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On an asymptotic property of Toeplitz operators

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Abstract. We establish an asymptotic property of Toeplitz operators on the Dirichlet space and on a weighted Bergman space. Additionally, we determine a sufficient condition for a self-commutator inequality to hold in a specific case.

1. Introduction

The study of commutators $T^*T - TT^*$ on Hilbert spaces plays a significant role in operator theory. The spectral theorem reduces the analysis of normal operators to multiplication operators and measure theory. Toeplitz operators, which are compressions of multiplication operators, offer new insights into the properties of commutators. These operators are typically studied on Hilbert spaces of analytic functions, where many properties of the operators are closely tied to the properties of their symbols.

Numerous studies have been conducted in this area, particularly focusing on normality and hyponormality, where the commutator is positive. Such studies often demonstrate how certain properties of the symbol imply corresponding properties of the commutator and vice versa. A substantial amount of results has been published on this topic; here, we cite only a few key contributions. The first study on the Bergman space appears in [11]. Notable results are due to P. Ahern and Z. Cuckovic [1], as well as S. Axler and Z. Cuckovic [2]. Additional contributions can be found in [9,10], with a recent result by R. Curto and Z. Cuckovic [5]. Some work on nonharmonic symbols is presented in [7]. Earlier studies on the Hardy space include the foundational work of C. C. Cowen [3,4].

In this work, we investigate the matrices of $T_\varphi^* T_\varphi - T_{\overline{\varphi}}^* T_{\overline{\varphi}}$ on the Dirichlet space and on a weighted Bergman space. In both cases, we demonstrate that the matrices exhibit an asymptotic property. Since on the Bergman space we have $T_\varphi^* T_\varphi - T_{\overline{\varphi}}^* T_{\overline{\varphi}} = T_{\overline{\varphi}} T_\varphi - T_\varphi T_{\overline{\varphi}}$, we will call these operators commutators for simplicity sake.

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The paper is organized as follows:

- In the first section, we introduce basic notations and definitions for the Dirichlet space before proving the asymptotic property.
- In the second section, we provide foundational definitions and notations for the weighted Bergman space, then establish an asymptotic property of the commutator. This property is subsequently used to deduce a necessary condition for hyponormality.
- In the third section, we identify a sufficient condition, in a specific case, for a commutator inequality to hold on the weighted Bergman space.

2. An asymptotic property of commutators on the Dirichlet space

Let $U = \{z \in \mathbb{C} \mid |z| < 1\}$ denote the unit disk, and let $W^{2,1}(U)$ be the Sobolev space, defined as the completion of the space of smooth functions satisfying

$$\left|\int_U f(z)dA(z)\right|^2 + \int_U \left(\left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial f}{\partial \overline{z}}\right|^2\right) dA(z) < \infty$$

where $dA(z)$ is the Lebesgue area on U. Here, the partial derivatives are given by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

The subspace of $W^{2,1}(U)$ consisting of analytic functions on U is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_U f(z)dA(z)\int_U \overline{g(z)}dA(z) + \int_U \left(\frac{\partial f}{\partial z}\overline{\frac{\partial g}{\partial z}} + \frac{\partial f}{\partial \overline{z}}\overline{\frac{\partial g}{\partial \overline{z}}}\right)dA(z).$$

This space is called the Dirichlet space and is denoted by D^2 [12]. It can also be defined as the Hilbert space of analytic functions satisfying

$$\|f\|^2_{H^2} + \|f'\|^2_{L^2a} < \infty,$$

where H^2 and L^2_a are the Hardy space and Bergman space, respectively.

For the defined inner product, the orthonormal basis is given by

$$\{e_n\}_{n\geq 0} = \left\{1, \frac{z^n}{\sqrt{n}}, n \geq 1\right\}.$$

For a bounded and measurable function φ on U, we define the Toeplitz operator T_φ on D^2 as

$$T_\varphi(f) = P(\varphi f),$$

where P is the orthogonal projection of $W^{2,1}(U)$ onto D^2. The reproducing kernel of the Dirichlet space D^2 is given by

$$K_z(w) = 1 + \sum_{n=1}^{\infty} \frac{(w\overline{z})^n}{n} = 1 + \log\left(\frac{1}{1-\overline{z}w}\right)$$

and we have

$$P(f)(z) = \int_U f dA(w) + \int_U \frac{z}{1-z\overline{w}} \frac{\partial f}{\partial w}(w) dA(w).$$

We now establish an asymptotic property of the matrix of $T_\varphi^* T_\varphi - T_{\overline{\varphi}}^* T_{\overline{\varphi}}$ for an analytic function φ with a bounded derivative on U.

