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Regular Strichartz estimates in Lorentz-type spaces with application to the H^s-critical inhomogeneous biharmonic NLS equation

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Abstract. In this paper, we investigate the Cauchy problem for the H^s-critical inhomogeneous biharmonic nonlinear Schrödinger (IBNLS) equation

$$iu_t \pm \Delta^2 u = \lambda |x|^{-b}|u|^\sigma u, \quad u(0) = u_0 \in H^s(\mathbb{R}^d),$$

where $\lambda \in \mathbb{C}$, $d \geq 3$, $1 \leq s < \frac{d}{2}$, $0 < b < \min\{4, 2 + \frac{d}{2} - s\}$ and $\sigma = \frac{8-2b}{d-2s}$. First, we study the properties of Lorentz-type spaces such as Besov–Lorentz spaces and Triebel–Lizorkin–Lorentz spaces. We then derive the regular Strichartz estimates for the corresponding linear equation in Lorentz-type spaces. Using these estimates, we establish the local well-posedness as well as the small data global well-posedness and scattering in H^s for the H^s-critical IBNLS equation under less regularity assumption on the nonlinear term than in the recent work by An–Kim–Ryu [Discrete Contin. Dyn. Syst. Ser. B 29 (2024), 3326–3345]. This result also extends the ones of Saanouni–Ghanmi [Adv. Oper. Theory 9 (2024), article no. 1] and Saanouni–Peng [Mediterr. J. Math. 20 (2023), article no. 170] by extending the validity of d, b, and s. Finally, we give the well-posedness result in the homogeneous Sobolev spaces $\dot{H}^s$.

1. Introduction

In this paper, we are concerned with the Cauchy problem for the inhomogeneous biharmonic nonlinear Schrödinger (IBNLS) equation

$$\begin{cases} iu_t \pm \Delta^2 u = \lambda |x|^{-b}|u|^\sigma u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x) \in H^s(\mathbb{R}^d), \end{cases} \tag{1.1}$$

where $d \in \mathbb{N}$, $s \geq 0$, $0 < b < 4$, $\lambda \in \mathbb{C}$, and $0 < \sigma < \infty$. The case $b = 0$ is the well-known classic biharmonic nonlinear Schrödinger equation (also called fourth-order NLS equation), which has been widely studied during the last two decades. See, for example, [12–14, 21, 22, 25, 28] and the references therein. Recently, the inhomogeneous biharmonic NLS equation (1.1) with $b > 0$ has also attracted a lot of interest. The local well-posedness as well as the small data global well-posedness and scattering in H^s with

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$s \geq 0$ were studied by [2, 5–7, 17, 24, 34, 35]. Meanwhile, the global existence/scattering and blow-up of H^2-solution have also been studied by several authors. See, for example, [9, 10, 15, 18, 24, 31–33].

The IBNLS equation (1.1) is invariant under scaling

$$u_\alpha(t,x) = \alpha^{\frac{4-b}{\sigma}} u(\alpha^4 t, \alpha x), \quad \alpha > 0.$$

An easy computation shows that

$$\|u_\alpha(t)\|_{\dot{H}^s} = \alpha^{s+\frac{4-b}{\sigma}-\frac{d}{2}} \|u(t)\|_{\dot{H}^s}.$$

We thus define the critical Sobolev index

$$s_c := \frac{d}{2} - \frac{4-b}{\sigma}.$$

Putting

$$\sigma_c(s) := \begin{cases} \frac{8-2b}{d-2s} & \text{if } s < \frac{d}{2}, \\ \infty & \text{if } s \geq \frac{d}{2}, \end{cases} \tag{1.2}$$

we can easily see that $s > s_c$ is equivalent to $\sigma < \sigma_c(s)$. If $s < \frac{d}{2}$, then $s = s_c$ is equivalent to $\sigma = \sigma_c(s)$.

For initial data $u_0 \in H^s(\mathbb{R}^d)$, we say that the Cauchy problem (1.1) is H^s-critical (for short, critical) if $0 \leq s < \frac{d}{2}$ and $\sigma = \sigma_c(s)$. If $s \geq 0$ and $\sigma < \sigma_c(s)$, then the problem (1.1) is said to be H^s-subcritical (for short, subcritical).

Throughout the paper, (q,r) is said to be biharmonic admissible (for short, $(q,r) \in B$) if

$$\begin{cases} 2 \leq r \leq \frac{2d}{d-4} & \text{if } d \geq 5, \\ 2 \leq r < \infty & \text{if } d \leq 4 \end{cases} \tag{1.3}$$

and

$$\frac{4}{q} = \frac{d}{2} - \frac{d}{r}. \tag{1.4}$$

In addition, $(q,r) \in B_0$ means that $(q,r) \in B$ with $\max\{\frac{d-4}{2d}, 0\} < \frac{1}{r} < \frac{1}{2}$.

A pair (q,r) is said to be Schrödinger admissible (for short, $(q,r) \in S$) if

$$\begin{cases} 2 \leq r \leq \frac{2d}{d-2} & \text{if } d \geq 3, \\ 2 \leq r < \infty & \text{if } d \leq 2 \end{cases} \tag{1.5}$$

and

$$\frac{2}{q} = \frac{d}{2} - \frac{d}{r}. \tag{1.6}$$

It is well known that the notions of the biharmonic admissible pairs and the Schrödinger admissible pairs arise from the Strichartz estimates for the biharmonic Schrödinger semigroup $S(t) = e^{\pm it\Delta^2}$. Using the argument of Keel–Tao [23], Pausader [30] obtained the classical Strichartz estimates for the biharmonic admissible pairs and the regular Strichartz estimates for the Schrödinger admissible pairs in Sobolev spaces (cf. [30, Proposition 3.1]). These admissible pairs also play an important role in this paper (see, e.g., Lemma 2.9 and Proposition 3.11).

In this paper, we focus on the H^s-critical case, i.e., $\sigma = \frac{8-2b}{d-2s}$ with $0 < s < \frac{d}{2}$. This case was already studied by [6, 34, 35]. They studied the local well-posedness as well as the small data global well-posedness in H^s by using different methods from each other. Saanouni–Peng [35] used the Strichartz estimates in some weighted Lebesgue spaces adapted to handle the inhomogeneous term $|x|^{-b}$. Saanouni–Ghanmi [34] used the fractional Hardy-type inequality, following the idea of [3]. Recently, the authors in [6] give the unified proof for both of H^s-subcritical case and H^s-critical case by using the Sobolev–Lorentz spaces theory . More precisely, they proved that the Cauchy problem (1.1) with $d \in \mathbb{N}$, $0 \le s < \min\{2 + \frac{d}{2}, d\}$, $0 < b < \min\{4, d - s, 2 + \frac{d}{2} - s\}$ and $0 < \sigma \le \sigma_c(s)$ is locally well posed in H^s if either σ is an even integer or[1]

$$\sigma > \lceil s \rceil - 1. \tag{1.7}$$

The condition (1.7) was assumed to ensure the regularity of the nonlinear term $|u|^\sigma u$ when σ is not an even integer. Obviously, this condition becomes trivial when $s \le 1$. However, when $s > 1$, this condition is not so good as in H^s-subcritical case obtained by [5, 7, 24] and leads to restrictive assumption on d and b. In particular, when we study the H^2-theory for (1.1), this condition leads to too restrictive assumption: $d \le 11$ and $b < 6 - \frac{d}{2}$.

