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Incidence estimates for tubes in complex space

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Abstract. In this paper, we prove a complex version of the incidence estimate of Guth, Solomon and Wang (2019) for tubes obeying certain strong spacing conditions, and we use one of our new estimates to resolve a discretized variant of Falconer's distance set problem in $\mathbb{C}^2$.

1. Introduction

It has long been known that in Fourier analysis and related fields, only some of the results true in the real space hold in the complex space as well. For instance,

- the Kakeya problem in dimension three under the Wolff axioms is true over $\mathbb{R}$ [8] but false over $\mathbb{C}$ because of the Heisenberg example [4];
- the Falconer distance problem and its cousins, the Erdős ring problem and the Furstenberg problem, have counterexamples over $\mathbb{C}$ because of the existence of the quadratic subfield $\mathbb{R}$ [9], but are still open over $\mathbb{R}$;
- the Szemerédi–Trotter theorem is true over both $\mathbb{R}$ [6] and $\mathbb{C}$ [7];
- the Erdős distinct distances problem, with an ε-loss, is true over both $\mathbb{R}$ [2] and $\mathbb{C}$ [5].

The results in this paper are complex analogues of the incidence estimates and distance estimates obtained in [3], giving some more examples of problems that have positive answers over both $\mathbb{R}$ and $\mathbb{C}$.

By "complex lines", we mean the translations of 1-dimensional vector subspaces of $\mathbb{C}^n$ (over the field $\mathbb{C}$), and "complex tubes" refers to small neighbourhoods of complex line segments. By "the angle between the tubes", we mean the angle between the line segments generating the tubes. One intricacy of this paper is in defining the angle between any two complex directions. There are two common ways to do so, one via an explicit metric on $\mathbb{CP}^{n-1}$, and another via a variational formula in $\mathbb{R}^{2n}$. The two turn out to be equivalent, and details are spelled out in Section 2. We implicitly invoke the metric definition when discussing the partition of $\mathbb{CP}^{n-1}$ into tiny parts.

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Some more terminology we are using: A *δ-ball* is a ball of radius δ, a *δ-tube* is the δ-neighbourhood of a unit-length line segment. We call a subset $\theta \subset \mathbb{CP}^{n-1}$ an *almost-δ-cap* if there exists a point $\boldsymbol{u} \in \mathbb{CP}^{n-1}$ such that

$$B_{\mathbb{CP}^{n-1}}\left(\boldsymbol{u}, \frac{1}{2}\delta\right) \subseteq \theta \subseteq B_{\mathbb{CP}^{n-1}}(\boldsymbol{u}, 2\delta).$$

We say two solids $A_1, A_2 \subseteq \mathbb{C}^n \simeq \mathbb{R}^{2n}$ (e.g., balls, complex tubes, etc.) essentially intersect if their intersection is at least half as large as the maximum intersection between all rigid transformations of the objects, i.e.,

$$|A_1 \cap A_2| \geqslant \frac{1}{2} \max_{\sigma_1, \sigma_2 \in \mathrm{SE}(2n)} |\sigma_1(A_1) \cap \sigma_2(A_2)|.$$

Two solids of the same kind are said to be essentially distinct if they do not essentially intersect, or else they are said to be essentially the same. For example, two δ-tubes T_1 and T_2 are essentially distinct if the volume of $T_1 \cap T_2$ is no more than half the volume of T_1. One solid A_1 is said to essentially contain another solid A_2 if A_1 contains at least half of A_2, i.e.,

$$|A_2 \setminus A_1| < \frac{1}{2}|A_2|.$$

Following common conventions, we write $X \lesssim_\alpha Y$ to mean that there exists a positive constant $C_{\alpha,n}$ depending only on the parameter α and the dimension n such that $X \leqslant C_{\alpha,n} Y$. The dependency on the dimension n is often suppressed in the notation. We write $X \sim_\alpha Y$ to mean that $X \lesssim_\alpha Y \lesssim_\alpha X$. Plus, we write $X(\delta) \ll Y(\delta)$ if $\lim_{\delta \to 0} \frac{X(\delta)}{Y(\delta)} = 0$.

Given a collection $\mathbb{T}$ of solids of the same kind (e.g., complex tubes), we denote by $P_r(\mathbb{T})$ a maximal[1] collection of essentially distinct r-rich δ-balls. Now, we are ready to describe our main theorems. The precise spacing condition is stated later when we prove the theorems. (See the statements of Theorems 4.1, 4.2 and 5.2, which correspond to Theorems 1.1, 1.2 and 1.3, respectively.)

Theorem 1.1. *Let Θ be a partition of $\mathbb{CP}^{n-1}$ into almost-δ-caps and suppose that $1 \leqslant W \leqslant \delta^{-1}$ and $1 \leqslant N \leqslant W^{-1}\delta^{-1}$. For each $\theta \in \Theta$, let $\{T_{\theta,j}\}_{1 \leqslant j \lesssim N W^{2(n-1)}}$ be a family of essentially distinct complex δ-tubes contained in the unit ball $B_{\mathbb{C}^n}(\mathbf{0}, 1)$*

[1] That is to say, we first consider the collection of all the δ-balls that are essentially contained in the intersection of at least r of the solids $T \in \mathbb{T}$, i.e., r-rich, and then take a subcollection $P_r(\mathbb{T})$ consisting of essentially distinct δ-balls such that every other r-rich δ-ball in the super-collection is essentially the same as one of the δ-balls in the subcollection. There can be a little ambiguity as to what exactly this subcollection $P_r(\mathbb{T})$ is; but since all the maximal subcollections have about the same size, we just fix any one of them.

with direction in θ such that every complex W^{-1}-tube contained in the unit ball with direction in θ essentially contains N of the complex tubes $T_{\theta,j}$. Let $\mathbb{T}$ denote the set of all the tubes $T_{\theta,j}$. Then, for richness $r \gtrsim_\varepsilon \delta^{2(n-1)-\varepsilon}|\mathbb{T}|$,

$$|P_r(\mathbb{T})| \lesssim_\varepsilon \delta^{-\varepsilon} W^{-2(n-1)} r^{-2} |\mathbb{T}|^2.$$

Theorem 1.2. *Let $n = 2$ or 3, $1 \leqslant W \leqslant \delta^{-1}$, and $\mathbb{T}$ be a collection of complex δ-tubes essentially contained in $B_{\mathbb{C}^n}(\mathbf{0}, 1)$. Suppose each complex W^{-1}-tube essentially contained in $B_{\mathbb{C}^n}(\mathbf{0}, 1)$ contains N_0 complex δ-tube $T \in \mathbb{T}$. Then, for*

$$r \geqslant \max\{\delta^{2(n-1)-\varepsilon}|\mathbb{T}|, 1 + N_0\},$$

we have

$$|P_r(\mathbb{T})| \lesssim_{\varepsilon,n,N} \delta^{-\varepsilon} |\mathbb{T}|^{\frac{n}{n-1}} r^{-\frac{n+1}{n-1}}.$$

Corollary 1.3. *Fix $2 < s < 4$. Let E be a set of $\sim \delta^{-s}$ many δ-balls in $B_{\mathbb{C}^2}(\mathbf{0}, 1)$ with at most N δ-balls in each ball of radius $\delta^{s/4}$. Then, the number of disjoint δ-balls needed to cover the difference set*

$$\Delta(E) := \left\{(x_1 - x_2)^2 + (y_1 - y_2)^2 \in \mathbb{C} : \vec{p}_1 = (x_1, y_1), \vec{p}_2 = (x_2, y_2) \in \bigcup E\right\}$$
$$\gtrsim_{\varepsilon,s,N} \delta^{-s+\varepsilon}$$

for all $\varepsilon > 0$.

In Section 2, we review some definitions and conventions that are used in the rest of the paper. In Section 3, we prove a lemma which allows us to capture the behaviour of a large number of small balls using a few slightly larger balls, and thus lays the foundation for induction on scale. In Section 4, we consider two strong spacing conditions on complex tubes, and prove an upper bound on the number of rich balls in each case. And lastly, in Section 5, we derive a lower bound on the number of distinct complex distances for certain sparse sets in $\mathbb{C}^2$ by reducing the problem to one about incidences between complex tubes.

2. Definitions, notations and conventions

As mentioned in the introduction, there are two equivalent ways to define the angle between two complex lines, say

$$\ell_1 = \{z\boldsymbol{u}_1 + \boldsymbol{t}_1 \in \mathbb{C}^n : z \in \mathbb{C}\} \quad \text{and} \quad \ell_2 = \{z\boldsymbol{u}_2 + \boldsymbol{t}_2 \in \mathbb{C}^n : z \in \mathbb{C}\},$$

with $\|\boldsymbol{u}_1\| = \|\boldsymbol{u}_2\| = 1$:

(1) Treat the complex lines as 2-planes in $\mathbb{R}^{2n}$, and define the angle between the complex lines to be the first principal angle between the planes

$$\min_{z_1,z_2\in\mathbb{C}:|z_1|=|z_2|=1} \arccos\big(\iota(z_1\boldsymbol{u}_1)\cdot\iota(z_2\boldsymbol{u}_2)\big),$$

where $\iota(\boldsymbol{u})$ denotes the usual embedding of $\boldsymbol{u}\in\mathbb{C}^n$ into $\mathbb{R}^{2n}$;

(2) Define the angle numerically to be $\arccos|\langle\boldsymbol{u}_1,\boldsymbol{u}_2\rangle|$, where $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{C}^n$; this is also the Fubini–Study distance, up to a scaling constant, between the equivalent classes of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in $\mathbb{CP}^{n-1}$.

Note that for 2-planes in $\mathbb{R}^{2n}$ corresponding to complex tubes, the two principal angles are equal[2]. Hence, the study of incidences between complex lines in $\mathbb{C}^n$ is different from the study of incidences between general 2-planes in $\mathbb{R}^{2n}$.

To see the equivalence, write

$$\boldsymbol{u}_1=(a_1+b_1i,c_1+d_1i)\text{ and }\boldsymbol{u}_2=(a_2+b_2i,c_2+d_2i),$$

with $a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2\in\mathbb{R}$. Then, for any $z=e^{i\zeta}$ with $\zeta\in\mathbb{R}$,

$$\begin{aligned}\iota(\boldsymbol{u}_1)\cdot\iota(z\boldsymbol{u}_2)=(a_1a_2+b_1b_2+c_1c_2+d_1d_2)\cos\zeta&\\ +(-a_1b_2+a_2b_1-c_1d_2+c_2d_1)\sin\zeta.&\end{aligned}$$

Hence,

$$\begin{aligned}&\min_{z_1,z_2\in\mathbb{C}:|z_1|=|z_2|=1}\arccos\big(\iota(z_1\boldsymbol{u}_1)\cdot\iota(z_2\boldsymbol{u}_2)\big)\\ &\quad=\arccos\big(\max_{z\in\mathbb{C}:|z|=1}\iota(\boldsymbol{u}_1)\cdot\iota(z\boldsymbol{u}_2)\big)\\ &\quad=\arccos\sqrt{(a_1a_2+b_1b_2+c_1c_2+d_1d_2)^2+(-a_1b_2+a_2b_1-c_1d_2+c_2d_1)^2}\\ &\quad=\arccos|\langle\boldsymbol{u}_1,\boldsymbol{u}_2\rangle|.\end{aligned}$$

The following proposition implies that two tubes with the same centre are essentially distinct only if the angle between them is $\gtrsim\delta$. More generally, if tubes T_1 and T_2 intersect with

$$|T_1\cap T_2|>\frac{1}{2}\max_{\boldsymbol{t}\in\mathbb{C}^n}(|T_1\cap(T_2+\boldsymbol{t})|),$$

then T_1 and T_2 are essentially distinct only if the angle between T_1 and T_2 is $\gtrsim\delta$.

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[2]Let's assume without loss of generality that $\boldsymbol{t}_1=\boldsymbol{t}_2=\boldsymbol{0}$. Suppose the first principal angle is attained for the pair of unit vectors $(\boldsymbol{v}_1,\boldsymbol{v}_2)\in e^{i\mathbb{R}}\boldsymbol{u}_1\times e^{i\mathbb{R}}\boldsymbol{u}_2$. Since the orthogonal complement of $\iota(\boldsymbol{v}_j)$ in $\iota(\ell_j)$ is $\mathrm{span}_{\mathbb{R}}(\iota(i\boldsymbol{v}_j))$ for each $j\in\{1,2\}$, the second principal angle between $\iota(\ell_1)$ and $\iota(\ell_2)$ is $\arccos(\iota(i\boldsymbol{v}_1)\cdot\iota(i\boldsymbol{v}_2))=\arccos(\iota(\boldsymbol{v}_1)\cdot\iota(\boldsymbol{v}_2))$.

Proposition 2.1. *Consider δ-neighbourhoods of the lines*

$$\ell_1 = \{z\boldsymbol{u}_1 : z \in \mathbb{C}\} \quad \textit{and} \quad \ell_2 = \{z\boldsymbol{u}_2 : z \in \mathbb{C}\}$$

for some direction vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ that form angle $\theta > 0$ to each other, according to the definition above. Let $\iota : \mathbb{C}^n \to \mathbb{R}^{2n}$ be the usual embedding. Then

$$|\iota(N_\delta(\ell_1)) \cap \iota(N_\delta(\ell_2))| \sim \frac{\delta^{2n}}{\sin^2\theta}. \tag{2.1}$$

We prove this proposition in the appendix.

Given any shape $A \subseteq \mathbb{C}^n \simeq \mathbb{R}^{2n}$ and any scalar $b \in [0, \infty)$, we denote by bA the uniformly scaled shape $\{b(\boldsymbol{\alpha} - \boldsymbol{c}_A) + \boldsymbol{c}_A : \boldsymbol{\alpha} \in A\}$, with $\boldsymbol{c}_A$ being the centre of mass of A.

Given any shape $A \subseteq \mathbb{C}^n$, we define its dual, denoted A^*, as follows:

$$A^* := \left\{\boldsymbol{x} \in \mathbb{C}^n : |\boldsymbol{x} \cdot (\boldsymbol{y} - \boldsymbol{c}_A)| < 1 \text{ for all } \boldsymbol{y} \in A\right\}.$$

For example, consider a tube of radius $\sim \delta$ and length ~ 1 in $\mathbb{R}^n$, not necessarily centred at the origin. Its dual is a slab of radius $\sim \delta^{-1}$ and thickness ~ 1 centred at the origin.

By fixing a bump function for the unit ball, and then pre-composing the bump with translation and isotropic scaling (and rotation if we wish), we can construct smooth bump functions for balls in $\mathbb{C}^n$ in a uniform way. In a similar manner, we can uniformly construct bump functions for complex tubes as well. These uniformly constructed bump functions are called "the" bump functions for the balls and the tubes.

3. A bridge between different scales

To study the high-frequency part of a later integral, we need the following geometric fact.

Fact 3.1. Let $\mathbb{T}$ be any collection of essentially distinct complex tubes of length ~ 1 and radius $\sim D$, and $\boldsymbol{\omega}$ be any point in $B_{\mathbb{C}^n}(\mathbf{0}, \lambda) \setminus B_{\mathbb{C}^n}(\mathbf{0}, \rho)$, with $\lambda D^{-1} \ll \lambda^{-1} \ll \rho \ll 1$ (say $\lambda := D^{\frac{\varepsilon}{100n}}$ and $\rho := D^{\varepsilon^3}\lambda^{-1}$). Then, the number of enlarged dual complex slabs λT^* (centred at the origin of width $\sim \lambda$ and thickness $\sim \lambda D^{-1}$) passing through the point $\boldsymbol{\omega}$, not counting any multiplicity[3], is

$$\lesssim \lambda^2 \|\boldsymbol{\omega}\|^{-2} D^{2(n-2)}.$$

[3]For example, imagine two complex tubes T_1 and T_2 such that T_2 is a translate of a tube that is not essentially distinct from T_1. Then, the two dual complex slabs T_1^* and T_2^* are not essentially distinct. If both λT_1^* and λT_2^* pass through the point $\boldsymbol{\omega}$, they would count as one enlarged dual slab rather than two.

Proof. It is not hard to see from symmetry that the maximum is no more than a constant multiple of the average number of enlarged dual slabs in a maximal D^{-1}-separated collection that pass through a point on the sphere $\partial B_{\mathbb{C}^n}(\mathbf{0}, \|\boldsymbol{\omega}\|)$. A double-counting argument then gives

$$\begin{aligned}\#\{\text{enlarged dual slabs passing through } \boldsymbol{\omega}\} &\sim \frac{D^{2(n-1)} \cdot (\lambda D^{-1})^2 \|\boldsymbol{\omega}\|^{2(n-2)}}{\|\boldsymbol{\omega}\|^{2(n-1)}} \\ &= \lambda^2 \|\boldsymbol{\omega}\|^{-2} D^{2(n-2)}. \end{aligned}$$ ∎

In the proof below, we do not make a clear distinction between integrals over (subsets of) $\mathbb{C}^n$ and integrals over (subsets of) $\mathbb{R}^{2n}$. The readers shall treat all integrals as over $\mathbb{R}^{2n}$, especially when Fourier transforms are involved.