Lemma 2.1. *For positive integers m and n, the projection satisfies*

$$P(z^n\overline{z}^m) = \begin{cases} z^{n-m}, & \text{if } n \geq m+1, \\ \frac{1}{n+1}, & \text{if } n = m, \\ 0, & \text{if } n \leq m-1. \end{cases}$$

Proof. Note that

$$\int_U w^n \overline{w}^m dA(w) = 0 \quad \text{if } n \neq m.$$

We compute

$$\begin{aligned} P(z^n\overline{z}^n) &= \int_U w^n \overline{w}^n dA(w) + n \int_U \frac{z}{1-z\overline{w}} w^{n-1}\overline{w}^n dA(w) \\ &= \frac{1}{n+1} + nz \sum_{k\geq 0} z^k \int_U w^{n-1}\overline{w}^{n+k} dA(w) \\ &= \frac{1}{n+1}. \end{aligned}$$

For $n \leq m-1$, we have

$$\int_U \frac{z}{1-z\overline{w}} w^{n-1}\overline{w}^m dA(w) = nz \sum_{k\geq 0} z^k \int_U w^{n-1}\overline{w}^{m+k} dA(w) = 0.$$

For $n \geq m+1$, we obtain

$$\int_U \frac{z}{1-z\overline{w}} w^{n-1}\overline{w}^m dA(w) = nz \sum_{k\geq 0} z^k \int_U w^{n-1}\overline{w}^{m+k} dA(w) = z^{n-m}. \qquad \blacksquare$$

We now deduce the matrix of $T_\varphi^* T_\varphi - T_{\overline{\varphi}}^* T_{\overline{\varphi}}$ for $\varphi(z) = \sum_{n\geq 1} \varphi_n z^n$, an analytic function on U with a bounded derivative.

Proposition 2.2. *The matrix elements $\Gamma_{i,j}$ of $T_\varphi^* T_\varphi - T_{\overline{\varphi}}^* T_{\overline{\varphi}}$ in the orthonormal basis $(e_n)_{n\geq 0}$ are given by*

$$\begin{aligned}\Gamma_{i,j} = &\sum_{m\geq \max(1,j-i+1)} \frac{(i+m+1)\varphi_{m+i-j}\overline{\varphi_m}}{\sqrt{i}\sqrt{j}}\\ &- \frac{\overline{\varphi_j}\varphi_i}{\sqrt{i}\sqrt{j}(i+1)(j+1)}\\ &- \sum_{\max(1,i-j+1)\leq m\leq i} \frac{(i-m+1)\overline{\varphi_{m+j-i}}\varphi_m}{\sqrt{i}\sqrt{j}}.\end{aligned}$$

Proof. We compute

$$\left\langle T_\varphi \frac{z^j}{\sqrt{j}}, T_\varphi \frac{z^i}{\sqrt{i}} \right\rangle = \sum_{m,n\geq 1} \frac{1}{\sqrt{i}\sqrt{j}} \overline{\varphi_m}\varphi_n \langle z^{n+j}, z^{m+i}\rangle.$$

Rewriting this, we obtain

$$\sum_{m\geq \max(1,j-i+1)} \frac{\varphi_{m+i-j}\overline{\varphi_m}(i+m+1)}{\sqrt{i}\sqrt{j}}.$$

Similarly, we compute

$$\left\langle T_{\overline{\varphi}} \frac{z^j}{\sqrt{j}}, T_{\overline{\varphi}} \frac{z^i}{\sqrt{i}} \right\rangle = \sum_{m\leq j, n\leq i} \frac{1}{\sqrt{i}\sqrt{j}} \overline{\varphi_n}\varphi_m \langle P(\overline{z}^n z^j), P(\overline{z}^m z^i)\rangle.$$

This simplifies to

$$\frac{\overline{\varphi_j}\varphi_i}{\sqrt{i}\sqrt{j}(i+1)(j+1)} + \sum_{\max(1,i-j+1)\leq m\leq i} \overline{\varphi_{m+j-i}}\varphi_m \frac{i-m+1}{\sqrt{i}\sqrt{j}}.$$

