Port. Math. 83 (2026), 335–365
DOI 10.4171/PM/2160

Decay of correlations via induced weak Gibbs–Markov maps for non-Hölder observables

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Abstract. We obtain estimates on the decay of correlations, central limit theorem and large deviations for dynamical systems admitting an induced weak Gibbs–Markov map, for larger classes of observables with weaker regularity than Hölder, characterized by suitable moduli of continuity.

1. Introduction

Statistical properties for Hölder observables are well understood for a large variety of dynamical systems. This includes the decay of correlations, central limit theorem (CLT), large deviations, almost sure invariance principle, etc. For instance, in [40, 41] the decay of correlations and CLT are obtained for nonuniformly hyperbolic systems. The large deviations and almost sure invariance principles for nonuniformly expanding systems are discussed in [30, 31]. We also refer to [9, 11, 14, 17, 22] where these statistical properties for distinct classes of dynamical systems are analysed. Note that in all of the above references, observables are assumed to be Hölder.

There are several works dealing with classes of observables strictly larger than Hölder. For example, the mixing rate of the equilibrium state of one-sided shift is discussed in [10] for non-Hölder potentials of summable variations, while the results for two-sided subshift of finite type are discussed in [35]. In addition, the polynomial decay of correlations and CLT are obtained in [18] for the equilibrium state of a one-sided shift map on two symbols with non-Hölder potentials. We also would like to mention some work beyond shift spaces. In [36], a class of observables which contains all piecewise Lipschitz functions is considered, in order to get the decay of correlation for certain nonuniformly expanding systems. Estimates for the rates of mixing for observables with weaker regularity than Hölder are given in [16] for one-dimensional expanding Lorenz-like map. Observables with various moduli of continuity were also

Mathematics Subject Classification 2020: 37A25 (primary); 37A05, 37A50 (secondary).
Keywords: weak Gibbs–Markov map, Young tower, decay of correlations, central limit theorem, large deviations.

considered in [19, 20, 24], obtaining decay of correlations in the context of expanding maps of the circle with an indifferent fixed point, including the Manneville–Pomeau family of maps. We refer to [37, 42] for invertible maps with non-Hölder observables. More generally, in [27], decay of correlations and CLT are obtained for those dynamical systems that admit an induced full branch map, referred to as Gibbs–Markov map, for much larger classes of observables. More precisely, the results obtained in [27] are an extension of [41] in the sense of considering bigger classes of observables.

There have been numerous other extensions of [40, 41] in different directions. For instance, in [7], the results of [40] were extended by weakening some of the assumptions, thereby allowing invertible maps exhibiting slow decay rates. In [25], mixing rates were obtained with uniform control on various constants for nonuniformly expanding and hyperbolic maps similar to those considered in [40, 41]. The estimates obtained in [41] were shown to be optimal in [21, 38]. In [32], results on the decay of correlations were obtained for nonuniformly expanding systems with a *good inducing scheme*. In particular, these results were applied to the systems considered in [40, 41]. In [39], the authors generalised the results of [41] under weaker assumptions on the dynamical system, in which the induced map was not necessarily full branch. More precisely, it was shown that decay of correlations, a central limit theorem, and large deviation estimates hold for dynamical systems admitting an induced *weak Gibbs–Markov* (WGM) map for Hölder observables.

This work aims to extend the results of [39] to dynamical systems admitting induced WGM maps, for observables with weaker regularity than Hölder.

Let us mention some works that consider WGM maps. Some ergodic properties of *Markov fibred systems*, which in particular hold for WGM maps, were discussed in [1]. In [43], exponential decay of correlations was obtained for uniformly expanding Markov maps on compact metric spaces with the finite images property, for Lipschitz observables under additional assumptions. In [21], sharp polynomial bounds for the decay of correlations and the central limit theorem were obtained for topologically mixing Markov maps for Hölder observables, using the method of operator renewal theory. In [13, 28], the decay of correlations for mixing tower maps was addressed for Hölder observables with respect to the combinatorial distance on the tower, using Birkhoff's cones and projective metrics. In [23], an induced WGM map was constructed for skew product systems, and it was proved that such systems admit a unique absolutely continuous ergodic measure, which also satisfies several limit theorems for Lipschitz observables.

The setting considered in this paper differs from the above-mentioned works on WGM maps. Here, we assume the existence of an induced WGM map defined with respect to a reference measure that is not necessarily invariant. We obtain general results for dynamical systems admitting such induced WGM maps for a larger class of observables with weaker regularity than Hölder. Furthermore, the compactness assumption, the finite images condition, and the requirement that the observables be supported on the inducing region are not imposed in the present work.

The goal of the present work is to extend the results of [39] by considering the broader classes of observables studied in [27]. More precisely, we investigate the decay of correlations, the central limit theorem, and large deviations for classes of observables with weaker regularity than Hölder, characterized by suitable moduli of continuity, in the setting of dynamical systems admitting an induced WGM map (that is not necessarily full branch; see Definition 2.1). Our approach is to construct a mixing tower over an induced WGM map and then transfer the results obtained to the original dynamical system. We extend the coupling arguments used in [39] to establish decay of correlations for the tower system for these broader classes of observables. As an immediate consequence, we also obtain the central limit theorem and large deviation estimates.

There are many works devoted to the construction of induced Gibbs–Markov (full branch) maps, e.g., [4, 8, 15, 33, 34], and several others addressing the consequences of having such induced maps, e.g., [5, 6, 12, 26, 31]. These results could be further generalised by considering WGM maps and potentially extended to include non-Hölder observables. From the perspective of future applications, we believe that the present work provides a useful framework to extend several results currently known for Hölder observables. For instance, in [21, 38], lower bounds for correlation functions were obtained for Hölder observables. It would be interesting to investigate whether these and other related results can be extended to non-Hölder observables in the absence of the full branch property.

This article is organised as follows: In Section 2, we give the necessary definitions and state the main results. In Section 3, we obtain the decay of correlations for the tower map and transfer the estimates to the original dynamical system.

2. Preliminaries and statement of main results

We begin by introducing some basic notions and describing the dynamical systems and classes of observables under consideration, followed by the statement of the main results. Consider a finite measure space $(\Delta_0, \mathcal{A}, m)$, with $m(\Delta_0) > 0$, a measurable map $F\colon \Delta_0 \to \Delta_0$ and an (m mod 0) finite or countable partition $\mathcal{P}_0$ of Δ_0 into invertibility domains of F, that is, F is a bijection from each $\omega \in \mathcal{P}_0$ to $F(\omega)$, with measurable inverse. For each $n \geq 0$ we set

$$F^{-n}(\mathcal{P}_0) = \{F^{-n}(\omega) : \omega \in \mathcal{P}_0\}.$$

Assuming $F_*m \ll m$, we have that $F^{-n}(\mathcal{P}_0)$ is an (m mod 0) partition of Δ_0, for all $n \geq 0$. In this case, we consider the sequence $(\mathcal{P}_0^n)_n$ of (m mod 0) partitions given by

$$\mathcal{P}_0^n = \bigvee_{i=0}^{n-1} F^{-i}(\mathcal{P}_0) = \{\omega_0 \cap F^{-1}(\omega_1) \cap \cdots \cap F^{-n+1}(\omega_{n-1}) : \omega_0, \ldots, \omega_{n-1} \in \mathcal{P}_0\},$$

for $n \geq 1$, and the (m mod 0) partition

$$\mathcal{P}_0^\infty = \bigvee_{i=0}^{\infty} F^{-i}(\mathcal{P}_0) = \{\omega_0 \cap F^{-1}(\omega_1) \cap \cdots : \omega_n \in \mathcal{P}_0 \text{ for all } n \geq 0\}.$$

We say that the sequence $(\mathcal{P}_0^n)_n$ is *a basis of* Δ_0 if it generates $\mathcal{A}$ (m mod 0) and $\mathcal{P}_0^\infty$ is the partition into single points.

Definition 2.1. We say that $F\colon \Delta_0 \to \Delta_0$ is a weak Gibbs–Markov (WGM) map, with respect to the partition $\mathcal{P}_0$, if the following hold:

(W1) Markov: F maps each $\omega \in \mathcal{P}_0$ bijectively to an (m mod 0) union of elements of $\mathcal{P}_0$.

(W2) Separability: the sequence $(\mathcal{P}_0^n)_n$ is a basis of Δ_0.

(W3) Nonsingular: there exists a strictly positive measurable function J_F defined on Δ_0 such that for each $A \subset \omega \in \mathcal{P}_0$,

$$m(F(A)) = \int_A J_F \, dm.$$

(W4) Gibbs: there exist $C_F > 0$ and $0 < \beta < 1$ such that for all $\omega \in \mathcal{P}_0$ and $x, y \in \omega$,

$$\log \frac{J_F(x)}{J_F(y)} \leq C_F \beta^{s(F(x),F(y))},$$

where

$$s(x, y) = \min\{n \geq 0 : F^n(x) \text{ and } F^n(y) \text{ lie in distinct elements of } \mathcal{P}_0\}.$$

(W5) Long branches: there exists $\delta_0 > 0$ such that $m(F(\omega)) \geq \delta_0$, for all $\omega \in \mathcal{P}_0$.

The term *weak* in the above definition refers that in (W1) we do not require the *full branches* property (that is, $F(\omega) = \Delta_0$ for all $\omega \in \mathcal{P}_0$). In the case of full branches, F is called a *Gibbs–Markov* map. We adopt the terminology used in [2].

Consider a measure space $(M, \mathcal{B}, m)$, a measurable map $f\colon M \to M$ and a measurable set $\Delta_0 \subset M$ with $m(\Delta_0) > 0$. For simplicity we denote the restriction of m to Δ_0 also by m. We say that $F\colon \Delta_0 \to \Delta_0$ is an *induced map for* f if there exist a countable (m mod 0) partition $\mathcal{P}_0$ of Δ_0 and a measurable function $R\colon \Delta_0 \to \mathbb{N}$, constant on each element of $\mathcal{P}_0$, such that

$$F|_\omega = f^{R(\omega)}|_\omega.$$

We formally denote the induced map F by f^R. We say that an induced map $f^R\colon \Delta_0 \to \Delta_0$ is *aperiodic* if for all $\omega_1, \omega_2 \in \mathcal{P}_0$ there exists $k_0 \in \mathbb{N}$ such that for all $n \geq k_0$,

$$m(\omega_1 \cap (f^R)^{-n}(\omega_2)) > 0.$$

We say that an induced map f^R has a *coprime block* if there exist $N \geq 2$ and $\omega_1, \omega_2, \ldots, \omega_N \in \mathcal{P}_0$ such that $\gcd\{R(\omega_i)\}_i = 1$ and for all $i = 1, \ldots, N$,

$$f^R(\omega_i) \supseteq \omega_1 \cup \omega_2 \cup \cdots \cup \omega_N \quad (m \text{ mod } 0).$$

Assume moreover that M is a metric space with metric d. We say that an induced map f^R is *expanding* if there are $0 < \beta < 1$ and $C > 0$ such that, for all $\omega \in \mathcal{P}_0$ and $x, y \in \omega$,

(i) $d(f^R(x), f^R(y)) \leq C\beta^{s(f^R(x), f^R(y))}$,

(ii) $d(f^j(x), f^j(y)) \leq Cd(f^R(x), f^R(y))$, for all $0 \leq j \leq R$.

