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Rates of (T-)asymptotic regularity of the generalized Krasnoselskii–Mann-type iteration

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Abstract. In this paper, we use proof mining methods to compute rates of (T-)asymptotic regularity of the generalized Krasnoselskii–Mann-type iteration associated to a nonexpansive mapping $T : X \to X$ in a uniformly convex normed space X. For special choices of the parameter sequences, we obtain quadratic rates.

1. Introduction

Let X be a normed space and $T : X \to X$ be a nonexpansive mapping with fixed points. The *generalized Krasnoselskii–Mann-type iteration*, studied by Kanzow and Shehu [6] in Hilbert spaces and by Zhang, Guo, Wang [20] in classes of uniformly convex Banach spaces, is defined as follows:

$$x_0 = x \in X, \quad x_{n+1} = \alpha_n x_n + \beta_n T x_n + r_n, \tag{1.1}$$

where $(r_n)_{n\in\mathbb{N}} \subseteq X$ and $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}} \subseteq [0,1]$ satisfy $\alpha_n + \beta_n \leq 1$ for all $n \in \mathbb{N}$.

Obviously, if $r_n = 0$ and $\alpha_n + \beta_n = 1$ for all $n \in \mathbb{N}$, then (x_n) becomes the well-known Krasnoselskii–Mann iteration [5, 13, 19]. As pointed out in [6, 20], by letting $\alpha_n = 1 - \beta_n$ and $r_n = \beta_n e_n$, where $(e_n) \subseteq X$ is a sequence of error terms, one obtains the inexact Krasnoselskii–Mann iteration studied by Combettes [3] in Hilbert spaces, by Kim and Xu [7] in Banach spaces and, more recently, by Liang, Fadili and Peyré [17].

Kanzow and Shehu [6] prove, in Hilbert spaces, the weak convergence of the generalized Krasnoselskii–Mann-type iteration (x_n) to a fixed point of T under some hypotheses on the parameter sequences (α_n), (β_n), (r_n). Zhang, Guo, Wang [20] generalize this weak convergence result to uniformly convex Banach spaces that satisfy additional properties. By analyzing the proofs from [20] one can see that, as it happens for numerous weak/strong convergence proofs of nonlinear iterations, an essential intermediate step is to obtain the (T-)asymptotic regularity [1, 2] of (x_n).

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We say that (x_n) is T-asymptotically regular if $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and that (x_n) is asymptotically regular if $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. A mapping $\Phi : \mathbb{N} \to \mathbb{N}$ is said to be

(i) a rate of T-asymptotic regularity of (x_n) if Φ is a rate of convergence towards 0 of the sequence $(\|x_n - Tx_n\|)$;

(ii) a rate of asymptotic regularity of (x_n) if Φ is a rate of convergence towards 0 of the sequence $(\|x_{n+1} - x_n\|)$.

We refer to Section 3 for the definition of a rate of convergence.

If Φ is a rate of (T-)asymptotic regularity of (x_n), we also say that (x_n) is (T-)asymptotically regular with rate Φ.

In this paper, we apply methods of proof mining [10, 11] developed by the second author for the Ishikawa iteration of a nonexpansive mapping [15, 16] to compute for the first time uniform rates of (T-)asymptotic regularity of the generalized Krasnoselskii–Mann-type iteration (x_n) in uniformly convex normed spaces. Furthermore, quadratic such rates are obtained for special choices of the parameter sequences.

Notations: $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\mathbb{R}_+ = [0,\infty)$. If $m, n \in \mathbb{N}$, $n \geq m$, we write $[m;n] = \{m, m+1, \ldots, n\}$.

2. Generalized Krasnoselskii–Mann-type iteration

In the following, X is a normed space, $T : X \to X$ a nonexpansive mapping and (x_n) is the generalized Krasnoselskii–Mann-type iteration defined by (1.1). We denote by $\mathrm{Fix}(T)$ the set of fixed points of T.

For every $z \in X$, let $(K_{z,n})$ be a sequence of real numbers defined as follows:

$$K_{z,0} = \|x - z\|,$$

$$K_{z,n} = \|x - z\| + \sum_{i=0}^{n-1} \beta_i \|Tz - z\| + \sum_{i=0}^{n-1}(1 - \alpha_i - \beta_i)\|z\| + \sum_{i=0}^{n-1}\|r_i\| \quad \text{for } n \geq 1.$$

Thus, for all $n \geq 0$,

$$K_{z,n+1} = K_{z,n} + \beta_n\|Tz - z\| + (1 - \alpha_n - \beta_n)\|z\| + \|r_n\|.$$

Lemma 2.1. *For all $z \in X$ and all $n \in \mathbb{N}$,*

$$\|x_{n+1} - z\| \leq (\alpha_n + \beta_n)\|x_n - z\| + \beta_n\|Tz - z\| + (1 - \alpha_n - \beta_n)\|z\| + \|r_n\|, \tag{2.1}$$

$$\|x_n - z\| \leq K_{z,n}, \tag{2.2}$$

$$\|x_{n+1} - x_n\| \le 2K_{z,n+1}, \tag{2.3}$$

$$\|x_n - Tx_n\| \le 2\|x_n - z\| + \|z - Tz\| \le 2K_{z,n} + \|z - Tz\|. \tag{2.4}$$

Proof. We have that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n Tx_n + r_n - z\| \\
&= \|\alpha_n(x_n - z) + \beta_n(Tx_n - z) + r_n - (1 - \alpha_n - \beta_n)z\| \\
&\le \alpha_n\|x_n - z\| + \beta_n\|Tx_n - z\| + \|r_n - (1 - \alpha_n - \beta_n)z\| \\
&\le \alpha_n\|x_n - z\| + \beta_n\|Tx_n - Tz\| + \beta_n\|Tz - z\| \\
&\quad + \|r_n\| + \|(1 - \alpha_n - \beta_n)z\| \\
&\le (\alpha_n + \beta_n)\|x_n - z\| + \beta_n\|Tz - z\| + (1 - \alpha_n - \beta_n)\|z\| + \|r_n\|,
\end{aligned}$$

as T is nonexpansive and $\alpha_n + \beta_n \le 1$. Thus, (2.1) holds.

(2.2) is proved by induction on n. The case $n = 0$ is obvious.

