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Variation operators associated with semigroups generated by Hardy operators involving fractional Laplacians in a half space

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Abstract. We represent by $\{W^\alpha_{\lambda,t}\}_{t>0}$ the semigroup generated by $-\mathbb{L}^\alpha_\lambda$, where $\mathbb{L}^\alpha_\lambda$ is a Hardy operator on a half space. The operator $\mathbb{L}^\alpha_\lambda$ includes a fractional Laplacian and it is defined by

$$\mathbb{L}^\alpha_\lambda = (-\Delta)^{\alpha/2}_{\mathbb{R}^d_+} + \lambda x_d^{-\alpha}, \quad \alpha \in (0,2], \lambda \geq 0.$$

We prove that, for every $k \in \mathbb{N}$, the ρ-variation operator $\mathcal{V}_\rho(\{t^k \partial_t^k W^\alpha_{\lambda,t}\})$ is bounded on $L^p(\mathbb{R}^d_+, w)$ for each $1 < p < \infty$ and $w \in A_p(\mathbb{R}^d_+)$, with $A_p(\mathbb{R}^d_+)$ being the Muckenhoupt p-class of weights on $\mathbb{R}^d_+$.

1. Introduction

In this paper, we consider the non-local Hardy-type operator $\mathbb{L}^\alpha_\lambda$ defined, in a formal way, by

$$\mathbb{L}^\alpha_\lambda := (-\Delta)^{\alpha/2}_{\mathbb{R}^d_+} + \lambda x_d^{-\alpha},$$

on $\mathbb{R}^d_+ := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_d > 0\}$. Here, $\alpha \in (0,2]$ and $\lambda \geq \lambda_*$, where

$$\lambda_* = -\frac{\Gamma\big(\frac{\alpha+1}{2}\big)}{\pi}\bigg(\Gamma\Big(\frac{\alpha+1}{2}\Big) - \frac{2^{\alpha-1}\sqrt{\pi}}{\Gamma\big(1-\frac{\alpha}{2}\big)}\bigg).$$

In order to give a precise definition of the operator $\mathbb{L}^\alpha_\lambda$, we consider the following quadratic forms. We define, for every $u, v \in C_c^1(\mathbb{R}^d_+)$ (the space of continuously differentiable functions having compact support on $\mathbb{R}^d_+$), when $\alpha \in (0,2)$,

$$Q^\alpha_\lambda(u,v) = \frac{1}{2}\mathcal{A}(d,\alpha)\int_{\mathbb{R}^d_+\times\mathbb{R}^d_+} \frac{(u(x)-u(y))\overline{(v(x)-v(y))}}{|x-y|^{d+\alpha}}\,dx\,dy + \lambda\int_{\mathbb{R}^d_+} \frac{u(x)\overline{v(x)}}{x_d^\alpha}\,dx,$$

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where $\mathcal{A}(d,\alpha) := \frac{-\alpha}{2^{1+\alpha}\pi^{d/2}} \frac{\Gamma((d-\alpha)/2)}{\Gamma(1+\frac{\alpha}{2})}$, and, when $\alpha = 2$,

$$Q_\lambda^2(u,v) = \int_{\mathbb{R}_+^d} \nabla u(x)\overline{\nabla v(x)}\,dx + \lambda \int_{\mathbb{R}_+^d} \frac{u(x)\overline{v(x)}}{x_d^2}\,dx.$$

By using the classical Hardy inequality when $\alpha = 2$ and those proved by Bogdan and Dyda [7, Theorem 1], we have that, for every $\alpha \in (0,2]$,

$$Q_\lambda^\alpha(u,u) \geq 0, \quad u \in C_c^1(\mathbb{R}_+^d),$$

provided that $\lambda \geq \lambda_*$. Note that Hardy inequalities proved in [7, Theorem 1] are sharp.

According to Friedrichs's Extension Theorem, the quadratic form Q_λ^α defines a self-adjoint and non-negative operator L_λ^α in $L^2(\mathbb{R}_+^d)$ for which $C_c^1(\mathbb{R}_+^d)$ is a form core. When f is smooth enough in $\mathbb{R}_+^d$,

$$\mathbb{L}_\lambda^\alpha f = L_\lambda^\alpha f.$$

The operator $-L_\lambda^\alpha$ generates in $L^2(\mathbb{R}_+^d)$ a contractive analytic C_0-semigroup of angle $\frac{\pi}{2}$ [42, Corollary 2.28]. We denote by $\{W_{\lambda,z}^\alpha\}_{\mathrm{Re}(z)>0}$ the semigroup generated by $-L_\lambda^\alpha$ in $L^2(\mathbb{R}_+^d)$. From [3, Theorem 3.1], for every $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, $W_{\lambda,z}^\alpha$ is an integral operator whose kernel, denoted by $W_{\lambda,z}^\alpha(x,y)$, is analytic in $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ for every $x, y \in \mathbb{R}_+^d$. In Section 2, we establish some estimates involving the kernel $W_{\lambda,z}^\alpha(x,y)$ that will be useful to prove the main results.

In [16, 17], we can find how the operators L_0^α appear related to censored processes and the two-sided estimates for the heat kernels associated with the semigroup generated by $-L_0^\alpha$ (see [17, Theorem 1.1] and [16, Theorem 1.1]).

We will now define the variation operator. Let $\rho > 0$ and suppose that $\{a_t\}_{t>0}$ is a set of complex numbers. We define the ρ-variation $\mathcal{V}_\rho(\{a_t\}_{t>0})$ as follows:

$$\mathcal{V}_\rho(\{a_t\}_{t>0}) = \sup_{0<t_0<\cdots<t_n,\ n\in\mathbb{N}} \left(\sum_{j=0}^{n-1} |a_{t_{j+1}} - a_{t_j}|^\rho \right)^{1/\rho}.$$

Assume that, for some $p \in (1,\infty)$, T_t is a bounded operator on $L^p(\Omega,\mu)$ for every $t > 0$, where (Ω,μ) is a measure space. The variation operator $\mathcal{V}_\rho(\{T_t\}_{t>0})$ is defined by

$$\mathcal{V}_\rho(\{T_t\}_{t>0})(f)(x) := \mathcal{V}_\rho(\{T_t(f)(x)\}_{t>0}), \quad f \in L^p(\Omega,\mu).$$

In order to have measurability of the function $\mathcal{V}_\rho(\{T_t\}_{t>0})(f)$, we need to have some continuity property in the time variable for $T_t(f)$ (see the comments after [14, Theorem 1.2]). Boundedness properties of ρ-variation operator (also oscillation operators and λ-jump counting function) give us quantitative measures for pointwise convergence of the family $\{T_t(f)\}_{t>0}$.

Lépingle [31] studied ρ-variations of martingales. He established strong L^p inequalities and weak-type $(1,1)$ estimates for the ρ-variation operator when $\rho > 2$. In [24], an example with $\rho = 2$ shows that $\rho > 2$ is the best exponent in the above estimate (see also [41]). Variational inequalities for martingales can be seen as an extension of Doob's maximal inequalities.

Bourgain, in a series of papers (see [8–10]), proved L^p variational inequalities in ergodic settings. Bourgain's ideas were the starting point for using the variation operator in the ergodic theory [26, 28, 29, 34, 43, 44], probability theory [1, 36, 40], and harmonic analysis [4, 6, 14, 15, 19, 20, 23, 27, 32, 33, 37, 39].

Our objective is to prove weighted L^p inequalities for the ρ-variation operator associated with $\{t^k \partial_t^k W_{\lambda,t}^\alpha\}_{t>0}$, that is, $\mathcal{V}_\rho(\{t^k \partial_t^k W_{\lambda,t}^\alpha\}_{t>0})$, where $k \in \mathbb{N}$. Our study is motivated by those ones in [2, 38], where the Schrödinger operator in $\mathbb{R}^d$ with inverse square potentials is considered. As it was mentioned, $\{W_{\lambda,t}^\alpha\}_{t>0}$ can be seen as the semigroup generated by the Hardy operator type in a half space $\mathbb{L}_\lambda^\alpha$. Frank and Merz [22] recently have proved that the scales of homogeneous Sobolev spaces generated by $\mathbb{L}_\lambda^\alpha$ and by $(-\Delta)_{\mathbb{R}_+^d}^\alpha$ are comparable.

Our first result is the following.

Theorem 1.1. *Let $0 < \alpha \le 2$, $\rho > 2$, $\lambda \ge 0$, and $1 < p < \infty$. Then, the ρ-variation operator $\mathcal{V}_\rho(\{t^k \partial_t^k W_{\lambda,t}^\alpha\}_{t>0})$ is bounded on $L^p(\mathbb{R}_+^d)$ for each $k \in \mathbb{N}$.*

For the proof of this theorem, we will start with the case $\lambda = 0$. We denote by $\{\mathbb{W}_t^\alpha\}_{t>0}$ the semigroup of operators generated by $-(-\Delta)^\alpha$ on $\mathbb{R}^d$. We have that

$$0 < W_{0,t}^\alpha(x,y) = W_{0,t}^\alpha(y,x) \le \mathbb{W}_t^\alpha(x,y), \quad x, y \in \mathbb{R}_+^d, t > 0.$$

Then,

$$\int_{\mathbb{R}_+^d} W_{0,t}^\alpha(x,y)\,dy = \int_{\mathbb{R}_+^d} W_{0,t}^\alpha(x,y)\,dx \le \int_{\mathbb{R}^d} \mathbb{W}_t^\alpha(x,y)\,dy = 1, \quad x \in \mathbb{R}_+^d,\ t > 0.$$

We deduce that, for every $t > 0$, $W_{0,t}^\alpha$ is a contraction in $L^1(\mathbb{R}_+^d)$ and in $L^\infty(\mathbb{R}_+^d)$. According to [30, Corollary 6.1], the ρ-variation operator $\mathcal{V}_\rho(\{t^k \partial_t^k W_{0,t}^\alpha\}_{t>0})$ is bounded on $L^p(\mathbb{R}_+^d)$ for every $1 < p < \infty$ and $k \in \mathbb{N}$.

In a second step, we prove that the ρ-variation operator

$$\mathcal{V}_\rho\big(\{t^k \partial_t^k (W_{\lambda,t}^\alpha - W_{0,t}^\alpha)\}_{t>0}\big)$$

is bounded on $L^p(\mathbb{R}_+^d)$ for every $1 < p < \infty$ and $k \in \mathbb{N}$. In order to do so, we will consider operators that control this ρ-variation operator.

Suppose that $F : (0,\infty) \to \mathbb{C}$ is a differentiable function on $(0,\infty)$. For $n \in \mathbb{N}$, let $0 < t_0 < \cdots < t_n$. We have that

$$\left(\sum_{j=0}^{n-1} |F(t_{j+1}) - F(t_j)|^\rho\right)^{1/\rho} \le \sum_{j=0}^{n-1} \left|\int_{t_j}^{t_{j+1}} F'(s)\,ds\right| \le \int_0^\infty |F'(s)|\,ds.$$

Thus,

$$\mathcal{V}_\rho(\{F(t)\}_{t>0}) \leq \int_0^\infty |F'(s)|\, ds.$$

From this inequality, we deduce that

$$\mathcal{V}_\rho\big(\{t^k \partial_t^k (W_{\lambda,t}^\alpha - W_{0,t}^\alpha)\}_{t>0}\big)(f)(x) \leq \int_{\mathbb{R}_+^d} f(y) K_\lambda^\alpha(x,y)\, dy, \quad x \in \mathbb{R}_+^d, \tag{1.1}$$

where

$$K_\lambda^\alpha(x,y) := \int_0^\infty \big|\partial_t\big(t^k \partial_t^k \big(W_{\lambda,t}^\alpha(x,y) - W_{0,t}^\alpha(x,y)\big)\big)\big|\, dt, \quad x,y \in \mathbb{R}_+^d.$$

For $m \in \mathbb{N}$, we consider the operator $T_\lambda^{\alpha,m}$ defined by

$$T_\lambda^{\alpha,m}(f)(x) = \int_{\mathbb{R}_+^d} T_\lambda^{\alpha,m}(x,y) f(y)\, dy, \quad x \in \mathbb{R}_+^d,$$

with

$$T_\lambda^{\alpha,m}(x,y) = \int_0^\infty \big|t^m \partial_t^{m+1}\big(W_{\lambda,t}^\alpha(x,y) - W_{0,t}^\alpha(x,y)\big)\big|\, dt, \quad x,y \in \mathbb{R}_+^d.$$

Taking into account (1.1), to see that $\mathcal{V}_\rho(\{t^k \partial_t^k (W_{\lambda,t}^\alpha - W_{0,t}^\alpha)\}_{t>0})$ is bounded on $L^p(\mathbb{R}_+^d)$, it will be sufficient to show that the operator $T_\lambda^{\alpha,m}$ has this property for $m = 0$ when $k = 0$, and for $m = k-1$ and $m = k$ when $k \geq 1$.