Condition (i) can be obtained from the usual expansion property of f^R. More precisely, if there exists $\lambda > 1$ such that, for any two points x, y in the same partition element $\omega \in \mathcal{P}_0$,

$$d(f^R(x), f^R(y)) \geq \lambda d(x, y),$$

then

$$d(f^R(x), f^R(y)) \leq C\beta^{s(f^R(x), f^R(y))},$$

where $C = \lambda \operatorname{diam}(\Delta_0)$ and $\beta = \lambda^{-1}$. This conclusion motivates the definition of an expanding map that we adopt here.

For a continuous function $\varphi\colon M \to \mathbb{R}$ and $\epsilon > 0$ we set

$$\mathcal{R}_\epsilon(\varphi) := \sup\{|\varphi(x) - \varphi(y)| : d(x, y) \leq \epsilon\}.$$

Note that $\mathcal{R}_\epsilon(\varphi) \to 0$ as $\epsilon \to 0$ if and only if φ is uniformly continuous. We consider the following classes of observables, associated with different moduli of continuity, as defined in [27]:

$$\begin{aligned}
(R1, \tau) &= \{\varphi : \mathcal{R}_\epsilon(\varphi) = \mathcal{O}(\epsilon^\tau)\}, && \tau \in (0, 1);\\
(R2, \tau) &= \{\varphi : \mathcal{R}_\epsilon(\varphi) = \mathcal{O}(e^{-|\log \epsilon|^\tau})\}, && \tau \in (0, 1);\\
(R3, \tau) &= \{\varphi : \mathcal{R}_\epsilon(\varphi) = \mathcal{O}(e^{-(\log|\log \epsilon|)^\tau})\}, && \tau > 1;\\
(R4, \tau) &= \{\varphi : \mathcal{R}_\epsilon(\varphi) = \mathcal{O}(|\log \epsilon|^{-\tau})\}, && \tau > 1.
\end{aligned}$$

The correlation sequence of two observable functions $\varphi \in \mathcal{H}$ and $\psi \in L^{\infty}(M,m)$, for some Banach space $\mathcal{H} \subset L^{\infty}(M,m)$, with respect to an f-invariant probability μ, is defined for all $n \in \mathbb{N}$ by

$$\mathrm{Cor}_{\mu}(\varphi, \psi \circ f^n) = \left| \int \varphi(\psi \circ f^n) d\mu - \int \varphi d\mu \int \psi d\mu \right|.$$

We can now state our result concerning the decay of correlations.

Theorem A. *Consider a measure space $(M, \mathcal{B}, m)$, endowed with some metric, and a measurable map $f\colon M \to M$ satisfying $f_* m \ll m$. Let $f^R\colon \Delta_0 \to \Delta_0$ be an aperiodic induced WGM expanding map with a coprime block, $R \in L^1(m)$ and let μ be the unique ergodic f-invariant probability measure μ such that $\mu \ll m$ and $\mu(\Delta_0) > 0$. Then,*

(1) *If $m\{R > n\} \le C n^{-a}$ for some $C > 0$ and $a > 1$, then, for a given $\kappa > 0$, there is $0 < \zeta < 1$ such that for all $\varphi \in (R4, \tau)$, with $\tau > \frac{2}{\zeta}$ and $\mathcal{R}_{\infty}(\varphi) < \kappa$, and for all $\psi \in L^{\infty}(M,m)$ we have*

(i) *if $\tau = \frac{a+1}{\zeta}$, there is some $C' > 0$ such that*

$$\mathrm{Cor}_{\mu}(\varphi, \psi \circ f^n) \le C'(n^{1-a} \log n);$$

(ii) *otherwise, there is some $C' > 0$ such that*

$$\mathrm{Cor}_{\mu}(\varphi, \psi \circ f^n) \le C' \max\{n^{1-a}, n^{2-\zeta\tau}\}.$$

(2) *If $m\{R > n\} \le C e^{-cn}$ for some C, $c > 0$, then*

(i) *for all $\varphi \in (R1, \tau)$ and $\psi \in L^{\infty}(M,m)$ there are C', $c' > 0$ such that*

$$\mathrm{Cor}_{\mu}(\varphi, \psi \circ f^n) \le C' e^{-c'n};$$

(ii) *for all $\varphi \in (R2, \tau)$ and $\psi \in L^{\infty}(M,m)$ there is $C' > 0$ such that, for every $\tau' < \tau$,*

$$\mathrm{Cor}_{\mu}(\varphi, \psi \circ f^n) \le C' e^{-n^{\tau'}};$$

(iii) *for all $\varphi \in (R3, \tau)$ and $\psi \in L^{\infty}(M,m)$ there is $C' > 0$ such that, for every $\tau' < \tau$,*

$$\mathrm{Cor}_{\mu}(\varphi, \psi \circ f^n) \le C' e^{-(\log n)^{\tau'}};$$

(iv) *for given $\kappa > 0$ there is $0 < \zeta < 1$ such that for all $\varphi \in (R4, \tau)$, with $\tau > \frac{1}{\zeta}$ and $\mathcal{R}_{\infty}(\varphi) < \kappa$, and for all $\psi \in L^{\infty}(M,m)$, there is $C' > 0$ such that*

$$\mathrm{Cor}_{\mu}(\varphi, \psi \circ f^n) \le C' n^{1-\zeta\tau}.$$

We remark that this extends the results of [39]. More precisely, Theorem A establishes the decay of correlations for the same dynamical system considered therein, but for larger classes of observables with weaker regularity than Hölder continuity. The existence of such a measure μ was already established in [39], and the proof of the remaining parts can be deduced from Proposition 3.4 (see Remark 3.5). We recall that if $f^R\colon \Delta_0 \to \Delta_0$ is an induced WGM map for f, then f^R admits an invariant probability measure $\nu_0 \ll m$ (see, e.g., [2, Theorem 3.13]). We can define a measure $\tilde{\mu}$ on M by

$$\tilde{\mu} = \sum_{j=0}^{\infty} f_*^j(\nu_0 \mid \{R > j\}).$$

If $R \in L^1(m)$ and $f_* m \ll m$, then by standard results, $\mu = \tilde{\mu}/\tilde{\mu}(M)$ is an f-invariant probability measure such that $\mu \ll m$ and $\mu(\Delta_0) > 0$. The assumptions of aperiodicity and the coprime block condition on the induced WGM map imply that the invariant measure μ is mixing; for this we refer to [39, Sections 3 and 4]. As a further remark, we note that the expanding property of the induced WGM map is required only to ensure that the lifted observables lie in suitable spaces; in particular, Lemma 3.2 relies on this assumption.

Let μ be an ergodic f-invariant probability measure. We say that an observable $\varphi\colon M \to \mathbb{R}$ with $\int \varphi d\mu = 0$ satisfies the *CLT* if $\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ f^i$ converges in law (or in distribution) to a normal distribution $\mathcal{N}(0, \sigma)$, for some $\sigma > 0$. We may also consider observables of non-zero expectation by replacing φ with $\varphi - \int \varphi d\mu$. In this situation, an observable φ satisfies the CLT if there exists $\sigma > 0$ such that for every interval $J \subset \mathbb{R}$,

$$\mu\left\{x : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left(\varphi(f^i(x)) - \int \varphi d\mu\right) \in J\right\} \to \frac{1}{\sigma\sqrt{2\pi}} \int_J e^{-\frac{t^2}{2\sigma^2}} dt, \qquad \text{as } n \to \infty.$$

A function φ is *coboundary* if there exists a measurable function g such that $\varphi \circ f = g \circ f - g$.

Corollary B (Central limit theorem). *Consider a measure space $(M, \mathcal{B}, m)$, endowed with some metric, and a measurable map $f\colon M \to M$ satisfying $f_* m \ll m$. Let $f^R\colon \Delta_0 \to \Delta_0$ be an aperiodic induced WGM expanding map with a coprime block such that $R \in L^1(m)$, and let μ be the unique ergodic f-invariant probability measure μ such that $\mu \ll m$ and $\mu(\Delta_0) > 0$. If $m\{R > n\} \leq Cn^{-a}$ for some $C > 0$ and $a > 2$, then the CLT is satisfied for all $\varphi \in (R4, \tau)$, some $\tau > 0$ sufficiently large if and only if φ is not coboundary.*

The proof of Corollary B is given in Section 3.5.

Given $\epsilon > 0$ we define the *large deviation at time n* of the time average of an observable $\varphi: M \to \mathbb{R}$ from the spatial average as

$$LD_\mu(\varphi, \epsilon, n) := \mu\left\{\left|\frac{1}{n}\sum_{i=0}^{n-1} \varphi \circ f^i - \int \varphi d\mu\right| > \epsilon\right\}.$$

Since the decay of correlations is helpful in deducing some statistical properties, in particular, by combining Theorem A and [3, 29], we obtain the following immediate corollary.

Corollary C (Large Deviations). *Consider a measure space $(M, \mathcal{B}, m)$, endowed with some metric, and a measurable map $f: M \to M$ satisfying $f_* m \ll m$. Let $f^R: \Delta_0 \to \Delta_0$ be an aperiodic induced WGM expanding map with a coprime block such that $R \in L^1(m)$, and let μ be the unique ergodic f-invariant probability measure μ such that $\mu \ll m$ and $\mu(\Delta_0) > 0$. Then,*

(1) *if $m\{R > n\} \le Cn^{-a}$ for some $C > 0$ and $a > 1$, then, for a given $\kappa > 0$, there is $0 < \zeta < 1$, such that for all $\epsilon > 0$ and $\varphi \in (R4, \tau)$, with $\frac{2}{\zeta} < \tau \neq \frac{a+1}{\zeta}$ and $\mathcal{R}_\infty(\varphi) < \kappa$, there is some $C' = C'(\epsilon, \varphi) > 0$ such that*

$$LD_\mu(\varphi, \epsilon, n) \le C' \max\{n^{-a+1}, n^{2-\zeta\tau}\};$$

(2) *if $m\{R > n\} \le Ce^{-cn}$ for some $C, c > 0$, then*

(i) *for all $\epsilon > 0$ and $\varphi \in (R1, \tau)$, there is some $C' = C'(\epsilon, \varphi) > 0$ and $c' = c'(c, \varphi, \epsilon, \tau)$ such that*

$$LD_\mu(\varphi, \epsilon, n) \le C' e^{-c' n^{\frac{1}{3}}};$$

(ii) *for all $\epsilon > 0$ and $\varphi \in (R2, \tau)$, there is some $C' = C'(\epsilon, \varphi) > 0$ and $c' = c'(\varphi, \epsilon)$ such that, for every $\tau' < \tau$,*

$$LD_\mu(\varphi, \epsilon, n) \le C' e^{-c' n^{\frac{\tau'}{\tau'+2}}};$$

(iii) *for given $\kappa > 0$ there is $0 < \zeta < 1$ such that for all $\epsilon > 0$ and $\varphi \in (R4, \tau)$ with $\tau > \frac{1}{\zeta}$ and $\mathcal{R}_\infty(\varphi) < \kappa$, there is some $C' = C'(\epsilon, \varphi) > 0$, such that*

$$LD_\mu(\varphi, \epsilon, n) \le C' n^{1-\zeta\tau}.$$

Now, we apply our main results to the following example with polynomial return time decay, introduced in [23].