Lemma 3.2. *Suppose P is a set of essentially distinct unit balls in $B_{\mathbb{C}^n}(0, D)$, and $\mathbb{T}$ is a set of essentially distinct tubes of length D and radius 1 in $B_{\mathbb{C}^n}(0, D)$ such that each ball of P lies in $\sim E$ (more specifically, at least E and less than $2E$) tubes of $\mathbb{T}$. Then, for any small $\varepsilon > 0$ and $1 \ll \rho^{-1} \ll \lambda \ll D$ (say $\lambda := D^{\frac{\varepsilon}{100n}}$ and $\rho := D^{\varepsilon^3}\lambda^{-1}$), at least one of the following is true:*

1. *(Thin case) $|P| \lesssim (\log D)\lambda^2 \rho^{-2} E^{-2} D^{2(n-1)} |\mathbb{T}|$.*
2. *(Thick case) There is a set of finitely overlapping 2λ-balls Q_j such that*
   - (a) *$\bigcup_j Q_j$ contains a $\gtrsim (\log D)^{-1}$ fraction of the balls $q \in P$; and*
   - (b) *each Q_j intersects $\gtrsim E\lambda^{-2}\rho^{-2n}$ tubes $T \in \mathbb{T}$.*

Proof. Let us begin with some pigeonholing. For each dyadic $E \lesssim k \lesssim D^{2(n-1)}$, let $P_k \subset P$ be the set of balls contained in $\sim k$ of the thickened tubes $\mathcal{N}_\lambda(T)$. Then, there exists an index k_0 for which

$$|P_{k_0}| \gtrsim_n (\log D)^{-1} |P|.$$

We study the subset $P' := P_{k_0}$ instead of P.

Define

$$f := \sum_{q \in P'} \psi_q, \qquad g := \sum_{T \in \mathbb{T}} \psi_T,$$

and

$$I(P', \mathbb{T}) := \left\{ (q, T) \in P' \times \mathbb{T} : |q \cap T| \geqslant \frac{1}{2}|q| \right\}.$$

Then, treating f and g as functions on $\mathbb{R}^{2n}$,

$$I(P', \mathbb{T}) \lesssim \int fg = \int \widehat{f}\,\overline{\widehat{g}} = \int \eta \widehat{f}\,\overline{\widehat{g}} + \int (1 - \eta)\widehat{f}\,\overline{\widehat{g}},$$

where η is "the" smooth bump function approximating $\chi_{B(\mathbf{0},\rho)}$.

In the low frequency case,

$$I(P', \mathbb{T}) \lesssim \int \eta \widehat{f}\, \overline{\widehat{g}} = \int f(g * \widecheck{\eta}) = \sum_{q \in P'} \sum_{T \in \mathbb{T}} \int \psi_q(\psi_T * \widecheck{\eta}).$$

Since $\widecheck{\eta}$ decays rapidly outside of $B(\mathbf{0}, \rho^{-1})$, $\lambda \gg \rho^{-1} \gg 1$ and $\operatorname{supp} \psi_T \subseteq 2T$,

$$|\psi_T * \widecheck{\eta}| \lesssim \sup_{\boldsymbol{\omega} \in \mathbb{C}^n} |B(\boldsymbol{\omega}, \lambda) \cap 2T| \cdot \|\widecheck{\eta}\|_{L^\infty} \sim \lambda^2 \cdot \rho^{2n}.$$

Hence, for each $q \in P'$,

$$\sum_{T \in \mathbb{T}} \int \psi_q(\psi_T * \widecheck{\eta}) \lesssim \lambda^2 \rho^{2n} \cdot \#\big\{T \in \mathbb{T} : q \cap \mathcal{N}_\lambda(T) \neq \varnothing\big\}.$$

It follows that

$$E|P'| \sim I(P', \mathbb{T}) \lesssim \sum_{q \in P'} \lambda^2 \rho^{2n} \cdot \#\big\{T \in \mathbb{T} : q \cap \mathcal{N}_\lambda(T) \neq \varnothing\big\}.$$

Thus, for balls $q \in P' = P_{k_0}$, we have

$$\#\big\{T \in \mathbb{T} : q \cap \mathcal{N}_\lambda(T) \neq \varnothing\big\} \gtrsim E\lambda^{-2}\rho^{-2n}.$$

Note that

$$\big\{T \in \mathbb{T} : q \cap \mathcal{N}_\lambda(T) \neq \varnothing\big\} = \big\{T \in \mathbb{T} : T \cap \mathcal{N}_\lambda(q) \neq \varnothing\big\},$$

and each neighbourhood $\mathcal{N}_\lambda(q)$ can be covered by a 2λ-ball.

As for the high frequency case,

$$I(P', \mathbb{T}) \lesssim \int (1-\eta) \widehat{f}\, \overline{\widehat{g}} \leqslant \Big(\int (1-\eta)|\widehat{f}|^2\Big)^{\frac{1}{2}} \Big(\int (1-\eta)|\widehat{g}|^2\Big)^{\frac{1}{2}}$$

where

$$\Big(\int (1-\eta)|\widehat{f}|^2\Big)^{\frac{1}{2}} \leqslant \|\widehat{f}\|_{L^2} = \|f\|_{L^2} \sim |P'|^{\frac{1}{2}}.$$

Group the tubes in $\mathbb{T}$ according to their directions: Let Θ be a partition of $\mathbb{CP}^{n-1}$ into almost-D^{-1}-caps θ. Define $\mathbb{T}_\theta$ to be the set of tubes in $\mathbb{T}$ with direction in θ, and set

$$g_\theta := \sum_{T \in \mathbb{T}_\theta} \psi_T.$$

For each $\theta \in \Theta$, fix a direction $\boldsymbol{v}_\theta \in \theta$ and a tube T_θ centred at the origin of length ~ 1 and radius $\sim D^{-1}$ in direction $\boldsymbol{v}_\theta$ which is contained in all tubes centred at the origin of length ~ 1 and radius $\sim D^{-1}$ with direction in θ. Then, for each $\theta \in \Theta$ and $T \in \mathbb{T}_\theta$, $\widehat{\psi_T}$ decays rapidly outside of T^*_θ. So,

$$\begin{aligned}
&\int (1-\eta)|\widehat{g}|^2 \\
&\quad = \int (1-\eta)\Big|\sum_{\theta\in\Theta} \widehat{g_\theta}\Big|^2 \\
&\quad \sim \int (1-\eta(\boldsymbol{\omega}))\Big|\sum_{\theta:\lambda T^*_\theta \ni \boldsymbol{\omega}} \widehat{g_\theta}(\boldsymbol{\omega})\Big|^2 d\boldsymbol{\omega} + \int \big(1-\eta(\boldsymbol{\omega})\big)\Big|\sum_{\theta:\lambda T^*_\theta \not\ni \boldsymbol{\omega}} \widehat{g_\theta}(\boldsymbol{\omega})\Big|^2 d\boldsymbol{\omega} \\
&\quad \lesssim \int_{\mathbb{C}^n \setminus B_{\mathbb{C}^n}(\mathbf{0},\rho)} \big(1-\eta(\boldsymbol{\omega})\big)\cdot \#\{\theta\in\Theta : \lambda T^*_\theta \ni \boldsymbol{\omega}\}\cdot \sum_{\theta\in\Theta} |\widehat{g_\theta}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} \\
&\quad \lesssim \lambda^2\rho^{-2}D^{2(n-2)}\sum_{\theta\in\Theta}\int |\widehat{g_\theta}|^2.
\end{aligned}$$

where, in the last inequality, we used Fact 3.1, and

$$\sum_{\theta\in\Theta}\int |\widehat{g_\theta}|^2 = \sum_{\theta\in\Theta}\int |g_\theta|^2 \lesssim \sum_{T\in\mathbb{T}}\int |\psi_T|^2 \sim |\mathbb{T}|\cdot D^2.$$

Putting things together, we get

$$E|P'| \sim I(P',\mathbb{T}) \lesssim |P|^{\frac{1}{2}}\cdot \lambda\rho^{-1}D^{n-2}\cdot|\mathbb{T}|^{\frac{1}{2}}D,$$

which reduces to

$$|P'| \lesssim E^{-2}\lambda^2\rho^{-2}D^{2(n-1)}|\mathbb{T}|. \qquad \blacksquare$$

4. Proof of the main theorems

We rephrase here Theorem 1.1.

Theorem 4.1. *Suppose that $1 \leqslant W \leqslant \delta^{-1}$ and $1 \leqslant N \leqslant W^{-1}\delta^{-1}$. Let Θ be a partition of $\mathbb{CP}^{n-1}$ into almost-caps θ of radius $\sim \delta$ and for each $\theta \in \Theta$, let*

$$\{T_{\theta,j} : 1 \leqslant j \lesssim W^{2(n-1)}N\}$$

be a family of essentially distinct complex tubes of radius δ and length 1 essentially contained in $B_{\mathbb{C}^n}(\mathbf{0},1)$ with the property that, for each direction θ and for a fixed maximal collection of essentially distinct complex tube of radius W^{-1} and length 1 essentially contained in $B_{\mathbb{C}^n}(\mathbf{0},1)$ with direction in θ, each of the complex W^{-1}-tube contains, essentially, $\sim N$ of the tubes $T_{\theta,j}$. Let $\mathbb{T}$ denote the set of all the *tubes* $T_{\theta,j}$. *Then, for each* $\varepsilon > 0$, *there exist constants* $c_1(\varepsilon)$ *and* $c_2(\varepsilon)$ *such that, for* $r \geqslant c_1(\varepsilon)\,\delta^{2(n-1)-\varepsilon}|\mathbb{T}|$,

$$|P_r(\mathbb{T})| \leqslant c_2(\varepsilon)\,\delta^{-\varepsilon} W^{-2(n-1)} r^{-2} |\mathbb{T}|^2.$$

Proof. Let us start with two base cases:

(I) If $c_0(n,\varepsilon) < \delta < 1$, where c_0 is a sufficiently small[4] constant, then $P_r(\mathbb{T}) \neq \varnothing$ only if $r \lesssim 1$, under which condition both $|P_r(\mathbb{T})|$ and $W^{-2(n-1)} r^{-2} |\mathbb{T}|^2$ are ~ 1.

(II) If $r \geqslant \alpha_n\,\delta^{-2(n-1)}$, where α_n is a sufficiently large dimensional constant, then $P_r(\mathbb{T}) = \varnothing$, and the claimed inequality is trivially true.

Now, assume that the following estimate holds when either $\widetilde{r} \geqslant 2r$ or $\widetilde{\delta} \geqslant \delta^{-\varepsilon^{10}}\delta$ or both, and $\widetilde{r} \geqslant c_1(\varepsilon)\,\widetilde{\delta}^{2(n-1)-\varepsilon}|\widetilde{\mathbb{T}}|$:

$$|P_{\widetilde{r}}(\widetilde{\mathbb{T}})| \leqslant c_2(\varepsilon)\,\widetilde{\delta}^{-\varepsilon}\,\widetilde{W}^{-2(n-1)}\widetilde{r}^{-2} |\widetilde{\mathbb{T}}|^2.$$

We can assume in addition and without loss of generality that $W \lesssim_{n,\varepsilon} \delta^{-1+\frac{\varepsilon}{2(n-1)}}$ since otherwise

$$\delta^{2(n-1)-\varepsilon}|\mathbb{T}| \gtrsim_n \delta^{2(n-1)-\varepsilon} W^{2(n-1)} \delta^{-2(n-1)} \gtrsim_{n,\varepsilon} \delta^{-2(n-1)-\varepsilon} \gg \delta^{-2(n-1)},$$

and it would follow from the second base case that $P_r(\mathbb{T}) = \varnothing$. This assumption implies $W \ll (\lambda\delta)^{-1}$, where $\lambda := \delta^{-\frac{\varepsilon}{100n}}$.

Let $P := P_r(\mathbb{T}) \setminus P_{2r}(\mathbb{T})$. If $|P_r(\mathbb{T})| \geqslant 10|P|$, then by our induction hypothesis

$$\begin{aligned} |P_r(\mathbb{T})| &\leqslant \frac{10}{9}|P_{2r}(\mathbb{T})| \leqslant \frac{10}{9} c_2(\varepsilon)\,\delta^{-\varepsilon} W^{-2(n-1)} (2r)^{-2} |\mathbb{T}|^2 \\ &\leqslant c_2(\varepsilon)\,\delta^{-\varepsilon} W^{-2(n-1)} r^{-2} |\mathbb{T}|^2. \end{aligned}$$

Thus, we may also assume $|P_r(\mathbb{T})| < 10|P|$.

If we are in the thin case, then, taking $E := r$ in Lemma 3.2, we get

$$\begin{aligned} |P| &\lesssim (-\log\delta)\delta^{-\frac{\varepsilon}{25n}+2\varepsilon^3} r^{-2} \delta^{-2(n-1)} |\mathbb{T}| \\ &\lesssim (-\log\delta)\delta^{-\frac{\varepsilon}{25n}+2\varepsilon^3} r^{-2} W^{-2(n-1)} |\mathbb{T}|^2 \\ &= (-\log\delta)\delta^{(1-\frac{1}{25n})\varepsilon+2\varepsilon^3} \cdot \delta^{-\varepsilon} r^{-2} W^{-2(n-1)} |\mathbb{T}|^2, \end{aligned}$$

where, in the second inequality, we used the fact that $|\mathbb{T}| \sim W^{2(n-1)}\delta^{-2(n-1)} N$.

[4]Being sufficiently small here means being small enough that all the expressions of the form $\delta^{\varepsilon \cdot h(\varepsilon)}$, with $h(\varepsilon)$ a real-coefficient polynomial, appearing later in this section, are as small or as large as needed.

It follows that for δ sufficiently small,

$$|P_r(\mathbb{T})| < 10|P| \leqslant c_2(\varepsilon)\,\delta^{-\varepsilon} W^{-2(n-1)} r^{-2} |\mathbb{T}|^2.$$

If we are in the thick case, then there exists a collection $\widetilde{P}$ of essentially distinct $2\lambda\delta$-balls Q_j such that $\bigcup_j Q_j$ contains a $\gtrsim (-\log\delta)^{-1}$ fraction of the δ-balls in P, and each Q_j intersects $\gtrsim r\delta^{-\frac{n-1}{50n}\varepsilon+2n\varepsilon^3}$ tubes of $\mathbb{T}$. Thicken each δ-tube in $\mathbb{T}$ to a $\lambda\delta$-tube, and call the new collection $\widetilde{\mathbb{T}}$. For each $1 \leqslant M \lesssim \lambda^{2(n-1)}$, let $\widetilde{\mathbb{T}}_M$ be the set of $2\lambda\delta$-tubes in $\widetilde{\mathbb{T}}$ which contain $\sim M$ tubes[5] of $\mathbb{T}$. By dyadic pigeonholing, we can find a particular M_0 such that $\widetilde{\mathbb{T}}_{M_0}$ contains a $\gtrsim ((n-1)\log\lambda)^{-1}$ fraction of the incidences between $\widetilde{\mathbb{T}}$ and $\widetilde{P}$. Observe that for each $2\lambda\delta$-tube $\widetilde{T}$, every $2\lambda\delta$-tube that is not essentially distinct from $\widetilde{T}$ lies completely in the tube $10\widetilde{T}$. Let $\widetilde{\mathbb{T}}_{M_0,\max} \subseteq \widetilde{\mathbb{T}}_{M_0}$ be a maximal subset of essentially distinct $2\lambda\delta$-tubes. Then,

$$I(10\widetilde{\mathbb{T}}_{M_0,\max}, \widetilde{P}) \supseteq I(\widetilde{\mathbb{T}}_{M_0}, \widetilde{P}),$$

where $10\widetilde{\mathbb{T}}_{M_0,\max} := \{10\widetilde{T} : \widetilde{T} \in \widetilde{\mathbb{T}}_{M_0,\max}\}$. Further, since each δ-tube is contained in $\lesssim_n 1$ essentially distinct $2\lambda\delta$-tubes,

$$|\mathbb{T}| \gtrsim_n M_0 |10\widetilde{\mathbb{T}}_{M_0,\max}| = M_0 |\widetilde{\mathbb{T}}_{M_0,\max}|.$$

From now on, we write $\widetilde{\widetilde{\mathbb{T}}} := 10\widetilde{\mathbb{T}}_{M_0,\max}$.

