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Entire functions with an arithmetic sequence of exponents

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Abstract. For a given entire function $f(z) = \sum_{j=0}^{\infty} a_j z^j$, we study the zero distribution of $f_r(z) = \sum_{j \equiv r \pmod m} a_j z^j$, where $m \in \mathbb{N}$ and $0 \le r < m$. We find conditions under which the zeros of $f_r(z)$ lie on m radial rays defined by $\mathrm{Im} z^m = 0$ and $\Re z^m \le 0$.

1. Introduction

The study of zero distribution of polynomials has a long history which spans at least from a theorem of René Descartes on the rule of signs to the modern theory of linear operators preserving zeros of polynomials on a circular domain developed by J. Borcea and P. Brändén [1]. On one end, the Descartes' rule of signs, whose proof can be found in [5], asserts that the number of positive zeros of a real polynomial is equal to or less than the number of sign changes of its coefficients by an even number. This rule will play an important role in our paper later in locating the final zero of our polynomials. On the other end, a special linear operator has a connection with the famous Hermite–Biehler theorem [3, Theorem 6.3.4] via the linear transformation which transforms a polynomial

$$P(z) = \sum_{j=0}^{n} a_j z^j$$

to the polynomial

$$P_r(z) = \sum_{\substack{j \equiv r \pmod m \\ 0 \le j \le n}} a_j z^j$$

for $m \in \mathbb{N}$ and $0 \le r < m$. If we define the sequence

$$\gamma_j = \begin{cases} 1 & \text{if } j \equiv r \pmod m, \\ 0 & \text{else,} \end{cases}$$

then

$$P_r(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.$$

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By Pólya and Schur's characterization [4], the sequence $\{\gamma_j\}_{j=0}^{\infty}$ is not a multiplier sequence, a sequence in which if the zeros of $\sum_{j=0}^{n} a_j z^j$ are real, then so are those of $\sum_{j=0}^{n} \gamma_j a_j z^j$. However, Hermite–Biehler theorem asserts that if $m = 2$ and the zeros of $P(z)$ lie on the open left half-plane, then the zeros of $P_r(z)$, $0 \leq r \leq 1$, are purely imaginary. It is natural to ask for similar results for other values of m. Theorems 1 and 2 of this paper give an answer to this question for the case $m = 3$ and 4. Finally, Theorems 3 and 4 extend these results to special classes of entire functions of orders 1 and 2, respectively. These classes of entire functions are different from the Laguerre–Pólya class [2, Definitions 2 and 3] in the sense that the zeros of these functions lie either on a sector of the plane or a half-plane (instead of on the real line).

In the whole paper, for a complex number z, we use the principal angle which is restricted by $-\pi < \operatorname{Arg} z \leq \pi$. With this convention, we provide formal statements of our theorems in this paper.

Theorem 1. *Suppose $P(z) = \sum_{i=0}^{n} a_i z^i \in \mathbb{R}[z]$ is a real polynomial whose zeros lie on the sector of the left half plane defined by the complex argument inequalities $2\pi/3 < |\operatorname{Arg} z| \leq \pi$. Then, for any $0 \leq r < 3$, the zeros of the polynomial*

$$P_r(z) = \sum_{\substack{j \equiv r \pmod 3 \\ 0 \leq j \leq n}} a_j z^j$$

lie on the 3 radial rays: $\operatorname{Im} z^3 = 0$ and $\Re z^3 \leq 0$.

Theorem 2. *Suppose $P(z) = \sum_{i=0}^{n} a_i z^i \in \mathbb{R}[z]$ is a real polynomial whose zeros lie on the open left half plane $\Re z < 0$. Then, for any $0 \leq r < 4$, the zeros of the polynomial*

$$P_r(z) = \sum_{\substack{j \equiv r \pmod 4 \\ 0 \leq j \leq n}} a_j z^j$$

lie on the 4 radial rays: $\operatorname{Im} z^4 = 0$ and $\Re z^4 \leq 0$.

For an example of Theorems 1 and 2, we consider

$$\begin{aligned} P(z) &= (z+1+i)^2(z+1-i)^2(z+1)^3(z+4+2i)(z+4-2i) \\ &= z^9 + 15z^8 + 99z^7 + 369z^6 + 876z^5 + 1392z^4 + 1492z^3 \\ &\quad + 1044z^2 + 432z + 80. \end{aligned}$$

In the case $(m, r) = (3, 0)$ or $(m, r) = (4, 0)$, the polynomial $P_r(z)$ is

$$z^9 + 369z^6 + 1492z^3 + 80$$

or

$$80 + 1392z^4 + 15z^8,$$

respectively. The zeros of $P_r(z)$ and the rays containing these zeros are plotted in Figure 1.1.

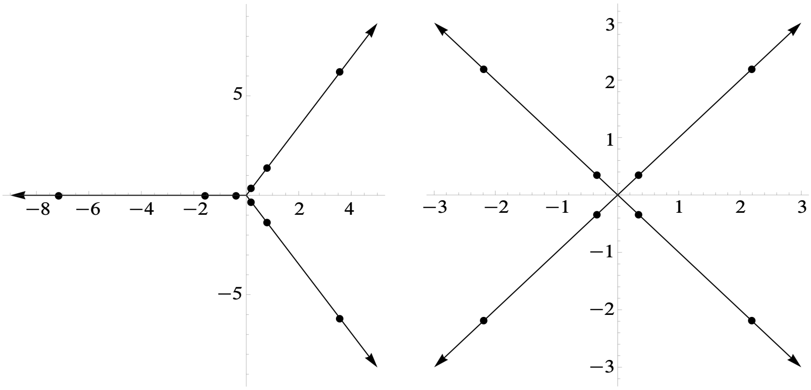

Figure 1.1. Zeros of $P_r(z)$ when $(m, r) = (3, 0)$ (left) and $(m, r) = (4, 0)$ (right).