Example 2.2. Let $M = S^1 \times [0, 1]$ and let m denote the Lebesgue measure on M. We consider the map $f: M \to M$ defined by

$$f(\theta, x) = (g(\theta), f_{\alpha(\theta)}(x)),$$

where $g(\theta) = 4\theta \pmod 1$, $\alpha: S^1 \to (0,1)$ is a C^1 map that has minimum $\alpha_{\min}$ and maximum $\alpha_{\max}$, with $\alpha_{\min} < \alpha_{\max}$ and

$$f_\alpha(\theta)(x) = \begin{cases} x(1 + 2^{\alpha(\theta)} x^{\alpha(\theta)}) & \text{if } 0 \le x \le \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

In [23], a partition $\mathcal{P}_0$ of $\Delta_0 = S^1 \times (\frac{1}{2}, 1]$ with a certain return time R was given in such a way that the induced map f^R is an aperiodic expanding WGM map (without the full branches property), admitting a coprime block. Moreover, we have $m\{R > n\} \le Cn^{-a}$ for some $C > 0$, with $a = 1/\alpha_{\max}$. Since $\alpha_{\max} < 1$ we have $R \in L^1(m)$. For completeness, we will provide further details on the construction of induced WGM maps later in this example.

By Theorem A there exists a unique ergodic f-invariant probability measure μ such that $\mu \ll m$. Moreover, for a given $\kappa > 0$, there is $0 < \zeta < 1$ such that for all $\varphi \in (R4, \tau)$ with $\tau > \frac{2}{\zeta}$ and $\mathcal{R}_\infty(\varphi) < \kappa$, and for all $\psi \in L^\infty(M, m)$ we have: if $\tau = \frac{a+1}{\zeta}$, there is some $C' > 0$ such that

$$\operatorname{Cor}_\mu(\varphi, \psi \circ f^n) \le C'(n^{-a+1} \log n),$$

otherwise, there is some $C' > 0$ such that

$$\operatorname{Cor}_\mu(\varphi, \psi \circ f^n) \le C' \max\{n^{-a+1}, n^{2-\zeta\tau}\}.$$

By Corollary B, if $\alpha_{\max} < \frac{1}{2}$ (thus $a > 2$) then CLT is satisfied for all $\varphi \in (R4, \tau)$, that is not coboundary with some $\tau > 0$ sufficiently large, and by Corollary C we also have

$$LD_\mu(\varphi, \epsilon, n) \le C'' \max\{n^{-a+1}, n^{2-\zeta\tau}\}$$

for all $\varphi \in (R4, \tau)$, with $\tau \ne \frac{a+1}{\zeta}$ and some $C'' = C''(\epsilon, \varphi) > 0$.

We now recall the construction of the induced map f^R, outlining the main steps and properties. For further details and proofs, we refer the reader to [23].

Let $\Delta_0 = S^1 \times (\frac{1}{2}, 1] \subset M$. We aim to construct an induced map $f^R: \Delta_0 \to \Delta_0$ for f. To this end, we define a partition $\mathcal{P}_0$ of Δ_0 and the return time function $R: \Delta_0 \to \mathbb{N}$. For all $\theta \in S^1$, we define $X_0(\theta) = 1$, $X_1(\theta) = \frac{1}{2}$, and for $n \ge 2$,

$$X_n(\theta) = f^{-1}_{\alpha(\theta)}(X_{n-1}(g(\theta))) \in \left[0, \frac{1}{2}\right].$$

Consider $Y_n(\theta) = f^{-1}_{\alpha(\theta)}(X_{n-1}(g(\theta))) \in [\frac{1}{2}, 1]$ and $J_n(\theta) = [Y_{n+1}(\theta), Y_n(\theta)]$ for all $\theta \in S^1$. Let $q > 0$ be large enough to be chosen appropriately, and define

$$A_{s_0,n} = \left[\frac{s_0}{4^{q+n}}, \frac{s_0 + 1}{4^{q+n}}\right] \times J_n,$$

for all $n \in \mathbb{N}$ and $0 \le s_0 \le 4^{q+n} - 1$. In the following, if I is an interval of S^1, then $I \times [Y_{n+1}, Y_n]$ stands for

$$\{(\theta, x) : \theta \in I,\ x \in [Y_{n+1}(\theta), Y_n(\theta)]\}.$$

Consider $\mathcal{P}_0 = \{A_{s_0,n}\}$ and define $R|_{A_{s_0,n}} = n$. For $0 \le i \le 4^q - 1$ set

$$K_i = \left[\frac{i}{4^q}, \frac{i+1}{4^q}\right] \times \left[\frac{1}{2}, 1\right].$$

Clearly, f^R maps each $A_{s_0,n}$ bijectively onto some K_i. Consequently, the induced map $f^R \colon \Delta_0 \to \Delta_0$ satisfies the Markov condition (W1) with respect to the partition $\mathcal{P}_0$, and it also has the long branch property (W5). We now verify that f^R is aperiodic and has a coprime block. Since

$$(f^R)^2(A_{s_0,n}) = \Delta_0$$

for each $A_{s_0,n} \in \mathcal{P}_0$, it follows that the induced map f^R is aperiodic. Next, for $N \ge 2$ pick

$$A_{0,1}, A_{0,2}, A_{0,3}, \ldots, A_{0,N} \in \mathcal{P}_0.$$

We have $R(A_{0,i}) = i$, which implies that $\gcd\{R(A_{0,i})\}_i = 1$. By the construction of the partition elements $A_{0,i}$, we have

$$f^R(A_{0,i}) = \left[\frac{0}{4^q}, \frac{1}{4^q}\right] \times \left[\frac{1}{2}, 1\right] \supseteq A_{0,i} \quad \text{for all } i,$$

and hence the induced map f^R has a coprime block.

For (θ_1, x_1) and $(\theta_2, x_2) \in S^1 \times [0, 1]$, set

$$d'((\theta_1, x_1), (\theta_2, x_2)) = a|x_1 - x_2| + |\theta_1 - \theta_2|,$$

for some appropriate $a > 0$. It is shown in [23, Proposition 2.6] that there exist $\lambda > 1$ and $C > 0$ such that, for every (θ_1, x_1) and $(\theta_2, x_2) \in A_{s_0,n}$, we have

$$d'(f^R(\theta_1, x_1), f^R(\theta_2, x_2)) \ge C\lambda d'((\theta_1, x_1), (\theta_2, x_2)). \tag{2.1}$$

In particular, this implies the separability condition (W2). We emphasize that the notion of expansion used in our work is not the same as that in [23, Proposition 2.6], which we reproduce here for clarity. To this end, we need to address conditions (i) and (ii), which here refer to f^R being expanding.

Condition (ii) is automatically satisfied in this case, since there is no contraction. Now, for (θ_1, x_1) and $(\theta_2, x_2) \in A_{s_0,n}$, let $s((\theta_1, x_1), (\theta_2, x_2)) = N$, for some $N \in \mathbb{N}$. Then it follows from (2.1) that

$$d'((f^R)^N(\theta_1, x_1), (f^R)^N(\theta_2, x_2)) \ge C\lambda^{N-1} d'(f^R(\theta_1, x_1), f^R(\theta_2, x_2)).$$

Therefore, $d'(f^R(\theta_1, x_1), f^R(\theta_2, x_2)) \leq C_1(\frac{1}{\lambda})^{s((\theta_1,x_1),(\theta_2,x_2))}$ for some $C_1 > 0$. Hence, condition (i) is verified.

It follows from [23, Proposition 2.9] that the map satisfies a distortion control property, implying the Gibbs property (W4).

In addition, [23, Proposition 2.2] shows that there exists $C > 0$ such that, for all $n \in \mathbb{N}$ and for all $\theta \in S^1$,

$$\frac{1}{Cn^{1/\alpha_{\min}}} \leq X_n(\theta) \leq \frac{C}{n^{1/\alpha_{\max}}}.$$

Therefore, we have

$$m(R > n) = m\Big(S^1 \times \Big[\frac{1}{2}, Y_{n+1}\Big]\Big) = \frac{1}{2}m(S^1 \times [0, X_n]) \leq \frac{1}{2}\frac{C}{n^{1/\alpha_{\max}}}.$$

Since $\alpha_{\max} < 1$, it follows that $\sum_{n\geq 0} m(R > n) < \infty$, and hence $R \in L^1(m)$. Moreover, $m(R > n) \leq Cn^{-a}$ with $a = 1/\alpha_{\max}$ and some $C > 0$. Hence, we conclude that f^R is an aperiodic induced WGM expanding map with $R \in L^1(m)$ and a coprime block.

3. Decay of correlations for tower maps

In this section, we consider the tower map associated with an induced WGM map $f^R: \Delta_0 \to \Delta_0$, where $f: M \to M$, and recall the existence of a mixing invariant probability measure for the tower map. We then formulate the problem of decay of correlations for this dynamical system, considering broad classes of observables, and devote the remainder of the section to addressing this problem, concluding with a discussion of the CLT for the tower map and how it leads to the proof of Corollary B.

