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Hausdorff dimension and Galois orbits

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Abstract. The aim of this paper is to investigate fractals and their Hausdorff dimensions in a non-Archimedean setting. We are interested in fractals like equilibrated and fundamental sets of the Tate fields and, in particular, in fractals like Galois orbits of generic elements of infinite Galois extensions of p-adic fields. Also, we study the invariance of the Hausdorff dimension of Galois orbits defined by integral transcendental elements of $\mathbb{C}_p$.

1. Introduction

The notions of Hausdorff measure and dimension defined over the field of real numbers (see the classical books of Falconer [5, 6]) can be defined in the same way over any metric space. These concepts are fundamental objects in fractal geometry, which play a key role in the study of the geometric structure of general sets. The aim of this paper is to investigate these notions in a p-adic setting. Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$, and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic absolute value $|\cdot|$. Let $O(T)$ denote the Galois orbit of an element $T \in \mathbb{C}_p$ with respect to the absolute Galois group $Gal_{\mathrm{cont}}(\mathbb{C}_p/\mathbb{Q}_p)$.

The paper is organized as follows. In Section 2 we introduce notation and some preliminary results. In Section 3 we discuss the class of equilibrated and fundamental sets of a metric space (M, d). We provide a few formulas for the Hausdorff measures and obtain the Hausdorff dimensions for the sets of this class in Theorem 3.1, which generalizes the well-known results of Abercrombie [1] and Barnea and Shalev [4] (see Corollary 3.2). In Application 1, we calculate the Hausdorff dimension and the Hausdorff measure for the ring of integers of a finite field extension of $\mathbb{Q}_p$ and then, in Application 2, we show that the non-Archimedean Cantor set (see [11]) has Hausdorff dimension $s_0 = \frac{\log 2}{\log 3}$ and the s_0-dimensional Hausdorff measure equal to 1, which is a fractal as in the classical case. The mentioned applications are consequences of Theorem 3.1 without iterated function systems (see [5, 6, 11]). We note that in a p-adic setting, the Falconer Distance Conjecture [7] fails (see Remarks 4 and 5).

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In Section 4, we discuss the case of an arbitrary infinite Galois extension of $\mathbb{Q}_p$. We present a large class of fractals. More precisely, we show that for any $\xi \in [0, 1]$, there exists an integral transcendental element T_ξ of $\mathbb{C}_p$, such that $\dim_{\mathrm{H}} O(T_\xi) = \xi$ (see Theorem 4.1). If K is an infinite Galois extension of $\mathbb{Q}_p$, then $\tilde{K}$ (which denotes the completion of K with respect to the p-adic absolute value $|\cdot|$) has a transcendental generic element $T \in \mathbb{C}_p$, that is, $\tilde{K} = \widetilde{\mathbb{Q}_p[T]}$ (see [8]). In Theorem 4.2 we obtain formulas for the Hausdorff measures and the Hausdorff dimension of the Galois orbit of the generator T. Moreover, if $G_K = Gal(K/\mathbb{Q}_p) \cong Gal_{\mathrm{cont}}(\tilde{K}/\mathbb{Q}_p)$ and H is an open subgroup of G_K, then the s-dimensional Hausdorff measure of H, and respectively, the Hausdorff dimension of H is related to that of G_K, via the natural topological isomorphism between G_K and $O(T)$ (see Theorem 4.4).

Section 5 is dedicated to the study of the invariance of the Hausdorff dimension. Given T, an integral transcendental element of $\mathbb{C}_p$ and $U \in \widetilde{\mathbb{Z}_p[T]} \setminus \overline{\mathbb{Q}_p}$, our aim is to obtain a relation between the s-dimensional Hausdorff measures of the orbits $O(T)$ and $O(U)$. Then we show in Theorem 5.1 that these orbits have the same Hausdorff dimension. In particular, the Hausdorff dimension of the orbit of any generic element of $\widetilde{\mathbb{Z}_p[T]}$ is one and the same for all, so it is an invariant of $\widetilde{\mathbb{Z}_p[T]}$ (see Corollary 5.2).

2. Notation and preliminary results

Let (M, d) be a metric space and X an arbitrary non-empty subset of M. For any subset E of M, we define the *diameter* of E by $\operatorname{dia} E = \sup\{d(x, y) : x, y \in E\}$. By a (closed) *ball* in M of radius $r \geq 0$ and centered at a we mean a set of the form $B[a, r] = \{x \in M : d(x, a) \leq r\}$. Let ε be an arbitrary positive real number. If $\{B_i\}_i$ is a countable collection of sets of diameter at most ε that cover X, that is, $X \subset \cup_{i=1}^{\infty} B_i$ with $0 < \operatorname{dia} B_i \leq \varepsilon$ for each i, we say that $\{B_i\}_i$ is an *ε-cover* of X.

Let us suppose that X is a subset of M and s is a non-negative real number. For any $\varepsilon > 0$, we define

$$\mathcal{H}^s_\varepsilon(X) := \inf\left\{\sum_{i=1}^{\infty}(\operatorname{dia} B_i)^s : \{B_i\}_i \text{ is an } \varepsilon\text{-cover of } X\right\}. \tag{2.1}$$

The function $\varepsilon \longmapsto \mathcal{H}^s_\varepsilon(X)$ decreases as $\varepsilon \to 0$, hence the following limit exists:

$$\mathcal{H}^s(X) = \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(X). \tag{2.2}$$

This is called the s-dimensional Hausdorff (outer) measure of X. In what follows, we define the Hausdorff dimension (or *Hausdorff–Besicovitch dimension*) of X by

$$\dim_{\mathrm{H}} X := \inf\{s : \mathcal{H}^s(X) = 0\} = \sup\{s : \mathcal{H}^s(X) = \infty\}, \tag{2.3}$$

which is always a real number in the interval $[0, \infty]$.

We remark that if $X \subset M$ is an arbitrary subset with a compact closure, we can use in (2.1) an ε-*cover* with a finite collection of closed balls of diameter at most ε. If $\dim_{\mathrm{H}} X$ is non-integral, then we say that X is a *fractal* (see Mandelbrot [12]).

