})$ as the closure of $\mathcal{A} \otimes \mathbb{F}$ in $\mathrm{BUC}(\mathbb{R}^m;\mathbb{F})$. Then it holds that (see [24]), for any $f \in \mathcal{A}(\mathbb{R}^m;\mathbb{F})$, the set $\{L(f) : L \in \mathbb{F}'$ with $\|L\|_{\mathbb{F}'} \leq 1\}$ is relatively compact in $\mathcal{A}$.

Let us note that we may still define the space $\mathcal{A}(\mathbb{R}^m;\mathbb{F})$ where $\mathbb{F}$ in this case is a Fréchet space. In that case, we replace the norm by the family of seminorms defining the topology of $\mathbb{F}$.

Now, let $f \in \mathcal{A}(\mathbb{R}^m;\mathbb{F})$. Then, defining $\|f\|_{\mathbb{F}}$ by $\|f\|_{\mathbb{F}}(y) = \|f(y)\|_{\mathbb{F}}$ $(y \in \mathbb{R}^m)$, we have that $\|f\|_{\mathbb{F}} \in \mathcal{A}$. Similarly we can define (for $0 < p < \infty$) the function $\|f\|_{\mathbb{F}}^p$ and $\|f\|_{\mathbb{F}}^p \in \mathcal{A}$. This allows us to define the Besicovitch seminorm on $\mathcal{A}(\mathbb{R}^m;\mathbb{F})$ as follows: for $1 \leq p < \infty$, we define the Marcinkiewicz-type space $\mathfrak{M}^p(\mathbb{R}^m;\mathbb{F})$ to be the vector space of functions $u \in L^p_{\mathrm{loc}}(\mathbb{R}^m;\mathbb{F})$ such that

$$\|u\|_p = \left(\limsup_{R\to\infty} ⨍_{B_R} \|u(y)\|_{\mathbb{F}}^p dy\right)^{\frac{1}{p}} < \infty$$

where B_R is the open ball in $\mathbb{R}^m$ centered at the origin and of radius R. Under the seminorm $\|\cdot\|_p$, $\mathfrak{M}^p(\mathbb{R}^m;\mathbb{F})$ is a complete seminormed space with the property that $\mathcal{A}(\mathbb{R}^m;\mathbb{F}) \subset \mathfrak{M}^p(\mathbb{R}^m;\mathbb{F})$ since $\|u\|_p < \infty$ for any $u \in \mathcal{A}(\mathbb{R}^m;\mathbb{F})$. We therefore define the vector-valued generalized Besicovitch space $B^p_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F})$ as the closure of $\mathcal{A}(\mathbb{R}^m;\mathbb{F})$ in $\mathfrak{M}^p(\mathbb{R}^m;\mathbb{F})$. The following hold true [24, 30]:

(i) The space $\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F}) = B^p_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F})/\mathcal{N}$ (where $\mathcal{N} = \{u \in B^p_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F}) : \|u\|_p = 0\}$) is a Banach space under the norm $\|u + \mathcal{N}\|_p = \|u\|_p$ for $u \in B^p_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F})$.

(ii) The mean value $M : \mathcal{A}(\mathbb{R}^m;\mathbb{F}) \to \mathbb{F}$ extends by continuity to a continuous linear mapping (still denoted by M) on $B^p_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F})$ satisfying

$$L(M(u)) = M(L(u)) \quad \text{for all } L \in \mathbb{F}' \text{ and } u \in B^p_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F}). \tag{2.1}$$

Moreover, for $u \in B^p_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F})$ we have

$$\|u\|_p = [M(\|u\|_{\mathbb{F}}^p)]^{\frac{1}{p}} \equiv \left[\lim_{R\to\infty} ⨍_{B_R} \|u(y)\|_{\mathbb{F}}^p dy\right]^{\frac{1}{p}},$$

and for $u \in \mathcal{N}$ one has $M(u) = 0$.

It is worth noticing that $\mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^m;H)$ (when $\mathbb{F} = H$ is a Hilbert space) is a Hilbert space with inner product

$$(u,v)_2 = M[(u,v)_H] \quad \text{for } u,v \in \mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^m;H),$$

$(\ ,\ )_H$ denoting the inner product in H and $(u,v)_H$ the function $y \mapsto (u(y),v(y))_H$ from $\mathbb{R}^m$ to $\mathbb{R}$, which belongs to $\mathcal{B}^1_{\mathcal{A}}(\mathbb{R}^m;\mathbb{R})$.

We also define the Sobolev–Besicovitch-type spaces as follows:

$$B_{\mathcal{A}}^{1,p}(\mathbb{R}^m;\mathbb{F}) = \left\{u \in B_{\mathcal{A}}^{p}(\mathbb{R}^m;\mathbb{F}) : \nabla_y u \in (B_{\mathcal{A}}^{p}(\mathbb{R}^m;\mathbb{F}))^m\right\},$$

endowed with the seminorm

$$\|u\|_{1,p} = (\|u\|_p^p + \|\nabla_y u\|_p^p)^{\frac{1}{p}},$$

which is a complete seminormed space. The Banach counterpart of $B_{\mathcal{A}}^{1,p}(\mathbb{R}^m;\mathbb{F})$ denoted by $\mathcal{B}_{\mathcal{A}}^{1,p}(\mathbb{R}^m;\mathbb{F})$ is defined by replacing $B_{\mathcal{A}}^{p}(\mathbb{R}^m;\mathbb{F})$ by $\mathcal{B}_{\mathcal{A}}^{p}(\mathbb{R}^m;\mathbb{F})$ and $\partial/\partial y_i$ by $\overline{\partial}/\partial y_i$, where $\overline{\partial}/\partial y_i$ is defined by

$$\frac{\overline{\partial}}{\partial y_i}(u + \mathcal{N}) := \frac{\partial u}{\partial y_i} + \mathcal{N} \quad \text{for } u \in B_{\mathcal{A}}^{1,p}(\mathbb{R}^m;\mathbb{F}). \tag{2.2}$$

It is important to note that $\overline{\partial}/\partial y_i$ is also defined as the infinitesimal generator in the ith direction coordinate of the strongly continuous group $\mathcal{T}(y) : \mathcal{B}_{\mathcal{A}}^{p}(\mathbb{R}^m;\mathbb{F}) \to \mathcal{B}_{\mathcal{A}}^{p}(\mathbb{R}^m;\mathbb{F})$; $\mathcal{T}(y)(u + \mathcal{N}) = u(\cdot + y) + \mathcal{N}$. Let us denote by $\varrho : B_{\mathcal{A}}^{p}(\mathbb{R}^m;\mathbb{F}) \to \mathcal{B}_{\mathcal{A}}^{p}(\mathbb{R}^m;\mathbb{F}) = B_{\mathcal{A}}^{p}(\mathbb{R}^m;\mathbb{F})/\mathcal{N}$, $\varrho(u) = u + \mathcal{N}$, the canonical surjection. We remark that if $u \in B_{\mathcal{A}}^{1,p}(\mathbb{R}^m;\mathbb{F})$ then $\varrho(u) \in \mathcal{B}_{\mathcal{A}}^{1,p}(\mathbb{R}^m;\mathbb{F})$ with further

$$\frac{\overline{\partial}\varrho(u)}{\partial y_i} = \varrho\left(\frac{\partial u}{\partial y_i}\right),$$

as seen above in (2.2). We may also define higher order derivatives $\overline{D}_y^{\alpha} = \frac{\overline{\partial}^{|\alpha|}}{\partial y_1^{\alpha_1} \cdots \partial y_m^{\alpha_m}}$ where $\frac{\overline{\partial}^{\alpha_j}}{\partial y_i^{\alpha_j}} = \frac{\overline{\partial}}{\partial y_i} \circ \cdots \circ \frac{\overline{\partial}}{\partial y_i}$, j times.

We set $\mathcal{D}_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F}) = \varrho(\mathcal{A}^{\infty}(\mathbb{R}^m;\mathbb{F}))$ (and merely $\mathcal{D}_{\mathcal{A}}(\mathbb{R}^m) = \mathcal{D}_{\mathcal{A}}(\mathbb{R}^m;\mathbb{R})$), and we define the space of vector-valued distributions on $\mathcal{A}$, denoted by $\mathcal{D}'_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F})$, as the space of continuous linear functionals $L : \mathcal{D}_{\mathcal{A}}(\mathbb{R}^m) \to \mathbb{F}$. Let $1 \leq p < \infty$; for $u \in \mathcal{B}_{\mathcal{A}}^{p}(\mathbb{R}^m;\mathbb{F})$, define $L_u : \mathcal{D}_{\mathcal{A}}(\mathbb{R}^m) \to \mathbb{F}$ by

$$\langle L_u, \varphi\rangle = M(\varphi u), \quad \text{for all } \varphi \in \mathcal{D}_{\mathcal{A}}(\mathbb{R}^m).$$

Then it is easy to see that $L_u \in \mathcal{D}'_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F})$, so that $\mathcal{B}_{\mathcal{A}}^{p}(\mathbb{R}^m;\mathbb{F}) \hookrightarrow \mathcal{D}'_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F})$. The mapping $u \mapsto L_u$ is linear continuous and injective (this can be proven by proceeding as in [19]). This allows us to view u as an element of $\mathcal{D}'_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F})$ (say L_u), so that $\langle L_u, \varphi\rangle = \langle u, \varphi\rangle = M(\varphi u)$ for all $\varphi \in \mathcal{D}_{\mathcal{A}}(\mathbb{R}^m)$. Especially, for $\varphi = 1$, we have $M(u) = \langle u, 1\rangle$. We may therefore define the mean value of a distribution $L \in \mathcal{D}'_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F})$ accordingly,

$$M(L) = \langle L, 1\rangle. \tag{2.3}$$

For $L \in \mathcal{D}'_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F})$ and $\alpha \in \mathbb{N}^m$, we define the partial derivative $\overline{D}_{y}^{\alpha} L \in \mathcal{D}'_{\mathcal{A}}(\mathbb{R}^m;\mathbb{F})$ as follows:

$$\langle \overline{D}_{y}^{\alpha} L, \varphi\rangle = (-1)^{|\alpha|}\langle L, \overline{D}_{y}^{\alpha}\varphi\rangle \quad \forall \varphi \in \mathcal{D}_{\mathcal{A}}(\mathbb{R}^m). \tag{2.4}$$

From (2.3) and (2.4), we observe that

$$M(\overline{D}_y^{\alpha} L) = 0 \quad \text{for all } \alpha \in \mathbb{N}^m \backslash \{0\}.$$

We define a further notion by restricting ourselves to the case $\mathbb{F} = \mathbb{R}$. We say that the algebra $\mathcal{A}$ is ergodic if any $u \in \mathcal{B}_{\mathcal{A}}^1(\mathbb{R}^m; \mathbb{R})$ that is invariant under $(\mathcal{T}(y))_{y \in \mathbb{R}^m}$ is a constant in $\mathcal{B}_{\mathcal{A}}^1(\mathbb{R}^m; \mathbb{R})$: this amounts to, if $\mathcal{T}(y)u = u$ in $\mathcal{B}_{\mathcal{A}}^1(\mathbb{R}^m; \mathbb{R})$ for every $y \in \mathbb{R}^m$, then $u = c$ in $\mathcal{B}_{\mathcal{A}}^1(\mathbb{R}^m; \mathbb{R})$ in the sense that $\|u - c\|_1 = 0$, c being a constant.

We end this subsection by defining the *corrector function* spaces. We are concerned with two special choices of the space $\mathbb{F}$: $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = W^{1,p}(G)$, G being an open subset of $\mathbb{R}^N$ (integer $N \geq 1$).

(1) If $\mathbb{F} = \mathbb{R}$, we denote by $B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^m) \equiv B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^m; \mathbb{R})$ the *corrector function* space defined by

$$B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^m) = \left\{u \in W_{\mathrm{loc}}^{1,p}(\mathbb{R}^m) : \nabla u \in B_{\mathcal{A}}^p(\mathbb{R}^m)^m \text{ and } M(\nabla u) = 0\right\}.$$

In $B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^m)$ we identify two elements by their gradients: $u = v$ in $B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^m)$ iff $\nabla_y(u - v) = 0$, i.e., $\|\nabla_y(u - v)\|_p = 0$. We may therefore equip $B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^m)$ with the gradient norm $\|u\|_{\#,p} = \|\nabla_y u\|_p$. This defines a Banach space [11, Theorem 3.12] containing $B_{\mathcal{A}}^{1,p}(\mathbb{R}^m; \mathbb{R}) = B_{\mathcal{A}}^{1,p}(\mathbb{R}^m)$ as a subspace.

(2) For $\mathbb{F} = W^{1,p}(G)$, we define the *corrector function* space $B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^m; W^{1,p}(G))$ by

$$\begin{aligned} &B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^m; W^{1,p}(G)) \\ &\quad = \Big\{u \in W_{\mathrm{loc}}^{1,p}(\mathbb{R}^m; W^{1,p}(G)) : \nabla u \in B_{\mathcal{A}}^p(\mathbb{R}^m; L^p(G))^{m+N} \\ &\qquad \text{and } \int_G M(\nabla u(\cdot, \zeta)) d\zeta = 0\Big\}, \end{aligned}$$

where in this case $\nabla = (\nabla_y, \nabla_\zeta)$, ∇_y (resp. ∇_ζ) being the gradient operator with respect to the variable $y \in \mathbb{R}^m$ (resp. $\zeta \in \mathbb{R}^N$). As in $B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^m)$, we still identify two elements by their gradients in the sense that: $u = v$ in $B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^m; W^{1,p}(G))$ iff $\nabla(u - v) = 0$, i.e., $\int_G \|\nabla(u(\cdot, \zeta) - v(\cdot, \zeta))\|_p^p d\zeta = 0$. The space $B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^m; W^{1,p}(G))$ is therefore a Banach space under the norm $\|u\|_{\#,p} = (\int_G \|\nabla u(\cdot, \zeta)\|_p^p d\zeta)^{1/p}$.

3. Sigma-convergence for thin heterogeneous domains

3.1. Sigma-convergence in thin heterogeneous domains with flat lateral boundaries

The concept of sigma-convergence was introduced in [23] in order to tackle two-scale phenomena occurring in media with microstructures that are distributed inside in a general deterministic way such as the periodic distribution, the almost periodic one and others. The concept was concerned with two-scale phenomena taking place in all space dimensions. In the special case of periodic structures, it has been generalized to thin heterogeneous media [22].

Our aim in this work is to provide a systematic study of the concept of sigma-convergence applied to thin heterogeneous domains whose heterogeneous structure is of general deterministic type including the periodic one and the almost periodic one as special cases. The compactness results obtained here generalize therefore those in [22] which are concerned only with periodic structures.

More precisely, let $d \geq 2$ be a given integer with $d = d_1 + d_2$, $d_1, d_2 \geq 1$ being integers. Let G_1 and G_2 be open bounded sets in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. We assume that $0 \in \overline{G}_2$ (the closure G_2 in $\mathbb{R}^{d_2}$), where 0 stands for the origin in $\mathbb{R}^{d_2}$. For $\varepsilon > 0$ a given small parameter, we define our thin domain by

$$G_\varepsilon = G_1 \times \varepsilon G_2.$$

When $\varepsilon \to 0$, G_ε shrinks to the "interface"

$$G_0 = G_1 \times \{0\}.$$

We note that in the definition of G_ε, the structure is heterogeneous in d_1 space dimension only.

The space $\mathbb{R}^m_\xi$ is the numerical space $\mathbb{R}^m$ of generic variable ξ. In this regard we set $\mathbb{R}^{d_1} = \mathbb{R}^{d_1}_{\overline{x}}$ or $\mathbb{R}^{d_1}_{\overline{y}}$ and $\mathbb{R}^{d_2} = \mathbb{R}^{d_2}_{\zeta}$, so that $x \in \mathbb{R}^d$ writes $(\overline{x}, \zeta)$. We identify G_0 with G_1 so that the generic element in G_0 is also denoted by $\overline{x}$ instead of $(\overline{x}, 0)$. Finally, we set $G = G_1 \times G_2 \equiv G_{\varepsilon=1}$.

This being so, let $\mathcal{A}$ be an algebra with mean value on $\mathbb{R}^{d_1}$. We denote by M the mean value on $\mathcal{A}$ as well as its extension on the associated generalized Besicovitch spaces $B^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2))$ and $\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2))$, $1 \leq p < \infty$.

We can now introduce the concept of sigma-convergence for thin heterogeneous domains.

Definition 3.1. (1) A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(G_\varepsilon)$ $(1 \leq p < \infty)$ is said to weakly Σ-converge in $L^p(G_\varepsilon)$ to some $u_0 \in L^p(G_0; \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)))$ if as $\varepsilon \to 0$, we have

$$\varepsilon^{-d_2}\int_{G_\varepsilon} u_\varepsilon(x) f\left(\overline{x},\frac{x}{\varepsilon}\right)dx \to \int_{G_0}\int_{G_2} M(u_0(\overline{x},\cdot,\zeta)f(\overline{x},\cdot,\zeta))d\zeta d\overline{x}$$

for any $f \in L^{p'}(G_0;\mathcal{A}(\mathbb{R}^{d_1};L^{p'}(G_2)))$ ($\frac{1}{p'}=1-\frac{1}{p}$); we denote this by "$u_\varepsilon \to u_0$ in $L^p(G_\varepsilon)$-weak $\Sigma_{\mathcal{A}}$".

(2) The sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(G_\varepsilon)$ is said to strongly Σ-converge in $L^p(G_\varepsilon)$ to some $u_0 \in L^p(G_0;\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1};L^p(G_2)))$ if it is weakly Σ-convergent to u_0 and further satisfies, as $\varepsilon \to 0$,

$$\varepsilon^{-\frac{d_2}{p}}\|u_\varepsilon\|_{L^p(G_\varepsilon)} \to \|u_0\|_{L^p(G_0;\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1};L^p(G_2)))}. \tag{3.1}$$

We express this by writing "$u_\varepsilon \to u_0$ in $L^p(G_\varepsilon)$-strong $\Sigma_{\mathcal{A}}$".

Remark 3.2. It is easy to see that if $u_0 \in L^p(G_0;\mathcal{A}(\mathbb{R}^{d_1};L^p(G_2)))$ then (3.1) is equivalent to

$$\varepsilon^{-\frac{d_2}{p}}\|u_\varepsilon - u_0^\varepsilon\|_{L^p(G_\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0, \tag{3.2}$$

where $u_0^\varepsilon(x) = u_0(\overline{x}, x/\varepsilon)$ for $x \in G_\varepsilon$.

Before we state the first compactness result for this section, we need a further notation. Throughout the work, the letter E will stand for any ordinary sequence $(\varepsilon_n)_{n\geq 1}$ with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ when $n \to \infty$. The generic term of E will be merely denoted by ε and $\varepsilon \to 0$ will mean $\varepsilon_n \to 0$ as $n \to \infty$. This being so, the following result holds true.

Theorem 3.3. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(G_\varepsilon)$ ($1 < p < \infty$) such that*

$$\sup_{\varepsilon\in E} \varepsilon^{-\frac{d_2}{p}}\|u_\varepsilon\|_{L^p(G_\varepsilon)} \leq C$$

where C is a positive constant independent of ε. Then there exists a subsequence E' of E such that the sequence $(u_\varepsilon)_{\varepsilon\in E'}$ weakly Σ-converges in $L^p(G_\varepsilon)$ to some $u_0 \in L^p(G_0;\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1};L^p(G_2)))$.

The proof of the above theorem relies on the following two lemmas.

Lemma 3.4. *Let $1 \leq p < \infty$. For any $f \in L^p(G_0;\mathcal{A}(\mathbb{R}^{d_1};L^p(G_2)))$ one has*

$$\varepsilon^{-d_2}\int_{G_\varepsilon}\left|f\left(\overline{x},\frac{x}{\varepsilon}\right)\right|^p dx \leq \|f\|^p_{L^p(G_0;\mathcal{A}(\mathbb{R}^{d_1};L^p(G_2)))} \tag{3.3}$$

and

$$\varepsilon^{-d_2}\int_{G_\varepsilon}\left|f\left(\overline{x},\frac{x}{\varepsilon}\right)\right|^p dx \to \int_{G_0}\int_{G_2} M(|f(\overline{x},\cdot,\zeta)|^p)d\zeta d\overline{x}. \tag{3.4}$$

Proof. The proof of (3.3) is obvious by making the change of variables $\zeta = \frac{\widehat{x}}{\varepsilon} \in G_2$, where $\widehat{x} = (x_{d_1+1}, \ldots, x_d)$ for $x = (x_1, \ldots, x_{d_1}, x_{d_1+1}, \ldots, x_d)$. Let us now turn our attention on (3.4). To this end, let $f \in L^p(G_0; \mathcal{A}(\mathbb{R}^{d_1}; L^p(G_2)))$. Then for a.e. $\overline{x} \in G_0$, the function $f(\overline{x})$ belongs to $\mathcal{A}(\mathbb{R}^{d_1}; L^p(G_2))$, so that, defining $g : G_0 \times \mathbb{R}^{d_1} \to \mathbb{R}$ by $g(\overline{x}, y) = \|f(\overline{x}, y, \cdot)\|^p_{L^p(G_2)}$, we have $g \in L^1(G_0; \mathcal{A})$; this stems from the definition of $\mathcal{A}(\mathbb{R}^{d_1}; L^p(G_2))$. It follows from the mean value property that

$$\int_{G_0} g\left(\overline{x}, \frac{\overline{x}}{\varepsilon}\right) d\overline{x} \to \int_{G_0} M(g(\overline{x}, \cdot)) d\overline{x},$$

and

$$\begin{aligned} \int_{G_0} M(g(\overline{x}, \cdot)) d\overline{x} &= \int_{G_0} M\left(\int_{G_2} |f(\overline{x}, \cdot, \zeta)|^p d\zeta\right) d\overline{x} \\ &= \int_{G_0} \int_{G_2} M(|f(\overline{x}, \cdot, \zeta)|^p) d\zeta d\overline{x}, \end{aligned} \tag{3.5}$$

where for the last equality above we have used the continuity of the mean value operator.

On the other hand, it holds that

$$\begin{aligned} \varepsilon^{-d_2} \int_{G_\varepsilon} \left| f\left(\overline{x}, \frac{x}{\varepsilon}\right) \right|^p dx &= \int_{G_0} \int_{G_2} \left| f\left(\overline{x}, \frac{\overline{x}}{\varepsilon}, \zeta\right) \right|^p d\overline{x} d\zeta \\ &= \int_{G_0} \left\| f\left(\overline{x}, \frac{\overline{x}}{\varepsilon}, \cdot\right) \right\|^p_{L^p(G_2)} d\overline{x} = \int_{G_0} g\left(\overline{x}, \frac{\overline{x}}{\varepsilon}\right) d\overline{x}. \end{aligned}$$

Property (3.4) therefore follows readily from (3.5) and the last series of equalities above. ■

Lemma 3.5 ([24, Proposition 3.2]). *Let X be a subspace (not necessarily closed) of a reflexive Banach space Y and let $f_n : X \to \mathbb{R}$ be a sequence of linear functionals (not necessarily continuous). Assume there exists a constant $C > 0$ such that*

$$\limsup_n |f_n(x)| \le C \|x\|_Y \quad \textit{for all } x \in X.$$

Then there exist a subsequence $(f_{n_k})_k$ of (f_n) and a functional $f \in Y'$ such that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ *for all $x \in X$.*

We are now able to prove Theorem 3.3.

