

On the uniform Besov regularity of local times of general processes

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Abstract. The α -local nondeterminism notion (α -LND, for short) has been introduced by the authors [Stoch. Partial Differ. Equ. Anal. Comput. 11 (2023), 388–425] to investigate the existence, joint continuity, and uniform Hölder continuity, i.e., Hölder continuity in the time variable t uniformly in the space variable x , for local times $L(x, t)$ of general processes. In the present paper, we aim to use the α -LND property to improve this uniform Hölder continuity of local times to a uniform Besov regularity, i.e., Besov regularity in the time variable t uniformly in the space variable x and in p (the index of the Besov space $\mathbf{B}_{p,\infty}^{\nu}(I; \mathbb{R})$). The Besov regularity of local times, in the time variable t for fixed space variable x , has never been treated in the literature even for Gaussian or stable processes. We also extend the classical Adler’s theorem, i.e., Theorem 8.7.1 [in: The Geometry of Random Fields (1981)] to the Besov spaces case. These results are then exploited to study the uniform (in p) Besov irregularity of the sample paths of the underlying processes. As applications, we get sharp uniform Besov irregularity results for a class of Gaussian processes and the solutions of systems of non-linear stochastic heat equations. The uniform Besov regularity of their corresponding local times is also obtained.

1. Introduction and main results

The local times of d -dimensional paths have gotten much interest during the last few decades by the analytical and probabilistic communities. The occupation measure basically measures the amount of time the path spends in a given set. The local time is defined as the Radon–Nikodym derivative of the occupation measure with respect to the Lebesgue measure. It is of importance in both theory and applications to investigate sample path properties of stochastic processes. One way to study the irregularity properties of the sample paths of stochastic processes is by analyzing the smoothness of their local times. S. Berman initiated this approach in a series of papers [3–5] by using Fourier analytic methods to Gaussian processes. Furthermore, he has introduced the concept of local nondeterminism (LND) for Gaussian processes to

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investigate the existence of jointly continuous local times. Since then, there have been a wide variety of extensions of the notion of local nondeterminism, notably for stable processes [28] and Gaussian random fields [14, 16, 29]; see [18, 20, 40–42] for further details. It is worth mentioning that these definitions of LND, given by Berman, Pitt, and Cuzick, are insufficient to establish fine regularity properties, such as the modulus of continuity for local times or the law of the iterated logarithm. In order to investigate these and other related properties for Gaussian random fields, the concept of *strong local nondeterminism* (SLND) has proven to be more appropriate; we refer to the non-exhaustive list [13, 15, 17, 26, 27, 32, 33, 36–39, 43]. However, it has been well known that the Brownian sheet does not satisfy the SLND property, which led Khoshnevisan and Xiao [22] to introduce the so-called sectorial local-nondeterminism and subsequently show that the Brownian sheet satisfies this property.

In the Gaussian case, the exponential form of the characteristic function allows expressing the LND property in terms of a condition on the variance. Nevertheless, the unknown form of the characteristic functions of general processes leads to difficulties in extending the LND condition beyond the Gaussian framework. Consequently, the LND property used in the Gaussian context should be replaced, for general processes, by fine estimations on the characteristic function of the increments. Recently, based on the conditional Malliavin calculus, Lou and Ouyang [24] have established an upper bound of Gaussian type for the partial derivatives of the n -point joint density of the solution to a stochastic differential equation driven by fractional Brownian motion. They have used this result as an alternative to the LND condition. Due to this, the authors in [24] have shown the existence and regularity of the local times of stochastic differential equations driven by fractional Brownian motions. In [9], a new condition, called α -local nondeterminism (α -LND for short), has been introduced to investigate the existence and joint regularity of the local times of the solutions to systems of nonlinear stochastic heat equations—which are neither Gaussian nor stable processes. Although the α -LND condition has been specifically designed for non-Gaussian processes, in a Gaussian setting, the α -LND property can be seen as a weaker condition than the classical LND (see Remark 2.9(ii)). Generally, the proof of the α -LND condition relies on the technique of integration by parts derived from the Malliavin calculus. Roughly speaking, we believe that the approach presented in [9] can be used to establish the α -LND for a class of adapted stochastic processes that are smooth in the Malliavin sense.

Consider $(X_t)_{t \in [0,1]}$ an $\mathbb{R}^d$ -valued continuous stochastic process, with $X(0) = 0$. Assume that X satisfies the α -LND with $\alpha \in (0, 1)$, see Definition 2.8. Let us state the following hypothesis on X :

H. *There exists $p_0 > \frac{1}{\alpha}$ and $K > 0$ such that for all $0 \leq s \leq t \leq 1$,*

$$\mathbb{E}[\|X_t - X_s\|^{p_0}] \leq K|t - s|^{p_0\alpha}. \quad (1.1)$$

One can get by the same calculations as in [9] (see [9, Remark 5.6]) that, when $\alpha < \frac{1}{d}$, X has a jointly continuous version of the local time, $L(x, t)$, satisfying almost surely a Hölder condition of order γ , for every $\gamma < 1 - d\alpha$, in the time variable t uniformly in the space variable x , i.e., there exist two random variables δ and η which are almost surely in $(0, \infty)$ such that, for all $t, t + h \in [0, 1]$ and all $|h| < \delta$,

$$\sup_{x \in \mathbb{R}^d} |L(x, t + h) - L(x, t)| \leq \eta |h|^\gamma. \quad (1.2)$$

It is well known that when the local time $L(x, t)$ has a jointly continuous version, still denoted by $L(x, t)$, then $L(x, \bullet)$ can be naturally extended to a finite Borel measure supported on the level set:

$$X^{-1}(x) = \{t \in [0, 1], X_t = x\}, \quad (1.3)$$

see [1, 38] for further details. Hence, local times are usually used as natural measures on $X^{-1}(x)$. Consequently, $L(x, t)$ is useful in investigating fractal properties—such as Hausdorff dimension—of level sets, and we have the following result:

Theorem 1.1. *Let $X = (X_t)_{t \in [0, 1]}$ be an $\mathbb{R}^d$ -valued continuous stochastic process which is α -LND with $\alpha \in (0, \frac{1}{d})$. Assume also that X verifies **H**. Hence, the following holds with probability one:*

$$\dim_H \{t \in [0, 1], X_t = x\} = 1 - d\alpha, \quad (1.4)$$

for any $x \in \mathbb{R}^d$ such that $L(x, [0, 1]) > 0$, where $\dim_H$ denotes the Hausdorff dimension.

One of the main aims of this article is to improve this uniform Hölder continuity of $L(x, \bullet)$ to regularity in the Besov spaces—we refer to Subsection 2.1 for some notions on Besov spaces. There are well known Besov regularity results, in the space variable x for fixed t , of the local times $L(x, t)$ of some classical Gaussian and stable processes, e.g. [11, 12]. However, to the best of our knowledge, there is no work in the literature treating the Besov regularity of $L(x, \bullet)$ even for the Gaussian or stable processes. To fill this gap, we investigate the Besov regularity, in the time variable t uniformly in the space variable x and in p (the index of the Besov space $\mathbf{B}_{p, \infty}^\nu(I; \mathbb{R})$), for local times of continuous general processes that satisfy the α -LND condition. Our first main result is the next theorem.

Theorem 1.2. *Let $X = (X_t)_{t \in [0, 1]}$ be an $\mathbb{R}^d$ -valued continuous stochastic process which is α -LND with $\alpha \in (0, \frac{1}{d})$. Assume also that X verifies **H**. Denote by $L(x, t)$ the jointly continuous version of the local time of X . Then, almost surely, for any $1 \leq p < \infty$,*

$$\sup_{0 < t \leq 1} t^{-(1-d\alpha)} \sup_{|h| \leq t} \|r \mapsto \sup_{x \in \mathbb{R}^d} |L(x, r + h) - L(x, r)|\|_{L^p(I(h); \mathbb{R})} < \infty. \quad (1.5)$$

In particular, we have the following Besov regularity

$$\mathbb{P}[L(x, \bullet) \in \mathbf{B}_{p,\infty}^{1-d\alpha}(I; \mathbb{R}), \text{ for all } x \in \mathbb{R}^d \text{ and } p \in [1, \infty)] = 1, \quad (1.6)$$

where $I = [0, 1]$, $I(h) = \{s \in I; s + h \in I\}$, and $L(x, \bullet) : t \in I \mapsto L(x, t)$.

As mentioned above, we know that the regularity of the local time is linked to the irregular behaviour of the sample paths of the corresponding process. Recall that Adler's theorem, [1, Theorem 8.7.1], establishes a connection between the Hölder continuity, in the time variable t uniformly on the space variable x , of the local time and the Hölder irregularity of its underlying function, as follows.

Theorem 1.3 (Adler). *Let $(f(t))_{t \in [0,1]}$ be an $\mathbb{R}^d$ -valued continuous function possessing a local time, $L(x, t)$, satisfying: there exist c and ρ positive and finite constants, such that for all $t, t + h \in [0, 1]$ and all $|h| < \rho$,*

$$\sup_{x \in \mathbb{R}^d} |L(x, t + h) - L(x, t)| \leq c|h|^\mu, \quad (1.7)$$

for $0 < \mu < 1$. Then, all coordinate functions of f are nowhere Hölder continuous of order greater than $(1 - \mu)/d$.

One of the goals of this article is to provide a similar theorem to that of Adler in the case of Besov spaces. We have the following theorem.

Theorem 1.4. *Let $(f(t))_{t \in [0,1]}$ be an $\mathbb{R}^d$ -valued continuous function possessing a local time, $L(x, t)$, satisfying for some $\mu \in (0, 1)$ and $p \in (d/(1 - \mu), \infty)$,*

$$\sup_{0 < t \leq 1} t^{-\mu/d} \sup_{|h| \leq t} \|s \mapsto \sup_{x \in \mathbb{R}^d} |L(x, s + h) - L(x, s)|^{\frac{1}{d}}\|_{L^{\frac{p}{(p-1)}}(I(h); \mathbb{R})} < \infty. \quad (1.8)$$

Then, the function f does not belong to the Besov space $\mathbf{B}_{p,\infty}^{(1-\mu)/d,0}(I, \mathbb{R}^d)$, where $I = [0, 1]$ and $I(h) = \{s \in I; s + h \in I\}$.

Based on the above results, we note that the Besov regularity, in the time variable t uniformly on the space variable x and in p , of the local times $L(x, t)$ associated with α -LND stochastic processes is valuable, as this knowledge can be applied towards uniform (in p) Besov irregularity of the underlying processes. Therefore, as a consequence of Theorems 1.2 and 1.4, we obtain the following theorem:

Theorem 1.5. *Let $(X_t)_{t \in [0,1]}$ be an $\mathbb{R}^d$ -valued continuous stochastic process with $X(0) = 0$, which is α -LND with $\alpha \in (0, \frac{1}{d})$. Assume also that X verifies **H**. Then*

$$\mathbb{P}[X(\bullet) \in \mathbf{B}_{p,\infty}^{\alpha,0}(I, \mathbb{R}^d), \text{ for some } p \in (1/\alpha, \infty)] = 0, \quad (1.9)$$

where $I = [0, 1]$ and $X(\bullet) : t \in I \mapsto X_t \in \mathbb{R}^d$.

According to the following continuous injections:

$$\mathbf{B}_{p,q}^\alpha(I, \mathbb{R}^d) \hookrightarrow \mathbf{B}_{p,\infty}^{\alpha,0}(I, \mathbb{R}^d), \quad 1 \leq q < \infty,$$

we get the following corollary.

