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Assouad and lower dimensions of graph-directed Bedford–McMullen carpets

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Abstract. We calculate the Assouad and the lower dimensions of graph-directed Bedford–McMullen carpets, which reflect the extreme local scaling laws of the sets, in contrast to known results on Hausdorff and box dimensions. We also investigate the relationship between distinct dimensions. In particular, we identify an equivalent condition when the box and the Assouad dimensions coincide, and show that under this condition, the Hausdorff dimension attains the same value.

1. Introduction

Since the mid 80s, the study of self-affine sets has emerged as an independent research field, in which the class of Bedford–McMullen carpets [2, 23] plays a prominent role. There has been significant interest in calculating the dimensions of these sets as well as their generalizations, including the Lalley–Gatzouras class [18], Barański carpets [1], Feng–Wang box-like sets [7, 9, 11], Kenyon–Peres ($\times m$, $\times n$) carpets [14–16], and their high-dimensional analogs [5, 13, 16, 17, 24]. Unlike the case of self-similar sets, a key feature in this study is the provision of plenty of examples generated by recursive constructions, which exhibit distinct Hausdorff and box dimensions.

In 2011, Mackay [22] initiated the study of Assouad dimension of sets generated by Lalley–Gatzouras construction, including Bedford–McMullen carpets. This investigation was later extended by Fraser [10] to Barański carpets, which are generated by a box-like affine construction with a more flexible grid structure. As a natural dual, the exact value of the lower dimension (introduced by Larman [19]) for sets in this scenario was also determined in [10]. As a rapidly expanding branch of dimension theory on fractals, Assouad and lower dimensions focus on the local geometric information of fractals, reflecting the scaling laws of the thickest or thinnest parts of the sets. For further background and details on these two dimensions, refer to [12]. A fascinating

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observation in [10, 22] for the Lalley–Gatzouras class is the dichotomy in which the Hausdorff, box, Assouad and lower dimensions are either all distinct or all equal, i.e.,

$$\begin{aligned} &\text{either} \quad \dim_L X < \dim_H X < \dim_B X < \dim_A X, \\ &\text{or} \quad \dim_L X = \dim_H X = \dim_B X = \dim_A X. \end{aligned} \tag{1.1}$$

In this note, we extend the consideration of Assouad and lower dimensions to graph-directed Bedford–McMullen carpets, which serve as a typical class of $(\times m, \times n)$ carpets introduced by Kenyon–Peres [15, 16].

Let us begin with a *finite directed graph*, denoted as $G := (V, E)$, where V represents the *vertex set* and E denotes the *directed edge set*, allowing loops and multiple edges. For each edge e in E, let $i(e)$ be the *initial vertex* of e, $t(e)$ be the *terminal vertex* of e, and denote the edge as $v \xrightarrow{e} v'$. We always assume that for each $v \in V$, there exists an edge $e \in E$ such that $i(e) = v$. Given two positive integers $n > m$, we associate each edge e in E with an affine contraction $\psi_e : [0,1]^2 \to [0,1]^2$ given by:

$$\psi_e \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} \xi_1 + x_e \\ \xi_2 + y_e \end{pmatrix}, \quad \text{for } (\xi_1, \xi_2) \text{ in } [0,1]^2,$$

where $(x_e, y_e) \in \{0, 1, \dots, n-1\} \times \{0, 1, \dots, m-1\}$. Let $\Psi = \{\psi_e\}_{e \in E}$ be the collection of all these contractions. Then the triple (V, E, Ψ) becomes a *graph-directed iterated function system*. It is well known that there exists a unique family of attractors $\{X_v\}_{v \in V}$ contained in $[0,1]^2$ satisfying

$$X_v = \bigcup_{e \in E : i(e) = v} \psi_e(X_{t(e)}), \quad \text{for all } v \in V.$$

We refer to $\{X_v\}_{v \in V}$ as a *graph-directed Bedford–McMullen* $(\times m, \times n)$*-carpet family* generated by (V, E, Ψ) and $X := \bigcup_{v \in V} X_v$ as a *graph-directed Bedford–McMullen* $(\times m, \times n)$*-carpet*.

Now, let us define

$$E^\infty := \big\{\omega = \omega_1 \omega_2 \cdots : \omega_i \in E, t(\omega_i) = i(\omega_{i+1}) \text{ for all } i \geq 1\big\},$$

as the collection of *infinite admissible words* along G. For a word $\omega \in E^\infty$, denote $i(\omega) := i(\omega_1)$. It is straightforward to observe that

$$X_v = \left\{ \left(\sum_{i=1}^\infty \frac{x_{\omega_i}}{n^i}, \sum_{i=1}^\infty \frac{y_{\omega_i}}{m^i} \right) : \omega \in E^\infty, i(\omega) = v \right\}, \quad \text{for all } v \in V.$$

It is worth noting that X is a sofic $(\times m, \times n)$-invariant set [14, 15] corresponding to the sofic subshift

$$\mathcal{I} = \big\{(x_{\omega_1}, y_{\omega_1})(x_{\omega_2}, y_{\omega_2}) \cdots : \omega \in E^\infty\big\} \subseteq \big\{\{0, \dots, n-1\} \times \{0, \dots, m-1\}\big\}^{\mathbb{N}}. \tag{1.2}$$

Throughout this note, in accordance with standard conventions, we write $\dim_H$, $\underline{\dim}_B$, $\overline{\dim}_B$, $\dim_A$, $\dim_L$ for the Hausdorff, lower box, upper box, Assouad and lower dimensions, respectively. If $\underline{\dim}_B$ and $\overline{\dim}_B$ coincide, we simply refer to the common value as the box dimension, denoted by $\dim_B$. For further details on these dimensions, refer to [6, 12].

Denote $\pi : \mathbb{R}^2 \to \mathbb{R}$ the projection map onto the second coordinate axis, i.e., $\pi(\xi_1, \xi_2) = \xi_2$. We introduce two sequences $\{\alpha_k\}_{k\geq 1}$, $\{\beta_k\}_{k\geq 1}$ whose values depend on $\{\dim_B \pi(X_v)\}_{v\in V}$, see the details in Subsection 2.2. Taking

$$\alpha = \lim_{k\to\infty} (\alpha_k)^{1/k} \quad \text{and} \quad \beta = \liminf_{k\to\infty} (\beta_k)^{1/k},$$

we prove the next theorem.

Theorem 1.1. *Let $\{X_v\}_{v\in V}$ be a graph-directed Bedford–McMullen $(\times m, \times n)$-carpet family. Write $X = \bigcup_{v\in V} X_v$. We have*

$$\dim_A X = \frac{\log \alpha}{\log n} \quad \textit{and} \quad \dim_L X = \frac{\log \beta}{\log n}. \tag{1.3}$$

The result in Theorem 1.1 can be seen as complementary to the Hausdorff and box dimensions of X considered in [14, 15], noting that Assouad and lower dimensions are more sensitive to the extreme local structure of fractals, whereas Hausdorff and box dimensions reveal more global geometric information.

Remark 1.2. (a) We should point out that once Theorem 1.1 is established, we can obtain the Assouad dimension for each X_v, $v \in V$, since $\dim_A X_v = \dim_A \bigcup_{v'\in V_v} X_{v'}$ noticing that $\{X_{v'}\}_{v'\in V_v}$ is a carpet family generated by the IFS (V_v, E_v, Ψ_v) with $V_v := \{v' \in V : v \to \cdots \to v'\} \cup \{v\}$, $E_v := \{e \in E : i(e) \in V_v, t(e) \in V_v\}$ and $\Psi_v := \{\psi_e \in \Psi : e \in E_v\}$. For example, consider a directed graph as shown in Figure 1, where $V_{v_2} = \{v_2, v_3\}$, $E_{v_2} = \{e_4, e_5, e_6, e_7\}$, and $\Psi_{v_2} = \{\psi_e \in \Psi : e \in E_{v_2}\}$. The Assouad dimension of X_{v_2} can be obtained by applying Theorem 1.1 to the subsystem $(V_{v_2}, E_{v_2}, \Psi_{v_2})$.

(b) In previous results [2, 15, 23], the *rectangular open set condition (ROSC)* is always assumed, i.e., it holds that

$$\bigcup_{e\in E: i(e)=v} \psi_e\big((0,1)^2\big) \subseteq (0,1)^2, \qquad \text{for all } v \in V,$$

and the union is disjoint (see [7]). In [14, 16], for the Hausdorff and box dimensions of sofic $(\times m, \times n)$-invariant sets, the ROSC is dropped. In our setting, the ROSC is also not required.

(c) The symbolic space $\mathcal{I}$ in (1.2) is a sofic subshift, i.e., there is a directed graph (V, E) and a labeling $\mathcal{L} : E \to \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\}$ mapping each

$e \in E$ to (x_e, y_e) such that $\mathcal{I} = \{\mathcal{L}(\omega_1)\mathcal{L}(\omega_2)\cdots : \omega \in E^\infty\}$. We say that $(V, E, \mathcal{L})$ is *right resolving* if at every vertex all outgoing edges have different labels. It is known that every sofic subshift has a right resolving presentation [21, Theorem 3.3.2], meaning that there exist another triple $(V', E', \mathcal{L}')$ which is right resolving, $\mathcal{L}'$ maps to the same set of labels, and the subshift generated by the new graph is the same as the original one. It can be directly seen that (V', E', Ψ') satisfies the ROSC, where Ψ' is the collection of contractions corresponding to $\mathcal{L}'$.

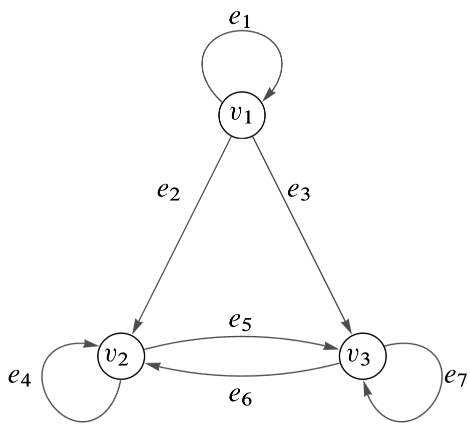


Figure 1. An example of a directed graph (V, E) with sets $V = \{v_1, v_2, v_3\}$ and $E = \{e_1, e_2, \ldots, e_7\}$.

In practice, the values α and β in Theorem 1.1 are difficult to calculate. Recently, Fraser and Jurga [14, Theorem 1.2] derived the box dimension of X via the topological entropy and a projection topological entropy of the sofic subshift corresponding to X. The formula for the Assouad dimension can also be expressed in terms of these topological entropies instead of α.

To illustrate this, let us introduce some additional notation. For $\mathcal{I}$ defined in (1.2), we denote

$$\pi\mathcal{I} = \{y_{\omega_1} y_{\omega_2} \cdots : \omega \in E^\infty\} \subseteq \{0, \ldots, m-1\}^{\mathbb{N}}.$$

Here, somewhat abusing the notation, we continue to use π to represent the "*projection*" from $\mathcal{I}$ onto $\pi\mathcal{I}$, i.e., we write $\pi((x_{\omega_1}, y_{\omega_1})(x_{\omega_2}, y_{\omega_2})\cdots) = y_{\omega_1} y_{\omega_2} \cdots$.

Let $\sigma : \mathcal{I} \to \mathcal{I}$ (resp. $\tilde{\sigma} : \pi\mathcal{I} \to \pi\mathcal{I}$) denote the left shift map. Clearly, $\sigma(\mathcal{I}) \subseteq \mathcal{I}$ and $\tilde{\sigma}(\pi\mathcal{I}) \subseteq \pi\mathcal{I}$. Regard $(\mathcal{I}, \sigma)$ and $(\pi\mathcal{I}, \tilde{\sigma})$ as two topological dynamics and $\pi : (\mathcal{I}, \sigma) \to (\pi\mathcal{I}, \tilde{\sigma})$ as a projection satisfying $\tilde{\sigma} \circ \pi = \pi \circ \sigma$. In a standard manner, we use $h_{\text{top}}(K)$ to denote the *topological entropy* of a compact set K in $\mathcal{I}$ (resp. $\pi\mathcal{I}$). (In our context, we always omit σ (resp. $\tilde{\sigma}$).)