Proof of Theorem 3.3. In Lemma 3.5 we set $Y = L^{p'}(G_0; \mathcal{B}^{p'}_{\mathcal{A}}(\mathbb{R}^{d_1}; L^{p'}(G_2)))$, $X = L^{p'}(G_0; \mathcal{A}(\mathbb{R}^{d_1}; L^{p'}(G_2)))$ and define the mapping

$$L_\varepsilon(f) = \varepsilon^{-d_2} \int_{G_0} u_\varepsilon(x) f\left(\overline{x}, \frac{x}{\varepsilon}\right) dx, \quad f \in X.$$

Then

$$\limsup_{\varepsilon\to 0}|L_\varepsilon(f)| \le C\|f\|_Y \quad \text{for all } f\in X.$$

Indeed, one has the inequality (arising from Hölder's inequality)

$$|L_\varepsilon(f)| \le \varepsilon^{-\frac{d_2}{p}}\|u_\varepsilon\|_{L^p(G_\varepsilon)}\left(\varepsilon^{-d_2}\int_{G_\varepsilon}\left|f\left(\overline{x},\frac{x}{\varepsilon}\right)\right|^{p'}dx\right)^{\frac{1}{p'}}.$$

Thus, letting $\varepsilon\to 0$ we get with the help of Lemma 3.4,

$$\begin{aligned}\varepsilon^{-d_2}\int_{G_\varepsilon}\left|f\left(\overline{x},\frac{x}{\varepsilon}\right)\right|^{p'}dx &\to \int_{G_0}\int_{G_2}M(|f(\overline{x},\cdot,\zeta)|^{p'})d\zeta d\overline{x}\\ &= \|f\|_Y^{p'}.\end{aligned}$$

We therefore apply Lemma 3.5 with the above notation to derive the existence of a subsequence E' of E and of a unique $u_0\in Y' = L^p(G_0;\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1};L^p(G_2)))$ such that

$$L_\varepsilon(f)\to\int_{G_0}\int_{G_2}M(u_0(\overline{x},\cdot,\zeta)f(\overline{x},\cdot,\zeta))d\zeta d\overline{x}\quad\text{for all } f\in X. \qquad \blacksquare$$

Remark 3.6. In the proof Theorem 3.3, there is no separability assumption on the algebra with mean value $\mathcal{A}$. Thus it applies either for $\mathcal{A} = \mathcal{C}_{\text{per}}(Y)$ ($Y = (0,1)^{d_1}$) or for $\mathcal{A} = \text{AP}(\mathbb{R}^{d_1})$ (which is not separable). Our result generalizes the one in [22] (see, for instance, [22, Proposition 4.2] that corresponds to the special case $\mathcal{A} = \mathcal{C}_{\text{per}}(Y)$ of our results here for $d_1 = d-1$ and $G_2 = (-1,1)$).

Before we state the next compactness result, we need however some preliminary results. For a function $\mathbf{g} = (g_1,\dots,g_{d_1},g_{d_1+1},\dots,g_d)\in[\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1};L^p(G_2))]^d$ we define the divergence $\overline{\text{div}}_{\overline{y},\zeta}\mathbf{g}$ by

$$\overline{\text{div}}_{\overline{y},\zeta}\mathbf{g} := \sum_{i=1}^{d_1}\frac{\overline{\partial}g_i}{\partial y_i}+\sum_{i=1}^{d_2}\frac{\partial g_{d_1+i}}{\partial\zeta_i},$$

that is, for any $\Phi = (\phi_i)_{1\le i\le d}\in[\mathcal{B}^{1,p'}_{\mathcal{A}}(\mathbb{R}^{d_1};W^{1,p'}(G_2))]^d$,

$$\begin{aligned}\langle\overline{\text{div}}_{\overline{y},\zeta}\mathbf{g},\Phi\rangle = &-\sum_{i=1}^{d_1}\int_{G_2}M\left(g_i(\cdot,\zeta)\frac{\overline{\partial}\phi_i}{\partial y_i}(\cdot,\zeta)\right)d\zeta\\ &-\sum_{i=1}^{d_2}\int_{G_2}M\left(g_{d_1+i}(\cdot,\zeta)\frac{\partial\phi_{d_1+i}}{\partial y_{d_1+i}}(\cdot,\zeta)\right)d\zeta.\end{aligned}$$

Any function belonging to an algebra with mean value on $\mathbb{R}^m$ will be considered as defined on the numerical space $\mathbb{R}^m$ of generic variables $\overline{y} = (y_1,\dots,y_m)$.

This being so, the first preliminary result is the following one.

Proposition 3.7. *Let $1 < p < \infty$ and let $\mathcal{A}$ be an ergodic algebra with mean value on $\mathbb{R}^{d_1}$. Finally, let L be a bounded linear functional on $[\mathcal{B}_{\mathcal{A}}^{1,p'}(\mathbb{R}^{d_1}; W^{1,p'}(G_2))]^d$ ($\frac{1}{p'} = 1 - \frac{1}{p}$) that vanishes on the kernel of the divergence, that is,*

$$L(\Psi) = 0 \quad \textit{for all } \Psi \in \mathcal{V}_{\mathrm{div}} = \{\Phi \in [\mathcal{D}_{\mathcal{A}}(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2))]^d : \overline{\mathrm{div}}_{\overline{y},\zeta}\Phi = 0\}.$$

Then there exists a function $f \in \mathcal{B}_{\mathcal{A}}^p(\mathbb{R}^{d_1}; L^p(G_2))$ such that $L = \overline{\nabla}_{\overline{y},\zeta} f$ (where $\overline{\nabla}_{\overline{y},\zeta} = (\overline{\nabla}_{\overline{y}}, \nabla_\zeta)$), i.e.,

$$L(\mathbf{g}) = -\int_{G_2} M(f(\cdot,\zeta)\overline{\mathrm{div}}_{\overline{y},\zeta}\mathbf{g}(\cdot,\zeta))d\zeta, \quad \textit{for all } \mathbf{g} \in [\mathcal{B}_{\mathcal{A}}^{1,p'}(\mathbb{R}^{d_1}; W^{1,p'}(G_2))]^d.$$

Moreover, f is unique up to an additive constant, provided that G_2 is connected.

Since the proof of the above proposition is similar to that of [25, Theorem 2.1], we will only sketch the proof. Before we can do that, let us first give some preliminaries.

For $u \in \mathcal{A}^\infty \otimes \mathcal{C}_0^\infty(G_2)$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{d_1}) \otimes \mathcal{C}_0^\infty(G_2)$ one may easily show that $u * \varphi \in \mathcal{A}^\infty \otimes \mathcal{C}_0^\infty(G_2)$ (see, e.g., [33]). Using the density of $\mathcal{A}^\infty \otimes \mathcal{C}_0^\infty(G_2)$ (resp. $\mathcal{C}_0^\infty(\mathbb{R}^{d_1}) \otimes \mathcal{C}_0^\infty(G_2)$) in $\mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2))$ (resp. $\mathcal{C}_0^\infty(\mathbb{R}^{d_1} \times G_2)$), we show that $u * \varphi \in \mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2))$ for $u \in \mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2))$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{d_1} \times G_2)$. Here $*$ stands for the usual convolution operator. Using also a density argument, we may define $u * \varphi$ for $u \in B_{\mathcal{A}}^p(\mathbb{R}^{d_1}; L^p(G_2))$ $(1 \le p < \infty)$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{d_1} \times G_2)$, and we have $u * \varphi \in B_{\mathcal{A}}^p(\mathbb{R}^{d_1}; L^p(G_2))$. From the equality $D_{\overline{y},\zeta}^\alpha(u * \varphi) = u * D_{\overline{y},\zeta}^\alpha \varphi$, we deduce that in fact $u * \varphi \in \mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}^\infty(\overline{G}_2))$. Moreover, it holds that

$$\|u * \varphi\|_p \le |\mathrm{supp}\,\varphi|^{1/p} \|\varphi\|_{L^{p'}(\mathbb{R}^{d_1} \times G_2)} \|u\|_p, \tag{3.6}$$

where $\mathrm{supp}\,\varphi$ denotes the support of φ and $|\mathrm{supp}\,\varphi|$ its Lebesgue measure. To see this, we have

$$\begin{aligned}\|u * \varphi\|_p^p &= \int_{G_2} M(|u * \varphi|^p) d\zeta = \int_{G_2} \left(\lim_{R\to\infty} \fint_{B_R} |(u * \varphi)(\overline{y},\zeta)|^p d\overline{y} \right) d\zeta \\ &= \lim_{R\to\infty} \int_{G_2} \fint_{B_R} |(u * \varphi)(\overline{y},\zeta)|^p d\overline{y} d\zeta.\end{aligned}$$

But

$$\begin{aligned}&\int_{G_2} \int_{B_R} |(u * \varphi)(\overline{y},\zeta)|^p d\overline{y} d\zeta \\ &\quad \le \left(\int_{G_2} \int_{B_R} |\varphi| d\overline{y} d\zeta \right)^p \iint_{B_R \times G_2} |u|^p d\overline{y} d\zeta \\ &\quad \le |(B_R \times G_2) \cap \mathrm{supp}\,\varphi| \|\varphi\|_{L^{p'}(B_R \times G_2)}^p \iint_{B_R \times G_2} |u|^p d\overline{y} d\zeta,\end{aligned}$$

from which (3.6).

From the obvious inequality $\|\varphi\|_{L^{p'}(B_R\times G_2)} \leq |\mathrm{supp}\,\varphi|^{1/p'}\|\varphi\|_\infty$ we infer from (3.6) that

$$\|u*\varphi\|_p \leq \|u\|_p\|\varphi\|_\infty \quad \forall u \in B^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)),\ \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{d_1}\times G_2). \tag{3.7}$$

This being so, let us sketch the proof of Proposition 3.7.

Proof of Proposition 3.7. Let $u \in \mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2))$ be freely fixed, and define $L_u : \mathcal{D}(\mathbb{R}^{d_1}\times G_2)^d \to \mathbb{R}$ by $L_u(\varphi) = L(\varrho(u*\varphi))$ for $\varphi = (\varphi_i)_{1\leq i\leq d}$, where ϱ is defined in Section 2; see the lines after (2.2). Then since $L \in ([\mathcal{B}_A^{1,p'}(\mathbb{R}^{d_1}; W^{1,p'}(G_2))]^d)'$, one has

$$|L_u(\varphi)| \leq \|L\|\|u*\varphi\|_{1,p'} \leq \|L\|\|u\|_p\|\varphi\|_\infty,$$

where the last inequality above stems from both (3.7) and the equality $\nabla_{\overline{y},\zeta}(u*\varphi) = (\nabla_{\overline{y},\zeta}u)*\varphi$. So L_u defines a distribution on $\mathcal{D}(\mathbb{R}^{d_1}\times G_2)^d$. In addition, if $\mathrm{div}_{\overline{y},\zeta}\,\varphi = 0$, then $L_u(\varphi) = 0$, i.e., L_u vanishes on the kernel of the divergence in $\mathcal{D}(\mathbb{R}^{d_1}\times G_2)^d$. Appealing to the usual De Rham theorem, we get the existence of a distribution $S(u) \in \mathcal{D}'(\mathbb{R}^{d_1}\times G_2)$ such that $L_u = \nabla_{\overline{y},\zeta}S(u)$, thereby defining an operator

$$S : \mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2)) \to \mathcal{D}'(\mathbb{R}^{d_1}\times G_2); \quad u \mapsto S(u).$$

The operator S enjoys the properties:

(i) $S(u(\cdot + y)) = S(u)(\cdot - y)\forall y = (\overline{y},\zeta) \in \mathbb{R}^d, \forall u \in \mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2))$;

(ii) S maps continuously and linearly $\mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2))$ into $L^{p'}_{\mathrm{loc}}(\mathbb{R}^{d_1}\times G_2)$;

(iii) it holds that

$$\|S(u)\|_{L^{p'}(B_R\times G_2)} \leq C_R\|L\||B_R\times G_2|^{1/p'}\|u\|_{p'}$$

where $C_R > 0$ is a locally bounded function of $R > 0$.

Properties (i), (ii) and (iii) are easily obtained by following the same line of reasoning as for their homologues in [25]. Let us just point out that, for $\varphi \in \mathcal{D}(\mathbb{R}^{d_1}\times G_2)^d$ with $\mathrm{supp}\,\varphi_i \subset B_R\times G_2$ for all $1\leq i\leq d$, one has

$$|L_u(\varphi)| \leq \max_{1\leq i\leq d}|\mathrm{supp}\,\varphi_i|^{1/p'}\|L\|\|u\|_{p'}\|\varphi\|_{W^{1,p}(B_R\times G_2)},$$

so that, because of the fact that $\mathrm{supp}\,\varphi_i \subset B_R\times G_2$ $(1\leq i\leq d)$,

$$\|L_u\|_{W^{-1,p'}(B_R\times G_2)} \leq \|L\||B_R\times G_2|^{1/p'}\|u\|_{p'}.$$

It therefore follows that there exists $C = C(p,R) > 0$ such that

$$\|S(u)\|_{L^{p'}(B_R\times G_2)} \leq C\|L\||B_R\times G_2|^{1/p'}\|u\|_{p'};$$

for the last inequality above, see the proof of [25, Theorem 2.6]. Hence we obtain, as in [25, Theorem 2.6], that $S(u) \in \mathcal{C}^\infty(\mathbb{R}^{d_1} \times G_2)$ with $D_y^\alpha S(u) = S(D_y^\alpha u)$ for all $\alpha \in \mathbb{N}^d$, so that

$$|S(u)(0)| \leq C\|L\|\|u\|_{p'} \quad \text{for all } u \in \mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2)).$$

So we define $\widetilde{S} : \mathcal{D}_{\mathcal{A}}(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2)) \to \mathbb{R}$ by $\widetilde{S}(\varrho(u)) = S(u)(0)$ for $u \in \mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2))$. Then $\widetilde{S}$ is linear and satisfies

$$|\widetilde{S}(\varrho(u))| \leq C\|L\|\|u\|_{p'} \quad \text{for all } u \in \mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2)). \tag{3.8}$$

We derive from (3.8) together with the density of $\mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2))$ in $B_{\mathcal{A}}^{p'}(\mathbb{R}^{d_1}; L^{p'}(G_2))$ that there exists $f \in \mathcal{B}_{\mathcal{A}}^p(\mathbb{R}^{d_1}; L^p(G_2))$ such that

$$\widetilde{S}(v) = \int_{G_2} M(fv)d\zeta \quad \forall v \in \mathcal{B}_{\mathcal{A}}^{p'}(\mathbb{R}^{d_1}; L^{p'}(G_2))$$

and

$$\|f\|_p \leq C\|L\|.$$

As in [25], we obtain that $L = \overline{\nabla}_{\overline{y},\zeta} f$, and since $\mathcal{A}$ is ergodic and G_2 is connected, f is unique up to addition of a constant. ∎

The next corollary is of interest in the forthcoming compactness result.

Corollary 3.8. *Let $1 < p < \infty$ and let $\mathbf{f} \in [\mathcal{B}_{\mathcal{A}}^p(\mathbb{R}^{d_1}; L^p(G_2))]^d$ be such that*

$$\int_{G_2} M(\mathbf{f}(\cdot, \zeta) \cdot \mathbf{g}(\cdot, \zeta))d\zeta = 0 \quad \textit{for all } \mathbf{g} \in \mathcal{V}_{\text{div}},$$

where $\mathcal{V}_{\text{div}}$ is defined as in Proposition 3.7 and where G_2 is a connected open subset of $\mathbb{R}^{d_2}$. Then there exists a function $u \in B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^{d_1}; W^{1,p}(G_2))$, uniquely determined modulo constants, such that $\mathbf{f} = \nabla_{\overline{y},\zeta} u$, where $\nabla_{\overline{y},\zeta} = (\nabla_{\overline{y}}, \nabla_\zeta)$.

Proof. Let us first recall that $B_{\#\mathcal{A}}^{1,p}(\mathbb{R}^{d_1}; W^{1,p}(G_2))$ is the space of functions $u \in W_{\text{loc}}^{1,p}(\mathbb{R}^{d_1}; W^{1,p}(G_2))$ satisfying $\nabla_{\overline{y},\zeta} u \in (B_{\mathcal{A}}^p(\mathbb{R}^{d_1}; L^p(G_2)))^d$ and

$$\int_{G_2} M(\nabla_{\overline{y},\zeta} u(\cdot, \zeta))d\zeta = 0.$$

This being so, let $(\varphi_n)_{n\geq 1} \subset \mathcal{C}_0^\infty(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ be a mollifier satisfying $\varphi_n(-\overline{y}, -\zeta) = \varphi_n(\overline{y}, \zeta)$ for all $(\overline{y}, \zeta) \in \mathbb{R}^d$. We extend $\mathbf{f}$ by 0 outside $\mathbb{R}^{d_1} \times G_2$ and we still denote by $\mathbf{f}$ its extension on $\mathbb{R}^d$. We define the convolution product $\mathbf{f}_n := \mathbf{f} \circledast \varphi_n \equiv (f_i \circledast \varphi_n)_{1\leq i\leq d}$ as follows: let $\mathbf{f}_0$ be a representative of $\mathbf{f}$, that is $\mathbf{f} = \mathbf{f}_0 + \mathcal{N}$ where

$\mathbf{f}_0 \in (B^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)))^d$; we know that $\mathbf{f}_0 * \varphi_n$ is well defined as an element of $(B^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)))^d$ (see, e.g., [25, page 9]). We therefore set

$$\mathbf{f} \circledast \varphi_n := \mathbf{f}_0 * \varphi_n + \mathcal{N} = \varrho(\mathbf{f}_0 * \varphi_n).$$

As in [25, page 9] we can easily show that $\mathbf{f}_n \equiv \mathbf{f} \circledast \varphi_n \in [\mathcal{D}_{\mathcal{A}}(\mathbb{R}^{d_1}; \mathcal{C}^\infty(\overline{G}_2))]^d$ with $\overline{D}^{\alpha}_{\overline{y},\zeta}\mathbf{f}_n = \varrho(\mathbf{f}_0 * D^{\alpha}_{\overline{y},\zeta}\varphi_n)$ for all $\alpha \in \mathbb{N}^d$. Moreover, from the convergence result

$$\mathbf{f}_0 * \varphi_n \to \mathbf{f}_0 \quad \text{in } (B^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)))^d \text{ as } n \to \infty,$$

we infer

$$\mathbf{f}_n \to \mathbf{f} \quad \text{in } (\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)))^d \text{ when } n \to \infty.$$

It further holds (using the equality $\varphi_n(-\overline{y}, -\zeta) = \varphi_n(\overline{y}, \zeta)$) that

$$\int_{G_2} M(\mathbf{f}_n(\cdot, \zeta) \cdot \mathbf{g}(\cdot, \zeta))d\zeta = \int_{G_2} M(\mathbf{f}(\cdot, \zeta) \cdot (\mathbf{g} \circledast \varphi_n)(\cdot, \zeta))d\zeta$$

for any $\mathbf{g} \in [\mathcal{D}_{\mathcal{A}}(\mathbb{R}^{d_1}; \mathcal{C}^\infty_0(G_2))]^d$, so that, if $\overline{\operatorname{div}}_{\overline{y},\zeta}\mathbf{g} = 0$, then $\int_{G_2} M(\mathbf{f}_n(\cdot, \zeta) \cdot \mathbf{g}(\cdot, \zeta))d\zeta = 0$.

Now we define the mapping

$$\mathbf{g} \mapsto \int_{G_2} M(\mathbf{f}_n(\cdot, \zeta) \cdot \mathbf{g}(\cdot, \zeta))d\zeta,$$

which is easily seen to belong to $([\mathcal{B}^{1,p'}_{\mathcal{A}}(\mathbb{R}^{d_1}; W^{1,p'}(G_2))]^d)'$. We deduce from Proposition 3.7 the existence of $u_n \in \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2))$ such that

$$\mathbf{f}_n = \overline{\nabla}_{\overline{y},\zeta} u_n. \tag{3.9}$$

Since $\mathbf{f}_n \in [\mathcal{D}_{\mathcal{A}}(\mathbb{R}^{d_1}; \mathcal{C}^\infty(\overline{G}_2))]^d$, it follows that $u_n \in \mathcal{D}_{\mathcal{A}}(\mathbb{R}^{d_1}; \mathcal{C}^\infty(\overline{G}_2))$. Hence, identifying u_n with any of its representative in $\mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}^\infty(\overline{G}_2))$ and using the uniqueness of its gradient, we get that $u_n \in B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^{d_1}; W^{1,p}(G_2))$. The sequence $(\mathbf{f}_n)_n$ being convergent in the norm topology of $(\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)))^d$, the sequence $(u_n)_n$ is a Cauchy sequence in $B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^{d_1}; W^{1,p}(G_2))$ for if

$$\begin{aligned}\|u_n - u_m\|^p_{\#,p} &= \int_{G_2} \|\nabla_{\overline{y},\zeta} u_n(\cdot, \zeta) - \nabla_{\overline{y},\zeta} u_m(\cdot, \zeta)\|^p_p d\zeta \\ &= \int_{G_2} \|\mathbf{f}_n(\cdot, \zeta) - \mathbf{f}_m(\cdot, \zeta)\|^p_p d\zeta \to 0 \quad \text{when } n, m \to \infty.\end{aligned}$$

It follows that there exists $u \in B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^{d_1}; W^{1,p}(G_2))$ such that $u_n \to u$ in $B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^{d_1}; W^{1,p}(G_2))$. From (3.9) we get readily $\mathbf{f} = \overline{\nabla}_{\overline{y},\zeta} u$. ∎

We are now able to state and prove the next compactness result dealing with the convergence of the gradient.

Theorem 3.9. *Assume that $\mathcal{A}$ is an ergodic algebra with mean value on $\mathbb{R}^{d_1}$ and that G_2 is connected. Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $W^{1,p}(G_\varepsilon)$ $(1 < p < \infty)$ such that*

$$\sup_{\varepsilon\in E}\left(\varepsilon^{-d_2/p}\|u_\varepsilon\|_{L^p(G_\varepsilon)} + \varepsilon^{-d_2/p}\|\nabla u_\varepsilon\|_{L^p(G_\varepsilon)}\right) \le C \tag{3.10}$$

where $C > 0$ is independent of ε. Then there exist a subsequence E' of E and a couple (u_0, u_1) with $u_0 \in W^{1,p}(G_0)$ and $u_1 \in L^p(G_0; B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^{d_1}; W^{1,p}(G_2)))$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \qquad \text{in } L^p(G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}}, \tag{3.11}$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial \overline{y}_i} \qquad \text{in } L^p(G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}},\ 1 \le i \le d_1, \tag{3.12}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_{d_1+i}} \to \frac{\partial u_1}{\partial \zeta_i} \qquad \text{in } L^p(G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}},\ 1 \le i \le d_2. \tag{3.13}$$

Remark 3.10. If we set $\zeta = (y_{d_1+1}, \dots, y_d)$ (so that $(\overline{y}, \zeta) = y$) and

$$\nabla_{\overline{x}} u_0 = \left(\frac{\partial u_0}{\partial x_1}, \dots, \frac{\partial u_0}{\partial x_{d_1}}, 0, \dots, 0\right),$$

then (3.12) and (3.13) are equivalent to

$$\nabla u_\varepsilon \to \nabla_{\overline{x}} u_0 + \nabla_y u_1 \qquad \text{in } L^p(G_\varepsilon)^d\text{-weak } \Sigma_{\mathcal{A}}.$$

Proof of Theorem 3.9. In view of the assumption (3.10), we appeal to Theorem 3.3 to derive the existence of a subsequence E' of E and $u_0 \in L^p(G_0; \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)))$ and $\mathbf{v} = (v_i)_{1\le i\le d} \in [L^p(G_0; \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)))]^d$ such that

$$u_\varepsilon \to u_0 \quad \text{in } L^p(G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}}, \tag{3.14}$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \to v_i \quad \text{in } L^p(G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}},\ 1 \le i \le d_1, \tag{3.15}$$

and

$$\frac{\partial u_\varepsilon}{\partial \zeta_i} \to v_{d_1+i} \quad \text{in } L^p(G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}},\ 1 \le i \le d_2, \tag{3.16}$$

where for $x = (x_1, \dots, x_{d_1}, x_{d_1+1}, \dots, x_d)$ we set $x = (\overline{x}, \zeta)$ with $\overline{x} = (x_i)_{1\le i\le d_1}$ and $\zeta = (x_{d_1+i})_{1\le i\le d_2}$ and thus, $\nabla = (\nabla_{\overline{x}}, \nabla_\zeta)$. Let us first show that u_0 does not depend on $(\overline{y}, \zeta)$. To that end, let $\Phi \in (\mathcal{C}_0^\infty(G_0) \otimes \mathcal{A}^\infty \otimes \mathcal{C}_0^\infty(G_2))^{d_1}$. One has

$$\begin{aligned}
&\varepsilon^{-d_2}\int_{G_\varepsilon} \varepsilon \nabla_{\overline{x}} u_\varepsilon(x) \cdot \Phi\left(\overline{x}, \frac{x}{\varepsilon}\right) dx \\
&\quad = -\int_{G_\varepsilon} \varepsilon^{-d_2} u_\varepsilon(x)\left(\varepsilon(\operatorname{div}_{\overline{x}} \Phi)\left(\overline{x}, \frac{x}{\varepsilon}\right) + (\operatorname{div}_{\overline{y}} \Phi)\left(\overline{x}, \frac{x}{\varepsilon}\right)\right) dx.
\end{aligned}$$

Letting $E' \ni \varepsilon \to 0$ and using (3.14)–(3.15), we get

$$\int_{G_0}\int_{G_2} M(u_0(\overline{x},\cdot,\zeta)\,\mathrm{div}_{\overline{y}}\,\Phi(\overline{x},\cdot,\zeta))d\zeta d\overline{x} = 0.$$

This shows that $\overline{\nabla}_{\overline{y}}u_0(\overline{x},\cdot,\zeta) = 0$ for a.e. $(\overline{x},\zeta)$, which amounts to $u_0(\overline{x},\cdot,\zeta)$ is an invariant function. Since the algebra $\mathcal{A}$ is ergodic, $u_0(\overline{x},\cdot,\zeta)$ does not depend on $\overline{y}$, that is, $u_0(\overline{x},\cdot,\zeta) = u_0(\overline{x},\zeta)$.