We will split the operator $T_\lambda^{\alpha,m}$ into two parts, determined by the sets

$$\begin{aligned} G &= G_1 \cup G_2 \\ &:= \{(x,y,t) \in \mathbb{R}_+^d \times \mathbb{R}_+^d \times (0,\infty) : x_d \vee y_d \leq t^{1/\alpha}\} \\ &\quad \cup \{(x,y,t) \in \mathbb{R}_+^d \times \mathbb{R}_+^d \times (0,\infty) : x_d \vee y_d > t^{1/\alpha},\ 2|x-y| \geq x_d \wedge y_d\} \end{aligned} \tag{1.2}$$

and

$$L := \big(\mathbb{R}_+^d \times \mathbb{R}_+^d \times (0,\infty)\big) \setminus G. \tag{1.3}$$

Here, the letter G stands for *global* and the letter L means *local*.

Hence, for $m \in \mathbb{N}$, we decompose $T_\lambda^{\alpha,m}$ as follows:

$$T_\lambda^{\alpha,m} = T_{\lambda,\mathrm{loc}}^{\alpha,m} + T_{\lambda,\mathrm{glob}}^{\alpha,m},$$

where

$$T_{\lambda,\mathrm{loc/glob}}^{\alpha,m}(f)(x) = \int_{\mathbb{R}_+^d} T_{\lambda,\mathrm{loc/glob}}^{\alpha,m}(x,y) f(y)\, dy, \quad x \in \mathbb{R}_+^d,$$

being

$$T_{\lambda,\mathrm{loc}}^{\alpha,m}(x,y) = \int_0^\infty \big|t^m \partial_t^{m+1}\big(W_{\lambda,t}^\alpha(x,y) - W_{0,t}^\alpha(x,y)\big)\big| \chi_L(x,y,t)\, dt, \quad x,y \in \mathbb{R}_+^d,$$

and

$$T^{\alpha,m}_{\lambda,\mathrm{glob}}(x,y) = T^{\alpha,m}_{\lambda}(x,y) - T^{\alpha,m}_{\lambda,\mathrm{loc}}(x,y), \quad x,y \in \mathbb{R}^d_+.$$

Actually, the cancellation in $W^{\alpha}_{\lambda,t}(x,y) - W^{\alpha}_{0,t}(x,y)$ is only relevant for $T^{\alpha,m}_{\lambda,\mathrm{loc}}$; in the global operator, this cancellation does not play any role.

In Section 3, we will prove the L^p boundedness of the operators $T^{\alpha,m}_{\lambda,\mathrm{loc}}$ and $T^{\alpha,m}_{\lambda,\mathrm{glob}}$ for $\alpha \in (0,2)$, and we will deal with the case $\alpha = 2$ in Section 4. We distinguish these two cases since the estimations for the heat kernel $W^{\alpha}_{\lambda,t}(x,y)$, $x,y \in \mathbb{R}^d_+$ and $t > 0$, are different (see Propositions 2.1 and 2.2).

We now establish the weighted L^p inequalities for $\mathcal{V}_\rho(\{t^k\partial_t^k W^{\alpha}_{\lambda,t}\}_{t>0})$. First of all, we recall the definitions of the Muckenhoupt classes in $\mathbb{R}^d_+$. A weight in $\mathbb{R}^d_+$ is a non-negative measurable function defined on $\mathbb{R}^d_+$. A weight w is said to be in $A_p(\mathbb{R}^d_+)$ with $1 < p < \infty$ when

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w^{-1/(p-1)}(x)\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^d_+$. The p-Lebesgue space with weight w is denoted by $L^p(\mathbb{R}^d_+, w)$.

The following boundedness result for the weighted ρ-variation operator will be proved in Section 5.

Theorem 1.2. *Let $0 < \alpha \le 2$, $\rho > 2$, $\lambda \ge 0$, $k \in \mathbb{N}$, and $1 < p < \infty$. Then, the ρ-variation operator $\mathcal{V}_\rho(\{t^k\partial_t^k W^{\alpha}_{\lambda,t}\}_{t>0})$ is bounded on $L^p(\mathbb{R}^d_+, w)$ provided that $w \in A_p(\mathbb{R}^d_+)$.*

Throughout this paper, c and C will always represent positive constants that may vary on each occurrence. The expression $a \lesssim b$ will indicate that $a \le Cb$ for some positive constant C, whilst $a \sim b$ means that both $a \lesssim b$ and $b \lesssim a$ hold.

2. Heat kernel estimates

Our goal in this section is to establish estimates for the time derivatives of the heat kernel $W^{\alpha}_{\lambda,t}$ that will be useful in the sequel.

We define the exponent $\mathfrak{p}$ as in [22]. We consider

$$M_\alpha = \begin{cases} \alpha & \text{if } \alpha \in (0,2), \\ \infty & \text{if } \alpha = 2, \end{cases}$$

and, for every $\alpha \in (0,2]$, we define

$$\Omega_\alpha : (-1, M_\alpha) \to \mathbb{R}$$

$$s \mapsto \Omega_\alpha(s) = \frac{1}{\pi}\Big(\Gamma(\alpha)\sin\Big(\frac{\pi\alpha}{2}\Big) + \Gamma(1+s)\Gamma(\alpha-s)\sin\Big(\frac{\pi(2s-\alpha)}{2}\Big)\Big).$$

We understand $\Omega_2(s) = s(s-1)$, for $s \in (-1,\infty)$, and $\Omega_1(s) = \frac{1}{\pi}(1 - \pi s \cot(\pi s))$, for $s \in (-1,1)$. The main properties of the function Ω_α can be found in [22, Appendix A]. For every $\lambda \geq \lambda_*$, there exists a unique $\mathfrak{p} \in [\frac{\alpha-1}{2}, M_\alpha)$ such that

$$\Omega_\alpha(\mathfrak{p}) = \lambda.$$

Note that $\mathfrak{p}$ depends on λ and α, but we write $\mathfrak{p}$ instead of $\mathfrak{p}(\lambda,\alpha)$. We have that

(a) $\mathfrak{p} = \max\{\alpha - 1, 0\} := (\alpha-1)_+$, when $\lambda = 0$;

(b) $\mathfrak{p} > (\alpha-1)_+$, when $\lambda > 0$;

(c) $\mathfrak{p} = \frac{1}{2}(1 + \sqrt{1+4\lambda})$, when $\alpha = 2$.

For simplicity, we denote by $\gamma = (\alpha-1)_+$ for any $\alpha \in (0,2]$.

The following pointwise bounds for the heat kernel $W^\alpha_{\lambda,t}(x,y)$, $x,y \in \mathbb{R}^d_+$ and $t > 0$, were established in [22, Theorems 9 and 10].

Proposition 2.1. *We have the following.*

(a) *Let $\alpha \in (0,2)$ and $\lambda \geq 0$. Then, for every $x,y \in \mathbb{R}^d_+$ and $t > 0$,*

$$W^\alpha_{\lambda,t}(x,y) \sim \Big(1 \wedge \frac{x_d}{t^{1/\alpha}}\Big)^{\mathfrak{p}} \Big(1 \wedge \frac{y_d}{t^{1/\alpha}}\Big)^{\mathfrak{p}} t^{-d/\alpha} \Big(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\Big)^{d+\alpha}.$$

(b) *Let $\lambda \geq -1/4$. Then, for every $x,\ y \in \mathbb{R}^d_+$ and $t > 0$,*

$$W^2_{\lambda,t}(x,y) \asymp \Big(1 \wedge \frac{x_d}{t^{1/2}}\Big)^{\mathfrak{p}} \Big(1 \wedge \frac{y_d}{t^{1/2}}\Big)^{\mathfrak{p}} t^{-d/2} e^{-\frac{c|x-y|^2}{t}}.$$

Here, the symbol $\asymp$ means the same as $\sim$ but admitting different values of c in the exponential function in the upper and lower bounds.

Our next objective is to extend the upper bounds in Proposition 2.1 to the complex plane. Note first that from (a) we deduce that

$$W^\alpha_{\lambda,t}(x,y) \lesssim \Big(\frac{x_d}{x_d + t^{1/\alpha}}\Big)^{\mathfrak{p}} \Big(\frac{y_d}{y_d + t^{1/\alpha}}\Big)^{\mathfrak{p}} t^{-d/\alpha} \Big(\frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|}\Big)^{d+\alpha} \tag{2.1}$$

for every $x,\ y \in \mathbb{R}^d_+$ and $t > 0$.

Since the function

$$\phi(s) = \frac{s}{s+a}, \quad s > -a,$$

is increasing for every $a > 0$, it follows that

$$W^\alpha_{\lambda,t}(x,y) \lesssim \Big(\frac{|x|}{|x| + t^{1/\alpha}}\Big)^{\mathfrak{p}} \Big(\frac{|y|}{|y| + t^{1/\alpha}}\Big)^{\mathfrak{p}} t^{-d/\alpha} \Big(\frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|}\Big)^{d+\alpha}, \tag{2.2}$$

for every $x,\ y \in \mathbb{R}^d_+$ and $t > 0$.

By using [13, Proposition 2.2 (a)], we obtain that if $\alpha \in (0,2)$, $\lambda \geq 0$, and $\varepsilon \in (0,1)$, then for every $z \in \mathbb{C}$, $|\arg(z)| < \varepsilon\pi/4$, and $x, y \in \mathbb{R}^d_+$,

$$W^\alpha_{\lambda,z}(x,y) \lesssim \Big(\frac{|x|}{|x|+|z|^{1/\alpha}}\Big)^{\mathfrak{p}} \Big(\frac{|y|}{|y|+|z|^{1/\alpha}}\Big)^{\mathfrak{p}} |z|^{-d/\alpha} \Big(\frac{|z|^{1/\alpha}}{|z|^{1/\alpha}+|x-y|}\Big)^{(d+\alpha)(1-\varepsilon)}.$$

By using the Cauchy integral formula, we deduce that, if $\alpha \in (0,2)$, $\lambda \geq 0$ and $k \in \mathbb{N}$,

$$|t^k \partial^k_t W^\alpha_{\lambda,t}(x,y)| \lesssim \Big(\frac{|x|}{|x|+t^{1/\alpha}}\Big)^{\mathfrak{p}} \Big(\frac{|y|}{|y|+t^{1/\alpha}}\Big)^{\mathfrak{p}} t^{-d/\alpha} \Big(\frac{t^{1/\alpha}}{t^{1/\alpha}+|x-y|}\Big)^{(d+\alpha)(1-\varepsilon)}$$

for every $x, y \in \mathbb{R}^d_+$ and $t > 0$.

In a similar way, we get that if $\alpha = 2$, $\lambda \geq -1/4$ and $k \in \mathbb{N}$, then

$$|t^k \partial^k_t W^2_{\lambda,t}(x,y)| \lesssim \Big(\frac{|x|}{|x|+t^{1/2}}\Big)^{\mathfrak{p}} \Big(\frac{|y|}{|y|+t^{1/2}}\Big)^{\mathfrak{p}} t^{-d/2} e^{-\frac{c|x-y|^2}{t}} \tag{2.3}$$

for every $x, y \in \mathbb{R}^d_+$ and $t > 0$.

However, we need to improve (2.2) and (2.3) by replacing $|x|$ and $|y|$ by x_d and y_d, respectively.