Corollary 1.6. *Let $(X_t)_{t \in [0,1]}$ be an $\mathbb{R}^d$ -valued continuous stochastic process with $X(0) = 0$, which is α -LND with $\alpha \in (0, \frac{1}{d})$. Assume also that X verifies **H**. Then*

$$\mathbb{P}[X(\bullet) \in \mathbf{B}_{p,q}^\alpha(I, \mathbb{R}^d)], \text{ for some } p \in (1/\alpha, \infty) \text{ and } q \in [1, \infty) = 0, \quad (1.10)$$

where $I = [0, 1]$ and

$$X(\bullet) : t \in I \mapsto X_t \in \mathbb{R}^d.$$

In the literature, there were a few separate attempts to study the Besov irregularity of some real valued Gaussian processes, we mention [8, 10, 31]. In contrast to the proof of Theorem 1.5, which relies on local times, the proof of these articles is based on a straightforward calculation. In Section 4, we illustrate Theorem 1.5 and 1.2 with some examples. We infer the uniform Besov regularity of the local times, as well as the uniform Besov irregularity, of the following α -LND processes: First, we consider an $\mathbb{R}^d$ -valued Gaussian process $(Y_t)_{t \in [0,1]}$, with $Y_t = (Y_t^1, \dots, Y_t^d)$, where $Y^1, \dots, Y^d$ are independent copies of a real-valued LND Gaussian process $(Y_t^0)_{t \in [0,1]}$, satisfying for some $\alpha \in (0, 1)$ and finite constants $c, C > 0$,

$$c(t-s)^{2\alpha} \leq \text{Var}(Y_t^0 - Y_s^0) \leq C(t-s)^{2\alpha}, \quad (1.11)$$

for every $0 \leq s < t \leq 1$. Notice that the d -dimensional bifractional Brownian motion verifies the above conditions. Secondly, as an example of non-Gaussian and non-stable processes, we regard systems of non-linear stochastic heat equations.

In this article, to deal with Besov spaces, we are based on representations of the Besov norms in terms of dyadic expansion coefficients of a given function. These descriptions of the Besov norms are derived from [23, 3.b.9 Corollary]. To the best of our knowledge, the treatise of König [23] has been used for the first time to investigate Besov regularity of stochastic processes by Hytönen and Veraar [21].

The rest of the paper is organized as follows. In the second section, we write some preliminary results on Besov spaces and local times. The third section is devoted to the proofs of the main results. In the fourth section, we give some examples.

Finally, we point out that constants in our proofs may change from line to line. For a process $X = (X_t)_{t \in [0,1]}$, sometimes if necessary, we write $X(t)$ instead of X_t .

2. Preliminaries

2.1. Besov spaces

For the definition of the real-valued Besov spaces, we refer to [34], and for the vector-valued Besov spaces, we suggest the treatise [23]. Let $I = [0, 1]$, for any $h \in \mathbb{R}$, we put $I(h) = \{s \in I; s + h \in I\}$. Let $1 \leq p < \infty$ and $v \in (0, 1)$, the modulus of continuity of a function $f \in L^p(I; \mathbb{R}^d)$ is defined by

$$\omega_p(f, t) = \sup_{|h| \leq t} \|x \mapsto f(x + h) - f(x)\|_{L^p(I(h); \mathbb{R}^d)}. \quad (2.1)$$

We define the vector-valued Besov space $\mathbf{B}_{p,\infty}^v(I; \mathbb{R}^d)$ as the space of all functions $f \in L^p(I; \mathbb{R}^d)$ such that the seminorm $\mathcal{N}_{v,p}(f) := \sup_{0 < t \leq 1} t^{-v} \omega_p(f, t)$ is finite. $\mathbf{B}_{p,\infty}^v(I; \mathbb{R}^d)$ endowed with the sum of the L^p -norm and the seminorm $\mathcal{N}_{v,p}$ is a Banach space. Let $\mathbf{B}_{p,\infty}^{v,0}(I, \mathbb{R}^d)$ be the space of all functions $f \in \mathbf{B}_{p,\infty}^v(I; \mathbb{R}^d)$ for which $\lim_{t \rightarrow 0^+} t^{-v} \omega_p(f, t) = 0$. Using a dyadic approximation argument (see the lemma in page 173 and Corollary 3.b.9 in [23]) one has the following theorem.

Theorem 2.1. *Let $1 \leq p < \infty$ and $v \in (0, 1)$. We have*

(i) *The seminorm $\mathcal{N}_{v,p}$ is equivalent to*

$$\|f\|_{v,p} := \sup_{j \geq 0} 2^{jv} \|x \mapsto f(x + 2^{-j}) - f(x)\|_{L^p(I(2^{-j}); \mathbb{R}^d)}. \quad (2.2)$$

(ii) *Let $f \in L^p(I; \mathbb{R}^d)$. Then f is in $\mathbf{B}_{p,\infty}^{v,0}(I, \mathbb{R}^d)$ if and only if*

$$\lim_{j \rightarrow \infty} 2^{jv} \|x \mapsto f(x + 2^{-j}) - f(x)\|_{L^p(I(2^{-j}); \mathbb{R}^d)} = 0. \quad (2.3)$$

(iii) *Let $1 \leq p < \infty$ and $0 < v < 1$. Then there exists a positive and finite constant c such that for all g from $\mathbb{R}^d \times [0, 1]$ to $\mathbb{R}$ jointly continuous function with compact support*

$$\begin{aligned} & c^{-1} \sup_{0 < t \leq 1} t^{-v} \sup_{|h| \leq t} \|r \mapsto \sup_{x \in \mathbb{R}^d} |g(x, r + h) - g(x, r)|\|_{L^p(I(h); \mathbb{R})} \\ & \leq \sup_{j \geq 0} 2^{jv} \|r \mapsto \sup_{x \in \mathbb{R}^d} |g(x, r + 2^{-j}) - g(x, r)|\|_{L^p(I(2^{-j}); \mathbb{R})} \\ & \leq c \sup_{0 < t \leq 1} t^{-v} \sup_{|h| \leq t} \|r \mapsto \sup_{x \in \mathbb{R}^d} |g(x, r + h) - g(x, r)|\|_{L^p(I(h); \mathbb{R})}. \end{aligned} \quad (2.4)$$

(iv) *Let $1 \leq p < \infty$, $0 < v < 1$, and f be an $\mathbb{R}^d$ -valued continuous function. Then*

$$\lim_{t \rightarrow 0^+} t^{-v} \sup_{|h| \leq t} \|x \mapsto \sup_{y, z \in [0, 1]} \|f(hz + x) - f(hy + x)\|\|_{L^p(I(h), \mathbb{R})} = 0,$$

if and only if

$$\lim_{j \rightarrow \infty} 2^{j\nu} \|x\| \mapsto \sup_{y, z \in [0, 1]} \|f(2^{-j}z + x) - f(2^{-j}y + x)\|_{L^p(I(2^{-j}); \mathbb{R})} = 0.$$

Now we introduce the spaces $\mathbf{b}_{p, \infty}^{v, 0}(I; \mathbb{R}^d)$, which plays a key role in the proof of Theorem 1.4. Let $I = [0, 1]$, $1 \leq p < \infty$, and $v \in (0, 1)$, $\mathbf{b}_{p, \infty}^{v, 0}(I; \mathbb{R}^d)$ is defined as the space of all functions $f \in C(I; \mathbb{R}^d)$, where $C(I; \mathbb{R}^d)$ is the space of $\mathbb{R}^d$ -valued continuous functions, such that

$$\lim_{j \rightarrow \infty} 2^{j\nu} \|x\| \mapsto \sup_{y, z \in [0, 1]} \|f(2^{-j}z + x) - f(2^{-j}y + x)\|_{L^p(I(2^{-j}); \mathbb{R})} = 0, \quad (2.5)$$

here $I(2^{-j}) = \{x \in I; x + 2^{-j} \in I\}$. In the below theorem, we give the relation between the spaces $\mathbf{b}_{p, \infty}^{v, 0}(I; \mathbb{R}^d)$ and the classical spaces $\mathbf{B}_{p, \infty}^{v, 0}(I, \mathbb{R}^d)$.

Theorem 2.2. *Let $I = [0, 1]$, $v \in (0, 1)$, and $\frac{1}{v} < p < \infty$. Then*

$$\mathbf{b}_{p, \infty}^{v, 0}(I; \mathbb{R}^d) = C(I; \mathbb{R}^d) \cap \mathbf{B}_{p, \infty}^{v, 0}(I, \mathbb{R}^d).$$

At this point, to prove the above theorem, we need the Garsia–Rodemich–Rumsey inequality [19].

Lemma 2.3. *Let $\Psi(u)$ and $p(u)$ be non-negative even functions respectively on $\mathbb{R}$ and $[-1, 1]$ with $p(0) = 0$ and $\Psi(\infty) = \infty$. Assume that $p(u)$ and $\Psi(u)$ are non decreasing for $u \geq 0$ and $p(u)$ is continuous. Let $g(x)$ be continuous on $[0, 1]$ and suppose that*

$$\int_0^1 \int_0^1 \Psi\left(\frac{g(v) - g(w)}{p(v - w)}\right) dv dw \leq B < \infty.$$

Then, for all $z, y \in [0, 1]$,

$$|g(z) - g(y)| \leq 8 \int_0^{|z-y|} \Psi^{-1}\left(\frac{4B}{u^2}\right) dp(u).$$

Proof of Theorem 2.2. We just need to prove that

$$C(I; \mathbb{R}^d) \cap \mathbf{B}_{p, \infty}^{v, 0}(I, \mathbb{R}^d) \subseteq \mathbf{b}_{p, \infty}^{v, 0}(I; \mathbb{R}^d),$$

since the reverse inclusion is trivial. Let $f = (f_1, \dots, f_d) \in C(I; \mathbb{R}^d) \cap \mathbf{B}_{p, \infty}^{v, 0}(I, \mathbb{R}^d)$, $\Psi(u) = |u|^p$, and $p(u) = |u|^{v+\frac{\beta}{p}}$, where $\beta \in [0, 1]$ such that $vp - 1 > 1 - \beta$. Hence by Fubini's theorem, changes of variables, and the fact that $f \in \mathbf{B}_{p, \infty}^{v, 0}(I, \mathbb{R}^d)$, we have for any $1 \leq i \leq d$,

$$B(i, j, x) := \int_0^1 \int_0^1 \frac{|f_i(2^{-j}v + x) - f_i(2^{-j}w + x)|^p}{|v - w|^{vp+\beta}} dv dw < \infty. \quad (2.6)$$

Therefore, by Lemma 2.3 we get for all $1 \leq i \leq d$ and $z, y \in [0, 1]$,

$$\begin{aligned} & |f_i(2^{-j}z + x) - f_i(2^{-j}y + x)|^p \\ & \leq C_{v,\beta,p} |z - y|^{vp+\beta-2} \int_0^1 \int_0^1 \frac{|f_i(2^{-j}v + x) - f_i(2^{-j}w + x)|^p}{|v - w|^{vp+\beta}} dv dw \\ & \leq C_{v,\beta,p} \int_0^1 \int_0^1 \frac{|f_i(2^{-j}v + x) - f_i(2^{-j}w + x)|^p}{|v - w|^{vp+\beta}} dv dw, \end{aligned}$$

where $C_{v,\beta,p} = 8^p 4p^p / (\beta + pv - 2)^p$. Hence, for all $1 \leq i \leq d$,

$$\begin{aligned} & \sup_{z,y \in [0,1]} |f_i(2^{-j}z + x) - f_i(2^{-j}y + x)|^p \\ & \leq C_{v,\beta,p} \int_0^1 \int_0^1 \frac{|f_i(2^{-j}v + x) - f_i(2^{-j}w + x)|^p}{|v - w|^{vp+\beta}} dv dw, \end{aligned}$$