The main purpose of this paper is to improve the condition (1.7) in the H^s-critical case with $s > 1$.

As mentioned above, in the H^s-subcritical case $\sigma < \sigma_c(s)$, the condition (1.7) was already improved by [5, 7, 24]. However, the methods of [5, 7, 24] cannot be applied in the H^s-critical case. To overcome this difficulty, we apply the theory of Besov–Lorentz spaces and Triebel–Lizorkin–Lorentz spaces to derive the regular Strichartz estimates for the biharmonic Schrödinger semi-group $S(t) = e^{\pm it\Delta^2}$ in the framework of Lorentz-type spaces. Using this regular Strichartz estimates and the contraction mapping principle, we have the following improved result about the local well-posedness as the small data global well-posedness and scattering in H^s for the critical IBNLS equation (1.1).

Theorem 1.1 (Well-posedness in H^s). *Let $d \ge 3$, $1 \le s < \frac{d}{2}$, $0 < b < \min\{4, 2 + \frac{d}{2} - s\}$ and $\sigma = \frac{8-2b}{d-2s}$. If σ is not an even integer, assume further*

$$\sigma > \lceil s \rceil - 2. \tag{1.8}$$

Then, for any $u_0 \in H^s(\mathbb{R}^d)$, there exist $T_{\max} = T_{\max}(u_0) > 0$ and $T_{\min} = T_{\min}(u_0) > 0$ such that (1.1) has a unique, maximal solution satisfying

$$u \in C((-T_{\min}, T_{\max}), H^s) \cap L^q_{\mathrm{loc}}((-T_{\min}, T_{\max}), H^s_{r,2})$$

[1] For $s \in \mathbb{R}$, $\lceil s \rceil$ denotes the minimal integer which is larger than or equal to s.

for any $(q, r) \in B$. *If* $\|u_0\|_{\dot{H}^s}$ *is sufficiently small, then the above solution is global and scatters.*

Remark 1.2. The authors in [6] established the well-posedness under the assumption that $0 < b < \min\{4, d - s, 2 + \frac{d}{2} - s\}$ and (1.7) are satisfied. Obviously, Theorem 1.1 improves these conditions.

As an immediate consequence of Theorem 1.1, we have the following well-posedness result in the energy space H^2.

Corollary 1.3 (Well-posedness in H^2). *Let* $d \geq 5$, $0 < b < \min\{4, \frac{d}{2}\}$ *and* $\sigma = \frac{8-2b}{d-4}$. *Then, for any* $u_0 \in H^2(\mathbb{R}^d)$, *there exist* $T_{\max} = T_{\max}(u_0) > 0$ *and* $T_{\min} = T_{\min}(u_0) > 0$ *such that* (1.1) *has a unique, maximal solution satisfying*

$$u \in C((-T_{\min}, T_{\max}), H^2) \cap L^q_{\text{loc}}((-T_{\min}, T_{\max}), H^2_{r,2})$$

for any $(q, r) \in B$. *Moreover, if* $\|u_0\|_{\dot{H}^2}$ *is sufficiently small, then the above solution is global and scatters.*

Remark 1.4. (1) In Corollary 1.3, the restriction $d \geq 5$ is natural, since Corollary 1.3 only covers the H^2-critical case. Furthermore, the assumption (1.8) becomes trivial, due to the fact that $s = 2$.

(2) Corollary 1.3 also improves the results of [34, 35] by extending the validity of d and b. In fact, Saanouni–Ghanmi [34] assumed $5 \leq d \leq 11$ and $0 < b < \min\left\{4, \frac{d}{2}, \frac{12+d}{d}\right\}$. And Saanouni–Peng [35] assumed $0 < b < \min\left\{4, \frac{d}{2}, \frac{d(4+d)}{4(d-2)}\right\}$.

We have the following well-posedness result in the homogeneous Sobolev spaces $\dot{H}^s$.

Theorem 1.5 (Well-posedness in $\dot{H}^s$). *Let* $d \geq 3$, $1 \leq s < \frac{d}{2}$, $0 < b < \min\{4, 2 + \frac{d}{2} - s\}$ *and* $\sigma = \frac{8-2b}{d-2s}$. *If* σ *is not an even integer, assume further*

$$\sigma > \lceil s \rceil - 1. \tag{1.9}$$

Then, for any $u_0 \in \dot{H}^s(\mathbb{R}^d)$, *there exist* $T_{\max} = T_{\max}(u_0) > 0$ *and* $T_{\min} = T_{\min}(u_0) > 0$ *such that* (1.1) *has a unique, maximal solution satisfying*

$$u \in C((-T_{\min}, T_{\max}), \dot{H}^s) \cap L^q_{\text{loc}}((-T_{\min}, T_{\max}), \dot{H}^s_{r,2})$$

for any $(q, r) \in B$. *If* $\|u_0\|_{\dot{H}^s}$ *is sufficiently small, then the above solution is global and scatters.*

Remark 1.6. One can easily verify that the result of Theorem 1.5 still holds for the H^s-subcritical case $\sigma < \frac{8-2b}{d-2s}$ by modifying the proof in Section 5 slightly.

This paper is organized as follows. In Section 2, we give some preliminary results related to our problem. In Section 3, we study the properties of Lorentz-type spaces and derive the regular Strichartz estimates for $S(t) = e^{\pm it\Delta^2}$ in the framework of Lorentz-type spaces. In Section 4, we prove Theorem 1.1. Theorem 1.5 is proved in Section 5.

2. Preliminaries

First of all, we introduce some notation used in this paper. Throughout the paper, $\mathcal{F}$ denotes the Fourier transform, and the inverse Fourier transform is denoted by $\mathcal{F}^{-1}$; $C > 0$ stands for a positive universal constant, which may be different at different places; $a \lesssim b$ means $a \le Cb$ for some constant $C > 0$; $a \sim b$ expresses $a \lesssim b$ and $b \lesssim a$. Given normed spaces X and Y, $X \hookrightarrow Y$ means that X is continuously embedded in Y. For $p \in [1, \infty]$, p' denotes the dual number of p, i.e., $1/p + 1/p' = 1$.

Let us recall the definitions and the properties of some function spaces.