$n \Rightarrow n + 1$:

$$\begin{aligned}
\|x_{n+1} - z\| &\overset{(2.1)}{\le} (\alpha_n + \beta_n)\|x_n - z\| + \beta_n\|Tz - z\| + (1 - \alpha_n - \beta_n)\|z\| + \|r_n\| \\
&\le \|x_n - z\| + \beta_n\|Tz - z\| + (1 - \alpha_n - \beta_n)\|z\| + \|r_n\| \\
&\qquad \text{as } \alpha_n + \beta_n \le 1 \\
&\le K_{z,n} + \beta_n\|Tz - z\| + (1 - \alpha_n - \beta_n)\|z\| + \|r_n\| \\
&\qquad \text{by the induction hypothesis} \\
&= K_{z,n+1}.
\end{aligned}$$

(2.3) and (2.4) follow easily:

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \|x_{n+1} - z\| + \|x_n - z\| \overset{(2.2)}{\le} K_{z,n+1} + K_{z,n} \le 2K_{z,n+1}, \\
\|x_n - Tx_n\| &\le \|x_n - z\| + \|z - Tz\| + \|Tz - Tx_n\| \le 2\|x_n - z\| + \|z - Tz\| \\
&\overset{(2.2)}{\le} 2K_{z,n} + \|z - Tz\|.
\end{aligned}$$

∎

A very useful immediate consequence of (2.4) is the following.

Corollary 2.2. *Assume that $z \in \mathrm{Fix}(T)$. Then for all $n \in \mathbb{N}$,*

$$\|x_n - Tx_n\| \le 2\|x_n - z\| \le 2K_{z,n}.$$

Lemma 2.3. *For all $n \in \mathbb{N}$,*

$$\|x_{n+1} - x_n\| \le \beta_n\|x_n - Tx_n\| + (1 - \alpha_n - \beta_n)\|x_n\| + \|r_n\|, \tag{2.5}$$

$$\|x_{n+1} - Tx_{n+1}\| \le \|x_n - Tx_n\| + 2(1 - \alpha_n - \beta_n)\|x_n\| + 2\|r_n\|. \tag{2.6}$$

Proof. Let $n \in \mathbb{N}$. We get that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\beta_n(Tx_n - x_n) - (1-\alpha_n-\beta_n)x_n + r_n\| \\ &\leq \beta_n\|x_n - Tx_n\| + (1-\alpha_n-\beta_n)\|x_n\| + \|r_n\|\end{aligned}$$

and

$$\begin{aligned}&\|Tx_{n+1} - x_{n+1}\| \\ &\quad = \|Tx_{n+1} - Tx_n + (1-\beta_n)(Tx_n - x_n) + (1-\alpha_n-\beta_n)x_n - r_n\| \\ &\quad \leq \|Tx_{n+1} - Tx_n\| + (1-\beta_n)\|Tx_n - x_n\| + (1-\alpha_n-\beta_n)\|x_n\| + \|r_n\| \\ &\quad \leq \|x_{n+1} - x_n\| + (1-\beta_n)\|Tx_n - x_n\| + (1-\alpha_n-\beta_n)\|x_n\| + \|r_n\| \\ &\quad \stackrel{(2.5)}{\leq} \|x_n - Tx_n\| + 2(1-\alpha_n-\beta_n)\|x_n\| + 2\|r_n\|.\end{aligned}$$

■

2.1. A useful lemma in uniformly convex normed spaces

Let us recall that a normed space X is uniformly convex if for all $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $x, y \in X$,

$$\|x\| \leq 1, \quad \|y\| \leq 1 \quad \text{and} \quad \|x-y\| \geq \varepsilon \quad \text{imply} \quad \left\|\frac{1}{2}(x+y)\right\| \leq 1-\delta. \tag{2.7}$$

A modulus of uniform convexity of X is a mapping $\eta : (0, 2] \to (0, 1]$ providing a $\delta := \eta(\varepsilon)$ satisfying (2.7) for given $\varepsilon \in (0, 2]$. Thus, X is uniformly convex if and only if X has a modulus of uniform convexity η.

As pointed out in [12, p. 3457] (see also [18, p. 63]), the uniformly convex Banach spaces L_p $(1 < p < \infty)$ have a modulus of uniform convexity η_p given by

$$\eta_p(\varepsilon) = \begin{cases} \frac{(p-1)\varepsilon^2}{8} & \text{if } 1 < p < 2, \\ \frac{\varepsilon^p}{p \cdot 2^p} & \text{if } 2 \leq p < \infty. \end{cases}$$

Thus, $\eta(\varepsilon) = \frac{\varepsilon^2}{8}$ is a modulus of uniform convexity for a Hilbert space.

In the sequel, we assume that X is uniformly convex and η is a modulus of uniform convexity of X.

Lemma 2.4. *Let $\varepsilon \in (0, 2]$, $r > 0$ and $a, x, y \in X$ be such that*

$$\|x-a\| \leq r, \quad \|y-a\| \leq r \quad \textit{and} \quad \|x-y\| \geq \varepsilon r.$$

Then for all $\lambda \in [0, 1]$,

$$\|(1-\lambda)x + \lambda y - a\| \leq (1 - 2\lambda(1-\lambda)\eta(\varepsilon))r.$$

Proof. Apply [9, Lemma 3.3] with $x := \frac{1}{r}(y-a)$ and $y := \frac{1}{r}(x-a)$. ■

The following lemma is the main tool in proving the results from Subsection 6.1.

Lemma 2.5. *Let z be a fixed point of T and $n \in \mathbb{N}$ be such that $\alpha_n + \beta_n > 0$.*

(i) *If $\gamma, \delta, \theta > 0$ satisfy*

$$\gamma \le \|x_n - z\| \le \delta \quad \text{and} \quad \|x_n - Tx_n\| \ge \theta,$$

then

$$\|\alpha_n x_n + \beta_n T x_n - (\alpha_n + \beta_n) z\| \le \|x_n - z\| - \frac{2\gamma \alpha_n \beta_n}{\alpha_n + \beta_n} \eta\left(\frac{\theta}{\delta}\right).$$

(ii) *Assume that η can be written as $\eta(\varepsilon) = \varepsilon \cdot \widetilde{\eta}(\varepsilon)$ with $\widetilde{\eta}$ increasing. If $\delta^*, \theta > 0$ are such that*

$$\|x_n - z\| \le \delta^* \quad \text{and} \quad \|x_n - Tx_n\| \ge \theta,$$

then

$$\|\alpha_n x_n + \beta_n T x_n - (\alpha_n + \beta_n) z\| \le \|x_n - z\| - \frac{2\theta \alpha_n \beta_n}{\alpha_n + \beta_n} \widetilde{\eta}\left(\frac{\theta}{\delta^*}\right).$$

Proof. (i) We have that

$$\alpha_n x_n + \beta_n T x_n = \frac{\alpha_n}{\alpha_n + \beta_n}(\alpha_n + \beta_n) x_n + \frac{\beta_n}{\alpha_n + \beta_n}(\alpha_n + \beta_n) T x_n$$

and

$$\|(\alpha_n + \beta_n) T x_n - (\alpha_n + \beta_n) z\| \le (\alpha_n + \beta_n) \|x_n - z\|,$$

$$\|(\alpha_n + \beta_n) x_n - (\alpha_n + \beta_n) T x_n\| \ge (\alpha_n + \beta_n)\theta \ge (\alpha_n + \beta_n) \|x_n - z\| \frac{\theta}{\delta}.$$

Obviously, $\frac{\|x_n - Tx_n\|}{\|x_n - z\|} > 0$ and, by Corollary 2.2, $\frac{\theta}{\delta} \le \frac{\|x_n - Tx_n\|}{\|x_n - z\|} \le 2$.