Return to the directed graph $G = (V, E)$. If for each pair $v, v' \in V$, there is a *path* from v to v' (i.e., $v \to \cdots \to v'$), we say G is an *irreducible directed graph*. An *irreducible component* H of G is a maximal subgraph of G such that H is irreducible. Let $\{H_i := (V_i, E_i)\}_{i=1}^r$ denote all the irreducible components of G. For $i = 1, \ldots, r$, denote

$$\mathcal{I}_{H_i} = \{(x_{\omega_1}, y_{\omega_1})(x_{\omega_2}, y_{\omega_2}) \cdots : \omega \in E_i^\infty\} \subseteq \mathcal{I},$$

where E_i^∞ represents the collection of infinite admissible words along H_i. Accordingly, write $\pi \mathcal{I}_{H_i} = \pi(\mathcal{I}_{H_i})$. Let $\{i\}^+$ be the collection of $1 \le j \le r$ such that there is a path from a vertex in H_i to a vertex in H_j. Say H_i is a *source* if $\{i\}^+ = \{1, \ldots, r\}$, and a *sink* if $i \in \{j\}^+$ for all $1 \le j \le r$. Fraser and Jurga [14, Theorem 1.2] derived the following formula for the box dimension of X,

$$\dim_B X = \max_{1 \le i \le r} \left\{ \frac{h_{\mathrm{top}}(\mathcal{I}_{H_i})}{\log n} + \max_{j \in \{i\}^+} h_{\mathrm{top}}(\pi \mathcal{I}_{H_j}) \left(\frac{1}{\log m} - \frac{1}{\log n} \right) \right\}. \tag{1.4}$$

Roughly, it is the maximal box dimension of carpets among all irreducible components. The following theorem says that the Assouad dimension of X has a similar formula.

Theorem 1.3. *Let X be the same as in Theorem 1.1. We have*

$$\dim_A X = \max_{1 \le i \le r} \left\{ \sup_{y \in \pi \mathcal{I}_{H_i}} \frac{h_{\mathrm{top}}(\pi^{-1}(y) \cap \mathcal{I}_{H_i})}{\log n} + \max_{j \in \{i\}^+} \frac{h_{\mathrm{top}}(\pi \mathcal{I}_{H_j})}{\log m} \right\}.$$

Further, we have a simpler formula for $\dim_A X$ in the case where a source or sink exists and has certain entropy maximizing properties, analogous to [14, Corollary 3.1] for $\dim_B X$.

Corollary 1.4. *Let X be the same as in Theorem 1.1. Suppose that one of the following two conditions holds:*

(a) *for some $1 \le i \le r$, H_i is a source and*

$$\sup_{y \in \pi \mathcal{I}_{H_i}} h_{\mathrm{top}}\big(\pi^{-1}(y) \cap \mathcal{I}_{H_i}\big) = \sup_{y \in \pi \mathcal{I}} h_{\mathrm{top}}\big(\pi^{-1}(y)\big), \tag{1.5}$$

(b) *for some $1 \le i \le r$, H_i is a sink and $h_{\mathrm{top}}(\pi \mathcal{I}_{H_i}) = h_{\mathrm{top}}(\pi \mathcal{I})$.*

Then

$$\dim_A X = \sup_{y \in \pi \mathcal{I}} \frac{h_{\mathrm{top}}(\pi^{-1}(y))}{\log n} + \frac{h_{\mathrm{top}}(\pi \mathcal{I})}{\log m}.$$

Next, we use Theorem 1.3 to explore whether a dichotomy like (1.1) holds for X. To this end, we establish an equivalent condition (see (1.6)) under which the box and Assouad dimensions of X coincide.

Corollary 1.5. *Let X be the same as in Theorem 1.1. We have $\dim_B X = \dim_A X$ if and only if there exists $i \in \{1, \dots, r\}$ such that*

$$\begin{aligned} \dim_B X &= \frac{h_{\mathrm{top}}(\mathcal{I}_{H_i})}{\log n} + h_{\mathrm{top}}(\pi \mathcal{I}_{H_i})\Big(\frac{1}{\log m} - \frac{1}{\log n}\Big), \\ \dim_A X &= \sup_{y \in \pi \mathcal{I}_{H_i}} \frac{h_{\mathrm{top}}(\pi^{-1}(y) \cap \mathcal{I}_{H_i})}{\log n} + \frac{h_{\mathrm{top}}(\pi \mathcal{I}_{H_i})}{\log m}, \\ h_{\mathrm{top}}(\mathcal{I}_{H_i}) &= \sup_{y \in \pi \mathcal{I}_{H_i}} h_{\mathrm{top}}(\pi^{-1}(y) \cap \mathcal{I}_{H_i}) + h_{\mathrm{top}}(\pi \mathcal{I}_{H_i}). \end{aligned} \tag{1.6}$$

Furthermore, if $\dim_B X = \dim_A X$, then

$$\dim_H X = \dim_B X = \dim_A X.$$

However, there is no general dichotomy as in (1.1), since we can provide an example of X satisfying

$$\dim_L X < \dim_H X = \dim_B X < \dim_A X.$$

Theorem 1.6. *If G is an irreducible directed graph, then for any graph-directed Bedford–McMullen carpet X, we have*

$$\dim_H X = \dim_B X \quad \text{implies} \quad \dim_B X = \dim_A X.$$

If G is not irreducible, then there is a graph-directed Bedford–McMullen carpet X such that

$$\dim_H X = \dim_B X < \dim_A X.$$

At last, we provide a simplified expression for the Assouad dimension of irreducible X under the ROSC. Let us define A as the $\#V \times \#V$ *adjacency matrix* of the directed graph $G = (V, E)$, where $A(v, v')$ represents the number of edges in E from v to v' for v, v' in V; and for $0 \le j < m$, let A_j be a $\#V \times \#V$ matrix defined by

$$A_j(v, v') = \#\{e \in E : v \xrightarrow{e} v', y_e = j\}, \qquad \text{for all } v, v' \in V.$$

Note that here $A = \sum_{j=0}^{m-1} A_j$. The next corollary suggests that, similar to [10, 22], in the graph-directed setting the Assouad dimension of X should be still the sum of $\dim_B \pi(X)$ and the maximal dimension of slices orthogonal to $\pi(X)$.

Corollary 1.7. *Let X be the same as in Theorem 1.1. Assume that ROSC holds and G is irreducible. We have*

$$\dim_A X = \dim_B \pi(X) + \frac{1}{\log n} \lim_{k \to \infty} \frac{1}{k} \log \max_{y_1, \dots, y_k \in \{0, 1, \dots, m-1\}} \|A_{y_1} \cdots A_{y_k}\|,$$

where $\|\cdot\|$ represents any matrix norm.

The organization of this note is as follows. In Section 2, we introduce necessary notations and define the numbers α and β. In Section 3, we prove Theorem 1.1 as well as Corollary 1.7 for the Assouad dimension. In Section 4, we prove Theorem 1.1 for the lower dimension. Finally, in Section 5, we prove Theorem 1.3, Corollaries 1.4, 1.5 and Theorem 1.6.

2. Preliminary

Let $\{X_v\}_{v\in V}$ be the same as in Theorem 1.1. In this section, we mainly introduce two numbers α, β which appear in the expression of Assouad and lower dimensions.

2.1. Notations and background

First, let us introduce some notations. Denote

$$E^* := \{w = w_1 \cdots w_k : \omega \in E^\infty, w_i = \omega_i \text{ for all } 1 \le i \le k, k \in \mathbb{N}\} \cup \{\emptyset\}$$

the collection of *finite admissible words*. For $w = w_1 \cdots w_k \in E^*$, denote $|w| = k$ the *length* of w, $i(w) := i(w_1)$, $t(w) := t(w_k)$ the *initial* and *terminal* vertices of w, and write $i(w) \xrightarrow{w} t(w)$. For $l \le k$, denote $w|_l = w_1 \cdots w_l$. Write

$$\psi_w = \psi_{w_1} \circ \psi_{w_2} \circ \cdots \circ \psi_{w_k}, \quad x_w = x_{w_1} x_{w_2} \cdots x_{w_k}, \quad y_w = y_{w_1} y_{w_2} \cdots y_{w_k}$$

for short and denote $\psi_\emptyset = id$ (the identity map) by convention. Denote E^k the collection of admissible words of *length* k. For $w, w' \in E^*$ with $i(w') = t(w)$, write ww' the *concatenation* of w and w'.

For $v, v' \in V$, we say there exists a *directed path* from v to v' if there exists $w \in E^*$ satisfying $v \xrightarrow{w} v'$ (write simply $v \to v'$ when we do not emphasize w). Note that $G = (V, E)$ is irreducible if and only if $v \to v'$ for each pair $v, v' \in V$.

For $w, w' \in E^*$, we write $w \sim w'$ if and only if $\psi_w = \psi_{w'}$. Denote $[w] = \{w' \in E^* : w' \sim w\}$ the equivalence class of w. We use $[E^*]$ to denote the collection of all equivalence classes with respect to "$\sim$", i.e.,

$$[E^*] = \big\{[w] : w \in E^*\big\}.$$

In a same way, denote $[E^k] = \{[w] : w \in E^k\}$. Denote

$$t(v, [w]) = \{t(w') : w' \in [w], i(w') = v\}$$

the collection of terminal vertices of $[w]$ from v and write $t([w]) = \bigcup_{v\in V} t(v, [w])$.

Write $\Sigma_X = \{0, 1, \ldots, n-1\}$ and $\Sigma_Y = \{0, 1, \ldots, m-1\}$, the *alphabets* along the first and second coordinate axis, respectively. Denote Σ_X^k (resp. Σ_Y^k) the collection of corresponding words of length k, and write Σ_X^0 (resp. Σ_Y^0) $= \{\emptyset\}$ for convenience. Let $\Sigma_X^* = \bigcup_{k\geq 0} \Sigma_X^k$ and $\Sigma_Y^* = \bigcup_{k\geq 0} \Sigma_Y^k$. Clearly, for $w \in E^k$, $x_w \in \Sigma_X^k$, $y_w \in \Sigma_Y^k$.

For $e \in E$, define

$$\phi_e(\cdot) := \frac{1}{m}(\cdot + y_e).$$

For $w \in E^k$, write $\phi_w = \phi_{w_1} \circ \phi_{w_2} \circ \cdots \circ \phi_{w_k}$ for short. It is easy to see that $\phi_w = \phi_{w'}$ for $w, w' \in E^k$ with $y_w = y_{w'}$.

Proposition 2.1. *For each $v \in V$, the box dimension of $\pi(X_v)$ exists, and*

$$\dim_H \pi(X_v) = \dim_B \pi(X_v). \tag{2.1}$$

In particular, if G is irreducible, we have

$$\dim_H \pi(X_v) = \dim_B \pi(X_v) = \dim_H \pi(X_{v'}) = \dim_B \pi(X_{v'}), \quad \textit{for } v, v' \in V. \tag{2.2}$$

Proof. Note that $\{\pi(X_v)\}_{v\in V}$ is a family of graph-directed self-similar sets generated by a graph-directed iterated function system (V, E, Φ), where

$$\Phi = \{\phi_e : e \in E\}.$$

By [4, Theorem 2.7], (V, E, Φ) satisfies a finite type overlapping condition. Combining this with [4, Theorem 1.1], we see that $\dim_B \bigcup_{v\in V} \pi(X_v)$ exists and

$$\dim_B \bigcup_{v\in V} \pi(X_v) = \dim_H \bigcup_{v\in V} \pi(X_v). \tag{2.3}$$

For each $v \in V$, if v is a *root* of G (i.e., $v \to v'$ for all other vertices v' in V), we have

$$\dim_H \bigcup_{v'\in V} \pi(X_{v'}) = \dim_H \pi(X_v) \leq \overline{\dim}_B \pi(X_v) \leq \dim_B \bigcup_{v'\in V} \pi(X_{v'}).$$

So, $\dim_B \pi(X_v)$ exists and (2.1) holds, by (2.3).

If v is not a root of G, define

$$\begin{aligned} V_v &= \{v' \in V : v \to v'\} \cup \{v\}, \\ E_v &= \{e \in E : i(e), t(e) \in V_v\}, \\ \Phi_v &= \{\phi_e \in \Phi : e \in E_v\}. \end{aligned}$$

Clearly, $\pi(X_v)$ belongs to the family of attractors generated by (V_v, E_v, Φ_v), and v is a root of (V_v, E_v). By a same argument as above, we still get (2.1).

If G is irreducible, it is straightforward to see (2.2). ∎

2.2. Two numbers α and β

Let $\eta : V \times \bigcup_{k\geq 0}(\Sigma_X^k \times \Sigma_Y^k) \to \{\dim_B \pi(X_v) : v \in V\} \cup \{0\}$ be a function defined by

$$\eta(v,x,y) = \begin{cases} \dim_B \pi(X_v \cap \psi_w((0,1)^2)) & \text{if } i(w) = v, (x_w, y_w) = (x,y) \\ & \qquad \text{for some } w \in E^*, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Clearly, $\eta(v,\emptyset,\emptyset) = \dim_B \pi(X_v)$, $\eta(v,x_w,y_w) = \eta(v,x_{w'},y_{w'})$ for $w \sim w'$, and

$$\eta(v,x_w,y_w) = \dim_B \bigcup_{v'\in t(v,[w])} \pi(X_{v'}) = \max_{v'\in t(v,[w])} \dim_B \pi(X_{v'}). \tag{2.5}$$

For $w, w' \in E^*$ with $t(w) = i(w')$, it is not hard to check that

$$\eta\big(i(w),x_w,y_w\big) \geq \eta\big(i(w),x_{ww'},y_{ww'}\big) = \max_{v\in t(i(w),[w])} \eta(v,x_{w'},y_{w'}). \tag{2.6}$$

Define

$$\tilde{V} = \big\{v \in V : \text{ there exists } w \in E^* \text{ with } |w| \geq \#V \text{ and } t(w) = v\big\}$$

the collection of vertices in V which are terminal vertices of some admissible words with length at least $\#V$.