Proposition 2.2. *We have the following.*

(a) *Let $\alpha \in (0,2)$, $\lambda \geq 0$, $\varepsilon \in (0,1)$, and $k \in \mathbb{N}$. For every $x, y \in \mathbb{R}^d_+$ and $t > 0$,*

$$\begin{aligned}&|t^k \partial^k_t W^\alpha_{\lambda,t}(x,y)| \\ &\quad \lesssim \Big(\frac{x_d}{x_d+t^{1/\alpha}}\Big)^{\mathfrak{p}} \Big(\frac{y_d}{y_d+t^{1/\alpha}}\Big)^{\mathfrak{p}} t^{-d/\alpha} \Big(\frac{t^{1/\alpha}}{t^{1/\alpha}+|x-y|}\Big)^{(d+\alpha)(1-\varepsilon)}.\end{aligned}$$

(b) *Let $\lambda \geq -1/4$ and $k \in \mathbb{N}$. For every $x, y \in \mathbb{R}^d_+$ and $t > 0$,*

$$|t^k \partial^k_t W^2_{\lambda,t}(x,y)| \lesssim \Big(\frac{x_d}{x_d+t^{1/2}}\Big)^{\mathfrak{p}} \Big(\frac{y_d}{y_d+t^{1/2}}\Big)^{\mathfrak{p}} t^{-d/2} e^{-\frac{c|x-y|^2}{t}}.$$

Proof. We prove (a) first. According to (2.2) and proceeding as in the proof of [12, Proposition 3.4] (see also [13, Proposition 2.2 (a)]), we deduce that

$$|w(x_d,z) W^\alpha_{\lambda,z}(x,y) w(y_d,z)| \leq \frac{C}{|z|^{d/\alpha}}, \tag{2.4}$$

for $x, y \in \mathbb{R}^d_+$ and $z \in \mathbb{C}$, $|\arg(z)| \leq \pi/4$, where

$$w(s,z) = \Big(\frac{s}{s+z^{1/\alpha}}\Big)^{-\mathfrak{p}},$$

for $s > 0$ and $z \in \mathbb{C}$, $\arg(z) \leq \pi/2$. Indeed, since the operator L^α_λ is self-adjoint and non-negative, for every $s > 0$, the operator $W^\alpha_{\lambda,is}$ is contractive in $L^2(\mathbb{R}^d_+)$. In order to prove this, it is sufficient to see that

$$\|W^\alpha_{\lambda,t}\|_{L^1(\mathbb{R}^d_+,w^{-1}) \hookrightarrow L^2(\mathbb{R}^d_+)} + \|W^\alpha_{\lambda,t}\|_{L^2(\mathbb{R}^d_+) \hookrightarrow wL^\infty(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2\alpha}}, \quad t > 0,$$

where $wL^\infty(\mathbb{R}^d) = \{f \text{ measurable in } \mathbb{R}^d : wf \in L^\infty(\mathbb{R}^d)\}$.

Suppose $f \in L^2(\mathbb{R}^d_+)$. According to (2.1) and using [12, (18)], we get

$$\begin{aligned}|W^\alpha_{\lambda,t}(f)(x)|w(x_d,t) &\lesssim \int_{\mathbb{R}^d_+} \left(\frac{y_d}{y_d + t^{1/\alpha}}\right)^{\mathfrak{p}} \frac{t}{t^{1/\alpha} + |x-y|^{d+\alpha}} |f(y)|dy \\ &\lesssim \left(\int_{\mathbb{R}^d_+} \left|\left(\frac{|y|}{|y| + t^{1/\alpha}}\right)^{\mathfrak{p}} \frac{t}{t^{1/\alpha} + |x-y|^{d+\alpha}}\right|^2 dy\right)^{1/2} \|f\|_{L^2(\mathbb{R}^d_+)} \\ &\lesssim t^{-\frac{d}{2\alpha}} \|f\|_{L^2(\mathbb{R}^d_+)},\end{aligned}$$

for $x \in \mathbb{R}^d_+$ and $t > 0$.

In a similar way, we can see that

$$\|W^\alpha_{\lambda,t} f\|_{L^2(\mathbb{R}^d_+)} \lesssim t^{-\frac{d}{2\alpha}} \|f\|_{L^1(\mathbb{R}^d_+, w^{-1})},$$

for $f \in L^1(\mathbb{R}^d_+, w^{-1})$ and $t > 0$. We conclude that (2.4) holds.

On the other hand, according to (2.1), it follows that

$$\begin{aligned}|w(x_d,t)W^\alpha_{\lambda,t}(x,y)w(y_d,t)| &\lesssim t^{-d/\alpha}\left(\frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|}\right)^{d+\alpha} \\ &= t^{-d/\alpha}\left(1 + \frac{|x-y|}{t^{1/\alpha}}\right)^{-d-\alpha} \\ &= t^{-d/\alpha}\left(1 + \left(\frac{|x-y|^\alpha}{t}\right)^{1/\alpha}\right)^{-d-\alpha},\end{aligned}$$

for $x, y \in \mathbb{R}^d_+$ and $t > 0$.

Let us define the function

$$b(s) = (1 + s^{1/\alpha})^{-d-\alpha}, \quad s > 0.$$

It is clear that b is bounded and decreasing in $(0, \infty)$. Then, by [21, Proposition 3.3], for each $\varepsilon \in (0, 1)$ and $\theta \in (0, \varepsilon\pi/2)$, there exists $C > 0$ such that

$$|W^\alpha_{\lambda,z}(x,y)| \leq C\left|\frac{x_d}{x_d + z^{1/\alpha}}\right|^{\mathfrak{p}}\left|\frac{y_d}{y_d + z^{1/\alpha}}\right|^{\mathfrak{p}} (\mathrm{Re}(z))^{-d/\alpha} b\left(\frac{|x-y|^\alpha}{|z|}\right)^{1-\varepsilon},$$

for $x, y \in (0, \infty)$ and $z \in \mathbb{C}$ such that $|\arg(z)| \leq \theta$.

If $z \in \mathbb{C}$ and $\arg(z)| \leq \theta$, with $0 < \theta < \pi/2$, then

$$|a + z| = \sqrt{(a + \mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2} \geq \sqrt{a^2 + |z|^2} \gtrsim a + |z|, \quad a > 0.$$

We conclude that, for each $\varepsilon \in (0, 1)$ and $\theta \in (0, \varepsilon\pi/2)$,

$$|W^\alpha_{\lambda,z}(x,y)| \lesssim \left(\frac{x_d}{x_d + |z|^{1/\alpha}}\right)^{\mathfrak{p}}\left(\frac{y_d}{y_d + |z|^{1/\alpha}}\right)^{\mathfrak{p}} |z|^{-d/\alpha}\left(\frac{|z|^{1/\alpha}}{|z|^{1/\alpha} + |x-y|}\right)^{(d+\alpha)(1-\varepsilon)}, \tag{2.5}$$

where $z \in \mathbb{C}$ and $|\arg(z)| \le \theta$. Note that

$$\mathrm{Re}(z) \simeq |z|$$

for $z \in \mathbb{C}$ with $|\arg(z)| \le \theta$.

Estimates like (2.5) can also be proved by using the methods in [35, Theorem 2.1].

The Cauchy integral formula allows us to obtain (a). In order to prove (b), we can proceed in a similar way. We can also prove it using Davies method as in the proof of [18, Theorem 3.4.8] and the Cauchy integral formula. ∎

3. Proof of Theorem 1.1 for $\alpha \in (0, 2)$

3.1. Global part

We define, for $k \in \mathbb{N}$, $k \ge 1$,

$$J_\lambda^{\alpha,k}(x,y) = \int_0^\infty \left|t^{k-1}\partial_t^k (W_{\lambda,t}^\alpha(x,y) - W_{0,t}^\alpha(x,y))\right| \chi_G(x,y,t)\,dt, \quad x,y \in \mathbb{R}_+^d, \tag{3.1}$$

where G is as in (1.2).

Lemma 3.1. *Let $k \in \mathbb{N}$, $k \ge 1$, $\lambda \ge 0$, and $\alpha \in (0,2)$. For every $0 \le \gamma \le (\alpha-1)_+$,*

$$\begin{aligned}\left|J_\lambda^{\alpha,k}(x,y)\right| \lesssim{}& \frac{(x_d y_d)^\gamma}{(|x-y| \vee x_d \vee y_d)^{d+2\gamma}} \\ &+ \chi_{\{|x-y| \ge \frac{x_d \wedge y_d}{2}\}}(x,y)\frac{(x_d \wedge y_d)^\gamma (x_d \vee y_d)^{(d+\alpha)(1-\varepsilon)-d-\gamma}}{|x-y|^{(d+\alpha)(1-\varepsilon)}},\end{aligned} \tag{3.2}$$

whenever $0 < \varepsilon < (\alpha - 2\gamma)/(d+\alpha)$.

Proof. By using Proposition 2.2 (a), we get

$$\begin{aligned}\left|J_\lambda^{\alpha,k}(x,y)\right| &\le \int_0^\infty \left(\left|t^{k-1}\partial_t^k W_{\lambda,t}^\alpha(x,y)\right| + \left|t^{k-1}\partial_t^k W_{0,t}^\alpha(x,y)\right|\right)\chi_G(x,y,t)\,dt \\ &\lesssim \int_0^\infty \frac{1}{t^{1+d/\alpha}}\left(\frac{x_d}{x_d + t^{1/\alpha}}\right)^\gamma \left(\frac{y_d}{y_d + t^{1/\alpha}}\right)^\gamma \\ &\quad \times \left(\frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|}\right)^{(d+\alpha)(1-\varepsilon)} \chi_G(x,y,t)\,dt, \quad x,y \in \mathbb{R}_+^d,\end{aligned}$$

where $0 < \varepsilon < 1$ and $0 \le \gamma \le (\alpha-1)_+$. Here, we can actually use the fact that the power $\mathfrak{p}$ can be replaced by any non-negative number less than or equal to $\mathfrak{p}$ in the estimate given in Proposition 2.2 (a).

We will split the last integral over G_1 and G_2 and denote

$$\begin{aligned}S_{\varepsilon,j}^\alpha(x,y) = \int_0^\infty &\frac{1}{t^{1+d/\alpha}}\left(\frac{x_d}{x_d + t^{1/\alpha}}\right)^\gamma \left(\frac{y_d}{y_d + t^{1/\alpha}}\right)^\gamma \\ &\times \left(\frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|}\right)^{(d+\alpha)(1-\varepsilon)} \chi_{G_j}(x,y,t)\,dt, \quad j = 1,2.\end{aligned}$$

From the definition of G_1, we can write, for $x, y \in \mathbb{R}^d_+$,

$$
\begin{aligned}
S^{\alpha}_{\varepsilon,1}(x,y) &= \int_{(x_d \vee y_d)^\alpha}^{\infty} \left(\frac{x_d}{x_d + t^{1/\alpha}}\right)^{\gamma} \left(\frac{y_d}{y_d + t^{1/\alpha}}\right)^{\gamma} \\
&\quad \times \left(\frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|}\right)^{(d+\alpha)(1-\varepsilon)} \frac{dt}{t^{1+d/\alpha}} \\
&\leq \int_{(x_d \vee y_d)^\alpha}^{\infty} \left(\frac{x_d}{t^{1/\alpha}}\right)^{\gamma} \left(\frac{y_d}{t^{1/\alpha}}\right)^{\gamma} \left(\frac{t^{1/\alpha}}{x_d \vee y_d \vee |x-y|}\right)^{(d+\alpha)(1-\varepsilon)} \frac{dt}{t^{1+d/\alpha}} \\
&\sim \frac{(x_d y_d)^{\gamma} (x_d \vee y_d)^{(d+\alpha)(1-\varepsilon)-d-2\gamma}}{(x_d \vee y_d \vee |x-y|)^{(d+\alpha)(1-\varepsilon)}} \\
&\lesssim \frac{(x_d y_d)^{\gamma}}{(|x-y| \vee x_d \vee y_d)^{d+2\gamma}}
\end{aligned}
$$

provided that $(d+\alpha)(1-\varepsilon) - d - 2\gamma > 0$, that is, whenever $0 < \varepsilon < (\alpha - 2\gamma)/(d+\alpha)$. Note that this is possible since $\gamma = (\alpha - 1)_+$ implies $\alpha - 2\gamma > 0$.