Let X be any non-empty bounded subset of M and, for any $\varepsilon > 0$, let $N_\varepsilon(X)$ be the minimal number of closed balls of radius ε that cover X. The lower and upper box dimensions of X are

$$\underline{\dim}_{\mathrm{B}} X = \liminf_{\varepsilon \to 0} \frac{\log N_\varepsilon(X)}{-\log \varepsilon}, \quad \overline{\dim}_{\mathrm{B}} X = \limsup_{\varepsilon \to 0} \frac{\log N_\varepsilon(X)}{-\log \varepsilon}$$

and the box dimension is

$$\dim_{\mathrm{B}} X = \lim_{\varepsilon \to 0} \frac{\log N_\varepsilon(X)}{-\log \varepsilon},$$

if this limit exists. For further details on fractals, see [5, 6].

For a prime number p, we let $\mathbb{Q}_p$ be the field of p-adic numbers; let $\overline{\mathbb{Q}_p}$ be a fixed algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}_p}$ with respect to the p-adic valuation v (see [2, 8, 10]). The field $\mathbb{C}_p$ is known as the Tate field or the field of p-adic complex numbers. Denote by $|\cdot|$ the p-adic module on $\mathbb{C}_p$. Let G be the Galois group $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ endowed with the Krull topology. We know that G is canonically isomorphic to $Gal_{\mathrm{cont}}(\mathbb{C}_p/\mathbb{Q}_p)$, which is the group of all continuous automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$. We identify these two groups.

For any closed subgroup H of G, we denote

$$Fix(H) := \{T \in \mathbb{C}_p : \sigma(T) = T \text{ for all } \sigma \in H\}.$$

Then $Fix(H)$ is a closed subfield of $\mathbb{C}_p$. If $T \in \mathbb{C}_p$, denote $H(T) = \{\sigma \in G : \sigma(T) = T\}$. Then $H(T)$ is a subgroup of G, and $Fix(H(T)) = \widetilde{\mathbb{Q}_p[T]}$, which is the closure of the polynomial ring $\mathbb{Q}_p[T]$ in $\mathbb{C}_p$. We say that T is a *topological generic element* of $\widetilde{\mathbb{Q}_p[T]}$. Any closed subfield K of $\mathbb{C}_p$ has a topological generic element, that is, there exists $T \in K$ such that $K = \widetilde{\mathbb{Q}_p[T]} = \widetilde{\mathbb{Q}_p(T)}$ (see [2, 8]).

Let T be a transcendental element of $\mathbb{C}_p$ and $O(T) = \{\sigma(T) : \sigma \in G\}$ the Galois orbit of T. The map $\sigma \rightsquigarrow \sigma(T)$ from G to $O(T)$ is continuous and it defines a homeomorphism from $G/H(T)$ to $O(T)$. Then $O(T)$ is a compact and totally disconnected subspace of $\mathbb{C}_p$, and the group G acts continuously on $O(T)$. Thus, if $\sigma \in G$ and $\tau(T) \in O(T)$, then $\sigma \times \tau(T) = (\sigma\tau)(T)$.

Definition 2.1. Let X be a non-empty compact subset of a metric space (M, d). We say that X is *equilibrated* if for any $\varepsilon > 0$, and any $\varepsilon' > 0$ with $\varepsilon \geq \varepsilon' > 0$, each ball of radius ε in X has a minimal decomposition in the same number of balls of radius ε'. Also, we say that X is a *fundamental set*, if the distance set

$$\Delta(X) := \{d(x, y) : x, y \in X\}$$

is a sequence $\{\varepsilon_n\}_{n\geq 1}$ that is strictly decreasing to zero. This sequence is called *the fundamental sequence associated with* X.

Remark 1. In fact, a fundamental and equilibrated subset X of M is a compact set that is an inverse limit space of a sequence of finite discrete spaces, and by this, it is a profinite space (also known as a Stone space).

2.1. Basic examples

Here are a few examples of equilibrated and fundamental sets.

1) The closed subgroups of a profinite group are equilibrated and fundamental; see the proof of Corollary 3.2.

2) Let K be a finite algebraic extension of $\mathbb{Q}_p$ and O_K its ring of integers. Then O_K is a fundamental and equilibrated set. In particular, $\mathbb{Z}_p$ and $\mathbb{Z}_p^\times$ are fundamental and equilibrated sets (see [10, 14] and Application 1).

3) The non-Archimedean Cantor set

$$\mathcal{C}_3 = \left\{ a \in \mathbb{Z}_3 : a = \sum_{i=0}^{\infty} a_i 3^i,\ a_i \in \{0, 2\} \text{ for any } i \geq 0 \right\}$$

is an equilibrated and fundamental set (see [11] and Application 2).

4) Another key example of a fundamental and equilibrated set is $O(T)$, the Galois orbit of a transcendental element $T \in \mathbb{C}_p$ (see [14]), that is, $O(T) = \{\sigma(T) : \sigma \in G\}$.

5) If $T \in \mathbb{C}_p$ is transcendental and B is a closed ball in $\mathbb{C}_p$, then $B \cap O(T)$ is also a fundamental and equilibrated set.

Remark 2. Any subset of $\mathbb{C}_p$, which is fundamental and equilibrated, is a set without isolated points. In particular, a set of the form $O(\alpha) \cup O(T)$, where α is an algebraic element of $\overline{\mathbb{Q}}_p$ and T is a transcendental element of $\mathbb{C}_p$, is not fundamental and equilibrated.

Let $\{\varepsilon_n\}_{n\geq 1}$ be the fundamental sequence associated with a fundamental and equilibrated set X of M. Let N_n be the number of the closed balls of a minimal decomposition of X with closed balls of radius ε_n. Because X is equilibrated, it is plain that N_n is a divisor of N_{n+1}, for any $n \geq 1$.

A subset $D \subseteq \mathbb{C}_p$ is *G-equivariant*, or *Galois equivariant*, provided that $\sigma(x) \in D$ for any $\sigma \in G$, and any $x \in D$. An example is the Galois orbit $O(T)$, where $T \in \mathbb{C}_p$. Another example is

$$B\left[O(T), |p|^{1+\varepsilon}\right] = \left\{z \in \mathbb{C}_p \colon \mathrm{dist}(z, O(T)) \leq |p|^{1+\varepsilon}\right\}$$

for any $\varepsilon > 0$.

A function $f : D \to \mathbb{C}_p$, where D is a Galois equivariant subset of $\mathbb{C}_p$, is called *G-equivariant*, or *Galois equivariant*, if $f(\sigma x) = \sigma f(x)$ for any $\sigma \in G$ and any $x \in D$.