Therefore, for all $1 \leq i \leq d$,

$$\begin{aligned} & 2^{jpv} \int_{I(2^{-j})} \sup_{z,y \in [0,1]} |f_i(2^{-j}z + x) - f_i(2^{-j}y + x)|^p dx \\ & \leq C_{v,\beta,p} \int_{[0,1]^2} \frac{2^{jpv}}{|v - w|^{vp+\beta}} \\ & \quad \times \int_0^{1-2^{-j}} |f_i(2^{-j}v + x) - f_i(2^{-j}w + x)|^p dx dv dw \\ & = C_{v,\beta,p} \int_{[0,1]^2} \frac{2^{jpv}}{|v - w|^{vp+\beta}} \\ & \quad \times \int_{2^{-j}w}^{1-2^{-j}+2^{-j}w} |f_i(x + 2^{-j}(v - w)) - f_i(x)|^p dx dv dw \\ & \leq C_{v,\beta,p} \int_{[0,1]^2} \frac{2^{jpv}}{|v - w|^{vp+\beta}} \\ & \quad \times \int_{I(2^{-j}(v-w))} |f_i(x + 2^{-j}(v - w)) - f_i(x)|^p dx dv dw, \quad (2.7) \end{aligned}$$

where $I(2^{-j}(v - w)) = \{x \in [0, 1]; x + 2^{-j}(v - w) \in [0, 1]\}$. By the definition of $f \in \mathbf{B}_{p,\infty}^{v,0}(I, \mathbb{R}^d)$, we have for all $v, w \in [0, 1]$ with $v \neq w$,

$$\lim_{j \rightarrow \infty} \frac{2^{jpv}}{|v - w|^{vp+\beta}} \int_{I(2^{-j}(v-w))} |f_i(x + 2^{-j}(v - w)) - f_i(x)|^p dx = 0.$$

Therefore, by Lebesgue's dominated convergence theorem the right-hand side of (2.7) converges to 0. Which concludes the proof of Theorem 2.2. $\blacksquare$

Remark 2.4. Let $\mathbf{B}_{p,q}^\nu(I, \mathbb{R}^d)$, for $1 \leq p, q < \infty$ and $\nu \in (0, 1)$, be the Besov spaces defined as follows:

$$\mathbf{B}_{p,q}^\nu(I, \mathbb{R}^d) := \{f \in L^p(I, \mathbb{R}^d); \|f\|_{\nu,p,q} < \infty\},$$

where by [23, 3.b.9 Corollary] we have

$$\|f\|_{\nu,p,q} := \left\{ \sum_{j \geq 0} 2^{jq\nu} \|x \mapsto f(x + 2^{-j}) - f(x)\|_{L^p(I(2^{-j}); \mathbb{R}^d)}^q \right\}^{\frac{1}{q}}.$$

Therefore, we have the following continuous injection: for all $1 \leq p, q < \infty$ and $\nu \in (0, 1)$,

$$\mathbf{B}_{p,q}^\nu(I, \mathbb{R}^d) \hookrightarrow \mathbf{B}_{p,\infty}^{\nu,0}(I, \mathbb{R}^d). \quad (2.8)$$

2.2. The local times

This section is devoted to give some aspects of the theory of local times. For more details on the subject, we refer to the survey of Geman and Horowitz [20].

Let $(\varphi_t)_{t \in [0, T]}$ be an $\mathbb{R}^d$ -valued Borel function. For any Borel set $B \subseteq [0, T]$, the occupation measure of φ on B is given by the following measure on $\mathbb{R}^d$:

$$\nu_B(\bullet) = \lambda\{t \in B; \varphi_t \in \bullet\},$$

where λ is the Lebesgue measure. When ν_B is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, λ_d , we say that the local time of φ on B exists and it is defined as the Radon–Nikodym derivative of ν_B with respect to λ_d , i.e., for almost every x ,

$$L(x, B) = \frac{d\nu_B}{d\lambda_d}(x).$$

In the above, we call B the time variable and x the space variable. We write $L(x, t)$ and $L(x)$ instead of respectively $L(x, [0, t])$ and $L(x, [0, T])$.

The local time fulfills the following occupation formula: for any Borel set $B \subseteq [0, T]$, and for every measurable bounded function $f : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\int_B f(\varphi_s) ds = \int_{\mathbb{R}^d} f(x) L(x, B) dx. \quad (2.9)$$

The deterministic function φ can be chosen to be the sample path of a separable stochastic process $(X_t)_{t \in [0, T]}$ with $X(0) = 0$ a.s. In this regard, we state that the process X has a local time (resp. square integrable local time) if for almost all ω , the trajectory $t \mapsto X_t(\omega)$ has a local time (resp., square integrable local time).

We study the local time through Berman's approach. The idea is to derive properties of the local time, $L(\bullet, B)$, from the integrability properties of the Fourier transform of the sample paths of the process X .

Let us state the following hypotheses.

A1. We have

$$\int_{\mathbb{R}^d} \int_0^T \int_0^T \mathbb{E}[e^{i\langle u, X_t - X_s \rangle}] dt ds du < \infty,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^d$.

A2. For every even integer $m \geq 2$,

$$\int_{(\mathbb{R}^d)^m} \int_{[0, T]^m} \left| \mathbb{E} \left[\exp \left(i \sum_{j=1}^m \langle u_j, X_{t_j} \rangle \right) \right] \right| \prod_{j=1}^m dt_j \prod_{j=1}^m du_j < \infty.$$

Recall the following essential result that we can find in [3].

Theorem 2.5. Assume **A1**. Hence the process X has a square integrable local time. Furthermore, we have almost surely, for all Borel set $B \subseteq [0, T]$, and for almost every x ,

$$L(x, B) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle u, x \rangle} \int_B e^{i\langle u, X_t \rangle} dt du. \quad (2.10)$$

Remark that $L(x, B)$, given by (2.10), is not a stochastic process. We follow Berman [4] to create a version of the local time, which is a stochastic process. The following theorem is given in Berman [4, Theorem 4.1] for $d = 1$ and $m = 2$, so we omit its proof.

Theorem 2.6. Assume **A1** and **A2**. Put for all integer $N \geq 1$,

$$L_N(x, t) = \frac{1}{(2\pi)^d} \int_{[-N, N]^d} e^{-i\langle u, x \rangle} \int_0^t e^{i\langle u, X_s \rangle} ds du.$$

Therefore, there exists a stochastic process $\tilde{L}(x, t)$ separable in the x -variable, such that for each even integer $m \geq 2$,

$$\lim_{N \rightarrow \infty} \sup_{(x, t) \in \mathbb{R}^d \times [0, T]} \mathbb{E}[|L_N(x, t) - \tilde{L}(x, t)|^m] = 0. \quad (2.11)$$

Theorem 2.7 ([4], Theorem 4.3). Let $\tilde{L}(x, t)$ be given by (2.11). If the stochastic process $\{\tilde{L}(x, t), x \in \mathbb{R}^d\}$ is almost surely continuous, hence it is a continuous (in the x -variable) version of the local time on $[0, t]$.

In order to prove Theorem 1.2, we need to estimate the moments of the increments of $\tilde{L}$. For this end, we have by equation (2.11), for all $x, y \in \mathbb{R}^d$, $t, h \in [0, T]$ such that $t + h \in [0, T]$, and even integer $m \geq 2$,

$$\mathbb{E}[\tilde{L}(x + y, t + h) - \tilde{L}(x, t + h) - \tilde{L}(x + y, t) + \tilde{L}(x, t)]^m \quad (2.12)$$

$$\begin{aligned} &= \frac{1}{(2\pi)^{md}} \times \int_{(\mathbb{R}^d)^m} \int_{[t, t+h]^m} \prod_{j=1}^m (e^{-i\langle u_j, x+y \rangle} - e^{-i\langle u_j, x \rangle}) \\ &\quad \times \mathbb{E}[e^{i\sum_{j=1}^m \langle u_j, X_{t_j} \rangle}] \prod_{j=1}^m dt_j \prod_{j=1}^m du_j \\ &= \frac{1}{(2\pi)^{md}} \int_{(\mathbb{R}^d)^m} \int_{[t, t+h]^m} \prod_{j=1}^m (e^{-i\langle v_j - v_{j+1}, x+y \rangle} - e^{-i\langle v_j - v_{j+1}, x \rangle}) \\ &\quad \times \mathbb{E}[e^{i\sum_{j=1}^m \langle v_j, X_{t_j} - X_{t_{j-1}} \rangle}] \prod_{j=1}^m dt_j \prod_{j=1}^m dv_j, \quad (2.13) \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}[\tilde{L}(x, t + h) - \tilde{L}(x, t)]^m \\ &= \frac{1}{(2\pi)^{md}} \int_{(\mathbb{R}^d)^m} \int_{[t, t+h]^m} e^{-i\sum_{j=1}^m \langle u_j, x \rangle} \mathbb{E}[e^{i\sum_{j=1}^m \langle u_j, X_{t_j} \rangle}] \prod_{j=1}^m dt_j \prod_{j=1}^m du_j \\ &= \frac{1}{(2\pi)^{md}} \int_{(\mathbb{R}^d)^m} \int_{[t, t+h]^m} e^{-i\langle v_1, x \rangle} \mathbb{E}[e^{i\sum_{j=1}^m \langle v_j, X_{t_j} - X_{t_{j-1}} \rangle}] \prod_{j=1}^m dt_j \prod_{j=1}^m dv_j, \quad (2.14) \end{aligned}$$

where $t_0 = 0$, and the last equality in (2.12) (resp., (2.14)) holds through suitable changes of variables as follows:

$$u_j = v_j - v_{j+1}, \quad j = 1, \dots, m, \quad \text{with} \quad v_{m+1} = 0.$$

In order to estimate (2.12) and (2.14), we need first to manage the following characteristic function $\mathbb{E}[e^{i\sum_{j=1}^m \langle v_j, X_{t_j} - X_{t_{j-1}} \rangle}]$. Therefore, we use the below condition called α -local nondeterminism (α -LND), which was introduced for the first time in [9].

Definition 2.8. Let $X = (X_t)_{t \in [0, T]}$ be an $\mathbb{R}^d$ -valued stochastic process, J a subinterval of $[0, T]$ and $\alpha \in (0, 1)$. X is said to satisfy the α -LND property on J , if for every non-negative integers $m \geq 2$, and $k_{j,l}$, for $j = 1, \dots, m$, $l = 1, \dots, d$, there exist positive constants c and ε , both may depend on m and $k_{j,l}$, such that

$$|\mathbb{E}[e^{i\sum_{j=1}^m \langle v_j, X_{t_j} - X_{t_{j-1}} \rangle}]| \leq \frac{c}{\prod_{j=1}^m \prod_{l=1}^d |v_{j,l}|^{k_{j,l}} (t_j - t_{j-1})^{\alpha k_{j,l}}}, \quad (2.15)$$

for all $v_j = (v_{j,l}; 1 \leq l \leq d) \in (\mathbb{R} \setminus \{0\})^d$, for $j = 1, \dots, m$, and for every ordered points $t_1 < \dots < t_m$ in J with $t_m - t_1 < \varepsilon$ and $t_0 = 0$.

Remark 2.9. (i) It is well known that the concept of local nondeterminism in the Gaussian framework means that “the value of the process at a given time point is relatively unpredictable based on a finite set of observations from the immediate past”. In the Gaussian context, Berman uses conditional variance to express this property. However, beyond the Gaussian case, conditional variance cannot be used, since in a general framework it is not deterministic. So, Berman has introduced the local g -nondeterminism concept for general processes by replacing the incremental variance, which is a measure of local unpredictability, by a measure of local predictability, namely, the value of the incremental density function at the origin, see [6, Definition 5.1]. By the Fourier inversion theorem, it is easy to see that the condition in Definition 2.8 implies the local g -nondeterminism condition. On the other hand, Nolan has introduced the notion of *characteristic function locally approximately independent increments* (see [28, Definition 3.1]), which is equivalent in the Gaussian and stable context to the classical LND condition. The condition in Definition 2.8 ($d = 1$) is an extension of Nolan’s notion by replacing the characteristic functions $|\mathbb{E}[e^{icm u_j (X(t_j) - X(t_{j-1}))}]|$ in the right-hand side of [28, Inequality (3.3)] by $c |u_j|^{-k_j} (t_j - t_{j-1})^{-\alpha k_j}$.

(ii) Although the α -LND has been specifically created for non-Gaussian processes, a large class of Gaussian processes possesses this property. Let

$$Y^0 = (Y_t^0)_{t \in [0, T]}$$

be a real-valued centred Gaussian process satisfying the classical local nondeterminism (LND) property on $J \subseteq [0, T]$ (see [7, Lemma 2.3]). Assume also that there exists a positive constant K , such that for every $s, t \in J$ with $s < t$,

$$K(t - s)^{2\alpha} \leq \text{Var}(Y_t^0 - Y_s^0).$$

Define $Y_t = (Y_t^1, \dots, Y_t^d)$, where $Y^1, \dots, Y^d$ are independent copies of Y^0 . Then Y is α -LND on J .

