Port. Math. 83 (2026), 367–383
DOI 10.4171/PM/2161

Li–Yorke and distributional chaos for weighted shifts on directed trees

Chol San Kim, Hyon Hui Ju, Jun Hyok Choe, and Jin Myong Kim

Abstract. We study Li–Yorke and distributional chaos for weighted shifts on directed trees. We give a complete characterization of Li–Yorke chaotic weighted shifts on rooted directed trees. Also we show some sufficient conditions for distributional chaos of these operators. Our results generalize the previous results on Li–Yorke and distributionally chaotic unilateral and bilateral weighted shifts. It also has relations with some papers in which other important dynamical properties were investigated for weighted shifts on trees.

1. Introduction

The study of chaos has attracted the attention of many authors in recent years. The mathematical term "chaos" first appeared in 1975 by Li and Yorke [19] in the context of compact dynamics. It was in [12] that this term was adopted in linear dynamics by Godefroy and Shapiro. Their notion is known as chaos in the sense of Devaney. Since then, there have been an intensive study on Devaney chaos as it contains hypercyclicity as one of its key ingredients, which is the main issue of linear dynamics and also intimately related to the invariant subset problem – an open problem in operator theory.

In recent years, the study on other notions of chaos, such as Li–Yorke chaos and distributional chaos, was initiated in the framework of linear dynamics. As to Li–Yorke chaos, which is the earliest notion of chaos, several authors recently started to investigate it in the setting of linear dynamics; see [3, 5, 7, 10, 25, 26]. In particular, in [3] the authors gave a characterization of Li–Yorke chaotic unilateral weighted shifts on certain sequence spaces. On the other hand, distributional chaos was introduced as a natural extension of Li–Yorke chaos by Schweizer and Smítal in the context of compact dynamics ([24]) and also adopted in linear dynamics; see, e.g., [3,4,6,22].

Mathematics Subject Classification 2020: 47A16 (primary); 47B38, 37B99 (secondary).
Keywords: weighted shifts, linear dynamics, Li–Yorke chaos, distributional chaos, directed trees.

It was in [3, 22] where some sufficient conditions for distributional chaos were shown for unilateral weighted shifts.

Some of the most famous yet simplest operators are weighted shifts. They serve as a "testing ground" for operator-theorists and provide a great deal of examples and counterexamples to distinguish several dynamical properties. A key step in the research on the dynamical properties of weighted shifts was made by Salas in [23] where he characterized hypercyclicity for weighted shifts on $l^2(\mathbb{N})$ and $l^2(\mathbb{Z})$. Later on, his study on weighted shifts was continued by many authors; see [2, 9, 11, 13, 17, 27].

However, some traditional examples constructed by weighted shifts are rather technical and complicated, or even completely fail to classify some important dynamical properties. Thus it is of great significance to consider a larger class of operators that provide the operator-theorists with more detailed and sophisticated examples. This can be achieved by modifying the set of indices which weighted shifts act on. More specifically, one can replace the index sets $\mathbb{N}$ or $\mathbb{Z}$ with a more general *directed tree*.

In [18], the authors initiated the study of weighted shifts on general directed trees from the perspective of the operator theory. The dynamical properties of these operators were firstly investigated by Martínez-Avendaño in [21] where he gave some conditions for the hypercyclicity of weighted shifts on directed trees. This paper was followed by some other important results by several authors. In [14], the boundedness, hypercyclicity, weak mixing and mixing property of weighted shifts on directed trees were characterized while Devaney chaos and $\mathcal{F}$-transitivity for the same operators were characterized in [15] and [1], respectively. In particular, [20] settled an open problem posed in [8] by constructing a counterexample based on weighted shifts on directed trees. They also showed that this cannot be done by traditional unilateral and bilateral weighted shifts.

Our main purpose in this paper is to study Li–Yorke and distributional chaos for weighted shifts on directed trees. One can relate our results to aforementioned papers in which several other dynamical properties such as hypercyclicity, $\mathcal{F}$-transitivity and Devaney chaos were investigated for these operators, and also to those that dealt with Li–Yorke and distributional chaos for traditional weighted shifts.

This paper is organized as follows. In Section 2, we give some preliminaries and fix the notations that will be used throughout this paper. In Section 3, we study Li–Yorke chaos for weighted shifts on directed trees. We first characterize Li–Yorke chaotic weighted shifts on rooted directed trees. Then we show that the existence of a nonzero orbital limit point guarantees Li–Yorke chaos for these operators. As a special case, we observe an unweighted shift on an unweighted tree and show that there exists a Li–Yorke chaotic shift on a tree that has a free right end. In Section 4, we study distributional chaos for weighted shifts on directed trees. First we give some sufficient conditions for distributional chaos for these operators. These results generalize some previous results on unilateral and bilateral weighted shifts. Also we show that the existence of a nontrivial periodic point implies distributional chaos, yet it does not guarantee Devaney chaos as shown in [15].

2. Preliminaries

2.1. Sequence spaces on directed trees

A *directed tree* $T = (V, E)$ is a connected directed graph consisting of a countable set V of vertices and a set $E \subset (V \times V) \setminus \{(v, v) : v \in V\}$ of *directed edges* that has no cycles and for which each vertex $v \in V$ has at most one *parent*, that is, a vertex $w \in V$ with $(w, v) \in E$; the parent is denoted by $\mathrm{par}(v)$. We write

$$\mathrm{par}^n(v) = \mathrm{par}(\mathrm{par}(\cdots \mathrm{par}(v))) \quad (n \text{ times}).$$

There is at most one vertex with no parent, called the *root*. The root is denoted by v_0.

The set $\mathrm{Chi}(v)$ is the set of all *children* of $v \in V$, that is, the vertices with v as a parent. More generally, for $n \geq 1$,

$$\mathrm{Chi}^n(v) = \mathrm{Chi}(\mathrm{Chi}(\cdots (\mathrm{Chi}(v)))), \quad (n \text{ times}),$$

with $\mathrm{Chi}^0(v) = \{v\}$. A vertex with no children is called a *leaf*.