3. Hausdorff dimension of equilibrated and fundamental sets

Theorem 3.1. *Let X be an equilibrated and fundamental set of a metric space (M, d). Let $\{\varepsilon_n\}_{n\geq 1}$ be the fundamental sequence associated with X and, for any $n \geq 1$, let N_n be the number of the closed balls of a minimal covering of X by balls of radius ε_n. Then, for any $s \geq 0$, we have*

$$\mathcal{H}^s(X) = \liminf_{n\to\infty} N_n \varepsilon_n^s \quad \text{and} \quad \dim_H X = \underline{\dim}_B X.$$

Proof. For any $k \geq 1$, let $\varepsilon_{n_1} > \varepsilon_{n_2} > \cdots > \varepsilon_{n_k}$ be an arbitrary finite subsequence of the fundamental sequence of X, where $n_1 < n_2 < \cdots < n_k$. We cover X with closed balls of radii $\varepsilon_{n_1}, \varepsilon_{n_2}, \ldots, \varepsilon_{n_k}$. Let a_i be the number of closed balls of radius ε_{n_i} from the cover, $1 \leq i \leq k$, and let $a = a_1 + a_2 + \cdots + a_k$. For the sake of simplicity, we suppose that $B_1, B_2, \ldots, B_{a_1}$ are the closed balls of radius ε_{n_1}; then $B_{a_1+1}, B_{a_1+2}, \ldots, B_{a_1+a_2}$ are the closed balls of radius ε_{n_2}; and so on. Thus, $\cup_{j=1}^{a} B_j$ is a cover of X with closed balls of radii $\varepsilon_{n_1} > \varepsilon_{n_2} > \cdots > \varepsilon_{n_k}$. It results

$$\sum_{j=1}^{a} (\text{dia } B_j)^s = \sum_{i=1}^{k} a_i \varepsilon_{n_i}^s, \tag{3.1}$$

where dia $B_j = \sup\{d(x, y) : x, y \in B_j\}$ is the diameter of the ball B_j. The diameter of B_j is precisely its radius by Remark 1. There exists i_0, $1 \leq i_0 \leq k$, such that $\min_{1\leq i\leq k} N_{n_i} \varepsilon_{n_i}^s = N_{n_{i_0}} \varepsilon_{n_{i_0}}^s$. By (3.1) we deduce

$$\sum_{j=1}^{a} (\text{dia } B_j)^s \geq \sum_{i=1}^{k} a_i \times \frac{N_{n_{i_0}}}{N_{n_i}} \times \varepsilon_{n_{i_0}}^s \geq N_{n_{i_0}} \varepsilon_{n_{i_0}}^s, \tag{3.2}$$

because $\sum_{i=1}^{k} \frac{a_i}{N_{n_i}} \geq 1$, for any covering. Otherwise, if the sum is strictly less than 1, we obtain

$$N_{n_k} > \sum_{i=1}^{k} a_i \frac{N_{n_k}}{N_{n_i}}.$$

However, as X is equilibrated, the right-hand side is counting the number of balls of radius ε_{n_k} in a covering of X induced by the initial covering considered, while N_{n_k} is the minimal number of balls in such a covering. These yield to a contradiction. Then, for any $m \geq n \geq 1$, by (3.2), we deduce that

$$\inf\Big\{\sum(\text{dia } B_i)^s : \{B_i\}_i \text{ is a cover of } X \text{ with closed balls,} \\ \text{of radii } \rho_i \text{ such that } \varepsilon_m \leq \rho_i \leq \varepsilon_n\Big\} = \inf_{\varepsilon_m \leq \varepsilon_k \leq \varepsilon_n} N_k \varepsilon_k^s. \tag{3.3}$$

In (3.3), ρ_i's are radii from the fundamental sequences associated with X, and the equality follows from (3.2) and the fact that the minimal covering with balls of radius ε_k yields exactly the number $N_k \varepsilon_k^s$. The sequence in (3.3) is decreasing as m tends to infinity, so by passing to the limit on m, we get

$$\inf\Big\{\sum(\text{dia } B_i)^s : \{B_i\}_i \text{ is a cover of } X \text{ with closed balls} \\ \text{of radii } \rho_i \leq \varepsilon_n\Big\} = \inf_{k \geq n} N_k \varepsilon_k^s. \tag{3.4}$$

By the definition of the s-Hausdorff measure, we obtain

$$\mathcal{H}^s(X) = \lim_{n\to\infty} \mathcal{H}^s_{\varepsilon_n}(X) = \liminf_{n\to\infty} N_n \varepsilon_n^s, \tag{3.5}$$

and taking logarithms in (3.5), we derive

$$\log \mathcal{H}^s(X) = \liminf_{n\to\infty} \log \varepsilon_n \left(s - \frac{\log N_n}{-\log \varepsilon_n}\right). \tag{3.6}$$

Denote $s_0 = \liminf_{n\to\infty} \frac{\log N_n}{-\log \varepsilon_n}$. We claim that $\dim_{\mathrm{H}} X = s_0$. The following three cases may appear:

a) $\boxed{s_0 = \infty}$ Then, for any $0 \leq s < \infty$, by (3.6),

$$\lim_{n\to\infty} \log \varepsilon_n \left(s - \frac{\log N_n}{-\log \varepsilon_n}\right) = \infty,$$

hence $\mathcal{H}^s(X) = \infty$, which yields $\dim_{\mathrm{H}} X = \infty$.

b) $\boxed{s_0 = 0}$ Then, for any $s > 0$, by (3.6) we find that $\mathcal{H}^s(X) = 0$, and consequently $\dim_{\mathrm{H}} X = 0$.

c) $\boxed{s_0 \in (0, \infty)}$ If $0 \leq s < s_0$ and $0 < \varepsilon < s_0 - s$, then $\frac{\log N_n}{-\log \varepsilon_n} \geq s_0 - \varepsilon$ for n large enough. This implies $s - \frac{\log N_n}{-\log \varepsilon_n} \leq s - s_0 + \varepsilon < 0$, from which, for n large enough as well, it follows $(\log \varepsilon_n)(s - \frac{\log N_n}{-\log \varepsilon_n}) \geq (s - s_0 + \varepsilon) \log \varepsilon_n$. Then again by (3.6), we obtain $\mathcal{H}^s(X) = \infty$. If $s > s_0$, then there exists a sequence $\{n_k\}_{k\geq 1}$ such that $\lim_{k\to\infty} \frac{\log N_{n_k}}{-\log \varepsilon_{n_k}} = s_0$. Further, $\lim_{k\to\infty}(\log \varepsilon_{n_k})(s - \frac{\log N_{n_k}}{-\log \varepsilon_{n_k}}) = -\infty$, which implies $\liminf_{n\to\infty}(\log \varepsilon_n)(s - \frac{\log N_n}{-\log \varepsilon_n}) = -\infty$. Therefore, by (3.6) it follows $\mathcal{H}^s(X) = 0$, which means that $\dim_{\mathrm{H}} X = s_0$.