(iii) The question of whether or not the α -LND is strictly weaker than the classical local nondeterminism in the Gaussian framework is an open problem that is currently the subject of our investigation.

3. Proof of the main results

Our aim in this section is to prove Theorems 1.2, 1.5, and 1.4. We need first the following preliminary lemmas.

Lemma 3.1. *Let $1 \leq p < \infty$, $0 < \nu < 1$, and $I = [0, 1]$. Then, for all $\mathbb{R}$ -valued jointly continuous function g with compact support defined on $\mathbb{R}^d \times [0, 1]$,*

$$\begin{aligned} \sup_{j \geq 0} 2^{jpv} \|r \mapsto \sup_{x \in \mathbb{R}^d} |g(x, r + 2^{-j}) - g(x, r)|\|_{L^p(I(2^{-j}); \mathbb{R})}^p \\ = \sup_{j \geq 0} 2^{jpv-j} \int_0^1 \sum_{k=1}^{2^j-1} \sup_{x \in \mathbb{R}^d} |g(x, 2^{-j}(s+k)) - g(x, 2^{-j}(s+k-1))|^p ds. \end{aligned} \quad (3.1)$$

Proof. Denote

$$Z_j = 2^{jpv} \|r \mapsto \sup_{x \in \mathbb{R}^d} |g(x, r + 2^{-j}) - g(x, r)|\|_{L^p(I(2^{-j}); \mathbb{R})}^p,$$

here $I(2^{-j}) = \{t \in [0, 1]; t + 2^{-j} \in [0, 1]\}$. We have

$$\begin{aligned} Z_j &= 2^{jpv} \int_0^{1-2^{-j}} \sup_{x \in \mathbb{R}^d} |g(x, r + 2^{-j}) - g(x, r)|^p dr \\ &= 2^{jpv} \sum_{k=1}^{2^j-1} \int_{(k-1)2^{-j}}^{k2^{-j}} \sup_{x \in \mathbb{R}^d} |g(x, r + 2^{-j}) - g(x, r)|^p dr \\ &= 2^{jpv-j} \int_0^1 \sum_{k=1}^{2^j-1} \sup_{x \in \mathbb{R}^d} |g(x, 2^{-j}(s+k)) - g(x, 2^{-j}(s+k-1))|^p ds, \end{aligned}$$

where we have used the change of variables $s = 2^j(r - 2^{-j}(k-1))$. Which finishes the proof of Lemma 3.1. $\blacksquare$

Lemma 3.2. *Let $X = (X_t)_{t \in [0,1]}$ be an $\mathbb{R}^d$ -valued continuous stochastic process which is α -LND with $\alpha \in (0, \frac{1}{d})$. Denote by $L(x, t)$ the jointly continuous version of the local time of X , then*

- *For any positive integer q , there exists a positive constant $K = K(q, d, \alpha)$ such that for all integer $j \geq 1$ and $x \in \mathbb{R}^d$,*

$$\mathbb{P} \left[\sum_{k=1}^{2^j-1} |L(x + X(d_{j,k}), D_{j,k})|^q \geq 2^{-jq(1-d\alpha)+j} \right] \leq K 2^{-jd\alpha}; \quad (3.2)$$

- *Let a and q be positive integers and $0 < \theta < \{(\frac{1}{\alpha} - d)/2\} \wedge 1$, then there exists a positive constant $K = K(q, a, d, \alpha, \theta)$, such that for all integers $j, h \geq 1$ and any $x, y \in \mathbb{R}^d$ and $\gamma > 0$,*

$$\mathbb{P} \left[\sum_{k=1}^{2^j-1} |A_{j,k,x,y}|^q \geq 2^{-jq(1-d\alpha-\theta\alpha)+j} \|x - y\|^{q\theta} 2^{\gamma h} \right] \leq K 2^{-jd\alpha} 2^{-2^a \gamma h}, \quad (3.3)$$

where $d_{j,k} = 2^{-j}(k-1)$, $D_{j,k} = [2^{-j}(k-1), 2^{-j}(k+1)]$, and

$$A_{j,k,x,y} = L(x + X(d_{j,k}), D_{j,k}) - L(y + X(d_{j,k}), D_{j,k}). \quad (3.4)$$

Proof. We only prove (3.3). We have by Hölder's inequality, for all positive integer a ,

$$\begin{aligned} \mathbb{E} \left[\left\{ \sum_{k=1}^{2^j-1} |A_{j,k,x,y}|^q \right\}^{2^a} \right] &= \sum_{k_1, \dots, k_{2^a}=1}^{2^j-1} \mathbb{E} \left[\prod_{i=1}^{2^a} |A_{j,k_i,x,y}|^q \right] \\ &\leq \left\{ \sum_{k=1}^{2^j-1} \mathbb{E} [|A_{j,k,x,y}|^{q2^a}]^{1/2^a} \right\}^{2^a}. \end{aligned} \quad (3.5)$$

On the other hand, let $Y_t = X(t) - X(d_{j,k})$. The occupation measure of Y is just the occupation measure of X translated by the (random) constant $X(d_{j,k})$. Since the occupation measure of X has a jointly continuous density, the occupation measure of Y has also a jointly continuous density given by $L_Y(x, t) = L_X(x + X(d_{j,k}), t)$. Put $m = q2^a$, hence by (3.4) and (2.12) we obtain

$$\begin{aligned} &\mathbb{E} [|A_{j,k,x,y}|^m] \\ &= E [|L(x + X(d_{j,k}), D_{j,k}) - L(y + X(d_{j,k}), D_{j,k})|^m] \\ &= E [|L_Y(x, D_{j,k}) - L_Y(y, D_{j,k})|^m] \\ &= \frac{1}{(2\pi)^{md}} \int_{(\mathbb{R}^d)^m} \int_{(D_{j,k})^m} \prod_{n=1}^m (e^{-i\langle v_n - v_{n+1}, x \rangle} - e^{-i\langle v_n - v_{n+1}, y \rangle}) \\ &\quad \times \mathbb{E} [e^{i \sum_{n=1}^m \langle v_n, X_{t_n} - X_{t_{n-1}} \rangle}] \prod_{n=1}^m dt_n \prod_{n=1}^m dv_n, \end{aligned} \quad (3.6)$$

By (3.6) and the elementary inequality $|1 - e^{i\rho}| \leq 2^{1-\theta} |\rho|^\theta$ for any $0 < \theta < 1$ and $\rho \in \mathbb{R}$, we get

$$\mathbb{E} [|A_{j,k,x,y}|^m] \leq 2^{-m(d+\theta-1)} \pi^{-md} \|x - y\|^{m\theta} \mathcal{J}(m, \theta), \quad (3.7)$$

where

$$\begin{aligned} &\mathcal{J}(m, \theta) \\ &= \int_{(D_{j,k})^m} \int_{(\mathbb{R}^d)^m} \prod_{n=1}^m \|v_n - v_{n+1}\|^\theta |\mathbb{E} [e^{i \sum_{n=1}^m \langle v_n, X_{t_n} - X_{t_{n-1}} \rangle}]| \prod_{n=1}^m dv_n \prod_{n=1}^m dt_n. \end{aligned}$$

We replace the integration over the domain $(D_{j,k})^m$ by the integration over the subset $\Lambda_{j,k} = \{2^{-j}(k-1) \leq t_1 < \dots < t_m \leq 2^{-j}(k+1)\}$, hence we obtain

$$\begin{aligned} &\mathcal{J}(m, \theta) \\ &= m! \int_{\Lambda_{j,k}} \int_{(\mathbb{R}^d)^m} \prod_{n=1}^m \|v_n - v_{n+1}\|^\theta |\mathbb{E} [e^{i \sum_{n=1}^m \langle v_n, X_{t_n} - X_{t_{n-1}} \rangle}]| \prod_{n=1}^m dv_n \prod_{n=1}^m dt_n, \end{aligned}$$

where $t_0 = 0$ and $v_{m+1} = 0$. By the fact that $\|b - c\|^\theta \leq \|b\|^\theta + \|c\|^\theta$ for each $0 < \theta < 1$ and $b, c \in \mathbb{R}^d$, it follows that

$$\prod_{n=1}^m \|v_n - v_{n+1}\|^\theta \leq \prod_{n=1}^m (\|v_n\|^\theta + \|v_{n+1}\|^\theta). \quad (3.8)$$

Remark that the right side of this last inequality is at most equal to a finite sum of terms each of the form $\prod_{n=1}^m \|v_n\|^{\epsilon_n \theta}$, where $\epsilon_n = 0, 1$, or 2 and $\sum_{n=1}^m \epsilon_n = m$. Hence

$$\begin{aligned} \mathcal{J}(m, \theta) &\leq m! \sum_{(\epsilon_1, \dots, \epsilon_m) \in \{0, 1, 2\}^m} \int_{\Lambda_{j,k}} \int_{(\mathbb{R}^d)^m} \prod_{n=1}^m \|v_n\|^{\epsilon_n \theta} \\ &\quad \times |\mathbb{E}[e^{i \sum_{n=1}^m \langle v_n, X_{t_n} - X_{t_{n-1}} \rangle}]| \prod_{n=1}^m dv_n \prod_{n=1}^m dt_n. \end{aligned} \quad (3.9)$$

On the other hand, by the α -LND property of the process X , we get for every nonnegative integers $m \geq 2$, $k_{n,l}$, for $n = 1, \dots, m$ and $l = 1, \dots, d$, there exists a constant $c = c(m, k_{n,l})$ such that

$$|\mathbb{E}[e^{i \sum_{n=1}^m \langle v_n, X_{t_n} - X_{t_{n-1}} \rangle}]| \leq \frac{c}{\prod_{n=1}^m \prod_{l=1}^d |v_{n,l}|^{k_{n,l}} (t_n - t_{n-1})^{\alpha k_{n,l}}}, \quad (3.10)$$

where $v_n = (v_{n,1}, \dots, v_{n,d})$. Let us put $I_1^n = [-1/(t_n - t_{n-1})^\alpha, 1/(t_n - t_{n-1})^\alpha]$ and $I_2^n = \mathbb{R} \setminus I_1^n$. Therefore

$$(\mathbb{R}^d)^m = \bigcup_{\substack{i_{n,l} \in \{1,2\} \\ n=1, \dots, m; l=1, \dots, d}} \prod_{n=1}^m \prod_{l=1}^d I_{i_{n,l}}^n. \quad (3.11)$$

Set, for $n = 1, \dots, m$ and $l = 1, \dots, d$,

$$k_{n,l}(i_{n,l}) = \begin{cases} 0 & \text{if } i_{n,l} = 1; \\ 4 & \text{if } i_{n,l} = 2. \end{cases}$$

Hence, by (3.9)–(3.11), we obtain

$$\begin{aligned} \mathcal{J}(m, \theta) &\leq m! c \sum_{\substack{i_{n,l} \in \{1,2\} \\ n=1, \dots, m; l=1, \dots, d}} \sum_{(\epsilon_1, \dots, \epsilon_m) \in \{0, 1, 2\}^m} \int_{\Lambda_{j,k}} \int_{\prod_{n=1}^m \prod_{l=1}^d I_{i_{n,l}}^n} \prod_{n=1}^m \|v_n\|^{\epsilon_n \theta} \\ &\quad \times \frac{\prod_{n=1}^m \|v_n\|^{\epsilon_n \theta}}{\prod_{n=1}^m \prod_{l=1}^d |v_{n,l}|^{k_{n,l}(i_{n,l})} (t_n - t_{n-1})^{\alpha k_{n,l}(i_{n,l})}} \prod_{n=1}^m dv_n \prod_{n=1}^m dt_n. \end{aligned} \quad (3.12)$$

We remark that

$$\prod_{n=1}^m \|v_n\|^{\epsilon_n \theta} \leq \prod_{n=1}^m (|v_{n,1}|^{\epsilon_n \theta} + \cdots + |v_{n,d}|^{\epsilon_n \theta}) = \sum_{l_1, \dots, l_d \in \{1, \dots, d\}} \prod_{n=1}^m |v_{n,l_n}|^{\epsilon_n \theta}.$$