Let us now show that u_0 is independent of ζ. Let this time $\Phi \in (\mathcal{C}_0^\infty(G_0) \otimes \mathcal{C}_0^\infty(G_2))^{d_2}$. It is easily seen that

$$\varepsilon^{-d_2}\int_{G_\varepsilon} \varepsilon\nabla_\zeta u_\varepsilon \cdot \Phi\left(\overline{x},\frac{\zeta}{\varepsilon}\right)dx = -\int_{G_\varepsilon}\varepsilon^{-d_2}u_\varepsilon(\mathrm{div}_\zeta\,\Phi)\left(\overline{x},\frac{\zeta}{\varepsilon}\right)dx.$$

Letting once again $E' \ni \varepsilon \to 0$ and using (3.14) and (3.16), we obtain

$$\int_{G_0}\int_{G_2} u_0(\overline{x},\zeta)\,\mathrm{div}_\zeta\,\Phi(\overline{x},\zeta)d\zeta d\overline{x} = 0,$$

which shows that u_0 is independent of ζ. Thus $u_0(\overline{x},\zeta) = u_0(\overline{x})$.

Next, let $\Phi_\varepsilon(x) = \varphi(\overline{x})\Psi(x/\varepsilon)$ $(x \in G_\varepsilon)$ with $\varphi \in \mathcal{C}_0^\infty(G_0)$ and $\Psi \in (\mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2))^d$ with $\mathrm{div}_{\overline{y},\zeta}\,\Psi = 0$. We set $\Psi = (\Psi_{\overline{x}},\Psi_\zeta)$ with $\Psi_{\overline{x}} = (\psi_j)_{1\le j\le d_1}$ and $\Psi_\zeta = (\psi_{d_1+j})_{1\le j\le d_2}$. We clearly have

$$\begin{aligned}\int_{G_\varepsilon}\varepsilon^{-d_2}\left(\nabla_{\overline{x}}u_\varepsilon(x)\cdot\Psi_{\overline{x}}\left(\frac{x}{\varepsilon}\right) + \nabla_\zeta u_\varepsilon(x)\cdot\Psi_\zeta\left(\frac{x}{\varepsilon}\right)\right)\varphi(\overline{x})dx \\ = -\int_{G_\varepsilon}\varepsilon^{-d_2}u_\varepsilon(x)\Psi_{\overline{x}}\left(\frac{x}{\varepsilon}\right)\cdot\nabla_{\overline{x}}\varphi(\overline{x})dx.\end{aligned} \tag{3.17}$$

Letting $E' \ni \varepsilon \to 0$ in (3.17) yields

$$\begin{aligned}\int_{G_0}\int_{G_2} M(\mathbf{v}(\overline{x},\cdot,\zeta)\cdot\Psi(\cdot,\zeta))\varphi(\overline{x})d\overline{x}d\zeta \\ = -\int_{G_0}\int_{G_2} u_0(\overline{x})M(\Psi_{\overline{x}}(\cdot,\zeta))\cdot\nabla_{\overline{x}}\varphi(\overline{x})d\overline{x}d\zeta.\end{aligned} \tag{3.18}$$

First, taking in (3.18) $\Psi = (\varphi\delta_{ij})_{1\le i\le d}$ (for each fixed $1 \le j \le d$) with $\varphi \in \mathcal{C}_0^\infty(G_0)$ and where δ_{ij} are the Kronecker delta, we obtain

$$\int_{G_0}\left(\int_{G_2} M(v_j(\overline{x},\cdot,\zeta))d\zeta\right)\varphi(\overline{x})d\overline{x} = -|G_2|\int_{G_0} u_0\frac{\partial\varphi}{\partial x_j}d\overline{x}, \tag{3.19}$$

where $\mathbf{v} = (v_j)_{1\le j\le d}$ and $|G_2|$ stands for the Lebesgue measure of G_2. Recalling that $v_j \in L^p(G_0; \mathcal{B}_{\mathcal{A}}^p(\mathbb{R}^{d_1}; L^p(G_2)))$, we infer that the function $\overline{x} \mapsto \int_{G_2} M(v_j(\overline{x},\cdot,\zeta))d\zeta$ belongs to $L^p(G_0)$, so that (3.19) yields $\partial u_0/\partial x_j \in L^p(G_0)$ for $1 \le j \le d_1$, where

$\partial u_0/\partial x_j$ is the distributional derivative of u_0 with respect to x_j. We deduce that $u_0 \in W^{1,p}(G_0)$. Coming back to (3.18) and integrating its right-hand side with respect to $\overline{x}$, we have

$$\begin{aligned}\int_{G_0}\int_{G_2} & M(\mathbf{v}(\overline{x},\cdot,\zeta)\cdot\Psi(\cdot,\zeta))\varphi(\overline{x})d\overline{x}d\zeta \\ &= \int_{G_0}\int_{G_2} \nabla_{\overline{x}}u_0(\overline{x})\cdot M(\Psi_{\overline{x}}(\cdot,\zeta))\varphi(\overline{x})d\overline{x}d\zeta \\ &= \int_{G_0}\int_{G_2} \nabla_{\overline{x}}u_0(\overline{x})\cdot M(\Psi(\cdot,\zeta))\varphi(\overline{x})d\overline{x}d\zeta,\end{aligned}$$

where the last equality above arises from the equality

$$\nabla_{\overline{x}}u_0 = \left(\frac{\partial u_0}{\partial x_1},\ldots,\frac{\partial u_0}{\partial x_{d_1}},0,\ldots,0\right).$$

We obtain readily

$$\int_{G_0}\left(\int_{G_2} M((\mathbf{v}(\overline{x},\cdot,\zeta)-\nabla_{\overline{x}}u_0(\overline{x}))\cdot\Psi(\cdot,\zeta))d\zeta\right)\varphi(\overline{x})d\overline{x} = 0. \tag{3.20}$$

From the arbitrariness of φ, (3.20) entails

$$\int_{G_2} M((\mathbf{v}(\overline{x},\cdot,\zeta)-\nabla_{\overline{x}}u_0(\overline{x}))\cdot\Psi(\cdot,\zeta))d\zeta = 0 \quad \text{for a.e. } \overline{x}\in G_0,$$

and for all $\Psi \in (\mathcal{A}^\infty(\mathbb{R}^{d_1};\mathcal{C}_0^\infty(G_2)))^d$ with $\operatorname{div}_{\overline{y},\zeta}\Psi = 0$. We make use of Corollary 3.8 to deduce the existence of $u_1(\overline{x},\cdot,\cdot)\in B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^{d_1};W^{1,p}(G_2))$ such that

$$\mathbf{v}(\overline{x},\cdot,\zeta)-\nabla_{\overline{x}}u_0(\overline{x}) = \nabla_{\overline{y},\zeta}u_1(\overline{x},\cdot,\cdot) \quad \text{for a.e. } \overline{x}\in G_0.$$

Hence the existence of a function $\overline{x}\mapsto u_1(\overline{x},\cdot,\cdot)$ from G_0 into $B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^{d_1};W^{1,p}(G_2))$, which belongs to $L^p(G_0;B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^{d_1};W^{1,p}(G_2)))$, such that $\mathbf{v} = \nabla_{\overline{x}}u_0 + \nabla_{\overline{y},\zeta}u_1$. ∎

The following result provides us with sufficient conditions for which the convergence result in (3.11) is strong.

Theorem 3.11. *The assumptions are those of Theorem 3.9. Assume that G_1 is regular enough so that the embedding $W^{1,p}(G_1)\hookrightarrow L^p(G_1)$ is compact and further G_2 is convex. Let (u_0,u_1) and E' be as in Theorem 3.9. Then, as $E'\ni\varepsilon\to 0$, the conclusions of Theorem 3.9 hold and further*

$$u_\varepsilon \to u_0 \quad \textit{in } L^p(G_\varepsilon)\textit{-strong } \Sigma_{\mathcal{A}}.$$

Proof. Let us first define the average M_ε in the thin directions as follows:

$$(M_\varepsilon u_\varepsilon)(\overline{x}) = \fint_{\varepsilon G_2} u_\varepsilon(\overline{x}, \zeta) d\zeta \quad \text{for } \overline{x} \in G_1.$$

Then the Lebesgue dominated convergence theorem yields $M_\varepsilon \nabla_{\overline{x}} = \nabla_{\overline{x}} M_\varepsilon$. It follows therefore that $M_\varepsilon u_\varepsilon \in W^{1,p}(G_1)$ with

$$\|M_\varepsilon u_\varepsilon\|_{W^{1,p}(G_1)} \leq C \tag{3.21}$$

where C is a positive constant independent of ε, the last inequality above being a consequence of (3.10). Next the following Poincaré–Wirtinger inequality holds:

$$\varepsilon^{-\frac{d_2}{p}} \|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^p(G_\varepsilon)} \leq C\varepsilon \|\nabla u_\varepsilon\|_{L^p(G_\varepsilon)}, \tag{3.22}$$

where $C > 0$ is independent of ε. Indeed, from the density of $\mathcal{C}^1(\overline{G_\varepsilon})$ in $W^{1,p}(G_\varepsilon)$, we may assume, without lost of generality, that u_ε is smooth enough. In that case, one has, for $\xi \in \varepsilon G_2$,

$$\begin{aligned} u_\varepsilon(\overline{x}, \xi) - M_\varepsilon u_\varepsilon(\overline{x}) &= \fint_{\varepsilon G_2} (u_\varepsilon(\overline{x}, \xi) - u_\varepsilon(\overline{x}, z)) dz \\ &= \fint_{\varepsilon G_2} \left(\int_0^1 \nabla_\zeta u_\varepsilon(\overline{x}, z + t(\xi - z)) \cdot (\xi - z) dt \right) dz, \end{aligned}$$

so that, using Young's and Hölder's inequalities,

$$\begin{aligned} |u_\varepsilon(\overline{x}, \xi) - M_\varepsilon u_\varepsilon(\overline{x})|^p &\leq \fint_{\varepsilon G_2} \int_0^1 |\nabla_\zeta u_\varepsilon(\overline{x}, z + t(\xi - z))|^p |\xi - z|^p dt dz \\ &\leq \fint_{\varepsilon G_2} |\xi - z|^p dz \left(\int_{\varepsilon G_2} |\nabla_\zeta u_\varepsilon(\overline{x}, \eta)|^p d\eta \right) \\ &\leq C\varepsilon^p \int_{\varepsilon G_2} |\nabla_\zeta u_\varepsilon(\overline{x}, \eta)|^p d\eta, \end{aligned}$$

where $C > 0$ depends only on G_2 and d_2. Integrating over G_ε, the last series of inequalities above gives (3.22).

With all that in mind, we infer from both (3.21) and the compactness of the embedding $W^{1,p}(G_1) \hookrightarrow L^p(G_1)$, the existence of a subsequence of E' not relabeled, such that, as $E' \ni \varepsilon \to 0$,

$$M_\varepsilon u_\varepsilon \to u_0 \quad \text{in } L^p(G_1)\text{-strong}. \tag{3.23}$$

Now, inequality (3.22) yields, as $E' \ni \varepsilon \to 0$,

$$\varepsilon^{-\frac{d_2}{p}} \|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^p(G_\varepsilon)} \to 0. \tag{3.24}$$

Next, we have

$$\varepsilon^{-\frac{d_2}{p}}\|u_\varepsilon - u_0\|_{L^p(G_\varepsilon)} \le \varepsilon^{-\frac{d_2}{p}}\|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^p(G_\varepsilon)} + \varepsilon^{-\frac{d_2}{p}}\|M_\varepsilon u_\varepsilon - u_0\|_{L^p(G_\varepsilon)},$$

and

$$\varepsilon^{-\frac{d_2}{p}}\|M_\varepsilon u_\varepsilon - u_0\|_{L^p(G_\varepsilon)} = |G_2|^{\frac{1}{p}}\|M_\varepsilon u_\varepsilon - u_0\|_{L^p(G_1)}.$$

It follows readily from (3.23) and (3.24) that, as $E' \ni \varepsilon \to 0$,

$$\varepsilon^{-\frac{d_2}{p}}\|u_\varepsilon - u_0\|_{L^p(G_\varepsilon)} \to 0.$$

The proof is complete. ∎

The next result and its corollary are proved exactly as their homologues in [31, Theorem 6 and Corollary 5] (see also [34]).

Theorem 3.12. *Let $1 < p, q < \infty$ and $r \ge 1$ be such that $1/r = 1/p + 1/q \le 1$. Assume $(u_\varepsilon)_{\varepsilon\in E} \subset L^q(G_\varepsilon)$ is weakly $\Sigma_{\mathcal{A}}$-convergent in $L^q(G_\varepsilon)$ to some $u_0 \in L^q(G_0; \mathcal{B}^q_{\mathcal{A}}(\mathbb{R}^{d_1}; L^q(G_2)))$, and $(v_\varepsilon)_{\varepsilon\in E} \subset L^p(G_\varepsilon)$ is strongly $\Sigma_{\mathcal{A}}$-convergent in $L^p(G_\varepsilon)$ to some $v_0 \in L^p(G_0; \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)))$. Then the sequence $(u_\varepsilon v_\varepsilon)_{\varepsilon\in E}$ is weakly $\Sigma_{\mathcal{A}}$-convergent in $L^r(G_\varepsilon)$ to $u_0 v_0$.*

Corollary 3.13. *Let $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(G_\varepsilon)$ and $(v_\varepsilon)_{\varepsilon\in E} \subset L^{p'}(G_\varepsilon) \cap L^\infty(G_\varepsilon)$ ($1 < p < \infty$ and $p' = p/(p-1)$) be two sequences such that:*

(i) *$u_\varepsilon \to u_0$ in $L^p(G_\varepsilon)$-weak $\Sigma_{\mathcal{A}}$;*

(ii) *$v_\varepsilon \to v_0$ in $L^{p'}(G_\varepsilon)$-strong $\Sigma_{\mathcal{A}}$;*

(iii) *$(v_\varepsilon)_{\varepsilon\in E}$ is bounded in $L^\infty(G_\varepsilon)$.*

Then $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^p(G_\varepsilon)$-weak $\Sigma_{\mathcal{A}}$.

Another important result is the following proposition.

Proposition 3.14. *Assume that $\mathcal{A}$ is an ergodic algebra with mean value on $\mathbb{R}^{d_1}$ and further that G_2 is connected. Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $W^{1,p}(G_\varepsilon)$ such that*

$$\sup_{\varepsilon\in E}\left(\varepsilon^{-\frac{d_2}{p}}\|u_\varepsilon\|_{L^p(G_\varepsilon)} + \varepsilon^{1-\frac{d_2}{p}}\|\nabla u_\varepsilon\|_{L^p(G_\varepsilon)}\right) \le C$$

where $C > 0$ is independent of ε. Then there exist a subsequence E' of E and a function $u \in L^p(G_0; B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^{d_1}; W^{1,p}(G_2)))$ with $u_0 = \varrho(u) \in L^p(G_0; \mathcal{B}^{1,p}_{\mathcal{A}}(\mathbb{R}^{d_1}; W^{1,p}(G_2)))$ such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u_0 \quad \text{in } L^p(G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}},$$

and

$$\varepsilon\nabla u_\varepsilon \to \nabla_{\overline{y},\zeta} u \quad \text{in } L^p(G_\varepsilon)^d\text{-weak } \Sigma_{\mathcal{A}}.$$

Proof. From Theorem 3.3, we can find a subsequence E' from E and a couple $(u_0, u_1) \in L^p(G_0; \mathcal{B}^p_A(\mathbb{R}^{d_1}; L^p(G_2))) \times L^p(G_0; \mathcal{B}^p_A(\mathbb{R}^{d_1}; L^p(G_2)))^d$ such that, as $E' \ni \varepsilon \to 0$,

$$\begin{aligned} u_\varepsilon &\to u_0 \quad \text{in } L^p(G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}}, \\ \varepsilon \nabla u_\varepsilon &\to u_1 \quad \text{in } L^p(G_\varepsilon)^d\text{-weak } \Sigma_{\mathcal{A}}. \end{aligned}$$

Let us characterize u_1 in terms of u_0. To that end, let $\Phi \in (\mathcal{C}_0^\infty(G_0) \otimes \mathcal{A}^\infty(\mathbb{R}^{d_1}; \mathcal{C}_0^\infty(G_2)))^d$; then we have

$$\varepsilon^{-d_2} \int_{G_\varepsilon} \varepsilon \nabla u_\varepsilon \cdot \Phi^\varepsilon dx = -\varepsilon^{-d_2} \int_{G_\varepsilon} u_\varepsilon \left[(\operatorname{div}_{\overline{x}} \Phi)^\varepsilon + \frac{1}{\varepsilon} (\operatorname{div}_y \Phi)^\varepsilon \right] dx.$$

Letting $E' \ni \varepsilon \to 0$, we get

$$\begin{aligned} &\int_{G_0} \int_{G_2} M(u_1(\overline{x}, \cdot, \zeta) \cdot \Phi(\overline{x}, \cdot, \zeta)) d\zeta d\overline{x} \\ &\qquad = - \int_{G_0} \int_{G_2} M(u_0(\overline{x}, \cdot, \zeta) \operatorname{div}_y \Phi(\overline{x}, \cdot, \zeta)) d\zeta d\overline{x}. \end{aligned} \tag{3.25}$$

This shows that $u_1 = \overline{\nabla}_{\overline{y}, \zeta} u_0$, so that $u_0 \in L^p(G_0; \mathcal{B}^{1,p}_{\mathcal{A}}(\mathbb{R}^{d_1}; W^{1,p}(G_2)))$.

Now, coming back to (3.25) and choosing Φ such that $\operatorname{div}_y \Phi = 0$, we readily get

$$\int_{G_0} \int_{G_2} M(u_1(\overline{x}, \cdot, \zeta) \cdot \Phi(\overline{x}, \cdot, \zeta)) d\zeta d\overline{x} = 0 \quad \text{for all such } \Phi.$$

Owing to Corollary 3.8, there exists $u \in L^p(G_0; B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^{d_1}; W^{1,p}(G_2)))$ such that $u_1 = \nabla_{\overline{y}, \zeta} u$. This yields, since $\mathcal{A}$ is ergodic and G_2 is connected, that $u_0 = \varrho(u) + c$ where c is a constant possibly depending on $\overline{x}$. This shows that u_0 actually belongs to $L^p(G_0; B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^{d_1}; W^{1,p}(G_2)))$ with $\overline{\nabla}_{\overline{y}, \zeta} u_0 = \nabla_{\overline{y}, \zeta} u$. This concludes the proof. ■

3.2. Sigma-convergence in thin heterogeneous domains with oscillating boundaries

For the sake of simplicity, we assume here that $d_1 = d - 1$ and $d_2 = 1$, where integer $d \geq 2$. Let $h_1, h_2 \in W^{1,\infty}(\mathbb{R}^{d-1})$ be two bounded Lipschitz continuous functions defined on $\mathbb{R}^{d-1}$ and satisfying $\max_{\mathbb{R}^{d-1}} h_1 < \min_{\mathbb{R}^{d-1}} h_2$. Let Ω be a bounded open Lipschitz domain in $\mathbb{R}^{d-1}$. We define the thin heterogeneous domain with oscillating boundaries, Ω^ε in $\mathbb{R}^d$, as follows:

$$\Omega^\varepsilon = \left\{ x = (\overline{x}, x_d) \in \mathbb{R}^d : \overline{x} \in \Omega \text{ and } \varepsilon h_1\left(\frac{\overline{x}}{\varepsilon}\right) < x_d < \varepsilon h_2\left(\frac{\overline{x}}{\varepsilon}\right) \right\}.$$

We set

$$h_1^- = \min_{\mathbb{R}^{d-1}} h_1 \quad \text{and} \quad h_2^+ = \max_{\mathbb{R}^{d-1}} h_2, \quad I = (h_1^-, h_2^+),$$

and we define $G_\varepsilon = \Omega \times (\varepsilon h_1^-, \varepsilon h_2^+)$. Then G_ε has flat lateral boundaries $y_d = \varepsilon h_1^-, \varepsilon h_2^+$, and further, $\Omega^\varepsilon \subset G_\varepsilon$. We also assume that $0 \in [h_1^-, h_2^+]$.

We set $\widetilde{u} = P_\varepsilon u := \chi_{\Omega^\varepsilon} u$ for $u \in L^p(\Omega^\varepsilon)$, where $\chi_{\Omega^\varepsilon}$ stands for the characteristic function of the set Ω^ε in G_ε. Then $\widetilde{u} \in L^p(G_\varepsilon)$. Finally, we assume that

$$h_1, h_2 \in \mathcal{A}, \tag{3.26}$$

where $\mathcal{A}$ is an algebra with mean value on $\mathbb{R}^{d-1}$. We define the set

$$\mathbb{J} = \{y = (\overline{y}, y_d) \in \mathbb{R}^d : \overline{y} \in \mathbb{R}^{d-1} \text{ and } h_1(\overline{y}) < y_d < h_2(\overline{y})\},$$

and we denote by $\chi_{(h_1,h_2)}$ the characteristic function of $\mathbb{J}$,

$$\chi_{\mathbb{J}}(y) = \chi_{(h_1(\overline{y}),h_2(\overline{y}))}(y_d) = \begin{cases} 1 & \text{if } y \in \mathbb{J}, \\ 0 & \text{elsewhere.} \end{cases}$$

The following result holds.

Theorem 3.15. *Let $(u_\varepsilon)_{\varepsilon \in E} \subset L^p(\Omega^\varepsilon)$ $(1 \leq p < \infty)$ be such that $P_\varepsilon u_\varepsilon \to u_0$ in $L^p(G_\varepsilon)$-weak $\Sigma_{\mathcal{A}}$ as $E \ni \varepsilon \to 0$, where $u_0 \in L^p(G_0; \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d-1}; L^p(I)))$. Then, as $E \ni \varepsilon \to 0$,*

$$\frac{1}{\varepsilon}\int_{\Omega^\varepsilon} u_\varepsilon(x) f\left(\overline{x}, \frac{x}{\varepsilon}\right) dx \to \int_{G_0}\int_I M(\chi_{(h_1,h_2)}(y_d) u_0(\overline{x}, \cdot, y_d) f(\overline{x}, \cdot, y_d)) dy_d d\overline{x} \tag{3.27}$$

for all $f \in L^{p'}(G_0; \mathcal{A}(\mathbb{R}^{d-1}; \mathcal{C}(\overline{I})))$, $1/p' = 1 - 1/p$.

Proof. The proof is done in two steps.

Step 1. Let p be as above. Let us first check that $\chi_{\mathbb{J}} \in B^p_{\mathcal{A}}(\mathbb{R}^{d-1}; L^p(I))$. To proceed with this, we need to check the following:

(1) The sequence $(\chi_{\mathbb{J}})^\varepsilon$ defined by $(\chi_{\mathbb{J}})^\varepsilon(x) = \chi_{(h_1(\overline{x}/\varepsilon),h_2(\overline{x}/\varepsilon))}(x_d/\varepsilon)$ for $x \in G_\varepsilon$ satisfies
$$\|(\chi_{\mathbb{J}})^\varepsilon\|_{L^p(G_\varepsilon)} \leq C\varepsilon^{\frac{1}{p}},$$
so that, up to a subsequence, it is weakly sigma-convergent towards some $u \in L^p(G_0; \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d-1}; L^p(I)))$;

(2) the limit u determined above, has the form $u = \varrho(\chi_{\mathbb{J}})$. As a result, $\chi_{\mathbb{J}} \in B^p_{\mathcal{A}}(\mathbb{R}^{d-1}; L^p(I))$.

Let us prove (1) above. We have

$$\begin{aligned}\|(\chi_{\mathbb{J}})^\varepsilon\|^p_{L^p(G_\varepsilon)} &= \int_\Omega \int_{\varepsilon I} \chi_{(h_1^\varepsilon(\overline{x}),h_2^\varepsilon(\overline{x}))}\left(\frac{x_d}{\varepsilon}\right) dx = \varepsilon \int_\Omega \int_I \chi_{(h_1^\varepsilon,h_2^\varepsilon)}(y_d) d\overline{x} dy_d \\ &\leq C\varepsilon, \quad C = C(\Omega, h_1^-, h_2^+) > 0 \quad \text{and} \quad h_i^\varepsilon(\overline{x}) = h_i(\overline{x}/\varepsilon).\end{aligned}$$

Thus, up to a subsequence of E not relabeled, we have that

$$(\chi_{\mathbb{J}})^\varepsilon \to u \quad \text{in } L^p(G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}},$$

where $u \in L^p(G_0; \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d-1}; L^p(I)))$.