These complete the proof of Theorem 3.1. ■

By Theorem 3.1, we obtain the results of Abercrombie [1, Proposition 2.6] and Barnea and Shalev [4, Theorem 2.4].

Corollary 3.2. *Let G be a profinite group with a filtration $\{G_n\}_{n=0}^{\infty}$ and let $H \leq_c G$ be a closed subgroup. Then*

$$\dim_H H = \underline{\dim}_B H = \liminf_{n\to\infty} \frac{\log|HG_n/G_n|}{\log|G/G_n|} = \liminf_{n\to\infty} \frac{\log[H : H \cap G_n]}{\log[G : G_n]},$$

where the Hausdorff dimension is computed with respect to the metric associated with the filtration $\{G_n\}$.

Proof. By hypothesis, let G be a profinite group and let $\{G_n\}_{n\geq 1}$ be a descending chain of normal subgroups that forms a base for the neighborhoods of the identity. Then, by setting $d(x, y) := \inf\{\frac{1}{[G:G_n]} : xy^{-1} \in G_n\}$, we obtain an invariant metric on G. It turns out that the closed subgroups of G are equilibrated (just use the fact that clopen balls are cosets of clopen subgroups and indices are multiplicative, that is, for $G \geq H \geq K$ we have $[G : K] = [G : H] \cdot [H : K]$). Moreover, the closed subgroups of G are fundamental because they are profinite. ■

Remark 3. If G is a p-adic analytic pro-p group and $H \leq_c G$ is a closed subgroup of G, then $\dim_{\mathrm{H}} H = \frac{\dim H}{\dim G}$, where $\dim X$ denotes the analytic dimension of a p-adic manifold X (see Theorems 2.4 and 1.1 of Barnea and Shalev [4]). Moreover, for p-adic analytic groups, the Hausdorff dimension coincides with the analytic dimension only for certain filtrations; for other filtrations many different results can be obtained (see [9]).

Application 1. Let K be a finite algebraic extension of degree $n = ef \geq 1$ of $\mathbb{Q}_p$, where e is the ramification index and f is the residual degree. Let O_K be the ring of integers of K over $\mathbb{Q}_p$, and π a uniformizer with $|\pi| = |p|^{\frac{1}{e}}$. It is known that any $x \in O_K$ is of the form $x = \sum_{i\geq 0} a_i \pi^i$, where $a_i^{p^f} = a_i$ are the Teichmüller digits. It is plain that O_K is an equilibrated and fundamental set of $\mathbb{C}_p$. The fundamental sequence associated with O_K is of the form $\varepsilon_m = |\pi|^m = (\frac{1}{p})^{m/e}$ and the number of closed balls of radius ε_m that cover O_K is $N_m = (p^f)^m = p^{mf}$, for any $m \geq 0$. Then we see that $\lim_{n\to\infty} \frac{\log N_n}{-\log \varepsilon_n} = ef = n$, therefore $\dim_{\mathrm{H}} O_K = n$ and $\mathcal{H}^n(O_K) = 1$. In particular, it follows that: $\dim_{\mathrm{H}} \mathbb{Z}_p = 1$ and $\mathcal{H}^1(\mathbb{Z}_p) = 1$.

Remark 4. The well-known Falconer Distance Conjecture [7] says that if $n \geq 2$ is an integer and X is a compact subset of $\mathbb{R}^n$ such that $\dim_{\mathrm{H}}(X) > \frac{n}{2}$, then $\mu(\Delta(X)) > 0$, where μ is the Lebesgue measure. In a p-adic setting this conjecture fails. Indeed, if we choose $X = O_K$, as in Application 1, then we notice that $\mu(\Delta(O_K)) = 0$ and

$\dim_{\mathrm{H}} O_K = [K : \mathbb{Q}_p]$, which can be arbitrarily large. Also, we have $\dim_{\mathrm{H}}(\Delta(X)) = \mu(\Delta(X)) = 0$, for any X that is a fundamental subset of $\mathbb{C}_p$.

Remark 5. In Remark 4 the number n is the topological dimension of $\mathbb{R}^n$, which coincides with the Hausdorff dimension of $\mathbb{R}^n$. If we want to consider the Falconer Distance Conjecture in a more general setting, such as for a metrizable topological group, which could be a subject for a forthcoming study, we have to compare the Hausdorff dimension of a subspace with the Hausdorff dimension of the ambient space to say something about the Lebesgue measure of the distance set of the subspace. To emphasize, it is enough to mention that the locally profinite groups or totally disconnected locally compact (TDLC) groups have the topological dimension zero. If G is a profinite group, then it is a topological group that is compact, Hausdorff, and totally disconnected. Also, G is an inverse limit of finite groups and it is metrizable. With respect to this metric defined on it, which depends on the filtration, for any $H \leq_c G$, which is a closed subgroup of G, $\Delta(H)$ has Lebesgue measure zero, so the conjecture failed in spite of the fact that the Hausdorff dimension of H could be a positive number. Essentially, Falconer Distance Conjecture fails for every profinite group G (as G itself has Hausdorff dimension 1, topological dimension 0 and countable distance set). A relevant example is Corollary 3.2, and for more, see [4].

If we consider G a TDLC group, then G is metrizable if and only if it is first countable, by the theorem of Birkhoff–Kakutani. Moreover, G has a basis of neighborhoods of the identity element formed by open-compact subgroups, see the well-known theorem of van Dantzig. These neighborhoods are metrizable, profinite, so the conjecture failed on these open-compact subgroups. We mention some examples, which are metrizable: the general linear group $GL_n(\mathbb{Q}_p)$ of $n \times n$ invertible matrices with entries in the p-adic numbers, the field of formal power series $\mathbb{F}_p((t))$, or the examples in Remark 4 and Application 2, to name but a few.