Thus, the result follows.
We rewrite $\Gamma_{i,j}$ in the form

$$\begin{aligned}\Gamma_{i,i+p} = &\sum_{m\geq 1} \frac{(i+m+p+1)\varphi_m\overline{\varphi_{m+p}}}{\sqrt{i}\sqrt{i+p}}\\ &- \frac{\overline{\varphi_{i+p}}\varphi_i}{\sqrt{i}\sqrt{i+p}(i+1)(i+p)}\\ &- \sum_{1\leq m\leq i} \frac{(i-m+1)\overline{\varphi_{m+p}}\varphi_m}{\sqrt{i}\sqrt{i+p}}.\end{aligned}$$

■

On the Dirichlet space, the well-known property of Toeplitz operators

$$(T_f)^* = T_{\overline{f}}$$

which holds on the Bergman space and the Hardy space, does not necessarily hold. This equality, in the case of harmonic symbols, holds on the Dirichlet space only if f is constant [6]. Additionally, on D^2, the equality

$$(T_f)^* = T_f$$

holds only when f is a real constant. Various other significant differences are discussed in [6]. On the Bergman space, we have for any bounded analytic function φ on U

$$T_\varphi^* T_\varphi - T_{\overline{\varphi}}^* T_{\overline{\varphi}} = T_{\overline{\varphi}} T_\varphi - T_\varphi T_{\overline{\varphi}}.$$

The following property of self-commutators on the Bergman space is well known [11].

Theorem 2.3. *Let f be a bounded and analytic function on the unit disk U and assume that f'^2. If $(m_{i,j})$ is the matrix of the commutator*

$$T_{\overline{f}} T_f - T_f T_{\overline{f}},$$

then we have

$$\lim_{i\to\infty} i^2 m_{i,i+p} = \theta_{s,s+p},$$

where $(\theta_{i,j})$ is the matrix of the Hardy space Toeplitz operator $T_{|f'^2}$.

Despite the absence of important product and adjoint properties of Toeplitz operators on the Dirichlet space, an asymptotic property of the matrix of the operator

$$T_\varphi^* T_\varphi - T_{\overline{\varphi}}^* T_{\overline{\varphi}},$$

similar to the one stated above for the Bergman space, still holds on the Dirichlet space.

Denote by $(\Delta_{i,j})_{i,j}$ the matrix of the Hardy space Toeplitz operator $T_{2\,\mathrm{Re}(\varphi'\overline{\varphi})}$. We obtain an asymptotic expression for $(\Gamma_{i,j})$.

Theorem 2.4. *The following holds:*

$$\lim_{i\to\infty} i\Gamma_{i,i+p} = \Delta_{i,i+p}.$$

Proof. We start with

$$\begin{aligned} i\Gamma_{i,i+p} = \sum_{1\le m\le i} & \frac{i}{\sqrt{i}\sqrt{i+p}}(2m+p)\overline{\varphi_{m+p}}\varphi_m \\ & - \frac{i\overline{\varphi_{i+p}}\varphi_i}{\sqrt{i}\sqrt{i+p}(i+1)(i+p+1)} \\ & + \sum_{m\ge i+1} \frac{\varphi_m\overline{\varphi_{m+p}}i(i+m+p+1)}{\sqrt{i}\sqrt{i+p}}. \end{aligned}$$

Define

$$F_i(m) = \chi_{\{1,\ldots,i\}}(m)\frac{i}{\sqrt{i}\sqrt{i+p}}(2m+p)\overline{\varphi_{m+p}}\varphi_m.$$

Then, we have the inequality

$$\begin{aligned}|F_i(m)| &\leq m|\varphi_m||\varphi_{m+p}| + (m+p)|\varphi_m||\varphi_{m+p}|\\ &\leq \frac{1}{2}((m^2+1)|\varphi_m|^2 + ((m+p)^2+1)|\varphi_{m+p}|^2).\end{aligned}$$

Since φ and φ' are bounded on U, the right-hand side of the above inequality is an integrable function of m with respect to the counting measure $d\nu$. By the dominated convergence theorem, we obtain

$$\begin{aligned}&\lim_{i\to\infty}\sum_{1\leq m\leq i}\frac{i}{\sqrt{i}\sqrt{i+p}}(2m+p)\overline{\varphi_{m+p}}\varphi_m\\ &\quad = \lim_{i\to\infty}\int_{\mathbb{N}} F_i(m)d\nu(m) = \sum_{m\geq 1}(2m+p)\overline{\varphi_{m+p}}\varphi_m.\end{aligned}$$