3.1. Tower maps

Assume that $f^R: \Delta_0 \to \Delta_0$ is an induced WGM map for $f: M \to M$. We define the *tower*

$$\Delta = \{(x, \ell) : x \in \Delta_0 \text{ and } R(x) > \ell \geq 0\}$$

and the *tower map* $T: \Delta \to \Delta$ given by

$$T(x, \ell) = \begin{cases} (x, \ell + 1), & \text{if } R(x) > \ell + 1, \\ (f^R(x), 0), & \text{if } R(x) = \ell + 1. \end{cases}$$

Note that we can naturally identify the set $\{(x, 0) : x \in \Delta_0\} \subset \Delta$ with Δ_0, and the induced map $T^R: \Delta_0 \to \Delta_0$ with f^R. For each $\ell \geq 0$, we define the *ℓth level* of the tower

$$\Delta_\ell = \{(x,\ell) : x \in \Delta_0\},$$

which is naturally identified with $\{R > \ell\} \subset \Delta_0$. In view of this, we may extend the σ-algebra $\mathcal{A}$ and the reference measure m on Δ_0 to a σ-algebra and a measure on Δ, that we still denote by $\mathcal{A}$ and m, respectively. Moreover, the countable partition $\mathcal{P}_0$ of Δ_0 naturally induces an (m mod 0) partition of each level, that is, if $\mathcal{P}_0 = \{\Delta_{0,i}\}_{i\in\mathbb{N}}$ is the partition of Δ_0, then $\{\Delta_{\ell,i}\}_{i\in\mathbb{N}}$, where

$$\Delta_{\ell,i} = \{(x,\ell) \in \Delta_\ell : (x,0) \in \Delta_{0,i}\},$$

forms a partition of Δ_ℓ. So, the set $\eta = \{\Delta_{\ell,i}\}_{\ell,i}$ is an (m mod 0) partition of Δ. For each $n \geq 1$, we introduce

$$\eta_n = \bigvee_{i=0}^{n-1} T^{-i}\eta.$$

We can extend the separation time to $\Delta \times \Delta$ in the following way: if $x, y \in \Delta_\ell$, then there are unique $x_0, y_0 \in \Delta_0$ such that $x = T^\ell(x_0, 0)$ and $y = T^\ell(y_0, 0)$, and in this case we set $s(x, y) = s(x_0, y_0)$, otherwise set $s(x, y) = 0$. It is straightforward to check that J_T is

$$J_T(x,\ell) = \begin{cases} 1, & \text{if } R(x) > \ell+1, \\ J_{f^{R(x)}}, & \text{if } R(x) = \ell+1. \end{cases}$$

Given $0 < \beta < 1$ we define the following spaces of densities for the tower:

$$\mathcal{F}_\beta(\Delta) = \left\{\varphi\colon \Delta \to \mathbb{R} : \exists C_\varphi > 0 : |\varphi(x) - \varphi(y)| \leq C_\varphi \beta^{s(x,y)},\ \forall x, y \in \Delta\right\}$$

and

$$\mathcal{F}_\beta^+(\Delta) = \left\{\varphi \in \mathcal{F}_\beta(\Delta) : \exists \widehat{C}_\varphi > 0 \text{ s.t. } \varphi(x) > 0 \text{ and } \left|\frac{\varphi(x)}{\varphi(y)} - 1\right| \leq \widehat{C}_\varphi \beta^{s(x,y)}, \right.$$
$$\left. \forall x, y \in \omega \in \eta\right\}.$$

The following result is already established in [39], which provides a mixing invariant probability measure for the tower map.

Theorem 3.1. *Let $T\colon \Delta \to \Delta$ be the tower map of an aperiodic induced WGM map f^R with a coprime block and $R \in L^1(m)$. Then T has a unique exact invariant probability measure $\nu \ll m$ with $\frac{d\nu}{dm} \in \mathcal{F}_\beta^+(\Delta)$, and there is $C_0 > 0$ such that $0 < \frac{d\nu}{dm} \leq C_0$. Moreover, there exists $C > 0$ such that for all $x, y \in \omega \in \eta$*

$$\left|\log \frac{\frac{d\nu}{dm}(x)}{\frac{d\nu}{dm}(y)}\right| \leq C\beta^{s(x,y)}.$$

We first establish decay of correlations for the tower map for classes $(V1)$–$(V4)$ of observables on Δ (defined just below), corresponding to classes $(R1)$–$(R4)$ on M. We then transfer this information to the original dynamical system $f\colon M \to M$ via a semi-conjugacy, so that the proof of Theorem A can be deduced from Proposition 3.4 (see Remark 3.5).

Let us define some classes of observables on the tower. Given a bounded function $\varphi\colon \Delta \to \mathbb{R}$, we define the variation for each $n \geq 0$ as

$$v_n(\varphi) = \sup\{|\varphi(x) - \varphi(y)| : s(x, y) \geq n\}.$$

Consider the following regularity classes:

$$\begin{aligned}
(V1, \tau) &= \{\varphi : \exists D > 0 \text{ s.t. } v_n(\varphi) \leq D\tau^n\}, && \tau \in (0, 1);\\
(V2, \tau) &= \{\varphi : \exists D > 0 \text{ s.t. } v_n(\varphi) \leq De^{-n^\tau}\}, && \tau \in (0, 1);\\
(V3, \tau) &= \{\varphi : \exists D > 0 \text{ s.t. } v_n(\varphi) \leq De^{-(\log n)^\tau}\}, && \tau > 1;\\
(V4, \tau) &= \{\varphi : \exists D > 0 \text{ s.t. } v_n(\varphi) \leq Dn^{-\tau}\}, && \tau > 1.
\end{aligned}$$

Now we define a measurable semi-conjugacy $\pi\colon \Delta \to M$ between the tower map T and the map f, by $\pi(x, \ell) = f^\ell(x)$. We have $\pi \circ T = f \circ \pi$ and $\pi_*\nu$ coincides with the f-invariant measure μ given by Theorem A. It is an immediate consequence that for all $\varphi, \psi\colon M \to \mathbb{R}$,

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \mathrm{Cor}_\nu(\varphi \circ \pi, \psi \circ \pi \circ T^n). \tag{3.1}$$

This shows that in order to deduce Theorem A from a tower map, for a given regularity of $\varphi\colon M \to \mathbb{R}$, expressed in terms of $R_\epsilon(\varphi)$, we aim for a suitable regularity for the lifted observable $\varphi \circ \pi\colon \Delta \to \mathbb{R}$. In particular, we want the observable classes $(R1)$–$(R4)$ on M to correspond to the classes $(V1)$–$(V4)$ on Δ. For this, we have the following lemma.

Lemma 3.2 ([27, Lemma 4]). *Let $T\colon \Delta \to \Delta$ be the tower map of an induced WGM expanding map $f^R\colon \Delta_0 \to \Delta_0$. Then,*

(i) $\varphi \in (Ri, \tau) \Rightarrow \varphi \circ \pi \in (Vi, \tau')$, *with $\tau' < \tau$, for $i = 1, 2, 3$;*

(ii) $\varphi \in (R4, \tau) \Rightarrow \varphi \circ \pi \in (V4, \tau)$.

We emphasize that assuming the induced WGM map f^R to be expanding ensures that the projection map $\pi\colon \Delta \to M$ is Hölder continuous with respect to the separation time s on Δ, that is, there exist $0 < \beta < 1$ and $C > 0$ such that, for all $x, y \in \Delta$,

$$d(\pi(x), \pi(y)) \leq C\beta^{s(x,y)}.$$

We note that the Hölder continuity of the projection map π is sufficient to establish Lemma 3.2.

Consider the following spaces relevant to the extended coupling argument:

$$\mathcal{I}(\Delta) = \{\varphi\colon \Delta \to \mathbb{R} : v_n(\varphi) \to 0\},$$

and

$$\mathcal{I}^+(\Delta) = \Big\{\varphi \in \mathcal{I}(\Delta) : \exists C'_\varphi > 0 \text{ s.t. } \varphi(x) > 0 \\ \text{and } \left|\frac{\varphi(x)}{\varphi(y)} - 1\right| \le C'_\varphi v_{s(x,y)}(\varphi) + C''\beta^{s(x,y)},\ \forall x, y \in \omega \in \eta\Big\},$$

where $C'' > 0$ is a fixed constant to be specified in Lemma 3.6.

Given $\varphi \in L^\infty(\Delta)$ we define

$$\varphi^* = \frac{1}{\int (\varphi + 2\|\varphi\|_\infty + 1) d\nu} (\varphi + 2\|\varphi\|_\infty + 1),$$

and given a signed measure $\upsilon \ll m$ on Δ we define its *total variation* as

$$|\upsilon| = \int_\Delta \left|\frac{d\upsilon}{dm}\right| dm.$$

Lemma 3.3 ([2, Lemma 3.49]). *For all $\varphi \in L^\infty(m)$ with $\varphi \neq 0$ we have*

(i) $\frac{1}{3} \le \varphi^* \le 3$;

(ii) *for all $\psi \in L^\infty(m)$,*

$$\mathrm{Cor}_\nu(\varphi, \psi \circ T^n) \le 3(\|\varphi\|_\infty + 1)\|\psi\|_\infty |T^n_* \lambda - \nu|,$$

where λ is the probability measure on Δ such that

$$\frac{d\lambda}{dm} = \varphi^* \frac{d\nu}{dm}.$$

We remark that in [2,27], the authors studied a tower map for an induced GM (full branches) map. However, the proofs of Lemma 3.2 and Lemma 3.3 do not require the full branches property, so they also hold for an induced WGM expanding map. From (ii) of Lemma 3.3, in order to obtain the decay of correlations for the tower map, it suffices to estimate $|T^n_*\lambda - \nu|$. For that we have the following.

Proposition 3.4. *Let $T\colon \Delta \to \Delta$ be the tower map of an aperiodic induced WGM map f^R, with a coprime block and $R \in L^1(m)$. If ν is the unique mixing T-invariant probability measure such that $\frac{d\nu}{dm} \in \mathcal{I}^+(\Delta)$, then*

(1) *if $m\{R > n\} \le Cn^{-a}$ for some $C > 0$ and $a > 1$, given $\kappa > 0$, there is $0 < \zeta < 1$ such that, for any probability measure λ satisfying $\frac{d\lambda}{dm} \in \mathcal{I}^+(\Delta)$, $v_n(\frac{d\lambda}{dm}) \le Dn^{-\tau}$, for some $\tau > \frac{2}{\zeta}$, $D > 0$, and $v_0(\frac{d\lambda}{dm}) < \kappa$, we have*

(i) *if $\tau = \frac{a+1}{\zeta}$, there is $D' > 0$ such that $|T^n_*\lambda - \nu| \le D'(n^{1-a}\log n)$;*

(ii) *otherwise, there is $D' > 0$ such that $|T^n_*\lambda - \nu| \le D' \max\{n^{1-a}, n^{2-\zeta\tau}\}$;*

(2) *if* $m\{R > n\} \le Ce^{-cn}$ *for some* $C, c > 0$*, then*

(i) *for any probability measure* λ*, with* $\frac{d\lambda}{dm} \in \mathcal{I}^+(\Delta)$ *and* $v_n(\frac{d\lambda}{dm}) \le D\tau^n$*, for some* $\tau \in (0,1)$*,* $D > 0$*, there are* $D', c' > 0$ *such that*

$$|T_*^n\lambda - \nu| \le D'e^{-c'n};$$

(ii) *for any probability measure* λ*, with* $\frac{d\lambda}{dm} \in \mathcal{I}^+(\Delta)$ *and* $v_n(\frac{d\lambda}{dm}) \le De^{-n^\tau}$*, for some* $\tau \in (0,1)$*,* $D > 0$*, there is* $D' > 0$ *such that*

$$|T_*^n\lambda - \nu| \le D'e^{-n^{\tau'}} \quad \text{for every } 0 < \tau' < \tau;$$

(iii) *for any probability measure* λ*, with* $\frac{d\lambda}{dm} \in \mathcal{I}^+(\Delta)$ *and* $v_n(\frac{d\lambda}{dm}) \le De^{-(\log n)^\tau}$*, for some* $\tau > 1$*,* $D > 0$*, there is* $D' > 0$ *such that*

$$|T_*^n\lambda - \nu| \le D'e^{-(\log n)^{\tau'}} \quad \text{for every } 0 < \tau' < \tau;$$

(iv) *given* $\kappa > 0$*, there is* $\zeta < 1$ *such that, for any probability measure* λ*, with* $\frac{d\lambda}{dm} \in \mathcal{I}^+(\Delta)$ *and* $v_n(\frac{d\lambda}{dm}) \le D(n^{-\tau})$*, for some* $\tau > \frac{1}{\zeta}$*,* $D > 0$*, and* $v_0(\frac{d\lambda}{dm}) < \kappa$*, there is* $D' > 0$ *such that*

$$|T_*^n\lambda - \nu| \le D'n^{1-\zeta\tau}.$$

Remark 3.5. We observe from (3.1) that, in order to obtain decay of correlations for (f, μ), it suffices to establish decay of correlations for (T, ν). Thus, by applying Lemma 3.2, Lemma 3.3, and Proposition 3.4, we obtain the proof of Theorem A. It therefore remains to prove Proposition 3.4, which is the subject of the remaining sections of this article.