Application 2. The non-Archimedean Cantor set $\mathcal{C}_3 \subset \mathbb{Z}_3$ (see [11]) is characterized by the 3-adic expansion of its elements as follows

$$\mathcal{C}_3 = \left\{ a \in \mathbb{Z}_3 : a = \sum_{i=0}^{\infty} a_i 3^i,\ a_i \in \{0, 2\} \text{ for any } i \geq 0 \right\},$$

which is totally disconnected, uncountably infinite and without isolated points. Moreover, the 3-adic Cantor set is an equilibrated and fundamental set, which is homeomorphic with the classical Cantor set. In the 3-adic case, the fundamental sequence associated with $\mathcal{C}_3$ is of the form $\varepsilon_n = (\frac{1}{3})^n$ and the number of closed balls of radius ε_n that cover $\mathcal{C}_3$ is $N_n = 2^n$. Therefore $s_0 = \lim_{n\to\infty} \frac{\log N_n}{-\log \varepsilon_n} = \frac{\log 2}{\log 3}$ and, by Theorem 3.1, it follows: $\dim_{\mathrm{H}} \mathcal{C}_3 = \frac{\log 2}{\log 3}$ and $\mathcal{H}^{s_0}(\mathcal{C}_3) = 1$, as in the classical Cantor set.

Example 1. We give an example of a compact subset X of $\mathbb{C}_p$, which is fundamental and non-equilibrated, such that $\dim_{\mathrm{H}} X \neq \underline{\dim}_{\mathrm{B}} X$. For any integer $m \geq 1$, there exists an mth root of p in the algebraic closure of $\mathbb{Q}_p$. We consider the element $p^{1/m}$ determined by a conjugacy modulo a root of unity of order m. Denote $\gamma_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$, for any $k \geq 1$. For any $n \geq 1$, we define $\alpha_n = \sum_{k=1}^{n} p^{\gamma_k}$, which is an element of $\overline{\mathbb{Q}}_p$, with p^{γ_k} defined above. Because $\gamma_n \to \infty$, the sequence $\{\alpha_n\}_{n\geq 1}$ is convergent and let T be its limit. The set we need is $X = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots\} \cup \{T\}$, which is compact and fundamental. The fundamental sequence associated with X is of the form $\varepsilon_n = (\frac{1}{p})^{\gamma_n}$. The set X is non-equilibrated because for any $1 \leq i < n$ the element α_i is in a single ball of radius ε_n and for any $i \geq n$ all the elements α_i are in the ball $B(T, \varepsilon_n)$. It follows that $N_n = n$, and

$$\dim_{\mathrm{B}} X = \lim_{n\to\infty} \frac{\log N_n}{-\log \varepsilon_n} = \lim_{n\to\infty} \frac{\log n}{\gamma_n \log p} = \frac{1}{\log p},$$

because $\lim_{n\to\infty}(\gamma_n - \log n) = \gamma$, where γ is the Euler–Mascheroni constant. It is plain that $\dim_{\mathrm{H}} X = 0 \neq \frac{1}{\log p} = \dim_{\mathrm{B}} X$.

Remark 6. There is some very recent result of Mayordomo and Nies [13], which provides an alternative proof of the coincidence between the Hausdorff dimension and the lower box dimension (in a different context, but based likewise on sets that are equilibrated and fundamental).

4. The case of infinite Galois extensions

Theorem 4.1. *Let p be an odd prime number. For each $\xi \in [0, 1]$ there exists an integral transcendental element T_ξ of $\mathbb{C}_p$ such that*

$$\dim_H O(T_\xi) = \xi.$$

In the case $p = 2$, the above equality holds for each $\xi \in [0, \frac{1}{2}]$. In both cases, for each $\xi \neq 0$, we have

$$\mathcal{H}^{\xi}\big(O(T_\xi)\big) = \frac{p-1}{p^2}\left(\frac{1}{p}\right)^{\frac{\xi}{p-1}}.$$

Proof. For any $n \geq 1$, we let ζ_{p^n} denote a primitive root of unity of order p^n. Next, we let $\alpha_n = \zeta_p + a_2\zeta_{p^2} + \cdots + a_n\zeta_{p^n}$, where $a_i \in \mathbb{Z}$ are such that $a_i \equiv 0 \pmod{p}$ for all i with $2 \leq i \leq n$. Note that for all i with $1 \leq i \leq n$, we can choose a_i such that $\deg \alpha_n = p^{n-1}(p-1)$ and

$$|a_{n+1}| < |a_n| \cdot \left(\frac{1}{p}\right)^{\frac{1}{p-1}} \tag{4.1}$$

for any $n \geq 1$ (see the Appendix in [14]). Denote $T = \lim_{n\to\infty} \alpha_n$, which is a transcendental element of $\mathbb{C}_p$. Then $\widetilde{\mathbb{Q}_p[T]} = \widetilde{\cup_n \mathbb{Q}_p[\zeta_{p^n}]}$ (see [8]). If $K = \cup_n \mathbb{Q}_p[\zeta_{p^n}]$, then K is an infinite Galois extension of $\mathbb{Q}_p$, which is totally ramified, and

$$G_K = Gal(K/\mathbb{Q}_p) \cong Gal_{\text{cont}}(\tilde{K}/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times.$$

By the Appendix in [14], the fundamental sequence associated with the orbit $O(T)$ is $\varepsilon_n = |a_n|(\frac{1}{p})^{\frac{1}{p-1}}$ and the sequence $\{N_n\}_{n\geq 1}$ of closed balls of radius ε_n that cover the orbit of T is given by $N_n = \varphi(p^{n-1}) = p^{n-2}(p-1)$, where φ denotes Euler's totient function.