In a rooted tree with root v_0, the set

$$\mathrm{Gen}_n = \mathrm{Chi}^n(v_0), \quad n \in \mathbb{N}$$

is called the *n-th generation*.

The *subtree* is a subset of a tree that is a tree in its own right without changing the parent-child relationship. An important subtree is

$$V_m = \bigcup_{n=0}^{m} \mathrm{Gen}_n$$

A tree V is called *symmetric* if $|\mathrm{Chi}(u)| = |\mathrm{Chi}(v)|$ whenever $u, v \in V$ belong to the same generation. Here $|\cdot|$ denotes a cardinality of the set. For a symmetric tree, we define

$$\gamma_n = |\mathrm{Chi}(v)|$$

when $v \in \mathrm{Gen}_n$.

A *path* is a sequence $(v_1, \ldots, v_n)$ in V with $v_{k-1} = \mathrm{par}(v_k)$ for $k = 2, \ldots, n$. A *branch* starting from a given vertex v_1 is a sequence $(v_1, v_2, \ldots)$ of maximal length in V with $v_{k-1} = \mathrm{par}(v_k)$ for $k \geq 2$. A family $\mu = (\mu_v)_{v \in V}$ of nonzero scalars is called a *weight* on V.

Now we recall the concept of some important sequence spaces on trees. Let $\mu = (\mu_v)_{v\in V}$ be a weight. We consider

$$l^p(V,\mu) = \left\{ f : V \to \mathbb{K} : \|f\|^p := \sum_{v\in V} |f(v)\mu_v|^p < \infty \right\}, \quad 1 \le p < \infty$$

and

$$c_0(V,\mu) = \{ f : V \to \mathbb{K} : \forall \varepsilon > 0, \exists F \subset V \text{ finite}, \forall v \in V \setminus F, |f(v)\mu_v| < \varepsilon \}.$$

which is endowed with the canonical norm on $l^\infty(V,\mu)$. If $\mu = 1$ for all $v \in V$, the corresponding unweighted spaces are denoted by

$$l^p(V)(1 \le p < \infty) \quad \text{and} \quad c_0(V).$$

2.2. Weighted backward shifts

Let V denote a tree. Let $\lambda = (\lambda_v)_{v\in V}$ be a family of nonzero scalars, that is, a weight. Then the *weighted backward shift* B_λ is defined by

$$(B_\lambda f)(v) = \sum_{u\in \mathrm{Chi}(v)} \lambda_u f(u), \quad v \in V,$$

where the empty sum is zero. If $\lambda = 1$ for all $v \in V$, the corresponding unweighted backward shift is denoted by B.

A weight $\lambda = (\lambda_v)_{v\in V}$ is called *symmetric* if $\lambda_u = \lambda_v$ whenever u and v belong to the same generation. Then we define $\lambda_n = \lambda_v$ when $v \in \mathrm{Gen}_n$. In this case, we call B_λ *symmetric*. Particular symmetric shifts are the *Rolewicz operators* λB, $\lambda \in \mathbb{K} \setminus \{0\}$.

Given a Fréchet sequence space X over V, we say that B_λ is *defined* on X if all the series defining $B_\lambda f$ converge unconditionally for any $f \in X$. By the Banach–Steinhaus theorem, B_λ is then a (continuous linear) operator from X to $\mathbb{K}^V$. On the other hand, we say that B_λ is *an operator on* X if $B_\lambda f \in X$ for all $f \in X$. In that case, by the closed graph theorem, B_λ is a (continuous linear) operator from X to X.

In connection with the iterates of weighted backward shifts the following notation is useful. If $\lambda = (\lambda_v)_{v\in V}$ is a weight and $(v_1, \ldots, v_n)$ is a path in V, then we set

$$\lambda(v_1 \to v_n) = \lambda_{v_2}\lambda_{v_3}\cdots\lambda_{v_n}.$$

Indeed, we have for any $v \in V$ and $n \ge 1$,

$$(B_\lambda^n f)(v) = \sum_{u\in \mathrm{Chi}^n(v)} \lambda(v \to u) f(u).$$

Note also that

$$\lambda(v \to u) = \lambda(v \to w)\lambda(w \to u)$$

if there is a path from v to u via w.

We recall here the well-known and very useful fact that every weighted backward shift is conjugate to an unweighted one if one modifies the underlying space. More precisely, let X be a Fréchet sequence space over V and $\mu = (\mu_v)_{v\in V}$ and $\lambda = (\lambda_v)_{v\in V}$ be weights. Define $X_\mu = \{f \in \mathbb{K}^V : (f(v)\mu_v)_{v\in V} \in X\}$, which turns canonically into a Fréchet sequence space. Then $\phi_\mu : X_\mu \to X,\ f \mapsto (f(v)\mu_v)_{v\in V}$ is an isometric isomorphism. And we have

$$B_\lambda \circ \phi_\mu = \phi_\mu \circ B$$

if the weights μ and λ are related via $\lambda_v = \frac{\mu_{\mathrm{par}(v)}}{\mu_v}$ for all $v \neq v_0$. Equivalently, we have

$$\mu_v = \frac{\prod_{k=0}^{m-1} \lambda_{\mathrm{par}^k(v_0)}}{\prod_{k=0}^{n-1} \lambda_{\mathrm{par}^k(v)}} \mu_{v_0} = \frac{\lambda(w \to v_0)}{\lambda(w \to v)} \mu_{v_0}, \quad v \in V$$

where w is a common ancestor of v and v_0, that is, $w = \mathrm{par}^n(v) = \mathrm{par}^m(v_0)$ for some $n, m \geq 0$.

2.3. Linear dynamics

We finally recall some basic notions from linear dynamics. An operator $T : X \to X$ on a separable Fréchet space X is called *hypercyclic* if there exists a vector $x \in X$ whose orbit $\{T^n x : n \geq 0\}$ is dense in X. T is called *topologically transitive* on X if for any nonempty open sets U and V in X, there exists some $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$. If X is a separable complete metric space without isolated points, T is hypercyclic if and only if it is topologically transitive by the Birkhoff transitivity theorem.