It is easy to show by contradiction that for δ sufficiently small, a $\gtrsim_\varepsilon \delta^{\varepsilon^7}$ fraction[6] of the $2\lambda\delta$-balls $Q_j \in \widetilde{P}$ intersects $\gtrsim \delta^{\varepsilon^5} \cdot r\delta^{-\frac{n-1}{50n}\varepsilon+2n\varepsilon^3} M_0^{-1}$ tubes of $\widetilde{\widetilde{\mathbb{T}}}$, with the implicit constant in the latter inequality being, say, $\widetilde{C} = \widetilde{C}(n,\varepsilon)$. Since each $2\lambda\delta$-ball contains $\lesssim_n \lambda^{2n}$ essentially distinct δ-balls,

$$|P| \lesssim (-\log\delta)\lambda^{2n}|\widetilde{P}| \lesssim (-\log\delta)\delta^{-\frac{\varepsilon}{50}} \cdot \delta^{-\varepsilon^7} |P_{\widetilde{r}}(\widetilde{\widetilde{\mathbb{T}}})|,$$

where $\widetilde{r} := \lfloor \widetilde{C} r \delta^{-\frac{n-1}{50n}\varepsilon+2n\varepsilon^3+\varepsilon^5} M_0^{-1} \rfloor$.

It follows from the given properties of $\mathbb{T}$ that $\widetilde{\widetilde{\mathbb{T}}}$, together with $\widetilde{\delta} := 2\lambda\delta$ and $\widetilde{W} := W$, meets the spacing condition in the induction hypothesis. Moreover,

$$\begin{aligned}
\widetilde{r} \gtrsim r\delta^{-\frac{n-1}{50n}\varepsilon+2n\varepsilon^3+\varepsilon^5} M_0^{-1} &\geqslant c_1(\varepsilon)\delta^{2(n-1)-\varepsilon}|\mathbb{T}| M_0^{-1} \delta^{-\frac{n-1}{50n}\varepsilon+2n\varepsilon^3+\varepsilon^5} \\
&\gtrsim_\varepsilon (\lambda\delta)^{2(n-1)-\varepsilon} \lambda^{-2(n-1)+\varepsilon} |\widetilde{\widetilde{\mathbb{T}}}| \delta^{-\frac{n-1}{50n}\varepsilon+2n\varepsilon^3+\varepsilon^5} \\
&\sim (\lambda\delta)^{2(n-1)-\varepsilon} |\widetilde{\widetilde{\mathbb{T}}}| \delta^{-\frac{1}{100n}\varepsilon^2+2n\varepsilon^3+\varepsilon^5},
\end{aligned}$$

[5] At least M and strictly fewer than $2M$ tubes, to be precise.

[6] The choice of the exponent 7 in the expression δ^{ε^7} and the exponent 5 in the expression δ^{ε^5} in the next line is somewhat arbitrary, other powers could work just as fine for our proof, as long as the latter exponent is sufficiently large and the former is even larger.

with $\delta^{-\frac{1}{100n}\varepsilon^2+2n\varepsilon^3+\varepsilon^5} \gg 1$. Thus, by the induction hypothesis,

$$\begin{aligned}|P_{\tilde{r}}(\tilde{\tilde{\mathbb{T}}})| &\leqslant c_2(\varepsilon)\,(2\lambda\delta)^{-\varepsilon} W^{-2(n-1)}\tilde{r}^{-2}|\tilde{\tilde{\mathbb{T}}}|^2 \\ &\leqslant c_2(\varepsilon)\,\delta^{-\varepsilon} W^{-2(n-1)} r^{-2}|\mathbb{T}|^2 \cdot \lambda^{-\varepsilon}\widetilde{C}^{-2}\delta^{\frac{n-1}{25n}\varepsilon-4n\varepsilon^3-2\varepsilon^5}.\end{aligned}$$

Hence,

$$\begin{aligned}|P_r(\mathbb{T})| &\lesssim |P| \\ &\lesssim (-\log\delta)\delta^{-\frac{\varepsilon}{50}-\varepsilon^7}|P_{\tilde{r}}(\tilde{\tilde{\mathbb{T}}})| \\ &\lesssim c_2(\varepsilon)\delta^{-\varepsilon}W^{-2(n-1)}r^{-2}|\mathbb{T}|^2 \cdot \widetilde{C}^{-2}(-\log\delta)\delta^{\frac{n-2}{50n}\varepsilon+\frac{1}{100n}\varepsilon^2-4n\varepsilon^3-2\varepsilon^5-\varepsilon^7},\end{aligned}$$

where $\widetilde{C}^{-2}(-\log\delta)\delta^{\frac{n-2}{50n}\varepsilon+\frac{1}{100n}\varepsilon^2-4n\varepsilon^3-2\varepsilon^5-\varepsilon^7} \ll 1$ when $n \geqslant 2$. This closes the induction. ∎

Below is a restatement of Theorem 1.2.

Theorem 4.2. *Let $n = 2$ or 3, $1 \leqslant W \leqslant \delta^{-1}$, N_0 be a positive integer independent of δ and W, and $\mathbb{T}$ be a collection of essentially distinct complex δ-tubes essentially contained in $B_{\mathbb{C}^n}(\mathbf{0},1)$. Suppose, for some maximal family of essentially distinct complex W^{-1}-tubes essentially contained in $B_{\mathbb{C}^n}(\mathbf{0},1)$, each of the W^{-1}-tubes contains exactly N_0 complex δ-tubes $T \in \mathbb{T}$. Then, for $r \geqslant \max\{C_\varepsilon\delta^{2(n-1)-\varepsilon}|\mathbb{T}|, 1+N_0\}$,*

$$|P_r(\mathbb{T})| \lesssim_{\varepsilon,n,N_0} \delta^{-\varepsilon}|\mathbb{T}|^{\frac{n}{n-1}} r^{-\frac{n+1}{n-1}}.$$

Note that, given any collection $\mathbb{T}$ of essentially distinct complex δ-tubes obeying the slightly looser spacing condition that each complex W^{-1}-tube in some fixed maximal family contains at most N_0 of the tubes from $\mathbb{T}$, we can always augment the collection $\mathbb{T}$ with extra δ-tubes if needed to obtain a collection $\mathbb{T}_{\text{aug}} \supseteq \mathbb{T}$ of essentially distinct complex δ-tubes such that each complex W^{-1}-tube in the fixed maximal family contains exactly N_0 of the complex δ-tubes in $\mathbb{T}_{\text{aug}}$. So we have the following quick corollary, which we use in the proof of Theorem 5.2.

Corollary 4.3. *Let $n = 2$ or 3, $1 \leqslant W \leqslant \delta^{-1}$, N_0 be a positive integer independent of δ and W, and $\mathbb{T}$ be a collection of essentially distinct complex δ-tubes essentially contained in $B_{\mathbb{C}^n}(\mathbf{0},1)$. Suppose, for some maximal family of essentially distinct complex W^{-1}-tubes essentially contained in $B_{\mathbb{C}^n}(\mathbf{0},1)$, each of the W^{-1}-tubes contains at most N_0 complex δ-tubes $T \in \mathbb{T}$. Then, for richness*

$$r \geqslant \max\{a_n C_\varepsilon \delta^{2(n-1)-\varepsilon} N_0 W^{4(n-1)}, 1+N_0\},$$

where a_n denotes the smallest dimensional constant that ensures $|\mathbb{T}| \leqslant a_n N_0 W^{4(n-1)}$, we have that

$$|P_r(\mathbb{T})| \lesssim_{\varepsilon,n,N_0} \delta^{-\varepsilon} W^{4n} r^{-\frac{n+1}{n-1}}.$$

Proof of Theorem 4.2. We again do a double induction on the scale δ and the richness r. Here are the base cases:

(I) Suppose $r \gtrsim_n N_0\delta^{-2n+2}$, with the implicit dimensional constant in the lower bound being sufficiently large. Then, $r \gtrsim_n N_0\delta^{-2n+2} \geqslant N_0 W^{2n-2}$ would imply that $P_r(\mathbb{T}) = \varnothing$, and the claimed inequality would be trivially true.

(II) Suppose $\delta \sim_n 1$. Then, on one hand,

$$|P_r(\mathbb{T})| \leqslant |P_{1+N_0}(\mathbb{T})| \lesssim_n (W^{2(2n-2)})^2 \cdot N_0^2 \leqslant \delta^{-(8n-8))} N_0^2 \sim_{n,N_0} 1.$$

On the other hand, $P_r(\mathbb{T})$ is nonempty only if $r \lesssim_n N_0 W^{2n-2}$, in which case

$$\begin{aligned}|\mathbb{T}|^{\frac{n}{n-1}} r^{-\frac{n+1}{n-1}} &\gtrsim_n (N_0 W^{2(2n-2)})^{\frac{n}{n-1}} (N_0 W^{2n-2})^{-\frac{n+1}{n-1}} \\ &\geqslant N_0^{-\frac{1}{n-1}} \delta^{-(2n-2)} \sim_{n,N_0} 1.\end{aligned}$$

Thus, the claimed inequality follows.

For the sake of induction, let us assume that the bound below holds when either $\widetilde{r} \geqslant 2r$ or $\widetilde{\delta} \geqslant \delta^{-\varepsilon^{10}}\delta$ or both, and $\widetilde{r} \geqslant \max\{C_\varepsilon \widetilde{\delta}^{2(n-1)-\varepsilon}|\widetilde{\mathbb{T}}|, 1 + N_0\}$:

$$|P_{\widetilde{r}}(\widetilde{\mathbb{T}})| \leqslant c_{n,\varepsilon,N_0}\, \delta^{-\varepsilon} |\widetilde{\mathbb{T}}|^{\frac{n}{n-1}} \widetilde{r}^{-\frac{n+1}{n-1}}.$$

We assume without loss of generality that $W < \delta^{-1+\frac{\varepsilon}{2n}} \ll (\lambda\delta)^{-1}$, with $\lambda := \delta^{-\frac{\varepsilon}{100n}}$.

By dyadic pigeonholing, we can find an $\alpha = \alpha(r) > W^{-1}$ such that the subcollection

$$P_{r,\alpha}(\mathbb{T}) := \big\{p \in P_r(\mathbb{T}) : \text{The maximal angle between complex tubes} \\ \text{of } \mathbb{T} \text{ passing through the } \delta\text{-ball } p \text{ is} \sim \alpha.\big\}$$

has size $\gtrsim (-\log\delta)^{-1}|P_r(\mathbb{T})|$. The lower bound W^{-1} comes from the requirement $r \geqslant 1 + N_0$, which infers that the complex δ-tubes passing through any r-rich δ-ball p cannot all lie in the same W^{-1}-tube.

Let us first consider two cases where $r < \delta^{-\varepsilon^3}$.

(a) Suppose $\alpha \leqslant \delta^{\frac{3}{2}\varepsilon^3}$. Let $\{\tau\} \subseteq \mathbb{CP}^{n-1}$ be a maximal set of α-separated directions. For each τ, cover $B_{\mathbb{C}^n}(\mathbf{0}, 1)$ with $\sim_n \alpha^{-2(n-1)}$ complex α-tubes $\square_\tau$ in the direction τ. Let $\mathbb{T}_{\square_\tau}$ denote the collection of complex tubes $T \in \mathbb{T}$ that are essentially contained in $\square_\tau$. Then, our assumption about the spacing of the complex tubes implies that $|\mathbb{T}_{\square_\tau}| \lesssim_n N_0(\alpha W)^{4(n-1)}$.

Now, fix a complex tube $\square_\tau$, and rescale $\square_\tau$ to essentially a unit ball. Then, the complex δ-tubes in $\mathbb{T}_{\square_\tau}$ become complex $\widetilde{\delta}$-tubes, the collection of which we call $\widetilde{\mathbb{T}}_{\square_\tau}$. Here, $\widetilde{\delta} := \frac{\delta}{\alpha} \geqslant \delta^{-\frac{3}{2}\varepsilon^3}\delta$. This collection $\widetilde{\mathbb{T}}_{\square_\tau}$ might not exactly match the spacing condition in the induction hypothesis with $\widetilde{W} := \alpha W$ as $\widetilde{\mathbb{T}}_{\square_\tau}$ could be sparse; but by the argument preceding Corollary 4.3, we still have the following bound at the richness

$$\widetilde{r} := \max\{\lceil a_n C_\varepsilon \widetilde{\delta}^{2(n-1)-\varepsilon} N_0 \widetilde{W}^{4(n-1)}\rceil, 1 + N_0\} \leqslant r$$

we have

$$|P_{\widetilde{r}}(\widetilde{\mathbb{T}}'_{\square_\tau})| \lesssim_{n,\varepsilon,N_0} \widetilde{\delta}^{-\varepsilon} \widetilde{W}^{4n} \widetilde{r}^{-\frac{n+1}{n-1}} < \widetilde{\delta}^{-\varepsilon} \widetilde{W}^{4n}. \tag{4.1}$$

Rescaling the unit ball back to the complex α-tube $\square_\tau$, we see that each $\widetilde{r}$-rich $\widetilde{\delta}$-balls becomes a complex tube of radius δ and length $\frac{\delta}{\alpha}$, which can be viewed as an almost disjoint union (or, more precisely, a finitely overlapping union) of $\sim_n \alpha^{-2}$ of the δ-balls in $P_{\widetilde{r},\alpha}(\mathbb{T})$. As a result,

$$\begin{aligned}
|P_{r,\alpha(r)}(\mathbb{T})| \leqslant |P_{\widetilde{r},\alpha(r)}(\mathbb{T})| &\lesssim_n \sum_{\tau} \sum_{\square_\tau} |P_{\widetilde{r}}(\widetilde{\mathbb{T}}_{\square_\tau})| \cdot \alpha^{-2} \\
&\lesssim_{n,\varepsilon,N_0} \alpha^{-4(n-1)} \cdot \left(\frac{\delta}{\alpha}\right)^{-\varepsilon} (\alpha W)^{4n} \cdot \alpha^{-2} \\
&\sim_{n,N_0} \delta^{-\varepsilon} |\mathbb{T}|^{\frac{n}{n-1}} \cdot \alpha^{2+\varepsilon},
\end{aligned}$$

where

$$\alpha^{2+\varepsilon} \leqslant \delta^{\frac{3}{2}\varepsilon^3 \cdot (2+\varepsilon)} \leqslant \delta^{\frac{n+1}{n-1}\varepsilon^3} \cdot \delta^{\frac{3}{2}\varepsilon^4} < r^{-\frac{n+1}{n-1}} \cdot \delta^{\frac{3}{2}\varepsilon^4}.$$

So, we can close the induction in the case where $r < \delta^{-\varepsilon^3}$ and $\alpha \leqslant \delta^{\frac{3}{2}\varepsilon^3}$.

(b) Next, suppose $\alpha > \delta^{\frac{3}{2}\varepsilon^3}$. Then by our assumption about r,

$$\delta^{-\varepsilon} |\mathbb{T}|^{\frac{n}{n-1}} r^{-\frac{n+1}{n-1}} \gtrsim_n \delta^{-\varepsilon} N_0^{\frac{n}{n-1}} W^{4n} \delta^{\frac{n+1}{n-1}\varepsilon^3}.$$

Thus, when $W > \delta^{-\frac{1}{2}+\frac{\varepsilon}{8}}$, we have

$$\delta^{-\varepsilon} |\mathbb{T}|^{\frac{n}{n-1}} r^{-\frac{n+1}{n-1}} \gtrsim_{N_0} \delta^{-2n-\frac{\varepsilon}{2}+\frac{n+1}{n-1}\varepsilon^3} \gg \delta^{-2n} \gtrsim |P_r(\mathbb{T})|.$$

For this reason, we may as well assume $W \leqslant \delta^{-\frac{1}{2}+\frac{\varepsilon}{8}}$ in the rest of part (b). Then, $W^{-2} \geqslant \delta^{-\frac{\varepsilon}{4}+1} \gg \delta$. Let $\widetilde{\mathbb{T}}$ be the collection of complex W^{-2}-tubes obtained by thickening each complex δ-tube in $\mathbb{T}$, and let $\widetilde{\mathbb{T}}_{\max} \subseteq \widetilde{\mathbb{T}}$ be a maximal subcollection of essentially distinct W^{-2}-tubes. Since $W^{-2} < W^{-1}$, the spacing condition of $\mathbb{T}$ implies that $|\widetilde{\mathbb{T}}_{\max}| \geqslant N_0^{-1} |\widetilde{\mathbb{T}}|$. Let Q be a minimal covering of $B_{\mathbb{C}^n}(\mathbf{0}, 1)$ consisting of W^{-2}-balls; and for dyadic $X, M \geqslant 1$, let $Q_{X,M}$ be the collection of W^{-2}-balls in Q that contains $\sim X$ of the δ-balls in $P_{r,\alpha}(\mathbb{T})$ and intersects $\sim M$ of[7] the complex δ-tubes in $\mathbb{T}$. We note that the set $Q_{X,M}$ is nonempty only if $X \lesssim (\frac{W^{-2}}{\delta})^{2n}$ and $r \leqslant M \lesssim W^{4(n-1)}$. Then, by dyadic pigeonholing twice, we can find some particular

[7] At least M and strictly less than $2M$ of the complex tubes, to be precise.