As in [8,37], for $s \in \mathbb{R}$, $1 \le q \le \infty$ and normed space X, the space $\dot{l}_q^s(X)$ denotes the space of all sequences $a = \{a_k\}_{k\in\mathbb{Z}} \subset X$ such that

$$\|a\|_{\dot{l}_q^s(X)} = \left(\sum_{k\in\mathbb{Z}}[2^{-ks}\|a_k\|_X]^q\right)^{\frac{1}{q}} < \infty,$$

with usual modification when $q = \infty$. In the case $X = \mathbb{C}$, we write $\dot{l}_q^s$ instead of $\dot{l}_q^s(\mathbb{C})$. Similarly, the space $l_q^s(X)$ denotes the space of all sequences $a = \{a_k\}_{k=0}^{\infty} \subset X$ such that

$$\|a\|_{l_q^s(X)} = \left(\sum_{k=0}^{\infty}[2^{-ks}\|a_k\|_X]^q\right)^{\frac{1}{q}} < \infty,$$

with usual modification when $q = \infty$.

As in [39], for $s \in \mathbb{R}$ and $1 < p < \infty$, we denote by $H_p^s(\mathbb{R}^d)$ and $\dot{H}_p^s(\mathbb{R}^d)$ the nonhomogeneous Sobolev space and homogeneous Sobolev space, respectively. The norms of these spaces are given as

$$\|f\|_{H_p^s(\mathbb{R}^d)} = \|(I-\Delta)^{s/2}f\|_{L^p(\mathbb{R}^d)}, \quad \|f\|_{\dot{H}_p^s(\mathbb{R}^d)} = \|(-\Delta)^{s/2}f\|_{L^p(\mathbb{R}^d)},$$

where $(I-\Delta)^{s/2}f = \mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F}f$ and $(-\Delta)^{s/2}f = \mathcal{F}^{-1}|\xi|^s\mathcal{F}f$. As usual, we abbreviate $H_2^s(\mathbb{R}^d)$ and $\dot{H}_2^s(\mathbb{R}^d)$ as $H^s(\mathbb{R}^d)$ and $\dot{H}^s(\mathbb{R}^d)$, respectively.

For $0 < p, q \le \infty$, we denote by $L^{p,q}(\mathbb{R}^d)$ the Lorentz space (cf. [16]). The quasi-norms of these spaces are given by

$$\|f\|_{L^{p,q}(\mathbb{R}^d)} = \left(\int_0^\infty (t^{\frac{1}{p}}f^*(t))^q\frac{dt}{t}\right)^{\frac{1}{q}}, \quad \text{when } 0 < q < \infty,$$
$$\|f\|_{L^{p,\infty}(\mathbb{R}^d)} = \sup_{t>0} t^{\frac{1}{p}}f^*(t), \quad \text{when } q = \infty,$$

where $f^*(t) = \inf\{\tau : M^d(\{x : |f(x)| > \tau\}) \le t\}$, with M^d being the Lebesgue measure in $\mathbb{R}^d$. Note that $L^{p,q}(\mathbb{R}^d)$ is a quasi-Banach space for $0 < p, q \le \infty$. When $1 < p < \infty$ and $1 \le q \le \infty$, $L^{p,q}(\mathbb{R}^d)$ can be turned into a Banach space via an equivalent norm. In particular, $L^{p,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$, while $L^{p,\infty}$ corresponds to weak L^p space. These spaces are natural in the context of (1.1) due to the fact that $|x|^{-b} \in L^{\frac{d}{b},\infty}(\mathbb{R}^d)$. In general, we have the embedding $L^{p,q} \hookrightarrow L^{p,r}$ for $q < r$. Note also that $\| |f|^r \|_{L^{p,q}} = \|f\|^r_{L^{pr,qr}}$ for $1 \le p,\ r < \infty$, $1 \le q \le \infty$. We also have the following Hölder's inequality in Lorentz spaces.

Lemma 2.1 (Hölder's inequality in Lorentz spaces [29]). *Assume that* $1 < p, p_1, p_2 < \infty$, $1 \le q, q_1, q_2 \le \infty$ *and*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then, we have

$$\|fg\|_{L^{p,q}} \lesssim \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.$$

As in [1,4,6,19], for $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \le q \le \infty$, the nonhomogeneous Sobolev–Lorentz space $H^s_{p,q}(\mathbb{R}^d)$ is defined as the set of tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $(I - \Delta)^{s/2} f \in L^{p,q}(\mathbb{R}^d)$, equipped with the norm

$$\|f\|_{H^s_{p,q}(\mathbb{R}^d)} = \|(I - \Delta)^{s/2} f\|_{L^{p,q}(\mathbb{R}^d)}.$$

The homogeneous Sobolev–Lorentz space $\dot{H}^s_{p,q}(\mathbb{R}^d)$ is defined as the set of equivalence classes of distribution $f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ such that $(-\Delta)^{s/2} f \in L^{p,q}(\mathbb{R}^d)$, equipped with the norm

$$\|f\|_{\dot{H}^s_{p,q}(\mathbb{R}^d)} = \|(-\Delta)^{s/2} f\|_{L^{p,q}(\mathbb{R}^d)},$$

where $\mathcal{P}(\mathbb{R}^d)$ denotes the set of polynomials with d variables.

Lemma 2.2 ([4, Lemma 2.7]). *Let* $s \ge 0$, $1 < p < \infty$ *and* $1 \le q_1 \le q_2 \le \infty$. *Then, the following statements hold:*

(a) $\dot{H}^s_{p,1} \hookrightarrow \dot{H}^s_{p,q_1} \hookrightarrow \dot{H}^s_{p,q_2} \hookrightarrow \dot{H}^s_{p,\infty}$,

(b) $\dot{H}^s_{p,p} = \dot{H}^s_p$.

Lemma 2.3 ([6, Lemma 3.4]). *Let* $s \ge 0$, $1 < p < \infty$ *and* $1 \le q \le \infty$. *Then, we have* $H^s_{p,q} = L^{p,q} \cap \dot{H}^s_{p,q}$ *with*

$$\|f\|_{H^s_{p,q}} \sim \|f\|_{L^{p,q}} + \|f\|_{\dot{H}^s_{p,q}}.$$

Lemma 2.4 ([6, Lemma 3.5]). *Let* $-\infty < s_2 \le s_1 < \infty$ *and* $1 < p_1 \le p_2 < \infty$ *with* $s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2}$. *Then, for any* $1 \le q \le \infty$, *there hold the embeddings*

$$\dot{H}^{s_1}_{p_1,q} \hookrightarrow \dot{H}^{s_2}_{p_2,q}, \qquad H^{s_1}_{p_1,q} \hookrightarrow H^{s_2}_{p_2,q}.$$

Lemma 2.5 (Fractional product rule under Lorentz norms, Theorem 6.1 of [11]). *Let* $s \ge 0$, $1 < p, p_1, p_2, p_3, p_4 < \infty$ *and* $1 \le q, q_1, q_2, q_3, q_4 \le \infty$. *Assume that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} + \frac{1}{q_4}.$$