As we saw in the statements of Theorems 1 and 2, the condition of the region (whether a sector of the plane or a half-plane) containing zeros of $P(z)$ depends on the parity of m. Indeed, the conclusion of Theorem 1 will not hold if we replace this region, $2\pi/3 < |\mathrm{Arg}\, z| \le \pi$, by the half-plane $\Re z < 0$. For example, the zeros of

$$\begin{aligned} P(z) = ((z+1)^2 + 10^2)^3 &= z^6 + 6z^5 + 315z^4 + 1220z^3 \\ &\quad + 31815z^2 + 61206z + 1030301 \end{aligned}$$

are $-1 \pm 10i$ which lie on the open left half plane. For $m = 3$ and $r = 0$, the zeros of

$$P_r(z) = z^6 + 1220z^3 + 1030301$$

do not lie on the given rays, as seen in Figure 1.2.

Note that, in the case $m = 2$, the Hermite–Biehler theorem states that the interlacing of the zeros of $P_0(z)$ and $P_1(z)$ on the two imaginary rays is also a sufficient condition for the zeros of $P(z)$ to lie on the open left half-plane. However, a similar statement may not hold for higher values of m. For example, in the case $m = 3$, the zeros of $P(z) = ((z+1)^2 + 3^2)^4$ lie outside the sector $2\pi/3 < |\mathrm{Arg}\, z| \le \pi$ and Figure 1.3 (left) shows that the zeros of $P_0(z)$ (●), $P_1(z)$ (■), and $P_2(z)$ (♦) interlace. Similarly, in the case $m = 4$, the zeros of $P(z) = ((z-1)^2 + 1)^5((z+1)^2 + 2^2)^3$ do not lie entirely on the open left half-plane while Figure 1.3 (right) shows that the zeros of $P_1(z)$ (■), $P_2(z)$ (♦), $P_3(z)$ (▲), and $P_0(z)$ (●) interlace.

We state the last two theorems of the paper and provide an example for each theorem.

Theorem 3. *Suppose $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $a_j \in \mathbb{R}$, is a real entire function whose zeros lie on the plane sector $2\pi/3 < |\mathrm{Arg}\, z| \le \pi$. If $f(z)$ has the Weierstrass factorization*

$$f(z) = z^p e^{Bz+C} \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) e^{z/z_j},$$

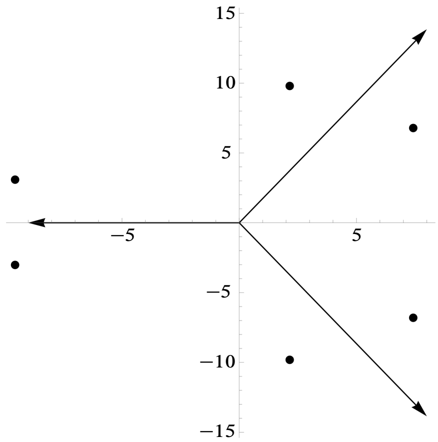

Figure 1.2. Zeros of $z^6 + 1220z^3 + 1030301$.

where $\sum_{j=1}^{\infty} 1/|z_j|$ *converges and* $B + \sum_{j=1}^{\infty} \frac{1}{z_j} > 0$, *then, for any* $0 \le r < 3$, *the zeros of the entire function*

$$f_r(z) = \sum_{j \equiv r \pmod 3} a_j z^j$$

lie on the 3 *radial rays:* $\operatorname{Im} z^3 = 0$ *and* $\Re z^3 \le 0$.

For an example, we consider

$$f(z) = e^{2z}(z+1+i)(z+1-i),$$

and note that the zeros $-1 \pm i$ of $f(z)$ lie on the sector $2\pi/3 < |\operatorname{Arg} z| \le \pi$ and

$$2 + \frac{1}{-1+i} + \frac{1}{-1-i} = 1 > 0.$$

The function $f_r(z)$ for $m = 3$ and $r = 0$ has infinitely many zeros, some of them are plotted in Figure 1.4.

Theorem 4. *Suppose* $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $a_j \in \mathbb{R}$, *is a real entire function whose zeros lie on the left half plane* $\Re z < 0$. *If* $f(z)$ *has the Weierstrass factorization*

$$f(z) = z^p e^{Az^2+Bz+C} \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) e^{z/z_j},$$

where $A \ge 0$, $\sum_{j=1}^{\infty} 1/|z_j|$ *converges, and* $B + \sum_{j=1}^{\infty} \frac{1}{z_j} > 0$, *then, for any* $0 \le r < 4$, *the zeros of the entire function*

$$f_r(z) = \sum_{j \equiv r \pmod 4} a_j z^j$$

lie on the 4 *radial rays:* $\operatorname{Im} z^4 = 0$ *and* $\Re z^4 \le 0$.