Let us check point (2) above, that is, $u = \varrho(\chi_{\mathbb{J}})$, where ϱ is the canonical mapping of $B^p_{\mathcal{A}}(\mathbb{R}^{d-1}; L^p(I))$ into $\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d-1}; L^p(I))$. To this end, let $f \in L^{p'}(G_0; \mathcal{A}(\mathbb{R}^{d-1}; \mathcal{C}(\overline{I})))$. Then, up to the same subsequence as above, we have

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_{G_\varepsilon} \chi_{(h_1^\varepsilon(\overline{x}), h_2^\varepsilon(\overline{x}))}\left(\frac{x_d}{\varepsilon}\right) f\left(\overline{x}, \frac{x}{\varepsilon}\right) dx \\
&\quad = \int_\Omega \int_{h_1^-}^{h_2^+} \chi_{(h_1^\varepsilon(\overline{x}), h_2^\varepsilon(\overline{x}))}(y_d) f\left(\overline{x}, \frac{\overline{x}}{\varepsilon}, y_d\right) dy_d\, d\overline{x} \\
&\quad = \int_\Omega \int_{h_1^\varepsilon(\overline{x})}^{h_2^\varepsilon(\overline{x})} f\left(\overline{x}, \frac{\overline{x}}{\varepsilon}, y_d\right) dy_d\, d\overline{x} \\
&\quad = \int_\Omega \int_0^1 f\left(\overline{x}, \frac{\overline{x}}{\varepsilon}, (1-t)h_1^\varepsilon(\overline{x}) + t h_2^\varepsilon(\overline{x})\right)(h_2^\varepsilon(\overline{x}) - h_1^\varepsilon(\overline{x}))\, dt\, d\overline{x} \\
&\quad \to \int_\Omega \int_0^1 M([f(\overline{x}, \cdot, (1-t)h_1 + t h_2)](h_2 - h_1))\, d\overline{x} \\
&\quad = \int_\Omega M\left(\int_{h_1}^{h_2} f(\overline{x}, \cdot, y_d)\, dy_d\right) d\overline{x} \\
&\quad = \int_\Omega M\left(\int_{h_1^-}^{h_2^+} \chi_{\mathbb{J}}(\cdot, y_d) f(\overline{x}, \cdot, y_d)\, dy_d\right) d\overline{x} \\
&\quad = \int_\Omega \int_I M(\chi_{\mathbb{J}}(\cdot, y_d) f(\overline{x}, \cdot, y_d))\, dy_d\, d\overline{x},
\end{aligned}
$$

where here above, we have used the fact that, for any $t \in (0,1)$, the function $\overline{y} \mapsto f(\cdot, \overline{y}, (1-t)h_1(\overline{y}) + t h_2(\overline{y}))$ belongs to $L^p(G_0; \mathcal{A})$ together with the property of the mean value to obtain the part "$\to$" and the property (2.1) (the interchangeability of the integral and the mean value).

We infer from the uniqueness of the limit that $u = \varrho(\chi_{\mathbb{J}})$ since $\chi_{\mathbb{J}} \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$. This gives at once $\chi_{\mathbb{J}} \in B^p_{\mathcal{A}}(\mathbb{R}^{d-1}; L^p(I))$, as $\chi_{\mathbb{J}}$ is independent of $\overline{x} \in G_0$. As a byproduct we have $\chi_{\mathbb{J}} \in B^\infty_{\mathcal{A}}(\mathbb{R}^{d-1}; L^\infty(I))$.

Step 2. Let $f \in \mathcal{C}(\overline{G}_0) \otimes \mathcal{A}(\mathbb{R}^{d-1}; \mathcal{C}(\overline{I}))$. Since $\chi_{\mathbb{J}} \in B^{p'}_{\mathcal{A}}(\mathbb{R}^{d-1}; L^{p'}(I))$, we have that $\chi_{\mathbb{J}} f \in \mathcal{C}(\overline{G}_0) \otimes B^{p'}_{\mathcal{A}}(\mathbb{R}^{d-1}; L^{p'}(I))$. It can therefore be taken as test function, so that

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_{\Omega^\varepsilon} u_\varepsilon(x) f\left(\overline{x}, \frac{x}{\varepsilon}\right) dx &= \frac{1}{\varepsilon}\int_{G_\varepsilon} (P_\varepsilon u_\varepsilon)(x)(\chi_{\mathbb{J}})\left(\frac{x}{\varepsilon}\right) f\left(\overline{x}, \frac{x}{\varepsilon}\right) dx \\
&\to \int_{G_0} \int_{h_1^-}^{h_2^+} M(u_0(\overline{x}, \cdot, y_d)\chi_{\mathbb{J}}(\cdot, y_d) f(\overline{x}, \cdot, y_d))\, dy_d\, d\overline{x}.
\end{aligned}
$$

The convergence result (3.27) follows from the last convergence result above associated to the density of $\mathcal{C}(\overline{G}_0) \otimes \mathcal{A}(\mathbb{R}^{d-1}; \mathcal{C}(\overline{I}))$ in $L^{p'}(G_0; \mathcal{A}(\mathbb{R}^{d-1}; \mathcal{C}(\overline{I})))$. ∎

Theorems 3.3 and 3.9 have their evolutionary counterparts. To see this, we first need to state the time-dependent version of the sigma-convergence concept for thin heterogeneous domains. The domain G_ε is defined as in the beginning of this section. Let T be a positive real number. All the notations are as in this section.

A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p((0,T)\times G_\varepsilon)$ is said to

(i) weakly Σ-converge in $L^p((0,T)\times G_\varepsilon)$ to $u_0 \in L^p((0,T)\times G_0; \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)))$ if as $\varepsilon \to 0$,

$$\varepsilon^{-d_2}\int_{(0,T)\times G_\varepsilon} u_\varepsilon(t,x) f\left(t,\overline{x},\frac{x}{\varepsilon}\right)dxdt \to \int_{(0,T)\times G_0}\int_{G_2} M(u_0(t,\overline{x},\cdot,\zeta) f(t,\overline{x},\cdot,\zeta))d\zeta d\overline{x}dt$$

for any $f \in L^{p'}((0,T)\times G_0; \mathcal{A}(\mathbb{R}^{d_1}; L^{p'}(G_2)))$; we denote this by "$u_\varepsilon \to u_0$ in $L^p((0,T)\times G_\varepsilon)$-weak $\Sigma_{\mathcal{A}}$";

(ii) strongly Σ-converge in $L^p((0,T)\times G_\varepsilon)$ to $u_0 \in L^p((0,T)\times G_0; \mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1}; L^p(G_2)))$ if it is weakly sigma-convergent and further

$$\varepsilon^{-d_2/p}\|u_\varepsilon\|_{L^p((0,T)\times G_\varepsilon)} \to \|u_0\|_{L^p((0,T)\times G_0;\mathcal{B}^p_{\mathcal{A}}(\mathbb{R}^{d_1};L^p(G_2)))};$$

we denote this by "$u_\varepsilon \to u_0$ in $L^p((0,T)\times G_\varepsilon)$-strong $\Sigma_{\mathcal{A}}$".

The time-dependent versions of Theorems 3.3 and 3.9 are stated here below, and are proven exactly in the same way:

- Any sequence $(u_\varepsilon)_{\varepsilon\in E}$ in $L^p((0,T)\times G_\varepsilon)$ $(1<p<\infty)$ such that

$$\sup_{\varepsilon\in E}\varepsilon^{-d_2/p}\|u_\varepsilon\|_{L^p((0,T)\times G_\varepsilon)} < \infty$$

possesses a weakly Σ-convergent subsequence.

- Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(0,T;W^{1,p}(G_\varepsilon))$ $(1<p<\infty)$ such that

$$\sup_{\varepsilon\in E}\left(\varepsilon^{-d_2/p}(\|u_\varepsilon\|_{L^p((0,T)\times G_\varepsilon)} + \|\nabla u_\varepsilon\|_{L^p((0,T)\times G_\varepsilon)})\right) < \infty.$$

Then there exist a subsequence E' of E and a couple (u_0,u_1) with $u_0 \in L^p(0,T; W^{1,p}(G_0))$ and $u_1 \in L^p((0,T)\times G_0; B^{1,p}_{\#\mathcal{A}}(\mathbb{R}^{d_1}; W^{1,p}(G_2)))$ such that, as $E' \ni \varepsilon \to 0$,

$$\begin{aligned} u_\varepsilon &\to u_0 && \text{in } L^p((0,T)\times G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}}, \\ \frac{\partial u_\varepsilon}{\partial \overline{x}_i} &\to \frac{\partial u_0}{\partial \overline{x}_i} + \frac{\partial u_1}{\partial \overline{y}_i} && \text{in } L^p((0,T)\times G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}},\ 1\le i\le d_1, \\ \frac{\partial u_\varepsilon}{\partial x_{d_1+i}} &\to \frac{\partial u_1}{\partial \zeta_i} && \text{in } L^p((0,T)\times G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}},\ 1\le i\le d_2. \end{aligned}$$

The above time-dependent properties have been proved in [10] (see also [28]).

4. Homogenization of the Darcy–Lapwood–Brinkmann equation in thin heterogeneous domain: Case of flat lateral boundaries

In this section, we deal with non oscillating boundaries.

4.1. Statement of the problem and a priori estimates

Let Ω be a bounded open connected Lipschitz subset in $\mathbb{R}^2$. For $\varepsilon > 0$, we define the thin heterogeneous domain Ω^ε in $\mathbb{R}^3$ by

$$\Omega^\varepsilon = \Omega \times (-\varepsilon, \varepsilon) \equiv \{(\overline{x}, x_3) \in \mathbb{R}^3 : \overline{x} \in \Omega \text{ and } -\varepsilon < x_3 < \varepsilon\}.$$

In the fracture Ω^ε, the flow of fluid at the micro-scale is described by the Darcy–Lapwood–Brinkmann (DLB) system

$$\begin{cases} -\operatorname{div}\left(A\left(\dfrac{x}{\varepsilon}\right)\nabla \boldsymbol{u}_\varepsilon\right) + \dfrac{\mu}{K_\varepsilon}\boldsymbol{u}_\varepsilon + \dfrac{\rho}{\phi^2}(\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon = \boldsymbol{f} \quad \text{in } \Omega^\varepsilon \\ \operatorname{div} \boldsymbol{u}_\varepsilon = 0 \quad \text{in } \Omega^\varepsilon \quad \text{and} \quad \boldsymbol{u}_\varepsilon = 0 \quad \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{4.1}$$

where

(A1) $A \in L^\infty(\mathbb{R}^3)^{3\times 3}$ is a symmetric matrix satisfying

$$\alpha|\lambda|^2 \leq A(y)\lambda \cdot \lambda \leq \beta|\lambda|^2 \quad \text{for all } \lambda \in \mathbb{R}^3 \text{ and a.e. } y \in \mathbb{R}^3,$$

α and β being two positive real numbers;

(A2) the right-hand side $\boldsymbol{f}$ has the form $\boldsymbol{f}(x) = (\boldsymbol{f}_1(\overline{x}), 0)$ for a.e. $x = (\overline{x}, x_3) \in \Omega \times (-1, 1)$, where $\boldsymbol{f}$ belongs to $(L^2(\Omega \times (-1, 1)))^3$;

(A3) $A \in (B^2_{\mathcal{A}}(\mathbb{R}^2; L^\infty(I)))^{3\times 3}$, where $I = (-1, 1)$.

In (4.1), $\boldsymbol{u}_\varepsilon$ and p_ε are respectively the velocity of the fluid and the pressure; ρ represents the fluid density while ϕ stands for the porosity of the medium; K_ε is the permeability of the porous medium and μ is the dynamic coefficient of the viscosity.

With the above assumptions (A1) and (A2) on A and $\boldsymbol{f}$ respectively, equation (4.1) possesses at least (for each fixed $\varepsilon > 0$) a solution $(\boldsymbol{u}_\varepsilon, p_\varepsilon) \in H^1_0(\Omega^\varepsilon)^3 \times L^2_0(\Omega^\varepsilon)$, where $L^2_0(\Omega^\varepsilon) = \{u \in L^2(\Omega^\varepsilon) : \int_{\Omega^\varepsilon} u dx = 0\}$.

For the sequel we adopt the following notation. If $A = (a_{ij})_{1\leq i,j\leq 3}$ and $B = (b_{ij})_{1\leq i,j\leq 3}$ we set

$$A \cdot B = \sum_{i,j=1}^{3} a_{ij}b_{ij},\ AB = (c_{ij})_{1\leq i,j\leq 3} \text{ with } c_{ij} = \sum_{k=1}^{3} a_{ik}b_{kj};$$

$$\text{for } x = (x_i)_{1\leq i\leq 3} \text{ and } y = (y_i)_{1\leq i\leq 3},\ x \cdot y = \sum_{i=1}^{3} x_i y_i.$$

The following technical result whose proof can be found in [20] will be useful in the sequel.

Lemma 4.1. *It holds that*

$$\|\boldsymbol{u}\|_{L^2(\Omega^\varepsilon)^3} \leq C\varepsilon\|\nabla\boldsymbol{u}\|_{L^2(\Omega^\varepsilon)^{3\times3}}, \tag{4.2}$$

$$\|\boldsymbol{u}\|_{L^4(\Omega^\varepsilon)^3} \leq C\varepsilon^{\frac{1}{2}}\|\nabla\boldsymbol{u}\|_{L^2(\Omega^\varepsilon)^{3\times3}} \tag{4.3}$$

for any $\boldsymbol{u} \in H_0^1(\Omega^\varepsilon)^3$, where C is a positive constant independent of $\varepsilon > 0$.

The following estimates hold for the velocity.

Proposition 4.2. *Let $\boldsymbol{u}_\varepsilon$ be determined by (4.1). Then for all ε,*

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq C\min(\varepsilon^{\frac{5}{2}}, \varepsilon^{\frac{3}{2}}K_\varepsilon^{\frac{1}{2}}), \tag{4.4}$$

$$\|\nabla\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times3}} \leq C\varepsilon^{\frac{3}{2}}, \tag{4.5}$$

where C is a positive constant independent of ε.

Proof. For any $\boldsymbol{\varphi} \in V = \{\boldsymbol{u} \in H_0^1(\Omega^\varepsilon)^3 : \operatorname{div}\boldsymbol{u} = 0\}$, we have

$$\int_{\Omega^\varepsilon} A\left(\frac{x}{\varepsilon}\right)\nabla\boldsymbol{u}_\varepsilon \cdot \nabla\boldsymbol{\varphi}dx + \frac{\mu}{K_\varepsilon}\int_{\Omega^\varepsilon}\boldsymbol{u}_\varepsilon\boldsymbol{\varphi}dx + \frac{\rho}{\phi^2}\int_{\Omega^\varepsilon}(\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}_\varepsilon\boldsymbol{\varphi}dx = \int_{\Omega^\varepsilon}\boldsymbol{f}\boldsymbol{\varphi}dx. \tag{4.6}$$

Choosing $\boldsymbol{\varphi} = \boldsymbol{u}_\varepsilon$ in (4.6) and using the properties of the matrix A, the fact that $\int_{\Omega^\varepsilon}(\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{u}_\varepsilon dx = 0$ and the expression of $\boldsymbol{f}$, we get

$$\alpha\int_{\Omega^\varepsilon}|\nabla\boldsymbol{u}_\varepsilon|^2dx + \frac{\mu}{K_\varepsilon}\int_{\Omega^\varepsilon}|\boldsymbol{u}_\varepsilon|^2dx \leq \int_{\Omega^\varepsilon}\boldsymbol{f}_1(\overline{x})\boldsymbol{u}_\varepsilon'(x)dx, \tag{4.7}$$

where $\boldsymbol{u}_\varepsilon'(x) = (u_{\varepsilon,i})_{1\leq i\leq 2}$. Now using the fact that $\boldsymbol{f}_1 \in L^2(\Omega)^2$, (4.2) and the Cauchy–Schwarz inequality, we obtain

$$\left|\int_{\Omega^\varepsilon}\boldsymbol{f}_1(\overline{x})\boldsymbol{u}_\varepsilon'(x)dx\right| \leq C\varepsilon^{\frac{3}{2}}\|\nabla\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times3}}.$$

It follows readily from (4.7) that $\|\nabla\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times3}} \leq C\varepsilon^{\frac{3}{2}}$, that is, (4.5). Using once again (4.2), we get

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq C\varepsilon^{\frac{5}{2}}. \tag{4.8}$$

Coming back to (4.7), we obtain

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq C\varepsilon^{\frac{3}{2}}K_\varepsilon^{\frac{1}{2}}. \tag{4.9}$$

Putting together (4.8) and (4.9) we are led to (4.4). This completes the proof. ∎

Remark 4.3. It follows from (4.4) that

(i) If $K_\varepsilon = O(\varepsilon^2)$ or if $K_\varepsilon \gg \varepsilon^2$, then $\|\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq C\varepsilon^{\frac{5}{2}}$.

(ii) If $K_\varepsilon \ll \varepsilon^2$, then $\|\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq C\varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}}$.

Next, we need to derive the estimates for the pressure. To this end, we need the following well-known result whose proof can be found in [20, Lemma 20].

Lemma 4.4 ([20, Lemma 20]). *For any $g_\varepsilon \in L_0^2(\Omega^\varepsilon)$, there exists a unique $\varphi_\varepsilon \in H_0^1(\Omega^\varepsilon)^3$ satisfying* $\operatorname{div}\varphi_\varepsilon = g_\varepsilon$ *and*

$$\|\varphi_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq C\|g_\varepsilon\|_{L^2(\Omega^\varepsilon)}, \quad \|\nabla\varphi_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}} \leq \frac{C}{\varepsilon}\|g_\varepsilon\|_{L^2(\Omega^\varepsilon)},$$

where the positive constant C is independent of ε.

The following result holds true.

Proposition 4.5. *Let $p_\varepsilon \in L_0^2(\Omega^\varepsilon)$ satisfy (4.1). Then*

(1) *If $K_\varepsilon = O(\varepsilon^2)$ or if $K_\varepsilon \gg \varepsilon^2$, then*

$$\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}. \tag{4.10}$$

(2) *If $K_\varepsilon \ll \varepsilon^2$, then*

$$\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)} \leq C\varepsilon^{\frac{3}{2}} K_\varepsilon^{-1/2}. \tag{4.11}$$

In (4.10) and (4.11), C is a positive constant independent of ε.

Proof. Since $p_\varepsilon \in L_0^2(\Omega^\varepsilon)$, we appeal to Lemma 4.4 to derive the existence of $\varphi_\varepsilon \in H_0^1(\Omega^\varepsilon)^3$ with $\operatorname{div}\varphi_\varepsilon = p_\varepsilon$ and

$$\|\varphi_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq C\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)}, \quad \|\nabla\varphi_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}} \leq \frac{C}{\varepsilon}\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)}.$$

We choose φ_ε as a test function in the variational form of (4.1) to get

$$\begin{aligned}\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 = -\langle\nabla p_\varepsilon, \varphi_\varepsilon\rangle &= \int_{\Omega^\varepsilon} A^\varepsilon \nabla\boldsymbol{u}_\varepsilon \cdot \nabla\varphi_\varepsilon dx + \frac{\mu}{K_\varepsilon}\int_{\Omega^\varepsilon} \boldsymbol{u}_\varepsilon \varphi_\varepsilon dx \\ &\quad + \frac{\rho}{\phi^2}\int_{\Omega^\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon \cdot \varphi_\varepsilon dx - \int_{\Omega^\varepsilon} \boldsymbol{f}\varphi_\varepsilon dx \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

One has

$$\begin{aligned}|I_1| &\leq C\|\nabla\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}}\|\nabla\varphi_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{1}{2}}\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)}, \\ |I_2| &\leq \frac{\mu}{K_\varepsilon}\|\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^3}\|\varphi_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq \frac{C}{K_\varepsilon}\min(\varepsilon^{\frac{5}{2}}, \varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}})\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)},\end{aligned}$$

$$\begin{aligned}|I_3| &\le \|\boldsymbol{u}_\varepsilon\|_{L^4(\Omega^\varepsilon)^3}\|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}}\|\varphi_\varepsilon\|_{L^4(\Omega^\varepsilon)^3}\\ &\le C\varepsilon\|\nabla \boldsymbol{u}_\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times 3}}\|\nabla\varphi_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}} \le C\varepsilon^3\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)},\end{aligned}$$

and

$$|I_4| \le C\varepsilon^{\frac{1}{2}}\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)}.$$

It follows that

$$\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C\left(\varepsilon^{\frac{1}{2}} + \varepsilon^3 + \frac{1}{K_\varepsilon}\min(\varepsilon^{\frac{5}{2}}, \varepsilon^{\frac{3}{2}}K_\varepsilon^{\frac{1}{2}})\right). \tag{4.12}$$

We note that the precise estimates should depend on the magnitude of K_ε with respect to ε. Precisely,

- if $K_\varepsilon = O(\varepsilon^2)$, then (4.12) yields
$$\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C\varepsilon^{\frac{1}{2}};$$
- if $K_\varepsilon \ll \varepsilon^2$, then $\varepsilon^{\frac{3}{2}}K_\varepsilon^{\frac{1}{2}} < \varepsilon^{\frac{5}{2}}$ and $\varepsilon^{\frac{1}{2}} < \varepsilon^{\frac{3}{2}}K_\varepsilon^{-\frac{1}{2}}$, so that
$$\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C(\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{3}{2}}K_\varepsilon^{-\frac{1}{2}}) \le C\varepsilon^{\frac{3}{2}}K_\varepsilon^{-\frac{1}{2}};$$
- if $K_\varepsilon \gg \varepsilon^2$, then
$$\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C\varepsilon^{\frac{1}{2}}.$$ ■

According to the estimates in Propositions 4.2 and 4.5, three different regimes should be considered: 1) $K_\varepsilon = O(\varepsilon^2)$, 2) $K_\varepsilon \ll \varepsilon^2$ and 3) $K_\varepsilon \gg \varepsilon^2$. Each of these special cases will be worked out separately in the following subsections.

For the sequel we identify Ω with $\Omega\times\{0\}$ so that any point $(\overline{x}, 0)$ should be merely written as $\overline{x}$. We also set $I = (-1, 1)$.