For $k \geq 1$, define

$$\alpha_k := \max_{v\in\tilde{V}} \max_{y\in\Sigma_Y^k} \sum_{\substack{[w]\in[E^k]:\\ i(w)=v, y_w=y}} n^{k\eta(v,x_w,y_w)}. \tag{2.7}$$

Lemma 2.2. *The limit of* $\{(\alpha_k)^{1/k}\}$ *exists and equals* $\inf_{k\geq 1}(\alpha^k)^{1/k}$.

Proof. For k, l in $\mathbb{N}$ with $k \geq \#V$, we have

$$\begin{aligned}
&\alpha_{k+l}\\
&= \max_{v\in\tilde{V}} \max_{y\in\Sigma_Y^{k+l}} \sum_{\substack{[w]\in[E^{k+l}]:\\ i(w)=v, y_w=y}} n^{(k+l)\eta(v,x_w,y_w)}\\
&\leq \max_{v\in\tilde{V}} \max_{y\in\Sigma_Y^k, y'\in\Sigma_Y^l} \sum_{\substack{[w]\in[E^k]:\\ i(w)=v, y_w=y}} n^{k\eta(v,x_w,y_w)} \sum_{\substack{[w']\in[E^l]:\\ i(w')\in t(v,[w]), y_{w'}=y'}} \max_{v'\in t(v,[w])} n^{l\eta(v',x_{w'},y_{w'})}\\
&\leq \max_{v\in\tilde{V}} \max_{y\in\Sigma_Y^k, y'\in\Sigma_Y^l} \sum_{\substack{[w]\in[E^k]:\\ i(w)=v, y_w=y}} n^{k\eta(v,x_w,y_w)} \sum_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^l]:\\ i(w')=v', y_{w'}=y'}} n^{l\eta(v',x_{w'},y_{w'})}\\
&\leq \#V \max_{v\in\tilde{V}} \max_{y\in\Sigma_Y^k} \sum_{\substack{[w]\in[E^k]:\\ i(w)=v, y_w=y}} n^{k\eta(v,x_w,y_w)} \max_{v'\in\tilde{V}} \max_{y'\in\Sigma_Y^l} \sum_{\substack{[w']\in[E^l]:\\ i(w')=v', y_{w'}=y'}} n^{l\eta(v',x_{w'},y_{w'})}\\
&= \#V\alpha_k\alpha_l,
\end{aligned}$$

where the third line follows from (2.6), and the fifth line follows from the fact that $t(v,[w]) \subseteq \tilde{V}$ since $|w| \geq \#V$. Therefore, $\{\alpha_k\}_{k\geq 1}$ is a submultiplicative sequence and the lemma follows. ∎

Write

$$\alpha = \lim_{k\to\infty} (\alpha_k)^{1/k}.$$

In Section 3, we prove that the Assouad dimension of X is $\frac{\log \alpha}{\log n}$.

Remark 2.3. (a) When the directed graph G is irreducible, it is easy to see that $\tilde{V} = V$ and $\dim_B \pi(X_v) = \dim_B \pi(X_{v'})$ for v, v' in V (by Proposition 2.1). Write $\lambda := \dim_B \pi(X_v)$ for $v \in V$. Then α_k defined in (2.7) is equal to

$$\alpha_k = n^{k\lambda} \cdot \max_{v\in V} \max_{y\in\Sigma_Y^k} \#\big\{[w] \in [E^k] : i(w) = v, y_w = y\big\}.$$

When the ROSC also holds, we can further simplify α_k to

$$\begin{aligned}\alpha_k &= n^{k\lambda} \cdot \max_{v\in V} \max_{y\in\Sigma_Y^k} \#\big\{w \in E^k : i(w) = v, y_w = y\big\} \\ &= n^{k\lambda} \cdot \max_{y_1,\ldots,y_k\in\{0,1,\ldots,m-1\}} \|A_{y_1}\cdots A_{y_k}\|_\infty,\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the maximum row sum norm.

(b) When G is not necessarily irreducible, let $\{H_i = (V_i, E_i)\}_{i=1}^r$ be the collection of irreducible components of G. Then, for v, v' in V_i, it is easy to see that $\dim_B \pi(X_v) = \dim_B \pi(X_{v'})$. So, for $i = 1,\ldots,r$, we can write $\lambda^{(i)} := \dim_B \pi(X_v)$ for $v \in V_i$. For $k \geq 1$, if we define

$$\alpha_k^{(i)} = n^{k\lambda^{(i)}} \cdot \max_{v\in V_i} \max_{y\in\Sigma_Y^k} \#\big\{[w] \in [E^k] : i(w) = v, t(w) \in V_i, y_w = y\big\},$$

then the limit $\lim_{k\to\infty}(\alpha_k^{(i)})^{1/k}$ exists for each i.

Lemma 2.4. *For each $i = 1,\ldots,r$, the limit $\alpha^{(i)} := \lim_{k\to\infty}(\alpha_k^{(i)})^{1/k}$ exists and*

$$\max_{i=1,\ldots,r} \alpha^{(i)} = \alpha.$$

Proof. Note that $\dim_B \pi(X_v) \geq \dim_B \pi(X_{v'})$ if $v \to v'$. It is not hard to see that there exists $C > 0$ such that for $k \geq 1$ v in $\tilde{V}$, y in Σ_Y^k,

$$\sum_{\substack{[w]\in[E^k]:\\ i(w)=v, y_w=y}} n^{k\eta(v,x_w,y_w)} \leq C \cdot \sum_{1\leq i_1,\ldots,i_j\leq r} \sum_{s_{i_1}+\cdots+s_{i_j}=k} \alpha_{s_{i_1}}^{(i_1)} \cdots \alpha_{s_{i_j}}^{(i_j)},$$

which immediately yields that $\alpha \le \max_{i=1,\dots,r} \alpha^{(i)}$ (by the same argument considering the convergence radius of $\sum_{k\ge 1} \alpha_k t^k$ in the proofs of [14, Lemma 3.4] and [15, Lemma 3.3]).

Conversely, since $\bigcup_{i=1}^{r} V_i \subseteq \tilde{V}$, for $i = 1, \dots, r$, we have

$$\alpha_k \ge \frac{1}{\#\tilde{V}} \sum_{v\in\tilde{V}} \max_{y\in\Sigma_Y^k} \sum_{\substack{[w]\in[E^k]:\\ i(w)=v, y_w=y}} n^{k\eta(v,x_w,y_w)} \ge \frac{1}{\#\tilde{V}} \alpha_k^{(i)}.$$

The lemma follows. ■

Let $\theta : V \times \bigcup_{k\ge 0}(\Sigma_X^k \times \Sigma_Y^k) \times \Sigma_Y^* \to \{\dim_B \pi(X_v) : v \in V\} \cup \{0\}$ be a function defined by

$$\theta(v,x,y,y') = \begin{cases} \eta(v,x,y) & \text{if } y' = \emptyset,\\ \dim_B \pi(X_v \cap \psi_w((0,1)^2)) \cap \phi_w(I_{y'}) & \text{if } y' \ne \emptyset,\\ 0 & \text{otherwise,} \end{cases}$$

where w is the same as that in the definition of η in (2.4) and

$$I_{y'} = \left(\sum_{i=1}^{l} \frac{y_i'}{m^i}, \sum_{i=1}^{l} \frac{y_i'}{m^i} + \frac{1}{m^l}\right), \quad \text{for } y' = y_1' \cdots y_l',\ l \ge 1. \tag{2.8}$$

Clearly,

$$\begin{aligned} \theta(v,x_w,y_w,y') &= \dim_B\Bigg(\bigcup_{\substack{w'\in E^l: y_{w'}=y',\\ i(w')\in t(v,[w])}} \phi_w \circ \phi_{w'}(\pi(X_{t(w')}))\Bigg)\\ &= \max_{\substack{w'\in E^l: y_{w'}=y',\\ i(w')\in t(v,[w])}} \dim_B \pi(X_{t(w')}). \end{aligned}$$

For $k \ge 1$, define

$$\beta_k = \min_{\substack{w\in E^*:\\ |w|\ge \#V}} \max_{v\in t([w])} \Bigg\{\min\Bigg\{ \sum_{\substack{[w']\in[E^k]:\\ i(w')=v, y_{w'}=y}} n^{k\theta(v,x_{w'},y_{w'},y')} \cdot 1_{\{\theta(v,x_{w'},y_{w'},y')>0\}} : \sum_{\substack{[w']\in[E^k]:\\ i(w')=v, y_{w'}=y}} \theta(v,x_{w'},y_{w'},y') > 0,\ y\in\Sigma_Y^k,\ y'\in\Sigma_Y^* \Bigg\}\Bigg\}.$$

Write

$$\beta = \liminf_{k\to\infty} (\beta_k)^{1/k}. \tag{2.9}$$

In Section 4, we prove that the lower dimension of X is $\frac{\log\beta}{\log n}$.

Remark 2.5. For $v \in V$, $w \in E^*$, $i(w) = v$, we have

$$\pi\big(X_v \cap \psi_w((0,1)^2)\big) \cap \phi_w(I_{y'}) \subseteq \pi\big(X_v \cap \psi_w([0,1]^2)\big),$$

which gives $\eta(v, x_w, y_w) = \theta(v, x_w, y_w, \emptyset) \geq \theta(v, x_w, y_w, y')$, for all y' in Σ_Y^*. In addition,

$$\alpha_k = \max_{v \in \tilde{V}} \max_{y \in \Sigma_Y^k} \max_{y' \in \Sigma_Y^*} \sum_{\substack{[w] \in [E^k]: \\ i(w)=v, y_w = y}} n^{k\theta(v, x_w, y_w, y')}.$$

2.3. Approximate squares

Before proceeding further, let us look at the definition of the Assouad dimension. Given a bounded set $F \subseteq \mathbb{R}^d$, the *Assouad dimension* of F is

$$\dim_A F = \inf\Big\{s \geq 0 : \exists C > 0, \text{ for any } 0 < r < R \leq 1 \text{ and } x \in F,$$
$$\mathcal{N}_r(F \cap B(x,R)) \leq C\Big(\frac{R}{r}\Big)^s\Big\},$$

where $\mathcal{N}_r(E)$ is the least number of balls of radius r covering the set E, and $B(x,R)$ is the ball with center x in $\mathbb{R}^d$ and radius $R > 0$. See [12] for more details.

An important concept in the dimension theory of self-affine sets is the so-called "approximate square". For $k \geq 1$, choose $l \leq k$ so that $n^l \leq m^k < n^{l+1}$, that is, $l = \lfloor k \log_n m \rfloor$, the maximal integer less than or equal to $k \log_n m$. For p, q in $\mathbb{Z}$, denote

$$Q_k(p,q) = \left[\frac{p}{n^l}, \frac{p+1}{n^l}\right] \times \left[\frac{q}{m^k}, \frac{q+1}{m^k}\right]$$

the *approximate square* of level k at (p,q).

For a bounded set $E \subseteq \mathbb{R}^2$, for $k \geq 1$, let $N_k(E)$ be the least number of elements in $\{Q_k(p,q) : p, q \in \mathbb{Z}\}$ covering the set E. In a standard way, we may replace $\mathcal{N}_r(X \cap B(x,R))$ with

$$N_{k'}\big(X \cap Q_k(p,q)\big)$$

in the definition of the Assouad dimension of X, i.e.,

$$\dim_A X = \inf\big\{s \geq 0 : \exists C > 0, \text{ for any } k' > k \text{ and } Q_k(p,q)$$
$$\text{with } X \cap Q_k(p,q) \neq \emptyset, N_{k'}(X \cap Q_k(p,q)) \leq C m^{(k'-k)s}\big\}.$$

Similar to the 2-dimensional case, for a bounded set $E \subseteq \mathbb{R}$, $k \geq 1$, we write $N_k(E)$ the least number of intervals in the form $[\frac{i}{m^k}, \frac{i+1}{m^k}]$ $(i \in \mathbb{Z})$ covering E.