Since we aim to apply Lemma 3.3 and Proposition 3.4 to prove Theorem A, we need to verify the necessary regularity of $\frac{d\lambda}{dm} = \varphi^*\frac{d\nu}{dm}$, for a given φ in one of the classes $(V1)$–$(V4)$.

Lemma 3.6. *If* $\varphi \in (Vj, \tau)$*, and* $\tau \in (0,1)$ *for* $j = 1, 2$ *or* $\tau > 1$ *for* $j = 3, 4$ *then* $\varphi^*\frac{d\nu}{dm} \in \mathcal{I}^+(\Delta)$.

Proof. Set $\rho = \frac{d\nu}{dm}$. Since $0 < \rho \le C_0$ and from Lemma 3.3 we have

$$\begin{aligned}|\varphi^*(x)\rho(x) - \varphi^*(y)\rho(y)| &\le |\varphi^*(x)(\rho(x) - \rho(y))| + |\rho(y)(\varphi^*(x) - \varphi^*(y))| \\ &\le 3|\rho(x) - \rho(y)| + C_0|\varphi(x) - \varphi(y)|.\end{aligned}$$

Thus $v_n(\varphi^*\rho) \le 3C_\rho\beta^n + C_0 v_n(\varphi)$. This implies that $\varphi^*\rho \in \mathcal{I}(\Delta)$. From Lemma 3.3, we have

$$\begin{aligned}\left|\frac{\varphi^*(x)}{\varphi^*(y)} - 1\right| &= \frac{1}{\varphi^*(y)}|\varphi^*(x) - \varphi^*(y)| \\ &\le 3|\varphi(x) - \varphi(y)| \le 3v_{s(x,y)}(\varphi). \qquad (3.2)\end{aligned}$$

Since $\frac{1}{9} \leq \frac{\varphi^*(x)}{\varphi^*(y)} \leq 9$, for all $x, y \in \Delta$, there exists $K_1' > 0$ such that

$$\left|\log \frac{\varphi^*(x)}{\varphi^*(y)}\right| \leq K_1' \left|\frac{\varphi^*(x)}{\varphi^*(y)} - 1\right|. \tag{3.3}$$

It follows from (3.2) and (3.3) that for all $x, y \in \omega \in \eta$,

$$\begin{aligned} \left|\log \frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)}\right| &\leq \left|\log \frac{\varphi^*(x)}{\varphi^*(y)}\right| + \left|\log \frac{\rho(x)}{\rho(y)}\right| \\ &\leq 3K_1' v_{s(x,y)}(\varphi) + C\beta^{s(x,y)}. \end{aligned} \tag{3.4}$$

Since $0 < \frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)} \leq 9e^C$, for all $x, y \in \omega \in \eta$, then we also have some uniform constant $K_2' > 0$ that for all $x, y \in \omega \in \eta$,

$$\left|\frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)} - 1\right| \leq K_2' \left|\log \frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)}\right| \leq K_2'(3K_1' v_{s(x,y)}(\varphi) + C\beta^{s(x,y)}).$$

Hence $\varphi^*\rho \in \mathcal{I}^+(\Delta)$ with $C'' = K_2' C$. ∎

We would like to point out a remark about $\varphi^* \frac{d\nu}{dm}$, which is important because we will assume this kind of regularity in the extended coupling argument.

Remark 3.7. From the previous proof we can see that there exists $C''_{\varphi^*\rho} > 0$ such that for all $x, y \in \omega \in \eta$,

$$\left|\log \frac{\varphi^*(x)\rho(x)}{\varphi^*(y)\rho(y)}\right| \leq C''_{\varphi^*\rho} v_{s(x,y)}(\varphi) + C\beta^{s(x,y)}.$$

3.2. Extended coupling

On the one hand, as pointed out in [39, Remark 5.15], the lack of full branches in the induced map has several implications, such as the fact that the density of the invariant measure ν for the tower map does not necessarily have a positive lower bound – a property that simplifies many arguments, as seen in [41] – and the failure of the symmetry condition found in [41, (1)], which in turn implies that the quantity I_2 in the proof of Proposition 3.22 does not vanish. On the other hand, the consideration of observable classes beyond Hölder regularity requires modifications to the standard coupling arguments. In this section, we address both of these extensions by combining the coupling arguments developed in [39] and [27].

Let λ_1 and λ_2 be probability measures on Δ with $\varphi_1 = \frac{d\lambda_1}{dm}, \varphi_2 = \frac{d\lambda_2}{dm} \in \mathcal{I}^+(\Delta)$. For the proof of Proposition 3.4, we first derive estimates for $P\{S > n\}$ in terms of $P\{R > n\}$, and later for $|T_*^n\lambda_1 - T_*^n\lambda_2|$ in terms of $P\{S > n\}$. Then, by setting $\lambda_1 = \lambda$ and $\lambda_2 = \nu$ in Sections 3.3.1 and 3.3.2, we obtain the required estimates for $|T_*^n\lambda - \nu|$.

3.2.1. Stopping times and simultaneous returns. Let $P = \lambda_1 \times \lambda_2$ be the product measure on $\Delta \times \Delta$. We consider the product transformation $T \times T\colon \Delta \times \Delta \to \Delta \times \Delta$, and let $\pi_1, \pi_2\colon \Delta \times \Delta \to \Delta$ be the projections onto the first and second coordinates, respectively. We use $\eta \times \eta$ to denote the product partition of $\Delta \times \Delta$ and

$$(\eta \times \eta)_n = \bigvee_{i=0}^{n-1} (T \times T)^{-i} (\eta \times \eta).$$

Notice that

$$T^n \circ \pi_1 = \pi_1 \circ (T \times T)^n \quad \text{and} \quad T^n \circ \pi_2 = \pi_2 \circ (T \times T)^n. \tag{3.5}$$

Let $\widehat{R}\colon \Delta \to \mathbb{N}$ be the return time to Δ_0, defined for $x \in \Delta$ by

$$\widehat{R}(x) = \min\{n \geq 0 : T^n(x) \in \Delta_0\}.$$

Observe that

$$m\{\widehat{R} > n\} = \sum_{\ell > n} m(\Delta_\ell) = \sum_{\ell > n} m\{R > \ell\}. \tag{3.6}$$

From (W5) and the mixing property of measure $\nu \ll m$, we may fix some integer $n_0 > 0$ such that for any $\omega \in \mathcal{P}_0$,

$$m(T^{-n}(\Delta_0) \cap f^R(\omega)) \geq \text{some } \gamma_0 > 0 \quad \text{for } n \geq n_0.$$

The above choice of n_0 is important for Lemma 3.11. Let us now introduce a sequence of *stopping times* $0 = \tau_0 < \tau_1 < \tau_2 \cdots$ on $\Delta \times \Delta$,

$$\begin{aligned}
\tau_1 &= n_0 + \widehat{R} \circ T^{n_0} \circ \pi_1, \\
\tau_2 &= n_0 + \tau_1 + \widehat{R} \circ T^{n_0 + \tau_1} \circ \pi_2, \\
\tau_3 &= n_0 + \tau_2 + \widehat{R} \circ T^{n_0 + \tau_2} \circ \pi_1, \\
\tau_4 &= n_0 + \tau_3 + \widehat{R} \circ T^{n_0 + \tau_3} \circ \pi_2, \\
&\vdots
\end{aligned}$$

We define the *simultaneous return* to the base Δ_0 $S\colon \Delta \times \Delta \to \mathbb{N}$ by

$$S(x, y) = \min_{i \geq 2}\{\tau_i(x, y) : (T^{\tau_i(x,y),}(x), T^{\tau_i(x,y),}(y)) \in \Delta_0 \times \Delta_0\}, \tag{3.7}$$

which is well defined $m \times m$ almost everywhere. Let

$$\xi_0 < \xi_1 < \xi_2 < \xi_3 \cdots$$

be an increasing sequence of partitions on $\Delta \times \Delta$ defined as follows. As usual, given a partition ξ, we denote $\xi(x)$ the element of ξ containing x. First we take $\xi_0 = \eta \times \eta$. Now we describe the general inductive step in the construction of partitions ξ_k. Assume that ξ_j has been constructed for all $j < k$. The definition of ξ_k depends on whether k is odd or even. For definiteness we assume that k is odd. The construction for k even is the same apart from the change in the role of the first and second components. We let $\xi_k = \{\xi_k(\overline{x}) : \overline{x} \in \Delta \times \Delta\}$, where

$$\xi_k(\overline{x}) = \bigvee_{i=0}^{\tau_k(\overline{x})-1} (T^{-i}(\eta))(x) \times \pi_2(\xi_{k-1}(\overline{x})).$$

From the construction of above sequence of partitions ξ_k, we have the following.

Remark 3.8. For any $\Gamma \in \xi_k$, we can find some $\omega_1, \omega_2 \in \mathcal{P}_0$ such that:

- if k is odd, $T^{\tau_k}(\pi_1(\Gamma)) = f^R(\omega_1) \subset \Delta_0$ and $T^{\tau_{k-1}}(\pi_2(\Gamma)) = f^R(\omega_2) \subset \Delta_0$;
- if k is even, $T^{\tau_{k-1}}(\pi_1(\Gamma)) = f^R(\omega_1) \subset \Delta_0$ and $T^{\tau_k}(\pi_2(\Gamma)) = f^R(\omega_2) \subset \Delta_0$.