4.2. Homogenization results in the case when $K_\varepsilon = O(\varepsilon^2)$

We assume that

$$\frac{K_\varepsilon}{\varepsilon^2} \to K \quad \text{when } \varepsilon \to 0, \text{ with } 0 < K < \infty. \tag{4.13}$$

According to Propositions 4.2 and 4.5, the following uniform estimates hold: there exists a positive constant C such that for all $\varepsilon > 0$,

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \le C\varepsilon^{\frac{5}{2}}, \quad \|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}} \le C\varepsilon^{\frac{3}{2}} \quad \text{and} \quad \|p_\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C\varepsilon^{\frac{1}{2}}. \tag{4.14}$$

Let $\mathcal{A}$ be an ergodic algebra with mean value on $\mathbb{R}^2$. In view of Proposition 3.14 and Theorem 3.3, given an ordinary sequence E, there exist a subsequence E' of

E and $\boldsymbol{u}_0 \in [L^2(\Omega; \mathcal{B}^{1,2}_{\mathcal{A}}(\mathbb{R}^2; H^1_0(I)))]^3$, $p_0 \in L^2(\Omega; \mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I)))$ such that, as $E' \ni \varepsilon \to 0$,

$$\frac{\boldsymbol{u}_\varepsilon}{\varepsilon^2} \to \boldsymbol{u}_0 \quad \text{in } L^2(\Omega^\varepsilon)^3\text{-weak } \Sigma_{\mathcal{A}}, \tag{4.15}$$

$$\frac{1}{\varepsilon}\nabla \boldsymbol{u}_\varepsilon \to \overline{\nabla}_y \boldsymbol{u}_0 \quad \text{in } L^2(\Omega^\varepsilon)^{3\times 3}\text{-weak } \Sigma_{\mathcal{A}}, \tag{4.16}$$

$$p_\varepsilon \to p_0 \quad \text{in } L^2(\Omega^\varepsilon)\text{-weak } \Sigma_{\mathcal{A}}, \tag{4.17}$$

where in (4.16) we put $y = (\overline{y}, y_3)$, so that $\overline{\nabla}_y = (\overline{\nabla}_{\overline{y}}, \partial/\partial y_3)$, $\overline{\nabla}_{\overline{y}} = (\overline{\partial}/\partial y_i)_{1\leq i\leq 2}$, $\overline{\partial}/\partial y_i$ being defined in (2.2). Since $\operatorname{div} \boldsymbol{u}_\varepsilon = 0$ in Ω^ε, it follows that $\overline{\operatorname{div}}_y \boldsymbol{u}_0 = 0$ in $\Omega \times \mathbb{R}^2 \times I$. Indeed, setting

$$\boldsymbol{u}'_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}),$$

we have, for $\boldsymbol{\varphi} \in \mathcal{C}^\infty_0(\Omega) \otimes \mathcal{A}^\infty(\mathbb{R}^2; \mathcal{C}^\infty_0(I))$,

$$\begin{aligned} 0 &= \int_{\Omega^\varepsilon} \operatorname{div} \boldsymbol{u}_\varepsilon(x) \boldsymbol{\varphi}\left(\overline{x}, \frac{x}{\varepsilon}\right) dx \\ &= -\int_{\Omega^\varepsilon} \boldsymbol{u}'_\varepsilon \cdot (\nabla_{\overline{x}} \boldsymbol{\varphi})^\varepsilon dx + \frac{1}{\varepsilon}\int_{\Omega^\varepsilon} \boldsymbol{u}_\varepsilon \cdot (\nabla_y \boldsymbol{\varphi})^\varepsilon dx, \end{aligned}$$

where $\boldsymbol{\varphi}^\varepsilon(x) = \boldsymbol{\varphi}(\overline{x}, \frac{x}{\varepsilon})$ for $x \in \Omega^\varepsilon$. Multiplying the last equality above by ε^{-2} and letting $E' \ni \varepsilon \to 0$ yields

$$\int_\Omega \int_{-1}^1 M(\boldsymbol{u}_0(\overline{x}, \cdot, y_3) \cdot \nabla_y \boldsymbol{\varphi}(\overline{x}, \cdot, y_3)) dy_3 d\overline{x} = 0.$$

This amounts to $\overline{\operatorname{div}}_y \boldsymbol{u}_0 = 0$ in $\Omega \times \mathbb{R}^2 \times I$, where $\overline{\operatorname{div}}_y \boldsymbol{u}_0 = \overline{\operatorname{div}}_{\overline{y}} \boldsymbol{u}'_0 + \frac{\partial u_{0,3}}{\partial y_3}$ with $\boldsymbol{u}'_0 = (u_{0,i})_{1\leq i\leq 2}$.

Now, set

$$\begin{aligned} \boldsymbol{u}(\overline{x}) &= \int_{-1}^1 M(\boldsymbol{u}_0(\overline{x}, \cdot, y_3)) dy_3 \quad \text{for } \overline{x} \in \Omega \\ &= (u_i(\overline{x}))_{1\leq i\leq 3}. \end{aligned} \tag{4.18}$$

Then $\boldsymbol{u} \in L^2(\Omega)^3$. Moreover,

$$\operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}' \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{4.19}$$

where ν is the outward unit normal to $\partial\Omega$. Here $\boldsymbol{u} = (\boldsymbol{u}', u_3)$. First of all, one has

$$u_3 = 0 \quad \text{in } \Omega. \tag{4.20}$$

Indeed, from the equality $\overline{\operatorname{div}}_y \boldsymbol{u}_0 = 0$ in $\Omega \times \mathbb{R}^2 \times I$, we have $M(\overline{\operatorname{div}}_y \boldsymbol{u}_0) = 0$, that is, $\frac{\partial}{\partial y_3} M(u_{0,3}(\overline{x}, \cdot, y_3)) = 0$. This shows that $u_{0,3}$ is independent of y_3. But $u_{\varepsilon,3} = 0$ on $\Omega \times \{\varepsilon\}$, so that $M(u_{0,3}(\overline{x}, \cdot, y_3)) = 0$ on $\Omega \times \{1\}$, i.e., $M(u_{0,3}(\overline{x}, \cdot)) = 0$ in Ω since $u_{0,3}$ does not depend on y_3. This shows that $\boldsymbol{u} = (\boldsymbol{u}', 0)$.

This being so, let us check (4.19). To that end, let $\varphi \in \mathcal{D}(\overline{\Omega})$. Using the Stokes formula together with the equality div $\boldsymbol{u}_\varepsilon = 0$ in Ω^ε, we obtain

$$\int_{\Omega^\varepsilon} \boldsymbol{u}'_\varepsilon(x) \cdot \nabla_{\overline{x}}\varphi(\overline{x})dx = 0.$$

Dividing the last equality above by ε^3 and letting $E' \ni \varepsilon \to 0$, we are led to

$$\int_{\Omega} \boldsymbol{u}'(x) \cdot \nabla_{\overline{x}}\varphi(\overline{x})d\overline{x} = 0.$$

This yields at once (4.19).

We are now able to pass to the limit in the critical case $K_\varepsilon = O(\varepsilon^2)$.

Theorem 4.6. *Let $\mathcal{A}$ be an ergodic algebra with mean value on $\mathbb{R}^2$. Assume that (A3) and (4.13) hold true. Let $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ be determined by (4.1). Then, when $E' \ni \varepsilon \to 0$, we have (4.15), (4.16) and (4.17). Moreover, there exists $q \in L^2(\Omega; \mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I)))$ such that $(\boldsymbol{u}_0, p_0, q)$ solves the equation*

$$\begin{cases} -\overline{\operatorname{div}}_y(A(y)\overline{\nabla}_y\boldsymbol{u}_0) + \dfrac{\mu}{K}\boldsymbol{u}_0 + \overline{\nabla}_y q = \boldsymbol{f}_1 - \nabla_{\overline{x}}p_0 & \text{in } \Omega\times\mathbb{R}^2\times I,\\ \overline{\operatorname{div}}_y\boldsymbol{u}_0 = 0 & \text{in } \Omega\times\mathbb{R}^2\times I,\\ \operatorname{div}_{\overline{x}}\left(\displaystyle\int_{-1}^1 M(\boldsymbol{u}_0(\overline{x},\cdot,y_3))dy_3\right) = 0 & \text{in } \Omega,\\ \left(\displaystyle\int_{-1}^1 M(\boldsymbol{u}_0(\overline{x},\cdot,y_3))dy_3\right)\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.21}$$

Proof. Let $(\boldsymbol{u}_0, p_0)$ be determined by (4.15)–(4.17) such that (4.18) and (4.19) are satisfied. Let us first show that p_0 is independent of $y = (\overline{y}, y_3)$. To that end, let $\boldsymbol{\varphi} \in (\mathcal{C}_0^\infty(\Omega)\otimes\mathcal{A}^\infty(\mathbb{R}^2;\mathcal{C}_0^\infty(I)))^3$. Testing (4.1) against $\boldsymbol{\varphi}^\varepsilon(x) = \boldsymbol{\varphi}(\overline{x}, x/\varepsilon)$ $(x \in \Omega^\varepsilon)$, we have

$$\begin{aligned} &\int_{\Omega^\varepsilon} A\left(\frac{x}{\varepsilon}\right)\nabla\boldsymbol{u}_\varepsilon\cdot\left[(\nabla_{\overline{x}}\boldsymbol{\varphi})^\varepsilon + \frac{1}{\varepsilon}(\nabla_y\boldsymbol{\varphi})^\varepsilon\right]dx + \mu\frac{\varepsilon^2}{K_\varepsilon}\int_{\Omega^\varepsilon}\frac{\boldsymbol{u}_\varepsilon}{\varepsilon^2}\boldsymbol{\varphi}^\varepsilon dx\\ &\quad + \frac{\rho}{\phi^2}\int_{\Omega^\varepsilon}(\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}_\varepsilon\boldsymbol{\varphi}^\varepsilon dx - \int_{\Omega^\varepsilon} p_\varepsilon\left[(\operatorname{div}_{\overline{x}}\boldsymbol{\varphi})^\varepsilon + \frac{1}{\varepsilon}(\operatorname{div}_y\boldsymbol{\varphi})^\varepsilon\right]dx\\ &= \int_{\Omega^\varepsilon}\boldsymbol{f}\boldsymbol{\varphi}^\varepsilon dx. \end{aligned} \tag{4.22}$$

Using the first two estimates in (4.14) together with (4.3) give

$$\begin{aligned} \left|\int_{\Omega^\varepsilon}(\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}_\varepsilon\boldsymbol{\varphi}^\varepsilon dx\right| &\le \|\boldsymbol{u}_\varepsilon\|_{L^4(\Omega^\varepsilon)}\|\nabla\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)}\|\boldsymbol{\varphi}^\varepsilon\|_{L^4(\Omega^\varepsilon)}\\ &\le C\varepsilon\|\nabla\boldsymbol{u}_\varepsilon\|^2_{L^2(\Omega^\varepsilon)}\|\nabla\boldsymbol{\varphi}^\varepsilon\|_{L^2(\Omega^\varepsilon)}\\ &\le C\varepsilon^3. \end{aligned}$$

Thus

$$\frac{1}{\varepsilon}\int_{\Omega^\varepsilon}(\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}_\varepsilon\boldsymbol{\varphi}^\varepsilon dx\to 0 \quad \text{when } E'\ni\varepsilon\to 0. \tag{4.23}$$

Hence, passing to the limit when $E'\ni\varepsilon\to 0$ in (4.22) using (4.15)–(4.17) leads to

$$\int_\Omega\int_{-1}^1 M(p_0(\overline{x},\cdot,y_3)\operatorname{div}_y\boldsymbol{\varphi}(\overline{x},\cdot,y_3))dy_3d\overline{x}=0,$$

which means that p_0 does not depend on $y=(\overline{y},y_3)$.

Now, coming back to (4.22) and choosing there $\boldsymbol{\varphi}$ such that $\operatorname{div}_y\boldsymbol{\varphi}=0$, and next dividing both sides of the resulting equation by ε, we get

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\Omega^\varepsilon}A\Big(\frac{x}{\varepsilon}\Big)\nabla\boldsymbol{u}_\varepsilon\cdot\Big[(\nabla_{\overline{x}}\boldsymbol{\varphi})^\varepsilon+\frac{1}{\varepsilon}(\nabla_y\boldsymbol{\varphi})^\varepsilon\Big]dx+\mu\frac{\varepsilon^2}{K_\varepsilon}\frac{1}{\varepsilon}\int_{\Omega^\varepsilon}\frac{\boldsymbol{u}_\varepsilon}{\varepsilon^2}\boldsymbol{\varphi}^\varepsilon dx\\+\frac{\rho}{\phi^2}\frac{1}{\varepsilon}\int_{\Omega^\varepsilon}(\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}_\varepsilon\boldsymbol{\varphi}^\varepsilon dx-\frac{1}{\varepsilon}\int_{\Omega^\varepsilon}p_\varepsilon(\operatorname{div}_{\overline{x}}\boldsymbol{\varphi})^\varepsilon dx=\frac{1}{\varepsilon}\int_{\Omega^\varepsilon}\boldsymbol{f}\boldsymbol{\varphi}^\varepsilon dx.\end{aligned} \tag{4.24}$$

We pass to the limit in (4.24) by considering each term separately. First, since $A\in(B^2_{\mathcal{A}}(\mathbb{R}^2;L^2(I))\cap L^\infty(\mathbb{R}^2\times I))^{3\times 3}$, the matrix-functions $A\nabla_{\overline{x}}\boldsymbol{\varphi}$ and $A\nabla_y\boldsymbol{\varphi}$ can be seen as test functions for the weak $\Sigma_{\mathcal{A}}$-convergence. Therefore, appealing to (4.15)–(4.17) in conjunction with (4.13), we pass to the limit when $E'\ni\varepsilon\to 0$ in (4.24) and obtain

$$\begin{aligned}\int_\Omega\int_{-1}^1 M(A\overline{\nabla}_y\boldsymbol{u}_0\cdot\nabla_y\boldsymbol{\varphi})dy_3d\overline{x}+\frac{\mu}{K}\int_\Omega\int_{-1}^1 M(\boldsymbol{u}_0\boldsymbol{\varphi})dy_3d\overline{x}\\-\int_\Omega\int_{-1}^1 p_0(\overline{x})M(\operatorname{div}_{\overline{x}}\boldsymbol{\varphi})dy_3d\overline{x}=\int_\Omega\int_{-1}^1 M(\boldsymbol{f}\boldsymbol{\varphi})dy_3d\overline{x}.\end{aligned} \tag{4.25}$$

Since (4.25) holds for every $\boldsymbol{\varphi}\in(\mathcal{C}_0^\infty(\Omega)\otimes\mathcal{A}^\infty(\mathbb{R}^2;\mathcal{C}_0^\infty(I)))^3$ with $\operatorname{div}_y\boldsymbol{\varphi}=0$, we deduce from Proposition 3.7 the existence of $q\in L^2(\Omega;\mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2;L^2(I)))$ such that (4.21) holds. The proof is therefore completed. ∎

We are now in a position to prove the main result of the current subsection, which besides, is one of the main results of the work.

Theorem 4.7. *Under the assumptions of Theorem 4.6, the sequence $(\boldsymbol{u}_\varepsilon/\varepsilon^2,p_\varepsilon)_{\varepsilon>0}$ weakly $\Sigma_{\mathcal{A}}$-converges in $L^2(\Omega^\varepsilon)^3\times L^2_0(\Omega^\varepsilon)$ towards $(\boldsymbol{u}_0,p_0)$ determined by (4.15)–(4.17). Moreover, $p_0\in H^1(\Omega)$ and, defining $\boldsymbol{u}=(\boldsymbol{u}',u_3)$ by (4.18), one has $u_3=0$ and $(\boldsymbol{u}',p_0)$ is the unique solution of the effective problem*

$$\begin{cases}\boldsymbol{u}'=\widehat{A}(\boldsymbol{f}_1-\nabla_{\overline{x}}p_0) & \text{in }\Omega,\\ \operatorname{div}_{\overline{x}}\boldsymbol{u}'=0 & \text{in }\Omega\quad\text{and}\quad \boldsymbol{u}'\cdot\nu=0\text{ on }\partial\Omega,\end{cases} \tag{4.26}$$

where $\widehat{A} = (\widehat{a}_{ij})_{1\le i,j\le 2}$ *is a symmetric, positive definite* 2×2 *matrix defined by its entries*

$$\widehat{a}_{ij} = \int_{-1}^{1} M(A\overline{\nabla}_y \boldsymbol{w}_i \cdot \overline{\nabla}_y \boldsymbol{w}_j)dy_3 + \frac{\mu}{K}\int_{-1}^{1} M(\boldsymbol{w}_i \boldsymbol{w}_j)dy_3, \quad 1\le i,j\le 2.$$

Here $\boldsymbol{w}_i$ $(1\le i\le 2)$ *is the unique solution in* $(\mathcal{B}^{1,2}_{\mathcal{A}}(\mathbb{R}^2; H^1_0(I)))^3$ *of the Stokes–Brinkmann system*

$$\begin{cases} -\overline{\operatorname{div}}_y(A(y)\overline{\nabla}_y \boldsymbol{w}_i) + \dfrac{\mu}{K}\boldsymbol{w}_i + \overline{\nabla}_y \pi_i = e_i & \text{in } \mathbb{R}^2\times I, \\ \overline{\operatorname{div}}_y \boldsymbol{w}_i = 0 & \text{in } \mathbb{R}^2\times I, \end{cases} \tag{4.27}$$

e_i *being the* i*th vector of the canonical basis in* $\mathbb{R}^3$.

Proof. First and foremost, let us prove that (4.27) possesses a unique solution $\boldsymbol{w}_i \in (\mathcal{B}^{1,2}_{\mathcal{A}}(\mathbb{R}^2; H^1_0(I)))^3$. It is a fact that (4.27) is equivalent to

$$\begin{cases} \displaystyle\int_{-1}^{1} M(A\overline{\nabla}_y \boldsymbol{w}_i \cdot \overline{\nabla}_y \boldsymbol{v})dy_3 + \frac{\mu}{K}\int_{-1}^{1} M(\boldsymbol{w}_i \boldsymbol{v})dy_3 = \int_{-1}^{1} M(\boldsymbol{v})e_i dy_3 \\ \text{for all } \boldsymbol{v} \in (\mathcal{B}^{1,2}_{\mathcal{A}}(\mathbb{R}^2; H^1_0(I)))^3 \text{ with } \overline{\operatorname{div}}_y \boldsymbol{v} = 0. \end{cases} \tag{4.28}$$

In view of assumption (A1) on A, (4.28) possesses a unique solution in $\mathcal{B}^{1,2}_{\mathrm{div}} = \{\boldsymbol{v} \in (\mathcal{B}^{1,2}_{\mathcal{A}}(\mathbb{R}^2; H^1_0(I)))^3 : \overline{\operatorname{div}}_y \boldsymbol{v} = 0\}$.

Next, we recall that by density, (4.25) still holds for $\boldsymbol{\varphi} \in L^2(\Omega; \mathcal{B}^{1,2}_{\mathrm{div}})$, so that, choosing in (4.25) the test function $\boldsymbol{\varphi} = \psi \otimes \boldsymbol{w}_i$ with $\psi \in \mathcal{C}^\infty_0(\Omega)$ and $\boldsymbol{w}_i$ $(1\le i\le 3)$ defined by (4.28), we obtain, after integrating by parts,

$$\int_{-1}^{1} M(A\overline{\nabla}_y \boldsymbol{u}_0 \cdot \overline{\nabla}_y \boldsymbol{w}_i)dy_3 + \frac{\mu}{K}\int_{-1}^{1} M(\boldsymbol{u}_0 \boldsymbol{w}_i)dy_3 + \nabla_{\overline{x}} p_0 \int_{-1}^{1} M(\boldsymbol{w}_i)dy_3$$
$$= \boldsymbol{f}(x)\int_{-1}^{1} M(\boldsymbol{w}_i)dy_3. \tag{4.29}$$

Taking in (4.28) the test function $\boldsymbol{u}_0(\overline{x},\cdot)$, we obtain

$$\begin{cases} \displaystyle\int_{-1}^{1} M(A\overline{\nabla}_y \boldsymbol{w}_i \cdot \overline{\nabla}_y \boldsymbol{u}_0)dy_3 + \frac{\mu}{K}\int_{-1}^{1} M(\boldsymbol{u}_0 \boldsymbol{w}_i)dy_3 = \int_{-1}^{1} M(\boldsymbol{u}_0)e_i dy_3 \\ \displaystyle\qquad = \int_{-1}^{1} M(u_{0,i})dy_3 = u_i(\overline{x}), \quad 1\le i\le 3. \end{cases} \tag{4.30}$$

Using the fact that A is symmetric, we obtain

$$\int_{-1}^{1} M(A\overline{\nabla}_y \boldsymbol{u}_0 \cdot \overline{\nabla}_y \boldsymbol{w}_i)dy_3 = \int_{-1}^{1} M(A\overline{\nabla}_y \boldsymbol{w}_i : \overline{\nabla}_y \boldsymbol{u}_0)dy_3,$$

so that, comparing (4.29) and (4.30), we get

$$u_i(\overline{x}) = \sum_{j=1}^{2} \left(\int_{-1}^{1} M(\boldsymbol{w}_i)e_j dy_3 \right)\left(f_{1,j}(\overline{x}) - \frac{\partial p_0}{\partial \overline{x}_j}(\overline{x}) \right). \tag{4.31}$$

But

$$\int_{-1}^{1} M(\boldsymbol{w}_i)e_j dy_3 = \int_{-1}^{1} M(A\overline{\nabla}_y \boldsymbol{w}_i : \overline{\nabla}_y \boldsymbol{w}_j)dy_3 + \frac{\mu}{K}\int_{-1}^{1} M(\boldsymbol{w}_i \boldsymbol{w}_j)dy_3,$$

in such a way that, setting

$$\widehat{a}_{ij} = \int_{-1}^{1} M(A\overline{\nabla}_y \boldsymbol{w}_i \cdot \overline{\nabla}_y \boldsymbol{w}_j)dy_3 + \frac{\mu}{K}\int_{-1}^{1} M(\boldsymbol{w}_i \boldsymbol{w}_j)dy_3,$$

and accounting of (4.20) (that is, $\int_{-1}^{1} M(u_{0,3})dy_3 = 0$), we deduce $\widehat{a}_{i3} = 0$ for $1 \leq i \leq 2$. Since $\widehat{a}_{ij} = \widehat{a}_{ji}$, we also deduce that $\widehat{a}_{3i} = 0$ for $1 \leq i \leq 2$. This shows that $\widehat{A} = (\widehat{a}_{ij})_{1\leq i,j\leq 2}$ is a 2×2 symmetric and positive definite matrix. From (4.31) we get at once

$$\boldsymbol{u}'(\overline{x}) = \widehat{A}(\boldsymbol{f}_1(\overline{x}) - \nabla_{\overline{x}} p_0(\overline{x})). \tag{4.32}$$

Now using (4.19) together with the fact that (4.32) holds in the classical sense of distributions in Ω, we get that p_0 solves the problem

$$\begin{aligned} \operatorname{div}_{\overline{x}}(\widehat{A}(\boldsymbol{f}_1(\overline{x}) - \nabla_{\overline{x}} p_0(\overline{x}))) = 0 \quad & \text{in } \Omega, \\ (\widehat{A}(\boldsymbol{f}_1(\overline{x}) - \nabla_{\overline{x}} p_0(\overline{x}))) \cdot \nu = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{4.33}$$

which shows that $p_0 \in H^1(\Omega)$ is uniquely determined by (4.33). The convergence of the whole sequence $(\boldsymbol{u}_\varepsilon/\varepsilon^2, p_\varepsilon)_{\varepsilon>0}$ is a consequence of the uniqueness of the solution to (4.33) (and hence to (4.26)). This completes the proof of the theorem. ■

4.3. Homogenization results: Case when $K_\varepsilon \gg \varepsilon^2$

In view of Propositions 4.2 and 4.5, it holds that

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq C\varepsilon^{\frac{5}{2}}, \quad \|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{3}{2}}$$
$$\text{and} \quad \|p_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq C\varepsilon^{\frac{1}{2}} \quad \text{for all } \varepsilon > 0,$$

where $C > 0$ is independent of $\varepsilon > 0$. Thus, given an ordinary sequence E, there exist a subsequence E' of E and $\boldsymbol{u}_0 \in (L^2(\Omega; \mathcal{B}_{\mathcal{A}}^{1,2}(\mathbb{R}^2; H_0^1(I))))^3$, $p_0 \in L^2(\Omega; \mathcal{B}_{\mathcal{A}}^2(\mathbb{R}^2; L^2(I)))$ such that, when $E' \ni \varepsilon \to 0$,

$$\frac{\boldsymbol{u}_\varepsilon}{\varepsilon^2} \to \boldsymbol{u}_0 \quad \text{in } L^2(\Omega^\varepsilon)^3\text{-weak } \Sigma_{\mathcal{A}}, \tag{4.34}$$

$$\frac{1}{\varepsilon}\nabla \boldsymbol{u}_\varepsilon \to \overline{\nabla}_y \boldsymbol{u}_0 \quad \text{in } L^2(\Omega^\varepsilon)^{3\times 3}\text{-weak } \Sigma_{\mathcal{A}} \tag{4.35}$$

and

$$p_\varepsilon \to p_0 \quad \text{in } L^2(\Omega^\varepsilon)\text{-weak } \Sigma_{\mathcal{A}}. \tag{4.36}$$

Defining the function $\boldsymbol{u}$ as in (4.18), we have that (4.19) holds and $u_3 = 0$. The following is the first main result when $K_\varepsilon \gg \varepsilon^2$.