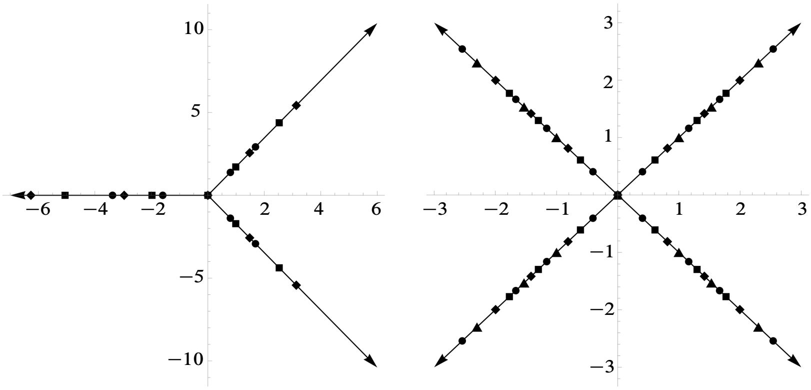

Figure 1.3. Zeros of $P_r(z)$ when $m = 3$ (left) and $m = 4$ (right).

For the plot of some zeros of $f_r(z)$ when $m = 4$, $r = 0$, and

$$f(z) = e^{z^2+2z}(z+1+i)(z+1-i),$$

see Figure 2.1.

2. Polynomials with arithmetic exponents

In this section, we will prove Theorems 1 and 2. We first note that it suffices to prove these theorems when the lead coefficient of $P(z)$ is 1. For any $m \in \mathbb{N}$ and $0 \le r < m$, if we let

$$P_r(z) = \sum_{\substack{j \equiv r \pmod m \\ 0 \le j \le n}} a_j z^j = \sum_{j=0}^{\lfloor n/m \rfloor} a_{mj+r} z^{mj+r},$$

then

$$e^{-r\pi i/m} P_r(e^{\pi i/m} z) = \sum_{j=0}^{\lfloor n/m \rfloor} (-1)^j a_{jm+r} z^{jm+r}. \tag{2.1}$$

Let $H(z)$ be the polynomial on the right-hand side of this equation, let $\omega_k = e^{(2k-1)\pi i/m}$, $0 \le k < m$, be the m-th roots of -1. We note from

$$P(z) = \sum_{j=0}^{n} a_j z^j$$

that

$$\sum_{k=0}^{m-1} \omega_k^{-r} P(\omega_k z) = \sum_{k=0}^{m-1} \sum_{j=0}^{n} \omega_k^{-r+j} a_j z^j. \tag{2.2}$$

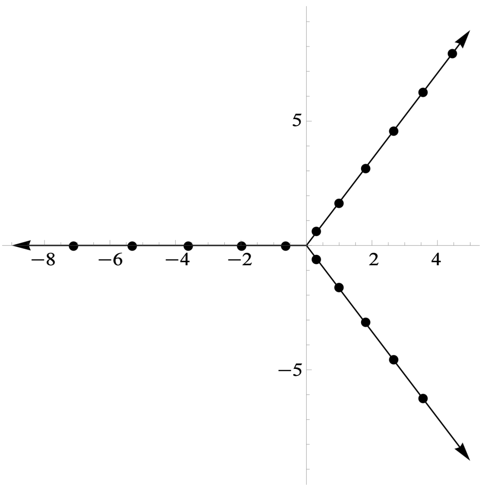

Figure 1.4. Zeros of $f_r(z)$ for $f(z) = e^{2z}(z+1+i)(z+1-i)$.

Since ω_k, $0 \le k < m$, are the m-th roots of -1, we have

$$\sum_{k=0}^{m-1} \omega_k^{j-r} = \begin{cases} 0 & \text{if } j \not\equiv r \mod m, \\ m(-1)^{(j-r)/m} & \text{if } j \equiv r \mod m. \end{cases}$$

We conclude from this equation, (2.1), and (2.2) that

$$\sum_{k=0}^{m-1} \omega_k^{-r} P(\omega_k z) = m \sum_{\substack{j \equiv r \pmod m \\ 0 \le j \le n}} (-1)^{(j-r)/m} a_j z^j = mH(z).$$

Since z_j, $1 \le j \le n$, are the zeros of the monic polynomial $P(z)$, we have

$$P(z) = \prod_{j=1}^{n} (z - z_j),$$

and consequently,

$$mH(z) = \sum_{k=0}^{m-1} \omega_k^{-r} P(\omega_k z) = \sum_{k=0}^{m-1} \omega_k^{-r} \prod_{j=1}^{n} (\omega_k z - z_j). \tag{2.3}$$

To prove Theorems 1 and 2, we will show that the zeros of

$$H(z) = e^{-r\pi i/m} P_r(e^{\pi i/m} z)$$

lie on the m radial rays $\operatorname{Im} z^m = 0$ and $\Re z^m \ge 0$. We note that the positive real ray is one of them. Our main strategy is to show that $H(z)$ has n zeros on these rays and the claim

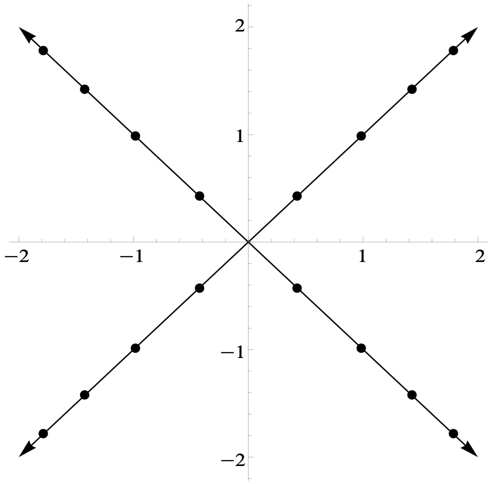

Figure 2.1. Zeros of $f_r(z)$ for $f(z) = e^{z^2+2z}(z+1+i)(z+1-i)$.

will follow from the fundamental theorem of algebra. From (2.1), we conclude that $H(z)$ has a zero at $z = 0$ or order r and if $z > 0$ is a positive real zero of $H(z)$, then so are $e^{k\pi i/m}z$, $0 \le k < m$. Thus, it suffices to show that $H(z)$ has $\lfloor n/m \rfloor$ positive real zeros.