We begin by establishing some bounded distortion properties for the tower map $T\colon \Delta \to \Delta$, which will be useful for the next few lemmas.

Lemma 3.9 ([2, Lemma 3.30]). *There exists $\overline{C}_T > 0$ such that for all $n \geq 1$, $\omega \in \eta_n$ and $x, y \in \omega$,*

$$\log \frac{J_{T^n}(x)}{J_{T^n}(y)} \leq \overline{C}_T \beta^{s(T^n(x), T^n(y))}.$$

Lemma 3.10. *There exists $D_1 > 0$, depending on C'_{φ_j} and $v_0(\varphi_j)$, for $j = 1, 2$, such that for all $n \geq 1$, $\omega \in \eta_n$ and $x, y \in T^n(\omega)$, we have*

$$\frac{dT^n_*(\lambda_j|\omega)}{dm}(x) \bigg/ \frac{dT^n_*(\lambda_j|\omega)}{dm}(y) \leq D_1.$$

Moreover, the dependence of D_1 on C'_{φ_j} and $v_0(\varphi_j)$ can be removed if the number of visits of ω to Δ_0 up to n is sufficiently large.

Proof. Let $x_0, y_0 \in \omega$ be such that $T^n(x_0) = x$ and $T^n(y_0) = y$. We have

$$\frac{dT^n_*(\lambda_j|\omega)}{dm}(x) \bigg/ \frac{dT^n_*(\lambda_j|\omega)}{dm}(y) = \frac{J_{T^n}(y_0)}{J_{T^n}(x_0)} \frac{\varphi_j(x_0)}{\varphi_j(y_0)}. \tag{3.8}$$

By Lemma 3.9, there exists $C'_T > 0$ such that

$$\frac{J_{T^n}(y_0)}{J_{T^n}(x_0)} \leq C'_T. \tag{3.9}$$

Considering $i = \#\{0 \le k \le n : T^k\omega \cap \Delta_0 \ne \emptyset\}$ to be the number of visits of ω to Δ_0 until time n, we have $s(x_0, y_0) \ge i$. Since $\varphi_j \in \mathcal{I}^+(\Delta)$, we get

$$\frac{\varphi_j(x_0)}{\varphi_j(y_0)} = 1 + \left|\frac{\varphi_j(x_0)}{\varphi_j(y_0)} - 1\right| \le 1 + C'_{\varphi_j} v_i(\varphi_j) + C''\beta^i. \tag{3.10}$$

From (3.8)–(3.10), we get

$$\frac{dT_*^n(\lambda_j|\omega)}{dm}(x) \bigg/ \frac{dT_*^n(\lambda_j|\omega)}{dm}(y) \le D_1,$$

where $D_1 = C'_T(1 + C'_{\varphi_j} v_i(\varphi_j) + C''\beta^i) \le C'_T(1 + C'_{\varphi_j} v_0(\varphi_j) + C'')$. Note that the constant C'_T does not depend on φ_j. So, the dependence of D_1 on C'_{φ_j} and $v_0(\varphi_j)$ can be removed if we take i sufficiently large. ∎

3.2.2. Estimating $P\{S > n\}$. The next couple of lemmas are crucial for estimating $P\{S > n\}$. Their proofs follow the same steps as those of [39, Lemmas 5.10 and 5.11], respectively, leading to $\epsilon_0 = \gamma_0/D_1^2 m(\Delta_0)$ and $D_2 = M_0 D_1^2/\delta_0$ in their statements. In both situations, it follows from Lemma 3.10 that the dependence of D_1 (hence ϵ_0 and D_2) on C'_{φ_j} and $v_0(\varphi_j)$, for $j = 1, 2$, can be removed if we take k large enough.

Lemma 3.11. *There exists $\epsilon_0 > 0$, depending on C'_{φ_j} and $v_0(\varphi_j)$, for $j = 1, 2$, such that for all $k \ge 1$ and $\Gamma \in \xi_k$ with $S|_\Gamma > \tau_{k-1}$, we have*

$$P(S = \tau_k|\Gamma) \ge \epsilon_0.$$

Moreover, the dependence of ϵ_0 on C'_{φ_j} and $v_0(\varphi_j)$, for $j = 1, 2$, can be removed if we take k sufficiently large.

Lemma 3.12. *There exists D_2, depending on C'_{φ_1}, C'_{φ_2}, $v_0(\varphi_1)$, $v_0(\varphi_2)$ such that for all $n, k \ge 0$ and for all $\Gamma \in \xi_k$, we have*

$$P(\tau_{k+1} - \tau_k > n + n_0|\Gamma) \le D_2 m\{\widehat{R} > n\}.$$

Moreover, the dependence of ϵ_0 on C'_{φ_j} and $v_0(\varphi_j)$, for $j = 1, 2$, can be removed if we take k sufficiently large.

Finally, we derive an estimate for $P\{S > n\}$ using equation (3.6), Lemma 3.11, Lemma 3.12, and [2, Propositions 3.46 and 3.48].

Corollary 3.13. *The following estimates hold:*

(i) *If $m\{R > n\} \le Cn^{-a}$ for some $C > 0$, $a > 1$, then $P\{S > n\} \le \widetilde{C}n^{-a+1}$ for some $\widetilde{C} > 0$.*

(ii) *If $m\{R > n\} \le Ce^{-cn}$ for some $C, c > 0$, then $P\{S > n\} \le \widetilde{C}e^{-c_0 n}$ for some $\widetilde{C}, c_0 > 0$.*

3.2.3. Preparatory results. Consider the induced map

$$\widetilde{T} = (T \times T)^S : \Delta \times \Delta \to \Delta \times \Delta,$$

with S as in (3.7), and the functions $0 = S_0 < S_1 < S_2 < \cdots$, given, for all $n \geq 1$, by

$$S_n = S_{n-1} + S \circ (T \times T)^{S_{n-1}}.$$

Note that we have

$$\widetilde{T}^n = (T \times T)^{S_n}.$$

Let $\widetilde{\xi}$ be the partition of $\Delta \times \Delta$ into the rectangles Ω on which S is constant and $\widetilde{T}$ maps Ω bijectively onto a union of elements of $\eta \times \eta|_{\Delta_0 \times \Delta_0}$. Without loss of generality, we assume that for any $\Omega \in \widetilde{\xi}|_{\Delta_0 \times \Delta_0}$, there exists $\omega_j \times \omega_{j'} \in \eta \times \eta|_{\Delta_0 \times \Delta_0}$ such that $\Omega \subset \omega_j \times \omega_{j'}$. For each $n \geq 1$, define

$$\widetilde{\xi}_n = \bigvee_{j=0}^{n-1} \widetilde{T}^{-j}(\widetilde{\xi}).$$

Each $\widetilde{\xi}_n$ is a partition into sets $\Omega \subset \Delta \times \Delta$ on which S_n is constant and $\widetilde{T}^n$ maps Ω bijectively onto an $m \times m$ mod 0 union of elements of $\eta \times \eta|_{\Delta_0 \times \Delta_0}$. Let us introduce a separation time in $\Delta \times \Delta$, defining, for each $u, v \in \Delta \times \Delta$,

$$\tilde{s}(u, v) = \min\{n \geq 0 : \widetilde{T}^n(u) \text{ and } \widetilde{T}^n(v) \text{ lie in distinct elements of } \widetilde{\xi}\}.$$

Let $\Phi(x, y) = \varphi_1(x)\varphi_2(y)$ and let C''_{φ_1} and C''_{φ_2} be constants such that, for $i = 1, 2$,

$$\left| \log \frac{\varphi_i(x)}{\varphi_i(y)} \right| \leq C''_{\varphi_i} v_{s(x,y)}(\varphi_i) \quad \text{for all } x, y \in \omega \in \eta. \tag{3.11}$$

Before proceeding, we recall some auxiliary results that will be useful later on. One of the concerns in this subsection is the dependence of the constants.

Lemma 3.14 ([2, Lemma 3.35]). *There exists $\bar{C}_{\widetilde{T}} > 0$ such that for all $n \geq 1$, $\Omega \in \widetilde{\xi}_n$ and $u, v \in \Omega$ we have*

$$\log \frac{J_{\widetilde{T}^n}(u)}{J_{\widetilde{T}^n}(v)} \leq \bar{C}_{\widetilde{T}} \beta^{\tilde{s}(\widetilde{T}^n(u), \widetilde{T}^n(v))}.$$

The following auxiliary lemma describes explicitly C_Φ and $v_n(\Phi)$.

Lemma 3.15. *For all $u, v \in \Omega \in \widetilde{\xi}$,*

$$\log \frac{\Phi(u)}{\Phi(v)} \leq C_\Phi v_{\tilde{s}(u,v)}(\Phi),$$

where $C_\Phi = C''_{\varphi_1} + C''_{\varphi_2}$ and $v_{\tilde{s}(u,v)}(\Phi) = \max\{v_{\tilde{s}(u,v)}(\varphi_1), v_{\tilde{s}(u,v)}(\varphi_2)\}$.

Proof. For all $u = (x_1, x_2)$ and $v = (y_1, y_2)$ in $\Delta \times \Delta$, we have $s(x_i, y_i) \geq \tilde{s}(u, v)$ for $i = 1, 2$. The result then follows directly from the definition of Φ and (3.11). ∎

Remark 3.16. Notice that if

$$\left|\log \frac{\varphi_1(x)}{\varphi_1(y)}\right| \le C''_{\varphi_1} v_{s(x,y)}(\varphi_1) + C''_{\varphi_2} v_{s(x,y)}(\varphi_2),$$

then Lemma 3.15 remains true with $C_\Phi = C''_{\varphi_1} + 2C''_{\varphi_2}$.

Lemma 3.17. *There exists $D_3 > 0$, depending on C_Φ and $v_0(\Phi)$, such that for all $n \ge 1, \Omega \in \widetilde{\xi}_n$ and $u, v \in \widetilde{T}^n(\Omega)$ we have*

$$\frac{d\widetilde{T}^n_*(P \mid \Omega)}{d(m \times m)}(u) \Big/ \frac{d\widetilde{T}^n_*(P \mid \Omega)}{d(m \times m)}(v) \le D_3.$$

Proof. The proof follows similarly to that of [39, Proposition 5.15], yielding the result with $D_3 = C'_{\widetilde{T}} e^{C_\Phi v_0(\Phi)}$, which depends on C_Φ and $v_0(\Phi)$. ■

The following result will be used in Proposition 3.22.

Proposition 3.18. *There exists $D_4 > 0$ depending on $C_\Phi, v_0(\Phi)$ such that for all $n, k \ge 0$, we have*

$$P\{S_{k+1} - S_k > n\} \le D_4(m \times m)\{S > n\}.$$

Proof. An argument analogous to that of [39, Proposition 5.15] yields the result with $D_4 = D_3^2/\delta_0^2$. ■

The following lemma plays an important role in this setting and originates from the coupling argument in [27]. We remark that (3.12) and (3.13) in Lemma 3.19 are important for Lemma 3.20 (see [27]), while (3.14) will be useful in Sections 3.3.1 and 3.3.2.