Lemma 2.6. *For any $\epsilon > 0$, there exists a constant $c_1 > 0$, such that for each $v \in V$, $w \in E^*$ with $i(w) = v$, $k \geq 1$ and $y \in \Sigma_Y^* \setminus \{\emptyset\}$, we have*

$$c_1^{-1} m^{k(\eta(v,x_w,y_w)-\epsilon)} \leq N_{k+|w|}\big(\pi(X_v \cap \psi_w((0,1)^2))\big) \leq c_1 m^{k(\eta(v,x_w,y_w)+\epsilon)} \tag{2.10}$$

and

$$\begin{aligned} c_1^{-1} m^{k(\theta(v,x_w,y_w,y)-\epsilon)} &\leq N_{k+|w|+|y|}\big(\pi(X_v \cap \psi_w((0,1)^2)) \cap \phi_w(I_y)\big) \\ &\leq c_1 m^{k(\theta(v,x_w,y_w,y)+\epsilon)}, \end{aligned} \tag{2.11}$$

for those

$$I_y = \Big(\sum_{i=1}^{|y|} \frac{y_i}{m^i}, \sum_{i=1}^{|y|} \frac{y_i}{m^i} + \frac{1}{m^{|y|}}\Big)$$

satisfying $\pi(X_v \cap \psi_w((0,1)^2)) \cap \phi_w(I_y) \neq \emptyset$.

Proof. For $v \in V$, write $\lambda_v := \dim_B \pi(X_v)$ for short. It follows immediately from the definition of box dimension that there exists a constant $C > 0$, such that for $k \geq 1$ we have

$$C^{-1} m^{k(\lambda_v - \epsilon)} \leq N_k\big(\pi(X_v)\big) \leq C m^{k(\lambda_v+\epsilon)}, \qquad \text{for all } v \in V. \tag{2.12}$$

Note that

$$\begin{aligned} N_{k+|w|}\big(\pi(X_v \cap \psi_w((0,1)^2)\big) &= N_{k+|w|}\Big(\phi_w\Big(\bigcup_{v' \in t(v,[w])} \pi(X_{v'})\Big)\Big) \\ &= N_k\Big(\bigcup_{v' \in t(v,[w])} \pi(X_{v'})\Big). \end{aligned}$$

Combining (2.5) and (2.12), we have

$$\begin{aligned} N_{k+|w|}\big(\pi(X_v \cap \psi_w((0,1)^2)\big) &\leq \#V \max_{v' \in t(v,[w])} N_k\big(\pi(X_{v'})\big) \\ &\leq \#V \max_{v' \in t(v,[w])} C m^{k(\lambda_{v'}+\epsilon)} \\ &= \#V \cdot C m^{k(\eta(v,x_w,y_w)+\epsilon)}, \end{aligned}$$

and

$$\begin{aligned} N_{k+|w|}\big(\pi(X_v \cap \psi_w((0,1)^2)\big) &\geq \max_{v' \in t(v,[w])} N_k\big(\pi(X_{v'})\big) \\ &\geq \max_{v' \in t(v,[w])} C^{-1} m^{k(\lambda_{v'}-\epsilon)} \\ &= C^{-1} m^{k(\eta(v,x_w,y_w)-\epsilon)}, \end{aligned}$$

which gives (2.10) by taking $c_1 = \#V \cdot C$.

Noticing that

$$\begin{aligned} N_{k+|w|+|y|}\big(\pi(X_v \cap \psi_w((0,1)^2) \cap \phi_w(I_y)\big) &= N_{k+|w|+|y|}\bigg(\phi_w\bigg(\bigcup_{\substack{w'\in E^l: y_{w'}=y,\\ i(w')\in t(v,[w])}} \phi_{w'} \circ \pi(X_{t(w')})\bigg)\bigg) \\ &= N_k\bigg(\bigcup_{\substack{w'\in E^l: y_{w'}=y,\\ i(w')\in t(v,[w])}} \pi(X_{t(w')})\bigg), \end{aligned}$$

equation (2.11) follows in a similar way. ∎

3. Assouad dimension

The main purpose of this section is to prove Theorem 1.1 and Corollary 1.7 for the Assouad dimension.

For $k \geq 1$, write

$$\tau_k = \max_{v\in V} \#\big\{[w] : w \in E^k, i(w) = v\big\}.$$

The following lemma is useful for the upper bound estimate of the Assouad dimension.

Lemma 3.1. *The limit of $\{(\tau_k)^{1/k}\}_{k\geq 1}$ exists and $\tau := \lim_{k\to\infty}(\tau_k)^{1/k} \leq \alpha$.*

Proof. The existence of the limit of $\{(\tau_k)^{1/k}\}_{k\geq 1}$ follows in a similar way as the proof of Lemma 2.2.

We prove $\tau \leq \alpha$ through the following steps: (i) $\frac{\log \tau}{\log n} \leq \dim_B X$, (ii) $\dim_B X \leq \frac{\log \alpha}{\log n}$ (the existence of $\dim_B X$ is ensured in [14]).

Step (i): $\frac{\log \tau}{\log n} \leq \dim_B X$.

By the definition of τ, for $\epsilon > 0$, there exists $C > 0$ such that for $l \geq 1$, we have

$$\max_{v\in V} \#\big\{[w] : w \in E^l, i(w) = v\big\} = \tau_l \geq C(\tau - \epsilon)^l.$$

For each $k \geq 1$, choosing $l = \lfloor k \log_n m \rfloor$, we have

$$N_k(X) \geq \max_{v\in V} N_k(X_v) \geq C(\tau - \epsilon)^l.$$

So $\dim_B X \geq \frac{\log(\tau-\epsilon)}{\log n}$, thus $\dim_B X \geq \frac{\log \tau}{\log n}$ by the arbitrary choice of ϵ.

Step (ii): $\dim_B X \le \frac{\log \alpha}{\log n}$. Write $\zeta = \dim_B X$. For each k, write $l(k) = \lfloor k \log_n m \rfloor$ and $l'(k) = \lfloor l(k) \log_n m \rfloor$. Note that for $\epsilon > 0$, there exists $C' > 0$, such that for any $k \ge 1$, we have

$$N_{l(k)}(X) \le C' m^{l(k)(\zeta+\epsilon)}. \tag{3.1}$$

Fix an integer k and choose a collection $\{Q_{l(k)}(p,q)\}$ that covers X such that $\#\{Q_{l(k)}(p,q)\} = N_{l(k)}(X)$. We now turn to estimate $N_k(X \cap Q^\circ_{l(k)}(p,q))$, where $Q^\circ_{l(k)}(p,q)$ denotes the interior of $Q_{l(k)}(p,q)$.

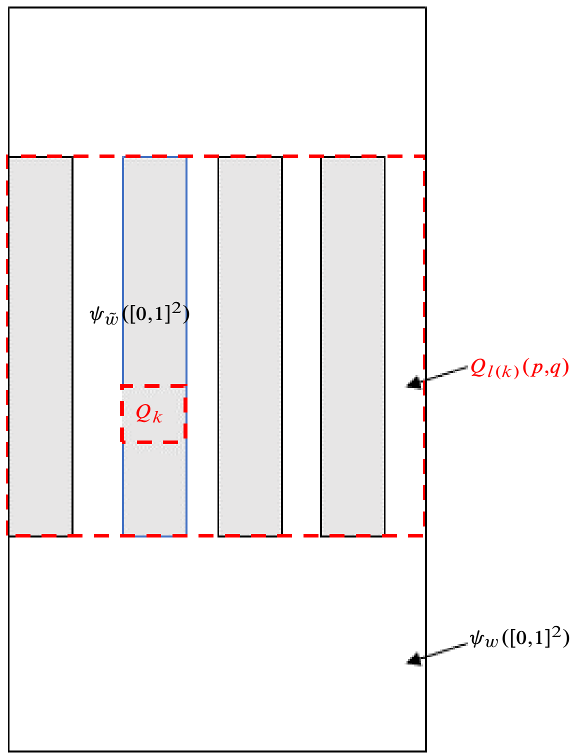


Figure 2. The location of $Q_{l(k)}(p,q)$ (resp. Q_k) in $\psi_w([0,1]^2)$ (resp. $\psi_{\tilde{w}}([0,1]^2)$).

For $v \in V$, assume that $X_v \cap Q^\circ_{l(k)}(p,q) \ne \emptyset$. Then there exists $w \in E^{l'(k)}$ with $i(w) = v$ such that $Q^\circ_{l(k)}(p,q) \subseteq \psi_w((0,1)^2)$. Note that for $\tilde{w} \in E^{l(k)}$, $\psi_{\tilde{w}}([0,1]^2)$ is a rectangle with width $n^{-l(k)}$ and height $m^{-l(k)}$, and the height of $Q_{l(k)}(p,q)$ is $m^{-l(k)}$. By first dividing $Q_{l(k)}(p,q)$ into rectangles in terms of $\psi_{\tilde{w}}([0,1]^2)$ with $\tilde{w} \in E^{l(k)}$, then covering each rectangle with approximate squares of level k (see

Figure 2 for an illustration), we have

$$
\begin{aligned}
&N_k\big(X_v \cap Q^\circ_{l(k)}(p,q)\big)\\
&\le \sum_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^{l(k)-l'(k)}]:i(w')=v',\\ \psi_w\circ\psi_{w'}((0,1)^2)\subseteq Q^\circ_{l(k)}(p,q)}} N_k\big(\psi_w(X_{v'}\cap\psi_{w'}((0,1)^2))\big)\\
&= \sum_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^{l(k)-l'(k)}]:i(w')=v',\\ \psi_w\circ\psi_{w'}((0,1)^2)\subseteq Q^\circ_{l(k)}(p,q)}} N_k\big(\phi_w\circ\pi(X_{v'}\cap\psi_{w'}((0,1)^2))\big)\\
&= \sum_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^{l(k)-l'(k)}]:i(w')=v',\\ \psi_w\circ\psi_{w'}((0,1)^2)\subseteq Q^\circ_{l(k)}(p,q)}} N_{k-l'(k)}\big(\pi(X_{v'}\cap\psi_{w'}((0,1)^2))\big)\\
&\le \sum_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^{l(k)-l'(k)}]:i(w')=v',\\ \psi_w\circ\psi_{w'}((0,1)^2)\subseteq Q^\circ_{l(k)}(p,q)}} c_1 m^{(k-l(k))(\eta(v',x_{w'},y_{w'})+\epsilon)} \qquad \text{by Lemma 2.6}\\
&\le c_1 n^{1+\epsilon} n^{(l(k)-l'(k))\epsilon}\cdot \#V \max_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^{l(k)-l'(k)}]:i(w')=v',\\ \psi_w\circ\psi_{w'}((0,1)^2)\subseteq Q^\circ_{l(k)}(p,q)}} n^{(l(k)-l'(k))\eta(v',x_{w'},y_{w'})}\\
&\le c_1 n^{1+\epsilon} n^{(l(k)-l'(k))\epsilon}\cdot \#V\alpha_{l(k)-l'(k)}. \qquad (3.2)
\end{aligned}
$$

Combining this with (3.1), we have

$$N_k(X) \le C' m^{l(k)(\zeta+\epsilon)}\cdot(\#V)^2 c_1 n^{1+\epsilon} n^{(l(k)-l'(k))\epsilon}\alpha_{l(k)-l'(k)},$$

which yields that

$$\dim_B X \le (\zeta+\epsilon)\log_n m + \left(\frac{\log\alpha}{\log n}+\epsilon\right)(1-\log_n m)$$

by using $l(k)/k \to \log_n m$ as $k\to\infty$. Therefore, $\dim_B X \le \frac{\log\alpha}{\log n} + \frac{1}{1-\log_n m}\epsilon$. So $\dim_B X \le \frac{\log\alpha}{\log n}$ by the arbitrary choice of ϵ. ∎

Now we turn to estimate the upper bound of the Assouad dimension of X.