A vector $x \in X$ is called a *periodic point* for T if $T^N x = x$ for some $N \geq 1$, and N is called a *period* of x.

An operator $T : X \to X$ is called *Denaney chaotic* if it is hypercyclic and has a dense set of periodic points.

Next we state the notion of Li–Yorke chaos. Let $T : X \to X$ be an operator. A pair $(x, y) \in X \times X$ is called a *Li–Yorke pair* for T if

$$\liminf_{n\to\infty} d(T^n x, T^n y) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(T^n x, T^n y) > 0.$$

A *scrambled set* for T is a subset S in X such that (x, y) is a Li–Yorke pair for T whenever x and y are distinct points in S. T is said to be (*densely*) *Li–Yorke chaotic* if there exists an uncountable (dense) scrambled set for T.

It is known that Li–Yorke chaos for linear operators is equivalent to the existence of an irregular vector. A vector $x \in X$ such that $\liminf_{n\to\infty} \|T^n x\| = 0$ and $\limsup_{n\to\infty} \|T^n x\| = \infty$ is called an *irregular vector* for T. A vector subspace Y of X is called an *irregular manifold* for $T \in L(X)$ if every vector $y \in Y \setminus \{0\}$ is irregular for T. Clearly, an irregular manifold for T is a scrambled set for T.

The notion of distributional chaos is a natural extension of Li–Yorke chaos. Let T be an operator on X. For each pair $x, y \in X$ and each $n \in \mathbb{N}$, the *distributional function* $F_{xy}^n : \mathbb{R}^+ \to [0, 1]$ is defined by

$$F_{xy}^n(\tau) := \frac{1}{n}|0 \le i \le n-1 : d(T^n x, T^n y) < \tau|.$$

And we define

$$F_{xy}(\tau) := \liminf_{n\to\infty} F_{xy}^n(\tau) \quad \text{and} \quad F_{xy}^*(\tau) := \limsup_{n\to\infty} F_{xy}^n(\tau).$$

T is called (*densely*) *distributionally chaotic* on X if there exist an uncountable (dense) set $\Gamma \subset X$ and $\varepsilon > 0$ such that for every $\tau > 0$ and each pair of distinct points $x, y \in \Gamma$, we have that

$$F_{xy}(\varepsilon) = 0 \quad \text{and} \quad F_{xy}^*(\tau) = 1.$$

Distributional chaos for linear operators has a close relation with the existence of a distributionally irregular vector. The orbit of $x \in X$ is said to be *distributionally near to* 0 if there exists $A \subset \mathbb{N}$ with $\overline{\text{dens}}(A) = 1$ such that $\lim_{n\in A} T^n x = 0$. And we say x has a *distributionally unbounded orbit* if there exists $B \subset \mathbb{N}$ with $\overline{\text{dens}}(B) = 1$ such that $\lim_{n\in B} \|T^n x\| = \infty$. A vector $x \in X$ is called a *distributionally irregular vector* for T if its orbit is both distributionally unbounded and distributionally near to 0.

A vector subspace Y of X is called a *distributionally irregular manifold* for $T \in L(X)$ if every vector $y \in Y \setminus \{0\}$ is a distributionally irregular vector for T.

3. Li–Yorke chaos for weighted shifts on directed trees

In this section, we study Li–Yorke chaos for weighted shifts on directed trees. As mentioned in Section 2, weighted shifts on unweighted trees and unweighted shifts on weighted trees are conjugate, which enables us to focus on one of these cases. For convenience, we observe unweighted shifts on weighted directed trees and then transfer the results to the weighted shifts on unweighted trees.

First we consider the case where the tree has a root v_0. In order to give a complete characterization of Li–Yorke chaos for weighted shifts on rooted directed trees, we need the following proposition that gives the norm of powers of B.

Proposition 3.1. *Let V be a tree with root v_0 and $\mu = (\mu_v)_{v\in V}$ a weight on V. The following hold:*

(1) *If B is an operator on $l^1(V, \mu)$, then*

$$\|B^n\| = \sup_{v\in V\setminus V_{n-1}} \left|\frac{\mu_{\text{par}^n(v)}}{\mu_v}\right|$$

for every $n \in \mathbb{N}$.

(2) *If B is an operator on $l^p(V,\mu)$ $(1 < p < \infty)$, then*

$$\|B^n\| = \sup_{v\in V}\Bigg(\sum_{u\in \mathrm{Chi}^n(v)}\Big|\frac{\mu_v}{\mu_u}\Big|^{p^*}\Bigg)^{\frac{1}{p^*}}$$

for every $n \in \mathbb{N}$.

(3) *If B is an operator on $c_0(V,\mu)$, then*

$$\|B^n\| = \sup_{v\in V}\Bigg(\sum_{u\in \mathrm{Chi}^n(v)}\Big|\frac{\mu_v}{\mu_u}\Big|\Bigg)$$

for every $n \in \mathbb{N}$.

Proof. We show that (2) holds. (1) and (3) are proved in a similar way. Let

$$M_n = \sup_{v\in V}\Bigg(\sum_{u\in \mathrm{Chi}^n(v)}\Big|\frac{\mu_v}{\mu_u}\Big|^{p^*}\Bigg)^{\frac{1}{p^*}},$$

then by the Hölder inequality, we have

$$\begin{aligned}\|B^n f\|^p &= \sum_{v\in V}|\mu_v(B^n f)(v)|^p = \sum_{v\in V}\Big|\mu_v\sum_{u\in \mathrm{Chi}^n(v)} f(u)\Big|^p\\ &= \sum_{v\in V}\Bigg(|\mu_v|^p\Big|\sum_{u\in \mathrm{Chi}^n(v)} f(u)\Big|^p\Bigg)\\ &\le \sum_{v\in V}\Bigg(|\mu_v|^p\Bigg(\sum_{u\in \mathrm{Chi}^n(v)}\Big|\frac{1}{\mu_u}\Big|^{p^*}\Bigg)^{\frac{p}{p^*}}\Bigg(\sum_{u\in \mathrm{Chi}^n(v)}|f(u)\mu_u|^p\Bigg)\Bigg)\\ &\le M_n^p\|f\|^p\end{aligned}$$

for every $f \in l^p(V,\mu)$. This implies $\|B^n\| \le M_n$.