Lemma 3.19 ([27, Proposition 1 and Lemma 1]). *Given a sequence $(v_i(\Phi))_i$, there exists a sequence $(\epsilon'_i)_i$, with $\epsilon'_i \le \frac{1}{2}$, such that*

$$v_i(\Phi) \prod_{j=1}^{i} (1 + \epsilon'_j) \le D_5, \tag{3.12}$$

$$\sum_{j=1}^{i} \left(\prod_{k=j}^{i} (1 + \epsilon'_k) \right) \beta^{i-j+1} \le D_5 \tag{3.13}$$

for some sufficiently large constant D_5 depending on $v_0(\Phi)$. Moreover, for any $\bar{D} > 1$ and $\bar{\delta} > 0$ with $\epsilon_i = \bar{\delta}\epsilon'_i$,

$$\prod_{j=1}^{i} \left(1 - \frac{\epsilon_j}{\bar{D}}\right) \le \widetilde{C} \max(v_i(\Phi)^{\frac{\bar{\delta}}{\bar{D}}}, \theta^i) \tag{3.14}$$

for some $\theta < 1$, depending only on T, and some $\widetilde{C} > 0$.

3.2.4. Densities in induced times and real times. We are going to define a sequence of densities $\widetilde{\Phi}_0 \geq \widetilde{\Phi}_1 \geq \widetilde{\Phi}_2 \geq \cdots$ in $\Delta \times \Delta$, for the total measure remaining in the system after n iterations by $\widetilde{T}$. Consider a constant $\bar{\delta} > 0$, depending on C_Φ and $v_0(\Phi)$ to be chosen properly such that defining

$$\widetilde{\Phi}_i(u) = \begin{cases} \Phi(u), & \text{if } i = 0, \\ \widetilde{\Phi}_{i-1}(u) - \epsilon_i J_{\widetilde{T}^i}(u) \min_{v \in \Omega_i(u)} \frac{\widetilde{\Phi}_{i-1}(v)}{J_{\widetilde{T}^i}(v)}, & \text{if } i > 0. \end{cases} \tag{3.15}$$

for all $i \geq 1$ with $\epsilon_i = \bar{\delta}\epsilon_i' > 0$, the following holds (for details, see [27, Lemma 2]).

Lemma 3.20. *There exists $D_6 > 1$, depending on C_Φ and $v_0(\Phi)$, such that for all $i \geq 1$ we have*

$$\widetilde{\Phi}_i \leq \left(1 - \frac{\epsilon_i}{D_6}\right)\widetilde{\Phi}_{i-1}.$$

Now we are going to define the corresponding densities in real time iterations under $T \times T$. Let us introduce functions $\Phi_0 \geq \Phi_1 \geq \Phi_2 \geq \cdots$ on $\Delta \times \Delta$ as follows: for $v \in \Delta \times \Delta$ we define

$$\Phi_n(v) = \widetilde{\Phi}_i(v), \quad \text{if } S_i(v) \leq n < S_{i+1}(v). \tag{3.16}$$

Since, by definition, $S_0 \leq i_1 \leq S_{i_1} < S_{i_1+1}$, this implies that for all $n \leq i_1$, $\Phi_n = \widetilde{\Phi}_j$ for some $0 \leq j < i_1 + 1$, and $\widetilde{\Phi}_j = \Phi$ for all $0 \leq j < i_1 + 1$, we have

$$\Phi_n = \Phi \quad \text{for all } n \leq i_1.$$

Note that for all $n \geq 1$ we can write

$$\Phi = \Phi_n + \sum_{k=1}^{n} (\Phi_{k-1} - \Phi_k). \tag{3.17}$$

For each $k \geq 1$, let $A_k = \bigcup_i A_{k,i}$, where

$$A_{k,i} = \{u \in \Delta \times \Delta : k = S_i(u)\}.$$

Note that $A_{k,i} \cap A_{k,j} = \emptyset$ for $i \neq j$ (because $S_i(u) \neq S_j(u)$ for $i \neq j$), and each $A_{k,i}$ is a union of elements of $\widetilde{\xi}_i$.

Remark 3.21. By definition, for any $\Omega \in \widetilde{\xi}_i|_{A_{k,i}}$, we have $S_{i-1}|_\Omega < S_i|_\Omega = k$, and $S_{i-1}|_\Omega \leq k - 1$. This implies that $\Phi_{k-1} - \Phi_k = \widetilde{\Phi}_{i-1} - \widetilde{\Phi}_i$ on $\Omega \in \widetilde{\xi}_i|_{A_{k,i}}$, and $\Phi_k = \Phi_{k-1}$ on $\Delta \times \Delta \setminus A_k$.

3.2.5. Estimating $|T_*^n \lambda_1 - T_*^n \lambda_2|$. Our goal is now to estimate $|T_*^n \lambda_1 - T_*^n \lambda_2|$ in terms of $P\{S > n\}$. We state the main result of this subsection, which constitutes an important step in the proof of Proposition 3.4.

Proposition 3.22. *There exists $D_7 > 0$, depending on $C_\Phi, v_0(\Phi)$, such that for all $n \geq 1$,*

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \leq 2P\{S > n\} + D_7 \sum_{i=1}^{n} \prod_{j=1}^{i} \left(1 - \frac{\epsilon_j}{D_6}\right)(i+1)(m \times m)\left\{S > \frac{n}{i+1}\right\}.$$

Proof. From (3.5) and (3.17), for each $n \geq 1$ we have

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \leq I_1 + I_2, \tag{3.18}$$

where

$$I_1 = |(\pi_{1*} - \pi_{2*})(T \times T)_*^n (\Phi_n (m \times m))|$$

and

$$I_2 = \sum_{k=1}^{n} \big|(\pi_{1*} - \pi_{2*})[(T \times T)_*^n ((\Phi_{k-1} - \Phi_k)(m \times m))]\big|.$$

On the one hand, we have

$$\begin{aligned} I_1 &= |(\pi_{1*} - \pi_{2*})(T \times T)_*^n (\Phi_n (m \times m))| \\ &\leq 2 \int_{\Delta \times \Delta} \Phi_n d(m \times m) \\ &= 2 \int_{\{S_1 > n\}} \Phi_n d(m \times m) + 2 \sum_{i=1}^{\infty} \int_{\{S_i \leq n < S_{i+1}\}} \Phi_n d(m \times m). \end{aligned} \tag{3.19}$$

As $\Phi_n|_{\{S_1 > n\}} = \Phi$, we have

$$\int_{\{S_1 > n\}} \Phi_n d(m \times m) = \int_{\{S_1 > n\}} \Phi d(m \times m) = P\{S_1 > n\}. \tag{3.20}$$

It follows from (3.16), (3.15) and Lemma 3.20 that we can get for all $i \geq 1$,

$$\begin{aligned} \int_{\{S_i \leq n < S_{i+1}\}} \Phi_n d(m \times m) &= \int_{\{S_i \leq n < S_{i+1}\}} \tilde{\Phi}_i d(m \times m) \\ &= \prod_{j=1}^{i} \left(1 - \frac{\epsilon_j}{D_6}\right) P\{S_i \leq n < S_{i+1}\}. \end{aligned} \tag{3.21}$$

By combining (3.19), (3.20) and (3.21), we obtain

$$I_1 \leq 2P\{S > n\} + 2 \sum_{i=1}^{\infty} \prod_{j=1}^{i} \left(1 - \frac{\epsilon_j}{D_6}\right) P\{S_i \leq n < S_{i+1}\}. \tag{3.22}$$

On the other hand,

$$
\begin{aligned}
I_2 &\le 2\sum_{k=1}^{n}\int_{\Delta\times\Delta}(\Phi_{k-1}-\Phi_k)d(m\times m)\\
&= 2\sum_{k=1}^{n}\int_{\Delta\times\Delta\setminus A_k}(\Phi_{k-1}-\Phi_k)d(m\times m)\\
&\quad + 2\sum_{k=1}^{n}\int_{A_k}(\Phi_{k-1}-\Phi_k)d(m\times m).
\end{aligned}
\tag{3.23}
$$

From Remark 3.21, we have

$$
2\sum_{k=1}^{n}\int_{\Delta\times\Delta\setminus A_k}(\Phi_{k-1}-\Phi_k)d(m\times m)=0,
\tag{3.24}
$$

and writing $A_k=\bigcup_{i=1}^{\infty}A_{k,i}$, we have

$$
\int_{A_k}(\widetilde{\Phi}_{k-1}-\widetilde{\Phi}_k)d(m\times m)=\sum_{i=1}^{\infty}\int_{A_{k,i}}(\widetilde{\Phi}_{i-1}-\widetilde{\Phi}_i)d(m\times m).
\tag{3.25}
$$

By (3.15) and Lemma 3.20 we have

$$
\begin{aligned}
\sum_{i=1}^{\infty}\int_{A_{k,i}}(\widetilde{\Phi}_{i-1}-\widetilde{\Phi}_i)d(m\times m) &= \sum_{i=1}^{\infty}\int_{A_{k,i}}\epsilon_i J_{\widetilde{T}^i}(u)\min_{v\in A_{k,i}}\frac{\widetilde{\Phi}_{i-1}(v)}{J_{\widetilde{T}^i}(v)}d(m\times m)\\
&\le \sum_{i=1}^{\infty}\int_{A_{k,i}}\epsilon_i\widetilde{\Phi}_{i-1}(u)d(m\times m)\\
&\le \sum_{i=1}^{\infty}\int_{A_{k,i}}\epsilon_i\prod_{j=1}^{i-1}\left(1-\frac{\epsilon_j}{D_6}\right)\Phi(u)d(m\times m)\\
&= \sum_{i=1}^{\infty}\epsilon_i\prod_{j=1}^{i-1}\left(1-\frac{\epsilon_j}{D_6}\right)P\{S_i=k\}.
\end{aligned}
\tag{3.26}
$$

It follows from (3.23), (3.24), (3.25) and (3.26) that

$$
I_2\le 2\sum_{i=1}^{\infty}\epsilon_i\prod_{j=1}^{i-1}\left(1-\frac{\epsilon_j}{D_6}\right)P\left(\bigcup_{k=1}^{n}\{S_i=k\}\right).
\tag{3.27}
$$

For each $1\le k\le n$, we have $\{S_i=k\}\subset\{S_i\le n<S_{i+1}\}$, and this implies that

$$
\bigcup_{k=1}^{n}\{S_i=k\}\subset\{S_i\le n<S_{i+1}\}.
$$

From (3.27) we have

$$I_2 \le \frac{2\bar{\delta} D_6}{2D_6 - \bar{\delta}} \sum_{i=1}^{\infty} \prod_{j=1}^{i} \left(1 - \frac{\epsilon_j}{D_6}\right) P\{S_i \le n < S_{i+1}\}. \tag{3.28}$$