Lemma 3.2. *For $\epsilon>0$, there exists a constant $C>0$ such that for $1\le k\le k'$ and any $p,q\in\mathbb{Z}$, we have*

$$N_{k'}\big(X\cap Q^\circ_k(p,q)\big) \le C m^{(k'-k)(\frac{\log\alpha+\epsilon}{\log n}+\epsilon)}.$$

Proof. Let $l=\lfloor k\log_n m\rfloor$ and $l'=\lfloor k'\log_n m\rfloor$. It suffices to show that for each $v\in V$, we have

$$N_{k'}\big(X_v\cap Q^\circ_k(p,q)\big) \le C n^{(l'-l)(\frac{\log\alpha+\epsilon}{\log n}+\epsilon)}, \qquad (3.3)$$

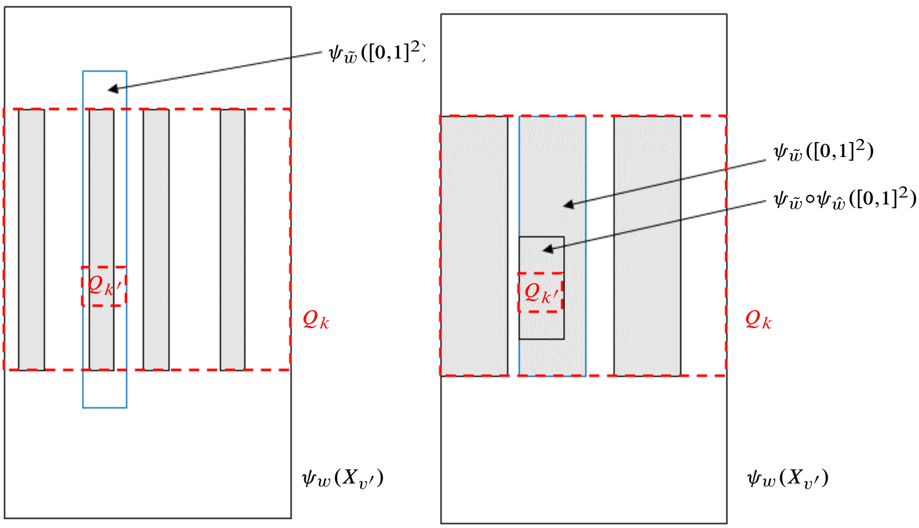
for all $k' \geq k$ with $k \geq \#V \frac{2}{\log_n m}$. Assume that $X_v \cap Q_k^\circ(p,q) \neq \emptyset$. There exists $w \in E^l$ such that $i(w) = v$, $Q_k^\circ(p,q) \subseteq \psi_w((0,1)^2)$. Recall that $\alpha = \lim_{s\to\infty}(\alpha_s)^{1/s}$, there exists S such that for $s \geq S$,

$$\frac{\log \alpha_s}{s} \leq \log \alpha + \epsilon.$$

Therefore, there exists a constant $C_1 > 0$ such that for $s \geq 1$,

$$\alpha_s \leq C_1 n^{s(\frac{\log \alpha + \epsilon}{\log n})}. \tag{3.4}$$

Consider the following two cases:



(i) (ii)

Figure 3. Covering $Q_k^\circ(p,q)$ with approximate squares of level k' in Cases (i), (ii).

Case (i): $l' < k$.

In this case, by first dividing $Q_k(p,q)$ into rectangles in terms of $\psi_{\tilde{w}}([0,1]^2) \cap Q_k(p,q)$ with $\tilde{w} \in E^{l'}$, then covering each rectangle with approximate squares of level k' (see Figure 3-(i)), we have

$$\begin{aligned} &N_{k'}\big(X_v \cap Q_k^\circ(p,q)\big) \\ &\leq \sum_{v' \in t(v,[w])} \sum_{\substack{[w'] \in [E^{l'-l}]: i(w') \in v', \\ \psi_w \circ \psi_{w'}((0,1)^2)) \cap Q_k^\circ(p,q) \neq \emptyset}} N_{k'}\big(\psi_w(X_{v'} \cap \psi_{w'}((0,1)^2)) \cap Q_k^\circ(p,q)\big) \end{aligned}$$

$$\begin{aligned}
&= \sum_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^{l'-l}]:i(w')=v',\\ \psi_w\circ\psi_{w'}((0,1)^2))\cap Q_k^\circ(p,q)\neq\emptyset}} N_{k'}\big(\phi_w\circ\pi(X_{v'}\cap\psi_{w'}((0,1)^2)\cap\psi_w^{-1}(Q_k^\circ(p,q)))\big)\\
&= \sum_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^{l'-l}]:i(w')=v',\\ \psi_w\circ\psi_{w'}((0,1)^2))\cap Q_k^\circ(p,q)\neq\emptyset}} N_{k'-l}\big(\pi(X_{v'}\cap\psi_{w'}((0,1)^2)\cap\psi_w^{-1}(Q_k^\circ(p,q)))\big)
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\leq \sum_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^{l'-l}]:i(w')=v',\\ \psi_w\circ\psi_{w'}((0,1)^2))\cap Q_k^\circ(p,q)\neq\emptyset}} c_1 m^{(k'-k)(\theta(v',x_{w'},y_{w'},y)+\epsilon)} \quad \text{by Lemma 2.6}\\
&\leq \sum_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^{l'-l}]:i(w')=v',\\ \psi_w\circ\psi_{w'}((0,1)^2))\cap Q_k^\circ(p,q)\neq\emptyset}} c_1 m^{(k'-k)(\eta(v',x_{w'},y_{w'})+\epsilon)}\\
&\leq c_1\#V\cdot\alpha_{l'-l}n^{(l'-l+1)\epsilon+1} \quad \text{since } l\geq k\log_n m-1\geq\#V\\
&\leq c_1\#V\cdot n^{\epsilon+1}C_1 n^{(l'-l)(\frac{\log\alpha+\epsilon}{\log n}+\epsilon)} \quad \text{by (3.4)}
\end{aligned}$$

where in the fourth line y is in $\Sigma_Y^{k-l'}$ so that

$$\phi_{w'}^{-1}\circ\pi\circ\psi_w^{-1}\big(Q_k^\circ(p,q)\big)=I_y, \quad \text{recall (2.8)}, \tag{3.6}$$

and the second to last line follows in a same way as (3.2).

Case (ii): $k\leq l'$.

In this case, by first dividing $Q_k(p,q)$ into rectangles in terms of $\psi_{\tilde w}([0,1]^2)$ with $\tilde w\in E^k$, second dividing each rectangle into smaller ones in terms of $\psi_{\tilde w}\circ\psi_{\hat w}([0,1]^2)$ with $\hat w\in E^{l'-k}$, then covering each smaller rectangle with approximate squares of level k' (see Figure 3-(ii)), we have

$$\begin{aligned}
&N_{k'}\big(X_v\cap Q_k^\circ(p,q)\big)\\
&\leq \sum_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^{k-l}]:i(w')=v',\\ \psi_w\circ\psi_{w'}((0,1)^2)\subseteq Q_k^\circ(p,q)}} N_{k'}\big(\psi_w(X_{v'}\cap\psi_{w'}((0,1)^2))\big)\\
&\leq \sum_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^{k-l}]:i(w')=v',\\ \psi_w\circ\psi_{w'}((0,1)^2)\subseteq Q_k^\circ(p,q)}} \sum_{v''\in t(v',[w'])} \sum_{\substack{[w'']\in[E^{l'-k}]:\\ i(w'')=v''}} N_{k'}\big(\psi_w\circ\psi_{w'}(X_{v''}\cap\psi_{w''}((0,1)^2))\big)\\
&= \sum_{v'\in t(v,[w])} \sum_{\substack{[w']\in[E^{k-l}]:i(w')=v',\\ \psi_w\circ\psi_{w'}([0,1]^2)\subseteq Q_k^\circ(p,q)}} \sum_{v''\in t(v',[w'])}
\end{aligned}$$

$$\sum_{\substack{[w'']\in[E^{l'-k}]:\\ i(w'')=v''}} N_{k'}\big(\phi_w\circ\phi_{w'}\circ\pi(X_{v''}\cap\psi_{w''}([0,1]^2))\big)$$

$$\begin{aligned}
&= \sum_{v'\in t(v,[w])}\ \sum_{\substack{[w']\in[E^{k-l}]:i(w')=v',\\ \psi_w\circ\psi_{w'}([0,1]^2)\subseteq Q_k^\circ(p,q)}}\ \sum_{v''\in t(v',[w'])}\ \sum_{\substack{[w'']\in[E^{l'-k}]:\\ i(w'')=v''}} N_{k'-k}\big(\pi(X_{v''}\cap\psi_{w''}([0,1]^2))\big)\\
&\le \sum_{v'\in t(v,[w])}\ \sum_{\substack{[w']\in[E^{k-l}]:i(w')=v',\\ \psi_w\circ\psi_{w'}([0,1]^2)\subseteq Q_k^\circ(p,q)}}\ \sum_{v''\in t(v',[w'])}\ \sum_{\substack{[w'']\in[E^{l'-k}]:\\ i(w'')=v''}} c_1 m^{(k'-l')(\eta(v'',x_{w''},y_{w''})+\epsilon)} \quad \text{by Lemma 2.6}\\
&\le \sum_{v'\in t(v,[w])}\ \sum_{\substack{[w']\in[E^{k-l}]:i(w')=v',\\ \psi_w\circ\psi_{w'}([0,1]^2)\subseteq Q_k^\circ(p,q)}} c_1 m^{(k'-l')(\eta(v',x_{w'},y_{w'}+\epsilon)}\cdot \#V\tau_{l'-k}\\
&\le c_1(\#V)^2\cdot\alpha_{k-l}n^{(k-l+1)\epsilon+1}\cdot\tau_{l'-k} \quad \text{since } l=\lfloor k\log_n m\rfloor\ge\#V.
\end{aligned}$$

On the other hand, by Lemma 3.1, $\log\tau\le\log\alpha$, so there exists $C_2>0$ such that for all $s\ge 1$,

$$\tau_s\le C_2 n^{s(\frac{\log\alpha+\epsilon}{\log n})}. \tag{3.7}$$

Therefore, combining the above estimate with (3.4) and (3.7), we have

$$\begin{aligned}
N_{k'}\big(X_v\cap Q_k^\circ(p,q)\big) &\le c_1 n^{\epsilon+1}(\#V)^2\cdot C_1 n^{(k-l)(\frac{\log\alpha+\epsilon}{\log n}+\epsilon)}\cdot C_2 n^{(l'-k)(\frac{\log\alpha+\epsilon}{\log n})}\\
&\le c_1 n^{\epsilon+1}(\#V)^2\cdot C_1C_2\cdot n^{(l'-l)(\frac{\log\alpha+\epsilon}{\log n}+\epsilon)},
\end{aligned}$$

which gives (3.3). ■

Next, we estimate the Assouad dimension of X from below.

Lemma 3.3. *For any small $\epsilon>0$ and for any $C>0$, there exist $k'\ge k\ge 1$, p,q in $\mathbb{Z}$, such that*

$$N_{k'}\big(X\cap Q_k^\circ(p,q)\big)\ge Cm^{(k'-k)(\frac{\log\alpha-\epsilon}{\log n}-2\epsilon)}.$$

Proof. By the definition of α, there exists $C_1>0$ such that for $s\ge 1$, we have

$$\alpha_s\ge C_1 n^{s(\frac{\log\alpha-\epsilon}{\log n})}. \tag{3.8}$$

Fix s, let $v\in\tilde V$, $y=y_1\cdots y_s\in\Sigma_Y^s$ such that

$$\alpha_s=\sum_{\substack{[w]\in[E^s]:\\ i(w)=v,y_w=y}} n^{s\eta(v,x_w,y_w)}.$$

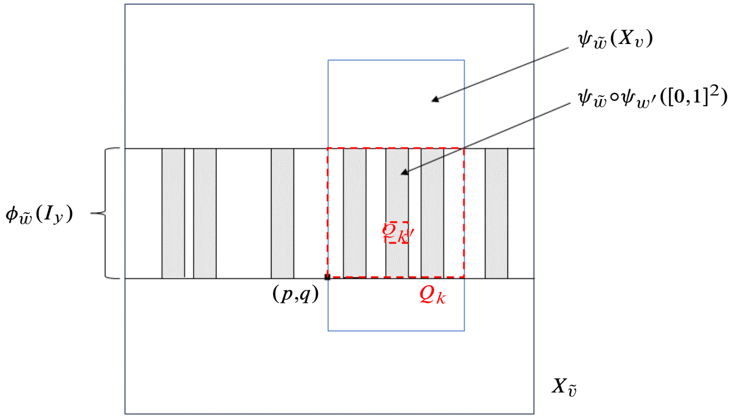


Figure 4. The choice of $Q_k(p,q)$ according to $s, v, \tilde{v}$ and y.