Additionally, we have

$$\left|\sum_{m\geq i+1}\frac{\varphi_m\overline{\varphi_{m+p}}i(i+m+p+1)}{\sqrt{i}\sqrt{i+p}}\right| \leq \sum_{m\geq i+1}(2m+p)|\overline{\varphi_{m+p}}\varphi_m|.$$

Applying the previous bounds and the fact that φ and φ' are bounded, we conclude

$$\lim_{i\to\infty}\sum_{m\geq i+1}\frac{\varphi_m\overline{\varphi_{m+p}}i(i+m+p+1)}{\sqrt{i}\sqrt{i+p}} = 0.$$

Since

$$\sum_{m\geq 1}(2m+p)\overline{\varphi_{m+p}}\varphi_m = \Delta_{i,i+p},$$

it follows that

$$\lim_{i\to\infty} i\,\Gamma_{i,i+p} = \Delta_{i,i+p}.$$ ■

3. The asymptotic property on a weighted Bergman space

We consider the Hilbert space of measurable functions on the unit disk U such that

$$\int_U |h(z)|^2 d\mu(z) < \infty,$$

equipped with the inner product

$$\langle f, g\rangle = \int_U f(z)\overline{g(z)}d\mu(z),$$

where the measure μ is defined by

$$d\mu(z) = \frac{4}{3\pi}|\log|z||^3 dA(z),$$

and $dA(z)$ denotes the Lebesgue area measure on the unit disk.

If h is analytic on U, then it admits a power series expansion

$$h(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and the norm is given by

$$\|h\|^2 = \int_U |h(z)|^2 d\mu(z) = \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^4}.$$

We denote by $L^2_{a,\mu}$ the closed subspace of $L^2(U, d\mu)$ consisting of analytic functions. An orthonormal basis for $L^2_{a,\mu}$ is given by

$$\{(n+1)^2 z^n, n \geq 0\}.$$

For a bounded measurable function f on U, the Toeplitz operator T_f is defined by

$$T_f(h) = P(fh),$$

where $h \in L^2_{a,\mu}$ and P denotes the orthogonal projection from $L^2(U, d\mu)$ onto $L^2_{a,\mu}$.

Similarly, the Hankel operator H_f is defined by

$$H_f(h) = (I - P)(fh),$$

for $h \in L^2_{a,\mu}$. In this section, we establish an asymptotic property of the matrix of the commutator

$$T_{\overline{f}} T_f - T_f T_{\overline{f}}$$

on the space $L^2_{a,\mu}$. We start by recalling some known properties of Toeplitz operators, which are analogous to their counterparts on the unweighted Bergman space [11].

Proposition 3.1. *Let f, g be in $L^\infty(U, d\mu)$. Then the following hold:*

(1) $T_{f+g} = T_f + T_g$.

(2) $T_f^* = T_{\overline{f}}$.

(3) $T_{\overline{f}} T_g = T_{\overline{f}g}$ *if f or g is analytic on U.*

(4) *For bounded and analytic f,*

$$T_{\overline{f}} T_f - T_f T_{\overline{f}} = H_{\overline{f}}^* H_{\overline{f}}.$$

(5) *For bounded and analytic f and g,*

$$T_{\overline{f}+g}T_{f+\overline{g}} - T_{f+\overline{g}}T_{\overline{f}+g} \geq 0$$

if and only if

$$T_{\overline{g}}T_g - T_gT_{\overline{g}} \leq T_{\overline{f}}T_f - T_fT_{\overline{f}}.$$

Inequality (5) defines the hyponormality of $T_{f+\overline{g}}$. The following lemma provides the key property of the projection, which is used for matrix computations of the commutator $T_{\overline{f}}T_f - T_fT_{\overline{f}}$.

Lemma 3.2. *Let s, t be positive integers. Then*

$$P(\overline{z}^t z^s) = \begin{cases} \frac{(s-t+1)^4}{(s+1)^4} z^{s-t}, & \text{if } s \geq t, \\ 0, & \text{if } s < t. \end{cases}$$

Proof. Assume first that $s \geq t$. For any integer $k \geq 0$, we have

$$\langle P(\overline{z}^t z^s), z^k \rangle = \langle \overline{z}^t z^s, z^k \rangle = \langle z^s, z^{t+k} \rangle.$$

Since $\{z^n\}_{n\geq 0}$ is an orthogonal family, the right-hand side vanishes unless $k = s - t$. Hence

$$P(\overline{z}^t z^s) = \lambda z^{s-t}$$

for some constant λ.