Theorem 4.8. *Let A be an ergodic algebra with mean value on $\mathbb{R}^2$. Assume that (A3) holds and $K_\varepsilon \gg \varepsilon^2$. Let $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ be a solution of (4.1). Then we have (4.34)–(4.36). Furthermore, there exists $q \in L^2(\Omega; \mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I)))$ such that $(\boldsymbol{u}_0, p_0, q)$ solves the system*

$$\begin{cases} -\overline{\operatorname{div}}_y(A(y)\overline{\nabla}_y \boldsymbol{u}_0) + \overline{\nabla}_y q = \boldsymbol{f}_1 - \nabla_{\overline{x}} p_0 & \text{in } \Omega \times \mathbb{R}^2 \times I, \\ \overline{\operatorname{div}}_y \boldsymbol{u}_0 = 0 & \text{in } \Omega \times \mathbb{R}^2 \times I, \\ \operatorname{div}_{\overline{x}} \left(\int_{-1}^{1} M(\boldsymbol{u}_0(\overline{x}, \cdot, y_3)) dy_3 \right) = 0 & \text{in } \Omega, \\ \left(\int_{-1}^{1} M(\boldsymbol{u}_0(\overline{x}, \cdot, y_3)) dy_3 \right) \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.37}$$

Proof. We proceed as in the proof of Theorem 4.6. Firstly, we see that p_0 is independent of y. Next, choosing in (4.22) a test function $\boldsymbol{\varphi}$ such that $\operatorname{div}_y \boldsymbol{\varphi} = 0$ and then multiplying the resulting equality by $1/\varepsilon$, we appeal once again to (4.34)–(4.36) to obtain

$$\begin{cases} \int_\Omega \int_{-1}^{1} M(A\overline{\nabla}_y \boldsymbol{u}_0 : \nabla_y \boldsymbol{\varphi}) dy_3 d\overline{x} - \int_\Omega \int_{-1}^{1} p_0(\overline{x}) M(\operatorname{div}_{\overline{x}} \boldsymbol{\varphi}) dy_3 d\overline{x} \\ \qquad = \int_\Omega \int_{-1}^{1} M(\boldsymbol{f} \boldsymbol{\varphi}) dy_3 d\overline{x}. \end{cases} \tag{4.38}$$

This yields as before, the existence of q such that (4.37) holds. ■

As in the previous subsection, let us consider the following Stokes system:

$$\begin{cases} \text{Find } \boldsymbol{w}_j \in B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2; H^1_0(I))^3 \text{ such that} \\ -\overline{\operatorname{div}}_y(A\overline{\nabla}_y \boldsymbol{w}_j) + \overline{\nabla}_y \pi_j = e_j \quad \text{in } \mathbb{R}^2 \times I, \\ \overline{\operatorname{div}}_y \boldsymbol{w}_j = 0 \quad \text{in } \mathbb{R}^2 \times I. \end{cases} \tag{4.39}$$

Then in view of [10, Lemma 4.1], system (4.39) possesses a unique solution $\boldsymbol{w}_j \in B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2; H^1_0(I))^3$. Also, proceeding as in Section 4.2, we are able to see that, defining $\boldsymbol{u}$ by (4.18) and setting

$$\widehat{A} = (\widehat{a}_{ij})_{1 \le i,j \le 2} \quad \text{with } \widehat{a}_{ij} = \int_{-1}^{1} M(A\overline{\nabla}_y \boldsymbol{w}_i : \overline{\nabla}_y \boldsymbol{w}_j) dy_3, \ 1 \le i, j \le 2, \tag{4.40}$$

($\widehat{A} = (\widehat{a}_{ij})_{1 \le i,j \le 2}$ is a symmetric positive definite 2×2 matrix), we have $u_3 = 0$ and

$$\begin{cases} \boldsymbol{u}' = \widehat{A}(\boldsymbol{f}_1 - \nabla_{\overline{x}} p_0) & \text{in } \Omega, \\ \operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 & \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}' \cdot \nu = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.41}$$

The homogenized result in this case is therefore stated as follows.

Theorem 4.9. *Under the assumptions of Theorem 4.8, the sequence* $(\boldsymbol{u}_\varepsilon/\varepsilon^2, p_\varepsilon)_{\varepsilon>0}$ *weakly* $\Sigma_{\mathcal{A}}$*-converges in* $L^2(\Omega^\varepsilon)^3 \times L_0^2(\Omega^\varepsilon)$ *towards* $(\boldsymbol{u}_0, p_0)$ *determined by* (4.34)–(4.36). *In addition,* $p_0 \in H^1(\Omega)$ *and, defining* $\boldsymbol{u} = (\boldsymbol{u}', u_3)$ *by* (4.18)*, one has* $u_3 = 0$ *and* $(\boldsymbol{u}', p_0)$ *is the unique solution of the effective problem* (4.41) *where* $\widehat{A}$ *is given by* (4.40).

4.4. Homogenization results: Case when $K_\varepsilon \ll \varepsilon^2$

Throughout this subsection, we assume that $\varepsilon^2/K_\varepsilon \to 0$ as $\varepsilon \to 0$. In that case, the following estimates hold:

$$\begin{aligned} &\|\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq C\varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}}, \quad \|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{3}{2}}, \\ &\|p_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \leq C\frac{\varepsilon^{\frac{3}{2}}}{K_\varepsilon^{\frac{1}{2}}}, \quad \text{for all } \varepsilon > 0, \end{aligned} \tag{4.42}$$

C being a positive constant independent of ε.

In view of (4.42), the following result holds.

Lemma 4.10. *Given an ordinary sequence* E*, there exist a subsequence* E' *of* E *and functions* $\boldsymbol{u}_0 \in (L^2(\Omega; \mathcal{B}_{\mathcal{A}}^2(\mathbb{R}^2; L^2(I))))^3$, $\boldsymbol{u}_1 \in (L^2(\Omega; B_{\#\mathcal{A}}^{1,2}(\mathbb{R}^2; H^1(I))))^3$ *and* $p_0 \in L^2(\Omega; \mathcal{B}_{\mathcal{A}}^2(\mathbb{R}^2; L^2(I)))$ *such that, as* $E' \ni \varepsilon \to 0$,

$$\frac{\boldsymbol{u}_\varepsilon}{\varepsilon K_\varepsilon^{\frac{1}{2}}} \to \boldsymbol{u}_0 \quad \textit{in } L^2(\Omega^\varepsilon)^3\textit{-weak } \Sigma_{\mathcal{A}}, \tag{4.43}$$

$$\frac{\nabla \boldsymbol{u}_\varepsilon}{\varepsilon} \to \nabla_y \boldsymbol{u}_1 \quad \textit{in } L^2(\Omega^\varepsilon)^{3\times 3}\textit{-weak } \Sigma_{\mathcal{A}}, \tag{4.44}$$

$$\frac{K_\varepsilon^{\frac{1}{2}}}{\varepsilon} p_\varepsilon \to p_0 \quad \textit{in } L^2(\Omega^\varepsilon)\textit{-weak } \Sigma_{\mathcal{A}}. \tag{4.45}$$

Moreover,

$$\overline{\operatorname{div}}_y \boldsymbol{u}_0 = 0 \quad \textit{and} \quad \operatorname{div}_y \boldsymbol{u}_1 = 0 \quad \textit{in } \Omega \times \mathbb{R}^2 \times I.$$

Proof. Applying Theorem 3.3, we derive the existence of a subsequence E' of E and of a triple $(\boldsymbol{u}_0, \boldsymbol{v}, p_0) \in (L^2(\Omega; \mathcal{B}_{\mathcal{A}}^{1,2}(\mathbb{R}^2; H^1(I))))^3 \times (L^2(\Omega; \mathcal{B}_{\mathcal{A}}^{1,2}(\mathbb{R}^2; H^1(I))))^{3\times 3}$ $\times L^2(\Omega; \mathcal{B}_{\mathcal{A}}^2(\mathbb{R}^2; L^2(I)))$ such that (4.43) and (4.45) hold, and further

$$\frac{\nabla \boldsymbol{u}_\varepsilon}{\varepsilon} \to \boldsymbol{v} \quad \text{in } L^2(\Omega^\varepsilon)^{3\times 3}\text{-weak } \Sigma_{\mathcal{A}}. \tag{4.46}$$

Let us characterize $\boldsymbol{v}$. To that end, let $\Phi \in (\mathcal{C}_0^\infty(\Omega) \otimes \mathcal{A}^\infty(\mathbb{R}^2; \mathcal{C}_0^\infty(I)))^3$ be such that $\operatorname{div}_y \Phi = 0$. Then

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\Omega^\varepsilon} \frac{1}{\varepsilon}\nabla \boldsymbol{u}_\varepsilon \cdot \Phi^\varepsilon dx &= -\frac{1}{\varepsilon}\int_{\Omega^\varepsilon}\frac{1}{\varepsilon}\boldsymbol{u}_\varepsilon (\operatorname{div}_{\overline{x}} \Phi)^\varepsilon dx \\ &= -K_\varepsilon^{\frac{1}{2}}\frac{1}{\varepsilon}\int_{\Omega^\varepsilon}\frac{\boldsymbol{u}_\varepsilon}{\varepsilon K_\varepsilon^{\frac{1}{2}}}(\operatorname{div}_{\overline{x}} \Phi)^\varepsilon dx \end{aligned} \tag{4.47}$$

where in (4.47), $\Phi^\varepsilon(x) = \Phi(\overline{x}, x/\varepsilon)$ for $x \in \Omega^\varepsilon$ and $\nabla \boldsymbol{u}_\varepsilon \cdot \Phi^\varepsilon$ is viewed as the vector $(\nabla u_{\varepsilon,i} \cdot \Phi^\varepsilon)_{1\leq i\leq 3}$ and $\boldsymbol{u}_\varepsilon(\operatorname{div}_{\overline{x}} \Phi)^\varepsilon$ is viewed as the vector $(u_{\varepsilon,i}(\operatorname{div}_{\overline{x}} \Phi)^\varepsilon)_{1\leq i\leq 3}$, with $\boldsymbol{u}_\varepsilon = (u_{\varepsilon,i})_{1\leq i\leq 3}$. Letting $E' \ni \varepsilon \to 0$ in (4.47) using (4.43) and (4.46) together with the fact that $K_\varepsilon \to 0$, we are led to

$$\int_\Omega\int_{-1}^1 M(\boldsymbol{v}(\overline{x}, \cdot, y_3) \cdot \Phi(\overline{x}, \cdot, y_3))dy_3 d\overline{x} = 0$$

for all $\Phi \in (\mathcal{C}_0^\infty(\Omega) \otimes \mathcal{A}^\infty(\mathbb{R}^2; \mathcal{C}_0^\infty(I)))^3$ with $\operatorname{div}_y \Phi = 0$. We infer from Corollary 3.8 the existence of $\boldsymbol{u}_1 \in (L^2(\Omega; B^{1,2}_{\#\mathcal{A}}(\mathbb{R}^2; H^1(I))))^3$ such that $\boldsymbol{v} = \nabla_y \boldsymbol{u}_1$. Next, from the equality $\operatorname{div} \boldsymbol{u}_\varepsilon = 0$, we deduce

$$\sum_{i=1}^3 \frac{1}{\varepsilon}\int_{\Omega^\varepsilon}\frac{1}{\varepsilon}\frac{\partial u_{\varepsilon,i}}{\partial x_i}(x)\psi\left(\overline{x}, \frac{x}{\varepsilon}\right)dx = 0 \quad \text{for all } \psi \in \mathcal{C}_0^\infty(\Omega) \otimes \mathcal{A}^\infty(\mathbb{R}^2; \mathcal{C}_0^\infty(I)),$$

so that, in view of (4.44), we have, as $E' \ni \varepsilon \to 0$,

$$\sum_{i=1}^3 \int_\Omega\int_{-1}^1 M\left(\frac{\partial u_{1,i}}{\partial y_i}\psi\right)dy_3 d\overline{x} = 0 \quad \text{for all } \psi \in \mathcal{C}_0^\infty(\Omega) \otimes \mathcal{A}^\infty(\mathbb{R}^2; \mathcal{C}_0^\infty(I)).$$

This amounts to $\operatorname{div}_y \boldsymbol{u}_1 = 0$ in $\Omega \times \mathbb{R}^2 \times I$. Also, the same equality $\operatorname{div} \boldsymbol{u}_\varepsilon = 0$ yields

$$\int_{\Omega^\varepsilon} \boldsymbol{u}_\varepsilon\left((\nabla_{\overline{x}}\psi)^\varepsilon + \frac{1}{\varepsilon}(\nabla_y\psi)^\varepsilon\right)dx = 0 \quad \text{for } \psi \text{ as above.} \tag{4.48}$$

Dividing both members of (4.48) by $\varepsilon K_\varepsilon^{1/2}$ and letting $E' \ni \varepsilon \to 0$ gives

$$\int_\Omega\int_{-1}^1 M(\boldsymbol{u}_0 \cdot \nabla_y \psi)dy_3 d\overline{x} = 0,$$

which amounts to $\overline{\operatorname{div}}_y \boldsymbol{u}_0 = 0$ in $\Omega \times \mathbb{R}^2 \times I$. This concludes the proof. ∎

Now we go back to the variational form (4.22), i.e.,

$$\left\{\begin{aligned} &\int_{\Omega^\varepsilon} A\left(\frac{x}{\varepsilon}\right)\nabla \boldsymbol{u}_\varepsilon \cdot \left[(\nabla_{\overline{x}}\boldsymbol{\varphi})^\varepsilon + \frac{1}{\varepsilon}(\nabla_y \boldsymbol{\varphi})^\varepsilon\right]dx + \frac{\mu}{K_\varepsilon}\int_{\Omega^\varepsilon} \boldsymbol{u}_\varepsilon \boldsymbol{\varphi}^\varepsilon dx \\ &\qquad + \frac{\rho}{\phi^2}\int_{\Omega^\varepsilon}(\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon \boldsymbol{\varphi}^\varepsilon dx - \int_{\Omega^\varepsilon} p_\varepsilon\left[(\operatorname{div}_{\overline{x}} \boldsymbol{\varphi})^\varepsilon + \frac{1}{\varepsilon}(\operatorname{div}_y \boldsymbol{\varphi})^\varepsilon\right]dx \\ &\quad = \int_{\Omega^\varepsilon} \boldsymbol{f}\boldsymbol{\varphi}^\varepsilon dx. \end{aligned}\right. \tag{4.49}$$

We assume that (4.43), (4.44) and (4.45) hold. Since $K_\varepsilon/\varepsilon^2 \to 0$ when $\varepsilon \to 0$, we use (4.43) to see that, as $E' \ni \varepsilon \to 0$,

$$\frac{K_\varepsilon}{\varepsilon^2}\frac{\mu}{K_\varepsilon}\int_{\Omega^\varepsilon} \boldsymbol{u}_\varepsilon \boldsymbol{\varphi}^\varepsilon dx = \frac{\mu}{\varepsilon^2}\int_{\Omega^\varepsilon} \boldsymbol{u}_\varepsilon \boldsymbol{\varphi}^\varepsilon dx = \mu\left(\frac{K_\varepsilon}{\varepsilon^2}\right)^{\frac{1}{2}}\int_{\Omega^\varepsilon} \frac{\boldsymbol{u}_\varepsilon}{\varepsilon K_\varepsilon^{\frac{1}{2}}}\boldsymbol{\varphi}^\varepsilon dx \to 0.$$

Thus, if we multiply (4.49) by $K_\varepsilon^{1/2}/\varepsilon$ and we let $E' \ni \varepsilon \to 0$ in the resulting identity, taking into account (4.43), (4.44) and (4.45) together with the fact that $K_\varepsilon \to 0$ as $\varepsilon \to 0$, we get

$$\int_\Omega \int_{-1}^1 M(p_0 \operatorname{div}_y \boldsymbol{\varphi}) dy_3 d\overline{x} = 0,$$

thereby showing that p_0 does not depend on y. Next, choosing in (4.49) a test function $\boldsymbol{\varphi}$ such that $\operatorname{div}_y \boldsymbol{\varphi} = 0$ and then multiplying the resulting equality by $K_\varepsilon^{1/2}/\varepsilon^2$, we appeal once again to (4.43)–(4.45) to obtain

$$\mu \int_\Omega \int_{-1}^1 M(\boldsymbol{u}_0 \boldsymbol{\varphi}) dy_3 d\overline{x} - \int_\Omega \int_{-1}^1 p_0(\overline{x}) M(\operatorname{div}_{\overline{x}} \boldsymbol{\varphi}) dy_3 d\overline{x} = 0.$$

This yields as before, the existence of $q \in L^2(\Omega; \mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I)))$ such that $(\boldsymbol{u}_0, p_0, q)$ solves the system

$$\begin{cases} \mu \boldsymbol{u}_0 + \overline{\nabla}_y q = -\nabla_{\overline{x}} p_0 & \text{in } \Omega \times \mathbb{R}^2 \times I, \\ \overline{\operatorname{div}}_y \boldsymbol{u}_0 = 0 & \text{in } \Omega \times \mathbb{R}^2 \times I, \\ \operatorname{div}_{\overline{x}}\left(\int_{-1}^1 M(\boldsymbol{u}_0(\overline{x}, \cdot, y_3)) dy_3\right) = 0 & \text{in } \Omega, \\ \left(\int_{-1}^1 M(\boldsymbol{u}_0(\overline{x}, \cdot, y_3)) dy_3\right) \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.50}$$

We have therefore proved the following result.

Theorem 4.11. *Let $\mathcal{A}$ be an ergodic algebra with mean value on $\mathbb{R}^2$. Assume that (A3) holds and $K_\varepsilon \ll \varepsilon^2$. Let $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ be a solution of (4.1). Then we have (4.43)–(4.45). Furthermore, there exists $q \in L^2(\Omega; \mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I)))$ such that $(\boldsymbol{u}_0, p_0, q)$ solves the system (4.50).*

In order to derive the effective equation in the case when $K_\varepsilon \ll \varepsilon^2$, we consider the local problems: For $1 \leq i \leq 3$, find $\boldsymbol{w}_i \in \mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I))^3$ such that

$$\mu \boldsymbol{w}_i + \overline{\nabla}_y \pi_i = e_i \quad \text{in } \mathbb{R}^2 \times I \quad \text{and} \quad \overline{\operatorname{div}}_y \boldsymbol{w}_i = 0 \quad \text{in } \mathbb{R}^2 \times I. \tag{4.51}$$

Then the existence and uniqueness of $\boldsymbol{w}_i$ satisfying (4.51) follow a two steps process described below:

(1) We approximate (4.51) as follows: for any integer $n \geq 1$, there exists a unique $\boldsymbol{w}_{i,n} \in \mathcal{B}^{1,2}_{\mathrm{div}}$ solution of

$$\begin{cases} -\dfrac{1}{n^2}\overline{\Delta}_y \boldsymbol{w}_{i,n} + \mu \boldsymbol{w}_{i,n} + \overline{\nabla}_y \pi_{i,n} = e_i & \text{in } \mathbb{R}^2 \times I, \\ \overline{\mathrm{div}}_y \boldsymbol{w}_{i,n} = 0 & \text{in } \mathbb{R}^2 \times I \end{cases} \tag{4.52}$$

in the sense of (4.28), where $\overline{\Delta}_y = \overline{\mathrm{div}}_y(\overline{\nabla}_y)$. Testing (4.52) with $\boldsymbol{w}_{i,n}$ yields the estimates

$$\sup_{n\geq 1}\left(\frac{1}{n}\|\overline{\nabla}_y \boldsymbol{w}_{i,n}\|_{\mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2;L^2(I))^{3\times 3}} + \|\boldsymbol{w}_{i,n}\|_{\mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2;L^2(I))^3}\right) \leq C, \tag{4.53}$$

where $C > 0$ is independent of n.

(2) We pass to the limit in the variational formulation

$$\frac{1}{n^2}\int_{-1}^{1} M(\overline{\nabla}_y \boldsymbol{w}_{i,n} \cdot \overline{\nabla}_y \boldsymbol{v})dy_3 + \mu \int_{-1}^{1} M(\boldsymbol{w}_{i,n}\boldsymbol{v})dy_3$$
$$= \int_{-1}^{1} M(\boldsymbol{v})e_i\,dy_3 \qquad \text{for all } \boldsymbol{v} \in \mathcal{B}^{1,2}_{\mathrm{div}}$$

using (4.53) and the reflexivity of the Hilbert space $\mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I))$ to derive the existence and uniqueness of a $\boldsymbol{w}_i \in (\mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I)))^3$ solution to (4.51).

Now, proceeding as in the proof of Theorem 4.7, we may choose in (4.50) the test function $\boldsymbol{\varphi} = \psi \otimes \boldsymbol{w}_i$; then

$$\mu \int_{-1}^{1} M(\boldsymbol{u}_0 \boldsymbol{w}_i)dy_3 + \nabla_{\overline{x}} p_0 \int_{-1}^{1} M(\boldsymbol{w}_i)dy_3 = 0. \tag{4.54}$$

Next, take $\boldsymbol{u}_0(\overline{x}, \cdot)$ as test function in (4.51),

$$\mu \int_{-1}^{1} M(\boldsymbol{u}_0 \boldsymbol{w}_i)dy_3 = \int_{-1}^{1} M(\boldsymbol{u}_0)e_i\,dy_3 \equiv u_i(\overline{x}). \tag{4.55}$$

Putting together (4.54) and (4.55) yields

$$u_i(\overline{x}) = -\nabla_{\overline{x}} p_0 \int_{-1}^{1} M(\boldsymbol{w}_i)dy_3 = -\sum_{j=1}^{3} \frac{\partial p_0}{\partial \overline{x}_j} \int_{-1}^{1} M(\boldsymbol{w}_i)e_j\,dy_3.$$

Still in (4.51) with $\boldsymbol{w}_j$ taken as test function, we are led to

$$\int_{-1}^{1} M(\boldsymbol{w}_i)e_j\,dy_3 = \mu \int_{-1}^{1} M(\boldsymbol{w}_i \boldsymbol{w}_j)dy_3.$$

So, setting

$$\widehat{A} = (\widehat{a}_{ij})_{1\leq i,j\leq 2} \qquad \text{with } \widehat{a}_{ij} = \mu \int_{-1}^{1} M(\boldsymbol{w}_i \boldsymbol{w}_j)dy_3, \tag{4.56}$$

we obtain a 2×2 symmetric matrix. We recall that the fact that $u_d = 0$ amounts to $\widehat{a}_{id} = 0$, and so $\widehat{a}_{di} = 0$ for $1 \leq i \leq 2$.

We have thus proved the following result, which besides is the main result of the work in the case when $K_\varepsilon \ll \varepsilon^2$.

Theorem 4.12. *Under the assumptions of Theorem 4.11, the sequence*

$$\left(\frac{\boldsymbol{u}_\varepsilon}{\varepsilon K_\varepsilon^{\frac{1}{2}}}, \frac{K_\varepsilon^{\frac{1}{2}}}{\varepsilon^2} p_\varepsilon \right)_{\varepsilon > 0}$$

weakly $\Sigma_{\mathcal{A}}$-converges in $L^2(\Omega^\varepsilon)^3 \times L_0^2(\Omega^\varepsilon)$ towards $(\boldsymbol{u}_0, p_0)$ determined by (4.43) and (4.45). Moreover, $p_0 \in H^1(\Omega)$ and, setting

$$\begin{aligned} \boldsymbol{u}(\overline{x}) &= \int_{-1}^{1} M(\boldsymbol{u}_0(\overline{x}, \cdot, y_3)dy_3 \quad (\overline{x} \in \Omega) \\ &= (u_i(\overline{x}))_{1 \leq i \leq 3} \quad \textit{and} \quad \boldsymbol{u}' = (u_i)_{1 \leq i \leq 2}, \end{aligned}$$

we have $u_3 = 0$ and $(\boldsymbol{u}', p_0)$ is a solution of the homogenized problem

$$\boldsymbol{u}' = -\widehat{A} \nabla_{\overline{x}} p_0 \quad \textit{in } \Omega, \quad \operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 \quad \textit{in } \Omega, \quad \textit{and} \quad \boldsymbol{u}' \cdot \nu = 0 \quad \textit{on } \partial\Omega,$$

where $\widehat{A}$ is defined by (4.56), $\boldsymbol{w}_i$ $(1 \leq i \leq 2)$ being the unique solution of (4.51).

5. Homogenization of the Darcy–Lapwood–Brinkmann equation in thin heterogeneous domain with highly oscillating boundaries

5.1. Introduction

In this section, we deal with problem (4.1), but this time, stated in the following domain

$$\Omega^\varepsilon = \left\{ x = (\overline{x}, x_3) \in \Omega \times \mathbb{R} : \varepsilon h_1\left(\frac{\overline{x}}{\varepsilon}\right) < x_3 < \varepsilon h_2\left(\frac{\overline{x}}{\varepsilon}\right) \right\}, \tag{5.1}$$

where $\varepsilon > 0$ is a small parameter, $\Omega \subset \mathbb{R}^2$ is a bounded open Lipschitz domain, h_1, $h_2 \in W^{1,\infty}(\mathbb{R}^2)$ are two bounded Lipschitz continuous functions on $\mathbb{R}^2$ and satisfying

$$\max_{\mathbb{R}^2} h_1 < \min_{\mathbb{R}^2} h_2; \tag{5.2}$$

$$h_1, h_2 \in \mathcal{A} \quad \text{with } M(h_2 - h_1) \neq 0. \tag{5.3}$$

We set

$$h_1^- = \min_{\mathbb{R}^2} h_1, \quad h_2^+ = \max_{\mathbb{R}^2} h_2, \quad \text{and} \quad I = (h_1^-, h_2^+),$$

and we define $G_\varepsilon = \Omega \times (\varepsilon h_1^-, \varepsilon h_2^+)$. Then, as seen in Section 3, G_ε has flat lateral boundaries $y_3 = \varepsilon h_1^-, \varepsilon h_2^+$, with $\Omega^\varepsilon \subset G_\varepsilon$. When $\varepsilon \to 0$, G_ε shrinks to $G_0 = \Omega \times \{0\}$ which can be identified to Ω through the identification $\overline{x} \equiv (\overline{x}, 0)$. This will be the case in the sequel.