In order to show $\|B^n\| \ge M_n$, we fix $v \in V$ and let $I : l^p(\mathrm{Chi}^n(v), \mu\chi_{\mathrm{Chi}^n(v)}) \to l^p(V,\mu)$ be a continuous embedding where $\mu\chi_{\mathrm{Chi}^n(v)}$ denotes the restriction of μ onto $\mathrm{Chi}^n(v)$. If we define

$$\phi_{v,n}(f) := \mu_v(B^n If)(v) = \sum_{u\in \mathrm{Chi}^n(v)}\mu_v f(u), \qquad f \in l^p(\mathrm{Chi}^n(v), \mu\chi_{\mathrm{Chi}^n(v)}),$$

then $\phi_{v,n}$ is a linear functional on $l^p(\mathrm{Chi}^n(v), \mu\chi_{\mathrm{Chi}^n(v)})$ and

$$|\phi_{v,n}(f)| = \|B^n If\|_{l^p(V,\mu)} \le \|B^n\|\|If\|_{l^p(V,\mu)} = \|B^n\|\|f\|_{l^p(\mathrm{Chi}^n(v),\mu\chi_{\mathrm{Chi}^n(v)})}.$$

On the other hand, the fact

$$\phi_{v,n} \in (l^p(\mathrm{Chi}^n(v), \mu\chi_{\mathrm{Chi}^n(v)}))^* = l^{p^*}\left(\mathrm{Chi}^n(v), \frac{1}{\mu}\chi_{\mathrm{Chi}^n(v)}\right)$$

and the definition of $\phi_{v,n}$ yield

$$\|\phi_{v,n}\| = \|\mu_v \chi_{\mathrm{Chi}^n(v)}\|_{l^{p^*}(\mathrm{Chi}^n(v), \frac{1}{\mu}\chi_{\mathrm{Chi}^n(v)})} = \left(\sum_{u\in\mathrm{Chi}^n(v)} \left|\frac{\mu_v}{\mu_u}\right|^{p^*}\right)^{\frac{1}{p^*}},$$

which in turn implies $\|B^n\| \geq (\sum_{u\in\mathrm{Chi}^n(v)} |\frac{\mu_v}{\mu_u}|^{p^*})^{\frac{1}{p^*}}$. Since v is arbitrary, we obtain $\|B^n\| \geq M_n$. This concludes the proof. ■

Now by using Proposition 3.1, we can give a complete characterization of Li–Yorke chaotic unweighted shifts on rooted weighted trees.

Theorem 3.2. *Let V be a tree with root v_0 and $\mu = (\mu_v)_{v\in V}$ a weight on V. If B is an operator on X, then the following assertions are equivalent:*

(1) *B is Li–Yorke chaotic on X;*

(2) *B is densely Li–Yorke chaotic on X;*

(3) *B admits a dense irregular manifold;*

(4) *if $X = l^1(V, \mu)$,*

$$\sup_{n\in\mathbb{N}} \sup_{v\in V\setminus V_{n-1}} \left|\frac{\mu_{\mathrm{par}^n(v)}}{\mu_v}\right| = \infty;$$

if $X = l^p(V, \mu)$ $(1 < p < \infty)$,

$$\sup_{n\in\mathbb{N}, v\in V} \sum_{u\in\mathrm{Chi}^n(v)} \left|\frac{\mu_v}{\mu_u}\right|^{p^*} = \infty;$$

if $X = c_0(V, \mu)$,

$$\sup_{n\in\mathbb{N}, v\in V} \sum_{u\in\mathrm{Chi}^n(v)} \left|\frac{\mu_v}{\mu_u}\right| = \infty.$$

Proof. Since $X_0 = \mathrm{span}\{e_v : v \in V\}$ is dense in X and $\lim_{n\to\infty} B^n x = 0$ for every $x \in X_0$, we have (1) ⇔ (2) ⇔ (3) by [5, Corollary 21]. And by combining [3, Proposition 2.1] with [3, Theorem 25], we also have (1) ⇔ (4). ■

It is known that the zero-one law holds true for weighted shifts on $\mathbb{N}$ and $\mathbb{Z}$, that is, the existence of a nonzero orbital limit point implies hypercyclicity; see [9]. However, in [1] it was shown that this is no longer valid for general directed trees. Then one might wonder whether the existence of a nonzero orbital limit point guarantees Li–Yorke chaos for these operators, which is a weaker notion than hypercyclicity. The following proposition gives a positive answer to this question.

Proposition 3.3. *Let V be a tree with root v_0 and $\mu = (\mu_v)_{v \in V}$ a weight on V. Assume that B is an operator on $X = l^p(V, \mu)$ $(1 \leq p < \infty)$ or $c_0(V, \mu)$. If B admits an orbit with a nonzero limit point, then B is Li–Yorke chaotic on X.*

Proof. We prove in the case where $X = l^p(V, \mu)$ $(1 < p < \infty)$. Suppose that B admits an orbit with a nonzero limit point, then by [1, Theorem 7.1] there exist a vertex $v \in V$ and a strictly increasing sequence of positive integers $(n_k)_{k \in \mathbb{N}}$ such that

$$\sum_{u \in \mathrm{Chi}^{n_k}(v)} \frac{1}{|\mu_u|^{p^*}} \to \infty \quad (k \to \infty),$$

which clearly implies

$$\sum_{u \in \mathrm{Chi}^{n_k}(v)} \left|\frac{\mu_v}{\mu_u}\right|^{p^*} = |\mu_v|^{p^*} \sum_{u \in \mathrm{Chi}^{n_k}(v)} \frac{1}{|\mu_u|^{p^*}} \to \infty \quad (k \to \infty).$$

By Theorem 3.2, B is Li–Yorke chaotic on $l^p(V, \mu)$. The proof proceeds similarly for the cases where $X = l^1(V, \mu)$ or $c_0(V, \mu)$. ■

Remark 3.1. The converse implication of Proposition 3.3 does not hold true in general. Since the existence of a nonzero orbital limit point is equivalent to hypercyclicity where $V = \mathbb{N}$ or $V = \mathbb{Z}$, one can simply pick a Li–Yorke chaotic unilateral weighted shift that is not hypercyclic; see [5, Remark 24].