For $j = 2$, we take into account that $|x-y| \geq (x_d \wedge y_d)/2$, so we have

$$
\begin{aligned}
S^{\alpha}_{\varepsilon,2}(x,y) &\lesssim \int_0^{(x_d \vee y_d)^\alpha} \left(\frac{x_d \wedge y_d}{x_d \wedge y_d + t^{1/\alpha}}\right)^{\gamma} \left(\frac{t^{1/\alpha}}{t^{1/\alpha} + |x-y|}\right)^{(d+\alpha)(1-\varepsilon)} \frac{dt}{t^{1+d/\alpha}} \\
&\lesssim \int_0^{(x_d \vee y_d)^\alpha} \left(\frac{x_d \wedge y_d}{t^{1/\alpha}}\right)^{\gamma} \left(\frac{t^{1/\alpha}}{|x-y|}\right)^{(d+\alpha)(1-\varepsilon)} \frac{dt}{t^{1+d/\alpha}} \\
&\sim \frac{(x_d \wedge y_d)^{\gamma} (x_d \vee y_d)^{(d+\alpha)(1-\varepsilon)-d-\gamma}}{|x-y|^{(d+\alpha)(1-\varepsilon)}},
\end{aligned}
$$

provided that $(d+\alpha)(1-\varepsilon) - d - \gamma > 0$, i.e., $0 < \varepsilon < (\alpha - \gamma)/(d+\alpha)$. Actually, this is possible since the restriction obtained in the case $j = 1$ implies this one.

Therefore, we get that (3.2) holds for every $0 < \varepsilon < (\alpha - 2\gamma)/(d+\alpha)$. ∎

Fix now $0 < \varepsilon < \alpha/(d+\alpha)$. We define now $\alpha' = (d+\alpha)(1-\varepsilon) - d$. Hence (see [22, p. 31]),

$$
\frac{1}{(|x-y| \vee x_d \vee y_d)^d} \lesssim \frac{(|x-y| \vee x_d \vee y_d)^{\alpha'}}{(|x-y| \vee (x_d \wedge y_d))^{d+\alpha'}}
$$

and

$$
\frac{(x_d \vee y_d)^{\alpha'}}{|x-y|^{d+\alpha'}} \chi_{\{|x-y| \geq \frac{x_d \wedge y_d}{2}\}}(x,y) \lesssim \frac{(|x-y| \vee x_d \vee y_d)^{\alpha'}}{(|x-y| \vee (x_d \wedge y_d))^{d+\alpha'}}.
$$

Therefore, by proceeding as in the proof of [22, Proposition 19, Step 1] and by using (3.2) with $\gamma = 0$, we deduce that

$$
\int_{\mathbb{R}^{d-1}} \left| J^{\alpha,k}_{\lambda}(x,y) \right| dy' \lesssim \frac{(x_d \vee y_d)^{\alpha'}}{(|x_d - y_d| \vee (x_d \wedge y_d))^{\alpha'+1}}, \tag{3.3}
$$

with $y' = (y_1, \ldots, y_{d-1})$.

By considering $w_\beta(x,y) = (\frac{x_d}{y_d})^\beta$, $x, y \in \mathbb{R}^d_+$ and $\beta > 0$, we get, as in [22, Proposition 19, Step 2] by invoking (3.3), that for any $1 < p < \infty$,

$$\sup_{x\in\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} w_\beta(x,y)^{1/p} \big|J^{\alpha,k}_\lambda(x,y)\big|\, dy < \infty, \qquad 0 < \beta/p < 1,$$

$$\sup_{x\in\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} w_\beta(x,y)^{-1/p'} \big|J^{\alpha,k}_\lambda(x,y)\big|\, dx < \infty, \qquad 0 < \beta/p' < 1.$$

According to Schur's test with weights (see, for instance, [25, Lemma 3.3]), we can conclude that the integral operator defined by

$$J^{\alpha,k}_\lambda(f)(x) = \int_{\mathbb{R}^d_+} J^{\alpha,k}_\lambda(x,y) f(y)\, dy, \quad x \in \mathbb{R}^d_+,$$

is bounded on $L^p(\mathbb{R}^d_+)$ for every $1 < p < \infty$.

3.2. Local part

According to Duhamel formula, we can write, for every $x, y \in \mathbb{R}^d_+$ and $t > 0$, that

$$\begin{aligned} D^{\alpha,0}_{\lambda,t}(x,y) &:= \lambda \int_0^t \int_{\mathbb{R}^d_+} W^\alpha_{0,t-s}(x,z) z_d^{-\alpha} W^\alpha_{\lambda,s}(z,y)\, dz ds \\ &= \lambda \int_0^{t/2} \int_{\mathbb{R}^d_+} W^\alpha_{0,t-s}(x,z) z_d^{-\alpha} W^\alpha_{\lambda,s}(z,y)\, dz ds \\ &\quad + \lambda \int_0^{t/2} \int_{\mathbb{R}^d_+} W^\alpha_{0,s}(x,z) z_d^{-\alpha} W^\alpha_{\lambda,t-s}(z,y)\, dz ds. \end{aligned} \tag{3.4}$$

We have that $D^{\alpha,0}_{\lambda,t}(x,y) = t^{-d/\alpha} F(t^{-1/\alpha}x, t^{-1/\alpha}y)$, $x, y \in \mathbb{R}^d_+$, $t > 0$ (see comments after [22, Theorem 9]) for a smooth function $F : \mathbb{R}^d_+ \times \mathbb{R}^d_+ \to \mathbb{R}$. Thus, for every $k \in \mathbb{N}$, there exists a smooth function $F_k : \mathbb{R}^d_+ \times \mathbb{R}^d_+ \to \mathbb{R}$ for which

$$t^k \partial_t^k D^{\alpha,0}_{\lambda,t}(x,y) = t^{-d/\alpha} F_k\big(t^{-1/\alpha}x, t^{-1/\alpha}y\big), \quad x, y \in \mathbb{R}^d_+, t > 0. \tag{3.5}$$

Given $k \in \mathbb{N}$, from (3.4), we can write

$$\begin{aligned} \partial_t^k D^{\alpha,0}_{\lambda,t}(x,y) &= \sum_{j=0}^{k-1} a_{j,k} \int_{\mathbb{R}^d_+} \partial_t^j W^\alpha_{0,t/2}(x,z) z_d^{-\alpha} \partial_t^{k-1-j} W^\alpha_{\lambda,t/2}(z,y)\, dz \\ &\quad + \lambda \int_0^{t/2} \int_{\mathbb{R}^d_+} \partial_t^k W^\alpha_{0,t-s}(x,z) z_d^{-\alpha} W^\alpha_{\lambda,s}(z,y)\, dz ds \\ &\quad + \lambda \int_0^{t/2} \int_{\mathbb{R}^d_+} W^\alpha_{0,s}(x,z) z_d^{-\alpha} \partial_t^k W^\alpha_{\lambda,t-s}(z,y)\, dz ds, \quad x, y \in \mathbb{R}^d_+,\ t > 0, \end{aligned}$$

where $a_{j,k} \in \mathbb{R}$ for each $j = 0, \dots, k-1$.

We define, for every $j = 0, \dots, k-1$, the integrals involved in the above summation as

$$H_j(x,y,t) = \int_{\mathbb{R}^d_+} \partial_t^j W^\alpha_{0,t/2}(x,z) z_d^{-\alpha} \partial_t^{k-1-j} W^\alpha_{\lambda,t/2}(z,y)\, dz, \quad x, y \in \mathbb{R}^d_+,\ t > 0,$$

and we will estimate H_j in the local region.

By taking into account the scaling property (3.5), we reduce the problem to estimating $H_j(x,y,1)$, $j = 0, \dots, k-1$, when $|x-y| \leq \frac{x_d \wedge y_d}{2}$ and $1 \leq x_d \vee y_d$.

Lemma 3.2. *Let $j = 0, \dots, k-1$, with $k \in \mathbb{N}$, $k \geq 1$, $\lambda \geq 0$ and $\alpha \in (0,2)$. Then,*

$$|H_j(x,y,1)| \lesssim \frac{1}{x_d^\alpha} \left(\frac{1}{1+|x-y|} \right)^{(d+\alpha)(1-\varepsilon)}, \quad |x-y| \leq \frac{x_d \wedge y_d}{2},\ 1 \leq x_d \vee y_d, \tag{3.6}$$

with $0 < \varepsilon < \alpha/(d+\alpha)$.

Proof. Assume that $x, y \in \mathbb{R}^d_+$ with $|x-y| \leq \frac{x_d \wedge y_d}{2}$ and $1 \leq x_d \vee y_d$. Notice that

$$x_d \sim y_d \gtrsim 1.$$

We decompose $H_j(x,y,1)$ as follows:

$$\begin{aligned} H_j(x,y,1) &= \int_{\{z \in \mathbb{R}^d_+ : z_d > x_d/2\}} \partial_t^j W^\alpha_{0,t/2}(x,z)\Big|_{t=1} z_d^{-\alpha} \partial_t^{k-1-j} W^\alpha_{\lambda,t/2}(z,y)\Big|_{t=1} dz \\ &\quad + \int_{\{z \in \mathbb{R}^d_+ : z_d \leq x_d/2\}} \partial_t^j W^\alpha_{0,t/2}(x,z)\Big|_{t=1} z_d^{-\alpha} \partial_t^{k-1-j} W^\alpha_{\lambda,t/2}(z,y)\Big|_{t=1} dz \\ &:= H_{j,1}(x,y,1) + H_{j,2}(x,y,1). \end{aligned}$$

By using Proposition 2.2 (a) and (2.1), for $\varepsilon \in (0,1)$, we have

$$\begin{aligned} |H_{j,1}(x,y,1)| \lesssim x_d^{-\alpha} \int_{\mathbb{R}^d_+} &\left(\frac{x_d}{x_d+1} \right)^{(\alpha-1)_+} \left(\frac{z_d}{z_d+1} \right)^{(\alpha-1)_+} \left(\frac{1}{1+|x-z|} \right)^{(d+\alpha)(1-\varepsilon)} \\ &\times \left(\frac{y_d}{y_d+1} \right)^{(\alpha-1)_+} \left(\frac{z_d}{z_d+1} \right)^{(\alpha-1)_+} \left(\frac{1}{1+|y-z|} \right)^{(d+\alpha)(1-\varepsilon)} dz. \end{aligned}$$

We define $\alpha' = \alpha(1-\varepsilon) - d\varepsilon$ and consider $0 < \varepsilon < \alpha/(\alpha+d)$. Then, $0 < \alpha' < \alpha < 2$, and we get

$$\begin{aligned} |H_{j,1}(x,y,1)| \lesssim x_d^{-\alpha} \int_{\mathbb{R}^d_+} &\left(\frac{x_d}{x_d+1} \right)^{(\alpha'-1)_+} \left(\frac{z_d}{z_d+1} \right)^{2(\alpha'-1)_+} \left(\frac{y_d}{y_d+1} \right)^{(\alpha'-1)} \\ &\times \left(\frac{1}{1+|x-z|} \right)^{d+\alpha'} \left(\frac{1}{1+|y-z|} \right)^{d+\alpha'} dz. \end{aligned}$$

According to [22, Theorem 9], we have

$$\begin{aligned}|H_{j,1}(x,y,1)| &\lesssim x_d^{-\alpha}\int_{\mathbb{R}^d_+} W^{\alpha'}_{0,1}(x,z)W^{\alpha'}_{0,1}(z,y)\,dz\\ &\sim x_d^{-\alpha}W^{\alpha'}_{0,2}(x,y)\\ &\lesssim x_d^{-\alpha}\left(\frac{x_d}{x_d+1}\right)^{(\alpha'-1)_+}\left(\frac{y_d}{y_d+1}\right)^{(\alpha'-1)}\left(\frac{1}{1+|x-y|}\right)^{d+\alpha'}.\end{aligned}$$

We now estimate $H_{j,2}$. If $z_d \le x_d/2$, since $|x-y| < (x_d \wedge y_d)/2$, we deduce $|y_d - z_d| \sim y_d \sim x_d$ and $|x_d - z_d| \sim x_d$. Then, if $z_d \le x_d/2$, we have $|x-z| \sim |x'-z'| + x_d$ and $|y-z| \sim |y'-z'| + x_d$. By [22, Lemma 22] with $N = d-1 \ge 1$, $\beta = \alpha(1-\varepsilon) + 1 - d\varepsilon$ and $r = s = 1 + x_d$, and using Proposition 2.2 (a), we get that whenever $z_d \le x_d/2$,