Therefore

$$\begin{aligned} \mathcal{J}(m, \theta) &\leq m!c \sum_{l_1, \dots, l_d \in \{1, \dots, d\}} \sum_{\substack{i_{n,l} \in \{1,2\} \\ n=1, \dots, m; l=1, \dots, d}} \sum_{(\epsilon_1, \dots, \epsilon_m) \in \{0,1,2\}^m} \int_{\Lambda_{j,k}} \int_{\prod_{n=1}^m \prod_{l=1}^d I_{i_{n,l}}^n} \\ &\quad \times \frac{\prod_{n=1}^m |v_{n,l_n}|^{\epsilon_n \theta}}{\prod_{n=1}^m \prod_{l=1}^d |v_{n,l}|^{k_{n,l}(i_{n,l})} (t_n - t_{n-1})^{\alpha k_{n,l}(i_{n,l})}} \prod_{n=1}^m dv_n \prod_{n=1}^m dt_n. \end{aligned}$$

By Fubini's theorem, the right side of the above expression is equal to

$$\begin{aligned} &m!c \sum_{l_1, \dots, l_d \in \{1, \dots, d\}} \sum_{\substack{i_{n,l} \in \{1,2\} \\ n=1, \dots, m; l=1, \dots, d}} \sum_{(\epsilon_1, \dots, \epsilon_m) \in \{0,1,2\}^m} \int_{\Lambda_{j,k}} \prod_{n=1}^m \int_{\prod_{l=1}^d I_{i_{n,l}}^n} \\ &\quad \times \frac{|v_{n,l_n}|^{\epsilon_n \theta}}{\prod_{l=1}^d |v_{n,l}|^{k_{n,l}(i_{n,l})} (t_n - t_{n-1})^{\alpha k_{n,l}(i_{n,l})}} dv_n \prod_{n=1}^m dt_n \\ &= m!c \sum_{l_1, \dots, l_d \in \{1, \dots, d\}} \sum_{\substack{i_{n,l} \in \{1,2\} \\ n=1, \dots, m; l=1, \dots, d}} \sum_{(\epsilon_1, \dots, \epsilon_m) \in \{0,1,2\}^m} \\ &\quad \times \int_{\Lambda_{j,k}} \prod_{n=1}^m \prod_{\substack{l=1 \\ l \neq l_n}}^d \int_{I_{i_{n,l}}^n} \frac{1}{|v_{n,l}|^{k_{n,l}(i_{n,l})} (t_n - t_{n-1})^{\alpha k_{n,l}(i_{n,l})}} dv_{n,l} \\ &\quad \times \int_{I_{i_{n,l_n}}^n} \frac{1}{|v_{n,l_n}|^{k_{n,l_n}(i_{n,l_n}) - \epsilon_n \theta} (t_n - t_{n-1})^{\alpha k_{n,l_n}(i_{n,l_n})}} dv_{n,l_n} \prod_{n=1}^m dt_n. \end{aligned} \tag{3.13}$$

- If $i_{n,l} = 1$ or 2 with $l \neq l_n$, then we have

$$\int_{I_{i_{n,l}}^n} \frac{1}{|v_{n,l}|^{k_{n,l}(i_{n,l})} (t_n - t_{n-1})^{\alpha k_{n,l}(i_{n,l})}} dv_{n,l} = \frac{K_1}{(t_n - t_{n-1})^\alpha},$$

where the constant K_1 depends only on $i_{n,l}$.

- If $i_{n,l_n} = 1$ or 2 , then we get

$$\int_{I_{i_{n,l_n}}^n} \frac{1}{|v_{n,l_n}|^{k_{n,l_n}(i_{n,l_n}) - \epsilon_n \theta} (t_n - t_{n-1})^{\alpha k_{n,l_n}(i_{n,l_n})}} dv_{n,l_n} = \frac{K_2}{(t_n - t_{n-1})^{\alpha(1 + \epsilon_n \theta)}},$$

where the constant K_2 depends on i_{n,l_n} , θ , and ϵ_n such that $\sup_{\theta, \epsilon_n} K_2 < \infty$.

Combining the above discussion with (3.13), we obtain

$$\begin{aligned}
 \mathcal{J}(m, \theta) &\leq m!c_1 \sum_{l_1, \dots, l_d \in \{1, \dots, d\}} \sum_{\substack{i_{n,l} \in \{1,2\} \\ n=1, \dots, m; l=1, \dots, d}} \sum_{(\epsilon_1, \dots, \epsilon_m) \in \{0,1,2\}^m} \\
 &\quad \times \int_{\Lambda_{j,k}} \prod_{n=1}^m \frac{1}{(t_n - t_{n-1})^{\alpha(d+\epsilon_n\theta)}} \prod_{n=1}^m dt_n \\
 &= m!c_2 \sum_{(\epsilon_1, \dots, \epsilon_m) \in \{0,1,2\}^m} \int_{\Lambda_{j,k}} \prod_{n=1}^m \frac{1}{(t_n - t_{n-1})^{\alpha(d+\epsilon_n\theta)}} \prod_{n=1}^m dt_n \\
 &= m!c_2 \sum_{(\epsilon_1, \dots, \epsilon_m) \in \{0,1,2\}^m} \int_{a_{j,k}}^{b_{j,k}} dt_1 \int_{t_1}^{b_{j,k}} dt_2 \cdots \int_{t_{m-1}}^{b_{j,k}} dt_m \\
 &\quad \times \prod_{n=1}^m \frac{1}{(t_n - t_{n-1})^{\alpha(d+\epsilon_n\theta)}}, \quad (3.14)
 \end{aligned}$$

where $a_{j,k} = 2^{-j}(k-1)$ and $b_{j,k} = 2^{-j}(k+1)$. Let $0 < \theta < \{(\frac{1}{\alpha} - d)/2\} \wedge 1$. We integrate in the order of $dt_m, dt_{m-1}, \dots, dt_1$, and use changes of variables in each step to construct Beta functions. Hence, (3.14) is equal to

$$\sum_{(\epsilon_1, \dots, \epsilon_m) \in \{0,1,2\}^m} K_\epsilon \int_{a_{j,k}}^{b_{j,k}} (b_{j,k} - t_1)^{(m-1)(1-d\alpha)-\alpha\theta} \sum_{i=0}^{m-2} \epsilon_{m-i} t_1^{-\alpha(d+\epsilon_1\theta)} dt_1, \quad (3.15)$$

where

$$K_\epsilon = m!c_2 \frac{1}{1-\alpha(d+\epsilon_m\theta)} \frac{\Gamma(2-\alpha(d+\epsilon_m\theta)) \prod_{i=1}^{m-2} \Gamma(1-\alpha(d+\epsilon_{m-i}\theta))}{\Gamma(1+(m-1)(1-d\alpha)-\theta \sum_{i=0}^{m-2} \epsilon_{m-i})}$$

and $\epsilon = (\epsilon_1, \dots, \epsilon_m)$. Recall that $\sum_{n=1}^m \epsilon_n = m$. Then

- If $k = 1$. According to (3.7) and (3.15) we get

$$\mathbb{E}[|A_{j,1,x,y}|^m] \leq K_1 \|x - y\|^{m\theta} 2^{-jm(1-d\alpha-\alpha\theta)}, \quad (3.16)$$

where K_1 is equal to

$$c \sum_{\epsilon \in \{0,1,2\}^m} K_\epsilon \frac{\Gamma(1+(m-1)(1-d\alpha)-\alpha\theta \sum_{i=0}^{m-2} \epsilon_{m-i}) \Gamma(1-\alpha(d+\epsilon_1\theta))}{\Gamma(1+m(1-d\alpha-\alpha\theta))}$$

with $c = 2^{-m(d+\theta-1)} \pi^{-md} 2^{m(1-d\alpha-\alpha\theta)}$.

- If $2 \leq k \leq 2^j - 1$. Therefore (3.15) is less than or equal to

$$\begin{aligned}
& \sum_{(\epsilon_1, \dots, \epsilon_m) \in \{0, 1, 2\}^m} K_\epsilon (b_{j,k} - a_{j,k})^{(m-1)(1-d\alpha) - \alpha\theta} \sum_{i=0}^{m-2} \epsilon_{m-i} \int_{a_{j,k}}^{b_{j,k}} t_1^{-\alpha(d+\epsilon_1\theta)} dt_1 \\
&= \sum_{(\epsilon_1, \dots, \epsilon_m) \in \{0, 1, 2\}^m} K_\epsilon 2^{(m-1)(1-d\alpha) - \alpha\theta} \sum_{i=0}^{m-2} \epsilon_{m-i} 2^{-jm(1-d\alpha - \alpha\theta)} \\
&\quad \times \int_{k-1}^{k+1} u^{-\alpha(d+\epsilon_1\theta)} du \\
&\leq \tilde{K}_2 2^{-jm(1-d\alpha - \alpha\theta)} \int_{k-1}^{k+1} u^{-\alpha d} du \leq \tilde{K}_2 2^{-jm(1-d\alpha - \alpha\theta)} (k-1)^{-\alpha d}.
\end{aligned} \tag{3.17}$$

Hence by (3.7) and (3.17), we have

$$\mathbb{E}[|A_{j,k,x,y}|^m] \leq K_2 \|x - y\|^{m\theta} 2^{-jm(1-d\alpha - \alpha\theta)} (k-1)^{-\alpha d}. \tag{3.18}$$

Now we return to estimate $\mathbb{E}[\{\sum_{k=1}^{2^j-1} |A_{j,k,x,y}|^q\}^{2^a}]$. Recall that $m = q2^a$. According to (3.5) and the convexity of the function $x \mapsto x^{2^a}$, we have

$$\begin{aligned}
\mathbb{E}\left[\left\{\sum_{k=1}^{2^j-1} |A_{j,k,x,y}|^q\right\}^{2^a}\right] &\leq \left\{\sum_{k=1}^{2^j-1} \mathbb{E}[|A_{j,k,x,y}|^{q2^a}]^{1/2^a}\right\}^{2^a} \\
&\leq 2^{a-1} \left(\mathbb{E}[|A_{j,1,x,y}|^{q2^a}] + \left\{\sum_{k=2}^{2^j-1} \mathbb{E}[|A_{j,k,x,y}|^{q2^a}]^{1/2^a}\right\}^{2^a}\right).
\end{aligned}$$

Hence, using (3.16) and (3.18), we derive that this last term is less than or equal to

$$\begin{aligned}
& \tilde{K} \|x - y\|^{q2^a\theta} 2^{-jq2^a(1-d\alpha - \alpha\theta)} \left(1 + \left\{\sum_{k=2}^{2^j-1} (k-1)^{-\alpha d/2^a}\right\}^{2^a}\right) \\
&\leq \tilde{K} \|x - y\|^{q2^a\theta} 2^{-jq2^a(1-d\alpha - \alpha\theta)} \left(1 + \left\{\sum_{k=2}^{2^j-1} 2 \int_{k-\frac{3}{2}}^{k-1} x^{-\alpha d/2^a} dx\right\}^{2^a}\right) \\
&\leq \tilde{K} \|x - y\|^{q2^a\theta} 2^{-jq2^a(1-d\alpha - \alpha\theta)} \left(1 + \left\{2 \int_{\frac{1}{2}}^{2^j-2} x^{-\alpha d/2^a} dx\right\}^{2^a}\right) \\
&\leq \hat{K} \|x - y\|^{q2^a\theta} 2^{-jq2^a(1-d\alpha - \alpha\theta)} (1 + 2^{2^a j - jd\alpha}).
\end{aligned}$$

Therefore

$$\mathbb{E}\left[\left\{\sum_{k=1}^{2^j-1} |A_{j,k,x,y}|^q\right\}^{2^a}\right] \leq K \|x - y\|^{q2^a\theta} 2^{-jq2^a(1-d\alpha - \alpha\theta)} 2^{j(2^a - \alpha d)}. \tag{3.19}$$

The remainder of the proof is by Chebyshev's inequality, i.e.,

$$\begin{aligned} \mathbb{P} \left[\sum_{k=1}^{2^j-1} |A_{j,k,x,y}|^q \geq 2^{-jq(1-d\alpha-\theta\alpha)+j} \|x-y\|^{q\theta} 2^{\gamma h} \right] \\ \leq 2^{j2^a q(1-d\alpha-\theta\alpha)-2^a j} \|x-y\|^{-q2^a \theta} 2^{-2^a \gamma h} \mathbb{E} \left[\left\{ \sum_{k=1}^{2^j-1} |A_{j,k,x,y}|^q \right\}^{2^a} \right]. \end{aligned}$$

Hence (3.19) concludes the proof of Lemma 3.2. $\blacksquare$

Lemma 3.3. *Let $(X_t)_{t \in [0,1]}$ be an $\mathbb{R}^d$ -valued continuous stochastic process that verifies **H**. Then, for all $0 < \delta < 1$, almost surely there exists $j_1 = j_1(\omega, \delta)$ such that*

$$\sup_{1 \leq k \leq 2^j-1} \sup_{t \in D_{j,k}} \|X(t) - X(d_{j,k})\| \leq 2^{-j(\alpha - \frac{2-\delta}{p_0})} \quad \text{for } j \geq j_1, \quad (3.20)$$

where $d_{j,k}$ and $D_{j,k}$ are as in Lemma 3.2.