We can apply Lemma 2.4 with

$$x := (\alpha_n + \beta_n) x_n, \qquad y := (\alpha_n + \beta_n) T x_n, \quad a := (\alpha_n + \beta_n) z,$$

$$r := (\alpha_n + \beta_n) \|x_n - z\|, \quad \lambda := \frac{\beta_n}{\alpha_n + \beta_n}, \qquad \varepsilon := \frac{\theta}{\delta}$$

to get that

$$\begin{aligned}
\|\alpha_n x_n + \beta_n T x_n - (\alpha_n + \beta_n) z\| &\le \left(1 - 2\frac{\alpha_n \beta_n}{(\alpha_n + \beta_n)^2} \eta\left(\frac{\theta}{\delta}\right)\right)(\alpha_n + \beta_n) \|x_n - z\| \\
&= (\alpha_n + \beta_n)\|x_n - z\| - \frac{2\alpha_n \beta_n}{\alpha_n + \beta_n} \eta\left(\frac{\theta}{\delta}\right) \|x_n - z\| \\
&\le \|x_n - z\| - \frac{2\alpha_n \beta_n}{\alpha_n + \beta_n} \eta\left(\frac{\theta}{\delta}\right) \gamma.
\end{aligned}$$

(ii) Remark first that, by Corollary 2.2 and hypothesis, we have that

$$\|x_n - z\| \geq \frac{\|x_n - Tx_n\|}{2} \geq \frac{\theta}{2} > 0.$$

Apply (i) with $\gamma := \delta := \|x_n - z\|$ to get that

$$\begin{aligned}
\|\alpha_n x_n + \beta_n T x_n - (\alpha_n + \beta_n) z\| &\leq \|x_n - z\| - \frac{2\|x_n - z\|\alpha_n\beta_n}{\alpha_n + \beta_n}\eta\left(\frac{\theta}{\|x_n - z\|}\right) \\
&= \|x_n - z\| - \frac{2\theta\alpha_n\beta_n}{\alpha_n + \beta_n}\widetilde{\eta}\left(\frac{\theta}{\|x_n - z\|}\right) \\
&\leq \|x_n - z\| - \frac{2\theta\alpha_n\beta_n}{\alpha_n + \beta_n}\widetilde{\eta}\left(\frac{\theta}{\delta^*}\right),
\end{aligned}$$

as $\frac{\theta}{\delta^*} \leq \frac{\theta}{\|x_n - z\|}$, so $\widetilde{\eta}(\frac{\theta}{\delta^*}) \leq \widetilde{\eta}(\frac{\theta}{\|x_n - z\|})$, since $\widetilde{\eta}$ is increasing. ∎

3. Quantitative notions, lemmas and hypotheses

Firstly, let us recall the quantitative notions used in the paper. Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence of real numbers. We say that a function $\varphi : \mathbb{N} \to \mathbb{N}$ is

(i) a rate of convergence of (a_n) towards $a \in \mathbb{R}$ if for all $k \in \mathbb{N}$,

$$|a_n - a| \leq \frac{1}{k+1} \quad \text{holds for all } n \geq \varphi(k).$$

(ii) a Cauchy modulus of (a_n) if for all $k \in \mathbb{N}$,

$$|a_{n+p} - a_n| \leq \frac{1}{k+1} \quad \text{holds for all } n \geq \varphi(k) \text{ and all } p \in \mathbb{N}.$$

If the series $\sum_{n=0}^{\infty} a_n$ converges, then a Cauchy modulus of the series is a Cauchy modulus of the sequence $(\widetilde{a}_n = \sum_{i=0}^{n} a_i)$ of partial sums. Furthermore, if $\sum_{n=0}^{\infty} a_n$ diverges, then $\theta : \mathbb{N} \to \mathbb{N}$ is a rate of divergence of the series if $\sum_{i=0}^{\theta(k)} a_i \geq k$ for all $k \in \mathbb{N}$.

The following useful lemmas are immediate.

Lemma 3.1. *Assume that $(a_n) \subseteq \mathbb{R}_+$ and that $\sum_{n=0}^{\infty} a_n$ converges with Cauchy modulus φ. Then*

(i) $\sum_{n=0}^{\infty} a_n \leq M$, *where $M \in \mathbb{N}^*$ is such that $M \geq \sum_{i=0}^{\varphi(0)} a_i + 1$.*

(ii) $\lim_{n\to\infty} a_n = 0$ *with rate of convergence $\psi(k) = \varphi(k) + 1$.*

Proof. See the proof of [4, Lemma 4.3 (ii)]. ∎

Lemma 3.2. *Assume that $a_n \in [0,1)$ for all $n \in \mathbb{N}$ and that $\sum_{n=0}^{\infty} a_n$ diverges with rate of divergence θ. Then $\theta(n) \geq n$ for all $n \in \mathbb{N}$.*

Proof. Suppose, by contradiction, that for some $n \in \mathbb{N}$, we have that $\theta(n) < n$, hence $\theta(n) \leq n-1$. Then $\sum_{i=0}^{\theta(n)} a_i \leq \sum_{i=0}^{n-1} a_i < n$, which is a contradiction. ∎

Lemma 3.3. *Let $(a_n), (b_n) \subseteq \mathbb{R}$, $s, t \in \mathbb{N}^*$ and $c_n = sa_n + tb_n$ for all $n \in \mathbb{N}$. Assume that φ_1 is a Cauchy modulus of (a_n) and φ_2 is a Cauchy modulus of (b_n). Define*

$$\varphi(k) = \max\{\varphi_1(2s(k+1)-1), \varphi_2(2t(k+1)-1)\}.$$

Then φ is a Cauchy modulus of (c_n).