Then, we have

$$\|fg\|_{\dot{H}^s_{p,q}} \lesssim \|f\|_{\dot{H}^s_{p_1,q_1}} \|g\|_{L^{p_2,q_2}} + \|f\|_{L^{p_3,q_3}} \|g\|_{\dot{H}^s_{p_4,q_4}}.$$

Lemma 2.6 (Fractional chain rule under Lorentz norms [1,6]). *Let us assume that $s>0$ and $F\in C^{\lceil s\rceil}(\mathbb{C}\to\mathbb{C})$. Then, for $1<p,p_{1_k},p_{2_k},p_{3_k}<\infty$ and $1\le q,q_{1_k},q_{2_k},q_{3_k}<\infty$ satisfying*

$$\frac{1}{p}=\frac{1}{p_{1_k}}+\frac{1}{p_{2_k}}+\frac{k-1}{p_{3_k}},\quad \frac{1}{q}=\frac{1}{q_{1_k}}+\frac{1}{q_{2_k}}+\frac{k-1}{q_{3_k}},\quad k=1,2,\dots,\lceil s\rceil,$$

we have

$$\|(-\Delta)^{s/2}F(u)\|_{L^{p,q}}\lesssim\sum_{k=1}^{\lceil s\rceil}\|F^{(k)}(u)\|_{L^{p_{1_k},q_{1_k}}}\|(-\Delta)^{s/2}u\|_{L^{p_{2_k},q_{2_k}}}\|u\|^{k-1}_{L^{p_{3_k},q_{3_k}}},$$

where the kth order derivative of $F(z)$ is defined under the identification $\mathbb{C}=\mathbb{R}^2$ (see, e.g., [3,4,6]).

Lemma 2.6 was first obtained by [1] only for $0<s\le1$ (see [1, Lemma 2.4]). Recently, it was generalized to $s>0$ by [6] (see [6, Lemma 3.11]).

As an immediate consequence of Lemma 2.6, we have the following useful nonlinear estimates in Sobolev–Lorentz spaces $\dot H^s_{p,q}$ with $s\ge0$.

Corollary 2.7. *Let $F(u)=|u|^\sigma u$ or $F(u)=|u|^{\sigma+1}$. Assume $s\ge0$ and $\sigma>0$. If σ is not an even integer, assume further $\sigma>\lceil s\rceil-1$. Then, for $1<p,p_1,p_2<\infty$ and $1\le q,q_1,q_2<\infty$ satisfying*

$$\frac{1}{p}=\frac{\sigma}{p_1}+\frac{1}{p_2},\quad \frac{1}{q}=\frac{\sigma}{q_1}+\frac{1}{q_2},$$

we have

$$\|F(u)\|_{\dot H^s_{p,q}}\lesssim\|u\|^\sigma_{L^{p_1,q_1}}\|u\|_{\dot H^s_{p_2,q_2}}.$$

We also recall the useful fact concerning the term $|x|^{-b}$ with $b>0$.

Lemma 2.8. *Let $b>0$, $s\ge0$ and $b+s<d$. Then, we have $|x|^{-b}\in\dot H^s_{d/(b+s),\infty}(\mathbb{R}^d)$.*

Proof. The result follows from the fact that $|x|^{-b}\in L^{\frac{d}{b},\infty}(\mathbb{R}^d)$ and

$$(-\Delta)^{s/2}(|x|^{-b})=C_{d,b}|x|^{-b-s}.$$

■

Finally, we recall the standard Strichartz estimates in Sobolev–Lorentz spaces.

Lemma 2.9 ([6,23]). *Let $S(t)=e^{\pm it\Delta^2}$ and $s\in\mathbb{R}$. Then, for any $(q,r)\in B$ and $(\bar q,\bar r)\in B$, we have*

$$\|S(t)\phi\|_{L^q(\mathbb{R},\dot H^s_{r,2})}\lesssim\|\phi\|_{\dot H^s},\tag{2.1}$$

$$\left\|\int_0^t S(t-\tau)f(\tau)d\tau\right\|_{L^q(\mathbb{R},\dot H^s_{r,2})}\lesssim\|f\|_{L^{\bar q'}(\mathbb{R},\dot H^s_{\bar r',2})}.\tag{2.2}$$

3. Regular Strichartz estimates in Lorentz-type spaces

In this section, we study the properties of Lorentz-type spaces and derive the regular Strichartz estimates in Besov–Lorentz and Sobolev–Lorentz spaces.

3.1. Besov–Lorentz and Triebel–Lizorkin–Lorentz spaces

We recall the definition of Besov–Lorentz and Triebel–Lizorkin–Lorentz spaces and study the properties of these spaces. The Besov–Lorentz and Triebel–Lizorkin–Lorentz spaces were initially introduced in Section 2.4 of Triebel [37]. Later, these spaces have been studied by several authors and applied to the study of partial differential equations such as Navier–Stokes equations and Euler equations. See, for example, [20, 26, 36, 40–43].

We first recall the definition of homogeneous Littlewood–Paley decomposition (cf. Wang et al. [39]). Let $\varphi : \mathbb{R}^d \to [0,1]$ be a smooth radial cut-off function such that $\varphi(\xi) = \frac{1}{2}$ for $|\xi| \le 1$ and $\varphi(\xi) = 0$ for $|\xi| \ge 2$. For $k \in \mathbb{Z}$, let $\varphi_k(\xi) = \varphi(2^{-k}\xi) - \varphi(2^{1-k}\xi)$. We define the homogeneous Littlewood–Paley decomposition operator $\{\dot{\Delta}_k\}_{k\in\mathbb{Z}}$ as follows:

$$\dot{\Delta}_k := \mathcal{F}^{-1}\varphi_k\mathcal{F}, \quad k \in \mathbb{Z}.$$

Combining the above homogeneous Littlewood–Paley decomposition operator $\{\dot{\Delta}_k\}_{k\in\mathbb{Z}}$ with the function spaces $\dot{l}^s_r(L^{p,q})$ and $L^{p,q}(\dot{l}^s_r)$, we can introduce homogeneous Besov–Lorentz spaces $\dot{B}^s_{(p,q),r}$ and Triebel–Lizorkin–Lorentz spaces $\dot{F}^s_{(p,q),r}$. See, for example, [26, 40–43].