For the reason mentioned above, we consider the polynomial $H(z)$ when z is a positive real number. For each $1 \le j \le n$ and $0 \le k < m$, and $z \in \mathbb{R}^+$, we let

$$q_{j,k} = \frac{\omega_k z - z_j}{\omega_0 z - \overline{z_j}} \tag{2.4}$$

and

$$\theta_j = \operatorname{Arg}(\omega_1 z - z_j), \tag{2.5}$$

where $\omega_k = e^{(2k-1)\pi i/m}$. Since

$$\Re(\omega_1 z - z_j) > 0,$$

we have

$$-\pi/2 < \theta_j < \pi/2. \tag{2.6}$$

We normalize $mH(z)$ by dividing the right side of (2.3) by $\prod_{j=1}^{n}(\omega_0 z - \overline{z_j})e^{i\theta_j}$ and conclude that this quotient is $\sum_{k=0}^{m-1} \omega_k^{-r} \prod_{j=1}^{n} q_{j,k} e^{-i\theta_j}$ or equivalently

$$g(\theta) := \sum_{k=0}^{m-1} \omega_k^{-r} e^{-in\theta} \prod_{j=1}^{n} q_{j,k}, \tag{2.7}$$

where $\theta := \sum_{j=1}^{n} \theta_j$. The two lemmas below show the first two terms (when $k = 0$ and $k = 1$) in (2.7) dominate the other terms in the sum.

Lemma 5. *Suppose $z \in \mathbb{R}^+$ and $q_{j,k}$, $1 \le j \le n$, $0 \le k < m$, are defined as in (2.4). For any $m \ge 2$, if $\pi - \pi/m < |\operatorname{Arg} z_j| \le \pi$ for all $1 \le j \le n$, then, for any $0 \le k < m$, we have*

$$\prod_{j=1}^{n} |q_{j,k}| \le 1. \tag{2.8}$$

Moreover, the equation occurs if and only if $k = 0$ or $k = 1$.

Proof. In the case $k = 1$, we have

$$\prod_{j=1}^{n} q_{j,1} = \prod_{j=1}^{n} \frac{\omega_1 z - z_j}{\omega_0 z - \overline{z_j}} = \prod_{j=1}^{n} e^{2i\theta_j} = e^{2i\theta}, \tag{2.9}$$

where the second equation comes from (2.5) and the fact that $\omega_1 z - z_j$ and $\omega_0 z - \overline{z_j}$ are two complex conjugates. Thus,

$$\prod_{j=1}^{n} |q_{j,1}| = 1.$$

If $k = 0$ and $z_j \in \mathbb{R}$, then $q_{j,0} = 1$. On the other hand, if $k = 0$ and $z_j \notin \mathbb{R}$, then both z_j and $\bar{z}_j$ are zeros of $P(z)$. As a consequence, the $\prod_{j=1}^{n} q_{j,0}$ contains the factor

$$\frac{\omega_0 z - z_j}{\omega_0 z - \overline{z_j}} \cdot \frac{\omega_0 z - \overline{z_j}}{\omega_0 z - z_j} = 1,$$

from which we deduce that

$$\prod_{j=1}^{n} q_{j,0} = 1. \tag{2.10}$$

To complete the lemma, it remains to show that

$$\prod_{j=1}^{n} |q_{j,k}| < 1$$

for $2 \le k < m$ and $m > 2$. If $z_j = r_j e^{i\phi_j}$, $\pi - \pi/m < |\phi_j| \le \pi$, then, for $2 \le k < m$,

$$\begin{aligned} |\omega_k z - z_j|^2 &= \left(z \cos \frac{(2k-1)\pi}{m} - r_j \cos \phi_j \right)^2 \\ &\quad + \left(z \sin \frac{(2k-1)\pi}{m} - r_j \sin \phi_j \right)^2 \\ &= z^2 + r_j^2 - 2 z r_j \cos\left(\phi_j - \frac{(2k-1)\pi}{m} \right) \\ &= z^2 + r_j^2 - 2 z r_j \cos\left(\phi_j^* - \frac{(2k-1)\pi}{m} \right), \end{aligned} \tag{2.11}$$

where

$$\phi_j^* = \begin{cases} \phi_j & \text{if } \phi_j \in (\pi/2, \pi], \\ \phi_j + 2\pi & \text{if } \phi_j \in (-\pi, -\pi/2). \end{cases}$$

Similarly,

$$|\omega_0 z - \overline{z_j}|^2 = z^2 + r_j^2 - 2zr_j \cos\left(\phi_j^* - \frac{\pi}{m}\right). \tag{2.12}$$