Proof. See the proof of [4, Lemma 3.2]. ∎

3.1. A quantitative lemma on sequences of nonnegative reals

A modulus of liminf [16] for a sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ is a mapping $\delta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ satisfying the following:

for all $k, L \in \mathbb{N}$ there exists $N \in [L; \delta(k, L)]$ such that $a_N < \frac{1}{k+1}$.

Clearly, $\liminf_{n\to\infty} a_n = 0$ if and only if (a_n) has a modulus of liminf.

The following quantitative lemma is one of the main tools in the proof of our main result, Theorem 4.1.

Lemma 3.4. *Assume that $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ satisfy the following:*

(i) $\liminf_{n\to\infty} a_n = 0$ *with modulus of liminf δ;*

(ii) $\sum_{n=0}^{\infty} b_n$ *converges with Cauchy modulus ψ;*

(iii) $a_{n+1} \leq a_n + b_n$ *for all $n \in \mathbb{N}$.*

Then $\lim_{n\to\infty} a_n = 0$ with rate of convergence φ defined by

$$\varphi(k) = \delta(2k+1, \psi(2k+1)+1).$$

Proof. First, let us remark that we get, by induction on m, that for all $n, m \in \mathbb{N}$,

$$a_{n+m} \leq a_n + \sum_{i=n}^{n+m-1} b_i.$$

Let $k \in \mathbb{N}$ and let us denote $\psi_k = \psi(2k+1)$. As δ is a modulus of liminf of (a_n), there exists $N \in [\psi_k + 1; \varphi(k)]$ and $a_N < \frac{1}{2(k+1)}$.

It follows that for all $n \geq \varphi(k)$,

$$a_n \leq a_N + \sum_{i=N}^{n-1} b_i < \frac{1}{2(k+1)} + \sum_{i=N}^{n-1} b_i.$$

As $n \geq \varphi(k) \geq N \geq \psi_k + 1$, we can apply the fact that ψ is a Cauchy modulus of $(\sum_{i=0}^{n} b_i)$ to get that

$$a_n < \frac{1}{2(k+1)} + \frac{1}{2(k+1)} = \frac{1}{k+1}. \qquad \blacksquare$$

3.2. Quantitative hypotheses on the parameter sequences

We consider the following quantitative hypotheses on the parameter sequences (α_n), (β_n), (r_n):

(C1) $\sum_{n=0}^{\infty}(1-\alpha_n-\beta_n) < \infty$ with Cauchy modulus σ_1,

(C2) $\alpha_n + \beta_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} \frac{\alpha_n\beta_n}{\alpha_n+\beta_n} = \infty$ with rate of divergence σ_2,

(C3) $\sum_{n=0}^{\infty} \|r_n\| < \infty$ with Cauchy modulus σ_3.

Assume that (C1) and (C3) hold. Then the series $\sum_{n=0}^{\infty}(1-\alpha_n-\beta_n)$, $\sum_{n=0}^{\infty}\|r_n\|$ are bounded.

In the sequel, $M_{\alpha,\beta}, M_r \in \mathbb{N}$ are such that

$$\sum_{n=0}^{\infty}(1-\alpha_n-\beta_n) \leq M_{\alpha,\beta} \quad \text{and} \quad \sum_{n=0}^{\infty}\|r_n\| \leq M_r. \tag{3.1}$$

If $\alpha_n + \beta_n = 1$ for all $n \in \mathbb{N}$, then $M_{\alpha,\beta} = 0$. Similarly, if $r_n = 0$ for all $n \in \mathbb{N}$, then $M_r = 0$. Otherwise, by Lemma 3.1, we can take

$$M_{\alpha,\beta} \geq \left\lceil \sum_{n=0}^{\sigma_1(0)}(1-\alpha_n-\beta_n) \right\rceil + 1 \quad \text{and} \quad M_r \geq \left\lceil \sum_{n=0}^{\sigma_3(0)} \|r_n\| \right\rceil + 1.$$

Lemma 3.5. *Assume that (C1) and (C3) hold and that $z \in \mathrm{Fix}(T)$. Then for all $n \in \mathbb{N}$,*

$$\|x_n - z\| \leq \|x - z\| + M_{\alpha,\beta}\|z\| + M_r.$$

Proof. By the definition of $K_{z,n}$, (2.2) and (3.1). $\blacksquare$

4. Main results

The main result of the paper computes uniform rates of (T-)asymptotic regularity of the generalized Krasnoselskii–Mann-type iteration.

Theorem 4.1. *Let X be a uniformly convex normed space, η a modulus of uniform convexity of X, $T : X \to X$ a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$, and (x_n) be given by*

$$x_0 = x \in X, \quad x_{n+1} = \alpha_n x_n + \beta_n T x_n + r_n,$$

where $(r_n) \subseteq X$ and $(\alpha_n), (\beta_n) \subseteq [0, 1]$ are such that $\alpha_n + \beta_n \leq 1$ for all $n \in \mathbb{N}$.

Suppose that (C1), (C2), (C3) hold, $M_{\alpha,\beta}, M_r \in \mathbb{N}$ satisfy (3.1) and $b, M_0, M \in \mathbb{N}^*$ are such that

$$b \geq \max\{\|x - z\|, \|z\|\}, \quad M_0 = b + M_{\alpha,\beta} b + M_r, \quad M = M_0 + b \tag{4.1}$$

for some fixed point z of T.

Define $\Phi, \Psi : \mathbb{N} \to \mathbb{N}$ by

$$\Phi(k) = \sigma_2\big(\Omega(2k+1) + \max\big\{\sigma_1(8M(k+1)-1), \sigma_3(8k+7)\big\} + 1\big), \tag{4.2}$$
$$\Psi(k) = \Phi(2k+1), \tag{4.3}$$

where

$$\Omega : \mathbb{N} \to \mathbb{N}, \quad \Omega(k) = \left\lceil \frac{(M_0 + M_{\alpha,\beta} b + M_r + 1)(k+1)}{\eta\left(\frac{1}{M_0(k+1)}\right)} \right\rceil. \tag{4.4}$$

Then (x_n) is T-asymptotically regular with rate Φ and asymptotically regular with rate Ψ.

One can get for (x_n), by using Lemma 2.5 (ii), a result similar to the one proved by Kohlenbach [9, Theorem 3.4] for the Krasnoselskii–Mann iteration.