Now, let $\theta \geq 1$ be a real number and let $p \geq 3$ be a prime number. For any $n \geq 2$, we choose $a_n = p^{[n\theta]}$, where $[\cdot]$ means the integer part. Then condition (4.1) is satisfied and, by Theorem 3.1, we obtain

$$\xi = \lim_{n\to\infty} \frac{\log N_n}{-\log \varepsilon_n} = \frac{1}{\theta} = \dim_{\mathrm{B}} O(T) = \dim_{\mathrm{H}} O(T).$$

This means that we can find transcendental elements $T \in \mathbb{C}_p$ such that the Hausdorff dimension of the orbit of T takes any positive real value in the interval $(0, 1]$. We remark that in the case $a_n = p^{n^2}$, we have $\dim_{\mathrm{H}} O(T) = 0$. In the case $p = 2$, the inequality (4.1) is satisfied for any $\theta \geq 2$, therefore we can find transcendental elements $T \in \mathbb{C}_p$ such that the Hausdorff dimension of the orbit of T takes any positive real value in the interval $(0, 1/2]$. In both cases, from Theorem 3.1 and Dirichlet's approximation theorem, we deduce $\mathcal{H}^\xi(O(T)) = \frac{p-1}{p^2}(\frac{1}{p})^{\frac{\xi}{p-1}}$. Indeed, we have

$$\begin{aligned}
\mathcal{H}^\xi\big(O(T)\big) &= \liminf_{n\to\infty} N_n \varepsilon_n^\xi \\
&= \liminf_{n\to\infty} p^{n-2}(p-1)|a_n|^\xi \left(\frac{1}{p}\right)^{\frac{\xi}{p-1}} \\
&= \frac{p-1}{p^2}\left(\frac{1}{p}\right)^{\frac{\xi}{p-1}} \liminf_{n\to\infty} p^n |a_n|^\xi \\
&= \frac{p-1}{p^2}\left(\frac{1}{p}\right)^{\frac{\xi}{p-1}} \liminf_{n\to\infty} p^{n-\xi[n/\xi]} \\
&= \frac{p-1}{p^2}\left(\frac{1}{p}\right)^{\frac{\xi}{p-1}},
\end{aligned}$$

which concludes the proof of Theorem 4.1. ■

Remark 7. By Theorem 4.1 we obtain a large class of fractals given by the Galois orbits of some integral transcendental elements of the Tate fields. It is worth mentioning that the Galois orbits $O(T_\xi)$ in Theorem 4.1 have *strong Hausdorff dimension* meaning the lower limit in Theorem 3.1 is an actual limit, which is not true in general for arbitrary equilibrated and fundamental subsets.

Remark 8. For each $\xi \in [0, 1]$, the integral transcendental element $T_\xi \in \mathbb{C}_p$ from Theorem 4.1 and its proof have the following properties:

i) The extension $\widetilde{\mathbb{Q}_p[T_\xi]}/\mathbb{Q}_p$ is normal, that is, $H(T_\xi) = Gal_{\text{cont}}(\mathbb{C}_p/\widetilde{\mathbb{Q}_p[T_\xi]})$ is a normal subgroup of $G = Gal_{\text{cont}}(\mathbb{C}_p/\mathbb{Q}_p)$.

ii) We have the following isomorphisms of groups:

$$O(T_\xi) \cong G/H(T_\xi) \cong Gal_{\text{cont}}\big(\widetilde{\mathbb{Q}_p[T_\xi]}/\mathbb{Q}_p\big) \cong Gal_{\text{cont}}\big(\widetilde{\cup_n \mathbb{Q}_p[\zeta_{p^n}]}/\mathbb{Q}_p\big) \cong \mathbb{Z}_p^\times.$$

iii) As groups, $O(T_\xi)$ and $\mathbb{Z}_p^\times$ are isomorphic and their Hausdorff dimensions are

$$\dim_{\mathrm{H}} O(T_\xi) = \xi \quad \text{and} \quad \dim_{\mathrm{H}} \mathbb{Z}_p^\times = \dim_{\mathrm{H}} \mathbb{Z}_p = 1.$$

Theorem 4.2. *Let K be a subfield of $\mathbb{C}_p$ that is, an infinite Galois extension of $\mathbb{Q}_p$. Then, there exists a generic element T of $\tilde{K}$, and a sequence $\{x_n\}_{n\geq 0}$ of elements of $\overline{\mathbb{Q}}_p$ that converges to T such that the finite field extension $\mathbb{Q}_p(x_{n-1}) \subset \mathbb{Q}_p(x_n)$ is normal and of prime number degree q_n for any $n \geq 1$. For any $s \geq 0$, the Hausdorff measure and the Hausdorff dimension of the Galois orbit of T are given by the following formulas:*

$$\mathcal{H}^s\big(O(T)\big) = \liminf_{n\to\infty} q_1 q_2 \cdots q_n \big(\overline{\omega}(x_{n+1})\big)^s$$

and

$$\dim_H O(T) = \liminf_{n\to\infty} \frac{\log(q_1 q_2 \cdots q_n)}{-\log(\overline{\omega}(x_{n+1}))},$$

where $\overline{\omega}(x_{n+1})$ is the distance between any two distinct conjugates of x_{n+1} over $\mathbb{Q}_p(x_n)$, which is the same.

Proof. Let $K \subseteq \mathbb{C}_p$ be a field that is an infinite Galois extension of $\mathbb{Q}_p$. Since $K/\mathbb{Q}_p$ is normal, it is also solvable. Then there exists a tower of fields $\{K_n\}_{n\geq 1}$ such that

i) $\mathbb{Q}_p = K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots \subset K = \cup_{n=1}^\infty K_n$, $[K_n : \mathbb{Q}_p] < \infty$
and

ii) $K_{n-1} \subset K_n$ is a Galois extension of degree a prime number q_n, for any $n \geq 1$. By considering

$$G_K = Gal(K/\mathbb{Q}_p) \cong Gal_{\text{cont}}(\tilde{K}/\mathbb{Q}_p)$$

and

$$H_n = Gal(K/K_n) \cong Gal_{\text{cont}}(\tilde{K}/K_n),$$

we see that $H_0 = G_K$, H_n is a normal subgroup of H_{n-1} and

$$|H_{n-1}/H_n| = [K_n : K_{n-1}] = q_n.$$

For any $\alpha \in \overline{\mathbb{Q}}_p$, we define

$$\omega(\alpha) := \min\{|\sigma(\alpha) - \alpha| : \sigma \in G_K,\ \sigma(\alpha) \neq \alpha\}$$

and

$$\Omega(\alpha) := \max\{|\sigma(\alpha) - \alpha| : \sigma \in G_K\}.$$

From the proof of Theorem 2 in [8] (see also [2]), we can find, for any $n \geq 1$, $a_n \in \mathbb{Z}_p$ small enough, and $\alpha_n \in \overline{\mathbb{Q}}_p$ such that if $x_n = 1 + \sum_{i=1}^{n} a_i \alpha_i$, $x_0 = 1$. Then we have $K_n = \mathbb{Q}_p(x_n)$ and the following conditions are fulfilled:

i) $|x_{n+1} - x_n| < \omega(x_n)$ and the sequence $\{|x_{n+1} - x_n|\}_{n \geq 1}$ is strictly decreasing to 0.

ii) $|a_n|\omega(\alpha_n) > |a_{n+1}|\Omega(\alpha_{n+1})$, for any $n \geq 1$.