Now it is easy to transfer our results to a weighted shift on an unweighted tree. By the conjugate relation mentioned in Section 2, we obtain the characterization of Li–Yorke chaotic weighted shifts on rooted directed trees.

Theorem 3.4. *Let V be a tree with root v_0 and $\lambda = (\lambda_v)_{v \in V}$ a weight on V. If B_λ is an operator on X, then the following assertions are equivalent:*

(1) *B_λ is Li–Yorke chaotic on X;*

(2) *B_λ is densely Li–Yorke chaotic on X;*

(3) *B_λ admits a dense irregular manifold;*

(4) *if $X = l^1(V)$,*

$$\sup_{n \in \mathbb{N}} \sup_{v \in V \setminus V_{n-1}} |\lambda(\mathrm{par}^n(v) \to v)| = \infty;$$

if $X = l^p(V)$ $(1 < p < \infty)$,

$$\sup_{n \in \mathbb{N}, v \in V} \sum_{u \in \mathrm{Chi}^n(v)} |\lambda(v \to u)|^{p^*} = \infty;$$

if $X = c_0(V)$,

$$\sup_{n\in\mathbb{N},v\in V} \sum_{u\in\mathrm{Chi}^n(v)} |\lambda(v \to u)| = \infty.$$

Next we turn to a special case where the shift and the tree are both unweighted. It is clear that the dynamical properties of the shift depend only on the structure of the tree in this case. This was studied by Martínez-Avendaño in [21], where he showed that an unweighted shift on an unweighted rooted tree is hypercyclic if and only if the tree has no free right ends. This arises a natural question. Does there exist a Li–Yorke chaotic shift on a tree that has a free right end? By using Theorem 3.2, we can also give an answer to this question.

Proposition 3.5. *Let V be a tree with root v_0. If B is an operator on $l^1(V)$, then B is not Li–Yorke chaotic on $l^1(V)$.*

Proof. Since $\mu_v = 1$ for every $v \in V$, the left-hand side of the first expression in condition (4) of Theorem 3.2 is 1, which shows B can never be Li–Yorke chaotic on $l^1(V)$. ■

Proposition 3.6. *Let V be a tree with root v_0 and B an operator on $X = l^p(V)$ $(1 < p < \infty)$ or $X = c_0(V)$. B is Li–Yorke chaotic on X if and only if*

$$\sup_{n\in\mathbb{N}} |\mathrm{Gen}_n| = \infty.$$

Proof. Since $\mu_v = 1$ for every $v \in V$, we can see that B is Li–Yorke chaotic if and only if

$$\sup_{n\in\mathbb{N},v\in V} |\mathrm{Chi}^n(v)| = \infty$$

by Theorem 3.2. This is equivalent to $\sup_{n\in\mathbb{N}} |\mathrm{Gen}_n| = \infty$. ■

Example 3.7. Here we give an example of a Li–Yorke chaotic shift on a tree that has a free right end. We consider a binary tree V with an extra branch sticking out from any fixed point. Clearly V has a free right end, but an unweighted shift on V is Li–Yorke chaotic by Proposition 3.6.

Finally, we consider the case where the tree is unrooted. One can easily notice that the fact that

$$X = \overline{\mathrm{span}\{e_v : v \in V\}} = \overline{\{x \in X : \lim_{n\to\infty} B^n x = 0\}}$$

plays a crucial role in the proof of Theorem 3.2. Hence by adopting the same discussion as in Theorem 3.2 and Theorem 3.4, we can give the following results on Li–Yorke chaotic backward shifts on unrooted trees.

Corollary 3.8. *Let V be an unrooted tree and $\mu = (\mu_v)_{v\in V}$ a weight on V such that*

$$\lim_{n\to\infty} \mu_{\mathrm{par}^n(v)} = 0.$$

If B is an operator on X, then the following assertions are equivalent:

(1) *B is Li–Yorke chaotic on X;*

(2) *B is densely Li–Yorke chaotic on X;*

(3) *B admits a dense irregular manifold;*

(4) *if $X = l^1(V,\mu)$,*

$$\sup_{n\in\mathbb{N}} \sup_{v\in V\setminus V_{n-1}} \left|\frac{\mu_{\mathrm{par}^n(v)}}{\mu_v}\right| = \infty;$$

if $X = l^p(V,\mu)$ $(1 < p < \infty)$,

$$\sup_{n\in\mathbb{N}, v\in V} \sum_{u\in \mathrm{Chi}^n(v)} \left|\frac{\mu_v}{\mu_u}\right|^{p^*} = \infty;$$

if $X = c_0(V,\mu)$,

$$\sup_{n\in\mathbb{N}, v\in V} \sum_{u\in \mathrm{Chi}^n(v)} \left|\frac{\mu_v}{\mu_u}\right| = \infty.$$

Corollary 3.9. *Let V be an unrooted tree and $\lambda = (\lambda_v)_{v\in V}$ a weight on V such that*

$$\lim_{n\to\infty} \lambda_v \lambda_{\mathrm{par}(v)} \cdots \lambda_{\mathrm{par}^n(v)} = 0.$$

If B_λ is an operator on X, then the following assertions are equivalent:

(1) *B_λ is Li–Yorke chaotic on X;*

(2) *B_λ is densely Li–Yorke chaotic on X;*

(3) *B_λ admits a dense irregular manifold;*

(4) *if $X = l^1(V)$,*

$$\sup_{n\in\mathbb{N}, v\in V} |\lambda(\mathrm{par}^n(v) \to v)| = \infty;$$

if $X = l^p(V)$ $(1 < p < \infty)$,

$$\sup_{n\in\mathbb{N}, v\in V} \sum_{u\in \mathrm{Chi}^n(v)} |\lambda(v \to u)|^{p^*} = \infty;$$

if $X = c_0(V)$,

$$\sup_{n\in\mathbb{N}, v\in V} \sum_{u\in \mathrm{Chi}^n(v)} |\lambda(v \to u)| = \infty.$$

4. Distributional chaos for weighted shifts on directed trees

The main aim of this section is to investigate distributional chaos for weighted shifts on directed trees. It is a tricky matter in general to completely characterize distributional chaos of an operator. However, we can give some sufficient conditions for a certain operator to be distributionally chaotic. We use the same techniques as in Section 3, that is, we first study unweighted shifts on weighted trees and transfer the results to the weighted shifts on unrooted trees.