Taking the inner product with z^{s-t} yields

$$\lambda \|z^{s-t}\|^2 = \langle \overline{z}^t z^s, z^{s-t} \rangle = \langle z^s, z^s \rangle = \|z^s\|^2.$$

Therefore,

$$\lambda = \frac{\|z^s\|^2}{\|z^{s-t}\|^2} = \frac{(s-t+1)^4}{(s+1)^4}.$$

If $s < t$, then $\langle \overline{z}^t z^s, z^k \rangle = 0$ for all $k \geq 0$, and hence $P(\overline{z}^t z^s) = 0$. ∎

This leads to the following matrix computation.

Lemma 3.3. *Let $f(z) = \sum_{m\geq 1} a_m z^m$ be bounded and analytic on U. The matrix of $H^*_{\overline{f}}H_{\overline{f}}$ in the orthonormal basis $\{(n+1)^2 z^n, n \geq 0\}$ is given by*

$$\begin{aligned} \alpha_{i,j} &= \sum_{m\geq \max(1, i-j+1)} a_m \overline{a_{m+j-i}} \frac{(i+1)^2(j+1)^2}{(j+m+1)^4} \\ &\quad - \sum_{m\geq \max(1, i-j+1)} a_m \overline{a_{m+j-i}} \frac{(j-m+1)^4}{(i+1)^4(j+1)^4}. \end{aligned}$$

Proof. Since

$$f(j+1)^2 z^j = \sum_{m\geq 1} (j+1)^2 a_m z^{m+j},$$

we obtain

$$\begin{aligned}&\langle f(j+1)^2 z^j, f(i+1)^2 z^i\rangle\\ &\quad = (i+1)^2(j+1)^2 \sum_{m\geq \max(1,i-j+1)} a_m \overline{a_{m+j-i}} \frac{1}{(m+j+1)^4}.\end{aligned}$$

Similarly, we show

$$\begin{aligned}&\langle T_f T_{\overline{f}} (j+1)^2 z^j, (i+1)^2 z^i\rangle\\ &\quad = (i+1)^2(j+1)^2 \sum_{m\geq n} a_n \overline{a_m} \langle P(\overline{z}^m z^j), P(\overline{z}^n z^i)\rangle.\end{aligned}$$

Thus, we conclude

$$\begin{aligned}\alpha_{ij} &= \sum_{m\geq n} a_m \overline{a_{m+j-i}} \frac{(i+1)^2(j+1)^2}{(j+m+1)^2}\\ &\quad - \sum_{m\geq \max(1,i-j+1)} a_{m+i-j} \overline{a_m} \frac{(j-m+1)^4}{(j+1)^2(i+1)^2}.\end{aligned}$$ ∎

Let us rewrite $\alpha_{i,j}$. We have for p non-negative integer

$$\begin{aligned}\alpha_{i,i+p} &= \sum_{1\leq m\leq i} a_m \overline{a_{m+p}} \left(\frac{(i+1)^2(i+p+1)^2}{(m+i+p+1)^4} - \frac{(i-m+1)^4}{(i+1)^2(i+p+1)^2} \right)\\ &\quad + \sum_{m\geq i+1} a_m \overline{a_{m+p}} \frac{(i+1)^2(i+p+1)^2}{(m+i+p+1)^4}.\end{aligned}$$

We deduce the following asymptotic property of $\alpha_{i,i+p}$.

Lemma 3.4. *Let f be a bounded and analytic function on U, such that f'^2. Let $(\omega_{i,j})$ be the matrix of the Toeplitz operator $T_{4|f'^2}$ on the Hardy space. Then,*

$$\lim_{i\to\infty} i^2 \alpha_{i,i+p} = \omega_{s,s+p}.$$

Proof. An elementary computation shows that the quotient

$$R^i_{m,p} = \frac{(i+1)^4(i+p+1)^4 - (i-m+1)^4(m+i+p+1)^4}{(m+i+p+1)^4(i+1)^2(i+p+1)^2}$$

can be reduced to

$$\frac{4m(m+p)i^6 + A^5_{m,p} i^5 + A^4_{m,p} i^4 + A^3_{m,p} i^3 + A^2_{m,p} i^2 + A^1_{m,p} i + A^0_{m,p}}{(m+i+p+1)^4(i+1)^2(i+p+1)^2}$$

for some coefficients $A^k_{m,p}$, $0 \leq k \leq 5$.