Proof. According to (1.1), we have almost surely

$$B := \int_0^1 \int_0^1 \frac{\|X(t) - X(s)\|^{p_0}}{|t-s|^{\alpha p_0 + \gamma}} dt ds < \infty. \quad (3.21)$$

Then by Lemma 2.3 with $\Psi(u) = |u|^{p_0}$ and $p(u) = |u|^{\alpha + \frac{\gamma}{p_0}}$ where $\frac{1}{p_0} < \alpha$ and $2 - \alpha p_0 < \gamma < 1$, we derive

$$\sup_{1 \leq k \leq 2^j-1} \sup_{t \in D_{j,k}} \|X(t) - X(d_{j,k})\|^{p_0} \leq C_{\alpha, p_0, \gamma} B 2^{-j(\alpha p_0 + \gamma - 2)}. \quad (3.22)$$

Therefore, by (1.1), (3.21), and (3.22) we get

$$\mathbb{E} \left[\sup_{1 \leq k \leq 2^j-1} \sup_{t \in D_{j,k}} \|X(t) - X(d_{j,k})\|^{p_0} \right] \leq K_{\alpha, p_0, \gamma} 2^{-j(\alpha p_0 + \gamma - 2)}. \quad (3.23)$$

Hence, let $0 < \delta < 1$ and $2 - \alpha p_0 < \gamma < 1$ such that $\delta < \gamma$, then by Chebyshev's inequality and (3.23) we write

$$\mathbb{P} \left[\sup_{1 \leq k \leq 2^j-1} \sup_{t \in D_{j,k}} \|X(t) - X(d_{j,k})\| \geq 2^{-j(\alpha - \frac{2-\delta}{p_0})} \right] \leq K_{\alpha, p_0, \gamma} 2^{-j(\gamma - \delta)}. \quad (3.24)$$

Therefore the Borel–Cantelli lemma concludes the proof of Lemma 3.3. $\blacksquare$

Lemma 3.4. *Let $(X_t)_{t \in [0,1]}$ be an $\mathbb{R}^d$ -valued continuous stochastic process which is α -LND with $\alpha \in (0, \frac{1}{d})$. Assume also that X verifies **H**. Then for all positive integer q , almost surely we have*

$$\sup_{j \geq 1} 2^{jq(1-d\alpha)-j} \int_0^1 \sum_{k=1}^{2^j-1} \sup_{x \in \mathbb{R}^d} |L(x, 2^{-j}(s+k)) - L(x, 2^{-j}(s+k-1))|^q ds < \infty. \quad (3.25)$$

Proof. Let $d_{j,k} = 2^{-j}(k-1)$ and $D_{j,k} = [2^{-j}(k-1), 2^{-j}(k+1)]$ for $j \geq 1$ and $1 \leq k \leq 2^j - 1$. It follows from Lemma 3.3 that for all $2 - \alpha p_0 < \delta < 1$, almost surely, there exists $j_1 = j_1(\omega, \delta)$ such that

$$\sup_{1 \leq k \leq 2^j - 1} \sup_{t \in D_{j,k}} \|X(t) - X(d_{j,k})\| \leq 2^{-j(\alpha - \frac{2-\delta}{p_0})} \quad \text{for } j \geq j_1, \quad (3.26)$$

Let $\beta_j = 2^{-j\alpha}$ and

$$G_j = \{x \in \mathbb{R}^d; \|x\| \leq 2^{-j(\alpha - \frac{2-\delta}{p_0})}, \quad x = \beta_j b \quad \text{for some } b \in \mathbb{Z}^d\},$$

where $\mathbb{Z}$ is the set of integers. The cardinality of G_j verifies

$$\#G_j \leq (2[2^{j(\frac{2-\delta}{p_0})}] + 1)^d \leq 3^d 2^{jd(\frac{2-\delta}{p_0})}, \quad (3.27)$$

where $[2^{j(\frac{2-\delta}{p_0})}]$ is the integral part of $2^{j(\frac{2-\delta}{p_0})}$. It follows from (3.27) and Lemma 3.2 that

$$\begin{aligned} \mathbb{P} \left[\sum_{k=1}^{2^j-1} |L(x + X(d_{j,k}), D_{j,k})|^q \geq 2^{-jq(1-d\alpha)+j} \quad \text{for some } x \in G_j \right] \\ \leq \#G_j K 2^{-jd\alpha} \leq 3^d K 2^{-jd(\alpha - \frac{2-\delta}{p_0})}. \end{aligned} \quad (3.28)$$

Since $2 - \alpha p_0 < \delta < 1$, it follows by the Borel–Cantelli lemma that almost surely there exists $j_2 = j_2(\omega, \delta)$ such that

$$\sum_{k=1}^{2^j-1} \sup_{x \in G_j} |L(x + X(d_{j,k}), D_{j,k})|^q \leq 2^{-jq(1-d\alpha)+j} \quad \text{for } j \geq j_2. \quad (3.29)$$

For any fixed integers $j, h \geq 1$ and any $x \in G_j$, define

$$F(j, h, x) = \left\{ y \in \mathbb{R}^d; y = x + \beta_j \sum_{n=1}^h \epsilon_n 2^{-n} \quad \text{for } \epsilon_n \in \{0, 1\}^d \right\}. \quad (3.30)$$

A pair of points $y_1, y_2 \in F(j, h, x)$ is said to be linked if $y_2 - y_1 = \beta_j \epsilon 2^{-h}$ for some $\epsilon \in \{0, 1\}^d$. Then by (3.3) with γ and a such that $\gamma < q\theta$ and $2^a > \frac{d}{\gamma}$, we have

$$\begin{aligned} \mathbb{P} \left[\sum_{k=1}^{2^j-1} |A_{j,k,y_1,y_2}|^q \geq 2^{-jq(1-d\alpha-\theta\alpha)+j} \|y_1 - y_2\|^{q\theta} 2^{\gamma h} \right. \\ \left. \text{for some } x \in G_j, h \geq 1 \text{ and some linked pair } y_1, y_2 \in F(j, h, x) \right] \\ \leq \#G_j \sum_{h=1}^{\infty} 2^{hd} K 2^{-jd\alpha} 2^{-2^a \gamma h} \leq K 3^d 2^{-jd(\alpha - \frac{2-\delta}{p_0})} \sum_{h=1}^{\infty} 2^{-h(2^a \gamma - d)}. \end{aligned}$$

As $2 - \alpha p_0 < \delta < 1$ and $2^a > \frac{d}{\gamma}$, it follows by the Borel–Cantelli lemma that almost surely there exists $j_3 = j_3(\omega, \delta, \gamma)$ such that for $j \geq j_3$

$$\begin{aligned} & \sum_{k=1}^{2^j-1} |L(y_1 + X(d_{j,k}), D_{j,k}) - L(y_2 + X(d_{j,k}), D_{j,k})|^q \\ & \leq 2^{-jq(1-d\alpha-\theta\alpha)+j} \|y_1 - y_2\|^{q\theta} 2^{\gamma h}, \end{aligned} \quad (3.31)$$

for all $x \in G_j$, $h \geq 1$ and any linked pair $y_1, y_2 \in F(j, h, x)$.

Let Ω_0 be the event that (3.26), (3.29) and (3.31) hold, hence $\mathbb{P}[\Omega_0] = 1$. Let $j \geq j_4 := \max\{j_1, j_2, j_3\}$ be fixed. For any $y \in \mathbb{R}^d$ with $\|y\| \leq 2^{-j(\alpha - \frac{2-\delta}{p_0})}$, we represent y in the form $y = \lim_{h \rightarrow \infty} y_h$, where

$$y_h = x + \beta_j \sum_{n=1}^h \epsilon_n 2^{-n} \quad (y_0 = x, \epsilon_n \in \{0, 1\}^d),$$

for some $x \in G_j$. Then each pair y_{h-1}, y_h is linked, so by (3.31) and the continuity of $L(\cdot, D_{j,k})$ we have

$$\begin{aligned} & \sum_{k=1}^{2^j-1} |L(y + X(d_{j,k}), D_{j,k}) - L(x + X(d_{j,k}), D_{j,k})|^q \\ & \leq 2^{-jq(1-d\alpha-\theta\alpha)+j} \sum_{h=1}^{\infty} |\beta_j 2^{-h}|^{q\theta} 2^{\gamma h} \\ & = 2^{-jq(1-d\alpha)+j} \sum_{h=1}^{\infty} 2^{-h(q\theta-\gamma)} \end{aligned}$$

Since $\gamma < q\theta$, we have almost surely for $j \geq j_4$,

$$\sum_{k=1}^{2^j-1} |L(y + X(d_{j,k}), D_{j,k}) - L(x + X(d_{j,k}), D_{j,k})|^q \leq C 2^{-jq(1-d\alpha)+j}. \quad (3.32)$$

for all $y \in \mathbb{R}^d$ with $\|y\| \leq 2^{-j(\alpha - \frac{2-\delta}{p_0})}$. It follows from (3.29) and (3.32) that almost surely for $j \geq j_4$,

$$\sum_{k=1}^{2^j-1} |L(y + X(d_{j,k}), D_{j,k})|^q \leq C_1 2^{-jq(1-d\alpha)+j}, \quad (3.33)$$

for all $y \in \mathbb{R}^d$ with $\|y\| \leq 2^{-j(\alpha - \frac{2-\delta}{p_0})}$.

On the other hand, we have almost surely for $j \geq j_4$,

$$\begin{aligned} & \int_0^1 \sum_{k=1}^{2^j-1} \sup_{x \in \mathbb{R}^d} |L(x, 2^{-j}(s+k)) - L(x, 2^{-j}(s+k-1))|^q ds \\ & \leq \sum_{k=1}^{2^j-1} \sup_{x \in \mathbb{R}^d} |L(x, D_{j,k})|^q. \end{aligned}$$

This last term is equal to

$$\begin{aligned} \sum_{k=1}^{2^j-1} \sup_{x \in X(D_{j,k})} |L(x, D_{j,k})|^q & \leq \sum_{k=1}^{2^j-1} \sup_{y \in V} |L(y + X(d_{j,k}), D_{j,k})|^q \\ & \leq C_1 2^{-jq(1-d\alpha)+j}, \end{aligned}$$

where

$$V = \{y \in \mathbb{R}^d; \|y\| \leq 2^{-j(\alpha - \frac{2-\delta}{p_0})}\}.$$

This completes the proof of Lemma 3.4. ■

Proof of Theorem 1.2. According to Lemma 3.4 and 3.1, and Theorem 2.1(iii), we conclude the proof. ■

Now we provide the proof of Theorem 1.4, which clearly explains that if a functions local time, $L(x, t)$, is Besov regular, in t uniformly in x , then this has a significant effect on the Besov irregularity of the function itself.