It follows from (3.18), (3.22) and (3.28), that

$$\begin{aligned} &|T_*^n \lambda_1 - T_*^n \lambda_2| \\ &\quad \le 2P\{S > n\} + 2\left(1 + \frac{\bar{\delta} D_6}{2D_6 - \bar{\delta}}\right) \sum_{i=1}^{\infty} \prod_{j=1}^{i} \left(1 - \frac{\epsilon_j}{D_6}\right) P\{S_i \le n < S_{i+1}\}. \end{aligned} \tag{3.29}$$

For each $i \ge 1$ we have

$$P\{S_i \le n < S_{i+1}\} \le \sum_{j=0}^{i} P\left\{S_{j+1} - S_j > \frac{n}{i+1}\right\}, \tag{3.30}$$

and from Proposition 3.18,

$$P\left\{S_{j+1} - S_j > \frac{n}{i+1}\right\} \le D_4 (m \times m)\left\{S > \frac{n}{i+1}\right\}. \tag{3.31}$$

Combining (3.30) and (3.31), we get

$$P\{S_i \le n < S_{i+1}\} \le D_4 (i+1)(m \times m)\left\{S > \frac{n}{i+1}\right\},$$

which, together with (3.29), implies

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \le 2P\{S > n\} + D_7 \sum_{i=1}^{n} \prod_{j=1}^{i} \left(1 - \frac{\epsilon_j}{D_6}\right)(i+1)(m \times m)\left\{S > \frac{n}{i+1}\right\},$$

with $D_7 = 2D_4(1 + \frac{\bar{\delta} D_6}{2D_6 - \bar{\delta}})$. We recall that $\bar{\delta}$, D_4 and D_6 (hence D_7) depend on $C_\Phi, v_0(\Phi)$. ■

3.3. Specific rates

Set $\zeta = \frac{\bar{\delta}}{D_6}$, which can be seen to depend only on C_Φ and $v_0(\Phi)$. In Sections 3.3.1 and 3.3.2, we let D generic constant, allowed to depend only on T and Φ.

3.3.1. Polynomial return time. In this subsection, we combine several estimates, from which we will conclude the first item of Proposition 3.4. If $m\{R > n\} \le Cn^{-a}$, for some $C > 0$ and $a > 1$, then by Corollary 3.13,

$$P\{S > n\} \le \widetilde{C} n^{-a+1}. \tag{3.32}$$

By similar arguments used to estimate $P\{S > n\}$, we have

$$(m \times m)\left\{S > \frac{n}{i+1}\right\} \le \widetilde{C}_1 \left(\frac{n}{i+1}\right)^{-a+1} \quad \text{for some } \widetilde{C}_1 > 0. \tag{3.33}$$

Assume that $v_i(\Phi) \le D i^{-\tau}$ for some $\tau > \frac{2}{\zeta}$ and $D > 0$. This corresponds to class $(V4, \tau)$. By Lemma 3.19, we have

$$\prod_{j=1}^{i} \left(1 - \frac{\epsilon_j}{D_6}\right) \le \widetilde{D} i^{-\zeta\tau}, \quad \text{for some } \widetilde{D} > 0. \tag{3.34}$$

By using (3.32), (3.33) and (3.34) in Proposition 3.22, we get

$$\begin{aligned} |T_*^n \lambda_1 - T_*^n \lambda_2| &\le 2\widetilde{C} n^{-a+1} + \widetilde{C}_1 \widetilde{D} D_7 \sum_{i=1}^{\infty} i^{-\zeta\tau}(i+1)\left(\frac{n}{i+1}\right)^{-a+1} \\ &\le 2\widetilde{C} n^{-a+1} + K_3' \widetilde{C}_1 \widetilde{D} D_7 n^{-a+1} \sum_{i=1}^{\infty} i^{-\zeta\tau} i^{a}, \quad \text{for some } K_3' > 0. \end{aligned}$$

We begin by estimating the second term. To that end, we consider the following cases:

(1) $\frac{2}{\zeta} < \tau < \frac{a+1}{\zeta}$. The sum is of order $n^{a+1-\zeta\tau}$, and so the whole term is $\le D' n^{2-\zeta\tau}$, for some $D' > 0$.

(2) $\tau = \frac{a+1}{\zeta}$. The sum is

$$\sum_{i=1}^{n} i^{-\zeta\tau+a} = \sum_{i=1}^{n} i^{-1} \le 1 + \int_1^n x^{-1}\,dx = 1 + \log(n) \le \bar{D} \log n,$$

for some $\bar{D} > 0$. So the whole term is $\le D' n^{1-a} \log n$, for some $D' > 0$.

(3) $\tau > \frac{a+1}{\zeta}$. Here, $a - \zeta\tau < -1$, so the sum is bounded from above independently of n, and the whole term is $\le D' n^{1-a}$, for some $D' > 0$. Consequently,

(i) if $\tau = \frac{a+1}{\zeta}$, then

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \le D' n^{1-a} \log n, \quad \text{for some } D' > 0;$$

(ii) otherwise,

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \le D' \max\{n^{1-a}, n^{2-\zeta\tau}\}, \quad \text{for some } D' > 0.$$

3.3.2. Exponential return time. In this subsection, we gather the necessary estimates to establish the second item of Proposition 3.4. If $m\{R > n\} \leq Ce^{-cn}$ for some $C, c > 0$, then by Lemma 3.13, we have

$$P\{S > n\} \leq \widetilde{C} e^{-c_0 n}. \tag{3.35}$$

By similar arguments used to estimate $P\{S > n\}$, we obtain

$$(m \times m)\left\{S > \frac{n}{i+1}\right\} \leq \widetilde{C}_2 e^{-\frac{c_0 n}{i+1}}, \quad \text{for some } \widetilde{C}_2 > 0. \tag{3.36}$$

Let us now discuss the situation for the different classes of observables.

Class $(V1, \tau)$: Assume that $v_i(\Phi) \leq D\tau^i$ for some $0 < \tau < 1$ and $D > 0$. By Lemma 3.19,

$$\prod_{j=1}^{i}\left(1 - \frac{\epsilon_j}{D_6}\right) \leq \widetilde{D}\theta_1^i, \tag{3.37}$$

for some $0 < \theta_1 < 1$ and $\widetilde{D} > 0$. Using (3.35)–(3.37) in Proposition 3.22, we get

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \leq 2\widetilde{C} e^{-c_0 n} + \widetilde{C}_2 \widetilde{D} D_7 \sum_{i=1}^{n} \theta_1^i (i+1) e^{-\frac{c_0 n}{i+1}} \leq D' e^{-c' n},$$

for some $D', c' > 0$. Following the same approach, we obtain the following estimates for the remaining classes of observables.

Class $(V2, \tau)$: Assume that $v_i(\Phi) \leq De^{-i^\tau}$ for some $\tau \in (0, 1)$ and $D > 0$. We have

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \leq 2\widetilde{C} e^{-c_0 n} + \widetilde{C}_2 \widetilde{D} D_7 \sum_{i=1}^{n} e^{-\xi i^\tau} (i+1) e^{-\frac{c_0 n}{i+1}} \leq D' e^{-n^{\tau'}},$$

for every $0 < \tau' < \tau$ and some $D' > 0$.

Class $(V3, \tau)$: Assume that $v_i(\Phi) \leq De^{-(\log i)^\tau}$ for some $\tau > 1$ and $D > 0$. We have

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \leq 2\widetilde{C} e^{-c_0 n} + \widetilde{C}_2 \widetilde{D} D_7 \sum_{i=1}^{n} e^{-\xi (\log i)^\tau} (i+1) e^{-\frac{c_0 n}{i+1}} \leq D' e^{-(\log n)^{\tau'}},$$

for every $0 < \tau' < \tau$ and some $D' > 0$.

Class $(V4, \tau)$: Assume that $v_i(\Phi) \leq Di^{-\tau}$ for some $\tau > \frac{1}{\xi}$ and $D > 0$. We have

$$|T_*^n \lambda_1 - T_*^n \lambda_2| \leq 2\widetilde{C} e^{-c_0 n} + \widetilde{C}_2 \widetilde{D} D_7 \sum_{i=1}^{n} i^{-\xi\tau} (i+1) e^{-\frac{c_0 n}{i+1}} \leq D' n^{1-\tau\xi},$$

for some $D' > 0$.

3.4. Finishing the proof of Proposition 3.4

We notice that from the proof of Lemma 3.6, we have

$$v_n(\varphi^*\rho) \le 3C_\rho\beta^n + C_0 v_n(\varphi),$$

and this implies that

$$v_n(\Phi) \le 3C_\rho\beta^n + C_0 v_n(\varphi).$$

Taking $\lambda_1 = \lambda$, where λ is the probability measure on Δ such that $\frac{d\lambda}{dm} = \varphi^*\rho$, and $\lambda_2 = \nu$ in Sections 3.3.1 and 3.3.2, we get the required estimates for $|T_*^n\lambda - \nu|$. Let us check the dependence of the constants. From (3.4), $C_\Phi = 3K_1' + 2C$. This means that the constant C_Φ does not depend on φ. Therefore the constant $\zeta = \frac{\bar{\delta}}{D_4}$ will only depend on $v_0(\Phi)$, which can be bounded with $v_0(\varphi)$, as $v_0(\Phi) \le 3C_\rho + C_0 v_0(\varphi)$.

3.5. CLT

We conclude with the proof of Corollary B. From [39, Proposition 6.2], we obtain the CLT for the tower map. In particular, if we assume that $m\{R > n\} \le Cn^{-a}$ for some $a > 2$ and $C > 0$, the CLT holds for all $\varphi \in (V4, \tau)$, with τ sufficiently large, if and only if φ is not a coboundary. Given that $\varphi \in (R4, \tau)$ for some $\tau > 1$, Lemma 3.2 ensures that $\varphi \circ \pi \in (V4, \tau)$. Moreover, φ is a coboundary if and only if $\varphi \circ \pi$ is a coboundary. The proof of Corollary B then follows from these observations and from the fact that $\mu = \pi_*\nu$, which implies that for every interval $J \subset \mathbb{R}$, we have

$$\mu\Big\{y \in M : \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\Big(\varphi(f^i(y)) - \int \varphi\, d\pi_*\nu\Big) \in J\Big\}$$
$$= \nu\Big\{x \in \Delta : \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\Big(\varphi\circ\pi(T^i(x)) - \int \varphi\circ\pi\, d\nu\Big) \in J\Big\}.$$

Funding. A. Ullah and H. Vilarinho were partially supported by Fundação para a Ciência e a Tecnologia (FCT) through Centro de Matemática e Aplicações (CMA-UBI), Universidade da Beira Interior, under the project UIDB/00212/2020. A. Ullah was also supported by FCT under the grant UI/BD/150796/2020.

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Received 2 June 2025; revised 7 November 2025.

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