Assume that $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \le q, r \le \infty$. The homogeneous Besov–Lorentz spaces $\dot{B}^s_{(p,q),r}$ and Triebel–Lizorkin–Lorentz spaces $\dot{F}^s_{(p,q),r}$ are defined, respectively, to be the sets of equivalence classes of distribution $f \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$ such that

$$\|f\|_{\dot{B}^s_{(p,q),r}} := \|\{\dot{\Delta}_k f\}_{k\in\mathbb{Z}}\|_{\dot{l}^s_r(L^{p,q})} < \infty$$

and

$$\|f\|_{\dot{F}^s_{(p,q),r}} := \|\{\dot{\Delta}_k f\}_{k\in\mathbb{Z}}\|_{L^{p,q}(\dot{l}^s_r)} < \infty.$$

Replacing $\dot{l}^s_q$-norms and the homogeneous Littlewood–Paley decomposition operator $\{\dot{\Delta}_k\}_{k\in\mathbb{Z}}$ by l^s_q-norms and the nonhomogeneous Littlewood–Paley decomposition operator $\{\Delta_k\}_{k=0}^{\infty}$ (cf. [38, 39]), we can define the nonhomogeneous Besov–Lorentz spaces $B^s_{(p,q),r}$ and Triebel–Lizorkin–Lorentz spaces $F^s_{(p,q),r}$. See, e.g., [20, 36] for the properties of these spaces.

In this paper, we only focus on the homogeneous Besov–Lorentz spaces and Triebel–Lizorkin–Lorentz spaces. If we replace the Lorentz-norms by the corresponding Lebesgue-norms, we get the well-known Besov spaces $\dot{B}^s_{p,r}$ and Triebel–Lizorkin spaces $\dot{F}^s_{p,r}$. More precisely, we have $\dot{B}^s_{(p,p),r} = \dot{B}^s_{p,r}$ and $\dot{F}^s_{(p,p),r} = \dot{F}^s_{p,r}$.

We recall the interpolation of Triebel–Lizorkin–Lorentz spaces.

Lemma 3.1 ([42]). *Let* $1 < p_0, p_1 < \infty$, $1 \le q, r, r_0, r_1 \le \infty$, $0 < \theta < 1$ *and* $s \in \mathbb{R}$. *If* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, *then we have the following real interpolation (see, e.g.,* [8, 37] *for the definition and properties of real interpolation):*

$$(\dot F^s_{(p_0,r_0),q}, \dot F^s_{(p_1,r_1),q})_{\theta,r} = \dot F^s_{(p,r),q}.$$

In particular, we have

$$(\dot F^s_{p_0,q}, \dot F^s_{p_1,q})_{\theta,r} = \dot F^s_{(p,r),q}.$$

We also have the following interpolation of Besov–Lorentz spaces.

Lemma 3.2. *Let* $1 < p_0, p_1 < \infty$, $1 \le q, q_0, q_1, r_0, r_1 < \infty$, $s, s_, s_1 \in \mathbb{R}$ *and* $0 < \theta < 1$. *If*

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

then we have the following real interpolation:

$$(\dot B^{s_0}_{(p_0,r_0),q_0}, \dot B^{s_1}_{(p_1,r_1),q_1})_{\theta,q} = \dot B^s_{(p,q),q}.$$

In particular, we have

$$(\dot B^{s_0}_{p_0,q_0}, \dot B^{s_1}_{p_1,q_1})_{\theta,q} = \dot B^s_{(p,q),q}.$$

Proof. Reference [8, Theorem 5.6.2] shows that[2]

$$(\dot l^{s_0}_{q_0}(A_0), \dot l^{s_1}_{q_1}(A_1))_{\theta,q} = \dot l^{s_1}_{q_1}((A_0, A_1)_{\theta,q}). \tag{3.1}$$

And Theorem 2 in Subsection 1.18.6 of [37] shows that

$$(L^{p_0,r_0}, L^{p_1,r_1})_{\theta,q} = L^{p,q}. \tag{3.2}$$

The results follow directly from (3.1) and (3.2). ∎

Using Lemma 3.1, we also have the following interesting result.

Lemma 3.3. *Let* $1 < p < \infty$, $1 \le q, r \le \infty$ *and* $s, \gamma \in \mathbb{R}$. *Then, we have*

$$\|(-\Delta)^{\gamma/2} f\|_{\dot F^s_{(p,r),q}} \sim \|f\|_{\dot F^{s+\gamma}_{(p,r),q}}.$$

Proof. Theorem 1 in Subsection 5.2.3 of [38] shows that

$$\|(-\Delta)^{\gamma/2} f\|_{\dot F^s_{p,q}} \sim \|f\|_{\dot F^{s+\gamma}_{p,q}} \tag{3.3}$$

[2]It is worth mentioning that the authors in [8] proved the real interpolation $(l^{s_0}_{q_0}(A_0), l^{s_1}_{q_1}(A_1))_{\theta,q} = l^{s_1}_{q_1}((A_0, A_1)_{\theta,q})$. But one can easily see that the proof, and so, the result still holds for the corresponding dotted spaces.

for any $1 < p < \infty$, $1 \leq q \leq \infty$, and $s, \gamma \in \mathbb{R}$. Equation (3.3) and Lemma 3.1 imply that

$$\|(-\Delta)^{\gamma/2} f\|_{\dot{F}^{s}_{(p,r),q}} \lesssim \|f\|_{\dot{F}^{s+\gamma}_{(p,r),q}} \tag{3.4}$$

for any $1 < p < \infty$, $1 \leq q, r \leq \infty$, and $s, \gamma \in \mathbb{R}$. Equation (3.4) shows that

$$\|(-\Delta)^{-\gamma/2} f\|_{\dot{F}^{s+\gamma}_{(p,r),q}} \lesssim \|f\|_{\dot{F}^{s}_{(p,r),q}}. \tag{3.5}$$

Taking $f = (-\Delta)^{\gamma/2} g$ in (3.5), we immediately get

$$\|g\|_{\dot{F}^{s+\gamma}_{(p,r),q}} \lesssim \|(-\Delta)^{\gamma/2} g\|_{\dot{F}^{s}_{(p,r),q}}. \tag{3.6}$$

Using (3.4) and (3.6), we immediately get the desired result. ∎

It is well known that $\|f\|_{\dot{H}^s_p} \sim \|f\|_{\dot{F}^s_{p,2}}$ for $1 < p < \infty$ and $s \in \mathbb{R}$. See, for example, [38, Subsection 5.2.3]. This result can be extended as follows.

Corollary 3.4. *Let $1 < p < \infty$, $1 \leq q < \infty$ and $s \in \mathbb{R}$. Then, we have*

$$\|f\|_{\dot{H}^s_{p,q}} \sim \|f\|_{\dot{F}^s_{(p,q),2}}.$$

Proof. The case $s = 0$ was proved in Theorem 5 of [40], i.e., we have

$$\|f\|_{L^{p,q}} \sim \|f\|_{\dot{F}^0_{(p,q),2}}.$$

Using this fact and Lemma 3.3, we immediately get the desired result. ∎

We have the following result for Besov–Lorentz spaces which is similar to Lemma 3.3.