Indeed, from the proof of Theorem 2 in [8], it follows that $\alpha_n \in \overline{\mathbb{Q}}_p$, $|\alpha_n| \leq 1$, and $K_n = \mathbb{Q}_p(\alpha_n) \subseteq K_{n+1} = \mathbb{Q}_p(\alpha_{n+1})$, for any $n \geq 1$.

By induction, we construct the sequence $\{a_n\}_{n \geq 1}$, with $a_n \in \mathbb{Z}_p$ small enough, such that $a_n \to 0$. Let us suppose that we have defined $a_1, a_2, \ldots, a_n$. Then, we let $a_{n+1} \in \mathbb{Z}_p$, be such that

$$|a_{n+1}| \cdot |\alpha_{n+1}| < \min\{\omega(x_n), |a_n| \cdot |\alpha_n|\}$$

and

$$|a_{n+1}| \cdot \Omega(\alpha_{n+1}) < |a_n| \cdot \omega(\alpha_n).$$

Then both conditions i) and ii) are fulfilled. Moreover, the element $T = \lim_{n\to\infty} x_n$ is a generic element of $\tilde{K}$, that is, $\tilde{K} = \widetilde{\mathbb{Q}_p[T]}$.

Denote $S_n = H_n \setminus H_{n+1}$ for any $n \geq 0$. By the above conditions, for any $\sigma \in S_n$, we have

$$\begin{aligned}
|\sigma(T) - T| &= |\sigma(a_{n+1}\alpha_{n+1}) - a_{n+1}\alpha_{n+1}| \\
&= |a_{n+1}| \cdot |\sigma(\alpha_{n+1}) - \alpha_{n+1}| \\
&= |\sigma(x_{n+1}) - x_{n+1}| \\
&= \overline{\omega}(x_{n+1}).
\end{aligned}$$

Note that this is independent of $\sigma \in S_n$ and, by all means, $\overline{\omega}(x_{n+1})$ is the distance between any two distinct roots of the minimal polynomial of x_{n+1} over K_n, which has prime degree q_{n+1}. We summarize the result that all these distances are equal to each other in the following lemma.

Lemma 4.3 ([15]). *Let (L, v) be a local field. Let $f \in L[X]$ be an irreducible polynomial, separable and of prime degree q. Then the distances between any two distinct roots of f are all the same.*

Now, because $G_K = \cup_{n \geq 0} S_n$, the fundamental sequence associated with T is of the form $\{\overline{\omega}(x_{n+1})\}_{n \geq 0}$, which is strictly decreasing to zero.

Indeed, we have $\overline{\omega}(x_{n+1}) = |a_{n+1}| \cdot |\sigma(\alpha_{n+1}) - \alpha_{n+1}| \to 0$ for $\sigma \in S_n$, because $|\sigma(\alpha_{n+1}) - \alpha_{n+1}| \leq 1$ (we have $|\alpha_i| \leq 1$ for any $i \geq 1$) and $|a_{n+1}| \to 0$.

The number of closed balls of radius $\varepsilon_{n+1} = \overline{\omega}(x_{n+1})$ that cover $O(T)$ coincides with the number of closed balls of radius ε_{n+1} that cover $O(x_n)$. We claim that this number is $N_{n+1} = \deg_{\mathbb{Q}_p} x_n = q_1 q_2 \cdots q_n$.

Indeed, since $|T - x_n| < \omega(x_n) \leq \overline{\omega}(x_n) = \varepsilon_n$ it follows $|T - x_n| \leq \varepsilon_{n+1}$. For any $\sigma, \tau \in \cup_{i=0}^{n-1} S_i$, we have $|\sigma(x_n) - x_n| > \varepsilon_{n+1}$ and

$$B\big[\sigma(x_n), \varepsilon_{n+1}\big] \cap B\big[\tau(x_n), \varepsilon_{n+1}\big] = \emptyset \quad \text{for } \sigma \neq \tau.$$

In conclusion, we deduce that $O(T) \subseteq \cup_{\sigma \in Gal(K_n/\mathbb{Q}_p)} B[\sigma(x_n), \varepsilon_{n+1}]$, which is a disjoint union of closed balls, thereby wrapping up the proof of the claim.

Summing up, via Theorem 3.1, we thus complete the proof of Theorem 4.2. ■

Theorem 4.4. *Let K be a subfield of $\mathbb{C}_p$ that is an infinite Galois extension of $\mathbb{Q}_p$, and let H be an open subgroup of $G_K = Gal(K/\mathbb{Q}_p)$. Then*

$$\mathcal{H}^s(H) = \frac{1}{[G_K : H]} \cdot \mathcal{H}^s(G_K) \quad \text{and} \quad \dim_H H = \dim_H G_K.$$

Proof. Let $K \subseteq \mathbb{C}_p$ be a field that is an infinite Galois extension of $\mathbb{Q}_p$ and T a generic transcendental element of $\tilde{K}$, that is, $\tilde{K} = \widetilde{\mathbb{Q}_p[T]}$, which always exists, as we know from [8, Theorem 2]. Denote $G_K = Gal(K/\mathbb{Q}_p) \cong Gal_{\text{cont}}(\tilde{K}/\mathbb{Q}_p)$. Let $H \leq_c G_K$ be a closed subgroup of G_K of finite index. We have that G_K acts in a natural way on $O(T)$. Moreover, G_K and $O(T)$ are topologically isomorphic and, via this isomorphism, H is identified with $H_T = \{\sigma(T) : \sigma \in H\} \subseteq O(T)$, which is in fact a closed ball centered at T of the orbit $O(T)$. On G_K, which is a profinite group, we consider the metric induced by the metric on $\mathbb{C}_p$, via the above topological isomorphism. We obtain an invariant metric on G_K. Then H and σH have the same fundamental sequence, for any $\sigma \in G_K$. By a simple calculation with respect to the above metric, we deduce that $\mathcal{H}^s(H) = \mathcal{H}^s(\sigma H)$, for any $\sigma \in G_K$, which means that the Hausdorff measure $\mathcal{H}^s$ is invariant under translations. Moreover, $\mathcal{H}^s(H) = \frac{1}{[G_K:H]} \cdot \mathcal{H}^s(G_K)$, which implies $\dim_{\mathrm{H}} H = \dim_{\mathrm{H}} G_K$. ■

We note that Theorem 4.4 is comparable to a remark that follows Theorems 2.4 regarding open subgroups in [4].