Choose an integer $k \geq 1$ such that $k = \lfloor k \log_n m \rfloor + s$ and let $l = \lfloor k \log_n m \rfloor$. Since $v \in \tilde{V}$, there exists $w \in E^*$ with $|w| \geq \#V$ such that $t(w) = v$. Noticing that $\{i(w_1), \ldots, i(w_{|w|}), t(w)\} \subseteq V$, there must exist $\tilde{w} \in E^l$ such that $t(\tilde{w}) = v$. Write $\tilde{v} = i(\tilde{w})$. Let $p = \sum_{i=1}^{l} x_{\tilde{w}_i} n^{l-i}$ and $q = \sum_{i=1}^{l} y_{\tilde{w}_i} m^{k-i} + \sum_{i=1}^{s} y_i m^{s-i}$, therefore

$$X_{\tilde{v}} \cap Q_k^\circ(p,q) \neq \emptyset.$$

We also choose a $k' \geq 1$ such that $k = \lfloor k' \log_n m \rfloor$. Since $X_{\tilde{v}} \subseteq X$, it suffices to estimate $N_{k'}(X_{\tilde{v}} \cap Q_k^\circ(p,q))$. Similar to the argument in Lemma 3.1 (see Figure 4 for an illustration), we have

$$\begin{aligned}
N_{k'}\big(X_{\tilde{v}} \cap Q_k^\circ(p,q)\big) &\geq N_{k'}\big(\psi_{\tilde{w}}(X_v) \cap Q_k^\circ(p,q)\big) \\
&= \sum_{\substack{[w'] \in [E^s]: \\ i(w')=v, y_{w'}=y}} N_{k'}\big(\psi_{\tilde{w}}(X_v) \cap \psi_{\tilde{w}} \circ \psi_{w'}((0,1)^2)\big) \\
&= \sum_{\substack{[w'] \in [E^s]: \\ i(w')=v, y_{w'}=y}} N_{k'-l}\big(\pi(X_v \cap \psi_{w'}((0,1)^2))\big) \\
&\geq \sum_{\substack{[w'] \in [E^s]: \\ i(w')=v, y_{w'}=y}} c_1^{-1} m^{(k'-k)(\eta(v,x_{w'},y_{w'})-\epsilon)} \qquad \text{by Lemma 2.6} \\
&\geq c_1^{-1} n^{\epsilon-1} n^{-s\epsilon} \alpha_s \geq c_1^{-1} n^{\epsilon-1} C_1 n^{s(\frac{\log\alpha-\epsilon}{\log n}-\epsilon)} \qquad \text{by (3.8)}.
\end{aligned}$$

For any $C > 0$, there exists large enough $s \geq 1$ such that

$$c_1^{-1} n^{\epsilon-1} C_1 n^{-(\frac{\log\alpha-\epsilon}{\log n}-2\epsilon)} n^{s\epsilon} > C.$$

Therefore,

$$\begin{aligned} N_{k'}(X \cap Q_k^\circ(p,q)) &\geq c_1^{-1} n^{\epsilon-1} C_1 n^{s\epsilon} m^{(k'-k-\log_m n)(\frac{\log\alpha-\epsilon}{\log n}-2\epsilon)} \\ &> C m^{(k'-k)(\frac{\log\alpha-\epsilon}{\log n}-2\epsilon)}, \end{aligned}$$

where k, k', p, q are chosen as before according to s. ■

Proof of Theorem 1.1 for the Assouad dimension. Combining Lemmas 3.2 and 3.3, letting ϵ go to 0, we finally obtain $\dim_A X = \frac{\log\alpha}{\log n}$. ■

Proof of Corollary 1.7. By assumption, $\dim_B \pi(X) = \dim_B \pi(X_v)$ for all v in V, and so

$$\eta(v, x_w, y_w) = \dim_B \pi(X),$$

for all $w \in E^*$ with $i(w) = v$. It then follows from Remark 2.3(a),

$$\alpha_k = n^{k \dim_B \pi(X)} \cdot \max_{y \in \Sigma_Y^k} \|A_{y_1} \cdots A_{y_k}\|_\infty.$$

Therefore,

$$\log\alpha = \dim_B \pi(X) \log n + \lim_{k\to\infty} \frac{1}{k} \log \max_{y\in\Sigma_Y^k} \|A_{y_1} \cdots A_{y_k}\|.$$

By Theorem 1.1, the expression of the Assouad dimension follows. ■

4. Lower dimension

In this section, we look at the lower dimension $\dim_L X$ of X, which equals

$$\begin{aligned} \dim_L X = \sup\big\{ s \geq 0 : \exists C > 0,\ \text{for any } k' > k \text{ and } Q_k(p,q) \\ \text{with } X \cap Q_k(p,q) \neq \emptyset, N_{k'}(X \cap Q_k(p,q)) \geq C m^{(k'-k)s} \big\}. \end{aligned}$$

The following lemma aims to the lower bound of lower dimension.

Lemma 4.1. *For any small $\epsilon > 0$, there exists a constant $C > 0$ such that for $k' \geq k \geq 1$ and any $Q_k(p,q)$ with $X \cap Q_k^\circ(p,q) \neq \emptyset$, we have*

$$N_{k'}\big(X \cap Q_k^\circ(p,q)\big) \geq C m^{(k'-k)(\frac{\log\beta-\epsilon}{\log n}-\epsilon)}.$$

Proof. Let $l = \lfloor k \log_n m \rfloor$ and $l' = \lfloor k' \log_n m \rfloor$. It suffices to show that

$$N_{k'}\big(X \cap Q_k^\circ(p,q)\big) \geq C n^{(l'-l)(\frac{\log\beta-\epsilon}{\log n}-\epsilon)},$$

for all $k' \geq k$ with $k \geq \#V \frac{2}{\log_n m}$. Since $X \cap Q_k^\circ(p,q) \neq \emptyset$, there exists $w \in E^l$ such that $Q_k^\circ(p,q) \subseteq \psi_w((0,1)^2)$. Note that $|w| \geq \#V$. According to (2.9), there exists $C_1 > 0$ such that for $s \geq 1$,

$$\beta_s \geq C_1 n^{s(\frac{\log \beta - \epsilon}{\log n})}. \tag{4.1}$$

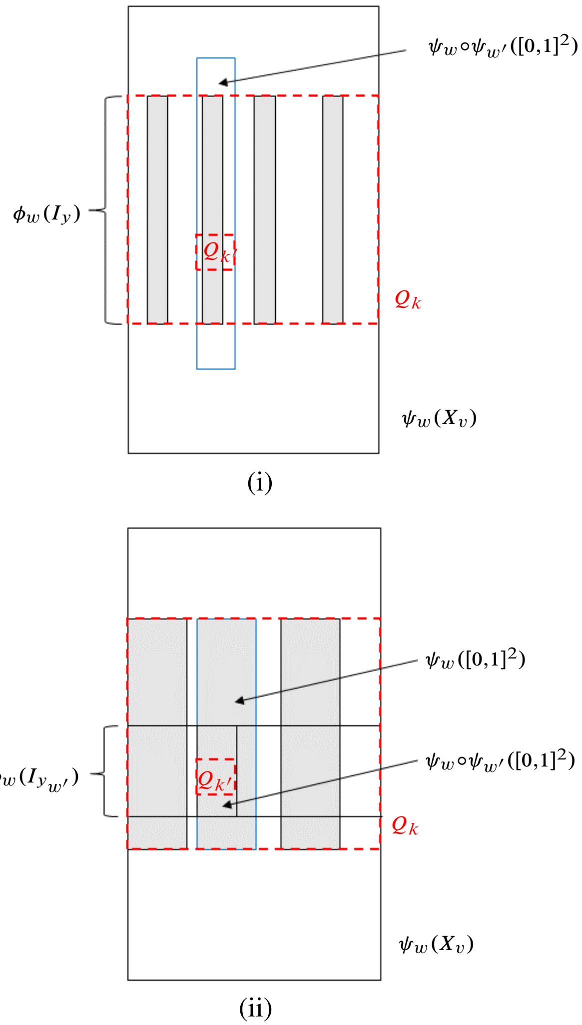


Figure 5. Covering $Q_k^\circ(p,q)$ with approximate squares of level k'.

Case (i): $l' < k$.

This case is similar to Case (i) in Lemma 3.2. By (3.5), we have (see Figure 5-(i))

$$N_{k'}\big(X \cap Q_k^\circ(p,q)\big)$$

$$
\begin{aligned}
&\geq \max_{v\in t([w])} N_{k'}\big(\psi_w(X_v)\cap Q_k^\circ(p,q)\big)\\
&\geq \max_{v\in t([w])} \sum_{\substack{[w']\in[E^{l'-l}]:i(w')=v,\\ \psi_w\circ\psi_{w'}((0,1)^2))\cap Q_k^\circ(p,q)\neq\emptyset}} N_{k'-l}\big(\pi(X_v\cap\psi_{w'}((0,1)^2)\cap\psi_w^{-1}(Q_k^\circ(p,q)))\big)\\
&\geq \max_{v\in t([w])} \sum_{\substack{[w']\in[E^{l'-l}]:i(w')=v,\\ \psi_w\circ\psi_{w'}((0,1)^2))\cap Q_k^\circ(p,q)\neq\emptyset}} c_1^{-1}m^{(k'-k)(\theta(v,x_{w'},y_{w'},y)-\epsilon)}\cdot 1_{\{\theta(v,x_{w'},y_{w'},y)>0\}}\\
&\geq c_1^{-1}n^{\epsilon-1}n^{-(l'-l)\epsilon}\beta_{l'-l}\geq c_1^{-1}n^{\epsilon-1}\cdot C_1 n^{(l'-l)(\frac{\log\beta-\epsilon}{\log n}-\epsilon)} \qquad \text{by (4.1)}
\end{aligned}
$$

where y is in $\Sigma_Y^{k-l'}$ satisfies (3.6) and the fourth line follows from Lemma 2.6.

Case (ii): $k\leq l'$.

Write $p=\sum_{i=1}^{l}p_i n^{l-i}$ and $q=\sum_{i=1}^{k}q_i m^{k-i}$ where p_i (resp. q_i) is in Σ_X (resp. Σ_Y). So that $q_1\cdots q_l=y_w$. Denote $\tilde{y}=q_{l+1}\cdots q_k$. Note that

$$
N_{k'}\big(X\cap Q_k^\circ(p,q)\big)=\sum_{y\in\Sigma_Y^{l'-k}}\ \sum_{\substack{[w']\in[E^{l'-l}]:\\ i(w')\in t([w]),y_{w'}=\tilde{y}y}} N_{k'}\big(X\cap\psi_w\circ\psi_{w'}((0,1)^2)\big).
$$

Write $\kappa(y)=\sum_{\substack{[w']\in[E^{l'-l}]:\\ i(w')\in t([w]),y_{w'}=\tilde{y}y}} N_{k'}(X\cap\psi_w\circ\psi_{w'}((0,1)^2))$, then (Figure 5-(ii))

$$
\begin{aligned}
&N_{k'}\big(X\cap Q_k^\circ(p,q)\big)\\
&\geq \#\big\{y\in\Sigma_Y^{l'-k}:\kappa(y)\neq 0\big\}\cdot\kappa(z)\\
&=N_{l'}\Bigg(\pi\circ\psi_w\Bigg(\bigcup_{v\in t([w])}X_v\Bigg)\cap\pi(Q_k^\circ(p,q))\Bigg)\cdot\kappa(z)\\
&\geq \max_{v\in t([w])} N_{l'-l}\big(\pi(X_v)\cap\pi\circ\psi_w^{-1}(Q_k^\circ(p,q))\big)\cdot\kappa(z)\\
&\geq \max_{v\in t([w])} c_1^{-1}m^{(l'-k)(\theta(v,\emptyset,\emptyset,\tilde{y})-\epsilon)}\cdot\sum_{\substack{[w']\in[E^{l'-l}]:\\ i(w')=v,y_{w'}=\tilde{y}z}} N_{k'}\big(\psi_w(X_v)\cap\psi_w\circ\psi_{w'}((0,1)^2)\big)\\
&\geq \max_{v\in t([w])} c_1^{-1}m^{(l'-k)(\theta(v,\emptyset,\emptyset,\tilde{y})-\epsilon)}\cdot\sum_{\substack{[w']\in[E^{l'-l}]:\\ i(w')=v,y_{w'}=\tilde{y}z}} c_1^{-1}m^{(k'-l')(\eta(v,x_{w'},y_{w'})-\epsilon)}\\
&\geq \max_{v\in t([w])} c_1^{-2}\sum_{\substack{[w']\in[E^{l'-l}]:\\ i(w')=v,y_{w'}=\tilde{y}z}} m^{(k'-k)(\theta(v,x_{w'},y_{w'},\emptyset)-\epsilon)}\\
&\geq c_1^{-2}n^{\epsilon-1}n^{-(l'-l)\epsilon}\beta_{l'-l}\geq c_1^{-2}n^{\epsilon-1}\cdot C_1 n^{(l'-l)(\frac{\log\beta-\epsilon}{\log n}-\epsilon)} \qquad \text{by (4.1)}
\end{aligned}
$$

for some z in $\Sigma_Y^{l'-k}$ with $\kappa(z)=\min\{\kappa(y):\kappa(y)\neq 0,\ y\in\Sigma_Y^{l'-k}\}$, where the fifth and sixth lines are both from Lemma 2.6. ■

Finally, we turn to the upper bound of the lower dimension.