From the assumption $\pi - \pi/m < |\phi_j| \le \pi$, one has $\pi - \pi/m < \phi_j^* < \pi + \pi/m$, and consequently,

$$\frac{m-2k+2}{m}\pi > \phi_j^* - \frac{(2k-1)\pi}{m} > \frac{m-2k}{m}\pi, \quad 2 \le k < m$$

and

$$\pi > \phi_j^* - \frac{\pi}{m} > \frac{m-2}{m}\pi.$$

Thus,

$$\cos\left(\phi_j^* - \frac{(2k-1)\pi}{m}\right) > \cos\left(\phi_j^* - \frac{\pi}{m}\right).$$

We compare (2.11) and (2.12) and deduce that $|q_{j,k}| = |\omega_k z - z_j|/|\omega_0 z - \overline{z_j}| < 1$ for all $1 \le j \le n$ and $2 \le k < m$. ■

In the case m is even, we can expand the sector containing z_j, $1 \le j \le n$, in the condition of Lemma 5 to a half plane in the lemma below.

Lemma 6. *Suppose $z \in \mathbb{R}^+$ and $q_{j,k}$, $1 \le j \le n$, $0 \le k < m$, are defined as in (2.4). If $\Re z_j < 0$ for all $1 \le j \le n$ and $m \ge 2$ is even, then, for any $0 \le k < m$, we have*

$$\prod_{j=1}^{n} |q_{j,k}| \le 1.$$

Moreover, the equation occurs if and only if $k = 0$ or $k = 1$.

Proof. For the case $k = 0$ and $k = 1$, we have (2.10) and (2.9). We now consider $2 \le k < m$. We note that if $\pi - \pi/m < |\arg z_j| \le \pi$, then, by the proof of Lemma 5, $|q_{j,k}| < 1$ for $2 \le k < m$. It remains to show that

$$\prod_{j} |q_{j,k}| < 1,$$

where the product runs over values of $1 \le j \le n$ such that $\pi/2 < |\arg z_j| < \pi - \pi/m$. Since z_j and $\overline{z_j}$ are zeros of $P(z)$, if the modulus of

$$q_{j,k} = \frac{\omega_k z - z_j}{\omega_0 z - \overline{z_j}}$$

is a term in this product, then so is the modulus of

$$\frac{\omega_k z - \overline{z_j}}{\omega_0 z - z_j}.$$

To complete the proof of this lemma, we will show that if $\pi/2 < \arg z_j < \pi - \pi/m$, then, for all $2 \le k < m$,

$$\left| \frac{\omega_k z - z_j}{\omega_0 z - \overline{z_j}} \frac{\omega_k z - \overline{z_j}}{\omega_0 z - z_j} \right| < 1$$

or equivalently

$$\left|\omega_k z - z_j\right|\left|\omega_k z - \overline{z_j}\right| < \left|\omega_0 z - z_j\right|\left|\omega_0 z - \overline{z_j}\right|.$$

From the property of complex conjugate that

$$\left|\omega_k z - z_j\right|\left|\omega_k z - \overline{z_j}\right| = \left|\overline{\omega_k} z - \overline{z_j}\right|\left|\overline{\omega_k} z - z_j\right|, \quad 0 \le k < m$$

and $\overline{\omega_0} = \omega_1$, it suffices to prove that

$$\left|\omega_k z - z_j\right|\left|\omega_k z - \overline{z_j}\right| < \left|\omega_1 z - z_j\right|\left|\omega_1 z - \overline{z_j}\right| \tag{2.13}$$

for any $2 \le k < m$ such that $\omega_k = e^{(2k-1)\pi i/m}$ lies on the open upper half plane ($\omega_k \ne -1$ since m is even). We note that the angles of those ω_k on the upper half plane and ω_1 belong to $[\pi/m, \pi - \pi/m]$. Thus, to prove (2.13), we will show that the function

$$f(\theta) = \left|ze^{i\theta} - z_j\right|^2 \left|ze^{i\theta} - \overline{z_j}\right|^2$$

attains is absolute maximum on $[\pi/m, \pi - \pi/m]$ only at $\theta = \pi/m$. We note that

$$\begin{aligned} f(\theta) &= \left|z^2 e^{2i\theta} - 2ze^{i\theta}\Re z_j + |z_j|^2\right|^2 \\ &= z^4 - \left(4z^3 + 4z|z_j|^2\right)\Re z_j \cos\theta + 4z^2(\Re z_j)^2 \\ &\quad + |z_j|^4 + 2z^2|z_j|^2 \cos 2\theta, \end{aligned} \tag{2.14}$$

from which we deduce that

$$f'(\theta) = \left(4z^3 + 4z|z_j|^2\right)\Re z_j \sin\theta - 4z^2|z_j|^2 \sin\theta\cos\theta.$$

Hence, $f(\theta)$ has at most one critical point on the interval $[\pi/m, \pi - \pi/m]$. We also note that for $\theta \in [\pi/m, \pi/2]$, $f'(\theta) < 0$ since $\Re z_j < 0$, $z > 0$, and $\cos\theta > 0$. Thus, $f(\theta)$ will not attain its maximum value at such critical point (if exists). We conclude that the maximum value of $f(\theta)$ occur at either $\theta = \pi/m$ or $\theta = \pi - \pi/m$. Since

$$\cos 2(\pi - \pi/m) = \cos(2\pi/m),$$

we deduce from (2.14) and $\Re z_j < 0$ that $f(\pi/m) > f(\pi - \pi/m)$. Hence, $f(\theta)$ attains its maximum value on $[\pi/m, \pi - \pi/m]$ only at π/m and the lemma follows. ∎