The problem to be investigated here is stated by (4.1) in the thin domain Ω^ε given above by (5.1). As in Section 4, the corresponding problem possesses at least a weak solution $\boldsymbol{u}_\varepsilon \in H_0^1(\Omega^\varepsilon)^3$. Moreover, to each $\boldsymbol{u}_\varepsilon$ is associated a unique $p_\varepsilon \in L_0^2(\Omega^\varepsilon)$ such that (4.1) is satisfied by the couple $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$. We extend $\boldsymbol{u}_\varepsilon$ to G_ε by zero off Ω^ε, and we still denote this extension by $\boldsymbol{u}_\varepsilon$; this is fully justified by the fact that $\boldsymbol{u}_\varepsilon = 0$ on $\partial\Omega^\varepsilon$. It is therefore easily seen that, proceeding exactly as in Section 4.1, we derive the existence of a positive constant C independent of ε such that

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^3} \le C \min(\varepsilon^{\frac{5}{2}}, \varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}}) \quad \text{and} \quad \|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times 3}} \le C\varepsilon^{\frac{3}{2}}, \tag{5.4}$$

and so,

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(G_\varepsilon)^3} \le C \min(\varepsilon^{\frac{5}{2}}, \varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}}) \quad \text{and} \quad \|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(G_\varepsilon)^{3\times 3}} \le C\varepsilon^{\frac{3}{2}}.$$

Concerning the pressure, since the domain Ω^ε satisfies the assumptions of [13, Corollary 3.4] (in fact, Ω is Lipschitz and connected), there exist functions $p_\varepsilon^0 \in H^1(\Omega)$ and $p_\varepsilon^1 \in L^2(\Omega^\varepsilon)$ such that

$$p_\varepsilon = p_\varepsilon^0 + \varepsilon p_\varepsilon^1 \quad \text{in } \Omega^\varepsilon, \tag{5.5}$$

and

$$\varepsilon^{\frac{3}{2}} \|p_\varepsilon^0\|_{H^1(\Omega)} + \varepsilon \|p_\varepsilon^1\|_{L^2(\Omega^\varepsilon)} \le C \|\nabla p_\varepsilon\|_{H^{-1}(\Omega^\varepsilon)^3}, \tag{5.6}$$

where the positive constant C in (5.6) is independent of ε. It remains to find a bound for $\|\nabla p_\varepsilon\|_{H^{-1}(\Omega^\varepsilon)^3}$. For that purpose, let $\boldsymbol{v} \in H_0^1(\Omega^\varepsilon)^3$; then appealing to $(4.1)_1$, we have

$$\begin{aligned}\langle \nabla p_\varepsilon, \boldsymbol{v}\rangle = &\int_{\Omega^\varepsilon} \boldsymbol{f} \cdot \boldsymbol{v} dx - \int_{\Omega^\varepsilon} A^\varepsilon \nabla \boldsymbol{u}_\varepsilon \cdot \nabla \boldsymbol{v} dx - \frac{\rho}{\phi^2} \int_{\Omega^\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} dx \\ &- \frac{\mu}{K_\varepsilon} \int_{\Omega^\varepsilon} \boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} dx. \end{aligned} \tag{5.7}$$

It is easy to see that

$$\left| \int_{\Omega^\varepsilon} \boldsymbol{f} \cdot \boldsymbol{v} dx \right| \le C\varepsilon^{\frac{3}{2}} \|\nabla \boldsymbol{v}\|_{L^2(\Omega^\varepsilon)^{3\times 3}},$$

$$\left| \int_{\Omega^\varepsilon} A^\varepsilon \nabla \boldsymbol{u}_\varepsilon \cdot \nabla \boldsymbol{v} dx \right| \le C\varepsilon^{\frac{3}{2}} \|\nabla \boldsymbol{v}\|_{L^2(\Omega^\varepsilon)^{3\times 3}},$$

and

$$\left| \int_{\Omega^\varepsilon} \boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} dx \right| \le C \min(\varepsilon^{\frac{5}{2}}, \varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}}) \|\nabla \boldsymbol{v}\|_{L^2(\Omega^\varepsilon)^{3\times 3}}.$$

As for the third term on the right-hand side of (5.7), one has

$$\begin{aligned}\left|\int_{\Omega^\varepsilon}(\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{u}_\varepsilon\cdot\boldsymbol{v}dx\right| &\leq \|\boldsymbol{u}_\varepsilon\|_{L^4(\Omega^\varepsilon)^3}\|\nabla\boldsymbol{u}_\varepsilon\|_{L^2(\Omega^\varepsilon)^{3\times3}}\|\boldsymbol{v}\|_{L^4(\Omega^\varepsilon)^3}\\ &\leq C\varepsilon\|\nabla\boldsymbol{u}_\varepsilon\|^2_{L^2(\Omega^\varepsilon)^{3\times3}}\|\nabla\boldsymbol{v}\|_{L^2(\Omega^\varepsilon)^{3\times3}}\\ &\leq C\varepsilon^4\|\nabla\boldsymbol{v}\|_{L^2(\Omega^\varepsilon)^{3\times3}},\end{aligned}$$

where we used Lemma 4.1 and inequality (5.4). It follows that

$$\|\nabla p_\varepsilon\|_{H^{-1}(\Omega^\varepsilon)^3} \leq C\left(\varepsilon^{\frac{3}{2}}+\varepsilon^4+\frac{\varepsilon}{K_\varepsilon}\min(\varepsilon^{\frac{5}{2}},\varepsilon^{\frac{3}{2}}K_\varepsilon^{\frac{1}{2}})\right).$$

Now, if $K_\varepsilon = O(\varepsilon^2)$ or $K_\varepsilon \gg \varepsilon^2$, then

$$\frac{\varepsilon}{K_\varepsilon}\min(\varepsilon^{\frac{5}{2}},\varepsilon^{\frac{3}{2}}K_\varepsilon^{\frac{1}{2}}) \leq C\frac{\varepsilon^{\frac{7}{2}}}{K_\varepsilon} \leq C\varepsilon^{\frac{3}{2}},$$

so that

$$\|\nabla p_\varepsilon\|_{H^{-1}(\Omega^\varepsilon)^3} \leq C\varepsilon^{\frac{3}{2}}.$$

If $K_\varepsilon \ll \varepsilon^2$, then

$$\varepsilon^{\frac{3}{2}}K_\varepsilon^{\frac{1}{2}} < \varepsilon^{\frac{5}{2}} \quad\text{and}\quad \varepsilon^{\frac{3}{2}} < \frac{\varepsilon^{\frac{5}{2}}}{K_\varepsilon^{\frac{1}{2}}},$$

so that

$$\|\nabla p_\varepsilon\|_{H^{-1}(\Omega^\varepsilon)^3} \leq C\left(\varepsilon^{\frac{3}{2}}+\frac{\varepsilon^{\frac{5}{2}}}{K_\varepsilon^{\frac{1}{2}}}\right) \leq C\frac{\varepsilon^{\frac{5}{2}}}{K_\varepsilon^{\frac{1}{2}}}.$$

We have shown that

$$\|\nabla p_\varepsilon\|_{H^{-1}(\Omega^\varepsilon)^3} \leq \begin{cases} C\varepsilon^{\frac{3}{2}} & \text{if } K_\varepsilon = O(\varepsilon^2) \text{ or } K_\varepsilon \gg \varepsilon^2,\\ C\frac{\varepsilon^{\frac{5}{2}}}{K_\varepsilon^{\frac{1}{2}}} & \text{if } K_\varepsilon \ll \varepsilon^2. \end{cases} \tag{5.8}$$

It follows from (5.6) and (5.8) that

$$\|p_\varepsilon^0\|_{H^1(\Omega)} \leq \begin{cases} C & \text{if } K_\varepsilon = O(\varepsilon^2) \text{ or } K_\varepsilon \gg \varepsilon^2,\\ C\frac{\varepsilon}{K_\varepsilon^{\frac{1}{2}}} & \text{if } K_\varepsilon \ll \varepsilon^2; \end{cases}$$

and

$$\|p_\varepsilon^1\|_{L^2(\Omega^\varepsilon)} \leq \begin{cases} C\varepsilon^{\frac{1}{2}} & \text{if } K_\varepsilon = O(\varepsilon^2) \text{ or } K_\varepsilon \gg \varepsilon^2,\\ C\frac{\varepsilon^{\frac{3}{2}}}{K_\varepsilon^{\frac{1}{2}}} & \text{if } K_\varepsilon \ll \varepsilon^2. \end{cases} \tag{5.9}$$

We extend p_ε^1 to G_ε by zero and we denote the corresponding extension by $\widetilde{p}_\varepsilon^1$ to see that inequality (5.9) holds mutatis mutandis (change p_ε^1 into $\widetilde{p}_\varepsilon^1$ and Ω^ε into G_ε). We summarize the estimates obtained above, in the following lines:

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(G_\varepsilon)^3} \leq \begin{cases} C\varepsilon^{\frac{5}{2}} & \text{if } K_\varepsilon = O(\varepsilon^2) \text{ or } K_\varepsilon \gg \varepsilon^2, \\ C\varepsilon^{\frac{3}{2}} K_\varepsilon^{\frac{1}{2}} & \text{if } K_\varepsilon \ll \varepsilon^2; \end{cases} \tag{5.10}$$

$$\|\nabla \boldsymbol{u}_\varepsilon\|_{L^2(G_\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{3}{2}}; \tag{5.11}$$

$$\|p_\varepsilon^0\|_{H^1(\Omega)} \leq \begin{cases} C & \text{if } K_\varepsilon = O(\varepsilon^2) \text{ or } K_\varepsilon \gg \varepsilon^2, \\ C\frac{\varepsilon}{K_\varepsilon^{\frac{1}{2}}} & \text{if } K_\varepsilon \ll \varepsilon^2; \end{cases} \tag{5.12}$$

and

$$\|\widetilde{p}_\varepsilon^1\|_{L^2(G_\varepsilon)} \leq \begin{cases} C\varepsilon^{\frac{1}{2}} & \text{if } K_\varepsilon = O(\varepsilon^2) \text{ or } K_\varepsilon \gg \varepsilon^2, \\ C\frac{\varepsilon^{\frac{3}{2}}}{K_\varepsilon^{\frac{1}{2}}} & \text{if } K_\varepsilon \ll \varepsilon^2. \end{cases} \tag{5.13}$$

In the light of the above estimates, we follow the same steps as in the preceding section. This is declined below in the following subsection.

5.2. Passage to the limit and proof of Theorem 1.5

5.2.1. Case when $K_\varepsilon = O(\varepsilon^2)$. We assume (4.13), i.e., $\frac{K_\varepsilon}{\varepsilon^2} \to K$ when $\varepsilon \to 0$, with $0 < K < \infty$. Still denoting by $\boldsymbol{u}_\varepsilon$ the extension of $\boldsymbol{u}_\varepsilon$ on G_ε, we have $\boldsymbol{u}_\varepsilon \in H_0^1(G_\varepsilon)^3$ with $(5.10)_1$, (5.11), $(5.12)_1$ and $(5.13)_1$. Let $\mathcal{A}$ be an algebra with mean value on $\mathbb{R}^2$. Given an ordinary sequence E, we derive the existence of a subsequence E' of E and of functions $\boldsymbol{u}_0 \in L^2(\Omega; \mathcal{B}_{\mathcal{A}}^{1,2}(\mathbb{R}^2; H_0^1(I)))^3$, $p_0 \in H^1(\Omega)$, $p_1^0 \in L^2(\Omega; B_{\#\mathcal{A}}^{1,2}(\mathbb{R}^2))$ and $p_1 \in L^2(\Omega; \mathcal{B}_{\mathcal{A}}^2(\mathbb{R}^2; L^2(I)))$ such that, when $E' \ni \varepsilon \to 0$,

$$\frac{\boldsymbol{u}_\varepsilon}{\varepsilon^2} \to \boldsymbol{u}_0 \qquad \text{in } L^2(G_\varepsilon)^3\text{-weak } \Sigma_{\mathcal{A}}, \tag{5.14}$$

$$\frac{1}{\varepsilon}\nabla \boldsymbol{u}_\varepsilon \to \overline{\nabla}_y \boldsymbol{u}_0 \qquad \text{in } L^2(G_\varepsilon)^{3\times 3}\text{-weak } \Sigma_{\mathcal{A}}, \tag{5.15}$$

$$p_\varepsilon^0 \to p_0 \qquad \text{in } H^1(\Omega)\text{-weak and in } L^2(\Omega)\text{-strong}, \tag{5.16}$$

$$\nabla_{\overline{x}} p_\varepsilon^0 \to \nabla_{\overline{x}} p_0 + \nabla_{\overline{y}} p_1^0 \qquad \text{in } L^2(\Omega)^2\text{-weak } \Sigma, \tag{5.17}$$

$$p_\varepsilon^1 \to p_1 \qquad \text{in } L^2(G_\varepsilon)\text{-weak } \Sigma_{\mathcal{A}}. \tag{5.18}$$

This being so, let us recall the definition of the set $\mathbb{J}$

$$\mathbb{J} = \{y = (\overline{y}, y_3) \in \mathbb{R}^3 : \overline{y} \in \mathbb{R}^2 \text{ and } h_1(\overline{y}) < y_3 < h_2(\overline{y})\}.$$

To any $u \in L_{\mathrm{loc}}^r(\mathbb{J})$ $(1 \leq r < \infty)$ is associated the transform u^b defined by

$$u^b(\overline{y}, t) = u(\overline{y}, (1-t)h_1(\overline{y}) + th_2(\overline{y})), \quad \overline{y} \in \mathbb{R}^2 \text{ and } 0 < t < 1,$$

so that $u^b \in L^r_{\mathrm{loc}}(\mathbb{R}^2; L^r(0,1))$. This allows us to define the following Besicovitch-type spaces: let $\mathcal{A}$ be an algebra with mean value on $\mathbb{R}^2$ such that $h_1, h_2 \in \mathcal{A}$. By $B^r_{\mathcal{A}}(\mathbb{J})$ we mean the space of those $u \in L^r_{\mathrm{loc}}(\mathbb{J})$ such that $u^b \in B^r_{\mathcal{A}}(\mathbb{R}^2; L^r(0,1))$. To $B^r_{\mathcal{A}}(\mathbb{J})$ we associate the Sobolev–Besicovitch space

$$B^{1,r}_{\mathcal{A}}(\mathbb{J}) = \{u \in B^r_{\mathcal{A}}(\mathbb{J}) : \nabla_y u \in B^r_{\mathcal{A}}(\mathbb{J})^3\}.$$

We recall that each of these spaces is a complete semi-normed space, the seminorm in $B^r_{\mathcal{A}}(\mathbb{J})$ being defined by

$$\|u\|_{B^r_{\mathcal{A}}(\mathbb{J})} = \|u^b\|_{B^r_{\mathcal{A}}(\mathbb{R}^2; L^r(0,1))}.$$

We also define the Banach counterpart of $B^r_{\mathcal{A}}(\mathbb{J})$ that we denote by $\mathcal{B}^r_{\mathcal{A}}(\mathbb{J})$ as follows: $u \in \mathcal{B}^r_{\mathcal{A}}(\mathbb{J})$ iff $u^b \in \mathcal{B}^r_{\mathcal{A}}(\mathbb{R}^2; L^r(0,1))$. Finally, the space

$$B^{1,r}_{\#}(\mathbb{J}) = \left\{u \in B^{1,r}_{\mathcal{A}}(\mathbb{J}) : u = 0 \text{ on } \partial\mathbb{J}\right\}$$

will be of special interest in the sequel. We recall that $\partial\mathbb{J} = \{y = (\overline{y}, y_3) : \overline{y} \in \mathbb{R}^2$ and $y_3 = h_i(\overline{y}), i = 1, 2\}$.

Proceeding as in Section 4.2, we see that $\overline{\operatorname{div}}_y \boldsymbol{u}_0 = 0$ in $\Omega \times \mathbb{J}$. Moreover, setting

$$\begin{aligned} \boldsymbol{u}(\overline{x}) &= M\left(\int_{h_1}^{h_2} \boldsymbol{u}_0(\overline{x}, \cdot, y_3)\,dy_3\right), \quad \overline{x} \in \Omega, \\ &= (u_i(\overline{x}))_{1\le i\le 3} \end{aligned} \tag{5.19}$$

we have $u_3 = 0$ and $\boldsymbol{u}' = (u_1, u_2) \in L^2(\Omega)^2$ with $\boldsymbol{u}' \cdot \nu = 0$ on $\partial\Omega$ and $\operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0$ in Ω.

With this in mind, the first homogenization result in this case reads as follows.

Theorem 5.1. *Let $\mathcal{A}$ be an ergodic algebra with mean value on $\mathbb{R}^2$. Assume that (A3), (4.13), (5.2) and (5.3). Let $(\boldsymbol{u}_\varepsilon, p_\varepsilon)$ be determined by (4.1). Then $E' \ni \varepsilon \to 0$, one has (5.14) to (5.18), where the quadruple $(\boldsymbol{u}_0, p_0, p_1^0, p_1)$ solves the system*

$$\begin{cases} -\overline{\operatorname{div}}_y(A\overline{\nabla}_y \boldsymbol{u}_0) + \dfrac{\mu}{K}\boldsymbol{u}_0 + \overline{\nabla}_y(p_1^0 + p_1) = f - \nabla_{\overline{x}} p_0 & \text{in } \Omega \times \mathbb{J}, \\ \overline{\operatorname{div}}_y \boldsymbol{u}_0 = 0 & \text{in } \Omega \times \mathbb{J}, \\ \operatorname{div}_{\overline{x}} M\left(\displaystyle\int_{h_1}^{h_2} \boldsymbol{u}_0(\overline{x}, \cdot, y_3)\,dy_3\right) = 0 & \text{in } \Omega, \\ M\left(\displaystyle\int_{h_1}^{h_2} \boldsymbol{u}_0(\overline{x}, \cdot, y_3)\,dy_3\right) \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.20}$$

Proof. Let the assumptions of Theorem 5.1 be in force. Let $\varphi \in (\mathcal{C}_0^\infty(\Omega) \otimes B^{1,2}_{\#}(\mathbb{J}))^3$. Defining $\varphi^\varepsilon \in H^1_0(\Omega^\varepsilon)^3$ by $\varphi^\varepsilon(x) = \varphi(\overline{x}, x/\varepsilon)$ $(x \in \Omega^\varepsilon)$ and choosing φ^ε as test function in (4.1), we get, after dividing both members of the resulting equality by ε,

$$\begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_{\Omega^\varepsilon} A^\varepsilon \nabla \boldsymbol{u}_\varepsilon \cdot \left[(\nabla_{\overline{x}}\varphi)^\varepsilon + \frac{1}{\varepsilon}(\nabla_y \varphi)^\varepsilon \right] dx + \mu \dfrac{\varepsilon^2}{K_\varepsilon}\dfrac{1}{\varepsilon}\displaystyle\int_{\Omega^\varepsilon} \frac{\boldsymbol{u}_\varepsilon}{\varepsilon^2}\varphi^\varepsilon dx \\ \qquad + \dfrac{\rho}{\phi^2}\dfrac{1}{\varepsilon}\displaystyle\int_{\Omega^\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon \varphi^\varepsilon dx + \frac{1}{\varepsilon}\int_{\Omega^\varepsilon} \nabla_{\overline{x}} p_\varepsilon^0 \cdot \varphi^\varepsilon dx \\ \qquad - \displaystyle\int_{\Omega^\varepsilon} p_\varepsilon^1 \left[(\operatorname{div}_{\overline{x}}\varphi)^\varepsilon + \frac{1}{\varepsilon}(\operatorname{div}_y \varphi)^\varepsilon \right] dx = \frac{1}{\varepsilon}\int_{\Omega^\varepsilon} \boldsymbol{f}\varphi^\varepsilon dx. \end{cases} \tag{5.21}$$

Let us consider each term in (5.21) separately. As for the first term on the left-hand side, it is equal to

$$\int_{\Omega^\varepsilon} A^\varepsilon \frac{\nabla \boldsymbol{u}_\varepsilon}{\varepsilon} \cdot (\nabla_{\overline{x}}\varphi)^\varepsilon dx + \frac{1}{\varepsilon}\int_{\Omega^\varepsilon} A^\varepsilon \frac{\nabla \boldsymbol{u}_\varepsilon}{\varepsilon} \cdot (\nabla_y \varphi)^\varepsilon dx = I_1 + I_2.$$

It is easy to see that $I_1 \to 0$ when $E' \ni \varepsilon \to 0$, while, for I_2, appealing to Theorem 3.15 associated to (5.15), we have

$$\begin{aligned} I_2 &\to \int_\Omega \int_{h_1^-}^{h_2^+} M(\chi_{\mathbb{J}}(\cdot, y_3) A(\cdot, y_3) \overline{\nabla}_y \boldsymbol{u}_0(\overline{x}, \cdot, y_3) \cdot \nabla_y \varphi(\overline{x}, \cdot, y_3)) dy_3 d\overline{x} \\ &= \int_\Omega M\left(\int_{h_1}^{h_2} A \overline{\nabla}_y \boldsymbol{u}_0 \cdot \nabla_y \varphi dy_3 \right) d\overline{x}. \end{aligned}$$

It is worth noting that in the last convergence above, we have used $A\nabla_y\varphi$ as test function; indeed, since $A \in (L^\infty(\mathbb{R}^3) \cap B^2_{\mathcal{A}}(\mathbb{R}^2; L^\infty(I)))^{3\times 3}$ (where here, $I = (h_1^-, h_2^+)$) and $L^\infty(\mathbb{R}^3) \cap B^2_{\mathcal{A}}(\mathbb{R}^2; L^\infty(I)) \hookrightarrow L^\infty(\mathbb{R}^2; L^\infty(I))$, we get that $A\nabla_y\varphi \in (\mathscr{C}_0^\infty(\Omega) \otimes B^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I)))^{3\times 3}$, so that it can be taken as test function in the $\Sigma_{\mathcal{A}}$-convergence. It follows that

$$\frac{1}{\varepsilon}\int_{\Omega^\varepsilon} A^\varepsilon \nabla \boldsymbol{u}_\varepsilon \cdot \left[(\nabla_{\overline{x}}\varphi)^\varepsilon + \frac{1}{\varepsilon}(\nabla_y\varphi)^\varepsilon \right] dx \to \int_\Omega M\left(\int_{h_1}^{h_2} A\overline{\nabla}_y \boldsymbol{u}_0 \cdot \nabla_y \varphi dy_3 \right) d\overline{x}. \tag{5.22}$$

It is a fact using (5.14) that

$$\frac{1}{\varepsilon}\int_{\Omega^\varepsilon} \frac{\boldsymbol{u}_\varepsilon}{\varepsilon^2}\varphi^\varepsilon dx \to \int_\Omega M\left(\int_{h_1}^{h_2} \boldsymbol{u}_0 \varphi dy_3 \right) d\overline{x}, \tag{5.23}$$

and, as in Section 4, we observe that

$$\left| \int_{\Omega^\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon \varphi^\varepsilon dx \right| \le C\varepsilon^3,$$

so that

$$\frac{1}{\varepsilon}\int_{\Omega^\varepsilon} (\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon \varphi^\varepsilon dx \to 0 \quad \text{when } E' \ni \varepsilon \to 0. \tag{5.24}$$

Concerning the terms involving the pressure, it holds that

$$\frac{1}{\varepsilon}\int_{\Omega^\varepsilon}\nabla_{\overline{x}}p_\varepsilon^0\cdot\varphi^\varepsilon dx\to\int_\Omega M\left(\int_{h_1}^{h_2}(\nabla_{\overline{x}}p_0+\nabla_{\overline{y}}p_1^0)\cdot\varphi dy_3\right)d\overline{x}, \tag{5.25}$$

and

$$\int_{\Omega^\varepsilon}p_\varepsilon^1\left[(\mathrm{div}_{\overline{x}}\,\varphi)^\varepsilon+\frac{1}{\varepsilon}(\mathrm{div}_y\,\varphi)^\varepsilon\right]dx\to\int_\Omega M\left(\int_{h_1}^{h_2}p_1\,\mathrm{div}_y\,\varphi dy_3\right)d\overline{x}. \tag{5.26}$$

Finally, one has, as $E'\ni\varepsilon\to0$,

$$\frac{1}{\varepsilon}\int_{\Omega^\varepsilon}f\varphi^\varepsilon dx\to\int_\Omega M\left(\int_{h_1}^{h_2}f\varphi dy_3\right)d\overline{x}. \tag{5.27}$$

Collecting the convergence results (5.22)–(5.27), we obtain (when passing to the limit in (5.21)) the following variational system:

$$\begin{cases}\displaystyle\int_\Omega M\left(\int_{h_1}^{h_2}A\overline{\nabla}_y\boldsymbol{u}_0\cdot\nabla_y\varphi dy_3\right)d\overline{x}+\frac{\mu}{K}\int_\Omega M\left(\int_{h_1}^{h_2}\boldsymbol{u}_0\varphi dy_3\right)d\overline{x}\\ \qquad\displaystyle+\int_\Omega M\left(\int_{h_1}^{h_2}\nabla_{\overline{x}}p_0\cdot\varphi dy_3\right)d\overline{x}\\ \qquad\displaystyle-\int_\Omega M\left(\int_{h_1}^{h_2}(p_1^0+p_1)\,\mathrm{div}_y\,\varphi dy_3\right)d\overline{x}\\ \qquad\displaystyle=\int_\Omega M\left(\int_{h_1}^{h_2}f\varphi dy_3\right)d\overline{x},\quad\text{for all }\varphi\in(\mathscr{C}_0^\infty(\Omega)\otimes B_\#^{1,2}(\mathbb{J}))^3.\end{cases} \tag{5.28}$$

By density, (5.28) holds true for every $\varphi\in L^2(\Omega;B_\#^{1,2}(\mathbb{J}))^3$.