Proof of Theorem 1.4. According to the occupation formula (2.9), we have for all $t \in (0, 1]$, $0 < h \leq t$, and $s \in [0, 1-h]$,

$$\begin{aligned} h &= \int_{f([s, s+h])} L(x, [s, s+h]) dx \\ &\leq \lambda_d(f([s, s+h])) \sup_{x \in \mathbb{R}^d} L(x, [s, s+h]) \\ &\leq \sup_{r, \tau \in [s, s+h]} \|f(r) - f(\tau)\|^d \sup_{x \in \mathbb{R}^d} L(x, [s, s+h]) \\ &= \sup_{r, \tau \in [0, 1]} \|f(hr + s) - f(h\tau + s)\|^d \sup_{x \in \mathbb{R}^d} L(x, [s, s+h]). \end{aligned}$$

Hence

$$h^{1/d} \leq \sup_{r, \tau \in [0, 1]} \|f(hr + s) - f(h\tau + s)\| \sup_{x \in \mathbb{R}^d} |L(x, [s, s+h])|^{1/d}.$$

Therefore, by Hölder's inequality, we derive that for all $t \in (0, 1]$ and $0 < h \leq t$,

$$\begin{aligned} (1-h)h^{1/d} &\leq \int_0^{1-h} \sup_{r, \tau \in [0, 1]} \|f(hr+s) - f(h\tau+s)\| \sup_{x \in \mathbb{R}^d} |L(x, [s, s+h])|^{1/d} ds \\ &\leq \|s \mapsto \sup_{r, \tau \in [0, 1]} \|f(hr+s) - f(h\tau+s)\|_{L^p(I(h), \mathbb{R})} \\ &\quad \times \left\{ \int_0^{1-h} \sup_{x \in \mathbb{R}^d} |L(x, [s, s+h])|^{\frac{p}{p-1}} ds \right\}^{\frac{p-1}{p}}. \end{aligned} \quad (3.34)$$

By the same calculations as above we get for all $t \in (0, 1]$ and $-t < h < 0$

$$\begin{aligned} (1-|h|)|h|^{1/d} &\leq \|s \mapsto \sup_{r, \tau \in [0, 1]} \|f(hr+s) - f(h\tau+s)\|_{L^p(I(h), \mathbb{R})} \\ &\quad \times \left\{ \int_{-h}^1 \sup_{x \in \mathbb{R}^d} |L(x, s) - L(x, s+h)|^{\frac{p}{p-1}} ds \right\}^{\frac{p-1}{p}}. \end{aligned} \quad (3.35)$$

According to (3.34), (3.35), and (1.8), we get for all $t \in (0, \frac{1}{2})$,

$$\frac{t^{1/d}}{2} \leq c t^{\mu/d} \sup_{|h| \leq t} \|s \mapsto \sup_{r, \tau \in [0, 1]} \|f(hr+s) - f(h\tau+s)\|_{L^p(I(h), \mathbb{R})}. \quad (3.36)$$

Therefore,

$$0 < \lim_{t \rightarrow 0^+} t^{-(1-\mu)/d} \sup_{|h| \leq t} \|s \mapsto \sup_{r, \tau \in [0, 1]} \|f(hr+s) - f(h\tau+s)\|_{L^p(I(h), \mathbb{R})}.$$

Then Theorems 2.1 (iv) and 2.2 conclude the proof of Theorem 1.4. $\blacksquare$

Proof of Theorem 1.5. For any $1 < p < \infty$, let $n = n(p)$ be a positive integer such that $d^n \geq \frac{p}{p-1}$. We have almost surely,

$$\begin{aligned} &\sup_{0 < t \leq 1} t^{-(1-\alpha d)/d} \sup_{|h| \leq t} \|s \mapsto \sup_{x \in \mathbb{R}^d} |L(x, s+h) - L(x, s)|^{\frac{1}{d}}\|_{L^{\frac{p}{p-1}}(I(h); \mathbb{R})} \\ &\leq \sup_{0 < t \leq 1} t^{-(1-\alpha d)/d} \sup_{|h| \leq t} \|s \mapsto \sup_{x \in \mathbb{R}^d} |L(x, s+h) - L(x, s)|^{\frac{1}{d}}\|_{L^{d^{n-1}}(I(h); \mathbb{R})}. \end{aligned}$$

Hence, Theorems 1.2 and 1.4 finish the proof of Theorem 1.5. $\blacksquare$

Proof of Corollary 1.6. The proof is a consequence of Theorem 1.5 and of the injections (2.8). $\blacksquare$

Proof of Theorem 1.1. The proof of the lower bound in (1.4) is a consequence of equation (1.2) and Frostman's theorem, see, e.g., [25, Theorem 8.8]. However, the upper bound follows from Hypothesis **H**, the joint continuity of the local time of X , and [1, Theorem 8.7.3]. $\blacksquare$

4. Examples

4.1. The Gaussian case

Let $Y^0 = (Y_t^0)_{t \in [0,1]}$ be a real-valued continuous centred Gaussian process, with $Y(0) = 0$, that satisfies the classical local nondeterminism (LND) property on $[0, 1]$. By [7, Lemma 2.3] we have for any $m \geq 2$, there exist two positive constants c_m and ε such that for every ordered points $0 = t_0 \leq t_1 < \dots < t_m \leq 1$ with $t_m - t_1 < \varepsilon$, and $(v_1, \dots, v_m) \in \mathbb{R}^m \setminus \{0\}$,

$$\text{Var}\left(\sum_{j=1}^m v_j (Y_{t_j}^0 - Y_{t_{j-1}}^0)\right) \geq c_m \sum_{j=1}^m v_j^2 \text{Var}(Y_{t_j}^0 - Y_{t_{j-1}}^0). \quad (4.1)$$

Assume also that Y^0 verifies (1.11), with some $\alpha \in (0, 1)$. Define $Y_t = (Y_t^1, \dots, Y_t^d)$, where $Y^1, \dots, Y^d$ are independent copies of Y^0 . Then Y is α -LND on $[0, 1]$. The following theorem is a consequence of Theorems 1.2 and 1.5, and Corollary 1.6.

Theorem 4.1. *Let $Y^0 = (Y_t^0)_{t \in [0,1]}$ be a real-valued continuous centred Gaussian process, with $Y(0) = 0$, that satisfies the classical local nondeterminism (LND) property on $[0, 1]$ and inequalities (1.11) with $0 < \alpha < \frac{1}{d}$. Let $Y^1, \dots, Y^d$ be independent copies of Y^0 and put $Y_t = (Y_t^1, \dots, Y_t^d)$. Denote by $L(x, t)$ the jointly continuous version of the local time of Y . Therefore*

$$\mathbb{P}[L(x, \bullet) \in \mathbf{B}_{p,\infty}^{1-d\alpha}(I; \mathbb{R}), \text{ for all } x \in \mathbb{R}^d \text{ and } p \in [1, \infty)] = 1; \quad (4.2)$$

$$\mathbb{P}[Y(\bullet) \in \mathbf{B}_{p,\infty}^{\alpha,0}(I, \mathbb{R}^d), \text{ for some } p \in (1/\alpha, \infty)] = 0; \quad (4.3)$$

$$\mathbb{P}[Y(\bullet) \in \mathbf{B}_{p,q}^{\alpha}(I, \mathbb{R}^d), \text{ for some } p \in (1/\alpha, \infty) \text{ and } q \in [1, \infty)] = 0, \quad (4.4)$$

where $I = [0, 1]$, $L(x, \bullet) : t \in I \mapsto L(x, t) \in \mathbb{R}$, and $Y(\bullet) : t \in I \mapsto Y_t \in \mathbb{R}^d$.

A particular example is $Y^0 = B^{H,K}$, a bifractional Brownian motion with $H \in (0, 1)$ and $K \in (0, 1]$; that is a real-valued centred Gaussian process, starting from zero, with covariance function

$$\mathbb{E}(B_t^{H,K} B_s^{H,K}) = \frac{1}{2K} [(t^{2H} + s^{2H})^K - |t - s|^{2HK}].$$

Notice that the case $K = 1$ corresponds to the fractional Brownian motion with Hurst parameter $H \in (0, 1)$. From [2, Lemma 3.3] we know that the bifractional Brownian motion is LND and by [2, equation (1)] the Hypothesis **H** holds with $\alpha = HK$. Therefore, The below corollary is a consequence of Theorem 4.1.

Corollary 4.2. *Let $(B_t^{H,K})_{t \in [0,1]}$ be a d -dimensional bifractional Brownian motion with $H \in (0, 1)$ and $K \in (0, 1]$, s.t. $HK < \frac{1}{d}$. Denote by $L(x, t)$ the jointly continuous*

version of the local time of $B^{H,K}$. Therefore

$$\mathbb{P}[L(x, \bullet) \in \mathbf{B}_{p,\infty}^{1-dHK}(I; \mathbb{R}), \text{ for all } x \in \mathbb{R}^d \text{ and } p \in [1, \infty)] = 1; \quad (4.5)$$

$$\mathbb{P}[B^{H,K}(\bullet) \in \mathbf{B}_{p,\infty}^{HK,0}(I, \mathbb{R}^d), \text{ for some } p \in (1/HK, \infty)] = 0; \quad (4.6)$$

$$\mathbb{P}[B^{H,K}(\bullet) \in \mathbf{B}_{p,q}^{HK}(I, \mathbb{R}^d), \text{ for some } p \in (1/HK, \infty) \text{ and } q \in [1, \infty)] = 0,$$

where $I = [0, 1]$, $L(x, \bullet) : t \in I \mapsto L(x, t) \in \mathbb{R}$, and $B^{H,K}(\bullet) : t \in I \mapsto B_t^{H,K} \in \mathbb{R}^d$.

Remark 4.3. A different approach, based on characterization of Besov spaces in terms of sequences spaces, has been used in [8] to investigate (4.6) for $d = 1$. However, to the best of our knowledge, the uniform Besov regularity results for the local times of Gaussian processes given by (4.2) and (4.5) are new and have never been considered in the literature.

4.2. The non-Gaussian case

Let us consider the following system of non-linear stochastic heat equations

$$\frac{\partial u_k}{\partial t}(t, x) = \frac{\partial^2 u_k}{\partial x^2}(t, x) + b_k(u(t, x)) + \sum_{l=1}^d \sigma_{k,l}(u(t, x)) \dot{W}^l(t, x), \quad (4.7)$$

with Neumann boundary conditions

$$u_k(0, x) = 0, \quad \frac{\partial u_k(t, 0)}{\partial x} = \frac{\partial u_k(t, 1)}{\partial x} = 0,$$

for $t \in [0, T]$, $1 \leq k \leq d$, $x \in [0, 1]$, where $u := (u_1, \dots, u_d)$. Let $\dot{W} = (\dot{W}^1, \dots, \dot{W}^d)$ be a vector of d -independent space-time white noises on $[0, T] \times [0, 1]$. Set $b = (b_k)_{1 \leq k \leq d}$ and $\sigma = (\sigma_{k,l})_{1 \leq k, l \leq d}$. We consider the below hypotheses on the coefficients $\sigma_{k,l}$ and b_k :

A1 For all $1 \leq k, l \leq d$, $\sigma_{k,l}$ and b_k are bounded and infinitely differentiable functions with their partial derivatives of all orders are bounded.

A2 The matrix σ is uniformly elliptic, i.e. there exists $\rho > 0$ such that for all $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ with $\|y\| = 1$, we get $\|\sigma(x)y\|^2 \geq \rho^2$ (where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$).

Following Walsh [35], a mild solution of (4.7) is a jointly measurable $\mathbb{R}^d$ -valued process $u = (u_1, \dots, u_d)$ such that for any $k \in \{1, \dots, d\}$, $t \in [0, T]$, and $x \in [0, 1]$,

$$\begin{aligned} u_k(t, x) = & \int_0^t \int_0^1 G_{t-r}(x, v) \sum_{l=1}^d \sigma_{k,l}(u(r, v)) W^l(dr, dv) \\ & + \int_0^t \int_0^1 G_{t-r}(x, v) b_k(u(r, v)) dv dr. \end{aligned} \quad (4.8)$$

According to [9, Theorem 5.4], we know that the solution to the system of non-linear stochastic heat equations (4.8), is $\frac{1}{4}$ -LND, and by [9, equation (2.12)] the hypothesis **H** holds with $\alpha = \frac{1}{4}$. Hence, Theorems 1.2 and 1.5, and Corollary 1.6 give the following theorem.