Theorem 4.1. *Let V be a tree with root v_0, $(\mu_v)_{v\in V}$ a weight on V and B an operator on X. Assume that there exists a branch $(v_0, v_1, v_2, \dots)$ and a set $S \subseteq \mathbb{N}$ with $\overline{\operatorname{dens}}(S) = 1$ such that*

$$\sum_{n\in S} |\mu_{v_n}|^p < \infty, \quad X = l^p(V,\mu) \quad (1 \le p < \infty);$$

$$\sup_{n\in S} |\mu_{v_n}| < \infty, \quad X = c_0(V,\mu),$$

then B is densely distributionally chaotic on X.

Proof. We only consider the case where $X = l^p(V,\mu)$ $(1 \le p < \infty)$. If we define $f_k = \sum_{n\in S, n\ge k} e_{v_n}$, then we have $f_k \in l^p(V,\mu)$ since $\|f_k\| = (\sum_{n\in S,n\ge k} |\mu_{v_n}|^p)^{\frac{1}{p}} < \infty$ for every $k \in \mathbb{N}$. Clearly we obtain $f_k \to 0$ and $\|B^n f_n\| = \|e_{v_0} + \cdots\| \ge |\mu_{v_0}|$ for every $n \in S$. Denoting S by a strictly increasing sequence $(N_k)_{k\in\mathbb{N}}$, we have

$$\lim_{k\to\infty} \frac{1}{N_k} |\{1 \le j \le N_k : \|B^j f_j\| > |\mu_{v_0}|\}| = 1.$$

By [4, Proposition 8], B admits a distributionally unbounded orbit, which completes the proof by [4, Theorem 15]. ∎

Corollary 4.2. *Let V be a tree with root v_0, $(\lambda_v)_{v\in V}$ a weight on V and B_λ an operator on X. Assume that there exists a branch $(v_0, v_1, v_2, \dots)$ and a set $S \subseteq \mathbb{N}$ with $\overline{\operatorname{dens}}(S) = 1$ such that*

$$\sum_{n\in S} |\lambda(v_0 \to v_n)|^{-p} < \infty, \quad X = l^p(V) \quad (1 \le p < \infty);$$

$$\inf_{n\in S} |\lambda(v_0 \to v_n)| > 0, \quad X = c_0(V),$$

then B_λ is densely distributionally chaotic on X.

Remark 4.1. If we set $V = \mathbb{N}$ in Theorem 4.1, then we obtain [4, Theorem 26].

In [15], the authors have indicated a marked difference between a general directed tree and the case $V = \mathbb{N}$ or $V = \mathbb{Z}$. Specifically, they noted that the existence of a nontrivial periodic point does not imply Devaney chaos for weighted shifts on directed trees; see [15, Examples 3.1, 3.2]. It reveals great contrast with traditional unilateral and bilateral weighted shifts where the existence of a nontrivial fixed point is equivalent to Devaney chaos. Here we show that the existence of a nontrivial fixed point guarantees distributional chaos for weighted shifts on rooted directed trees.

Theorem 4.3. *Let V be a tree with root v_0, $(\mu_v)_{v\in V}$ a weight on V and B an operator on $X = l^p(V,\mu)$ $(1 \le p < \infty)$ or $X = c_0(V,\mu)$. If B admits a nontrivial periodic point, then B is densely distributionally chaotic on X.*

Proof. Let f be a nontrivial periodic point of B with $B^p f = f$. If we put

$$M = \min\{\|f\|, \|Bf\|, \dots, \|B^{p-1} f\|\}$$

then we have $M > 0$ since $f \neq 0$. Defining $f_n = f - f\chi_{V_{n-1}}$ for every $n \in \mathbb{N}$, we clearly have $f_n \to 0$ and $B^n f_n = B^n f - B^n(f\chi_{V_{n-1}}) = B^n f = B^{pq+r} f = B^r f$ for some $0 \le r < p$. Hence it follows that

$$\lim_{n\to\infty} \frac{1}{n} |\{1 \le k \le n : \|B^k f_k\| \ge M\}| = 1.$$

This implies that B is densely distributionally chaotic by [4, Proposition 8]. ∎

Remark 4.2. The converse implication of Theorem 4.3 does not hold true in general. In fact, even in the case where $V = \mathbb{N}$ it is not true, since in this case the existence of a nontrivial periodic point is equivalent to Devaney chaos ([16, Theorem 4.8]) and there exists a unilateral weighted shift which is distributionally chaotic but not Devaney chaotic ([22, Example 10]). Furthermore, for weighted shifts on directed trees, the existence of a nontrivial periodic point does not imply Devaney chaos in general as shown in [15, Example 3.2]. So we can conclude we have the following chain of implications for weighted shifts on rooted directed trees where two converse implications are not valid:

Devaney chaos $\Rightarrow$ existence of a nontrivial periodic point $\Rightarrow$ distributional chaos.

We can give a sufficient condition for the existence of a dense irregular manifold from the perspective of the norm of B.