X_0 and M_0 such that $\bigcup_{q\in Q_{X_0,M_0}} q$ covers a $\gtrsim_n (-\log\delta)^{-2}$ fraction of the δ-balls in $P_{r,\alpha}(\mathbb{T})$. By definition, each δ-ball in $P_{r,\alpha}(\mathbb{T})$ is intersected by two complex tubes in $\mathbb{T}$ with directions differing by $\sim\alpha$. Hence, the number of δ-balls from $P_{r,\alpha}(\mathbb{T})$ contained in each q is $\lesssim \alpha^{-2}M_0^2$; and, consequently, we have

$$X_0 \lesssim \alpha^{-2}M_0^2 \lesssim \delta^{-3\varepsilon^3}M_0^2.$$

By the induction hypothesis,

$$|P_{M_0}(\widetilde{\mathbb{T}}_{\max})| \lesssim W^{2\varepsilon}|\widetilde{\mathbb{T}}|^{\frac{n}{n-1}}M_0^{-\frac{n+1}{n-1}}.$$

Therefore,

$$\begin{aligned}|P_r(\mathbb{T})| &\lesssim (-\log\delta)^2\cdot X_0\cdot|P_{M_0}(\widetilde{\mathbb{T}}_{\max})|\\ &\lesssim (-\log\delta)^2\delta^{-3\varepsilon^3}W^{2\varepsilon}|\widetilde{\mathbb{T}}_{\max}|^{\frac{n}{n-1}}M_0^{2-\frac{n+1}{n-1}}\\ &\leqslant (-\log\delta)^2\delta^{-3\varepsilon^3+2\varepsilon(-\frac{1}{2}+\frac{\varepsilon}{8})}|\mathbb{T}|^{\frac{n}{n-1}}M_0^{2-\frac{n+1}{n-1}}\\ &= (-\log\delta)^2\delta^{\frac{1}{4}\varepsilon^2-3\varepsilon^3}\cdot\delta^{-\varepsilon}|\mathbb{T}|^{\frac{n}{n-1}}\cdot M_0^{2-\frac{n+1}{n-1}},\end{aligned}$$

with $M_0^{2-\frac{n+1}{n-1}}\leqslant 1$ when $n=2$ or 3, and

$$(-\log\delta)^2\delta^{\frac{1}{4}\varepsilon^2-3\varepsilon^3} = (-\log\delta)^2\delta^{\frac{1}{4}\varepsilon^2-(3+\frac{n+1}{n-1})\varepsilon^3}\cdot\delta^{\frac{n+1}{n-1}\varepsilon^3} \ll r^{-\frac{n+1}{n-1}}.$$

This closes the induction in the case where $r<\delta^{-\varepsilon^3}$ and $\alpha>\delta^{\frac{3}{2}\varepsilon^3}$.

From now on, suppose that $r\geqslant\delta^{-\varepsilon^3}$. If $W\leqslant\delta^{-\varepsilon^4}$, then $P_r(\mathbb{T})\neq\varnothing$ only if $r\lesssim_n N_0W^{2n-2}$, in which case

$$\begin{aligned}|P_r(\mathbb{T})| \leqslant |P_{1+N_0}(\mathbb{T})| \lesssim_n (W^{2(2n-2)})^2\cdot N_0^2 &\sim_n |\mathbb{T}|^{\frac{n}{n-1}}(N_0W^{2(2n-2)})^{\frac{n-2}{n-1}}\\ &\lesssim_{n,N_0} |\mathbb{T}|^{\frac{n}{n-1}}r^{-\frac{n+1}{n-1}}W^{4(n-2)+2(n+1)}\\ &\leqslant |\mathbb{T}|^{\frac{n}{n-1}}r^{-\frac{n+1}{n-1}}\delta^{-(6n-6)\varepsilon^4}\\ &\ll |\mathbb{T}|^{\frac{n}{n-1}}r^{-\frac{n+1}{n-1}}\delta^{-\varepsilon}.\end{aligned}$$

So we conveniently assume $W>\delta^{-\varepsilon^4}$. Take $D:=\delta^{-\varepsilon^4}$, and let $\mathcal{Q}$ be a covering of $B_{\mathbb{C}^n}(\mathbf{0},2)$ consisting of $D\delta$-balls $\mathcal{q}$ such that no two of the balls $\frac{1}{2}\mathcal{q}$ overlap[8]. By our induction hypothesis, we may assume without loss of generality that $|P|\sim|P_r(\mathbb{T})|$,

[8] Basically, we find a packing of the set $B_{\mathbb{C}^n}(\mathbf{0},2)$ by $\frac{1}{2}D\delta$-balls and then double the radius of each ball to obtain a covering. Choosing such a covering helps us avoid ambiguity about where the chopped tubes come from later in the proof.

where $P := P_r(\mathbb{T}) \setminus P_{2r}(\mathbb{T})$. For any pair of intersecting T and $\mathcal{q}$, the intersection $T \cap \mathcal{q}$ is contained in a complex tube of length $2D\delta$ and radius δ, which we call $\mathring{T}$. Fix a maximal subset $\mathring{\mathbb{T}}_{\max} \subseteq \mathring{\mathbb{T}}$ consisting of essentially distinct chopped tubes that maximizes $I(P, \mathring{\mathbb{T}}_{\max})$. Then in particular,

$$I(P, \mathring{\mathbb{T}}_{\max}) \gtrsim (W^{-1}D)^{2(n-1)} I(P, \mathbb{T}).$$

Let $\mathring{\mathbb{T}}_{\mathcal{q},H}$ be the set of chopped complex tubes $\mathring{T} \in \mathring{\mathbb{T}}_{\max}$ which are essentially contained in $\mathcal{q}$ and $\sim H$ of the complex tubes $T \in \mathbb{T}$. Because of the spacing condition, $\mathring{\mathbb{T}}_{\mathcal{q},H} \neq \varnothing$ only if $H \lesssim (WD^{-1})^{2(n-1)}$. As in the proof of Theorem 4.1, we first use dyadic pigeonholing to pick a particular H_0 such that

$$\sum_{\mathcal{q}\in\mathcal{Q}} H_0 \cdot |I(P_{\mathcal{q}}, \mathring{\mathbb{T}}_{\mathcal{q},H_0})| \gtrsim_n (-\log\delta)^{-1} |I(P, \mathbb{T})|,$$

where $P_{\mathcal{q}}$ is the collection of δ-balls $p \in P$ that are essentially contained in $\mathcal{q}$. Let $P_{\mathcal{q},E}$ be the set of δ-balls in P which are essentially contained in $\mathcal{q}$ and $\sim E$ of the chopped tubes $\mathring{T} \in \mathring{\mathbb{T}}_{\mathcal{q},H_0}$. Again, by dyadic pigeonholing, we can find a particular E_0 such that

$$\sum_{\mathcal{q}\in\mathcal{Q}} H_0 \cdot |I(P_{\mathcal{q},E_0}, \mathring{\mathbb{T}}_{\mathcal{q},H_0})| \gtrsim_n (-\log\delta)^{-2} |I(P, \mathbb{T})|.$$

By the definition of $P_{\mathcal{q},E}$, we have $|I(P_{\mathcal{q},E_0}, \mathring{\mathbb{T}}_{\mathcal{q},H_0})| \sim E_0 |P_{\mathcal{q},E_0}|$; and by the definition of P, $|I(P, \mathbb{T})| \sim r|P|$. It then follows from our choice of E_0 that

$$\sum_{\mathcal{q}} |P_{\mathcal{q},E_0}| \gtrsim_n (-\log\delta)^{-2} \frac{r}{H_0 E_0} |P|. \tag{4.2}$$

On the other hand,

$$\sum_{\mathcal{q}} |P_{\mathcal{q},E_0}| \leqslant \sum_{\mathcal{q}} |P_{\mathcal{q}}| \leqslant 2|P|.$$

Hence,

$$H_0 E_0 \gtrsim |\log\delta|^{-2} r. \tag{4.3}$$

Further, since each $\mathcal{q} \in P_{\mathcal{q},E_0}$ is essentially contained in $\sim E_0$ of the chopped tubes $\mathring{T} \in \mathbb{T}_{\mathcal{q},H_0}$, and each $\mathring{T} \in \mathbb{T}_{\mathcal{q},H_0}$ is contained in $\sim H_0$ of the complex tubes $T \in \mathbb{T}$, each $\mathcal{q}$ is essentially contained in $\sim H_0 E_0$ of the complex tubes $T \in \mathbb{T}$. Recalling the definition $P = P_r(\mathbb{T}) \setminus P_{2r}(\mathbb{T})$, we deduce that

$$H_0 E_0 \lesssim 2r \sim r. \tag{4.4}$$

Thus, by (4.2),

$$|P| \lesssim_n |\log\delta|^2 \sum_{\mathcal{q}\in\mathcal{Q}} |P_{\mathcal{q},E_0}| \leqslant |\log\delta|^2 \Bigg(\sum_{\mathcal{q}\ \text{thin}} |P_{\mathcal{q},E_0}| + \sum_{\mathcal{q}\ \text{thick}} |P_{\mathcal{q},E_0}| \Bigg).$$

Now we apply Lemma 3.2 to each pair of P_{q,E_0} and $\dot{\mathbb{T}}_{q,H_0}$, rescaled and translated so that q becomes the unit ball.

(c) For each q falling into the thin case,

$$|P_{q,E_0}| \lesssim D^{\frac{\varepsilon}{25n}-2\varepsilon^3} E_0^{-2} D^{2(n-1)} |\dot{\mathbb{T}}_{q,H_0}|.$$

So, if the thin case dominates, then

$$|P| \lesssim_n |\log\delta|^2 \sum_{q \text{ thin}} |P_{q,E_0}| \lesssim |\log\delta|^2 \cdot D^{\frac{\varepsilon}{25n}-2\varepsilon^3} E_0^{-2} D^{2(n-1)} \sum_{q} |\dot{\mathbb{T}}_{q,H_0}|.$$

To estimate the term $\sum_q |\dot{\mathbb{T}}_{q,H_0}|$, let $\{\sigma\} \subseteq \mathbb{CP}^{n-1}$ be a maximal set of D^{-1}-separated directions, and for each σ, let $\{\square_\sigma\}$ be a minimal covering of $B_{\mathbb{C}^n}(\mathbf{0},1)$ consisting of complex D^{-1}-tubes $\square_\sigma$ in the direction σ. Let $\mathbb{T}_{\square_\sigma}$ denote the collection of complex tubes $T \in \mathbb{T}$ that are essentially contained in $\square_\sigma$. Each chopped complex tube $\dot{T} \in \dot{\mathbb{T}}_{q,H_0}$ intersects essentially $\sim H_0$ complex tubes of $\mathbb{T}$, all of which lie essentially in the same $\square_\sigma$. Rescale each $\square_\sigma$ to essentially a unit ball, and let $\widetilde{\mathbb{T}}_{\square_\sigma}$ be the resulting set of complex $\widetilde{\delta}$-tubes, where $\widetilde{\delta} := D\delta$. Then, $\widetilde{\mathbb{T}}_{\square_\sigma}$ meets the spacing condition with $\widetilde{W} := WD^{-1}$. Moreover,

$$\sum_q |\dot{\mathbb{T}}_{q,H_0}| \lesssim \sum_\sigma \sum_{\square_\sigma} |P_{\widetilde{r}}(\widetilde{\mathbb{T}}_{\square_\sigma})|,$$

where $\widetilde{r} \sim H_0$ (or, more precisely, $\widetilde{r}$ is a constant fraction of H_0). Since

$$\widetilde{\delta} = D\delta = \delta^{1-\varepsilon^4} \gg \delta^{1-\varepsilon^{10}},$$

we can apply our induction hypothesis and get

$$\begin{aligned} |P_{\widetilde{r}}(\widetilde{\mathbb{T}}_{\square_\sigma})| \lesssim \widetilde{\delta}^{-\varepsilon} |\widetilde{\mathbb{T}}_{\square_\sigma}|^{\frac{n}{n-1}} \widetilde{r}^{-\frac{n+1}{n-1}} &\sim (D\delta)^{-\varepsilon} (WD^{-1})^{4(n-1)\cdot\frac{n}{n-1}} H_0^{-\frac{n+1}{n-1}} \\ &\sim (D\delta)^{-\varepsilon} D^{-4n} |\mathbb{T}|^{\frac{n}{n-1}} H_0^{-\frac{n+1}{n-1}}, \end{aligned}$$

which implies

$$\begin{aligned} \sum_q |\dot{\mathbb{T}}_{q,H_0}| &\lesssim D^{4(n-1)} \cdot (D\delta)^{-\varepsilon} D^{-4n} |\mathbb{T}|^{\frac{n}{n-1}} H_0^{-\frac{n+1}{n-1}} \\ &= D^{-4} (D\delta)^{-\varepsilon} |\mathbb{T}|^{\frac{n}{n-1}} H_0^{-\frac{n+1}{n-1}}, \end{aligned}$$

and consequently

$$\begin{aligned} |P_r(\mathbb{T})| &\lesssim |\log\delta|^2 \cdot D^{2n-6+\frac{\varepsilon}{25n}-2\varepsilon^3} (D\delta)^{-\varepsilon} E_0^{-2} H_0^{-\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}} \\ &\lesssim |\log\delta|^{2+2\frac{n+1}{n-1}} \delta^{-\varepsilon} D^{2n-6+(\frac{1}{25n}-1)\varepsilon-2\varepsilon^3} E_0^{-2+\frac{n+1}{n-1}} r^{-\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}}, \end{aligned}$$

with the second inequality following from (4.3). When $-2 + \frac{n+1}{n-1} \geqslant 0$, i.e., when $n = 2$ or 3,

$$E_0^{-2+\frac{n+1}{n-1}} \lesssim (D^{2(n-1)})^{-2+\frac{n+1}{n-1}},$$

and hence

$$|P_r(\mathbb{T})| \lesssim |\log \delta|^{\frac{4n}{n-1}} D^{(\frac{1}{25n}-1)\varepsilon - 2\varepsilon^3} \cdot \delta^{-\varepsilon} r^{-\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}},$$

where $|\log \delta|^{\frac{4n}{n-1}} D^{(\frac{1}{25n}-1)\varepsilon - 2\varepsilon^3} \ll 1$.