Lemma 4.2. *For any $\epsilon > 0$, for any $C > 0$, there exist $k' \geq k \geq 1$, p, q in $\mathbb{Z}$, such that*

$$N_{k'}\big(X \cap Q_k^\circ(p,q)\big) \leq C m^{(k'-k)(\frac{\log \beta+\epsilon}{\log n}+2\epsilon)}.$$

Proof. It follows from (2.9), there exist $C_1 > 0$ and a sequence $\{s_j\}_{j\geq 1}$ such that

$$\beta_{s_j} \leq C_1 n^{s_j(\frac{\log \beta+\epsilon}{\log n})}. \tag{4.2}$$

Fix large j, let $w \in E^*$ with $|w| \geq \#V$, $y = y_1 \cdots y_{s_j} \in \Sigma_Y^{s_j}$ and $y' = y'_1 \cdots y'_h \in \Sigma_Y^h$ such that

$$\beta_{s_j} = \max_{v \in t([w])} \sum_{\substack{[w'] \in [E^{s_j}]: \\ i(w')=v, y_{w'}=y}} n^{s_j \theta(v,x_{w'},y_{w'},y')} \cdot 1_{\{\theta(v,x_{w'},y_{w'},y')>0\}}.$$

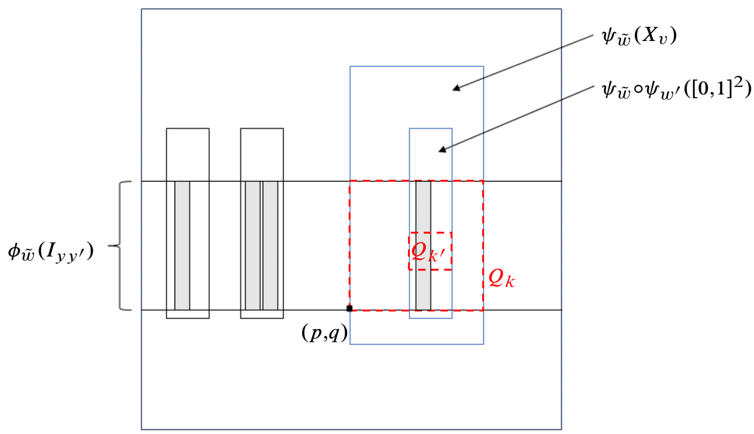


Figure 6. The choice of $Q_k(p,q)$ according to j, y and y'.

Choose a $k \geq 1$ such that $k = \lfloor k \log_n m \rfloor + s_j + h$ and let $l = \lfloor k \log_n m \rfloor$. Since $|w| \geq \#V$ and $\{i(w_1), \ldots, i(w_{|w|}), t(w)\} \subseteq V$, there must exist $\tilde{w} \in E^l$ such that $t(\tilde{w}) \in t([w])$. Let $p = \sum_{i=1}^{l} x_{\tilde{w}_i} n^{l-i}$ and

$$q = \sum_{i=1}^{l} y_{\tilde{w}_i} m^{k-i} + \sum_{i=1}^{s_j} y_i m^{h+s_j-i} + \sum_{i=1}^{h} y'_i m^{h-i}.$$

Clearly, $X \cap Q_k^\circ(p,q) \neq \emptyset$. Choose a $k' \geq 1$ such that $\lfloor k' \log_n m \rfloor = l + s_j$. Denote $l' = \lfloor k' \log_n m \rfloor$. Similar to Case (i) in Lemma 3.2, by (3.5), we have (see Figure 6)

$N_{k'}\big(X \cap Q_k^\circ(p,q)\big)$

$$
\begin{aligned}
&\le \#V \cdot \max_{v\in t([\tilde{w}])} N_{k'}\big(\psi_{\tilde{w}}(X_v) \cap Q_k^\circ(p,q)\big)\\
&\le \#V \cdot \max_{v\in t([\tilde{w}])} \sum_{\substack{[w']\in[E^{l'-l}]:i(w')=v,\\ \psi_{\tilde{w}}\circ\psi_{w'}((0,1)^2)\cap Q_k^\circ(p,q)\neq\emptyset}} N_{k'}\big(\psi_{\tilde{w}}(X_v \cap \psi_{w'}((0,1)^2)) \cap Q_k^\circ(p,q)\big)\\
&\le \#V \cdot \max_{v\in t([\tilde{w}])} \sum_{\substack{[w']\in[E^{l'-l}]:i(w')=v,\\ \psi_{\tilde{w}}\circ\psi_{w'}((0,1)^2)\cap Q_k^\circ(p,q)\neq\emptyset}} c_1 m^{(k'-k)(\theta(v,x_{w'},y_{w'},y')+\epsilon)} \cdot 1_{\{\theta(v,x_{w'},y_{w'},y')>0\}}\\
&\le \#V \cdot c_1 n^{\epsilon+1} n^{s_j\epsilon} \cdot \beta_{s_j} \le \#V \cdot c_1 C_1 n^{\epsilon+1} \cdot n^{s_j(\frac{\log\beta+\epsilon}{\log n}+\epsilon)} \quad \text{by (4.2).}
\end{aligned}
$$

For any $C > 0$, there exists $j \ge 1$ such that $\#V \cdot c_1 C_1 n^{\epsilon+1} n^{\frac{\log\beta+\epsilon}{\log n}+2\epsilon} n^{-s_j\epsilon} < C$, therefore,

$$
\begin{aligned}
N_{k'}\big(X \cap Q_k^\circ(p,q)\big) &\le \#V \cdot c_1 C_1 n^{\epsilon+1} n^{(\frac{\log\beta+\epsilon}{\log n}+2\epsilon)} n^{-s_j\epsilon} m^{(k'-k)(\frac{\log\beta+\epsilon}{\log n}+2\epsilon)}\\
&< C m^{(k'-k)(\frac{\log\beta+\epsilon}{\log n}+2\epsilon)},
\end{aligned}
$$

where k, k', p, q are chosen as before according to j. ∎

Proof of Theorem 1.1 for lower dimension. By combining Lemmas 4.1 and 4.2, we obtain that $\dim_L X = \frac{\log\beta}{\log n}$. ∎

5. Comparison between box and Assouad dimensions

In this section, we prove Theorems 1.3, 1.6 and Corollaries 1.4, 1.5. Recall that

$$
\mathcal{I} = \big\{(x_{\omega_1}, y_{\omega_1})(x_{\omega_2}, y_{\omega_2})\cdots : \omega \in E^\infty\big\} \subseteq \big\{\{0,\ldots,n-1\}\times\{0,\ldots,m-1\}\big\}^{\mathbb{N}}
$$

and

$$
\pi\mathcal{I} = \{y_{\omega_1} y_{\omega_2}\cdots : \omega \in E^\infty\} \subseteq \{0,\ldots,m-1\}^{\mathbb{N}}.
$$

For k in $\mathbb{N}$, denote $\mathcal{I}^k$ (resp. $\pi\mathcal{I}^k$) the collection of words with length k which appear in $\mathcal{I}$ (resp. $\pi\mathcal{I}$), that is

$$
\mathcal{I}^k = \big\{(x_{w_1}, y_{w_1})(x_{w_2}, y_{w_2})\cdots(x_{w_k}, y_{w_k}) : w \in E^k\big\}.
$$

Recall that the *topological entropy* of $\mathcal{I}$, $\pi\mathcal{I}$ and $\pi^{-1}(y)$ (for $y \in \pi\mathcal{I}$) are defined as

$$
\begin{aligned}
h_{\text{top}}(\mathcal{I}) &:= \lim_{k\to\infty} \frac{1}{k}\log\#\mathcal{I}^k,\\
h_{\text{top}}(\pi\mathcal{I}) &:= \lim_{k\to\infty} \frac{1}{k}\log\#\pi\mathcal{I}^k,\\
h_{\text{top}}(\pi^{-1}(y)) &:= \limsup_{k\to\infty} \frac{1}{k}\log\#\{x_w : w \in E^k, y_w = y|_k\}
\end{aligned}
$$

(the existence of the first two limits follow by a submultiplicativity argument). The following inequality is due to Bowen [3]:

$$h_{\text{top}}(\mathcal{I}) \leq h_{\text{top}}(\pi\mathcal{I}) + \sup_{y\in\pi\mathcal{I}} h_{\text{top}}\big(\pi^{-1}(y)\big). \tag{5.1}$$

First, let us look at the irreducible case.

Theorem 5.1. *Let X be the same as in Theorem 1.1. Assume that $G = (V, E)$ is irreducible. We have*

$$\dim_A X = \frac{h_{\text{top}}(\pi\mathcal{I})}{\log m} + \sup_{y\in\pi\mathcal{I}} \frac{h_{\text{top}}(\pi^{-1}(y))}{\log n}.$$

And the following three statements are equivalent:

(i) $\dim_B X = \dim_A X$*,*

(ii) $\dim_H X = \dim_B X$*,*

(iii) *equation* (5.1) *becomes an equality, i.e.,*

$$h_{\text{top}}(\mathcal{I}) = h_{\text{top}}(\pi\mathcal{I}) + \sup_{y\in\pi\mathcal{I}} h_{\text{top}}\big(\pi^{-1}(y)\big). \tag{5.2}$$

Before proving this theorem, we prepare some lemmas. The following lemma is a graph-directed version of the expression of $\dim_A X$ of Mackay [22].

Lemma 5.2. *When G is irreducible, we have*

$$\dim_A X = \frac{h_{\text{top}}(\pi\mathcal{I})}{\log m} + \sup_{y\in\pi\mathcal{I}} \frac{h_{\text{top}}(\pi^{-1}(y))}{\log n}.$$

Proof. For $k \geq 1$, $y \in \Sigma_Y^k$ and $v, v' \in V$, denote

$$\mathcal{I}_{v,v'}^k(y) = \big\{(x_{w_1}, y_{w_1}) \cdots (x_{w_k}, y_{w_k}) : w \in E^k, v \xrightarrow{w} v', y_w = y\big\},$$

$\mathcal{I}_v^k(y) = \bigcup_{v'\in V} \mathcal{I}_{v,v'}^k(y)$ and $\mathcal{I}^k(y) = \bigcup_{v\in V} \mathcal{I}_v^k(y)$. Since G is irreducible,

$$\alpha_k = \max_{v\in V} \max_{y\in\Sigma_Y^k} n^{k\dim_B \pi(X)} \cdot \#\mathcal{I}_v^k(y).$$

Noticing that $\dim_B \pi(X) = \lim_{k\to\infty} \frac{\log \#\pi\mathcal{I}^k}{k\log m} = \frac{h_{\text{top}}(\pi\mathcal{I})}{\log m}$, with Theorem 1.1 in hand, it suffices to prove that

$$\sup_{y\in\pi\mathcal{I}} h_{\text{top}}\big(\pi^{-1}(y)\big) = \lim_{k\to\infty} \max_{v\in V} \max_{y\in\Sigma_Y^k} \frac{\log \#\mathcal{I}_v^k(y)}{k}. \tag{5.3}$$

Since for $y \in \pi \mathcal{I}$,

$$h_{\text{top}}(\pi^{-1}(y)) = \limsup_{k\to\infty} \frac{\log \#\mathcal{I}^k(y|_k)}{k} \le \limsup_{k\to\infty} \max_{v\in V} \frac{\log \#\mathcal{I}_v^k(y|_k) + \log \#V}{k},$$

we have the direction " $\le$" in (5.3). On the other hand, let Δ be the right-hand side of equation (5.3). Then for $\epsilon > 0$, there exists N in $\mathbb{N}$ such that for all k in $\mathbb{N}$, there are v_k, v_k' in V, $w^{(k)}$ in E^{k+N} with $v_k \xrightarrow{w^{(k)}} v_k'$ and $y^{(k)} = y_{w^{(k)}}$, such that

$$\log \#\mathcal{I}_{v_k,v_k'}^{k+N}(y^{(k)}) \ge \log \#\mathcal{I}_{v_k}^{k+N}(y^{(k)}) - \log \#V \ge (k+N)(\Delta - \epsilon) - \log \#V. \quad (5.4)$$

Noticing that G is irreducible, there exists S in $\mathbb{N}$, such that for each distinct pair v, v' in V, there exists $w \in E^*$ with $|w| \le S$ and $v \xrightarrow{w} v'$. For any k in $\mathbb{N}$, pick a directed path $\tilde{w}^{(k)}$ from v_k' to v_{k+1} with length no more than S if $v_k' \ne v_{k+1}$; pick $\tilde{w}^{(k)} = \emptyset$ if $v_k' = v_{k+1}$. Write $\tilde{y}^{(k)} = y_{\tilde{w}^{(k)}}$, $\dot{y} = y^{(1)}\tilde{y}^{(1)}y^{(2)}\tilde{y}^{(2)}\cdots \in \pi\mathcal{I}$, $\dot{y}^{(k)} = y^{(1)}\tilde{y}^{(1)}\cdots y^{(k)}\tilde{y}^{(k)}$ and $s_k = |\dot{y}^{(k)}|$. Noticing that

$$\frac{k(k+1)}{2} + kN \le s_k \le \frac{k(k+1)}{2} + kN + kS,$$

by using (5.4), we have

$$\begin{aligned} h_{\text{top}}\big(\pi^{-1}(\dot{y})\big) &\ge \limsup_{k\to\infty} \frac{\log \#\mathcal{I}^{s_k}(\dot{y}^{(k)})}{s_k} \\ &\ge \limsup_{k\to\infty} \frac{\log \#\mathcal{I}_{v_1,v_1'}^{1+N}(y^{(1)}) + \cdots + \log \#\mathcal{I}_{v_k,v_k'}^{k+N}(y^{(k)})}{s_k} \ge \Delta - \epsilon, \end{aligned}$$

which gives the direction " $\ge$" in (5.3) by the arbitrary choice of ϵ. ■

Lemma 5.3. *When G is irreducible, equality* (5.2) *holds if and only if* $\dim_H X = \dim_B X$.