5. Invariance of Hausdorff dimension

Let T be an integral transcendental element of $\mathbb{C}_p$, let $O(T) = \{\sigma(T) : \sigma \in G\}$ be the Galois orbit of T, and $U \in \widetilde{\mathbb{Z}_p[T]} \setminus \overline{\mathbb{Q}}_p$. We want to find a relation between the Hausdorff measures of $O(T)$ and $O(U)$. Resulting from this, we point out a nice relation between $\dim_{\mathrm{H}} O(T)$ and $\dim_{\mathrm{H}} O(U)$.

First of all, because $U \in \widetilde{\mathbb{Z}_p[T]} \setminus \overline{\mathbb{Q}}_p$, there exists a polynomial $P_n \in \mathbb{Z}_p[X]$ with p-adic integer coefficients for any integer $n > 0$, such that

$$U = U(T) = \lim_{n\to\infty} P_n(T)$$

exists in $\mathbb{C}_p$. By [3, Theorem 3.1], the limit U defines a Galois equivariant continuous function on $O(T)$ that has a unique Galois equivariant analytic continuation $U(\cdot)$ to $B[O(T), |p|^{1+\varepsilon}]$ for any $\varepsilon > 0$. Also, it follows that $U(x) = \lim_{n\to\infty} P_n(x)$ and $U'(x) = \lim_{n\to\infty} P_n'(x)$ for any $x \in O(T)$. We have $|U'(T)| = |P_n'(T)|$ for n large enough, and because T is transcendental, it follows that $U'(T) \neq 0$. Moreover, since

$$P_n'(T) = \lim_{x\to T,\, x\in O(T)\setminus\{T\}} \frac{P_n(x) - P_n(T)}{x - T},$$

it follows that $|P_n(x) - P_n(T)| = |P_n'(T)| \cdot |x - T|$, if $|x - T|$ is small enough. In the limit, as n goes to infinity, we obtain $|U(x) - U(T)| = |U'(T)| \cdot |x - T|$, if $|x - T|$ is small enough.

Let $\{\varepsilon_n\}_{n\geq 1}$ be the fundamental sequence associated with the set $O(T)$. Then, the fundamental sequence associated with $O(U)$ contains a co-final part formed by the sequence $\{|U'(T)|\varepsilon_n\}_n$ for n large enough. Also, the number of closed balls of radius ε_n that cover $O(T)$ coincides with the number of closed balls of radius $|U'(T)|\varepsilon_n$ that cover $O(U)$ for n large enough. Then, by the definition of the Hausdorff measure, it follows that

$$\mathcal{H}^s\big(O(U)\big) = |U'(T)|^s \mathcal{H}^s\big(O(T)\big).$$

Because $|U'(T)| > 0$, we deduce that $\dim_{\mathrm{H}} O(T) = \dim_{\mathrm{H}} O(U)$.

We collect the above results in the following theorem.

Theorem 5.1. *Let T be an integral transcendental element of $\mathbb{C}_p$ and $U \in \widetilde{\mathbb{Z}_p[T]} \setminus \overline{\mathbb{Q}}_p$. Then*

$$\mathcal{H}^s\big(O(U)\big) = |U'(T)|^s \mathcal{H}^s\big(O(T)\big) \quad \text{and} \quad \dim_H O(T) = \dim_H O(U),$$

where s is an arbitrary non-negative real number and $U(\cdot)$ is the Galois equivariant analytic function defined by U as above.

Corollary 5.2. *Let T and U be integral transcendental elements of $\mathbb{C}_p$ such that $\widetilde{\mathbb{Z}_p[T]} = \widetilde{\mathbb{Z}_p[U]}$. Then* $\dim_H O(T) = \dim_H O(U)$.

Remark 9. By Corollary 5.2, the Hausdorff dimension of the orbit of any generic element of $\widetilde{\mathbb{Z}_p[T]}$ is the same, which means that it is an invariant of $\widetilde{\mathbb{Z}_p[T]}$.

Remark 10. Let p be an odd prime number. From Theorem 4.1 it follows that, for each $\xi \in [0, 1]$, there exists an integral transcendental element T_ξ of $\mathbb{C}_p$ such that $\dim_{\mathrm{H}} O(T_\xi) = \xi$ and, if $K = \cup_n \mathbb{Q}_p[\zeta_{p^n}]$, then $\widetilde{\mathbb{Q}_p[T_\xi]} = \tilde{K}$. Moreover, the Galois group $Gal(K/\mathbb{Q}_p)$ (endowed with the Krull topology) and $O(T_\xi)$ are topologically isomorphic. Thus, if $\xi_1, \xi_2 \in [0, 1]$ with $\xi_1 \neq \xi_2$, then $\widetilde{\mathbb{Q}_p[T_{\xi_1}]} = \widetilde{\mathbb{Q}_p[T_{\xi_2}]}$, $O(T_{\xi_1})$ is homeomorphic with $O(T_{\xi_2})$, and $\dim_{\mathrm{H}} O(T_{\xi_1}) \neq \dim_{\mathrm{H}} O(T_{\xi_2})$. It then follows that if $\xi_1, \xi_2 \in [0, 1]$ and $\xi_1 \neq \xi_2$, then $\widetilde{\mathbb{Z}_p[T_{\xi_1}]} \neq \widetilde{\mathbb{Z}_p[T_{\xi_2}]}$. Also, we deduce that Corollary 5.2 is not true if instead of $\mathbb{Z}_p$ we consider $\mathbb{Q}_p$. The case $p = 2$ is similar.

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