Theorem 4.4. *Let V be a tree with root v_0, $(\mu_v)_{v\in V}$ a weight on V and B an operator on $X = l^p(V,\mu)$ $(1 \le p < \infty)$ or $X = c_0(V,\mu)$. Assume that there exists a set $S \subseteq \mathbb{N}$ with $\overline{\mathrm{dens}}(S) = 1$ such that*

$$\sum_{n\in S} \inf_{v\in V\setminus V_{n-1}} \left|\frac{\mu_v}{\mu_{\mathrm{par}^n(v)}}\right| < \infty, \qquad X = l^1(V,\mu);$$

$$\sum_{n\in S}\left(\sup_{v\in V}\sum_{u\in \mathrm{Chi}^n(v)}\left|\frac{\mu_v}{\mu_u}\right|^{p^*}\right)^{-\frac{1}{p^*}}<\infty,\quad X=l^p(V,\mu)\quad(1<p<\infty);$$

$$\sum_{n\in S}\left(\sup_{v\in V}\sum_{u\in \mathrm{Chi}^n(v)}\left|\frac{\mu_v}{\mu_u}\right|\right)^{-1}<\infty,\quad X=c_0(V,\mu).$$

Then B admits a dense distributionally irregular manifold in X.

Proof. This is an immediate consequence of Proposition 3.1 and [3, Corollary 30]. ■

Corollary 4.5. *Let V be a tree with root v_0, $(\lambda_v)_{v\in V}$ a weight on V and B_λ an operator on $X=l^p(V)$ $(1\le p<\infty)$ or $X=c_0(V)$. Assume that there exists a set $S\subseteq\mathbb{N}$ with $\overline{\mathrm{dens}}(S)=1$ such that*

$$\sum_{n\in S}\inf_{v\in V\setminus V_{n-1}}\left|\frac{1}{\lambda(\mathrm{par}^n(v)\to v)}\right|<\infty,\quad X=l^1(V);$$

$$\sum_{n\in S}\left(\sup_{v\in V}\sum_{u\in \mathrm{Chi}^n(v)}|\lambda(v\to u)|^{p^*}\right)^{-\frac{1}{p^*}}<\infty,\quad X=l^p(V)\quad(1<p<\infty);$$

$$\sum_{n\in S}\left(\sup_{v\in V}\sum_{u\in \mathrm{Chi}^n(v)}|\lambda(v\to u)|\right)^{-1}<\infty,\quad X=c_0(V).$$

Then B_λ admits a dense distributionally irregular manifold in X.

We end this section with an observation on one special case. Rolewicz operators are a special case of a symmetric shift where $\lambda=(\lambda_n)_n$. In [15], it was shown that for these operators, Devaney chaos, mixing and hypercyclicity are equivalent. Here we show that even Li–Yorke chaos and distributional chaos are all equivalent to these properties. The following proposition establishes the equivalence of three notions of chaos for Rolewicz operators on directed trees.

Proposition 4.6. *Let V be a tree with root v_0, $\lambda\in\mathbb{K}\setminus\{0\}$ and λB a corresponding Rolewicz operator on $l^1(V)$. Then the following assertions are equivalent:*

(1) *λB is Li–Yorke chaotic on $l^1(V)$;*

(2) *λB is distributionally chaotic on $l^1(V)$;*

(3) *λB is Devaney chaotic on $l^1(V)$;*

(4) $|\lambda|>1$.

Proof. We have (3) $\Leftrightarrow$ (4) by [15, Theorem 9.3]. Also we obtain (3) $\Rightarrow$ (2) since $\mathrm{span}\{e_v : v\in V\}$ is dense in $l^1(V)$. (2) $\Rightarrow$ (1) is obvious. Hence it suffices to show that (1) $\Rightarrow$ (4) holds. Assume that λB is Li–Yorke chaotic on $l^1(V)$, then we have $\sup_{n\in\mathbb{N}}|\lambda|^n=\infty$ by Theorem 3.4. This implies $|\lambda|>1$. ■

Proposition 4.7. *Let V be a tree with root v_0 in which every vertex has exactly $N \geq 1$ children. If $\lambda \in \mathbb{K} \setminus \{0\}$ and λB is a corresponding Rolewicz operator on $X = l^p(V)$ $(1 < p < \infty)$ or $X = c_0(V)$, then the following assertions are equivalent:*

(1) *λB is Li–Yorke chaotic on X;*

(2) *λB is distributionally chaotic on X;*

(3) *λB is Devaney chaotic on X;*

(4) *$|\lambda| > N^{-\frac{1}{p^*}}$ if $X = l^p(V)$; $|\lambda| > N^{-1}$ if $X = c_0(V)$.*

Proof. We have (3) $\Leftrightarrow$ (4) and (3) $\Rightarrow$ (2) as in Proposition 4.6. As (2) $\Rightarrow$ (1) is clear, it remains to verify (1) $\Rightarrow$ (4). Assume that $X = l^p(V)$ $(1 < p < \infty)$. If λB is Li–Yorke chaotic, then we have $\sup_{n\in\mathbb{N}} N^n \lambda^{np^*} = \infty$ by Theorem 3.4. Thus we obtain $|\lambda| > N^{-\frac{1}{p^*}}$, which is to be shown. ∎

Finally, we mention that the results in this section can also be used in the unrooted case if the weight satisfies the condition stated in Corollary 3.8 or Corollary 3.9. In order to avoid unnecessary repetition, we do not write it here.

Acknowledgments. The authors would like to thank the reviewers for their helpful comments and suggestions.

Funding. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-prot sectors.