We recall that we want to show that $H(z)$ has $\lfloor n/m \rfloor$ positive real zeros. Since $g(\theta)$ defined as in (2.7) is the quotient of $mH(z)$ and $\prod_{j=1}^{n}(\omega_0 z - \overline{z_j})e^{i\theta_j}$, it suffices to show (i) $\theta = \theta(z)$ bijectively map the interval $z \in (0, \infty)$ to $(0, \pi/m)$ and (ii) $g(\theta)$ has $\lfloor n/m \rfloor$ zeros on $\theta \in (0, \pi/m)$. The first claim is proved in the lemma below.

Lemma 7. *For each $z \in \mathbb{R}^+$, and each $1 \le j \le n$, $0 \le k < m$, let $q_{j,k}$ and θ_j be defined as in (2.4) and (2.5). If for any $1 \le j \le n$, $z_j = r_j e^{i\phi_j}$, $\pi/2 < |\phi_j| \le \pi$, lie on the open left half plane, then*

$$\theta := \frac{1}{n}\sum_{j=1}^{n}\theta_j$$

is an increasing function in $z \in (0, \infty)$ and this function maps $(0, \infty)$ onto $(0, \pi/m)$.

Proof. We note that

$$\begin{aligned} n\frac{d\theta}{dz} &= \sum_{j=1}^{n}\frac{d\theta_j}{dz} \\ &= \sum_{j=1}^{n}\frac{d(\tan\theta_j)}{dz}\left(\frac{d(\tan\theta_j)}{d\theta_j}\right)^{-1} \\ &= \sum_{j=1}^{n}\frac{d(\tan\theta_j)}{dz}\cos^2\theta_j, \end{aligned} \tag{2.15}$$

where from (2.5), we have

$$\begin{aligned} \frac{d(\tan\theta_j)}{dz} &= \frac{d}{dz}\left(\frac{\operatorname{Im}(\omega_1 z - z_j)}{\Re(\omega_1 z - z_j)}\right) \\ &= \frac{d}{dz}\left(\frac{z\sin(\pi/m) - r_j\sin\phi_j}{z\cos(\pi/m) - r_j\cos\phi_j}\right) \\ &= \frac{r_j(\cos(\pi/m)\sin\phi_j - \sin(\pi/m)\cos\phi_j)}{(z\cos(\pi/m) - r_j\cos\phi_j)^2} \\ &= \frac{r_j\sin(\phi_j - \pi/m)}{(z\cos(\pi/m) - r_j\cos\phi_j)^2}. \end{aligned} \tag{2.16}$$

We recall that all z_j, $1 \le j \le n$, lie on the open left half plane and they are the zeros of the real polynomial $P(z)$. If $z_j \in \mathbb{R}^-$, then $\phi_j = \pi$ and the equation above implies $d(\tan\theta_j)/dz \ge 0$. On the other hand, if $z_j = r_j e^{i\phi_j} \notin \mathbb{R}^-$ is a zero of $P(z)$, then so is $\overline{z_j}$. Let $k \ne j$ be the index of such zero, i.e., $z_k = \overline{z_j}$. Since both z_j and z_k lie on the open left half plane, without loss of generality, we assume $\phi_k \in (\pi/2, \pi)$ and $\phi_j \in (-\pi, -\pi/2)$. The summation in (2.15) contains

$$\begin{aligned} &\frac{d(\tan\theta_j)}{dz}\cos^2\theta_j + \frac{d(\tan\theta_k)}{dz}\cos^2\theta_k \\ &= \left(\frac{d(\tan\theta_j)}{dz} + \frac{d(\tan\theta_k)}{dz}\right)\cos^2\theta_j + \frac{d(\tan\theta_k)}{dz}(\cos^2\theta_k - \cos^2\theta_j). \end{aligned} \tag{2.17}$$

To prove this sum is nonnegative, we note from $z_k = \overline{z_j}$ that $r_j = r_k$ and $\phi_j + \phi_k = 0$. Thus, (2.16) yields

$$\frac{d(\tan\theta_j)}{dz} + \frac{d(\tan\theta_k)}{dz} = \frac{2r_k \sin(-\pi/m)\cos(\phi_k)}{(z\cos(\pi/m) - r_k\cos\phi_k)^2} > 0.$$

Since $\phi_k \in (\pi/2, \pi)$, (2.16) implies that $d(\tan\theta_k)/dz > 0$. Consequently, to prove (2.17) is positive, it suffices to show that $\cos^2\theta_k - \cos^2\theta_j > 0$. Indeed, we have

$$\tan\theta_j = \frac{z\sin(\pi/m) - r_j\sin\phi_j}{z\cos(\pi/m) - r_j\cos\phi_j} > \frac{z\sin(\pi/m) - r_j\sin\phi_k}{z\cos(\pi/m) - r_j\cos\phi_k} = \tan\theta_k$$

since $\sin\phi_k = -\sin\phi_j > 0$ and $\cos\phi_j = \cos\phi_k < 0$. Hence,

$$\cos^2\theta_k = \frac{1}{1+\tan^2\theta_k} > \frac{1}{1+\tan^2\theta_j} = \cos^2\theta_j.$$