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}} \left|\partial_t^j W^{\alpha}_{0,t/2}(x,z)\big|_{t=1}\right|\left|\partial_t^{k-1-j} W^{\alpha}_{\lambda,t/2}(z,y)\big|_{t=1}\right| dz'\\ &\lesssim \left(\frac{x_d}{x_d+1}\right)^{(\alpha-1)_+}\left(\frac{z_d}{z_d+1}\right)^{(\alpha-1)_++\mathfrak{p}}\left(\frac{y_d}{y_d+1}\right)^{\mathfrak{p}}\\ &\quad\times\int_{\mathbb{R}^{d-1}}\left(\frac{1}{1+|x'-z'|+x_d}\right)^{(d+\alpha)(1-\varepsilon)}\left(\frac{1}{1+|y'-z'|+x_d}\right)^{(d+\alpha)(1-\varepsilon)} dz'\\ &\lesssim \left(\frac{x_d}{x_d+1}\right)^{(\alpha-1)_+}\left(\frac{z_d}{z_d+1}\right)^{(\alpha-1)_++\mathfrak{p}}\left(\frac{y_d}{y_d+1}\right)^{\mathfrak{p}}\frac{1}{x_d^{\alpha(1-\varepsilon)+1-d\varepsilon}}\\ &\quad\times\frac{1}{(1+x_d+|y'-x'|)^{(d+\alpha)(1-\varepsilon)}}\\ &\lesssim \left(\frac{x_d}{x_d+1}\right)^{(\alpha-1)_+}\left(\frac{z_d}{z_d+1}\right)^{(\alpha-1)_++\mathfrak{p}}\left(\frac{y_d}{y_d+1}\right)^{\mathfrak{p}}\frac{1}{x_d^{\alpha(1-\varepsilon)+1-d\varepsilon+(d+\alpha)(1-\varepsilon)}}.\end{aligned}$$

Since $x_d \gtrsim 1$, from [22, p. 26], we also have that

$$\int_0^{x_d/2}\left(\frac{z_d}{z_d+1}\right)^{(\alpha-1)_++\mathfrak{p}}\frac{dz_d}{z_d^{\alpha}} \sim \mathbf{1}_{\alpha\ge 1} + (\ln(1+x_d))\mathbf{1}_{\alpha=1} + x_d^{1-\alpha}\mathbf{1}_{\alpha<1}.$$

Then,

$$\begin{aligned}|H_{j,2}(x,y,1)| &\lesssim \frac{1}{x_d^{\alpha(1-\varepsilon)+1-d\varepsilon+(d+\alpha)(1-\varepsilon)}}\int_0^{x_d/2}\left(\frac{z_d}{z_d+1}\right)^{(\alpha-1)_++\mathfrak{p}}\frac{dz_d}{z_d^{\alpha}}\\ &\lesssim \frac{1}{x_d^{\alpha+(d+\alpha)(1-\varepsilon)}}\frac{1}{x_d^{1-(d+\alpha)\varepsilon}}\left(\mathbf{1}_{\alpha\ge 1} + (\ln(1+x_d))\mathbf{1}_{\alpha=1} + x_d^{1-\alpha}\mathbf{1}_{\alpha<1}\right)\\ &\lesssim \frac{1}{x_d^{\alpha+(d+\alpha)(1-\varepsilon)}} \lesssim \frac{1}{x_d^{\alpha}}\left(\frac{1}{1+|x-y|}\right)^{(d+\alpha)(1-\varepsilon)},\end{aligned}$$

provided that $0 < \varepsilon < \alpha/(d+\alpha)$. ∎

We consider now

$$H_k(x,y,t) = \int_0^{t/2}\int_{\mathbb{R}^d_+} \partial_t^k W^\alpha_{0,t-s}(x,z) z_d^{-\alpha} W^\alpha_{\lambda,s}(z,y)\,dz\,ds, \quad x,y\in\mathbb{R}^d_+,\ t>0,$$

with $k\in\mathbb{N}$.

As above, according to the scaling property (3.5), we estimate $H_k(x,y,1)$ when

$$|x-y| \le \frac{x_d \wedge y_d}{2} \quad \text{and} \quad 1 \le x_d \vee y_d.$$

Lemma 3.3. *Let $k\in\mathbb{N}$, $\lambda\ge 0$, and $\alpha\in(0,2)$. We have that*

$$H_k(x,y,1) \lesssim \frac{1}{x_d^\alpha}\left(\frac{1}{1+|x-y|}\right)^{(d+\alpha)(1-\varepsilon)}, \quad |x-y|\le \frac{x_d\wedge y_d}{2},\ 1\le x_d\vee y_d, \tag{3.7}$$

with $0<\varepsilon<\alpha/(d+\alpha)$.

Proof. Assume that $x,y\in\mathbb{R}^d$, $|x-y|\le\frac{x_d\wedge y_d}{2}$ and $1\le x_d\vee y_d$. Hence,

$$x_d \sim y_d \gtrsim 1.$$

First, we estimate the integrand in $H_k(x,y,1)$ by using (2.1) and Proposition 2.2 (a). Recalling that $\mathfrak{p}\ge(\alpha-1)_+$, we get

$$\begin{aligned}
&\left|\partial_t^k W^\alpha_{0,t-s}(x,z)\big|_{t=1} z_d^{-\alpha} W^\alpha_{\lambda,s}(z,y)\right| \\
&\lesssim (1-s)^{-k-d/\alpha}\left(\frac{x_d}{(1-s)^{1/\alpha}+x_d}\right)^{(\alpha-1)_+}\left(\frac{z_d}{(1-s)^{1/\alpha}+z_d}\right)^{(\alpha-1)_+} \\
&\quad\times\left(\frac{(1-s)^{1/\alpha}}{(1-s)^{1/\alpha}+|x-z|}\right)^{(d+\alpha)(1-\varepsilon)} z_d^{-\alpha}\left(\frac{y_d}{s^{1/\alpha}+y_d}\right)^{\mathfrak{p}}\left(\frac{z_d}{s^{1/\alpha}+z_d}\right)^{\mathfrak{p}}\frac{1}{s^{d/\alpha}} \\
&\quad\times\left(\frac{s^{1/\alpha}}{s^{1/\alpha}+|z-y|}\right)^{(d+\alpha)(1-\varepsilon)}.
\end{aligned}\tag{3.8}$$

We decompose $H_k(x,y,1)$ in the following way:

$$\begin{aligned}
H_k(x,y,1) &= \int_0^{1/2}\int_{\{z\in\mathbb{R}^d_+ : z_d > x_d/2\}} \partial_t^k W^\alpha_{0,t-s}(x,z)\Big|_{t=1} z_d^{-\alpha}W^\alpha_{\lambda,s}(z,y)\,dz\,ds \\
&\quad+ \int_0^{1/2}\int_{\{z\in\mathbb{R}^d_+ : z_d \le x_d/2\}} \partial_t^k W^\alpha_{0,t-s}(x,z)\Big|_{t=1} z_d^{-\alpha}W^\alpha_{\lambda,s}(z,y)\,dz\,ds \\
&:= H_{k,1}(x,y,t) + H_{k,2}(x,y,t), \quad x,y\in\mathbb{R}^d_+,\ t>0.
\end{aligned}$$

We define, as before, $\alpha' = \alpha(1-\varepsilon) - d\varepsilon$ for $0 < \varepsilon < \alpha/(d+\alpha)$, so $0 < \alpha' < \alpha < 2$. Thus, from Proposition 2.1 (a) with $\lambda = 0$, we have

$$
\begin{aligned}
&|H_{k,1}(x,y,1)| \\
&\lesssim \frac{1}{x_d^\alpha}\int_0^{1/2}\int_{\mathbb{R}^d_+}\left(\frac{x_d}{(1-s)^{1/\alpha}+x_d}\right)^{(\alpha'-1)_+}\left(\frac{z_d}{(1-s)^{1/\alpha}+z_d}\right)^{(\alpha'-1)_+} \\
&\quad\times\left(\frac{y_d}{s^{1/\alpha}+y_d}\right)^{(\alpha'-1)_+}\left(\frac{z_d}{s^{1/\alpha}+z_d}\right)^{(\alpha'-1)_+}\frac{1}{s^{d/\alpha}}\left(\frac{s^{1/\alpha}}{s^{1/\alpha}+|z-y|}\right)^{d+\alpha'} \\
&\quad\times\frac{1}{(1-s)^{d/\alpha}}\left(\frac{(1-s)^{1/\alpha}}{(1-s)^{1/\alpha}+|x-z|}\right)^{d+\alpha'}\, dzds \\
&\lesssim \frac{1}{x_d^\alpha}\int_0^{1/2}\int_{\mathbb{R}^d_+}\left(\frac{x_d}{(1-s)^{1/\alpha'}+x_d}\right)^{(\alpha'-1)_+}\left(\frac{z_d}{(1-s)^{1/\alpha'}+z_d}\right)^{(\alpha'-1)_+} \\
&\quad\times\frac{1}{s^{d/\alpha'}}\left(\frac{y_d}{s^{1/\alpha'}+y_d}\right)^{(\alpha'-1)_+}\left(\frac{z_d}{s^{1/\alpha'}+z_d}\right)^{(\alpha'-1)_+} \\
&\quad\times\left(\frac{(1-s)^{1/\alpha'}}{(1-s)^{1/\alpha'}+|x-z|}\right)^{d+\alpha'}\left(\frac{s^{1/\alpha'}}{s^{1/\alpha'}+|z-y|}\right)^{d+\alpha'}\, dzds \\
&\sim \frac{1}{x_d^\alpha}\int_0^{1/2}\int_{\mathbb{R}^d_+} W^{\alpha'}_{0,1-s}(x,z)W^{\alpha'}_{0,s}(z,y)\, dzds \\
&\sim \frac{1}{x_d^\alpha} W^{\alpha'}_{0,1}(x,y) \\
&\lesssim \frac{1}{x_d^\alpha}\left(\frac{x_d}{1+x_d}\right)^{(\alpha'-1)_+}\left(\frac{y_d}{1+y_d}\right)^{(\alpha'-1)_+}\left(\frac{1}{1+|x-y|}\right)^{\alpha'+d},
\end{aligned}
$$

where we used the fact that the function $\alpha \mapsto s^{-d/\alpha}(\frac{s^{1/\alpha}}{s^{1/\alpha}+u})^{d+\alpha'}$ is increasing for every $s \in (0,1)$ and $u > 0$.

In order to estimate $H_{k,2}$, observe that

$$
\begin{aligned}
&\int_{\mathbb{R}^{d-1}}\left(\frac{(1-s)^{1/\alpha}}{(1-s)^{1/\alpha}+|z-x|}\right)^{(d+\alpha)(1-\varepsilon)}\left(\frac{s^{1/\alpha}}{s^{1/\alpha}+|z-y|}\right)^{(d+\alpha)(1-\varepsilon)} dz' \\
&\lesssim \int_{\mathbb{R}^{d-1}}\left[\left(\frac{(1-s)^{1/\alpha}}{(1-s)^{1/\alpha}+x_d+|z'-x'|}\right)\left(\frac{s^{1/\alpha}}{s^{1/\alpha}+x_d+|z'-y'|}\right)\right]^{(d+\alpha)(1-\varepsilon)} dz' \\
&\lesssim (s(1-s))^{(d+\alpha)(1-\varepsilon)/\alpha}\int_{\mathbb{R}^{d-1}}\left(\frac{1}{(x_d+|z'-x'|)(x_d+|z'-y'|)}\right)^{(d+\alpha)(1-\varepsilon)} dz' \\
&\lesssim \frac{((1-s)s)^{(d+\alpha)(1-\varepsilon)/\alpha}}{x_d^{\alpha(1-\varepsilon)-d\varepsilon+1}}\frac{1}{(x_d+|x'-y'|)^{(d+\alpha)(1-\varepsilon)}},
\end{aligned}
$$

provided that $z_d \le x_d/2$, where we have used [22, Lemma 22] with $N = d-1$ and $\beta = (d+\alpha)(1-\varepsilon) - d + 1$.