(d) For each $\mathscr{q}$ falling into the thick case, there exists a collection $\widetilde{P}_{\mathscr{q}}$ of finitely overlapping balls $\widetilde{p}$ of width $\sim D^{\frac{\varepsilon}{100n}} \delta$ such that the union $\bigcup_{\widetilde{p} \in \widetilde{P}_{\mathscr{q}}} \widetilde{p}$ contains a $\gtrsim (\log D)^{-1}$ fraction of the balls $p \in P_{\mathscr{q}, E_0}$, and each ball $\widetilde{p}$ intersects $\gtrsim E_0 D^{\frac{n-1}{50n}\varepsilon - 2n\varepsilon^3}$ of the complex tubes $\dot{T} \in \dot{\mathbb{T}}_{\mathscr{q}, H_0}$. Set

$$\widetilde{P} := \bigcup_{\mathscr{q} \text{ thick}} \widetilde{P}_{\mathscr{q}}.$$

Then, each $\widetilde{p} \in \widetilde{P}$ intersects $\gtrsim N_0 E_0 D^{\frac{n-1}{50n}\varepsilon - 2n\varepsilon^3}$ of the complex tubes $T \in \mathbb{T}$. To apply the induction hypothesis, we fatten each complex δ-tube $T \in \mathbb{T}$ into a complex tube $\widetilde{T}$ of width $\sim D^{\frac{\varepsilon}{100n}} \delta$, and call the collection of all these complex tubes $\widetilde{\mathbb{T}}$. Then, $\widetilde{P} \subseteq P_{\widetilde{r}}(\widetilde{\mathbb{T}})$ for $\widetilde{r}$ a sufficiently small constant fraction of $H_0 E_0 D^{\frac{n-1}{50n}\varepsilon - 2n\varepsilon^3}$, and for such $\widetilde{r}$ we have

$$\begin{aligned}
\widetilde{r} \sim H_0 E_0 D^{\frac{n-1}{50n}\varepsilon - 2n\varepsilon^3} &\gtrsim (\log \delta)^{-2} r D^{\frac{n-1}{50n}\varepsilon - 2n\varepsilon^3} \gg r \\
&\geqslant \max\{C_\varepsilon \widetilde{\delta}^{2(n-1)-\varepsilon} |\widetilde{\mathbb{T}}|, 1 + N_0\}.
\end{aligned}$$

The collection $\widetilde{\mathbb{T}}$ meets the spacing condition with $\widetilde{\delta} := C D^{1+\frac{\varepsilon}{100n}} \delta$ and $\widetilde{W} := W$, where

$$\widetilde{W} = W \lesssim \delta^{-\frac{1}{2} - \frac{1}{4(n-1)} + \frac{\varepsilon}{4(n-1)}} \ll \delta^{-1+\varepsilon^4(1+\frac{\varepsilon}{100n})} \sim \widetilde{\delta}^{-1}.$$

Hence,

$$\begin{aligned}
\sum_{\mathscr{q} \text{ thick}} |\widetilde{P}_{\mathscr{q}}| \leqslant |P_{\widetilde{r}}(\widetilde{\mathbb{T}})| &\lesssim \widetilde{\delta}^{-\varepsilon} \widetilde{r}^{-\frac{n+1}{n-1}} |\widetilde{\mathbb{T}}|^{\frac{n}{n-1}} \\
&\lesssim \delta^{\varepsilon - \varepsilon^5(1+\frac{\varepsilon}{100n})} |\log \delta|^{\frac{2(n+1)}{n-1}} \cdot \delta^{-\varepsilon} r^{-\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}},
\end{aligned}$$

and

$$\begin{aligned}
|P_r(\mathbb{T})| \lesssim (\log \delta)^2 \sum_{\mathscr{q} \text{ thick}} |P_{\mathscr{q}, E_0}| &\lesssim (\log \delta)^2 (\log D) \sum_{\mathscr{q} \text{ thick}} |\widetilde{P}_{\mathscr{q}}| \\
&\lesssim \delta^{\varepsilon - \varepsilon^5(1+\frac{\varepsilon}{100n})} |\log \delta|^{3+\frac{2(n+1)}{n-1}} \cdot \delta^{-\varepsilon} r^{-\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}} \\
&\ll \delta^{-\varepsilon} r^{-\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}}.
\end{aligned}$$

The induction is now complete. ∎

5. Application to a variant of Falconer's distance set problem

In this section, we investigate the following question.

Question 5.1. Fix $2 < s < 4$. Let E be a collection of $\sim N\delta^{-s}$ many[9] δ-balls in $B_{\mathbb{C}^2}(\vec{0}, 1)$ with at most N δ-balls in each ball of radius $\delta^{s/4}$. How many finitely overlapping δ-balls are needed to cover the difference set

$$\Delta(E) := \left\{(x_1 - x_2)^2 + (y_1 - y_2)^2 \in \mathbb{C} : \vec{p}_1 = (x_1, y_1), \vec{p}_2 = (x_2, y_2) \in \bigcup E\right\}?$$

Our work builds up to a proof of Corollary 1.3, which we have restated below as Theorem 5.2 for convenience.

Theorem 5.2. *Let $2 < s < 4$, and E be a collection of $\sim N\delta^{-s}$ many δ-balls in $B_{\mathbb{C}^2}(\vec{0}, 1)$ with at most N δ-balls in each ball of radius $\delta^{s/4}$. Then, the number of distinct δ-balls needed to cover the difference set*

$$\Delta(E) := \left\{(x_1 - x_2)^2 + (y_1 - y_2)^2 \in \mathbb{C} : \vec{p}_1 = (x_1, y_1), \vec{p}_2 = (x_2, y_2) \in \bigcup E\right\}$$

is $\gtrsim_{\varepsilon,s,N} \delta^{-2+\varepsilon}$ for all $\varepsilon > 0$.

We emphasize that our complex analogue of the distance set is the set of squared sums of differences, rather than the set of squared sums of the norms of the differences.

Definition 5.3. Given any pair of points $\vec{p}_1 = (x_1, y_1)$ and $\vec{p}_2 = (x_2, y_2)$ in $\mathbb{C}^2$, define

$$\Delta(\vec{p}_1, \vec{p}_2) := (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

To build a connection between the difference set problem and the incidence estimates, we introduce an auxiliary line for each pair of points in $\mathbb{C}^2$.

Definition 5.4. Given any pair of points $\vec{p}_1 = (x_1, y_1)$ and $\vec{p}_2 = (x_2, y_2)$ in $\mathbb{C}^2$, define the auxiliary line

$$l_{\vec{p}_1,\vec{p}_2} := \left\{\left(\frac{x_1 + x_2}{2} + \frac{y_1 - y_2}{2} z, \frac{y_1 + y_2}{2} - \frac{x_1 - x_2}{2} z, z\right) : z \in \mathbb{C}\right\}.$$

To simplify future notation, let us write

$$\boldsymbol{v}_{\vec{p}_1,\vec{p}_2} := \left(\frac{y_1 - y_2}{2}, -\frac{x_1 - x_2}{2}, 1\right).$$

[9] Say $\frac{1}{2}|B_{\mathbb{C}^2}(\vec{0}, 1)|N\delta^{-s} < |E| < 2|B_{\mathbb{C}^2}(\vec{0}, 1)|N\delta^{-s}$.

Then,

$$l_{\vec{p}_1,\vec{p}_2} = \left\{ \left(\frac{\vec{p}_1 + \vec{p}_2}{2}, 0 \right) + z\boldsymbol{v}_{\vec{p}_1,\vec{p}_2} : z \in \mathbb{C} \right\}.$$

The intuition behind this definition comes from the Elekes–Sharir framework used in the distinct distance problem in $\mathbb{R}^2$ (see, e.g., [2]): Over $\mathbb{R}$ instead of $\mathbb{C}$, the orthogonal projection of $l_{\vec{p}_1,\vec{p}_2}$ onto the first two coordinates is the perpendicular bisector of the segment between the points $\vec{p}_1$ and $\vec{p}_2$ in $\mathbb{R}^2$, and the line $l_{\vec{p}_1,\vec{p}_2}$ itself parametrizes all the rigid motions mapping $\vec{p}_1$ to $\vec{p}_2$. Two things to note:

(1) For any four points $\vec{p}_j = (x_j, y_j) \in \mathbb{C}^2$, $j = 1, 2, 3, 4$, the auxiliary lines $l_{\vec{p}_1,\vec{p}_3}$ and $l_{\vec{p}_2,\vec{p}_4}$ intersect if and only if

$$\Delta(\vec{p}_1, \vec{p}_2) = \Delta(\vec{p}_3, \vec{p}_4).$$

We prove a more quantitative version of this statement in Proposition 5.7.

(2) For any two distinct points $\vec{p}_1, \vec{p}_2 \in \mathbb{C}^2$, the auxiliary lines $l_{\vec{p}_1,\vec{p}_2}$ and $l_{\vec{p}_2,\vec{p}_1}$ are different.

Definition 5.5. Given a collection E of δ-balls as described in Question 5.1, we can choose two balls B' and B'' of radius $C_1 \leqslant 0.01$ with centres at least $C_2 \geqslant 1.2$ apart so that each of the subcollections $E' := \{q \in E : q \subseteq B'\}$ and $E'' := \{q \in E : q \subseteq B''\}$ contains a $\gtrsim 1$ fraction of the balls in E.

Define $\mathbb{T}$ to be the collection of complex almost δ-tubes

$$\left\{ T_{q_1,q_2} : (q_1, q_2) \in (E' \times E'') \cup (E'' \times E') \right\},$$

where

$$T_{q_1,q_2} := \left(\bigcup_{(\vec{p}_1,\vec{p}_2) \in q_1 \times q_2} l_{\vec{p}_1,\vec{p}_2} \right) \cap B_{\mathbb{C}^3}(\mathbf{0}, 1).$$

The following proposition justifies our choice to call the auxiliary objects T_{q_1,q_2} "almost" tubes.

Proposition 5.6. *Let q_1 and q_2 be any two δ-balls contained in $B_{\mathbb{C}^2}(\vec{0}, 1)$, and let $\vec{\sigma}_1 = (x_{\vec{\sigma}_1}, y_{\vec{\sigma}_1})$ and $\vec{\sigma}_2 = (x_{\vec{\sigma}_2}, y_{\vec{\sigma}_2})$ denote the centres of q_1 and q_2, respectively. Then,*

(1) $T_{q_1,q_2} \subseteq \mathcal{N}_{2\delta}(l_{\vec{\sigma}_1,\vec{\sigma}_2}) \cap B_{\mathbb{C}^3}(\mathbf{0}, 1)$*; and*

(2) $T_{q_1,q_2} \supseteq \mathcal{N}_{\delta/2}(l_{\vec{\sigma}_1,\vec{\sigma}_2}) \cap B_{\mathbb{C}^3}(\mathbf{0}, 1)$.

The proof of this proposition involves nothing more than basic geometry and algebra, and is included only for the sake of completeness. Readers may trust their instinct and skip to the next page.

Proof. To prove (1), consider any two points $\vec{p}_1 = (x_1, y_1) \in q_1$ and $\vec{p}_2 = (x_2, y_2) \in q_2$, and any scalar $z \in \mathbb{C}$ such that

$$\left(\frac{x_1 + x_2}{2} + z\frac{y_1 - y_2}{2}, \frac{y_1 + y_2}{2} - z\frac{x_1 - x_2}{2}, z\right) \in B_{\mathbb{C}^3}(\mathbf{0}, 1).$$

Then, clearly, $|z| < 1$. The triangle inequality gives

$$\begin{aligned}
&\Bigg\| \left(\frac{x_1 + x_2}{2} + z\frac{y_1 - y_2}{2}, \frac{y_1 + y_2}{2} - z\frac{x_1 - x_2}{2}, z\right) \\
&\qquad - \left(\frac{x_{\vec{\sigma}_1} + x_{\vec{\sigma}_2}}{2} + z\frac{y_{\vec{\sigma}_1} - y_{\vec{\sigma}_2}}{2}, \frac{y_{\vec{\sigma}_1} + y_{\vec{\sigma}_2}}{2} - z\frac{x_{\vec{\sigma}_1} - x_{\vec{\sigma}_2}}{2}, z\right)\Bigg\| \\
&\leqslant \frac{\|\vec{p}_1 + \vec{p}_2 - \vec{\sigma}_1 - \vec{\sigma}_2\|}{2} + |z|\frac{\|\vec{p}_1 - \vec{p}_2 - \vec{\sigma}_1 + \vec{\sigma}_2\|}{2} \\
&\leqslant \frac{\|\vec{p}_1 - \vec{\sigma}_1\| + \|\vec{p}_2 - \vec{\sigma}_2\|}{2} + \frac{\|\vec{p}_1 - \vec{\sigma}_1\| + \|\vec{p}_2 - \vec{\sigma}_2\|}{2} < 2\delta.
\end{aligned}$$

To prove (2), consider any scalar $z \in \mathbb{C}$ and any point $(a, b, c) \in B_{\mathbb{C}^3}(\mathbf{0}, \frac{\delta}{2})$ such that

$$\left(\frac{x_{\vec{\sigma}_1} + x_{\vec{\sigma}_2}}{2} + z\frac{y_{\vec{\sigma}_1} - y_{\vec{\sigma}_2}}{2} + a, \frac{y_{\vec{\sigma}_1} + y_{\vec{\sigma}_2}}{2} - z\frac{x_{\vec{\sigma}_1} - x_{\vec{\sigma}_2}}{2} + b, z + c\right) \in B_{\mathbb{C}^2}(\mathbf{0}, 1).$$

We would like to solve for the following system under the additional requirement $(r_1, s_1), (r_2, s_2) \in B_{\mathbb{C}^2}(\vec{0}, \delta)$:

$$\begin{cases}
\dfrac{(x_{\vec{\sigma}_1} + r_1) + (x_{\vec{\sigma}_2} + r_2)}{2} + (z + c)\dfrac{(y_{\vec{\sigma}_1} + s_1) + (y_{\vec{\sigma}_2} + s_2)}{2} \\
\qquad\qquad = \dfrac{x_{\vec{\sigma}_1} + x_{\vec{\sigma}_2}}{2} + z\dfrac{y_{\vec{\sigma}_1} - y_{\vec{\sigma}_2}}{2} + a \\
\dfrac{(y_{\vec{\sigma}_1} + s_1) + (y_{\vec{\sigma}_2} + s_2)}{2} + (z + c)\dfrac{(x_{\vec{\sigma}_1} + r_1) + (x_{\vec{\sigma}_2} + r_2)}{2} \\
\qquad\qquad = \dfrac{x_{\vec{\sigma}_1} + x_{\vec{\sigma}_2}}{2} + z\dfrac{y_{\vec{\sigma}_1} - y_{\vec{\sigma}_2}}{2} + b
\end{cases}.$$

After rearrangement, the above system is equivalent to

$$\begin{cases}
\dfrac{r_1 + r_2}{2} + (z + c)\dfrac{s_1 - s_2}{2} = a - c\dfrac{y_{\vec{\sigma}_1} - y_{\vec{\sigma}_2}}{2} \\
\dfrac{s_1 + s_2}{2} + (z + c)\dfrac{r_1 - r_2}{2} = a - c\dfrac{x_{\vec{\sigma}_1} - x_{\vec{\sigma}_2}}{2}
\end{cases}.$$

Forcing $r_1 = r_2$ and $s_1 = s_2$, we find a solution

$$\begin{cases}
r_1 = r_2 = a - c\dfrac{y_{\vec{\sigma}_1} - y_{\vec{\sigma}_2}}{2} \\
s_1 = s_2 = b + c\dfrac{x_{\vec{\sigma}_1} - x_{\vec{\sigma}_2}}{2}
\end{cases},$$

which has the property that

$$\|(r_1, s_1)\| = \|(r_2, s_2)\| = \left\|(a,b) + c\left(-\frac{y_{\vec{\sigma}_1} - y_{\vec{\sigma}_2}}{2}, \frac{x_{\vec{\sigma}_1} - x_{\vec{\sigma}_2}}{2}\right)\right\|$$
$$\leqslant \|(a,b)\| + |c|\frac{\|\vec{\sigma}_1 - \vec{\sigma}_2\|}{2} < \frac{\delta}{2} + \frac{\delta}{2} \cdot 1 = \delta \qquad \blacksquare$$

Next, define Q to be the set of all quadruples $(q_1, q_2, q_3, q_4) \in E' \times E'' \times E'' \times E'$ such that

$$\delta > \operatorname{dist}\big(\Delta(q_1, q_2), \Delta(q_3, q_4)\big) := \min_{(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \in q_1 \times q_2 \times q_3 \times q_4} |\Delta(\vec{p}_1, \vec{p}_2) - \Delta(\vec{p}_3, \vec{p}_4)|.$$

The following two propositions then allow us to resolve Question 5.1.

Proposition 5.7. *If a quadruple of δ-balls $(q_1, q_2, q_3, q_4) \in E' \times E'' \times E'' \times E'$ is in Q, then the corresponding complex δ-tubes T_{q_1,q_3} and T_{q_2,q_4} intersect in a δ-ball.*

Proposition 5.8. *The collection $\mathbb{T}$ meets the spacing condition of Corollary 4.3, with $W \sim \delta^{-s/4}$ and $N_0 = N^2$.*

Remark 5.9. Here, we set $W \sim \delta^{-s/4}$ (rather than $W = \delta^{-s/4}$) to account for various implied constants that appear when proving that a spacing condition for the δ-balls results in a spacing condition for the associated tubes. Ultimately, we want to make W^{-1} a small positive multiple of $\delta^{s/4}$ (e.g., by $1/1000$); thus, we should think of W as a multiple of $\delta^{-s/4}$ by a constant > 1. One could alternatively read the proposition as the statement, "there exists $C > 1$ so that if $W = C\delta^{-s/4}$ and $N_0 = N^2$, then the collection $\mathbb{T}$ meets the spacing condition of Corollary 4.3."

Below, we give a proof of Theorem 5.2 assuming both Propositions 5.7 and 5.8. We later prove each of these propositions in a separate subsection.