Theorem 4.4. *Let u be given by (4.8). Assume that $d \leq 3$ and denote by $L(\xi, t)$ the jointly continuous version of the local time of the process $\{u(t, x), t \in [0, T]\}$ for x being fixed in $(0, 1)$. Then*

$$\mathbb{P}[L(\xi, \bullet) \in \mathbf{B}_{p,\infty}^{1-d/4}(I; \mathbb{R}), \text{ for all } \xi \in \mathbb{R}^d \text{ and } p \in [1, \infty)] = 1; \quad (4.9)$$

$$\mathbb{P}[u(\bullet, x) \in \mathbf{B}_{p,\infty}^{1/4,0}(I, \mathbb{R}^d), \text{ for some } p \in (4, \infty)] = 0; \quad (4.10)$$

$$\mathbb{P}[u(\bullet, x) \in \mathbf{B}_{p,q}^{1/4}(I, \mathbb{R}^d), \text{ for some } p \in (4, \infty) \text{ and } q \in [1, \infty)] = 0, \quad (4.11)$$

where $I = [0, 1]$, $L(\xi, \bullet) : t \in I \mapsto L(\xi, t) \in \mathbb{R}$, and $u(\bullet, x) : t \in I \mapsto u(t, x) \in \mathbb{R}^d$.

Remark 4.5. In [10], we have studied, for $d = 1$ and $\sigma = 1$, i.e., $u(t, x)$ is the solution to the linear stochastic heat equation, by a different method the following:

$$\mathbb{P}[u(\bullet, x) \in \mathbf{B}_{p,\infty}^{1/4,0}(I, \mathbb{R})] = 0.$$

However, (4.9) is new even in the linear case.

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References

- [1] R. J. Adler, *The geometry of random fields*. Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, Ltd., Chichester, 1981 Zbl 0478.60059 MR 0611857
- [2] M. Ait Ouahra, H. Ouahhabi, and A. Sghir, *Continuity in law of some additive functionals of bifractional Brownian motion*. *Stochastics* **91** (2019), no. 4, 613–628 Zbl 1495.60024 MR 3939348
- [3] S. M. Berman, *Harmonic analysis of local times and sample functions of Gaussian processes*. *Trans. Amer. Math. Soc.* **143** (1969), 269–281 Zbl 0186.50802 MR 0248905
- [4] S. M. Berman, *Local times and sample function properties of stationary Gaussian processes*. *Trans. Amer. Math. Soc.* **137** (1969), 277–299 Zbl 0184.40801 MR 0239652
- [5] S. M. Berman, *Gaussian sample functions: Uniform dimension and Hölder conditions nowhere*. *Nagoya Math. J.* **46** (1972), 63–86 Zbl 0246.60038 MR 0307320
- [6] S. M. Berman, *Local nondeterminism and local times of general stochastic processes*. *Ann. Inst. H. Poincaré Sect. B (N.S.)* **19** (1983), no. 2, 189–207 Zbl 0516.60047 MR 700709
- [7] S. M. Berman and R. Gettoor, *Local nondeterminism and local times of Gaussian processes*. *Indiana Univ. Math. J.* **23** (1973), no. 1, 69–94 Zbl 0264.60024 MR 0317397

- [8] B. Boufoussi and Y. Nachit, [On the Besov regularity of the bifractional Brownian motion](#). *Probab. Math. Statist.* **41** (2021), no. 2, 303–320 Zbl [1485.60041](#) MR [4359825](#)
- [9] B. Boufoussi and Y. Nachit, [Local times for systems of non-linear stochastic heat equations](#). *Stoch. Partial Differ. Equ. Anal. Comput.* **11** (2023), no. 1, 388–425 Zbl [1517.60073](#) MR [4563704](#)
- [10] B. Boufoussi and Y. Nachit, [On Besov regularity and local time of the solution to the stochastic heat equation](#). *Stochastics* **95** (2023), no. 6, 1120–1145 Zbl [1525.60043](#) MR [4608898](#)
- [11] B. Boufoussi and Y. Ouknine, [Régularité du temps local du processus symétrique stable en norme Besov](#). *Stochastics Stochastics Rep.* **66** (1999), no. 3-4, 167–175 Zbl [0929.60056](#) MR [1692888](#)
- [12] B. Boufoussi and B. Roynette, Le temps local brownien appartient p.s. à l'espace de Besov $\mathcal{B}_{p,\infty}^{1/2}$. *C. R. Acad. Sci. Paris Sér. I Math.* **316** (1993), no. 8, 843–848 Zbl [0788.46035](#) MR [1218273](#)
- [13] M. Csörgő, Z.-Y. Lin, and Q.-M. Shao, [On moduli of continuity for local times of Gaussian processes](#). *Stochastic Process. Appl.* **58** (1995), no. 1, 1–21 Zbl [0834.60088](#) MR [1341551](#)
- [14] J. Cuzick, [Local nondeterminism and the zeros of Gaussian processes](#). *Ann. Probability* **6** (1978), no. 1, 72–84 Zbl [0374.60051](#) MR [0488252](#)
- [15] J. Cuzick, [Continuity of Gaussian local times](#). *Ann. Probab.* **10** (1982), no. 3, 818–823 Zbl [0492.60033](#) MR [0659551](#)
- [16] J. Cuzick, [Multiple points of a Gaussian vector field](#). *Z. Wahrsch. Verw. Gebiete* **61** (1982), no. 4, 431–436 Zbl [0504.60052](#) MR [0682570](#)
- [17] J. Cuzick and J. P. DuPreez, [Joint continuity of Gaussian local times](#). *Ann. Probab.* **10** (1982), no. 3, 810–817 Zbl [0492.60032](#) MR [0659550](#)
- [18] M. Dozzi, [Occupation density and sample path properties of \$N\$ -parameter processes](#). In: *Topics in spatial stochastic processes (Martina Franca, 2001)*, pp. 127–166. Lecture Notes in Math. 1802, Springer, Berlin, 2003 Zbl [1042.60031](#) MR [1975519](#)
- [19] A. M. Garsia, E. Rodemich, H. Rumsey, and M. Rosenblatt, [A real variable lemma and the continuity of paths of some Gaussian processes](#). *Indiana Univ. Math. J.* **20** (1970), no. 6, 565–578 Zbl [0252.60020](#) MR [0267632](#)
- [20] D. Geman and J. Horowitz, [Occupation densities](#). *Ann. Probab.* **8** (1980), no. 1, 1–67 Zbl [0499.60081](#) MR [0556414](#)
- [21] T. P. Hytönen and M. C. Veraar, [On Besov regularity of Brownian motions in infinite dimensions](#). *Probab. Math. Statist.* **28** (2008), no. 1, 143–162 Zbl [1136.60358](#) MR [2445509](#)
- [22] D. Khoshnevisan and Y. Xiao, [Images of the Brownian sheet](#). *Trans. Amer. Math. Soc.* **359** (2007), no. 7, 3125–3151 Zbl [1124.60037](#) MR [2299449](#)
- [23] H. König, [Eigenvalue distribution of compact operators](#). Oper. Theory Adv. Appl. 16, Birkhäuser, Basel, 1986 Zbl [0618.47013](#) MR [0889455](#)
- [24] S. Lou and C. Ouyang, [Local times of stochastic differential equations driven by fractional Brownian motions](#). *Stochastic Process. Appl.* **127** (2017), no. 11, 3643–3660 Zbl [1381.60094](#) MR [3707240](#)

- [25] P. Mattila, *Geometry of sets and measures in Euclidean spaces: Fractals and rectifiability*. Cambridge Stud. Adv. Math. 44, Cambridge University Press, Cambridge, 1995
Zbl 0911.28005 MR 1333890
- [26] D. Monrad and L. D. Pitt, *Local nondeterminism and Hausdorff dimension*. Progr. Probab. Statist. 13, pp. 163–189. Birkhäuser Boston, Boston, MA, 1987 Zbl 0616.60049
MR 0902433
- [27] D. Monrad and H. Rootzén, *Small values of Gaussian processes, functional laws of the iterated logarithm*. *Probab. Theory Related Fields* **101** (1995), no. 2, 173–192
Zbl 0821.60043 MR 1318191
- [28] J. P. Nolan, *Local nondeterminism and local times for stable processes*. *Probab. Theory Related Fields* **82** (1989), no. 3, 387–410 Zbl 0659.60106 MR 1001520
- [29] L. D. Pitt, *Local times for Gaussian vector fields*. *Indiana Univ. Math. J.* **27** (1978), no. 2, 309–330 Zbl 0382.60055 MR 0471055
- [30] L. D. Pitt and L. T. Tran, *Local sample path properties of Gaussian fields*. *Ann. Probab.* **7** (1979), no. 3, 477–493 Zbl 0401.60035 MR 0528325
- [31] B. Roynette, *Mouvement brownien et espaces de Besov*. *Stochastics Stochastics Rep.* **43** (1993), no. 3-4, 221–260 Zbl 0808.60071
- [32] M. Talagrand, *Hausdorff measure of trajectories of multiparameter fractional Brownian motion*. *Ann. Probab.* **23** (1995), no. 2, 767–775 Zbl 0830.60034 MR 1334170
- [33] M. Talagrand, *Multiple points of trajectories of multiparameter fractional Brownian motion*. *Probab. Theory Related Fields* **112** (1998), no. 4, 545–563 Zbl 0928.60026
MR 1664704
- [34] H. Triebel, *Interpolation theory, function spaces, differential operators*. Johann Ambrosius Barth, Heidelberg, 1995 Zbl 0830.46028 MR 1328645
- [35] J. B. Walsh, *An introduction to stochastic partial differential equations*. In: *École d'été de probabilités de Saint-Flour, XIV–1984*, pp. 265–439. Lecture Notes in Math. 1180, Springer, Berlin, 1986 Zbl 0608.60060 MR 0876085
- [36] Y. Xiao, *Hausdorff measure of the sample paths of gaussian random fields*. *Osaka J. Math.* **33** (1996), no. 4, 895–913 Zbl 0872.60030 MR 1435460
- [37] Y. Xiao, *Hausdorff measure of the graph of fractional Brownian motion*. *Math. Proc. Cambridge Philos. Soc.* **122** (1997), no. 3, 565–576 Zbl 0897.60043 MR 1466658
- [38] Y. Xiao, *Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields*. *Probab. Theory Related Fields* **109** (1997), no. 1, 129–157
Zbl 0882.60035 MR 1469923
- [39] Y. Xiao, *Weak variation of Gaussian processes*. *J. Theoret. Probab.* **10** (1997), no. 4, 849–866 Zbl 0890.60035 MR 1481651
- [40] Y. Xiao, *Properties of local-nondeterminism of Gaussian and stable random fields and their applications*. *Ann. Fac. Sci. Toulouse Math. (6)* **15** (2006), no. 1, 157–193
Zbl 1128.60041 MR 2225751
- [41] Y. Xiao, *Strong local nondeterminism and the sample path properties of Gaussian random fields*. In: *Asymptotic theory in probability and statistics with applications*, Adv. Lect. Math. (ALM) 2, pp. 136–176. Int. Press, Somerville, MA, 2008 Zbl 1269.60043
MR 2466984

- [42] Y. Xiao, [Sample path properties of anisotropic Gaussian random fields](#). In: *A minicourse on stochastic partial differential equations*, Lecture Notes in Mathematics 1962, pp. 145–212. Springer, Berlin, 2009 Zbl [1167.60011](#) MR [2508776](#)
- [43] Y. Xiao and T. Zhang, [Local times of fractional Brownian sheets](#). *Probab. Theory Related Fields* **124** (2002), no. 2, 204–226 Zbl [1009.60024](#) MR [1936017](#)

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