Then, by taking $0 < \varepsilon < (\alpha - \mathfrak{p})/(d+\alpha)$ (which is possible since $\mathfrak{p} < \alpha$), from (3.8) we get

$$\begin{aligned}
|H_{k,2}(x,y,1)| &\lesssim \int_0^{1/2}\int_0^{x_d/2}\left(\frac{x_d}{(1-s)^{1/\alpha}+x_d}\right)^{(\alpha-1)_+}\left(\frac{z_d}{(1-s)^{1/\alpha}+z_d}\right)^{(\alpha-1)_+}\\
&\qquad\times z_d^{-\alpha}\left(\frac{y_d}{s^{1/\alpha}+y_d}\right)^{\mathfrak{p}}\left(\frac{z_d}{s^{1/\alpha}+z_d}\right)^{\mathfrak{p}}\frac{1}{s^{d/\alpha}}\\
&\qquad\times \frac{((1-s)s)^{(d+\alpha)(1-\varepsilon)/\alpha}}{x_d^{\alpha(1-\varepsilon)-d\varepsilon+1}}\frac{1}{(x_d+|x'-y'|)^{(d+\alpha)(1-\varepsilon)}}\,dz_d\,ds\\
&\lesssim \int_0^{x_d/2}\left(\int_0^{1/2}\left(\frac{z_d}{(1-s)^{1/\alpha}+z_d}\right)^{(\alpha-1)_+}\left(\frac{z_d}{s^{1/\alpha}+z_d}\right)^{\mathfrak{p}}\right.\\
&\qquad\qquad\left.\times\frac{((1-s)s)^{(d+\alpha)(1-\varepsilon)/\alpha}s^{-d/\alpha}}{(x_d+|x'-y'|)^{(d+\alpha)(1-\varepsilon)}}\,ds\right)\frac{z_d^{-\alpha}}{x_d^{\alpha(1-\varepsilon)-d\varepsilon+1}}\,dz_d\\
&\lesssim \int_0^{x_d/2}\frac{z_d^{-\alpha}}{x_d^{\alpha(1-\varepsilon)-d\varepsilon+1+(d+\alpha)(1-\varepsilon)}}(1\wedge z_d)^{\mathfrak{p}+(\alpha-1)_+}\,dz_d.
\end{aligned}$$

Here, we have used the fact that, for the values of ε indicated above,

$$\begin{aligned}
s^{(d+\alpha)(1-\varepsilon)/\alpha-d/\alpha}\left(1\wedge\frac{z_d}{s^{1/\alpha}}\right)^{\mathfrak{p}} &= \left(s^{(1-\varepsilon(d+\alpha)/\alpha)/\mathfrak{p}}\wedge\left(s^{(1-\varepsilon(d+\alpha)/\alpha)/\mathfrak{p}-1/\alpha}\right)z_d\right)^{\mathfrak{p}}\\
&\lesssim \left(1\wedge s^{(\alpha-\mathfrak{p}-\varepsilon(d+\alpha))/(\alpha\mathfrak{p})}z_d\right)^{\mathfrak{p}} \le (1\wedge z_d)^{\mathfrak{p}}.
\end{aligned}$$

Finally, since $0 < \varepsilon < \alpha/(d+\alpha)$, by proceeding as in the proof of Lemma 3.2, we get

$$|H_{k,2}(x,y,1)| \lesssim \frac{1}{x_d^\alpha}\left(\frac{1}{1+|x-y|}\right)^{(d+\alpha)(1-\varepsilon)}. \qquad \blacksquare$$

By proceeding as in the proof of the last result, we can also see that

$$|\widetilde{H}_k(x,y,1)| \lesssim \frac{1}{x_d^\alpha}\left(\frac{1}{1+|x-y|}\right)^{(d+\alpha)(1-\varepsilon)},\ |x-y|\le\frac{x_d\wedge y_d}{2},\ 1\le x_d\vee y_d, \tag{3.9}$$

where

$$\widetilde{H}_k(x,y,t) = \int_0^{t/2}\int_{\mathbb{R}^d_+} W^\alpha_{0,s}(x,z)z_d^{-\alpha}\partial_t^k W^\alpha_{\lambda,t-s}(z,y)\,dz\,ds,\quad x,y\in\mathbb{R}^d_+, t>0.$$

By putting together (3.6), (3.7), and (3.9), and using the scaling property (3.5), we obtain

$$\left|t^{k-1}\partial_t^k D^{\alpha,0}_{\lambda,t}(x,y)\right| \lesssim \frac{t^{-d/\alpha}}{(x_d\vee y_d)^\alpha}\left(\frac{t^{1/\alpha}}{t^{1/\alpha}+|x-y|}\right)^{(d+\alpha)(1-\varepsilon)},$$

for $0<\varepsilon<(\alpha-\mathfrak{p})/(d+\alpha)$.

Consequently, recalling the definition of the local part L in (1.3), for certain constant $C > 1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d_+}\int_0^\infty \left|t^{k-1}\partial_t^k D^{\alpha,0}_{\lambda,t}(x,y)\right|\chi_L(x,y,t)\,dt\,dy \\
&\lesssim \int_{\{y\in\mathbb{R}^d_+ : C^{-1}x_d\le y_d\le Cx_d\}}\int_0^{(x_d\vee y_d)^\alpha} \frac{t^{-d/\alpha}}{(x_d\vee y_d)^\alpha}\left(\frac{t^{1/\alpha}}{t^{1/\alpha}+|x-y|}\right)^{(d+\alpha)(1-\varepsilon)} dt\,dy \\
&\lesssim \int_0^{(Cx_d)^\alpha} t^{-d/\alpha}\int_{\{y\in\mathbb{R}^d_+ : C^{-1}x_d\le y_d\le Cx_d\}} x_d^{-\alpha}\left(\frac{t^{1/\alpha}}{t^{1/\alpha}+|x-y|}\right)^{(d+\alpha)(1-\varepsilon)} dy\,dt \\
&\lesssim x_d^{-\alpha}\int_0^{(Cx_d)^\alpha} t^{-d/\alpha}\int_{\mathbb{R}^d}\left(\frac{t^{1/\alpha}}{t^{1/\alpha}+|x-y|}\right)^{(d+\alpha)(1-\varepsilon)} dy\,dt \\
&\lesssim x_d^{-\alpha}\int_0^{(Cx_d)^\alpha}\int_{\mathbb{R}^d}\frac{1}{(1+|z|)^{(d+\alpha)(1-\varepsilon)}}\,dz\,dt \\
&\lesssim 1,
\end{aligned}$$

for each $x \in \mathbb{R}^d_+$, provided that $0 < \varepsilon < (\alpha - \mathfrak{p})/(d+\alpha)$.

By symmetry, we also deduce that

$$\int_{\mathbb{R}^d_+}\int_0^\infty \left|t^{k-1}\partial_t^k D^{\alpha,0}_{\lambda,t}(x,y)\right|\chi_L(x,y,t)\,dt\,dx \lesssim 1, \quad y\in\mathbb{R}^d_+,$$

when $0 < \varepsilon < \alpha/(d+\alpha)$.

Schur's test allows us to conclude that the integral operator

$$\mathbb{J}^{\alpha,k}_\lambda(f)(x) := \int_{\mathbb{R}^d_+}\int_0^\infty \left|t^{k-1}\partial_t^k D^{\alpha,0}_{\lambda,t}(x,y)\right|\chi_L(x,y,t)f(y)\,dt\,dy, \quad x\in\mathbb{R}^d_+,$$

is bounded on $L^p(\mathbb{R}^d_+)$ for every $1 \le p \le \infty$.

4. Proof of Theorem 1.1 for $\alpha = 2$

4.1. Global part

For the case $\alpha = 2$, we know, by Proposition 2.2 (b), that

$$\begin{aligned}
\left|t^k\partial_t^k W^2_{\lambda,t}(x,y)\right| &\lesssim \left(1\wedge\frac{x_d}{t^{1/2}}\right)^{\mathfrak{p}}\left(1+\frac{y_d}{t^{1/2}}\right)^{\mathfrak{p}}\frac{1}{t^{d/2}}e^{-\frac{c|x-y|^2}{t}} \\
&\lesssim \left(1\wedge\frac{x_d}{t^{1/2}}\right)^{\mathfrak{p}}\left(1+\frac{y_d}{t^{1/2}}\right)^{\mathfrak{p}}\frac{1}{t^{d/2}}\left(\frac{t^{1/2}}{t^{1/2}+|x-y|}\right)^{d+2},
\end{aligned}$$

for any $x, y \in \mathbb{R}^d_+, t > 0$.

With this estimate, we can proceed as in the case $\alpha \in (0,2)$ (see Section 3.1) and prove that, for every $k \in \mathbb{N}$, $k \geq 1$, the operator $J_\lambda^{2,k}$ given by

$$J_\lambda^{2,k}(f)(x) = \int_{\mathbb{R}_+^d} J_\lambda^{2,k}(x,y) f(y)\, dy, \quad x \in \mathbb{R}_+^d,$$

whose kernel $J_\lambda^{2,k}(x,y)$ is as in (3.1) with $\alpha = 2$, is bounded on $L^p(\mathbb{R}_+^d)$ for every $1 < p < \infty$.

4.2. Local part

When dealing with the local part and $\alpha = 2$, we keep the notation used in Section 3.2. We recall that in the local part, we are considering $x, y \in \mathbb{R}_+^d$ with $|x-y| \leq (x_d \wedge y_d)/2$ and $0 < t^{1/\alpha} \leq x_d \vee y_d$.

Lemma 4.1. *Let $j = 0, \dots, k-1$, with $k \in \mathbb{N}$ and $\lambda \geq 0$. Then,*

$$|H_j(x,y,1) \lesssim \frac{1}{x_d^\alpha}\left(\frac{1}{1+|x-y|}\right)^{d+2}, \quad |x-y| \leq \frac{x_d \wedge y_d}{2},\ 1 \leq x_d \vee y_d. \tag{4.1}$$

Proof. Assume that $|x-y| \leq \frac{x_d \wedge y_d}{2}$ and $1 \leq x_d \vee y_d$. Then, $1 \leq x_d \sim y_d$. According to Proposition 2.2 (b), we get the following estimate:

$$\begin{aligned}
|H_{j,1}(x,y,1)| &\lesssim x_d^{-2} \int_{\mathbb{R}_+^d} \left(\frac{x_d}{1+x_d}\right)\left(\frac{z_d}{1+z_d}\right)^{1+\mathfrak{p}}\left(\frac{y_d}{1+y_d}\right)^{\mathfrak{p}} e^{-c(|x-z|^2+|z-y|^2)}\, dz \\
&\lesssim x_d^{-2} \int_{\mathbb{R}_+^d} \left(\frac{x_d}{1+x_d}\right)\left(\frac{z_d}{1+z_d}\right)^{1+\mathfrak{p}}\left(\frac{y_d}{1+y_d}\right)^{\mathfrak{p}} e^{-\frac{c}{3}|x-y|^2} e^{-\frac{c}{3}|z-y|^2}\, dz \\
&\lesssim x_d^{-2} \left(\frac{x_d}{1+x_d}\right)\left(\frac{y_d}{1+y_d}\right)^{\mathfrak{p}} e^{-\frac{c}{3}|x-y|^2} \int_{\mathbb{R}^d} e^{-\frac{c}{3}|x-z|^2}\, dz \\
&\lesssim x_d^{-2} e^{-\frac{c}{3}|x-y|^2}.
\end{aligned}$$

Therefore, we get

$$|H_{j,1}(x,y,1)| \lesssim \frac{1}{x_d^2}\left(\frac{1}{1+|x-y|}\right)^{d+2}.$$

Similarly, we get

$$|H_{j,2}(x,y,1)| \lesssim \frac{1}{x_d^2}\left(\frac{1}{1+|x-y|}\right)^{d+2}. \qquad \blacksquare$$

Lemma 4.2. *Let $k \in \mathbb{N}$ and $\lambda \geq 0$. Then,*

$$|H_k(x,y,1)| \lesssim \frac{1}{x_d^2}\left(\frac{1}{1+|x-y|^2}\right)^{d+2}, \quad |x-y| \leq \frac{x_d \wedge y_d}{2},\ 1 \leq x_d \vee y_d. \tag{4.2}$$