Proposition 4.2. *In the hypotheses of Theorem 4.1, assume moreover that η can be written as $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $\tilde{\eta}$ increasing. Let $\tilde{\Omega} : \mathbb{N} \to \mathbb{N}$ be defined by*

$$\tilde{\Omega}(k) = \left\lceil \frac{(M_0 + M_{\alpha,\beta} b + M_r + 1)(k+1)}{2\tilde{\eta}\left(\frac{1}{M_0(k+1)}\right)} \right\rceil. \tag{4.5}$$

Then (x_n) is T-asymptotically regular with rate $\tilde{\Phi}$ and asymptotically regular with rate $\tilde{\Psi}$, where $\tilde{\Phi}$ is obtained from Φ by taking $\tilde{\Omega}$ instead of Ω in (4.2) and $\tilde{\Psi}(k) = \tilde{\Phi}(2k+1)$.

The proofs of Theorem 4.1 and Proposition 4.2 are given in Section 6.

In the following we give some consequences of our main results.

Corollary 4.3. *Assume that X is a Hilbert space. Then Proposition 4.2 holds with*

$$\tilde{\Omega}(k) = 4M_0(M_0 + M_{\alpha,\beta} b + M_r + 1)(k+1)^2.$$

Proof. Apply the fact that a modulus of uniform convexity of X is $\eta(\varepsilon) = \frac{\varepsilon^2}{8} = \varepsilon\tilde{\eta}(\varepsilon)$, where $\tilde{\eta}(\varepsilon) = \frac{\varepsilon}{8}$ is increasing. ∎

Corollary 4.4. *Assume that $r_n = 0$ for all $n \in \mathbb{N}$, hence $x_{n+1} = \alpha_n x_n + \beta_n T x_n$ for all $n \in \mathbb{N}$. Then Theorem 4.1 holds with $\sigma_3(k) = 0$, $M_r = 0$, $M_0 = b + M_{\alpha,\beta} b$.*

Applying Theorem 4.1 with $\alpha_n = 1 - \beta_n$, we get rates for a generalization of the inexact Krasnoselskii–Mann iteration.

Corollary 4.5. *Let (x_n) be the iteration defined as follows:*

$$x_0 = x \in X, \quad x_{n+1} = (1 - \beta_n)x_n + \beta_n T x_n + r_n.$$

Assume that (C2*) *and* (C3) *hold, where*

$$\text{(C2*)} \quad \sum_{n=0}^{\infty} \beta_n(1 - \beta_n) = \infty \text{ with rate of divergence } \sigma_2.$$

Define $\Omega^* : \mathbb{N} \to \mathbb{N}$,

$$\Omega^*(k) = \left\lceil \frac{(b + 2M_r + 1)(k+1)}{\eta\left(\frac{1}{(b+M_r)(k+1)}\right)} \right\rceil.$$

Then (x_n) is T-asymptotically regular with rate

$$\Phi^*(k) = \sigma_2(\Omega^*(2k+1) + \sigma_3(8k+7) + 1)$$

and asymptotically regular with rate $\Psi^*(k) = \Phi^*(2k+1)$.

Furthermore, if η can be written as $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $\tilde{\eta}$ increasing, then one can define Ω^* by

$$\Omega^*(k) = \left\lceil \frac{(b + 2M_r + 1)(k+1)}{2\tilde{\eta}\left(\frac{1}{(b+M_r)(k+1)}\right)} \right\rceil.$$

In particular, if X is a Hilbert space, then

$$\Omega^*(k) = 4(b + M_r)(b + 2M_r + 1)(k+1)^2. \tag{4.6}$$

Proof. Apply Theorem 4.1 with $\alpha_n = 1 - \beta_n$ for all $n \in \mathbb{N}$. As $1 - \alpha_n - \beta_n = 0$, (C1) obviously holds with $\sigma_1(k) = 0$ and $M_{\alpha,\beta} = 0$ and (C2) becomes (C2*). Then $M_0 = b + M_r$, $M = 2b + M_r$. If η can be written as $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $\tilde{\eta}$ increasing, apply Proposition 4.2. For the case when X is a Hilbert space, apply Corollary 4.3. ∎

By combining the previous two corollaries, we get rates of (T-)asymptotic regularity for the well-known Krasnoselskii–Mann iteration. The rate of T-asymptotic regularity that we obtain for the Krasnoselskii–Mann iteration is very similar to the one obtained by Kohlenbach [9, Theorem 3.4] by applying proof mining methods to a proof due to Groetsch [5].

Corollary 4.6. *Let (x_n) be the Krasnoselskii–Mann iteration*

$$x_0 = x \in X, \quad x_{n+1} = (1 - \beta_n)x_n + \beta_n T x_n.$$

Then Corollary 4.5 holds with $\sigma_3(k) = 0$ and $M_r = 0$.

Corollary 4.7. *Let (x_n) be defined by*

$$x_0 = x \in X, \quad x_{n+1} = \alpha_n x_n + \beta_n T x_n + (1 - \alpha_n - \beta_n)u, \tag{4.7}$$

where $u \in X$, $u \neq 0$.

Then Theorem 4.1 holds with $M_r = M_{\alpha,\beta}\lceil \|u\| \rceil$ and

$$\sigma_3(k) = \sigma_1(\lceil \|u\| \rceil (k+1) - 1).$$

Proof. Apply Theorem 4.1 with $r_n = (1 - \alpha_n - \beta_n)u$. ∎

Let us remark that an inexact form of (4.7) is studied by Kanzow and Shehu in [6, Section 4]. However, as pointed out in [6, p. 608], the proof methods and the conditions on the parameter sequences used in [6, Theorem 4.1] are different than the ones used in [6, Section 3] for the generalized Krasnoselskii–Mann-type iteration.

5. Examples

In the sequel, we show that one can compute quadratic rates of (T-)asymptotic regularity of the generalized Krasnoselskii–Mann-type iteration (x_n) for particular choices of the parameter sequences. The first example adapts to this iteration examples used in [9, 14] to compute quadratic rates of the Krasnoselskii–Mann iteration. The second example is much more interesting, as the chosen parameter sequences (α_n), (β_n) satisfy $\alpha_n + \beta_n < 1$ for all $n \in \mathbb{N}$.

Let us prove first a very useful lemma.

Lemma 5.1. *Let $t \in \mathbb{R}_+$, $L \in \mathbb{N}^*$ and define*

$$\varphi(k) = \lceil t \rceil (k+1), \quad \varphi^*(k) = \max\{\lceil t \rceil (k+1) - L, 0\},$$

$$M_t = \begin{cases} 0 & \text{if } t = 0, \\ \left\lceil t\left(\frac{1}{L} + \frac{1}{L^2}\right)\right\rceil & \text{if } t \neq 0. \end{cases}$$

Then

(i) *φ and φ^* are Cauchy moduli of $\sum_{n=0}^{\infty} \frac{t}{(n+L)^2}$.*

(ii) $\sum_{n=0}^{\infty} \frac{t}{(n+L)^2} \le M_t \le 2\lceil t \rceil$.