Proof of Theorem 5.2. Assuming Proposition 5.7, and counting the incidences, we have

$$|Q| \lesssim \sum_{r \geqslant 2 \text{ dyadic}} r^2 |P_r(\mathbb{T})|.$$

Assuming Proposition 5.8, we know from Theorem 4.2 that for $r \geqslant \delta^{4-\varepsilon}|\mathbb{T}|$,

$$|P_r(\mathbb{T})| \lesssim \delta^{-\varepsilon}|\mathbb{T}|^{\frac{3}{2}} r^{-2}.$$

The same bound holds true for $2 \leqslant r < \delta^{4-\varepsilon}|\mathbb{T}|$ since $|\mathbb{T}| \sim \delta^{-2s}$ and there are at most $\sim \delta^{-6}$ essentially distinct δ-balls in the unit ball $B_{\mathbb{C}^3}(\mathbf{0}, 1)$. Hence,

$$|Q| \lesssim |\log_2 \delta^4| \cdot \delta^{-\varepsilon}|\mathbb{T}|^{\frac{3}{2}} \sim |\log \delta| \cdot \delta^{-\varepsilon-3s}.$$

Finally, by the Cauchy–Schwarz inequality, the number of δ-balls required to cover the difference set $\Delta(E', E'') \subseteq \Delta(E)$ has the following lower bound on it:

$$\#\Delta(E', E'') \geqslant \frac{(\#E' \cdot \#E'')^2}{|Q|} \gtrsim |\log\delta|^{-1}\delta^{-s+\varepsilon} \geqslant \delta^{-2+\varepsilon}.$$

So we have proved Theorem 5.2. ∎

5.1. Proof of Proposition 5.7

Fix a quadruple of δ-balls $q_1 \times q_2 \times q_3 \times q_4 \in Q$ and a quadruple of points

$$(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \in q_1 \times q_2 \times q_3 \times q_4.$$

Claim 5.10. Given any $\vec{p}_1, \vec{p}_4 \in B'$ and $\vec{p}_2, \vec{p}_3 \in B''$, the angle between $l_{\vec{p}_1,\vec{p}_3}$ and $l_{\vec{p}_2,\vec{p}_4}$ is $\gtrsim 1$.

Proof. We recall from Section 2 the usual embedding $\iota : \mathbb{C}^n \to \mathbb{R}^{2n}$ splitting each complex number into its real and imaginary part, and note that ι commutes with addition and multiplication by real scalars. By the variational definition of the angle, all we need to show is that the angle between the two vectors $\iota(\boldsymbol{v}_{\vec{p}_1,\vec{p}_3})$ and $\iota(z\boldsymbol{v}_{\vec{p}_2,\vec{p}_4})$, which we denote by θ_z, is $\gtrsim 1$ for all $z \in \mathbb{C}$ with $|z| = 1$. The cosine law gives

$$\begin{aligned}
\cos\theta_z &= \frac{\|\iota(\boldsymbol{v}_{\vec{p}_1,\vec{p}_3})\|^2 + \|\iota(z\boldsymbol{v}_{\vec{p}_2,\vec{p}_4})\|^2 - \|\iota(\boldsymbol{v}_{\vec{p}_1,\vec{p}_3} - z\boldsymbol{v}_{\vec{p}_2,\vec{p}_4})\|^2}{2\|\iota(\boldsymbol{v}_{\vec{p}_1,\vec{p}_3})\|\|\iota(z\boldsymbol{v}_{\vec{p}_2,\vec{p}_4})\|} \\
&= \frac{\|\boldsymbol{v}_{\vec{p}_1,\vec{p}_3}\|^2 + \|\boldsymbol{v}_{\vec{p}_2,\vec{p}_4}\|^2 - \|\boldsymbol{v}_{\vec{p}_1,\vec{p}_3} - z\boldsymbol{v}_{\vec{p}_2,\vec{p}_4}\|^2}{2\|\boldsymbol{v}_{\vec{p}_1,\vec{p}_3}\|\|\boldsymbol{v}_{\vec{p}_2,\vec{p}_4}\|} \\
&= \frac{(\|\frac{\vec{p}_1-\vec{p}_3}{2}\|^2 + 1) + (\|\frac{\vec{p}_2-\vec{p}_4}{2}\|^2 + 1) - (\|\frac{\vec{p}_1-\vec{p}_3}{2} - z\frac{\vec{p}_2-\vec{p}_4}{2}\|^2 + |1-z|^2)}{2\sqrt{\|\frac{\vec{p}_1-\vec{p}_3}{2}\|^2 + 1}\sqrt{\|\frac{\vec{p}_2-\vec{p}_4}{2}\|^2 + 1}} \\
&< \frac{2((\frac{C_2+2C_1}{2})^2 + 1) - (\|\frac{\vec{p}_1-\vec{p}_3}{2} - z\frac{\vec{p}_2-\vec{p}_4}{2}\|^2 + |1-z|^2)}{2((\frac{C_2-2C_1}{2})^2 + 1)} \\
&= 1 + \frac{4C_1C_2 - (\|\frac{\vec{p}_1-\vec{p}_3}{2} - z\frac{\vec{p}_2-\vec{p}_4}{2}\|^2 + |1-z|^2)}{2((\frac{C_2-2C_1}{2})^2 + 1)},
\end{aligned}$$

so we just need to show that for some constant $\varepsilon_1 > 0$,

$$\frac{4C_1C_2 - (\|\frac{\vec{p}_1-\vec{p}_3}{2} - z\frac{\vec{p}_2-\vec{p}_4}{2}\|^2 + |1-z|^2)}{2((\frac{C_2-2C_1}{2})^2 + 1)} < -\varepsilon_1,$$

or in other words,

$$\left\|\frac{\vec{p}_1-\vec{p}_3}{2} - z\frac{\vec{p}_2-\vec{p}_4}{2}\right\|^2 + |1-z|^2 > 4C_1C_2 + 2\varepsilon_1\left(\left(\frac{C_2-2C_1}{2}\right)^2 + 1\right).$$

Since $C_2 < 2$ and $C_1 \leqslant 0.01$, it suffices to show for some $\varepsilon_1 > 0$

$$\left\| \frac{\vec{p}_1 - \vec{p}_3}{2} - z\frac{\vec{p}_2 - \vec{p}_4}{2} \right\|^2 + |1-z|^2 > 0.08 + 4\varepsilon_1.$$

Let us assume to the contrary that there exists some $z \in \mathbb{C}$ with $|z| = 1$ such that

$$\left\| \frac{\vec{p}_1 - \vec{p}_3}{2} - z\frac{\vec{p}_2 - \vec{p}_4}{2} \right\|^2 + |1-z|^2 \leqslant 0.09.$$

Then, on one hand,

$$|1-z| \leqslant 0.3;$$

while on the other hand,

$$\begin{aligned} 0.3 &\geqslant \left\| \frac{\vec{p}_1 - \vec{p}_3}{2} - z\frac{\vec{p}_2 - \vec{p}_4}{2} \right\| \\ &= \left\| (1+z)\frac{\vec{p}_1 - \vec{p}_3}{2} + z\frac{(\vec{p}_4 - \vec{p}_1) - (\vec{p}_2 - \vec{p}_3)}{2} \right\| \\ &\geqslant |1+z| \left\| \frac{\vec{p}_1 - \vec{p}_3}{2} \right\| - \left\| \frac{(\vec{p}_4 - \vec{p}_1) - (\vec{p}_2 - \vec{p}_3)}{2} \right\| \\ &> \frac{C_2 - 2C_1}{2}|1+z| - 2C_1 \geqslant \frac{1}{2}|1+z| - 0.02, \end{aligned}$$

which, after rearrangement, gives

$$0.64 > |1+z| = |z-(-1)|.$$

But then by the triangle inequality,

$$2 = |1-(-1)| \leqslant |1-z| + |z-(-1)| < 0.3 + 0.64 = 0.94,$$

which is contradictory.

To summarize, in order for θ_z to be small, the first two coordinates of $\boldsymbol{v}_{\vec{p}_1,\vec{p}_3}$ and of $\boldsymbol{v}_{\vec{p}_2,\vec{p}_4}$ force z to be closer to -1 while the last coordinate of $\boldsymbol{v}_{\vec{p}_1,\vec{p}_3}$ and of $\boldsymbol{v}_{\vec{p}_2,\vec{p}_4}$ force z to be closer to 1; yet the two cannot be achieved at the same time; thus θ_z cannot be too small. In conclusion,

$$\angle(l_{\vec{p}_1,\vec{p}_3}, l_{\vec{p}_2,\vec{p}_4}) = \min_{z\in\mathbb{C}:|z|=1} \theta_z \geqslant \arccos\left(1 - \frac{0.09-0.08}{4}\right) = \arccos\left(\frac{399}{400}\right). \qquad \blacksquare$$

Proving Proposition 5.7 now reduces to proving the following claim.

Claim 5.11. The distance between $l_{\vec{p}_1,\vec{p}_3}$ and $l_{\vec{p}_2,\vec{p}_4}$ is $< \delta/2$. Consequently, the complex tubes T_{q_1,q_3} and T_{q_2,q_4} intersect in (essentially) a δ-ball in $B_{\mathbb{C}_n}(\mathbf{0}, 1)$.

Let us consider the mapping $A : \mathbb{C}^2 \to l_{\vec{p}_1,\vec{p}_3} - l_{\vec{p}_2,\vec{p}_4} \subseteq \mathbb{C}^3$ given by

$$\boldsymbol{A}(t,t') := \left(\frac{\vec{p}_1 + \vec{p}_3}{2} - \frac{\vec{p}_2 + \vec{p}_4}{2}, 0\right) + t\boldsymbol{v}_{\vec{p}_1,\vec{p}_3} - t'\boldsymbol{v}_{\vec{p}_2,\vec{p}_4}.$$

Observe that

$$|(\vec{p}_1 - \vec{p}_3) - (\vec{p}_2 - \vec{p}_4)| = |(\vec{p}_1 + \vec{p}_4) - (\vec{p}_2 + \vec{p}_3)| \geqslant 2C_2 - 4C_1.$$

If $|x_1 - x_3 - x_2 + x_4| \geqslant |y_1 - y_3 - y_2 + y_4|$, then $|x_1 - x_3 - x_2 + x_4| \geqslant C_2 - 2C_1$ and by evaluating $\|\boldsymbol{A}\|$ at $t = t' = \frac{y_1+y_3-y_2-y_4}{x_1-x_3-x_2+x_4}$ we find that

$$\|\boldsymbol{A}(t,t')\|^2 = \frac{|\Delta(\vec{p}_1,\vec{p}_2) - \Delta(\vec{p}_3,\vec{p}_4)|^2}{(2|x_1 - x_3 - x_2 + x_4|)^2} < \left(\frac{\delta}{2C_2 - 4C_1}\right)^2 < \left(\frac{\delta}{2}\right)^2,$$

with $|t| = |t'| \leqslant \frac{4C_1}{C_2-2C_1} < \frac{1}{25}$. If $|x_1 - x_3 - x_2 + x_4| \leqslant |y_1 - y_3 - y_2 + y_4|$, then, evaluating $\|\boldsymbol{A}\|$ at $t = t' = \frac{x_1+x_3-x_2-x_4}{y_1-y_3-y_2+y_4}$, we again have

$$\|\boldsymbol{A}(t,t')\| < \frac{\delta}{2},$$

with $|t| = |t'| < \frac{1}{25}$. Thus, in either case, there exists a ball of radius $\frac{\delta}{4}$ which contains a point from $l_{\vec{p}_1,\vec{p}_3}$ and a point from $l_{\vec{p}_2,\vec{p}_4}$; moreover, this $\frac{\delta}{4}$-ball lies completely in the ball $B_{\mathbb{C}^3}(\mathbf{0},1)$ because

$$\left\|\frac{\vec{p}_1 + \vec{p}_3}{2}\right\|, \left\|\frac{\vec{p}_2 + \vec{p}_4}{2}\right\| < \sqrt{1 - \left(\frac{C_2 - 2C_1}{2}\right)^2} < \frac{\sqrt{3}}{2},$$

and

$$\left(\frac{\vec{p}_1+\vec{p}_3}{2},0\right) + tv_{\vec{p}_1,\vec{p}_3}, \left(\frac{\vec{p}_2+\vec{p}_4}{2},0\right) + t'v_{\vec{p}_2,\vec{p}_4} < \frac{\sqrt{3}}{2} + \frac{1}{25}\cdot\sqrt{\left(\frac{2}{2}\right)^2 + 1} < 0.93.$$

The δ-ball with the same centre witnesses the incidence between the complex tubes T_{q_1,q_3} and T_{q_2,q_4}. ∎

5.2. Proof of Proposition 5.8

Let $W^{-1} = c\delta^{s/4}$ for some small constant $c \in (0,1)$ to be determined. Let R be a complex W^{-1}-tube in $B_{\mathbb{C}^3}(\mathbf{0},1)$. We want to show that if c is sufficiently small, then R contains no more than N^2-many tubes of $\mathbb{T}$. To accomplish this, we partition $\mathbb{T}$ into at most N^2 parts and show that R contains at most 1 tube from each of these parts. To partition $\mathbb{T}$, we first partition E.

We can think of partitioning E as assigning a color to each δ-ball in E. We adapt a graph-theoretic argument that we first encountered in [1], in the proof of their Proposition 4.1. We build a graph whose vertex set is the set of centres of the δ-balls in E and place edges so that two vertices have an edge between them if and only if the corresponding δ-balls are both contained in a larger $\delta^{s/4}$-ball. The resulting graph has maximum degree $N-1$, so by Brooks' Theorem, the chromatic number of the graph is at most N; that is, we can color the δ-balls of E using N colors in a way that guarantees that no two δ-balls of the same color are in any $\delta^{s/4}$-ball.

For $j = 1, \dots, N$, let E_j be the set of δ-balls which were colored with color j. For $1 \leqslant j, k \leqslant N$, let $\mathbb{T}_{j,k} \subset \mathbb{T}$ be the collection

$$\mathbb{T}_{j,k} = \{T_{q_1,q_2} \in \mathbb{T} : q_1 \in E_j \text{ and } q_2 \in E_k\}.$$

Since the N^2 subsets $\{\mathbb{T}_{j,k}\}_{1 \leqslant j,k \leqslant N}$ partition $\mathbb{T}$, proving Proposition 5.8 now reduces to proving the following claim.

Claim 5.12. For any pair of indices j, k with $1 \leqslant j, k \leqslant N$, a W^{-1}-tube R contains at most 1 tube in $\mathbb{T}_{j,k}$.

We now prove the claim.

Proof. Fix a pair of indices (j, k). We consider two distinct δ-tubes in $\mathbb{T}_{j,k}$ and show that there is no W^{-1}-tube in $B_{\mathbb{C}^3}(\mathbf{0}, 1)$ that essentially contains both of these δ-tubes, provided that $W^{-1} = c\delta^{s/4}$ for a sufficiently small constant c. Although the proof is notation-heavy and may appear complicated, it ultimately reduces to an application of the parallelogram law.

Let $\vec{\sigma}_1 = (x_{\vec{\sigma}_1}, y_{\vec{\sigma}_1})$ and $\vec{\sigma}_2 = (x_{\vec{\sigma}_2}, y_{\vec{\sigma}_2})$ be the respective centres of δ-balls q_1 and q_2 in E_j, and let $\vec{\sigma}_3 = (x_{\vec{\sigma}_3}, y_{\vec{\sigma}_3})$ and $\vec{\sigma}_4 = (x_{\vec{\sigma}_4}, y_{\vec{\sigma}_4})$ be the centres of q_3 and q_4 in E_k. We want to show that there is no W^{-1}-tube that essentially contains both of the tubes[10] T_{q_1,q_3} and T_{q_2,q_4}. To accomplish this, we embed T_{q_1,q_3} and T_{q_2,q_4} into $\mathbb{R}^6$ and parameterize their respective images by functions $\boldsymbol{G}_{1,3}, \boldsymbol{G}_{2,4} : B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2}) \to \mathbb{R}^6$. We show that for all (α, β) in a set consisting of at least half the area of $B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2})$, the distance from the point $\boldsymbol{G}_{1,3}(\alpha, \beta)$ to the image of $\boldsymbol{G}_{2,4}$ exceeds W^{-1}.

[10] One should note that, despite the similarity in notation, the tubes T_{q_1,q_3} and T_{q_2,q_4} do *not* necessarily satisfy the same hypotheses as the tubes appearing in Proposition 5.7. In particular, in Proposition 5.7, we assume that $(q_1, q_2, q_3, q_4) \in E' \times E'' \times E'' \times E'$. In the context of our present proof, we merely know that the pair (q_1, q_3) is in one of $E' \times E''$ or $E'' \times E'$, but we do not know which. Similarly, we know that the pair (q_2, q_4) is in one of $E' \times E''$ or $E'' \times E'$, but we do not know which.