Lemma 3.5. *Let $1 < p < \infty$, $1 \leq q, r \leq \infty$ and $s, \gamma \in \mathbb{R}$. Then, we have*

$$\|(-\Delta)^{\gamma/2} f\|_{\dot{B}^s_{(p,r),q}} \sim \|f\|_{\dot{B}^{s+\gamma}_{(p,r),q}}.$$

Proof. Noticing that

$$\dot{\Delta}_k f = \sum_{l=-1}^{l=1} \dot{\Delta}_{k+l}(\dot{\Delta}_k f),$$

it suffices to prove that

$$\|\mathcal{F}^{-1}|\xi|^{\gamma}\varphi_{k+l}\mathcal{F} f\|_{L^{p,q}} \lesssim 2^{\gamma k}\|f\|_{L^{p,q}} \ (l = 0, \pm 1)$$

for any $1 < p < \infty$, $1 \leq q \leq \infty$, $\gamma \in \mathbb{R}$ and $k \in \mathbb{Z}$. And this result follows from the fact that

$$\||\xi|^{\gamma}\varphi_{k+l}\|_{M_p} \lesssim 2^{\gamma k},$$

and the real interpolation, where M_p denotes the set of all multipliers on L^p. See (5) in the proof of Lemma 6.2.1 on page 140 of [8]. This completes the proof. ∎

Let us consider the embeddings between the homogeneous Besov–Lorentz spaces, Triebel–Lizorkin–Lorentz spaces and Sobolev–Lorentz spaces. To this end, we recall the following embedding result[3].

Proposition 3.6 ([36]). *Let* $1 < p < \infty$, $1 \le q_0, q_1, r_0, r_1 < \infty$, $r_0 \le r_1$.

(1) *There holds the embedding* $\dot{l}^0_{q_0}(L^{p,r_0}) \hookrightarrow L^{p,r_1}(\dot{l}^0_{q_1})$, *if* $q_0 \le \min\{p, q_1, r_1\}$ *and one of the following conditions is satisfied:*
   - (i) $p \ne q_1$,
   - (ii) $p = q_1 \ge r_0$,
   - (iii) $q_0 < p = q_1 < r_0$.

(2) *There holds the embedding* $L^{p,r_0}(\dot{l}^0_{q_0}) \hookrightarrow \dot{l}^0_{q_1}(L^{p,r_1})$, *if* $q_1 \ge \min\{p, q_0, r_0\}$ *and one of the following conditions are satisfied:*
   - (i) $p \ne q_0$,
   - (ii) $p = q_0 \ge r_1$,
   - (iii) $r_1 < p = q_0 < q_1$.

Applying Proposition 3.6 to $\{2^{sk}\dot{\Delta}_k f\}_{k\in\mathbb{Z}}$, we have the following embedding result between the homogeneous Besov–Lorentz spaces and Triebel–Lizorkin–Lorentz spaces.

Corollary 3.7. *Let* $1 < p < \infty$, $1 \le q_0, q_1, r_0, r_1 < \infty$, $r_0 \le r_1$ *and* $s \in \mathbb{R}$.

(1) *There holds the embedding* $\dot{B}^s_{(p,r_0),q_0} \hookrightarrow \dot{F}^s_{(p,r_1),q_1}$, *if* $q_0 \le \min\{p, q_1, r_1\}$ *and one of the following conditions are satisfied:*
   - (i) $p \ne q_1$,
   - (ii) $p = q_1 \ge r_0$,
   - (iii) $q_0 < p = q_1 < r_0$.

(2) *There holds the embedding* $\dot{F}^s_{(p,r_0),q_0} \hookrightarrow \dot{B}^s_{(p,r_1),q_1}$, *if* $q_1 \ge \min\{p, q_0, r_0\}$ *and one of the following conditions are satisfied:*
   - (i) $p \ne q_0$,
   - (ii) $p = q_0 \ge r_1$,
   - (iii) $r_1 < p = q_0 < q_1$.

Using Corollaries 3.4 and 3.7, we can get the embedding result between the homogeneous Besov–Lorentz spaces and Sobolev–Lorentz spaces, which will be omitted. For later use, we only state the embedding result between the homogeneous Besov–Lorentz spaces $\dot{B}^s_{(p,r_0),2}$ and Sobolev–Lorentz spaces $\dot{H}^s_{p,r_1}$.

[3] In fact, the authors in [36] obtained the embedding result for the nonhomogeneous version. But one can see that the proof, and so, the result still holds for the homogeneous version.

Corollary 3.8. *Let* $1 < p < \infty$ *and* $s \in \mathbb{R}$.

(1) *There holds the embedding* $\dot{B}^s_{(p,r_0),2} \hookrightarrow \dot{H}^s_{p,r_1}$, *if one of the following conditions is satisfied:*

(i) $p > 2$ *and* $r_1 \geq 2$,

(ii) $p = 2$ *and* $r_1 \geq 2 \geq r_0$.

In particular, we have the embedding $\dot{B}^s_{(p,2),2} \hookrightarrow \dot{H}^s_{p,2}$, *if* $p \geq 2$.

(2) *There holds the embedding* $\dot{H}^s_{p,r_1} \hookrightarrow \dot{B}^s_{(p,r_0),2}$, *if one of the following conditions is satisfied:*

(i) $p < 2$ *and* $r_0 \leq 2$,

(ii) $p = 2$ *and* $r_1 \geq 2 \geq r_0$.

In particular, we have the embedding $\dot{B}^s_{(p,2),2} \hookrightarrow \dot{H}^s_{p,2}$, *if* $p \leq 2$.

3.2. Regular Strichartz estimates in Lorentz-type spaces

In this subsection, we derive the regular Strichartz estimates for $S(t) = e^{\pm it\Delta^2}$ in the framework of Lorentz-type spaces. First, we have the following time decay estimate for $S(t) = e^{\pm it\Delta^2}$ in Besov–Lorentz spaces.

Proposition 3.9. *Let* $d \in \mathbb{N}$, $2 \leq p < \infty$ *and* $S(t) = e^{\pm it\Delta^2}$. *Then, we have*

$$\|S(t)f\|_{\dot{B}^{s(p)}_{(p,2),2}} \lesssim |t|^{-d(1/2-1/p)} \|f\|_{\dot{B}^{-s(p)}_{(p',2),2}},$$

where $s(\cdot) = d(1/2 - 1/\cdot)$.

Proof. The case $p = 2$ is trivial, due to the fact that

$$\dot{B}^{s(p)}_{(p,2),2} = \dot{B}^{-s(p)}_{(p',2),2} = L^2.$$

When $p > 2$, the result follows from Lemma 3.2 and the following regular Strichartz estimates in Besov spaces (see, e.g., [39, Proposition 3.3]):

$$\|S(t)f\|_{\dot{B}^{s(p)}_{p,2}} \lesssim |t|^{-d(1/2-1/p)} \|f\|_{\dot{B}^{-s(p)}_{p',2}} \quad \forall p \in (2, \infty).$$

This completes the proof. ∎

To derive regular Strichartz estimates, we recall the following well-known result of Keel–Tao [23].