Proof. (i) As $\varphi \ge \varphi^*$, it is enough to prove that φ^* is a Cauchy modulus of $\sum_{n=0}^{\infty} \frac{t}{(n+L)^2}$. Let $k \in \mathbb{N}$ and $n \ge \varphi^*(k)$. It follows that $n + L \ge \lceil t \rceil (k+1) \ge$

$t(k+1)$. Then for all $p \in \mathbb{N}$,

$$\sum_{i=0}^{n+p} \frac{t}{(i+L)^2} - \sum_{i=0}^{n} \frac{t}{(i+L)^2} = \sum_{i=n+1}^{n+p} \frac{t}{(i+L)^2} \leq \sum_{i=n+1}^{n+p} \frac{t}{(i+L-1)(i+L)}$$
$$= \sum_{i=n+1}^{n+p} \left(\frac{t}{i+L-1} - \frac{t}{i+L} \right)$$
$$= \frac{t}{n+L} - \frac{t}{n+p+L} \leq \frac{t}{n+L} \leq \frac{1}{k+1}.$$

(ii) The case $t = 0$ is obvious. Assume that $t \neq 0$. Then for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} \frac{t}{(i+L)^2} = \frac{t}{L^2} + \sum_{i=1}^{n} \frac{t}{(i+L)^2} \leq \frac{t}{L^2} + \frac{t}{L} \leq \left\lceil t\left(\frac{1}{L} + \frac{1}{L^2}\right)\right\rceil = M_t.$$

As $\frac{1}{L} + \frac{1}{L^2} \leq 2$, we get that $M_t \leq 2\lceil t \rceil$. ∎

5.1. Example 1

Assume that for all $n \in \mathbb{N}$,

$$\alpha_n = 1 - \lambda, \quad \beta_n = \lambda, \quad r_n = \frac{1}{(n+L)^2} r^*,$$

where $\lambda \in (0, 1)$, $L \in \mathbb{N}^*$ and $r^* \in X$. Let us denote in the sequel

$$\Lambda = \left\lceil \frac{1}{\lambda(1-\lambda)} \right\rceil.$$

Proposition 5.2. *(x_n) is T-asymptotically regular with rate*

$$\Phi^*(k) = \Lambda(\Omega^*(2k+1) + 8\lceil \|r^*\| \rceil (k+1) + 1)$$

and asymptotically regular with rate

$$\Psi^*(k) = \Lambda(\Omega^*(4k+3) + 16\lceil \|r^*\| \rceil (k+1) + 1),$$

where Ω^* is defined as in Corollary 4.5 with $M_r = 2\lceil \|r^*\| \rceil$.

Proof. One can easily see that (C2*) holds with $\sigma_2(k) = k\Lambda$. Furthermore, by applying Lemma 5.1 with $t := \|r^*\|$ we get that (C3) holds with $\sigma_3(k) = \lceil \|r^*\| \rceil (k+1)$ and that one can take M_r as above. Apply now Corollary 4.5. ∎

In the setting of Hilbert spaces, we get quadratic rates of (T-)asymptotic regularity of (x_n).

Proposition 5.3. *Assume furthermore that X is a Hilbert space. Then (x_n) is T-asymptotically regular with rate*

$$\Phi^*(k) = 16\Lambda(b + 2\lceil\|r^*\|\rceil)(b + 4\lceil\|r^*\|\rceil + 1)(k + 1)^2 + 8\Lambda\lceil\|r^*\|\rceil(k + 1) + \Lambda$$

and asymptotically regular with rate

$$\Psi^*(k) = 64\Lambda(b + 2\lceil\|r^*\|\rceil)(b + 4\lceil\|r^*\|\rceil + 1)(k + 1)^2 \\ + 16\Lambda\lceil\|r^*\|\rceil(k + 1) + \Lambda.$$

Proof. Apply Proposition 5.2 with $\Omega^*(k)$ defined by (4.6). ∎

If, furthermore, $r^* = 0$, then (x_n) is the Krasnoselskii–Mann iteration and

$$\Phi^*(k) = 16\Lambda b(b + 1)(k + 1)^2 + \Lambda \quad \text{and} \quad \Psi^*(k) = 64\Lambda b(b + 1)(k + 1)^2 + \Lambda.$$

The rate Φ^* of T-asymptotic regularity is slightly worse (roughly, by a constant factor) than the ones obtained for the Krasnoselskii–Mann iteration by Kohlenbach [8, Corollary 1] in Hilbert spaces and by the second author [14, Corollary 20] in CAT(0) spaces.

5.2. Example 2

Assume that for all $n \in \mathbb{N}$,

$$\alpha_n = \lambda, \quad \beta_n = 1 - \lambda - \frac{1}{(n + J)^2}, \quad r_n = \frac{1}{(n + L)^2} r^*,$$

where $J \in \mathbb{N}$, $J \geq 2$, $L \in \mathbb{N}^*$, $\lambda \in (0, \frac{J^2-1}{J^2})$ and $r^* \in X$.

Remark that for all $n \in \mathbb{N}$, $\alpha_n + \beta_n = 1 - \frac{1}{(n+J)^2} < 1$ and $\beta_n \geq \beta_0 = \frac{J^2-1}{J^2} - \lambda > 0$.

Moreover, if we take $L = J$, one gets a particular case of the iteration from Corollary 4.7.

Proposition 5.4. *(x_n) is T-asymptotically regular with rate*

$$\Phi(k) = \Lambda\Omega(2k + 1) + 16\Lambda(2b + \lceil\|r^*\|\rceil)(k + 1) + 3\Lambda - 1$$

and asymptotically regular with rate

$$\Psi(k) = \Lambda\Omega(4k + 3) + 32\Lambda(2b + \lceil\|r^*\|\rceil)(k + 1) + 3\Lambda - 1,$$

where $\Lambda = \lceil\frac{1}{\lambda(1-\lambda)}\rceil$.

Proof. We can apply Lemma 5.1 to get that (C1) holds with $\sigma_1(k) = k + 1$, (C3) holds with $\sigma_3(k) = \lceil\|r^*\|\rceil(k + 1)$ and that one can take $M_{\alpha,\beta} = 2$, $M_r = 2\lceil\|r^*\|\rceil$.