Since q_1 and q_2 are both in E_j, it follows that $\|\vec{\sigma}_1 - \vec{\sigma}_2\| \geqslant \delta^{s/4}$. Similarly, since q_3 and q_4 are both in E_k, it follows that $\|\vec{\sigma}_3 - \vec{\sigma}_4\| \geqslant \delta^{s/4}$. To leverage this separation, we apply the parallelogram law to the vectors $\|\vec{\sigma}_1 - \vec{\sigma}_2\|$ and $\|\vec{\sigma}_3 - \vec{\sigma}_4\|$ to get

$$\begin{aligned} &2\|\vec{\sigma}_1 - \vec{\sigma}_2\|^2 + 2\|\vec{\sigma}_3 - \vec{\sigma}_4\|^2 \\ &\quad = \|(\vec{\sigma}_1 - \vec{\sigma}_2) + (\vec{\sigma}_3 - \vec{\sigma}_4)\|^2 + \|(\vec{\sigma}_1 - \vec{\sigma}_2) - (\vec{\sigma}_3 - \vec{\sigma}_4)\|^2 \\ &\quad = \|(\vec{\sigma}_1 + \vec{\sigma}_3) - (\vec{\sigma}_2 + \vec{\sigma}_4)\|^2 + \|(\vec{\sigma}_1 - \vec{\sigma}_3) - (\vec{\sigma}_2 - \vec{\sigma}_4)\|^2. \end{aligned} \tag{5.1}$$

We ultimately take cases according to which of the terms on the right-hand side of equality (5.1) is larger. In either case, the dominant term is comparable to

$$2\|\vec{\sigma}_1 - \vec{\sigma}_2\|^2 + 2\|\vec{\sigma}_3 - \vec{\sigma}_4\|^2,$$

which is, in turn, at least $4(\delta^{s/4})^2$.

The tube T_{q_1,q_3} is roughly the δ-neighbourhood of the complex line segment $\overline{l_{\vec{\sigma}_1,\vec{\sigma}_3}}$ defined by

$$\left\{\left(\frac{x_{\vec{\sigma}_1} + x_{\vec{\sigma}_3}}{2}, \frac{y_{\vec{\sigma}_1} + y_{\vec{\sigma}_3}}{2}, 0\right) + z\left(\frac{y_{\vec{\sigma}_1} - y_{\vec{\sigma}_3}}{2}, -\frac{x_{\vec{\sigma}_1} - x_{\vec{\sigma}_3}}{2}, 1\right) : z \in \mathbb{C}, |z| \leqslant \frac{1}{2}\right\},$$

and T_{q_2,q_4} is the δ-neighbourhood of the complex line segment

$$\begin{aligned} &\overline{l_{\vec{\sigma}_2,\vec{\sigma}_4}} \\ &= \left\{\left(\frac{x_{\vec{\sigma}_2} + x_{\vec{\sigma}_4}}{2}, \frac{y_{\vec{\sigma}_2} + y_{\vec{\sigma}_4}}{2}, 0\right) + z\left(\frac{y_{\vec{\sigma}_2} - y_{\vec{\sigma}_4}}{2}, -\frac{x_{\vec{\sigma}_2} - x_{\vec{\sigma}_4}}{2}, 1\right) : z \in \mathbb{C}, |z| \leqslant \frac{1}{2}\right\}. \end{aligned}$$

The complex lines $l_{\vec{\sigma}_1,\vec{\sigma}_3}$ and $l_{\vec{\sigma}_2,\vec{\sigma}_4}$ correspond[11] to real two-planes $P_{1,3}$ and $P_{2,4}$. If we write $x_{\vec{\sigma}_j} = a_j + b_j i$ and $y_{\vec{\sigma}_j} = c_j + d_j i$ with $a_j, b_j, c_j, d_j \in \mathbb{R}$ for $j = 1, \ldots, 4$, then we can define parameterizations $\boldsymbol{G}_{1,3}, \boldsymbol{G}_{2,4} : \mathbb{R}^2 \to \mathbb{R}^6$ by

$$\begin{aligned} \boldsymbol{G}_{1,3}(\alpha, \beta) = &\left(\frac{a_1 + a_3}{2}, \frac{b_1 + b_3}{2}, \frac{c_1 + c_3}{2}, \frac{d_1 + d_3}{2}, 0, 0\right) \\ &+ \alpha\left(\frac{c_1 - c_3}{2}, \frac{d_1 - d_3}{2}, -\frac{(a_1 - a_3)}{2}, -\frac{(b_1 - b_3)}{2}, 1, 0\right) \\ &+ \beta\left(-\frac{(d_1 - d_3)}{2}, \frac{c_1 - c_3}{2}, \frac{b_1 - b_3}{2}, -\frac{(a_1 - a_3)}{2}, 0, 1\right) \end{aligned}$$

[11]The correspondence is via the canonical embedding $\iota : \mathbb{C}^3 \to \mathbb{R}^6$. The real vectors $(\frac{c_1-c_3}{2}, \frac{d_1-d_3}{2}, -\frac{(a_1-a_3)}{2}, -\frac{(b_1-b_3)}{2}, 1, 0)$ and $(-\frac{(d_1-d_3)}{2}, \frac{c_1-c_3}{2}, \frac{b_1-b_3}{2}, -\frac{(a_1-a_3)}{2}, 0, 1)$ are the respective images of the complex vectors $(\vec{u}_1, 1)$ and $(i\vec{u}_1, i)$ under ι.

and

$$\boldsymbol{G}_{2,4}(\alpha, \beta) = \left(\frac{a_2 + a_4}{2}, \frac{b_2 + b_4}{2}, \frac{c_2 + c_4}{2}, \frac{d_2 + d_4}{2}, 0, 0\right) + \alpha\left(\frac{c_2 - c_4}{2}, \frac{d_2 - d_4}{2}, -\frac{(a_2 - a_4)}{2}, -\frac{(b_2 - b_4)}{2}, 1, 0\right) + \beta\left(-\frac{(d_2 - d_4)}{2}, \frac{c_2 - c_4}{2}, \frac{b_2 - b_4}{2}, -\frac{(a_2 - a_4)}{2}, 0, 1\right).$$

The complex line segments $\overline{l_{\vec{\sigma}_1,\vec{\sigma}_3}}$ and $\overline{l_{\vec{\sigma}_2,\vec{\sigma}_4}}$ correspond to the respective images under $\boldsymbol{G}_{1,3}$ and $\boldsymbol{G}_{2,4}$ of the disk $B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2}) = \{(\alpha, \beta) \in \mathbb{R}^2 : (\alpha^2 + \beta^2)^{1/2} \leqslant \frac{1}{2}\}$. We consider how the distance $d(\boldsymbol{G}_{1,3}(\alpha, \beta), P_{2,4})$ varies as (α, β) ranges over the ball $B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2})$. To show that no W^{-1}-box (essentially) contains both T_{q_1,q_3} and T_{q_2,q_4}, it is sufficient to show that the inequality

$$d\big(\boldsymbol{G}_{1,3}(\alpha, \beta), P_{2,4}\big) > W^{-1} + 4\delta \tag{5.2}$$

holds for all (α, β) in a set that accounts for at least half of the area of $B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2})$. Here,

$$d\big(\boldsymbol{G}_{1,3}(\alpha, \beta), P_{2,4}\big) = \min_{(\alpha', \beta') \in \mathbb{R}^2} \|\boldsymbol{G}_{1,3}(\alpha, \beta) - \boldsymbol{G}_{2,4}(\alpha', \beta')\|,$$

i.e., $d(\boldsymbol{G}_{1,3}(\alpha, \beta))$ represents the perpendicular distance from the point $\boldsymbol{G}_{1,3}(\alpha, \beta)$ to the plane $P_{2,4}$. No matter our eventual choice of c, we have $W^{-1} > 4\delta$ for δ sufficiently small, so (5.2) reduces to showing that

$$d\big(\boldsymbol{G}_{1,3}(\alpha, \beta), P_{2,4}\big) > 2W^{-1} \tag{5.3}$$

holds for all (α, β) in a set that accounts for at least half of the area of $B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2})$.

By Claim 5.13 – which is stated and proved below – the perpendicular distance from $\boldsymbol{G}_{1,3}(\alpha, \beta)$ to the plane $P_{2,4}$ is comparable to the distance from $\boldsymbol{G}_{1,3}(\alpha, \beta)$ to $\boldsymbol{G}_{2,4}(\alpha, \beta)$; specifically,

$$d\big(\boldsymbol{G}_{1,3}(\alpha, \beta), P_{2,4}\big) \geqslant \frac{1}{\sqrt{3}}\|\boldsymbol{G}_{1,3}(\alpha, \beta) - \boldsymbol{G}_{2,4}(\alpha, \beta)\|, \tag{5.4}$$

so it suffices to show that

$$\|\boldsymbol{G}_{1,3}(\alpha, \beta) - \boldsymbol{G}_{2,4}(\alpha, \beta)\| > 2\sqrt{3}W^{-1} \tag{5.5}$$

holds for all (α, β) in a set that accounts for at least half of the area of $B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2})$.

To estimate $\|\boldsymbol{G}_{1,3}(\alpha, \beta) - \boldsymbol{G}_{2,4}(\alpha, \beta)\|$, we begin by writing

$$\boldsymbol{G}_{1,3}(\alpha, \beta) - \boldsymbol{G}_{2,4}(\alpha, \beta) = \boldsymbol{u} + \alpha\boldsymbol{v} + \beta\boldsymbol{w},$$

where

$$\boldsymbol{u} = \Big(\frac{(a_1+a_3)-(a_2+a_4)}{2}, \frac{(b_1+b_3)-(b_2+b_4)}{2}, \frac{(c_1+c_3)-(c_2+c_4)}{2}, \frac{(d_1+d_3)-(d_2+d_4)}{2}, 0, 0\Big),$$

$$\boldsymbol{v} = \Big(\frac{(c_1-c_3)-(c_2-c_4)}{2}, \frac{(d_1-d_3)-(d_2-d_4)}{2}, \frac{-(a_1-a_3)+(a_2-a_4)}{2}, \frac{-(b_1-b_3)+(b_2-b_4)}{2}, 0, 0\Big),$$

and

$$\boldsymbol{w} = \Big(\frac{-(d_1-d_3)+(d_2-d_4)}{2}, \frac{(c_1-c_3)-(c_2-c_4)}{2}, \frac{(b_1-b_3)-(b_2-b_4)}{2}, \frac{-(a_1-a_3)+(a_2-a_4)}{2}, 0, 0\Big).$$

We note that

$$\|\boldsymbol{u}\| = \frac{1}{2}\|(\vec{\sigma}_1+\vec{\sigma}_3)-(\vec{\sigma}_2+\vec{\sigma}_4)\|.$$

Meanwhile, the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are orthogonal to each other and have the same magnitude; specifically,

$$\|\boldsymbol{v}\| = \|\boldsymbol{w}\| = \frac{1}{2}\|(\vec{\sigma}_1-\vec{\sigma}_3)-(\vec{\sigma}_2-\vec{\sigma}_4)\|.$$

This motivates us to take cases on the relative sizes of $\|(\vec{\sigma}_1+\vec{\sigma}_3)-(\vec{\sigma}_2+\vec{\sigma}_4)\|$ and $\|(\vec{\sigma}_1-\vec{\sigma}_3)-(\vec{\sigma}_2-\vec{\sigma}_4)\|$; specifically, we take cases according to whether

$$\|(\vec{\sigma}_1-\vec{\sigma}_3)-(\vec{\sigma}_2-\vec{\sigma}_4)\| \leqslant 2\sqrt{2}\|(\vec{\sigma}_1+\vec{\sigma}_3)-(\vec{\sigma}_2+\vec{\sigma}_4)\|.$$

(Here, the constant $2\sqrt{2}$ is chosen to ensure that the right-hand side of a later inequality – namely, inequality (5.9) – is a positive multiple of $\|(\vec{\sigma}_1-\vec{\sigma}_3)-(\vec{\sigma}_2-\vec{\sigma}_4)\|$.)

Case 1. First, suppose that

$$\|(\vec{\sigma}_1-\vec{\sigma}_3)-(\vec{\sigma}_2-\vec{\sigma}_4)\| \leqslant 2\sqrt{2}\|(\vec{\sigma}_1+\vec{\sigma}_3)-(\vec{\sigma}_2+\vec{\sigma}_4)\|. \tag{5.6}$$

By (5.1) and (5.6), we have that

$$4(\delta^{s/4})^2 \leqslant \big(1+(2\sqrt{2})^2\big)\|(\vec{\sigma}_1+\vec{\sigma}_3)-(\vec{\sigma}_2+\vec{\sigma}_4)\|^2 = 9\|(\vec{\sigma}_1+\vec{\sigma}_3)-(\vec{\sigma}_2+\vec{\sigma}_4)\|^2.$$

We consider the function

$$F(\alpha,\beta) := \|\boldsymbol{G}_{1,3}(\alpha,\beta)-\boldsymbol{G}_{2,4}(\alpha,\beta)\|^2$$

and show that, provided we take W^{-1} to be a sufficiently small multiple of $\delta^{s/4}$, we have

$$F(\alpha, \beta) > 12W^{-2}$$

for all (α, β) in a set consisting of at least half of $B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2})$. First, we note that

$$F(0,0) = \|\boldsymbol{u}\|^2 = \frac{1}{4}\|(\vec{\sigma}_1 + \vec{\sigma}_3) - (\vec{\sigma}_2 + \vec{\sigma}_4)\|^2 = \frac{1}{4}\left(\frac{4}{9}\delta^{s/2}\right) = \frac{1}{9}\delta^{s/2}.$$

It is natural to consider when we have

$$F(\alpha, \beta) \geqslant F(0,0).$$

To this end, note that F is a quadratic function of α and β. Specifically, using the fact that $\boldsymbol{v} \cdot \boldsymbol{w} = 0$, we compute that

$$\begin{aligned} F(\alpha, \beta) &= (\boldsymbol{u} + \alpha\boldsymbol{v} + \beta\boldsymbol{w}) \cdot (\boldsymbol{u} + \alpha\boldsymbol{v} + \beta\boldsymbol{w}) \\ &= \|\boldsymbol{v}\|^2\alpha^2 + \|\boldsymbol{w}\|^2\beta^2 + (\boldsymbol{v} \cdot \boldsymbol{u})\alpha + (\boldsymbol{w} \cdot \boldsymbol{u})\beta + \|\boldsymbol{u}\|^2. \end{aligned} \tag{5.7}$$

From this formula, it is apparent that we have $F(\alpha, \beta) > F(0,0)$ for all points (α, β) in a half-space. There are a couple of ways to see this, which we explain below:

(1) We observe that if $F(\alpha, \beta) < F(0,0)$, then we must have $(\boldsymbol{v} \cdot \boldsymbol{u})\alpha + (\boldsymbol{w} \cdot \boldsymbol{u})\beta < 0$, which implies that the antipodal point $(-\alpha, -\beta)$ satisfies $F(-\alpha, -\beta) > F(0,0)$.

(2) We observe that the graph of F over $\mathbb{R}^2$ is a paraboloid. If $(0,0)$ is the vertex of the paraboloid, then we have that $F(\alpha, \beta) \geqslant F(0,0)$ for all (α, β). Otherwise, if the vertex is not at $(0,0)$, let C be the unique level curve of F passing through $(0,0)$. Draw a line ℓ through $(0,0)$ tangent to C. Then for all points (α, β) lying on the opposite side of ℓ as the vertex, we must have $F(\alpha, \beta) \geqslant F(0,0)$. (Note that the collection of points (α, β) lying on the opposite side of ℓ as the vertex consists of half of the disk $B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2})$.)