Lemma 3.10 ([23]). *Let* $\alpha > 0$, H *be a Hilbert space and* B_0, B_1 *be Banach spaces. Suppose that for each time* t *we have an operator* $U(t) : H \to B_0^*$ *such that*

$$\|U(t)\|_{H \to B_0^*} \lesssim 1, \tag{3.7}$$

$$\|U(t)(U(s))^*\|_{B_1 \to B_1^*} \lesssim |t-s|^{-\alpha}, \tag{3.8}$$

where $(U(s))^*$ *denotes the adjoint of* $U(s)$ *and* $B_i^*(i = 0, 1)$ *denotes the dual space of* B_i. *Let* B_θ *denote the real interpolation space* $(B_0, B_1)_{\theta,2}$. *Then, we have the estimates*

$$\|U(t)f\|_{L^q(\mathbb{R},B_\theta^*)} \lesssim \|f\|_H,$$
$$\left\| \int_{s<t} U(t)(U(s))^* F(s)ds \right\|_{L^q(\mathbb{R},B_\theta^*)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R},B_{\tilde{\theta}})},$$

whenever $0 \le \theta \le 1$, $2 \le q = \frac{2}{\alpha\theta}$, $(q, \theta, \alpha) \ne (2, 1, 1)$, *and similarly for* $(\tilde{q}, \tilde{\theta})$.

Using Proposition 3.9 and Lemma 3.10, we have the following regular Strichartz estimates.

Proposition 3.11 (Regular Strichartz estimates in Lorentz-type spaces). *Let* $S(t) = e^{\pm it\Delta^2}$ *and* $s \in \mathbb{R}$. *Then, for any Schrödinger admissible pairs* $(q,r) \in S$ *and* $(\tilde{q}, \tilde{r}) \in S$, *we have*

$$\|S(t)\phi\|_{L^q(\mathbb{R},\dot{B}^{s+2/p}_{(r,2),2})} \lesssim \|\phi\|_{\dot{H}^s}, \tag{3.9}$$

$$\left\| \int_0^t S(t-\tau)f(\tau)d\tau \right\|_{L^q(\mathbb{R},\dot{B}^{s+2/q}_{(r,2),2})} \lesssim \|f\|_{L^{\tilde{q}'}(\mathbb{R},\dot{B}^{s-2/\tilde{q}}_{(\tilde{r}',2),2})}. \tag{3.10}$$

In (3.9), (3.10), *replacing homogeneous Besov–Lorentz spaces by corresponding Sobolev–Lorentz spaces, the results still hold.*

Proof. Noticing that

$$(\dot{B}^s_{p,2})^* = \dot{B}^{-s}_{p',2} \quad \forall p \in (1, \infty),\ \forall s \in \mathbb{R},$$

it follows from Lemma 3.2 and the duality theorem (see, e.g., [8, Theorem 3.7.1]) that

$$(\dot{B}^s_{(p,2),2})^* = \dot{B}^{-s}_{(p',2),2} \quad \forall p \in (1, \infty),\ \forall s \in \mathbb{R}.$$

Hence, putting $B_0 = L^2$ and $B_1 = \dot{B}^{-s(p)}_{(p',2),2}$, we have

$$B_{\tilde{\theta}} = \dot{B}^{-s(\tilde{r})}_{(\tilde{r}',2),2} = \dot{B}^{-2/\tilde{q}}_{(p',2),2}, \quad B_\theta^* = \dot{B}^{s(r)}_{(r,2),2} = \dot{B}^{2/q}_{(r,2),2},$$

where[4]

$$p = \begin{cases} \frac{2d}{d-2} & \text{if } d \ge 3, \\ \infty^- & \text{if } d = 1, 2 \end{cases}$$

and

$$\frac{1}{r} = \frac{1-\theta}{2} + \frac{1}{p}, \quad \frac{1}{\tilde{r}} = \frac{1-\tilde{\theta}}{2} + \frac{1}{\tilde{p}},$$

[4] $p = \infty^-$ denotes the fixed number $p > 0$ large enough.

it follows from Proposition 3.9 that $S(t)$ satisfies (3.7) and (3.8) with $\alpha = d(\frac{1}{2} - \frac{1}{p})$. We can also see that $2 \le q = \frac{2}{\alpha\theta}$ and $2 \le \tilde{q} = \frac{2}{\alpha\tilde{\theta}}$, due to the fact that $(q, r) \in S$ and $(\tilde{q}, \tilde{r}) \in S$. Hence, (3.9) and (3.10) follow from Lemmas 3.10 and 3.5. Using Corollary 3.8, we can see that (3.9) and (3.10) still hold if we replace homogeneous Besov–Lorentz spaces by the corresponding Sobolev–Lorentz spaces. This completes the proof. ∎

Remark 3.12. The regular Strichartz estimates in Besov or Sobolev spaces were initially obtained by [27, 30]. See [27, Proposition 2] and [30, Proposition 3.2]. Proposition 3.11 extends these results to the Lorentz-type spaces.

Corollary 3.13. *Let $S(t) = e^{\pm it\Delta^2}$ and $s \in \mathbb{R}$. Then, for any $(\bar{q}, \bar{r}) \in B$ and $(\tilde{q}, \tilde{r}) \in S$, we have*

$$\left\| \int_0^t S(t-\tau) f(\tau) d\tau \right\|_{L^{\bar{q}}(\mathbb{R}, \dot{H}^s_{\bar{r},2})} \lesssim \|f\|_{L^{\tilde{q}'}(\mathbb{R}, \dot{H}^{s-2/\tilde{q}}_{\tilde{r}',2})}. \tag{3.11}$$

Proof. Putting $\frac{d+1}{r} := \frac{d}{2} + \frac{d}{\bar{r}}$, we can see that $(\bar{q}, r) \in S$. Furthermore, we can see that $\dot{H}^{s+2/\bar{q}}_{r,2} \hookrightarrow \dot{H}^s_{\bar{r},2}$ by using Lemma 2.4. Hence, the result follows from Lemma 3.11. This completes the proof. ∎

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. To this end, we establish the following nonlinear estimates.

Lemma 4.1. *Let $d \ge 3$, $1 \le s < \frac{d}{2}$, $0 < b < \min\{4, 2 + \frac{d}{2} - s\}$ and $\sigma = \frac{8-2b}{d-2s}$. If σ is not an even integer, assume that $\sigma > \lceil s \rceil - 2$. Then, for any interval $I (\subset \mathbb{R})$, we have*

$$\| |x|^{-b} |u|^\sigma u \|_{L^{m_0'}(I, \dot{H}^{s-1}_{n_0',2})} \lesssim \|u\|^{\sigma+1}_{L^{q_0}(I, \dot{H}^s_{r_0,2})},$$

where $(m_0, n_0) \in S$ and $(q_0, r_0) \in B_0$ with $n_0 = \frac{2d}{d-2}$ and $\frac{1}{r_0} = \frac{1}{2} - \frac{2}{d(\sigma+1)}$.