Claim: (C2) holds with rate of divergence $\sigma_2(n) = \Lambda(n+2) - 1$.

Proof of claim: We have that for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{i=0}^{\sigma_2(n)} \frac{\alpha_i \beta_i}{\alpha_i + \beta_i} \\
&\geq \sum_{i=0}^{\sigma_2(n)} \alpha_i \beta_i \qquad \text{as } \alpha_i + \beta_i = 1 - \frac{1}{(i+J)^2} < 1 \\
&= \sum_{i=0}^{\sigma_2(n)} \lambda\left(1 - \lambda - \frac{1}{(i+J)^2}\right) = (\sigma_2(n)+1)\lambda(1-\lambda) - \lambda \sum_{i=0}^{\sigma_2(n)} \frac{1}{(i+J)^2} \\
&\geq (\sigma_2(n)+1)\lambda(1-\lambda) - 2\lambda, \qquad \text{by Lemma 5.1 (ii) with } t := 1 \text{ and } L := J \\
&= \Lambda(n+2)\lambda(1-\lambda) - 2\lambda \geq (n+2) - 2\lambda > n \qquad \text{as } \lambda < 1. \qquad \blacksquare
\end{aligned}$$

By (4.1), we get that $M_0 = 3b + 2\lceil \|r^*\| \rceil$, $M = 4b + 2\lceil \|r^*\| \rceil$. Apply now Theorem 4.1 to conclude the proof. ■

As an application of (the proof of) Proposition 5.4 and Corollary 4.3 we get quadratic rates of (T-)asymptotic regularity in the case of Hilbert spaces.

Proposition 5.5. *Assume furthermore that X is a Hilbert space. Then (x_n) is T-asymptotically regular with rate*

$$\Phi(k) = 16\Lambda M_1(k+1)^2 + 16\Lambda M_2(k+1) + 3\Lambda - 1$$

and asymptotically regular with rate

$$\Psi(k) = 64\Lambda M_1(k+1)^2 + 32\Lambda M_2(k+1) + 3\Lambda - 1,$$

where $M_1 = (3b + 2\lceil \|r^*\| \rceil)(5b + 4\lceil \|r^*\| \rceil + 1)$ and $M_2 = 2b + \lceil \|r^*\| \rceil$.

6. Proof of the main results

Lemma 6.1. *For all $n \in \mathbb{N}$, $\|x_n - z\| \leq M_0$ and $\|x_n\| \leq M$.*

Proof. Apply Lemma 3.5 and (4.1) to get that

$$\|x_n - z\| \leq \|x - z\| + M_{\alpha,\beta}\|z\| + M_r \leq b + M_{\alpha,\beta} b + M_r = M_0.$$

Furthermore, $\|x_n\| \leq \|x_n - z\| + \|z\| \leq M_0 + b = M$. ■

6.1. A modulus of liminf

Proposition 6.2. $\liminf_{n\to\infty}\|x_n - Tx_n\| = 0$ *with modulus of liminf* $\Delta : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ *defined by*

$$\Delta(k, L) = \sigma_2(\Omega(k) + L), \tag{6.1}$$

where Ω *is defined by* (4.4).

Proof. Assume by contradiction that Δ is not a modulus of liminf for $(\|x_n - Tx_n\|)$. Then there are $k, L \in \mathbb{N}$ such that

$$\|x_n - Tx_n\| \geq \frac{1}{k+1} \quad \text{for all } n \in [L; \Delta(k, L)].$$

Let $n \in [L; \Delta(k, L)]$ be arbitrary. Then $x_n \neq Tx_n$, so $x_n \neq z$, that is, $\|x_n - z\| > 0$. By (C2) and Lemma 6.1, we have that $\alpha_n + \beta_n > 0$ and $\|x_n - z\| \leq M_0$. Thus, we can apply Lemma 2.5 (i) with $\delta = M_0$, $\gamma = \|x_n - z\|$ and $\theta = \frac{1}{k+1}$ to get that

$$\|\alpha_n x_n + \beta_n Tx_n - (\alpha_n + \beta_n)z\| \leq \|x_n - z\| - \frac{2\|x_n - z\|\alpha_n\beta_n}{\alpha_n + \beta_n}\eta\left(\frac{1}{M_0(k+1)}\right).$$

Since, by Corollary 2.2, $2\|x_n - z\| \geq \|x_n - Tx_n\| \geq \frac{1}{k+1}$, we get that

$$\|\alpha_n x_n + \beta_n Tx_n - (\alpha_n + \beta_n)z\| \leq \|x_n - z\| - \frac{\alpha_n\beta_n}{(\alpha_n + \beta_n)(k+1)}\eta\left(\frac{1}{M_0(k+1)}\right).$$

As

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n Tx_n + r_n - z\| \\
&= \|(\alpha_n x_n + \beta_n Tx_n - (\alpha_n + \beta_n)z) - (1 - \alpha_n - \beta_n)z + r_n\| \\
&\leq \|\alpha_n x_n + \beta_n Tx_n - (\alpha_n + \beta_n)z\| + (1 - \alpha_n - \beta_n)\|z\| + \|r_n\|,
\end{aligned}$$

it follows that

$$\begin{aligned}
\|x_{n+1} - z\| \leq \|x_n - z\| &- \frac{\alpha_n\beta_n}{(\alpha_n + \beta_n)(k+1)}\eta\left(\frac{1}{M_0(k+1)}\right) \\
&+ (1 - \alpha_n - \beta_n)\|z\| + \|r_n\|.
\end{aligned}$$

Adding up the above for $n = L, \ldots, \Delta(k, L)$, we get that

$$\begin{aligned}
&\|x_{\Delta(k,L)+1} - z\| \\
&\quad \leq \|x_L - z\| - \frac{1}{k+1}\eta\left(\frac{1}{M_0(k+1)}\right)\sum_{n=L}^{\Delta(k,L)}\frac{\alpha_n\beta_n}{\alpha_n + \beta_n} \\
&\qquad + \|z\|\sum_{n=L}^{\Delta(k,L)}(1 - \alpha_n - \beta_n) + \sum_{n=L}^{\Delta(k,L)}\|r_n\|
\end{aligned}$$

$$\overset{(3.1)}{\leq} \|x_L - z\| - \frac{1}{k+1}\eta\left(\frac{1}{M_0(k+1)}\right)\sum_{n=L}^{\Delta(k,L)} \frac{\alpha_n\beta_n}{\alpha_n+\beta_n} + \|z\|M_{\alpha,\beta} + M_r$$

$$\leq M_0 - \frac{1}{k+1}\eta\left(\frac{1}{M_0(k+1)}\right)\sum_{n=L}^{\Delta(k,L)} \frac{\alpha_n\beta_n}{\alpha_n+\beta_n} + \|z\|M_{\alpha,\beta} + M_r$$

by Lemma 6.1

$$\overset{(4.1)}{\leq} (M_0 + M_{\alpha,\beta}b + M_r) - \frac{1}{k+1}\eta\left(\frac{1}{M_0(k+1)}\right)\sum_{n=L}^{\Delta(k,L)} \frac{\alpha_n\beta_n}{\alpha_n+\beta_n}.$$