We have shown that (2.15) is positive, and thus, θ is an increasing function in $z \in (0,\infty)$. To finish the proof of this lemma, it remains to show $\lim_{z\to 0}\theta = 0$ and $\lim_{z\to\infty}\theta = \pi/m$. From

$$\tan\theta_j = \frac{z\sin(\pi/m) - r_j\sin\phi_j}{z\cos(\pi/m) - r_j\cos\phi_j},$$

we conclude that

$$\lim_{z\to 0}\tan\theta_j = \tan\phi_j,$$
$$\lim_{z\to\infty}\tan\theta_j = \tan\frac{\pi}{m}.$$

We combine these two limits with the fact that $-\pi/2 < \theta_j < \pi/2$ (given in (2.6)) and $\pi/2 < |\phi_j| \le \pi$ to obtain

$$\lim_{z\to 0}\theta_j = \begin{cases} \phi_j - \pi & \text{if } \pi/2 < \phi_j \le \pi, \\ \phi_j + \pi & \text{if } -\pi < \phi_j < -\pi/2 \end{cases} \tag{2.18}$$

and

$$\lim_{z\to\infty}\theta_j = \frac{\pi}{m}.$$

The definition $\theta = \sum_{j=1}^{n}\theta_j$ yields $\lim_{z\to\infty}\theta = \pi/m$. Finally, this definition, (2.18), and the fact that both $r_je^{i\phi_j}$ and $r_je^{-i\phi_j}$ are zeros of $P(z)$ give $\lim_{z\to 0}\theta = 0$. ∎

Having proved that $\theta = \theta(z)$ bijectively map the interval $z \in (0,\infty)$ to $(0,\pi/m)$, we now turn our attention to proving that $g(\theta)$, defined as in (2.7), has $\lfloor n/m\rfloor$ zeros on the interval $(0,\pi/m)$. We recall that $g(\theta)$ is the quotient of $mH(z)$ and $\prod_{j=1}^{n}(\omega_0 z - \overline{z_j})e^{i\theta_j}$. From (2.5) and the fact that $\omega_1 z - z_j$ and $\omega_0 z - \overline{z_j}$ are complex conjugate, we conclude that $g(\theta) \in \mathbb{R}$ for $\theta \in (0,\pi/m)$. From (2.10) and (2.9), the sum of the first two terms of

$$g(\theta) = \sum_{k=0}^{m-1}\omega_k^{-r}e^{-in\theta}\prod_{j=1}^{n}q_{j,k}$$

when $k = 0$ and $k = 1$ is

$$e^{-in\theta - ri\pi/m} + e^{in\theta + ri\pi/m} = 2\cos\left(n\theta + \frac{r\pi}{m}\right).$$

From Lemmas 5 and 6, we obtain the following upper bound for the sum of the remainder terms when $2 \le k < m$:

$$\left|\sum_{k=2}^{m-1} \omega_k^{-r} e^{-in\theta} \prod_{j=1}^{n} q_{j,k}\right| \le \sum_{k=2}^{m-1} \prod_{j=1}^{n} |q_{j,k}| < m - 2.$$

When $m \le 4$, the right side is less than 2 and, thus, at the values of θ, where $\cos(n\theta + \frac{r\pi}{m}) = \pm 1$, $g(\theta)$ is positive/negative, respectively. Specifically, when

$$\theta_h = \frac{h - r/m}{n}\pi, \quad 1 \le h \le \lfloor n/m \rfloor, \tag{2.19}$$

the sign of $g(\theta_h)$ is $(-1)^h$. Moreover, $\theta_h > 0$ and

$$\theta_h \le \frac{n/m - r/m}{n}\pi \le \frac{\pi}{m}.$$

In the case $\theta_h = \pi/m$, we replace such value of θ_h by $(\pi/m)^-$ so that all θ_h defined in (2.19) lie on the interval $(0, \pi/m)$. By the intermediate value theorem, $g(\theta)$ has at least one zero on each interval (θ_h, θ_{h+1}), $1 \le h < \lfloor n/m \rfloor$. Thus, we obtain at least $\lfloor n/m \rfloor - 1$ zeros of $g(\theta)$ on $(0, \pi/m)$, each of which produces one positive real zero of $H(z)$ by Lemma 7.

We claim that $H(z)$ has exactly $\lfloor n/m \rfloor$ positive real zeros. Indeed, if $z_j \notin \mathbb{R}$, we pair the two terms $z - z_j$ and $z - \overline{z_j}$ on the right side of

$$P(z) = \prod_{j=1}^{n} (z - z_j)$$

and note that

$$(z - z_j)(z - \overline{z_j}) = z^2 - 2\Re z_j + |z_j|^2.$$

Since all the zeros of $P(z)$ lie on the left half plane, $P(z)$ is a product of polynomials with positive coefficients, and consequently, all the coefficients of $P(z)$ are positive. By Descartes' rule of signs, the number of positive zeros of

$$H(z) = \sum_{j=0}^{\lfloor n/m \rfloor} (-1)^j a_{jm+r} z^{jm+r}$$

is either $\lfloor n/m \rfloor$ or less than that by an even number. As we have shown that $H(z)$ has at least $\lfloor n/m \rfloor - 1$ positive real zeros, the number of such zeros must be $\lfloor n/m \rfloor$. This concludes the proofs of Theorems 1 and 2.