Proof. The "only if" part. Recall that from [14, Corollary 3.2], $\dim_H X = \dim_B X$ if and only if the measure of maximal entropy on $\mathcal{I}$ projects to the measure of maximal entropy on $\pi\mathcal{I}$, i.e., there exists a σ-invariant measure μ such that

$$h(\mu) = h_{\text{top}}(\mathcal{I}) \quad \text{and} \quad h(\mu \circ \pi^{-1}) = h_{\text{top}}(\pi\mathcal{I}), \quad (5.5)$$

where $h(\mu)$ denotes the *measure entropy* of μ.

It is due to Ledrappier and Walters [20] that there is a relative variational principle for (5.1) in the form

$$\sup_m h(m) = h(\nu) + \int_{\pi\mathcal{I}} h_{\text{top}}\big(\pi^{-1}(y)\big)d\nu(y), \quad (5.6)$$

where the supremum is taken over all the σ-invariant probability measure m satisfying $m \circ \pi^{-1} = \nu$. Choose μ the measure satisfying $h(\mu) = h_{\text{top}}(\mathcal{I})$. Let $\nu = \mu \circ \pi^{-1}$, then we have

$$\begin{aligned} h_{\text{top}}(\mathcal{I}) = \sup_m h(m) = h(\nu) + \int_{\pi \mathcal{I}} h_{\text{top}}\big(\pi^{-1}(y)\big) d\nu(y) \\ \le h_{\text{top}}(\pi \mathcal{I}) + \sup_{y \in \pi \mathcal{I}} h_{\text{top}}\big(\pi^{-1}(y)\big). \end{aligned}$$

Combining this with (5.2), we immediately get $h(\mu \circ \pi^{-1}) = h_{\text{top}}(\pi \mathcal{I})$, which gives $\dim_H X = \dim_B X$.

The "if" part. Since $\dim_H X = \dim_B X$, it follows from [8, Theorem 3.1] ($\mathcal{I}$ is a subshift satisfying the so-called "weak specification" in [8] since G is irreducible), there is a constant $C > 0$ such that

$$\begin{aligned} C^{-1} \#\{x_w : w \in E^k, y_w = y'\} \le \#\{x_w : w \in E^k, y_w = y\} \\ \le C \#\{x_w : w \in E^k, y_w = y'\} \end{aligned}$$

for all $y, y' \in \pi \mathcal{I}^k$ and $k \ge 1$. Therefore, for $y, y' \in \pi \mathcal{I}$,

$$h_{\text{top}}\big(\pi^{-1}(y)\big) = h_{\text{top}}\big(\pi^{-1}(y')\big).$$

Again, by $\dim_H X = \dim_B X$, we can choose a σ-invariant measure μ satisfying (5.5). Let $\nu = \mu \circ \pi^{-1}$, by (5.6), we have

$$\begin{aligned} h_{\text{top}}(\mathcal{I}) = \sup_{m:\, m \circ \pi^{-1} = \nu} h(m) = h(\nu) + \int_{\pi \mathcal{I}} h_{\text{top}}(\pi^{-1}(y)) d\nu(y) \\ = h_{\text{top}}(\pi \mathcal{I}) + \sup_{y \in \pi \mathcal{I}} h_{\text{top}}\big(\pi^{-1}(y)\big). \quad \blacksquare \end{aligned}$$

Proof of Theorem 5.1. By combining Lemmas 5.2 and 5.3, it suffices to prove that $\dim_B X = \dim_A X$ if and only if (5.2) holds. Since G is irreducible, the expression of $\dim_B X$ in (1.4) degenerates to the following form

$$\dim_B X = \frac{h_{\text{top}}(\mathcal{I})}{\log n} + h_{\text{top}}(\pi \mathcal{I}) \left(\frac{1}{\log m} - \frac{1}{\log n} \right).$$

Combining this with Lemma 5.2 and (5.2), the theorem follows. ■

Next, we consider the general case in which the graph G may be not irreducible. Let $\{H_i = (V_i, E_i)\}_{i=1}^r$ be the collection of irreducible components of G. Recall $\lambda^{(i)} = \dim_B \pi(X_v)$ for $v \in V_i$ in Remark 2.3(b). Recall $\{i\}^+$ denotes the collection of $1 \le j \le r$ such that there is a path from a vertex in H_i to a vertex in H_j.

Lemma 5.4. *For $i = 1, \ldots, r$, we have* $\lambda^{(i)} = \max_{j \in \{i\}^+} \frac{h_{\text{top}}(\pi \mathcal{I}_{H_j})}{\log m}$.

Proof. It is the same as in [14, Lemma 3.4], which follows by a same argument in Lemma 2.4. ∎

Proof of Theorem 1.3. Combining Lemmas 2.4, 5.4 and Theorem 5.1, we have

$$\begin{aligned}\dim_A X &= \max_{1\le i\le r}\left\{\sup_{y\in\pi\mathcal{I}_{H_i}}\frac{h_{\text{top}}(\pi^{-1}(y)\cap\mathcal{I}_{H_i})}{\log n}+\lambda^{(i)}\right\}\\ &= \max_{1\le i\le r}\left\{\sup_{y\in\pi\mathcal{I}_{H_i}}\frac{h_{\text{top}}(\pi^{-1}(y)\cap\mathcal{I}_{H_i})}{\log n}+\max_{j\in\{i\}+}\frac{h_{\text{top}}(\pi\mathcal{I}_{H_j})}{\log m}\right\}. \quad ∎\end{aligned}$$

Proof of Corollary 1.4. By Theorem 1.3, it is obvious that

$$\dim_A X \le \sup_{y\in\pi\mathcal{I}}\frac{h_{\text{top}}(\pi^{-1}(y))}{\log n}+\frac{h_{\text{top}}(\pi\mathcal{I})}{\log m}. \tag{5.7}$$

First, suppose condition (a) holds. By [14, Lemma 3.4], there exists $1\le j\le r$ such that $h_{\text{top}}(\pi\mathcal{I}_{H_j})=h_{\text{top}}(\pi\mathcal{I})$. Since H_i is a source, again using Theorem 1.3, noticing (1.5), we have

$$\begin{aligned}\dim_A X &\ge \sup_{y\in\pi\mathcal{I}_{H_i}}\frac{h_{\text{top}}(\pi^{-1}(y)\cap\mathcal{I}_{H_i})}{\log n}+\max_{l\in\{i\}+}\frac{h_{\text{top}}(\pi\mathcal{I}_{H_l})}{\log m}\\ &= \sup_{y\in\pi\mathcal{I}}\frac{h_{\text{top}}(\pi^{-1}(y))}{\log n}+\frac{h_{\text{top}}(\pi\mathcal{I})}{\log m}.\end{aligned}$$

Combining this with the upper bound (5.7) completes the proof.

Next, suppose condition (b) holds. Let H_i be the sink. For any $1\le j\le r$, since $h_{\text{top}}(\pi\mathcal{I}_{H_i})=h_{\text{top}}(\pi\mathcal{I})$, by Theorem 1.3, we have

$$\begin{aligned}\dim_A X &\ge \sup_{y\in\pi\mathcal{I}_{H_j}}\frac{h_{\text{top}}(\pi^{-1}(y)\cap\mathcal{I}_{H_j})}{\log n}+\max_{l\in\{j\}+}\frac{h_{\text{top}}(\pi\mathcal{I}_{H_l})}{\log m}\\ &= \sup_{y\in\pi\mathcal{I}_{H_j}}\frac{h_{\text{top}}(\pi^{-1}(y)\cap\mathcal{I}_{H_j})}{\log n}+\frac{h_{\text{top}}(\pi\mathcal{I})}{\log m}.\end{aligned}$$

Since the above inequality holds for every $1\le j\le r$, by (5.3) and the proof of Lemma 2.4, we obtain

$$\begin{aligned}\dim_A X &\ge \max_{1\le j\le r}\lim_{k\to\infty}\max_{v\in V_j}\max_{y\in\pi\mathcal{I}}\frac{\log\#\mathcal{I}^k_{v,j}(y|_k)}{k\log n}+\frac{h_{\text{top}}(\pi\mathcal{I})}{\log m}\\ &\ge \sup_{y\in\pi\mathcal{I}}\frac{h_{\text{top}}(\pi^{-1}(y))}{\log n}+\frac{h_{\text{top}}(\pi\mathcal{I})}{\log m},\end{aligned}$$

where $\mathcal{I}^k_{v,j}(y|_k)=\{(x_{w_1},y_{w_1})\cdots(x_{w_k},y_{w_k}): w\in E^k, i(w)=v, t(w)\in V_j, y_w=y|_k\}$. This completes the proof. ∎

Proof of Corollary 1.5. If (1.6) holds, it is direct to see that $\dim_B X = \dim_A X$. Conversely, assume that $\dim_B X = \dim_A X$. By (1.4), we can pick (i, j) with $j \in \{i\}^+$ such that

$$\dim_B X = \frac{h_{\mathrm{top}}(\mathcal{I}_{H_i})}{\log n} + h_{\mathrm{top}}(\pi \mathcal{I}_{H_j})\Big(\frac{1}{\log m} - \frac{1}{\log n}\Big).$$

Using (5.1) and $\max_{j' \in \{i\}^+} h_{\mathrm{top}}(\pi \mathcal{I}_{H_{j'}}) = h_{\mathrm{top}}(\pi \mathcal{I}_{H_j})$ we get

$$\begin{aligned}\dim_B X &\le \sup_{y \in \pi \mathcal{I}_{H_i}} \frac{h_{\mathrm{top}}(\pi^{-1}(y) \cap \mathcal{I}_{H_i})}{\log n} + \frac{h_{\mathrm{top}}(\pi \mathcal{I}_{H_j})}{\log m} + \frac{h_{\mathrm{top}}(\pi \mathcal{I}_{H_i}) - h_{\mathrm{top}}(\pi \mathcal{I}_{H_j})}{\log n} \\ &\le \dim_A X,\end{aligned}$$

which implies $h_{\mathrm{top}}(\pi \mathcal{I}_{H_i}) = h_{\mathrm{top}}(\pi \mathcal{I}_{H_j})$ and (1.6) follows.

Finally, let $\{X_v^{(i)}\}_{v \in V_i}$ be the graph-directed Bedford–McMullen $(\times m, \times n)$-carpets family associated with H_i and write $X^{(i)} = \bigcup_{v \in V_i} X_v^{(i)}$. If (1.6) holds, by Lemma 5.3, it holds that $\dim_H X^{(i)} = \dim_B X^{(i)}$ for the same i in the previous paragraph. Thus, we have

$$\dim_H X \ge \dim_H X^{(i)} = \dim_B X^{(i)} = \dim_B X,$$

giving that $\dim_H X = \dim_B X = \dim_A X$. ■

Proof of Theorem 1.6. The first part follows from Theorem 5.1. For the second part, we consider the example of graph-directed Bedford–McMullen carpet family $\{X_a, X_b\}$ generated in the way illustrated in Figure 7.

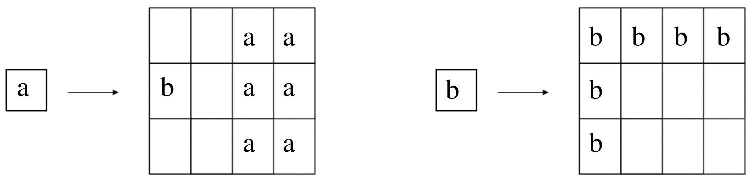


Figure 7. An example of graph-directed Bedford–McMullen carpet family.

At this time, $n = 4$ and $m = 3$. It is not hard to check that $\dim_B \pi(X_a) = \dim_B \pi(X_b) = 1$, and for $X = X_a \cup X_b$, $\dim_A X = 2$ and $\dim_L X = 1$, $\dim_B X = \dim_H X = \frac{3}{2}$. ■

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