Either argument shows that for all (α, β) in a region consisting of at least half of $B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2})$, we have

$$F(\alpha, \beta) \geqslant F(0,0) \geqslant \frac{1}{9}\delta^{s/2},$$

which means that

$$\|\boldsymbol{G}_{1,3}(\alpha, \beta) - \boldsymbol{G}_{2,4}(\alpha, \beta)\| \geqslant \frac{1}{3}\delta^{s/4}$$

Case 2. Now, suppose that

$$\|(\vec{\sigma}_1 - \vec{\sigma}_3) - (\vec{\sigma}_2 - \vec{\sigma}_4)\| \geqslant 2\sqrt{2}\|(\vec{\sigma}_1 + \vec{\sigma}_3) - (\vec{\sigma}_2 + \vec{\sigma}_4)\|. \tag{5.8}$$

In this case, since $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and $\|\boldsymbol{v}\| = \|\boldsymbol{w}\| = \frac{1}{2}\|(\vec{\sigma}_1 - \vec{\sigma}_3) - (\vec{\sigma}_2 - \vec{\sigma}_4)\|$, we have by the triangle inequality that

$$\begin{aligned}
&\|\boldsymbol{G}_{1,3}(\alpha, \beta) - \boldsymbol{G}_{2,4}(\alpha, \beta)\| \\
&\quad \geqslant \frac{|(\alpha, \beta)|}{2}\|(\vec{\sigma}_1 - \vec{\sigma}_3) - (\vec{\sigma}_2 - \vec{\sigma}_4)\| - \frac{1}{2}\|(\vec{\sigma}_1 + \vec{\sigma}_3) - (\vec{\sigma}_2 + \vec{\sigma}_4)\| \\
&\quad \geqslant \frac{|(\alpha, \beta)|}{2}\|(\vec{\sigma}_1 - \vec{\sigma}_3) - (\vec{\sigma}_2 - \vec{\sigma}_4)\| - \frac{1}{2}\left(\frac{1}{2\sqrt{2}}\|(\vec{\sigma}_1 - \vec{\sigma}_3) - (\vec{\sigma}_2 - \vec{\sigma}_4)\|\right).
\end{aligned}$$

It follows that if $|\alpha|^2 + |\beta|^2 \geqslant \frac{1}{2}$, then

$$\begin{aligned}
\|\boldsymbol{G}_{1,3}(\alpha, \beta) - \boldsymbol{G}_{2,4}(\alpha, \beta)\| &\geqslant \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\right)\|(\vec{\sigma}_1 - \vec{\sigma}_3) - (\vec{\sigma}_2 - \vec{\sigma}_4)\| \\
&= \frac{1}{4\sqrt{2}}\|(\vec{\sigma}_1 - \vec{\sigma}_3) - (\vec{\sigma}_2 - \vec{\sigma}_4)\|. \qquad (5.9)
\end{aligned}$$

By (5.1) and (5.8), we then have that

$$4(\delta^{s/4})^2 \leqslant \left(\frac{1}{8} + 1\right)\|(\vec{\sigma}_1 - \vec{\sigma}_3) - (\vec{\sigma}_2 - \vec{\sigma}_4)\|^2 = \frac{9}{8}\|(\vec{\sigma}_1 - \vec{\sigma}_3) - (\vec{\sigma}_2 - \vec{\sigma}_4)\|^2.$$

Consequently, if $|\alpha|^2 + |\beta|^2 \geqslant \frac{1}{2}$, then

$$\|\boldsymbol{G}_{1,3}(\alpha, \beta) - \boldsymbol{G}_{2,4}(\alpha, \beta)\| \geqslant \frac{1}{4\sqrt{2}}\left(\frac{4\sqrt{2}}{3}\right)\delta^{s/4} = \frac{1}{3}\delta^{s/4}. \qquad (5.10)$$

Conclusion: Combining Cases 1 and 2, we see that if we take $W^{-1} = c\delta^{s/4}$ for some $c < \frac{1}{6\sqrt{3}}$, then our desired inequality (5.5) holds for all (α, β) in a set that accounts for at least half of the area of $B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2})$. ■

Claim 5.13. Let the maps $\boldsymbol{G}_{1,3}$ and $\boldsymbol{G}_{2,4}$ be defined as above. Then for any point $(\alpha, \beta) \in B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2})$, we have that

$$d(\boldsymbol{G}_{1,3}(\alpha, \beta), P_{2,4}) \geqslant \frac{1}{\sqrt{3}}\|\boldsymbol{G}_{1,3}(\alpha, \beta) - \boldsymbol{G}_{2,4}(\alpha, \beta)\|. \qquad (5.11)$$

Proof. We consider the subspace $V = \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4) \subset \mathbb{R}^6$. Let $\theta_{1,3}$ denote the principal angle between V and $P_{1,3}$. Similarly, let $\theta_{2,4}$ denote the principal angle between V and $P_{2,4}$. We note that $\theta_{1,3}, \theta_{2,4} \gtrsim 1$; in particular,

$$\theta_{1,3}, \theta_{2,4} \geqslant \arctan\left(\frac{1}{\sqrt{2}}\right). \qquad (5.12)$$

To see why this is true, let

$$\vec{y}_1 = \left(\frac{y_{\vec{\sigma}_1} - y_{\vec{\sigma}_3}}{2}, -\frac{x_{\vec{\sigma}_1} - x_{\vec{\sigma}_3}}{2}\right) \quad \text{and} \quad \vec{y}_2 = \left(\frac{y_{\vec{\sigma}_2} - y_{\vec{\sigma}_4}}{2}, -\frac{x_{\vec{\sigma}_2} - x_{\vec{\sigma}_4}}{2}\right)$$

so that $(\vec{y}_1, 1)$ has the same direction as the complex line

$$l_{\vec{\sigma}_1,\vec{\sigma}_3} = \left\{\left(\frac{x_{\vec{\sigma}_1}+x_{\vec{\sigma}_3}}{2}, \frac{y_{\vec{\sigma}_1}+y_{\vec{\sigma}_3}}{2}, 0\right) + z\left(\frac{y_{\vec{\sigma}_1}-y_{\vec{\sigma}_3}}{2}, -\frac{x_{\vec{\sigma}_1}-x_{\vec{\sigma}_3}}{2}, 1\right) : z \in \mathbb{C}\right\},$$

and $(\vec{y}_2, 1)$ has the same direction as the complex line

$$l_{\vec{\sigma}_2,\vec{\sigma}_4} = \left\{\left(\frac{x_{\vec{\sigma}_2}+x_{\vec{\sigma}_4}}{2}, \frac{y_{\vec{\sigma}_2}+y_{\vec{\sigma}_4}}{2}, 0\right) + z\left(\frac{y_{\vec{\sigma}_2}-y_{\vec{\sigma}_4}}{2}, -\frac{x_{\vec{\sigma}_2}-x_{\vec{\sigma}_4}}{2}, 1\right) : z \in \mathbb{C}\right\}.$$

Then $\|\vec{y}_1\|, \|\vec{y}_2\| \lesssim 1$; by assuming that the points $(x_{\vec{\sigma}_1}, y_{\vec{\sigma}_1}), \ldots, (x_{\vec{\sigma}_4}, y_{\vec{\sigma}_4})$ are in $B_{\mathbb{C}^2}(\vec{0}, 1)$, it follows that $\|\vec{y}_1\|, \|\vec{y}_2\| \leqslant \sqrt{2}$. Here, we show the computation for $\|\vec{y}_1\|$:

$$\|\vec{y}_1\|^2 = \frac{1}{4}(|y_{\vec{\sigma}_1} - y_{\vec{\sigma}_3}|^2 + |x_{\vec{\sigma}_1} - x_{\vec{\sigma}_3}|^2) \leqslant \frac{1}{4}(2^2 + 2^2) \leqslant 2.$$

We note that each of $P_{1,3}$ and $P_{2,4}$ intersects any translate of V in a single point. This is because a basis for either, along with a basis for V, forms a basis for $\mathbb{R}^6$.

After fixing $(\alpha, \beta) \in B_{\mathbb{R}^2}(\vec{0}, \frac{1}{2})$, we consider the affine subspace

$$V_{\alpha,\beta} := V + (0, 0, 0, 0, \alpha, \beta).$$

To prove (5.11), we consider the right triangle $\triangle ABC$, where A is the point $\boldsymbol{G}_{1,3}(\alpha, \beta)$, B is the point $\boldsymbol{G}_{2,4}(\alpha, \beta)$, and C is the point in $P_{2,4}$ that attains the minimum distance from $\boldsymbol{G}_{1,3}(\alpha, \beta)$. Then we have that

$$\begin{aligned} d(\boldsymbol{G}_{1,3}(\alpha, \beta), P_{2,4}) = AB &= AC \sin(\angle ABC) \\ &\geqslant AC \sin(\theta_{2,4}) \geqslant \sin\left(\arctan \frac{1}{\sqrt{2}}\right) AC \\ &= \frac{1}{\sqrt{3}} AC = \frac{1}{\sqrt{3}} \|\boldsymbol{G}_{1,3}(\alpha, \beta) - \boldsymbol{G}_{2,4}(\alpha, \beta)\|. \end{aligned}$$

(Note that in order to go from the fourth to the fifth line, we have used the identity $\sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$.) ∎

6. Appendix

We return to the problem, introduced in Section 2, of estimating the volume of the intersection between the δ-neighbourhoods of two complex lines.

Proposition 6.1. *Suppose that $\ell_1, \ell_2 \subset \mathbb{C}^2$ are two complex lines through the origin that make angle $\theta > 0$ to each other. If we let $\iota : \mathbb{C}^2 \to \mathbb{R}^4$ denote the embedding that sends (z_1, z_2) to $(\mathrm{Re}(z_1), \mathrm{Im}(z_1), \mathrm{Re}(z_2), \mathrm{Im}(z_2))$, then*

$$|\iota(N_\delta(\ell_1)) \cap \iota((N_\delta(\ell_2)))| \sim \frac{\delta^4}{\sin^2\theta}. \tag{6.1}$$

More generally, if ℓ_1, ℓ_2 are lines through the origin in $\mathbb{C}^n$ which make angle $\theta > 0$ to each other, then

$$|\iota(N_\delta(\ell_1)) \cap \iota(N_\delta(\ell_2))| \sim \frac{\delta^{2n}}{\sin^2\theta}. \tag{6.2}$$

Proof. Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be direction vectors for the tubes' respective central axes. Suppose without loss of generality that $\boldsymbol{u}_1 = (1, 0)$. If $\boldsymbol{u}_2$ is parallel to $(0, 1)$, then $\theta = \frac{\pi}{2}$, and $|\iota(N_\delta(\ell_1)) \cap \iota(N_\delta(\ell_2))| = \delta^4$. Henceforth, we may assume that $\boldsymbol{u}_2 = \frac{1}{\sqrt{1+r^2}}(1, re^{i\zeta})$ for some $\zeta \in [0, 2\pi)$ and some real number $r > 0$. Note that in this case, we have that

$$\sin^2\theta = \frac{r^2}{1+r^2}.$$

Now, we set some notation. We define

$$\begin{aligned}
\boldsymbol{v}_1 &= \iota(\boldsymbol{u}_1) = (1, 0, 0, 0);\\
\boldsymbol{v}_2 &= \iota(i\boldsymbol{u}_1) = (0, 1, 0, 0);\\
\boldsymbol{w}_1 &= \iota(\boldsymbol{u}_2) = \frac{1}{\sqrt{1+r^2}}(1, 0, r\cos\zeta, r\sin\zeta);\\
\boldsymbol{w}_2 &= \iota(i\boldsymbol{u}_1) = \frac{1}{\sqrt{1+r^2}}(0, 1, -r\sin\zeta, r\cos\zeta).
\end{aligned}$$

We let $V = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ and $W = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ so that $\iota(N_\delta(\ell_1)) = N_\delta(V)$, and $\iota(N_\delta(\ell_2)) = N_\delta(W)$. We write a point in $\mathbb{R}^4$ as $(\vec{x}, \vec{y}) \in \mathbb{R}^2 \times \mathbb{R}^2$. Then, $N_\delta(V) = \{(\vec{x}, \vec{y}) : \|y\| \leqslant \delta\}$, and

$$\begin{aligned}
|\iota(T_1) \cap \iota(T_2)| = |N_\delta(V) \cap N_\delta(W)| &= \int_{\mathbb{R}^4} 1_{N_\delta(V)} 1_{N_\delta(W)}\, dV\\
&= \int_{B_{\mathbb{R}^2}(\vec{0},\delta)} \left(\int_{\infty}^{\infty} \int_{-\infty}^{\infty} 1_{N_\delta(W)}(\vec{x}, \vec{y})\, dA_{\vec{x}} \right) dA_{\vec{y}}.
\end{aligned} \tag{6.3}$$

By Lemma 6.2, we have $(\vec{x}, \vec{y}) \in N_\delta(W)$ if and only if $\vec{x}$ is in the ball of radius $\frac{\delta\sqrt{1+r^2}}{r}$ around the point $\frac{1}{r}R_\zeta(\vec{y}) = \frac{1}{r}(y_1\cos\zeta + y_2\sin\zeta, -y_1\sin\zeta + y_2\cos\zeta)$, where R_ζ denotes the rotation around the origin through an angle ζ in the plane. Thus, resuming from (6.3), we conclude that

$$\begin{aligned}
|\iota(T_1) \cap \iota(T_2)| &= \int_{B_{\mathbb{R}^2}(\vec{0},\delta)} \left(\int_{B_{\mathbb{R}^2}(\frac{1}{r}R_\zeta(\vec{y}), \frac{\delta\sqrt{1+r^2}}{r})} 1\, dA_{\vec{x}} \right) dA_{\vec{y}}\\
&\sim \int_{B_{\mathbb{R}^2}(\vec{0},\delta)} \frac{\delta^2(1+r^2)}{r^2}\, dA_{\vec{y}} \sim \frac{\delta^4(1+r^2)}{r^2} = \frac{\delta^4}{\sin^2\theta}.
\end{aligned}$$

This completes the proof of (6.1). For $n > 2$, the orthogonal complements of $\iota(\ell_1)$ and $\iota(\ell_1)$ intersect in a subspace of dimension $2n - 4$. Any slice of $\iota(N_\delta(\ell_1) \cap \iota(N_\delta(\ell_1))$

by a translation of this subspace is a figure like that of the C^2 case. We can integrate the 4-dimensional cross section volume over the common short directions to arrive at equation (6.2). ∎

Lemma 6.2. *Let* $\boldsymbol{v}_1 = \iota(\boldsymbol{u}_1) = (1,0,0,0)$ *and let* $\boldsymbol{v}_2 = \iota(i\boldsymbol{u}_1) = (0,1,0,0)$. *Fixing* $r > 0$ *and* $\zeta \in [0, 2\pi)$, *let*

$$\boldsymbol{w}_1 = \frac{1}{\sqrt{1+r^2}}(1, 0, r\cos\zeta, r\sin\zeta)$$

and let

$$\boldsymbol{w}_2 = \frac{1}{\sqrt{1+r^2}}(0, 1, -r\sin\zeta, r\cos\zeta).$$

Let $W = \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$. *Then we have*

$$(x_1, x_2, y_1, y_2) \in N_\delta(W) \iff (x_1, x_2) \in B_{\mathbb{R}^2}\left(\frac{1}{r}R_\zeta(y_1, y_2), \frac{\delta\sqrt{1+r^2}}{r}\right),$$

where $R_\zeta(y_1, y_2) = (y_1\cos\zeta + y_2\sin\zeta, -y_1\sin\zeta + y_2\cos\zeta)$ *is the rotation of* (y_1, y_2) *through angle* ζ.

Proof. Let Π_W denote the orthogonal projection onto W, and let $\Pi_{W^\perp}$ denote the orthogonal projection onto the orthogonal complement of W. The point $(\vec{x}, \vec{y}) = (x_1, x_2, y_1, y_2)$ is in $N_\delta(W)$ if and only if $\|\Pi_{W^\perp}(\vec{x}, \vec{y})\| < \delta$. We compute that

$$\begin{aligned}
\|\Pi_{W^\perp}(\vec{x}, \vec{y})\|^2 &= \|(\vec{x}, \vec{y}) - \Pi_{W^\perp}(\vec{x}, \vec{y})\|^2 \\
&= \|(\vec{x}, \vec{y}) - \big((\vec{x}, \vec{y}) \cdot \boldsymbol{w}_1\big)\boldsymbol{w}_1 - \big((\vec{x}, \vec{y}) \cdot \boldsymbol{w}_2\big)\boldsymbol{w}_2\|^2 \\
&= \frac{1}{1+r^2}\big((r^2x_1^2 + r_2x_2^2 + y_1^2 + y_2^2) \\
&\qquad - 2r((x_1y_1 + x_2y_2)\cos\zeta + (x_1y_2 - x_2y_1)\sin\zeta)\big) \\
&= \frac{1}{1+r^2}\big((rx_1 - (y_1\cos\zeta + y_2\sin\zeta))^2 + (rx_2 - (-y_1\sin\zeta + y_2\cos\zeta))^2\big) \\
&= \frac{r}{1+r^2}\left(\big(x_1 - \frac{1}{r}(y_1\cos\zeta + y_2\sin\zeta)\big)^2 + \big(x_2 - \frac{1}{r}(-y_1\sin\zeta + y_2\cos\zeta)\big)^2\right) \\
&= \frac{r}{1+r^2}\left\|\vec{x} - \frac{1}{r}R_\zeta(\vec{y})\right\|^2.
\end{aligned}$$

(Note that some steps are omitted between the second and third lines.)

We conclude that $\|\Pi_{W^\perp}(\vec{x}, \vec{y})\| < \delta$ if and only if (x_1, x_2) is within the ball of radius $\delta\sqrt{1+r^2}/r$ centred at the point

$$\frac{1}{r}R_\zeta(\vec{y}) = \frac{1}{r}(y_1\cos\zeta + y_2\sin\zeta, -y_1\sin\zeta + y_2\cos\zeta),$$

as claimed. ∎

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