3. Entire functions with arithmetic exponents

In this section, we will prove Theorems 3 and 4. Suppose $m = 3$ or 4 and $0 \leq r < m$. From an entire function $f(z) = \sum_{j=0}^{\infty} a_j z^j$, we define

$$f_r(z) = \sum_{j=0}^{\infty} a_{jm+r} z^{jm+r}.$$

We assume that

$$f(z) = z^p e^{Az^2+Bz+C} \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) e^{z/z_j},$$

where $\sum_{j=1}^{\infty} 1/z_j$ converges absolutely, $B + \sum_{j=1}^{\infty} \frac{1}{z_j} > 0$, and

$$\begin{cases} A \geq 0 \text{ and } \Re z_j < 0 & \text{if } m = 4, \\ A = 0 \text{ and } 2\pi/3 < |\operatorname{Arg} z_j| \leq \pi & \text{if } m = 3. \end{cases} \tag{3.1}$$

Since $\sum_{j=1}^{\infty} 1/z_j$ converges absolutely, we can collect the factors e^{z/z_j}, $1 \leq j < \infty$ to have

$$f(z) = z^p e^C \exp\left(Az^2 + \left(B + \sum_{j=1}^{\infty} \frac{1}{z_j}\right) z\right) \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right).$$

Under the condition (3.1), we will show that the zeros of $f_r(z)$ lie on the m radial rays: $\operatorname{Im} z^m = 0$ and $\Re z^m \leq 0$. With a similar argument as that in the previous section, it is equivalent to show that the zeros of

$$\begin{aligned} e^{-r\pi i/m} f_r(e^{\pi i/m} z) &= \sum_{j=0}^{\infty} (-1)^j a_{jm+r} z^{jm+r} \\ &= \frac{1}{m} \sum_{k=0}^{m-1} \omega_k^{-r} f(\omega_k z) \end{aligned}$$

lie on the m radial rays: $\operatorname{Im} z^m = 0$ and $\Re z^m \geq 0$. If we denote the polynomial above by $h(z)$, then it suffices to prove that the zeros of $h(z)$ lie on these rays on any compact set K. On that compact set, the sequence of polynomials

$$P_n(z) = z^p e^C \left(1 + \frac{1}{n}\left(Az^2 + \left(B + \sum_{j=1}^{\infty} \frac{1}{z_j}\right) z\right)\right)^n \prod_{j=1}^{n} \left(1 - \frac{z}{z_j}\right)$$

converges uniformly to $f(z)$ as $n \to \infty$. Consequently, the sequence of polynomials

$$\frac{1}{m} \sum_{k=0}^{m-1} \omega_k^{-r} P_n(\omega_k z) \tag{3.2}$$

converges uniformly on K to

$$\frac{1}{m}\sum_{k=0}^{m-1}\omega_k^{-r} f(\omega_k z) \tag{3.3}$$

as $n \to \infty$.

The zeros of $P_n(z)$ are z_j, $1 \le j < \infty$, and those of

$$Az^2 + \left(B + \sum_{j=1}^{\infty}\frac{1}{z_j}\right)z + n. \tag{3.4}$$

If $m = 3$, then $A = 0$ and all the zeros of $P_n(z)$ lie on the sector $2\pi/3 < |\operatorname{Arg} z| \le \pi$ since

$$B + \sum_{j=1}^{\infty}\frac{1}{z_j} > 0.$$

From the previous section, the zeros of the polynomial (3.2) lie on the m radial rays: $\operatorname{Im} z^m = 0$ and $\Re z^m \ge 0$. Since this sequence converges to (3.3) uniformly on K as $n \to \infty$, the intersection of the set of zeros of (3.3) and K must also be a subset of the same m radial rays. Since K is arbitrary, all the zeros of (3.3) must lie on these rays and Theorem 3 follows.

On the other hand, if $m = 4$, then the zeros of (3.4) lie on the left half-plane since the zeros are either

$$-n\left(B + \sum_{j=1}^{\infty}\frac{1}{z_j}\right)^{-1} < 0$$

in the case $A = 0$ or a pair of complex conjugates whose real part is

$$-\left(B + \sum_{j=1}^{\infty}\frac{1}{z_j}\right)\frac{1}{2A} < 0$$

in the case $A \ne 0$ by Vieta's formula. Theorem 4 follows from the same arguments as those in the case $m = 3$. We complete the proof of Theorems 3 and 4.

References

[1] J. Borcea and P. Brändén, Pólya–Schur master theorems for circular domains and their boundaries. *Ann. of Math. (2)* **170** (2009), no. 1, 465–492 Zbl 1184.30004 MR 2521123

[2] G. Csordas and T. Forgács, Multiplier sequences, classes of generalized Bessel functions and open problems. *J. Math. Anal. Appl.* **433** (2016), no. 2, 1369–1389 Zbl 1330.30015 MR 3398769

[3] Q. I. Rahman and G. Schmeisser, *Analytic theory of polynomials*. London Math. Soc. Monogr. (N.S.) 26, The Clarendon Press, Oxford University Press, Oxford, 2002 Zbl 1072.30006 MR 1954841

[4] J. Schur and G. Pólya, Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen. *J. Reine Angew. Math.* **144** (1914), 89–113 Zbl 45.0176.01 MR 1580897

[5] X. Wang, A simple proof of Descartes's rule of signs. *Amer. Math. Monthly* **111** (2004), no. 6, 525–526 Zbl 1080.26507

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