Proof. Suppose that $|x-y| \le \frac{x_d \wedge y_d}{2}$ and $1 \le x_d \vee y_d$. Proposition 2.2 (b) leads to

$$
\begin{aligned}
&|H_{k,1}(x,y,1)| \\
&\lesssim \int_0^{1/2} \int_{\{z\in\mathbb{R}^d_+ : z_d > x_d/2\}} (1-s)^{-k-d/2} \left(\frac{x_d}{(1-s)^{1/2}+x_d} \right) \left(\frac{z_d}{(1-s)^{1/2}+z_d} \right) \\
&\quad \times z_d^{-2} \left(\frac{y_d}{s^{1/2}+y_d} \right)^{\mathfrak{p}} \left(\frac{z_d}{s^{1/2}+z_d} \right)^{\mathfrak{p}} \frac{1}{s^{d/2}} e^{-c\frac{|x-z|^2}{1-s} - c\frac{|z-y|^2}{s}} \, dz ds \\
&\lesssim x_d^{-2} \int_0^{1/2} \frac{x_d}{(1-s)^{1/2}+x_d} \left(\frac{y_d}{s^{1/2}+y_d} \right)^{\mathfrak{p}} e^{-\frac{c}{3}\frac{|x-y|^2}{1-s}} \int_{\mathbb{R}^d} \frac{e^{-\frac{c}{3}\frac{|y-z|^2}{s}}}{s^{d/2}} \, dz ds \\
&\lesssim x_d^{-2} \int_0^{1/2} \frac{x_d}{(1-s)^{1/2}+x_d} \left(\frac{y_d}{s^{1/2}+y_d} \right)^{\mathfrak{p}} e^{-\frac{c}{3}\frac{|x-y|^2}{1-s}} \, ds \\
&\lesssim x_d^{-2} e^{-\frac{c}{3}|x-y|^2}.
\end{aligned}
\tag{4.3}
$$

We now study $H_{k,2}$. Using again Proposition 2.2 (b), it yields, as before,

$$
\begin{aligned}
&|H_{k,2}(x,y,1)| \\
&\lesssim \int_0^{1/2} \int_{\{z\in\mathbb{R}^d_+ : z_d \le x_d/2\}} (1-s)^{-k-d/2} \left(\frac{x_d}{(1-s)^{1/2}+x_d} \right) \left(\frac{z_d}{(1-s)^{1/2}+z_d} \right) \\
&\quad \times z_d^{-2} \left(\frac{y_d}{s^{1/2}+y_d} \right)^{\mathfrak{p}} \left(\frac{z_d}{s^{1/2}+z_d} \right)^{\mathfrak{p}} \frac{1}{s^{d/2}} e^{-c\frac{|x-z|^2}{1-s} - c\frac{|z-y|^2}{s}} \, dz ds.
\end{aligned}
$$

Note that, for each $s \in (0, \frac{1}{2})$, we can write

$$
\int_{\mathbb{R}^{d-1}} e^{-\frac{c}{2}\frac{|x'-z'|^2}{1-s} - \frac{c}{2}\frac{|z'-y'|^2}{s}} \, dz' \lesssim \int_{\mathbb{R}^{d-1}} e^{-\frac{c}{2}\frac{|w|^2}{s}} \, dw \lesssim s^{(d-1)/2}.
\tag{4.4}
$$

Since $s \le 1-s$, we have

$$
\begin{aligned}
\frac{|x-z|^2}{1-s} + \frac{|z-y|^2}{s} &\ge \frac{|x-z|^2+|z-y|^2}{2(1-s)} + \frac{|x-z|^2}{2(1-s)} + \frac{|z-y|^2}{2s} \\
&\ge \frac{|x-y|^2}{4(1-s)} + \frac{|x'-z'|^2}{2(1-s)} + \frac{|z'-y'|^2}{2s} + \frac{|x_d-z_d|^2}{2(1-s)} + \frac{|z_d-y_d|^2}{2s}.
\end{aligned}
$$

Using this inequality together with (4.4), we get

$$
\begin{aligned}
&\int_{\{z\in\mathbb{R}^d_+ : z_d \le x_d/2\}} \left(\frac{z_d}{(1-s)^{1/2}+z_d} \right) \left(\frac{z_d}{s^{1/2}+z_d} \right)^{\mathfrak{p}} z_d^{-2} e^{-c\frac{|x-z|^2}{1-s} - c\frac{|z-y|^2}{s}} \, dz \\
&\quad \lesssim \int_0^{x_d/2} \left(\frac{z_d}{(1-s)^{1/2}+z_d} \right) \left(\frac{z_d}{s^{1/2}+z_d} \right)^{\mathfrak{p}} z_d^{-2} s^{(d-1)/2} e^{-\frac{c}{4}\frac{|x-y|^2}{1-s} - c\frac{|z_d-y_d|^2}{2s}} \, dz_d.
\end{aligned}
$$

Combining the above estimates, we obtain

$$
\begin{aligned}
|H_{k,2}(x,y,1)| &\lesssim \int_0^{x_d/2}\int_0^{1/2} s^{(d-1)/2}\left(\frac{z_d}{(1-s)^{1/2}+z_d}\right)\left(\frac{z_d}{s^{1/2}+z_d}\right)^{\mathfrak{p}} z_d^{-2} \\
&\quad \times \left(\frac{(1-s)^{1/2}}{(1-s)^{1/2}+x_d+|y'-x'|}\right)^{d+4} ds dz_d \\
&\lesssim \int_0^{x_d/2}\int_0^{1/2} s^{(d-1)/2}\left(1\wedge\frac{z_d}{(1-s)^{1/2}}\right)\left(1\wedge\frac{z_d}{s^{1/2}}\right)^{\mathfrak{p}} ds \\
&\quad \times \frac{dz_d}{z_d^2}\left(\frac{1}{x_d+|y'-x'|}\right)^{d+4} \\
&\lesssim \int_0^{x_d/2}\int_0^{1/2} s^{(d-1)/2-1/4}(1\wedge z_d)^{1+1/4}\frac{dz_d}{z_d^2}\frac{1}{x_d^{d+4}} \\
&\lesssim \left(\int_0^1 z_d^{-3/4} dz_d + \int_1^\infty \frac{dz_d}{z_d^2}\right)\frac{1}{x_d^{d+4}} \\
&\lesssim \frac{1}{x_d^{d+4}} \\
&\lesssim \frac{1}{x_d^2}\left(\frac{1}{1+|x-y|}\right)^{d+2}. \qquad (4.5)
\end{aligned}
$$

From (4.3) and (4.5), we deduce (4.2). ■

Using the arguments from the last proof, we can also see that

$$
|\widetilde{H}_k(x,y,1)| \lesssim \frac{1}{x_d^2}\left(\frac{1}{1+|x-y|}\right)^{d+2}, \quad x_d \geq 1. \qquad (4.6)
$$

We now put (4.1), (4.2), and (4.6) together obtaining

$$
\left|\partial_t^k D_\lambda^{2,0}(x,y)\big|_{t=1}\right| \lesssim \frac{1}{x_d^2}\left(\frac{1}{1+|x-y|}\right)^{d+2}.
$$

Therefore,

$$
\left|t^{k-1}\partial_t^k D_\lambda^{2,0}(x,y)\right| \lesssim \frac{1}{t^{d/2}}\left(\frac{t^{1/2}}{t^{1/2}+|x-y|}\right)^{(d+2)(1-\varepsilon)}\frac{1}{(x_d\vee y_d)^2},
$$

with $0<\varepsilon<(2-\mathfrak{p})/(d+2)=(\alpha-\mathfrak{p})/(d+\alpha)$.

By proceeding as in the case $\alpha\in(0,2)$ (see Section 3.2), we can prove that the operator $\mathbb{J}_\lambda^{2,k}$ is bounded on $L^p(\mathbb{R}_+^d)$ for every $1\leq p\leq\infty$ and thus conclude the proof of Theorem 1.1 for $\alpha=2$.

5. Proof of Theorem 1.2

We proceed as in the proof of [38, Proposition 3.5] and of the weighted property in [2, Theorem 1.1] by using [11, Proposition 2.3] (see also [5, Theorem 6.6]). We define, for every $t > 0$,

$$A^{\alpha}_{\lambda,t} = (I - W^{\alpha}_{\lambda,t})^m,$$

for $m \in \mathbb{N}$ to be fixed later.

For every $x \in \mathbb{R}^d$, $r, s > 0$, let $B(x,r) = \{y \in \mathbb{R}^d : |x-y| < r\}$ be the usual ball in $\mathbb{R}^d$, $sB(x,r) = B(x,sr)$ and $\mathcal{B}(x,r) = B(x,r) \cap \mathbb{R}^d_+$. Also, let us consider, for a ball $B \subset \mathbb{R}^d$, the sets $S_0(B) = B$ and for $j \in \mathbb{N}$, $j \geq 1$, $S_j(B) = 2^j B \setminus 2^{j-1} B$. We denote also, for $B \subset \mathbb{R}^d$ and $j \in \mathbb{N}$, $S_j(\mathcal{B}) = S_j(B) \cap \mathbb{R}^d_+$.

From now on, we will assume that $1 \leq p < q < \infty$. Suppose that $\mathcal{B} = B \cap \mathbb{R}^d_+$, where $B = B(x_B, r_B)$ with $x_B \in \mathbb{R}^d_+$, $r_B > 0$ and that f is a smooth function supported in $\mathcal{B}$.

We consider first the case $\alpha = 2$. By using [2, (2.17)], from Proposition 2.2 (b), we deduce

$$\left(\frac{1}{|S_j(\mathcal{B})|}\int_{S_j(\mathcal{B})} |A^2_{\lambda,r_B^2}(f)(x)|^q\,dx\right)^{1/q} \leq C\frac{e^{-c2^{2j}}}{2^{j\alpha/q}}\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} |f(x)|^p\,dx\right)^{1/p},$$

for $j \in \mathbb{N}$, where $C > 0$ does not depend on j, β, or f.

On the other hand, the arguments in [2, pp. 13 and 14] allow us to obtain that

$$\left(\frac{1}{|S_j(\mathcal{B})|}\int_{S_j(\mathcal{B})} |V_\rho(\{t^k\partial_t^k W^w_{\lambda,t}\}_{t>0})(I - A^2_{\lambda,r_B^2})(f)(x)|^q\,dx\right)^{1/q} \lesssim 2^{-j(dq+2m)} f_{q,\mathcal{B}},$$

where $f_{q,\mathcal{B}} = (\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} |f(x)|^q\,dx)^{1/q}$ and the constant does not depend on j, $\mathcal{B}$, or f.

We choose now $m \in \mathbb{N}$ such that $m > d/2$. Then,

$$\sum_{j=1}^{\infty} 2^{-j(d/q+2m-d)} < \infty.$$

According to Theorem 1.1, the ρ-variation operator $\mathcal{V}_\rho(\{t^k\partial_t^k W^2_{\lambda,t}\}_{t>0})$ is bounded on $L^q(\mathbb{R}^d_+)$.

If $1 < q < \infty$, we say that a weight w belongs to the class $\mathrm{RH}_q(\mathbb{R}^d_+)$ if there exists $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B w^q(x)\,dx\right)^{1/q} \leq C\frac{1}{|B|}\int_B w(x)\,dx$$

for every ball B in $\mathbb{R}^d_+$.

By using [11, Proposition 2.3], we conclude that $\mathcal{V}_\rho(\{t^k\partial_t^k W^2_{\lambda,t}\}_{t>0})$ is bounded on $L^r(\mathbb{R}^d_+, w)$, provided that $p < r < q$ and $w \in A_{r/p}(\mathbb{R}^d_+) \cap \mathrm{RH}_{(q/r)'}(\mathbb{R}^d_+)$. As usual, in this context, for $1 < \sigma < \infty$, we denote by σ' the conjugate exponent, defined by $\frac{1}{\sigma} + \frac{1}{\sigma'} = 1$. By convention, the conjugate exponent of 1 is ∞, and the conjugate exponent of ∞ is 1.

Taking $p = 1$, we obtain that $\mathcal{V}_\rho(\{t^k \partial_t^k W^2_{\lambda,t}\}_{t>0})$ is bounded on $L^r(\mathbb{R}^d_+, w)$, provided that

$$w \in A_r(\mathbb{R}^d_+) \cap \mathrm{RH}_{s'}(\mathbb{R}^d_+), \quad 1 < r,\ s < \infty.$$

Since, if $w \in A_r(\mathbb{R}^d_+)$ with $1 < r < \infty$, there exists $\beta > 1$ such that $w \in \mathrm{RH}_\beta(\mathbb{R}^d_+)$, we conclude that $\mathcal{V}_\rho(\{t^k \partial_t^k W^2_{\lambda,t}\}_{t>0})$ is bounded on $L^r(\mathbb{R}^d_+, w)$ for every $1 < r < \infty$ and $w \in A_r(\mathbb{R}^d_+)$.