Thus,

$$\lim_{i\to\infty} i^2 R^i_{m,p} = 4m(m+p).$$

Now, let us write the sum

$$\sum_{1\le m\le i} a_m\overline{a_{m+p}}R^i_{m,p}$$

as an integral with respect to the counting measure $d\nu$, so we obtain

$$\sum_{1\le m\le i} a_m\overline{a_{m+p}}R^i_{m,p} = \int G_i(m)d\nu(m),$$

where

$$G_i(m) = \chi_{\{1,\ldots,i\}}(m)a_m\overline{a_{m+p}}Q^i_{m,p}.$$

We have

$$\lim_{i\to\infty} i^2 G_i(m) = 4m(m+p)a_m\overline{a_{m+p}}.$$

It is not difficult to see that the coefficients $A^k_{m,p}$, where $0\le k\le 5$, satisfy the inequality

$$|i^2 A^i_{m,p}| \le Cm(m+p)$$

for some constant C. Consequently,

$$|i^2 G_i(m)| \le \frac{C}{2}(m^2|a_m|^2 + (m+p)^2|a_{m+p}|^2).$$

Since f'^2, by the dominated convergence theorem, we get

$$\lim_{i\to\infty}\int i^2 G_i(m)d\nu(m) = 4\sum_{m\ge 1} m(m+p)a_m\overline{a_{m+p}}.$$

Since

$$|a_m\overline{a_{m+p}}|\frac{(i+1)^2(i+p+1)^2}{(m+i+p+1)^4} \le \frac{1}{2}(|a_m|^2 + |a_{m+p}|^2),$$

the sum

$$\sum_{m\ge i+1} a_m\overline{a_{m+p}}\frac{(i+1)^2(i+p+1)^2}{(m+i+p+1)^4}$$

converges to zero as $i\to\infty$.

Thus, we obtain

$$\lim_{i\to\infty} i^2\alpha_{i,i+p} = \omega_{s,s+p}. \qquad \blacksquare$$

This leads to a necessary condition for the inequality

$$T_{\overline{g}}T_g - T_g T_{\overline{g}} \leq T_{\overline{f}}T_f - T_f T_{\overline{f}}$$

to hold.

Theorem 3.5. *Let f and g be bounded analytic functions on U, such that f'^2. If*

$$T_{\overline{g}}T_g - T_g T_{\overline{g}} \leq T_{\overline{f}}T_f - T_f T_{\overline{f}}$$

on $L^2_{a,\mu}$, then g'^2 and $|g'| \leq |f'|$ a.e. on ∂U.

Proof. Let $(\beta_{i,j})$ denote the matrix of $H^*_{\overline{g}}H_{\overline{g}}$ in the orthonormal basis $\{(n+1)^2 z^n, n \geq 0\}$. Using the notation of the previous lemma and setting $g(z) = \sum_{n=1}^{\infty} b_n z^n$, we have

$$i^2\alpha_{i,i} = \sum_{m\leq i} |a_m|^2 i^2 R^i_{m,0} + \sum_{m>i} |a_m|^2 i^2 R^i_{m,0},$$

where

$$R^i_{m,p} = \frac{(i+1)^4(i+p+1)^4 - (i-m+1)^4(m+i+p+1)^4}{(m+i+p+1)^4(i+1)^2(i+p+1)^2}.$$

Similarly,

$$i^2\beta_{i,i} = \sum_{m\leq i} |b_m|^2 i^2 R^i_{m,0} + \sum_{m>i} |b_m|^2 i^2 R^i_{m,0}.$$

If

$$T_{\overline{g}}T_g - T_g T_{\overline{g}} \leq T_{\overline{f}}T_f - T_f T_{\overline{f}},$$

then we obtain the inequality

$$\sum_{m\leq i} |b_m|^2 i^2 R^i_{m,0} \leq i^2\alpha_{i,i}.$$

Defining

$$H_i(m) = \chi_{\{0,\ldots,i\}}(m)|b_m|^2 i^2 R^i_{m,0},$$

we observe that

$$H_i(m) \to 4m^2|b_m|^2 \quad \text{as } i \to \infty.$$

Using Fatou's lemma and the above inequality, we get

$$4\sum_{m\geq 1} m^2|b_m|^2 \leq 4\sum_{m\geq 1} m^2|a_m|^2.$$

Thus, g'^2, and from the previous lemma, we obtain

$$\lim_{i\to\infty} i^2(\alpha_{i,i+p} - \beta_{i,i+p}) = \tau_{s,s+p},$$

where the right-hand side is the matrix of the Hardy space Toeplitz operator $T_{4(|f'^2 - |g'^2)}$. A known property of Toeplitz forms [8] then implies that