References

[1] E. Abakumov and A. Abbar, On several dynamical properties of shifts acting on directed trees. 2024, arXiv:2402.05038v1

[2] F. Bayart and I. Z. Ruzsa, Difference sets and frequently hypercyclic weighted shifts. *Ergodic Theory Dynam. Systems* **35** (2015), no. 3, 691–709 Zbl 1355.37035 MR 3334899

[3] T. Bermúdez, A. Bonilla, F. Martínez-Giménez, and A. Peris, Li–Yorke and distributionally chaotic operators. *J. Math. Anal. Appl.* **373** (2011), no. 1, 83–93 Zbl 1214.47012 MR 2684459

[4] N. C. Bernardes, Jr., A. Bonilla, V. Müller, and A. Peris, Distributional chaos for linear operators. *J. Funct. Anal.* **265** (2013), no. 9, 2143–2163 Zbl 1302.47014 MR 3084499

[5] N. C. Bernardes, Jr., A. Bonilla, V. Müller, and A. Peris, Li–Yorke chaos in linear dynamics. *Ergodic Theory Dynam. Systems* **35** (2015), no. 6, 1723–1745 Zbl 1352.37100 MR 3377281

[6] N. C. Bernardes, Jr., A. Bonilla, A. Peris, and X. Wu, Distributional chaos for operators on Banach spaces. *J. Math. Anal. Appl.* **459** (2018), no. 2, 797–821 Zbl 1482.47009 MR 3732556

[7] N. C. Bernardes, Jr., U. B. Darji, and B. Pires, Li–Yorke chaos for composition operators on L^p-spaces. *Monatsh. Math.* **191** (2020), no. 1, 13–35 Zbl 1485.47012 MR 4054819

[8] N. C. Bernardes, Jr. and A. Peris, On shadowing and chain recurrence in linear dynamics. *Adv. Math.* **441** (2024), article no. 109539 Zbl 1540.37039 MR 4708149

[9] A. Bonilla, R. Cardeccia, K.-G. Grosse-Erdmann, and S. Muro, Zero-one law of orbital limit points for weighted shifts. *Proc. Edinb. Math. Soc. (2)* **68** (2025), no. 3, 945–978 Zbl 08086107 MR 4943976

[10] Y. Estaremi and S. Muhammad, Li–Yorke chaos for composition operators on Orlicz spaces. *Forum Math.* **36** (2024), no. 4, 973–981 MR 4767368

[11] N. S. Feldman, Hypercyclicity and supercyclicity for invertible bilateral weighted shifts. *Proc. Amer. Math. Soc.* **131** (2003), no. 2, 479–485 Zbl 1051.47005 MR 1933339

[12] G. Godefroy and J. H. Shapiro, Operators with dense, invariant, cyclic vector manifolds. *J. Funct. Anal.* **98** (1991), no. 2, 229–269 Zbl 0732.47016 MR 1111569

[13] K.-G. Grosse-Erdmann, Hypercyclic and chaotic weighted shifts. *Studia Math.* **139** (2000), no. 1, 47–68 Zbl 0991.47013 MR 1763044

[14] K.-G. Grosse-Erdmann and D. Papathanasiou, Dynamics of weighted shifts on directed trees. *Indiana Univ. Math. J.* **72** (2023), no. 1, 263–299 Zbl 07685829 MR 4557075

[15] K.-G. Grosse-Erdmann and D. Papathanasiou,Chaotic weighted shifts on directed trees. [v1] 2023, [v2] 2024 arXiv:2303.03980v2

[16] K.-G. Grosse-Erdmann and A. Peris Manguillot, *Linear chaos*. Universitext, Springer, London, 2011 Zbl 1246.47004 MR 2919812

[17] S. He, Y. Huang, and Z. Yin, $J^{\mathcal{F}}$-class weighted backward shifts. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **28** (2018), no. 6, article no. 1850076 Zbl 1395.47001 MR 3816610

[18] Z. J. Jabłoński, I. B. Jung, and J. Stochel, Weighted shifts on directed trees. *Mem. Amer. Math. Soc.* **216** (2012), article no. 1017 Zbl 1248.47033 MR 2919910

[19] T. Y. Li and J. A. Yorke, Period three implies chaos. *Amer. Math. Monthly* **82** (1975), no. 10, 985–992 Zbl 0351.92021 MR 0385028

[20] A. López-Martínez and D. Papathanasiou, Shifts on trees versus classical shifts in chain recurrence. *J. Differential Equations* **433** (2025), article no. 113230 Zbl 08031075 MR 4880974

[21] R. A. Martínez-Avendaño, Hypercyclicity of shifts on weighted L^p spaces of directed trees. *J. Math. Anal. Appl.* **446** (2017), no. 1, 823–842 Zbl 1346.05032 MR 3554758

[22] F. Martínez-Giménez, P. Oprocha, and A. Peris, Distributional chaos for backward shifts. *J. Math. Anal. Appl.* **351** (2009), no. 2, 607–615 Zbl 1157.47008 MR 2473967

[23] H. N. Salas, Hypercyclic weighted shifts. *Trans. Amer. Math. Soc.* **347** (1995), no. 3, 993–1004 Zbl 0822.47030 MR 1249890

[24] B. Schweizer and J. Smítal, Measures of chaos and a spectral decomposition of dynamical systems on the interval. *Trans. Amer. Math. Soc.* **344** (1994), no. 2, 737–754 Zbl 0812.58062 MR 1227094

[25] X. Wu and P. Zhu, On the equivalence of four chaotic operators. *Appl. Math. Lett.* **25** (2012), no. 3, 545–549 Zbl 1242.37009 MR 2856030

[26] X. Wu and P. Zhu, Li–Yorke chaos of backward shift operators on Köthe sequence spaces. *Topology Appl.* **160** (2013), no. 7, 924–929 Zbl 1286.54035 MR 3037883

[27] Z. Yin and Q. Yang, Distributionally n-scrambled set for weighted shift operators. *J. Dyn. Control Syst.* **23** (2017), no. 4, 693–708 Zbl 06793137 MR 3688890

Received 3 June 2025; revised 21 December 2025.

Chol San Kim
Faculty of Mathematics, Kim Il Sung University, Taesong District, Pyongyang, Democratic People's Republic of Korea; cs.kim0920@ryongnamsan.edu.kp

Hyon Hui Ju
Faculty of Mathematics, Kim Il Sung University, Taesong District, Pyongyang, Democratic People's Republic of Korea; hh.ju0524@ryongnamsan.edu.kp

Jun Hyok Choe
Faculty of Mathematics, Kim Il Sung University, Taesong District, Pyongyang, Democratic People's Republic of Korea; math3@ryongnamsan.edu.kp

Jin Myong Kim
Faculty of Mathematics, Kim Il Sung University, Taesong District, Pyongyang, Democratic People's Republic of Korea; jm.kim0211@ryongnamsan.edu.kp