Remark that

$$\sum_{n=L}^{\Delta(k,L)} \frac{\alpha_n\beta_n}{\alpha_n+\beta_n} = \sum_{n=L}^{\sigma_2(\Omega(k)+L)} \frac{\alpha_n\beta_n}{\alpha_n+\beta_n} = \sum_{n=0}^{\sigma_2(\Omega(k)+L)} \frac{\alpha_n\beta_n}{\alpha_n+\beta_n} - \sum_{n=0}^{L-1} \frac{\alpha_n\beta_n}{\alpha_n+\beta_n}$$

$$\geq (\Omega(k)+L) - \sum_{n=0}^{L-1} \frac{\alpha_n\beta_n}{\alpha_n+\beta_n} \qquad \text{by (C2)}$$

$$\geq (\Omega(k)+L) - L = \Omega(k), \qquad \text{as } \frac{\alpha_n\beta_n}{\alpha_n+\beta_n} \leq \alpha_n \leq 1.$$

Let us denote, for simplicity, $P = M_0 + M_{\alpha,\beta}b + M_r$. It follows that

$$\begin{aligned}\|x_{\Delta(k,L)+1} - z\| &\leq P - \frac{\Omega(k)}{k+1}\eta\left(\frac{1}{M_0(k+1)}\right)\\ &\overset{(4.4)}{=} P - \left\lceil \frac{(P+1)(k+1)}{\eta(\frac{1}{M_0(k+1)})}\right\rceil \frac{\eta(\frac{1}{M_0(k+1)})}{k+1}\\ &\leq P - (P+1) = -1,\end{aligned}$$

which is a contradiction. Thus, Δ is a modulus of liminf for $(\|x_n - Tx_n\|)$. ∎

Proposition 6.3. *Assume that η can be written as $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $\tilde{\eta}$ increasing. Then $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$ with modulus of liminf $\tilde{\Delta} : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ defined by*

$$\tilde{\Delta}(k, L) = \sigma_2(\tilde{\Omega}(k) + L),$$

where $\tilde{\Omega}$ is defined by (4.5).

Proof. Follow the proof of Proposition 6.2, but apply Lemma 2.5 (ii) instead of Lemma 2.5 (i). ∎

6.2. Proof of Theorem 4.1 and Proposition 4.2

By (2.6) and Lemma 6.1 we get that for all $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\leq \|x_n - Tx_n\| + 2(1-\alpha_n-\beta_n)\|x_n\| + 2\|r_n\| \\ &\leq \|x_n - Tx_n\| + 2M(1-\alpha_n-\beta_n) + 2\|r_n\|.\end{aligned}$$

Let us verify that the hypotheses of Lemma 3.4 hold with

$$a_n := \|x_n - Tx_n\| \quad \text{and} \quad b_n := 2M(1-\alpha_n-\beta_n) + 2\|r_n\|.$$

By Proposition 6.2, $\liminf_{n\to\infty} a_n = 0$ with modulus of liminf Δ given by (6.1). Furthermore, by (C1), (C3) and Lemma 3.3, we get that

$$\psi(k) = \max\{\sigma_1(4M(k+1)-1), \sigma_3(4k+3)\}$$

is a Cauchy modulus of (b_n).

Thus, we can apply Lemma 3.4 to get that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ with rate of convergence

$$\begin{aligned}\Phi(k) &= \Delta(2k+1, \psi(2k+1)+1) = \sigma_2\big(\Omega(2k+1) + \psi(2k+1) + 1\big) \\ &= \sigma_2\big(\Omega(2k+1) + \max\{\sigma_1(4M(2k+2)-1), \sigma_3(4(2k+1)+3)\} + 1\big) \\ &= \sigma_2\big(\Omega(2k+1) + \max\{\sigma_1(8M(k+1)-1), \sigma_3(8k+7)\} + 1\big).\end{aligned}$$

Let us prove now that Ψ defined by (4.3) is a rate of asymptotic regularity. First, one can easily see that $\frac{\alpha_n\beta_n}{\alpha_n+\beta_n} < 1$ for every $n \in \mathbb{N}$. Thus, by Lemma 3.2, $\sigma_2(k) \geq k$. We get that

$$\begin{aligned}\Psi(k) &= \Phi(2k+1) \\ &= \sigma_2\big(\Omega(4k+3) + \max\{\sigma_1(16M(k+1)-1), \sigma_3(16k+15)\} + 1\big) \\ &\geq \Omega(4k+3) + \max\{\sigma_1(16M(k+1)-1), \sigma_3(16k+15)\} + 1 \\ &\geq \max\{\sigma_1(16M(k+1)-1), \sigma_3(16k+15)\} + 1.\end{aligned}$$

Let $n \geq \Psi(k)$. It follows that

$$\begin{aligned}\|x_{n+1} - x_n\| &\overset{(2.5)}{\leq} \beta_n\|Tx_n - x_n\| + (1-\alpha_n-\beta_n)\|x_n\| + \|r_n\| \\ &\leq \|Tx_n - x_n\| + (1-\alpha_n-\beta_n)M + \|r_n\| \qquad \text{by Lemma 6.1} \\ &\leq \frac{1}{2(k+1)} + (1-\alpha_n-\beta_n)M + \|r_n\| \qquad \text{as } n \geq \Phi(2k+1) \\ &\leq \frac{1}{2(k+1)} + (1-\alpha_n-\beta_n)M + \frac{1}{16(k+1)} \\ &\qquad \text{by Lemma 3.1 (ii) and the fact that } n \geq \sigma_3(16k+15)+1\end{aligned}$$

$$\leq \frac{1}{2(k+1)} + \frac{1}{16(k+1)} + \frac{1}{16(k+1)} < \frac{1}{k+1}$$

by Lemma 3.1 (ii) and the fact that $n \geq \sigma_1(16M(k+1)-1)+1$.

Thus, Theorem 4.1 is proved.

The proof of Proposition 4.2 follows the same line, with the difference that we apply Proposition 6.3 instead of Proposition 6.2.

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