$$|g'| \leq |f'| \quad \text{a.e. on the unit circle.} \qquad \blacksquare$$

4. The sufficient condition

In this section, we consider the case $f = z^m$, where m is a positive integer, and we find a sufficient condition for the commutator inequality

$$T_{\overline{g}} T_g - T_g T_{\overline{g}} \leq T_{\overline{f}} T_f - T_f T_{\overline{f}},$$

to hold, where $g(z) = \sum_{n=1}^{\infty} b_n z^n$ is bounded on the unit disk. The following lemma is needed.

Lemma 4.1. *The matrix of $H^*_{\overline{z^m}} H_{\overline{z^m}}$ in the orthonormal basis $\{(i+1)^2 z^i, i \geq 0\}$ is diagonal and is given by*

$$d_i = \begin{cases} \frac{(i+1)^4}{(i+1+m)^4}, & \text{if } i < m, \\ \frac{(i+1)^4}{(i+1+m)^4} - \frac{(i+1-m)^4}{(i+1)^4}, & \text{if } i \geq m. \end{cases}$$

Our third main result is based on the following two propositions.

Proposition 4.2. *Let $n > m \geq 1$. Then $|\lambda|^2 H^*_{\overline{z^n}} H_{\overline{z^n}} \leq H^*_{\overline{z^m}} H_{\overline{z^m}}$ if and only if*

$$|\lambda| \leq \frac{m}{n}.$$

Proof. The previous lemma shows this is equivalent to the following three inequalities:

$$|\lambda|^2 \frac{(i+1)^4}{(i+1+n)^4} \leq \frac{(i+1)^4}{(i+1+m)^4}, \quad \text{for } i < m. \tag{1}$$

$$|\lambda|^2 \frac{(i+1)^4}{(i+1+n)^4} \leq \frac{(i+1)^4}{(i+1+m)^4} - \frac{(i+1-m)^4}{(i+1)^4}, \quad \text{for } m \leq i < n. \tag{2}$$

$$|\lambda|^2 \left(\frac{(i+1)^4}{(i+1+n)^4} - \frac{(i+1-n)^4}{(i+1)^4} \right) \leq \frac{(i+1)^4}{(i+1+m)^4} - \frac{(i+1-m)^4}{(i+1)^4}, \quad \text{for } i \geq n. \tag{3}$$

Clearly, inequality (1) is equivalent to

$$|\lambda|^2 \leq \min \left\{ \frac{(i+1+n)^4}{(i+1+m)^4}, i < m \right\}.$$

Since $m < n$, the function $S_{m,n,i} = \frac{(i+1+n)^4}{(i+1+m)^4}$ is decreasing in i, and the minimum is attained at $i = m - 1$. Thus, inequality (1) is equivalent to

$$|\lambda| \leq \frac{(m+n)^2}{(2m)^2} = C_{m,n}.$$

A computation shows that inequality (3) is equivalent to

$$|\lambda|^2 \leq \frac{Q_{m,i}}{Q_{n,i}}, \quad \text{for } i \geq n,$$

where

$$Q_{k,i} = \frac{(i+1)^8 - (i+1+k)^4(i+1-k)^4}{(i+1)^4(i+1+k)^4}.$$

Setting $x = i + 1$, we obtain, after simplifying, the function

$$F(x) = \frac{m^2(2x^2 - m^2)(x^4 + (x^2 - m^2)^2)}{n^2(2x^2 - n^2)(x^4 + (x^2 - n^2)^2)}.$$

Since $n \geq m$,

$$F(x) \geq \frac{m^2}{n^2}.$$

Also,

$$\lim_{x\to\infty} \frac{m^2(2x^2 - m^2)(x^4 + (x^2 - m^2)^2)}{n^2(2x^2 - n^2)(x^4 + (x^2 - n^2)^2)} = \frac{m^2}{n^2}.$$

Thus, since $S_{m,n,i}$ is decreasing with i and $\lim_{i\to\infty} S_{m,n,i} = 1$, inequality (3) is equivalent to

$$|\lambda| \leq \frac{m}{n}.$$

∎

The following definition was first introduced by C. Cowen [3] in the Hardy space setting. We denote by V the unit ball of $L^{2\perp}_{a,\mu}$.