Assume now $0 < \alpha < 2$. According to Proposition 2.2 (a) and by proceeding as in the proof of [12, Theorem 3.1], we deduce that if $1 \le p < q < \infty$, $\ell \in \mathbb{N}$ and $0 < \varepsilon < 1$, there exists $C > 0$ such that for every ball

$$\mathcal{B} = B(x_B, r_B) \cap \mathbb{R}^d_+,$$

with $x_B \in \mathbb{R}^d_+$ and $r_B > 0$, every $t > 0$ and $j \in \mathbb{N}$, we have that

$$\begin{aligned}
&\left(\frac{1}{|S_j(\mathcal{B})|}\int_{S_j(\mathcal{B})} |t^\ell \partial_t^\ell W^\alpha_{\lambda,t}(f)(x)|^q\,dx\right)^{1/q} \\
&\lesssim \max\left\{\left(\frac{r_B}{t^{1/\alpha}}\right)^{d/p}, \left(\frac{r_B}{t^{1/\alpha}}\right)^{d}\right\} \\
&\quad \times \left(1 + \frac{t^{1/\alpha}}{2^j r_B}\right)^{d/q}\left(1 + \frac{2^j r_B}{t^{1/\alpha}}\right)^{-(d+\alpha)(1-\varepsilon)}\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} |f(x)|^p\,dx\right)^{1/p}, \qquad (5.1)
\end{aligned}$$

for every $f \in L^p(\mathbb{R}^d_+)$ supported in $\mathcal{B}$, and

$$\begin{aligned}
&\left(\frac{1}{|S_j(\mathcal{B})|}\int_{S_j(\mathcal{B})} |t^\ell \partial_t^\ell W^\alpha_{\lambda,t}(f)(x)|^q\,dx\right)^{1/q} \\
&\lesssim \max\left\{\left(\frac{2^\ell r_B}{t^{1/\alpha}}\right)^{d/p}, \left(\frac{2^\ell r_B}{t^{1/\alpha}}\right)^{d}\right\} \\
&\quad \times \left(1+\frac{t^{1/\alpha}}{2^j r_B}\right)^{d/q}\left(1+\frac{2^j r_B}{t^{1/\alpha}}\right)^{-(d+\alpha)(1-\varepsilon)}\left(\frac{1}{|S_j(\mathcal{B})|}\int_{S_j(\mathcal{B})} |f(x)|^p\,dx\right)^{1/p}, \quad (5.2)
\end{aligned}$$

for every $f \in L^p(S_j(\mathcal{B}))$.

According to (1.1), we obtain

$$\mathcal{V}_\rho\big(\{t^k \partial_t^k W^\alpha_{\lambda,t}\}_{t>0}\big)(f)(x) \le \int_0^\infty \big|\partial_t (t^k \partial_t^k W^\alpha_{\lambda,t}(f)(x))\big|\,dt, \quad x \in \mathbb{R}^d_+.$$

Now, for $\ell \in \mathbb{R} \setminus \{0\}$, we consider the operator

$$\mathcal{T}_\ell(f)(x) = \int_0^\infty \big|t^{\ell-1}\partial_t^\ell W^\alpha_{\lambda,t}(f)(x)\big|\,dt,\ x \in \mathbb{R}^d.$$

Assume that $j \in \mathbb{N}$ and let $\mathcal{B} = B(x_B, r_B) \cap \mathbb{R}^d_+$, with $x_B \in \mathbb{R}^d_+$ and $r_B > 0$. By using Minkowski inequality, we get

$$\begin{aligned}
&\left(\frac{1}{S_j(\mathcal{B})|}\int_{S_j(\mathcal{B})} |\mathcal{T}_l((I - W^\alpha_{\lambda,r_B^\alpha})^m(f))(x)|^q dx\right)^{1/q}\\
&\le \int_0^\infty \left(\frac{1}{S_j(\mathcal{B})|}\int_{S_j(\mathcal{B})} |t^{\ell-1}\partial_t^l W^\alpha_{\lambda,t}((I - W^\alpha_{\lambda,r_B^\alpha})^m(f))(x)|^q dx\right)^{1/q} dt\\
&\le \left(\int_0^{r_B^\alpha} + \int_{r_B^\alpha}^\infty\right)\left(\frac{1}{S_j(\mathcal{B})|}\int_{S_j(\mathcal{B})} |t^{\ell-1}\partial_t^l W^\alpha_{\lambda,t}((I - W^\alpha_{\lambda,r_B^\alpha})^m(f))(x)|^q dx\right)^{1/q} dt\\
&:= \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

We now adapt the arguments in [38, pp. 15 and 16]. We begin analyzing I_2. First, observe that

$$(I - W^\alpha_{\lambda,r_B^\alpha})^m = \int_0^{r_B^\alpha}\cdots\int_0^{r_B^\alpha} \partial_{s_1}\cdots\partial_{s_m} W^\alpha_{\lambda,s_1+\cdots+s_m} d_{s_1}\cdots d_{s_m}.$$

Then, by (5.1) and calling $\mathfrak{s} = s_1 + \cdots + s_m$ and $d\mathfrak{s} = ds_1 \cdots ds_m$,

$$\begin{aligned}
\mathrm{I}_2 &\le \int_{r_B^\alpha}^\infty \int_0^{r_B^\alpha}\cdots\int_0^{r_B^\alpha}\left(\frac{1}{|S_j(\mathcal{B})|}\int_{S_j(\mathcal{B})} |t^{\ell-1}\partial_t^l\partial_{s_1}\cdots\partial_{s_m} W^\alpha_{\lambda,t+\mathfrak{s}}(f)(x)|^q dx\right)^{1/q} d\mathfrak{s}dt\\
&\le \int_{r_B^\alpha}^\infty \int_0^{r_B^\alpha}\cdots\int_0^{r_B^\alpha} \frac{1}{(t+\mathfrak{s})^{m+1}}\\
&\quad\times\left(\frac{1}{|S_j(\mathcal{B})|}\int_{S_j(\mathcal{B})} \left|u^{\ell+m}\partial_u^{\ell+m} W^\alpha_{\lambda,u}(f)(x)\big|_{u=t+\mathfrak{s}}\right|^q dx\right)^{1/q} d\mathfrak{s}dt\\
&\lesssim f_{q,\mathcal{B}}\Bigg(\int_{r_B^\alpha}^\infty \int_0^{r_B^\alpha}\cdots\int_0^{r_B^\alpha} \frac{1}{(t+\mathfrak{s})^{m+1}}\left(\frac{r_B}{(t+\mathfrak{s})^{1/\alpha}}\right)^{d/q}\left(1 + \frac{(t+\mathfrak{s})^{1/\alpha}}{2^j r_B}\right)^{d/q}\\
&\quad\times\left(1 + \frac{2^j r_B}{(t+\mathfrak{s})^{1/\alpha}}\right)^{-(d+\alpha)(1-\varepsilon)} d\mathfrak{s}dt\Bigg)\\
&\lesssim f_{q,\mathcal{B}}\Bigg(\int_{r_B^\alpha}^{(2^j r_B)^\alpha} \int_0^{r_B^\alpha}\cdots\int_0^{r_B^\alpha} \frac{1}{t^{m+1}}\left(\frac{r_B}{t^{1/\alpha}}\right)^{d/q}\left(\frac{t^{1/\alpha}}{2^j r_B}\right)^{d/q} d\mathfrak{s}dt\\
&\quad+ \int_{(2^j r_B)^\alpha}^\infty \int_0^{r_B^\alpha}\cdots\int_0^{r_B^\alpha} \frac{1}{t^{m+1}}\left(\frac{r_B}{t^{1/\alpha}}\right)^{d/q}\left(\frac{t^{1/\alpha}}{2^j r_B}\right)^{(d+\alpha)(1-\varepsilon)} d\mathfrak{s}dt\Bigg)\\
&\lesssim f_{q,\mathcal{B}}\Bigg(2^{-j(d+\alpha)(1-\varepsilon)}\int_{r_B^\alpha}^{(2^j r_B)^\alpha} t^{-m-1-\frac{d}{q\alpha}+\frac{(d+\alpha)(1-\varepsilon)}{\alpha}} dt\\
&\quad\times r_B^{\frac{d}{q}+m\alpha-(d+\alpha)(1-\varepsilon)} + \int_{(2^j r_B)^\alpha}^\infty 2^{-dj/q} r_B^{\alpha m} t^{-m-1} dt\Bigg)\\
&\lesssim f_{q,\mathcal{B}}\big(2^{-j(d+\alpha)(1-\varepsilon)} + 2^{-j(\alpha m+d/q)}\big)\\
&\lesssim f_{q,\mathcal{B}} 2^{-j(d+\alpha)(1-\varepsilon)},
\end{aligned}$$

provided that $m > \frac{(d+\alpha)(1-\varepsilon)}{\alpha} - \frac{d}{q^\alpha}$. Hence, if $m > \frac{d+\alpha}{\alpha} - \frac{dq}{q^\alpha}$, by choosing $0 < \varepsilon < 1$ such that $m > \frac{(d+\alpha)(1-\varepsilon)}{\alpha} - \frac{d}{q^\alpha}$, the estimates hold.

We can conclude that

$$\mathrm{I}_2 \leq C 2^{-j(d+\alpha)(1-\varepsilon)} f_{q,\mathcal{B}}.$$

On the other hand, we can prove by using (5.2) (see [38, pp. 15 and 16]) that

$$\mathrm{I}_1 \leq C 2^{-j(d+\alpha)(1-\varepsilon)} f_{q,\mathcal{B}},$$

where $0 < \varepsilon < 1$.

We obtain that

$$\Big(\frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |\mathcal{T}_\ell((I - W^\alpha_{\lambda, r_B^\alpha})^m)(f)(x)|^q \, dx \Big)^{1/q} \lesssim 2^{-j(d+\alpha)(1-\varepsilon)} f_{q,\mathcal{B}},$$

provided that $0 < \varepsilon < 1$ and $m > \frac{(d+\alpha)(1-\varepsilon)}{\alpha} - \frac{d}{q^\alpha}$.

Let $\varepsilon \in (0,1)$. According to (5.1), we deduce that

$$\begin{aligned}
&\Big(\frac{1}{|S_j(\mathcal{B})|} \int_{S_j(\mathcal{B})} |(I - A^\alpha_{\lambda, r^\alpha})(f)(x)|^q \Big)^{1/q} \\
&\lesssim \sum_{i=1}^m \Big(\frac{1}{|S_j(\mathcal{B})|} \int_{|S_j(\mathcal{B})|} |W^\alpha_{\lambda, r_B^{\alpha_i}}(f)(x)|^q \, dx \Big)^{1/q} \\
&\lesssim \sum_{i=1}^m \Big(\frac{1}{i^{1/\alpha}} \Big)^{1/p} \Big(1 + \frac{i^{1/\alpha}}{2^j} \Big)^{\alpha/q} \Big(1 + \frac{2^j}{i^{1/\alpha}} \Big)^{-(d+\alpha)(1-\varepsilon)} \Big(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |f(x)|^p \, dx \Big)^{1/p} \\
&\lesssim 2^{-j(d+\alpha)(1-\varepsilon)} \Big(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |f(x)|^p \, dx \Big)^{1/p}.
\end{aligned}$$

We define $\alpha_j = 2^{-j(d+\alpha)(1-\varepsilon)}$, $j \in \mathbb{N}$. We have that $\sum_{j=1}^\infty \alpha_j 2^{j\alpha} < \infty$ provided that $0 < \varepsilon < \alpha/(d+\alpha)$.

By [11, Proposition 2.3] (see also [5, Theorem 6.6]) and by proceeding as in the case $\alpha = 2$, we conclude that the operator $\mathcal{T}_\ell$ is bounded on $L^p(\mathbb{R}^d_+, w)$ for every $1 < p < \infty$ and $w \in A_p(\mathbb{R}^d_+)$. Hence, the ρ-variation operator $\mathcal{V}_\rho(\{t^k \partial_t^k W^\alpha_{\lambda,t}\}_{t>0})$ is also bounded on $L^p(\mathbb{R}^d_+, w)$ for every $1 < p < \infty$ and $w \in A_p(\mathbb{R}^d_+)$.

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