Also, proceeding as in Section 4, we derive the following properties for $\boldsymbol{u}_0$:

$$\mathrm{div}_{\overline{x}}\,M\left(\int_{h_1}^{h_2}\boldsymbol{u}_0(\overline{x},\cdot,y_3)dy_3\right)=0\quad\text{in }\Omega, \tag{5.29}$$

$$M\left(\int_{h_1}^{h_2}\boldsymbol{u}_0(\overline{x},\cdot,y_3)dy_3\right)\cdot\nu=0\quad\text{on }\partial\Omega. \tag{5.30}$$

We may also check that $p_0\in L_0^2(\Omega)$. Indeed, since $p_\varepsilon\in L_0^2(\Omega^\varepsilon)$, we have

$$0=\frac{1}{\varepsilon}\int_{\Omega^\varepsilon}p_\varepsilon dx=\int_\Omega\left(h_2\left(\frac{\overline{x}}{\varepsilon}\right)-h_1\left(\frac{\overline{x}}{\varepsilon}\right)\right)p_\varepsilon^0 d\overline{x}+\int_{\Omega^\varepsilon}p_\varepsilon^1 dx.$$

Letting $E'\ni\varepsilon\to0$ above yields $\int_\Omega M(h_2-h_1)p_0 d\overline{x}=0$, that is, $\int_\Omega p_0 d\overline{x}=0$, where we have taken (5.3) into account. This shows that $p_0\in H^1(\Omega)\cap L_0^2(\Omega)$.

Accounting of (5.28), (5.29) and (5.30), we get readily that the triple $(\boldsymbol{u}_0,p_0,q=p_1^0+p_1)$ solves the system (5.20). This completes the proof of the theorem. ∎

In order to derive the homogenized problem, we follow the same procedure as in the proof of Theorem 4.7. To this end, let $(e_i)_{1i\leq 3}$ be the canonical basis in $\mathbb{R}^3$. For $1 \leq i \leq 3$, consider the Stokes–Brinkmann system

$$\begin{cases} -\overline{\operatorname{div}}_y(A(y)\overline{\nabla}_y \boldsymbol{w}_i) + \dfrac{\mu}{K}\boldsymbol{w}_i + \overline{\nabla}_y \pi_i = e_i & \text{in } \mathbb{J}, \\ \overline{\operatorname{div}}_y \boldsymbol{w}_i = 0 & \text{in } \mathbb{J}, \\ \boldsymbol{w}_i = 0 & \text{on } \partial\mathbb{J}. \end{cases} \tag{5.31}$$

Then (5.31) possesses a unique solution in the space

$$B^{1,2}_{\#,\mathrm{div}}(\mathbb{J}) = \left\{\boldsymbol{u} \in B^{1,2}_{\#}(\mathbb{J})^3 : \overline{\operatorname{div}}_y \boldsymbol{u} = 0 \text{ in } \mathbb{J}\right\}.$$

With this in mind, if we choose in the variational form of (5.20) the test function $\varphi = \psi \otimes \boldsymbol{w}_i$ $(1 \leq i \leq 3)$ where $\boldsymbol{w}_i$ is determined by (5.31) and $\psi \in \mathcal{C}_0^\infty(\Omega)$, and then take in the variational form of (5.31), the test function $\boldsymbol{u}_0(\overline{x},\cdot)$, and finally we compare the resulting equalities, we get at once

$$\boldsymbol{u}'(\overline{x}) = \widehat{A}(\boldsymbol{f}_1(\overline{x}) - \nabla_{\overline{x}} p_0(\overline{x})), \qquad \text{a.e. } \overline{x} \in \Omega$$

where the matrix $\widehat{A} = (\widehat{a}_{ij})_{1\leq i,j\leq 2}$ is symmetric, positive definite and is defined by

$$\widehat{a}_{ij} = M\left(\int_{h_1}^{h_2} A\overline{\nabla}_y \boldsymbol{w}_i \cdot \overline{\nabla}_y \boldsymbol{w}_j \, dy_3\right) + \frac{\mu}{K} M\left(\int_{h_1}^{h_2} \boldsymbol{w}_i \cdot \boldsymbol{w}_j \, dy_3\right). \tag{5.32}$$

We have just derived the main result in the case when $K_\varepsilon = O(\varepsilon^2)$, and it reads as follows.

Theorem 5.2. *The assumptions are those of Theorem 5.1. For any $\varepsilon > 0$, let $(\boldsymbol{u}_\varepsilon, p_\varepsilon = p_\varepsilon^0 + \varepsilon p_\varepsilon^1) \in H_0^1(\Omega^\varepsilon)^3 \times L_0^2(\Omega^\varepsilon)$ be a solution of (4.1). Then, still denoting by $\boldsymbol{u}_\varepsilon$ and p_ε^1 the extension of $\boldsymbol{u}_\varepsilon$ and p_ε^1 by zero on G_ε, the sequence $(\boldsymbol{u}_\varepsilon/\varepsilon^2, p_\varepsilon)$ weakly $\Sigma_{\mathcal{A}}$-converges in $L^2(G_\varepsilon)^3 \times L^2(G_\varepsilon)$ towards $(\boldsymbol{u}_0, p_0)$ determined by (5.14)–(5.17). Defining $\boldsymbol{u} = (\boldsymbol{u}', u_3)$ by (5.19), we have $u_3 = 0$ and $(\boldsymbol{u}', p_0)$ is the unique solution of the homogenized problem*

$$\begin{cases} \boldsymbol{u}' = \widehat{A}(\boldsymbol{f}_1 - \nabla_{\overline{x}} p_0) & \text{in } \Omega, \\ \operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 & \text{in } \Omega, \quad \text{and} \quad \boldsymbol{u}' \cdot \nu = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where the matrix $\widehat{A}$ is defined by (5.32).

5.2.2. Case when $K_\varepsilon \ll \varepsilon^2$ or $K_\varepsilon \gg \varepsilon^2$. Following the same steps as in the previous section, we may proceed as in the previous case to obtain the homogenization result in the two other regimes. This is left to the reader.

6. Some concrete applications of Theorems 1.3 and 1.5

In the previous sections, we have used systematically the algebras with mean value in assumptions (A3) and (3.26) (or (5.3)) respectively on the coefficients of the diffusion operator $-\operatorname{div}(A^{\varepsilon}\nabla .)$ and on the functions h_1 and h_2. Assumption (A3) shows how the microstructures are distributed inside the domain Ω^{ε} while (3.26) deals with the way the lateral boundaries behave. We present here below a few concrete physical situations.

6.1. Problem 1: Pure periodic environment

We assume that the heterogeneities are uniformly distributed in Ω. This means that the distribution function of the microstructures is periodic, so that the matrix-function $\overline{y} \mapsto A(\overline{y}, y_3)$ is 1-periodic in each of its y_i $(i = 1, 2)$. The underlying algebra with mean value here is thus the algebra of Y-periodic continuous functions $\mathcal{A} = \mathcal{C}_{\text{per}}(Y)$, $Y = (0, 1)^2$. The mean value of a function $u \in \mathcal{C}_{\text{per}}(Y)$ is given by

$$M(u) = \int_Y u(y)dy.$$

The function spaces associated to $\mathcal{A}$ are as follows: $B^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I)) = L^2_{\text{per}}(Y; L^2(I))$ (the space of functions in $L^2_{\text{loc}}(\mathbb{R}^2; L^2(I))$ that are Y-periodic), $B^{1,2}_{\mathcal{A}}(\mathbb{R}^{d-1}; H^1(I)) = H^1_{\text{per}}(Y; H^1(I))$ (the subspace of $H^1_{\text{loc}}(Y; H^1(I))$ consisting of Y-periodic functions). It is worth noting that $B^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I)) = \mathcal{B}^2_{\mathcal{A}}(\mathbb{R}^2; L^2(I))$ since $L^2_{\text{per}}(Y; L^2(I))$ is a Banach space with the corresponding norm, and so $B^{1,2}_{\mathcal{A}}(\mathbb{R}^2; H^1(I)) = \mathcal{B}^{1,2}_{\mathcal{A}}(\mathbb{R}^2; H^1(I))$.

In this case, the sigma-convergence concept is merely the well-known two-scale convergence method for thin heterogeneous domains defined in [22] (see Remark 3.6) as follows: A sequence $(u_{\varepsilon})_{\varepsilon>0} \subset L^2(G_{\varepsilon})$ weakly two-scale converges in $L^2(G_{\varepsilon})$ towards $u_0 \in L^2(G_0; L^2_{\text{per}}(Y; L^2(I)))$ if, when $\varepsilon \to 0$,

$$\frac{1}{\varepsilon}\int_{G_{\varepsilon}} u_{\varepsilon}(x) f\left(\overline{x}, \frac{x}{\varepsilon}\right)dx \to \int_{G_0}\int_Z u_0(\overline{x}, y) f(\overline{x}, y)dyd\overline{x}$$

for any $f \in L^2(G_0; \mathcal{C}_{\text{per}}(Y; L^2(I)))$, where $Z = Y \times I$.

The above assumption on the matrix A amounts to $A \in L^2_{\text{per}}(Y; L^2(I))^{3\times 3}$. We also assume that $h_1, h_2 \in W^{1,\infty}(\mathbb{R}^2) \cap \mathcal{C}_{\text{per}}(Y)$. In order to state the periodic version of Theorem 1.5, we need to define some function spaces. First of all, let

$$\widetilde{Y} = \{y \in \mathbb{R}^3 : \overline{y} \in Y \text{ and } h_1(\overline{y}) < y_3 < h_2(\overline{y})\} \quad \text{and} \quad \Gamma = \{y \in \partial\mathbb{J} : \overline{y} \in Y\},$$

where we have assumed that the functions h_1 and h_2 are continuous Y-periodic functions: $h_1, h_2 \in \mathcal{C}_{\text{per}}(Y)$. The periodic counterparts of the spaces $B^2_{\mathcal{A}}(\mathbb{J})$, $B^{1,2}_{\mathcal{A}}(\mathbb{J})$ and

$B^{1,2}_{\#}(\mathbb{J})$, denoted below respectively by $L^2_{\rm per}(\widetilde{Y})$, $H^1_{\rm per}(\widetilde{Y})$ and $H^1_{\#}(\widetilde{Y})$ are defined as follows:

$$L^2_{\rm per}(\widetilde{Y}) = \Big\{ u \in L^2_{\rm loc}(\mathbb{J}) : \int_{\widetilde{Y}} |u|^2 dy < \infty \text{ and } u(\overline{y}+k, y_3) = u(\overline{y}, y_3) \quad \forall k \in \mathbb{Z}^2, \text{ a.e. } y \in \mathbb{J} \Big\},$$

$$H^1_{\rm per}(\widetilde{Y}) = \big\{ u \in H^1_{\rm loc}(\mathbb{J}) : u \in L^2_{\rm per}(\widetilde{Y}),\ \nabla_y u \in L^2_{\rm per}(\widetilde{Y})^3 \big\},$$

$$H^1_{\#}(\widetilde{Y}) = \big\{ u \in H^1_{\rm per}(\widetilde{Y}) : u = 0 \text{ on } \Gamma \big\}.$$

For the benefit of the reader, we restate the homogenization result in Theorem 1.5 in the periodic setting.

Theorem 6.1. *Assume that Ω^ε is given by (1.2) where the functions $h_1, h_2 \in \mathscr{C}_{\rm per}(Y)$. Let $(\boldsymbol{u}_\varepsilon, p_\varepsilon = p^0_\varepsilon + \varepsilon p^1_\varepsilon) \in H^1_0(\Omega^\varepsilon)^3 \times L^2_0(\Omega^\varepsilon)$ be a solution of (1.3). Assume that $A \in (L^2_{\rm per}(Y; L^\infty(I)))^{3\times 3}$. Then:*

(i) *If $K_\varepsilon = O(\varepsilon^2)$ with $K_\varepsilon/\varepsilon^2 \to K$ when $\varepsilon \to 0$, $0 < K < \infty$, then, still denoting by $\boldsymbol{u}_\varepsilon$ the extension by zero of $\boldsymbol{u}_\varepsilon$ on $G_\varepsilon = \Omega \times (\varepsilon h_1^-, \varepsilon h_2^+)$, one has*

$$\frac{\boldsymbol{u}_\varepsilon}{\varepsilon^2} \to \boldsymbol{u}_0 \quad \text{in } L^2(G_\varepsilon)^3\text{-weak } \Sigma_{\mathcal{A}},$$

and

$$p^0_\varepsilon \to p_0 \quad \text{in } H^1(\Omega)\text{-weak and in } L^2(\Omega)\text{-strong}.$$

Defining $\boldsymbol{u} = (\boldsymbol{u}', u_3)$ by $\boldsymbol{u}(\overline{x}) = \int_{\widetilde{Y}} \boldsymbol{u}_0(\overline{x}, y)dy$, we have $u_3 = 0$ and $(\boldsymbol{u}', p_0)$ is the unique solution of the homogenized problem

$$\begin{cases} \boldsymbol{u}' = \widehat{A}(\boldsymbol{f}_1 - \nabla_{\overline{x}} p_0) & \text{in } \Omega, \\ \operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 & \text{in } \Omega, \quad \text{and} \quad \boldsymbol{u}' \cdot \nu = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\widehat{A} = (\widehat{a}_{ij})_{1 \le i,j \le 2}$ is a symmetric, positive definite 2×2 matrix defined by its entries

$$\widehat{a}_{ij} = \int_{\widetilde{Y}} A \nabla_y \boldsymbol{w}_i \cdot \nabla_y \boldsymbol{w}_j dy + \frac{\mu}{K} \int_{\widetilde{Y}} \boldsymbol{w}_i \cdot \boldsymbol{w}_j dy.$$

Here $\boldsymbol{w}_i$ $(1 \le i \le 2)$ is the unique solution in $H^1_{\#}(\widetilde{Y})^3$ of the Stokes–Brinkmann system

$$\begin{cases} -\operatorname{div}_y(A(y)\nabla_y \boldsymbol{w}_i) + \dfrac{\mu}{K} \boldsymbol{w}_i + \nabla_y \pi_i = e_i & \text{in } \mathbb{J}, \\ \operatorname{div}_y \boldsymbol{w}_i = 0 & \text{in } \mathbb{J}, \\ \boldsymbol{w}_i = 0 & \text{on } \partial\mathbb{J}, \end{cases}$$

e_i being the ith vector of the canonical basis in $\mathbb{R}^3$.

(ii) *If $K_\varepsilon \ll \varepsilon^2$, then, up to a subsequence, one has*

$$\frac{\boldsymbol{u}_\varepsilon}{\varepsilon K_\varepsilon^{\frac{1}{2}}} \to \boldsymbol{u}_0 \quad \text{in } L^2(G_\varepsilon)^3\text{-weak } \Sigma_{\mathcal{A}},$$
$$\frac{K_\varepsilon^{\frac{1}{2}}}{\varepsilon} p_\varepsilon^0 \to p_0 \quad \text{in } H^1(\Omega)\text{-weak and in } L^2(\Omega)\text{-strong.}$$

Furthermore, defining $\boldsymbol{u}$ *as in* (i) *above, we have* $u_3 = 0$ *and* $(\boldsymbol{u}', p_0)$ *is a solution of*

$$\boldsymbol{u}' = -\widehat{A}\nabla_{\overline{x}} p_0 \quad \text{in } \Omega, \quad \operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 \quad \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}' \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where $\widehat{A}$ *is a symmetric matrix defined by*

$$\widehat{A} = (\widehat{a}_{ij})_{1\leq i,j\leq 2} \quad \text{with } \widehat{a}_{ij} = \mu \int_{\widetilde{Y}} \boldsymbol{w}_i \boldsymbol{w}_j \, dy,$$

$\boldsymbol{w}_i$ *($1 \leq i \leq 2$) being the unique solution in* $L^2_{\mathrm{per}}(\widetilde{Y})^3$ *of*

$$\mu \boldsymbol{w}_i + \nabla_y \pi_i = e_i \quad \text{in } \mathbb{J} \quad \text{and} \quad \operatorname{div}_y \boldsymbol{w}_i = 0 \quad \text{in } \mathbb{J}.$$

(iii) *If $K_\varepsilon \gg \varepsilon^2$, then, still denoting by $\boldsymbol{u}_\varepsilon$ and p_ε^1 the extension of $\boldsymbol{u}_\varepsilon$ and p_ε^1 by zero on G_ε,we have,*

$$\frac{\boldsymbol{u}_\varepsilon}{\varepsilon^2} \to \boldsymbol{u}_0 \quad \text{in } L^2(G_\varepsilon)^3\text{-weak } \Sigma_{\mathcal{A}},$$
$$\frac{1}{\varepsilon}\nabla \boldsymbol{u}_\varepsilon \to \overline{\nabla}_y \boldsymbol{u}_0 \quad \text{in } L^2(G_\varepsilon)^{3\times 3}\text{-weak } \Sigma_{\mathcal{A}},$$
$$p_\varepsilon^0 \to p_0 \quad \text{in } H^1(\Omega)\text{-weak and in } L^2(\Omega)\text{-strong,}$$

Still defining $\boldsymbol{u}$ *as in* (i) *above, it holds that*

$$\begin{cases} \boldsymbol{u}' = \widehat{A}(\boldsymbol{f}_1 - \nabla_{\overline{x}} p_0) & \text{in } \Omega, \\ \operatorname{div}_{\overline{x}} \boldsymbol{u}' = 0 & \text{in } \Omega \quad \text{and} \quad \boldsymbol{u}' \cdot \nu = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\widehat{A} = (\widehat{a}_{ij})_{1\leq i,j\leq 2}$ *is given by*

$$\widehat{a}_{ij} = \int_{\widetilde{Y}} A\nabla_y \boldsymbol{w}_i \cdot \nabla_y \boldsymbol{w}_j \, dy, \quad 1 \leq i, j \leq 2,$$

with $\boldsymbol{w}_i$ *($1 \leq i \leq 2$) being the unique solution in* $H^1_\#(\widetilde{Y})^3$ *of the Stokes system*

$$\begin{cases} -\operatorname{div}_y(A(y)\nabla_y \boldsymbol{w}_i) + \nabla_y \pi_i = e_i & \text{in } \mathbb{J}, \\ \operatorname{div}_y \boldsymbol{w}_i = 0 & \text{in } \mathbb{J}. \end{cases}$$

Proof. The result is a mere consequence of the equality $M(u) = \int_Y u(y)dy$ for any $u \in L^2_{\mathrm{per}}(Y)$. ∎

6.2. Problem 2: Asymptotic periodic framework

We assume that the functions h_1 and h_2 are asymptotic periodic, that is, each of these functions can be expressed as a sum of a continuous periodic function and of a continuous function which vanishes at infinity. This leads to the consideration of the algebra with mean value $\mathcal{A} = \mathcal{C}_{\mathrm{per}}(Y) + \mathcal{C}_0(\mathbb{R}^2) \equiv \mathcal{B}_{\infty,\mathrm{per}}(\mathbb{R}^2)$ [17, Section 5.2.3], where $\mathcal{C}_0(\mathbb{R}^2)$ stands for the Banach algebra of continuous functions vanishing at infinity. Since $\mathcal{C}_{\mathrm{per}}(Y) \subset \mathcal{C}_{\mathrm{per}}(Y) + \mathcal{C}_0(\mathbb{R}^2)$, we may assume either that $A \in L^2_{\mathrm{per}}(Y; L^2(I))^{3\times 3}$ or $A \in B^2_{\mathcal{B}_{\infty,\mathrm{per}}(\mathbb{R}^2)}(\mathbb{R}^2; L^2(I))^{3\times 3}$. Then the results in Theorems 1.3 and 1.5 are obtained with the algebra wmv $\mathcal{A} = \mathcal{C}_{\mathrm{per}}(Y) + \mathcal{C}_0(\mathbb{R}^2)$.

It is very important to note that we may assume different kind of behaviours on h_1 and on h_2. For example, if $h_1 \in \mathcal{C}_{\mathrm{per}}(Y)$ and $h_2 \in \mathcal{C}_{\mathrm{per}}(Y) + \mathcal{C}_0(\mathbb{R}^2)$, then we reach the same conclusion of Theorems 1.3 and 1.5 with $\mathcal{A} = \mathcal{C}_{\mathrm{per}}(Y) + \mathcal{C}_0(\mathbb{R}^2)$.

6.3. Problem 3: Almost periodic setting

We assume that the microstructures inside Ω are distributed in an almost periodic fashion, that is, the function $\overline{y} \mapsto A(\overline{y}, y_3)$ is almost periodic in the Besicovitch sense [6, 7]. The underlying algebra with mean value in $\mathbb{R}^2$ is the algebra of Bohr continuous almost periodic functions on $\mathbb{R}^2$ denoted by $\mathcal{A} = \mathrm{AP}(\mathbb{R}^2)$. It is worth recalling that $\mathrm{AP}(\mathbb{R}^2)$ [6, 7] is defined as the algebra of functions on $\mathbb{R}^2$ that are uniformly approximated by finite linear combinations of functions in the set $\{\cos(k\cdot), \sin(k\cdot) : k \in \mathbb{R}^{d-1}\}$ where $\cos(k\cdot)(y) = \cos(2\pi k \cdot y)$ and $\sin(k\cdot)(y) = \sin(2\pi k \cdot y)$, $y \in \mathbb{R}^2$. It is known that $\mathrm{AP}(\mathbb{R}^2)$ is an algebra wmv called the almost periodic algebra wmv on $\mathbb{R}^2$. The corresponding generalized Besicovitch space $B^p_A(\mathbb{R}^2)$ is precisely the Besicovitch space $\mathcal{B}^p(\mathbb{R}^2)$ defined in [6, 7].

We also assume that the functions h_1, h_2 belong to $\mathrm{AP}(\mathbb{R}^2)$. Then under these assumptions, the main results Theorems 1.3 and 1.5 are valid with the corresponding function spaces. It is well known from [7] that the mean value of a function $u \in \mathrm{AP}(\mathbb{R}^2)$ is the unique constant that belongs to the closed convex hull of the set of translates $\{u(\cdot + a) : a \in \mathbb{R}^2\}$ of u.

6.4. Problem 4: The asymptotic almost periodic environment

We may deal with the asymptotic almost periodic distribution of heterogeneities inside Ω with the corresponding algebra wmv $\mathcal{A} = \mathrm{AP}(\mathbb{R}^2) + \mathcal{C}_0(\mathbb{R}^2)$ [17, Section 5.2.3]. In this case, we may assume that the functions h_1 and h_2 are either in $\mathrm{AP}(\mathbb{R}^2)$ or in $\mathrm{AP}(\mathbb{R}^2) + \mathcal{C}_0(\mathbb{R}^2)$. We may also assume that $h_1 \in \mathrm{AP}(\mathbb{R}^2)$ and $h_2 \in \mathrm{AP}(\mathbb{R}^2) + \mathcal{C}_0(\mathbb{R}^2)$. All this leads to the validity of Theorems 1.3 and 1.5 with $\mathcal{A} = \mathrm{AP}(\mathbb{R}^2) + \mathcal{C}_0(\mathbb{R}^2)$.

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