Definition 4.3. Let $f \in L^2_{a,\mu}$. Define

$$\bigwedge_f = \{g \in L^2_{a,\mu} \mid \sup_{l\in V} |\langle \overline{g}, \overline{h}l\rangle| \leq \sup_{l\in V} |\langle \overline{f}, \overline{h}l\rangle|\},$$

for any bounded analytic function h on U.

As shown in [3], the set $\bigwedge_f$ is convex, balanced, and weakly closed in $L^2_{a,\mu}$. Finally, it is easy to see that the Toeplitz operator

$$T_{z^m+\lambda\overline{z}^m}$$

is hyponormal if and only if $|\lambda| \leq 1$.

Theorem 4.4. *Let $(\omega_n)_{n\geq m}$ be a sequence of complex numbers satisfying*

$$\sum_{n\geq m} |\omega_n| \leq 1,$$

and define

$$g_1(z) = \sum_{n\geq m} \omega_n \frac{m}{n} \overline{z}^n.$$

Then,

$$T_{\overline{g}_1} T_{g_1} - T_{g_1} T_{\overline{g}_1} \leq T_{\overline{z}^m} T_{z^m} - T_{z^m} T_{\overline{z}^m}$$

on $L^2_{a,\mu}$.

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References

[1] P. Ahern and Ž. Čučković, A mean value inequality with applications to Bergman space operators. *Pacific J. Math.* **173** (1996), no. 2, 295–305 Zbl 0962.47015 MR 1394391

[2] S. Axler and Ž. Čučković, Commuting Toeplitz operators with harmonic symbols. *Integral Equations Operator Theory* **14** (1991), no. 1, 1–12 Zbl 0733.47027 MR 1079815

[3] C. C. Cowen, Hyponormal and subnormal Toeplitz operators. In *Surveys of some recent results in operator theory, Vol. I*, pp. 155–167, Pitman Res. Notes Math. Ser. 171, Longman Sci. Tech., Harlow, 1988 Zbl 0677.47017 MR 0958573

[4] C. C. Cowen, Hyponormality of Toeplitz operators. *Proc. Amer. Math. Soc.* **103** (1988), no. 3, 809–812 Zbl 0668.47021 MR 0947663

[5] Ž. Čučković and R. E. Curto, A new necessary condition for the hyponormality of Toeplitz operators on the Bergman space. *J. Operator Theory* **79** (2018), no. 2, 287–300 Zbl 1424.47066 MR 3803558

[6] J. J. Duistermaat and Y. J. Lee, Toeplitz operators on the Dirichlet space. *J. Math. Anal. Appl.* **300** (2004), no. 1, 54–67 Zbl 1083.47024 MR 2100238

[7] M. Fleeman and C. Liaw, Hyponormal Toeplitz operators with non-harmonic symbol acting on the Bergman space. *Oper. Matrices* **13** (2019), no. 1, 61–83 Zbl 1448.47039 MR 3924372

[8] U. Grenander and G. Szegö, *Toeplitz forms and their applications*. California Monographs in Mathematical Sciences, University of California Press, Berkeley-Los Angeles, 1958 Zbl 0080.09501 MR 0094840

[9] I. S. Hwang, J. Lee, and S. W. Park, Hyponormal Toeplitz operators with polynomial symbols on weighted Bergman spaces. *J. Inequal. Appl.* (2014), article no. 335 Zbl 1447.47032 MR 3374726

[10] A. Phukon, Hyponormality of Toeplitz operators with polynomial symbols on the weighted Bergman space. *Arab. J. Math. (Springer)* **6** (2017), no. 2, 87–94 Zbl 06767420 MR 3661168

[11] H. Sadraoui, *Hyponormality of Toeplitz operators and composition operators*. Ph.D. thesis, Purdue University, 1992 MR 2687747

[12] Z. Wu, Function theory and operator theory on the Dirichlet space. In *Holomorphic spaces (Berkeley, CA, 1995)*, pp. 179–199, Math. Sci. Res. Inst. Publ. 33, Cambridge Univ. Press, Cambridge, 1998 Zbl 1128.47310 MR 1630650

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