Proof. An easy computation shows that

$$\frac{1}{n_0'} = \frac{1}{p_1} + \frac{b}{d} = \frac{1}{p_2} + \frac{b+s-1}{d}, \tag{4.1}$$

where

$$\frac{1}{p_1} = \frac{\sigma}{\alpha} + \frac{1}{\beta}, \quad \frac{1}{p_2} = \frac{\sigma+1}{\alpha}, \quad \frac{1}{\alpha} = \frac{1}{r_0} - \frac{s}{d}, \quad \frac{1}{\beta} = \frac{1}{r_0} - \frac{1}{d}. \tag{4.2}$$

We can also verify that

$$\frac{1}{r_0} > \frac{s}{d} \Leftrightarrow b < 2 + \frac{d}{2} - s.$$

Hence, using Lemma 2.4, it follows from the assumptions $b < 2 + \frac{d}{2} - s$ and $s \geq 1$ that

$$\dot{H}^s_{r_0,2} \hookrightarrow L^{\alpha,2}, \quad \dot{H}^s_{r_0,2} \hookrightarrow \dot{H}^{s-1}_{\beta,2}. \tag{4.3}$$

Using (4.1)–(4.3), Corollary 2.7, Lemmas 2.2, 2.5, and 2.8, we have

$$\begin{aligned}\||x|^{-b}|u|^\sigma u\|_{\dot{H}^{s-1}_{n_0',2}} &\lesssim \||x|^{-b}\|_{L^{d/b,\infty}} \||u|^\sigma u\|_{\dot{H}^{s-1}_{p_1,2}} + \||x|^{-b}\|_{\dot{H}^{s-1}_{d/(b+s-1),\infty}} \||u|^\sigma u\|_{L^{p_2,2}} \\ &\lesssim \|u\|^\sigma_{L^{\alpha,2(\sigma+1)}} \|u\|_{\dot{H}^{s-1}_{\beta,2(\sigma+1)}} + \|u\|^{\sigma+1}_{L^{\alpha,2(\sigma+1)}} \\ &\lesssim \|u\|^\sigma_{L^{\alpha,2}} \|u\|_{\dot{H}^{s-1}_{\beta,2}} + \|u\|^{\sigma+1}_{L^{\alpha,2}} \lesssim \|u\|^{\sigma+1}_{\dot{H}^s_{r_0,2}},\end{aligned}$$

where we used the fact that $\sigma > \lceil s \rceil - 2$ when σ is not an even integer. On the other hand, we can also see that

$$\frac{1}{m_0'} = \frac{\sigma+1}{q_0}. \tag{4.4}$$

Using (4.3), (4.4) and Hölder's inequality, we immediately get the desired result. ■

Lemma 4.2. *Let $d \in \mathbb{N}$, $0 \leq s < \frac{d}{2}$, $0 < b < \min\{4, d\}$ and $\sigma = \frac{8-2b}{d-2s}$. Then, for any interval $I(\subset \mathbb{R})$, there exist $(m_1, n_1) \in B_0$, $(q_1, r_1) \in B_0$ and $(q_2, r_2) \in B_0$ such that*

$$\||x|^{-b}|u|^\sigma v\|_{L^{m_1'}(I, L^{n_1',2})} \lesssim \|u\|^\sigma_{L^{q_1}(I, \dot{H}^s_{r_1,2})} \|v\|_{L^{q_2}(I, L^{r_2,2})}.$$

Proof. We first claim that there exist $(m_1, n_1) \in B_0$, $(q_1, r_1) \in B_0$ and $(q_2, r_2) \in B_0$ such that

$$\frac{1}{n_1'} = \sigma\Big(\frac{1}{r_1} - \frac{s}{d}\Big) + \frac{1}{r_2} + \frac{b}{d}, \quad \frac{1}{r_1} - \frac{s}{d} > 0. \tag{4.5}$$

In fact, we can see that there exists $(m_1, n_1) \in B_0$ satisfying (4.5) provided that we can prove that there exist $r_1, r_2 > 0$ such that

$$\begin{cases} \max\{\frac{d-4}{2d}, \frac{s}{d}\} < \frac{1}{r_1} < \frac{1}{2}, \\ \max\{\frac{d-4}{2d}, 0\} < \frac{1}{r_2} < \frac{1}{2}, \\ \frac{1}{2} < \sigma(\frac{1}{r_1} - \frac{s}{d}) + \frac{1}{r_2} + \frac{b}{d} < \min\{1, \frac{1}{2} + \frac{2}{d}\}. \end{cases} \tag{4.6}$$

We divide the study into two cases: $d \geq 4$ and $d < 4$. If $d < 4$, then we can easily check that there exists $r_2 > 0$ satisfying (4.6), if we can choose $r_1 > 0$ such that

$$\frac{\sigma s}{d} < \frac{\sigma}{r_1} < \min\Big\{\frac{\sigma}{2}, 1 + \frac{\sigma s}{d} - \frac{b}{d}\Big\}. \tag{4.7}$$

It is obvious that there exists $r_1 > 0$ satisfying (4.7), due to the fact that $s < \frac{d}{2}$ and $b < d$. If $d \geq 4$, then we can also see that there exists $r_2 > 0$ satisfying (4.6), if we can choose $r_1 > 0$ such that

$$\max\Big\{\frac{d-4}{2d}, \frac{s}{d}\Big\} < \frac{1}{r_1} < \min\Big\{\frac{1}{2}, \frac{s}{d} + \frac{4-b}{d\sigma}\Big\}. \tag{4.8}$$

We can easily check that there exists r_1 satisfying (4.8) by using the fact $s < \frac{d}{2}$, $b < 4$ and $\sigma = \frac{8-2b}{d-2s}$.

Using (4.5), Lemmas 2.1, 2.2, and 2.4, we have

$$\||x|^{-b}|u|^{\sigma}v\|_{L^{n_1',2}} \lesssim \||x|^{-b}\|_{L^{d/b,\infty}}\|u\|^{\sigma}_{L^{\bar{\alpha},2}}\|v\|_{L^{r_2,2}} \lesssim \|u\|^{\sigma}_{\dot{H}^{s}_{r_1,2}}\|v\|_{L^{r_2,2}}, \tag{4.9}$$

where $\frac{1}{\bar{\alpha}} := \frac{1}{r_1} - \frac{s}{d}$. Meanwhile, (4.5) is equivalent to

$$\frac{1}{m_1'} = \frac{\sigma}{q_1} + \frac{1}{q_2}. \tag{4.10}$$

Using (4.9), (4.10) and Hölder's inequality, we immediately get the desired result. ∎

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We first prove the local well-posedness result. Let $T > 0$ and $M > 0$ which will be chosen later. Given $I = [-T, T]$, we define the complete metric space (D_1, d) as follows:

$$D_1 = \left\{u \in \bigcap_{i=0}^{2} L^{q_i}(I, H^{s}_{r_i,2}) : \max_{i\in\{0,1,2\}} \|u\|_{L^{q_i}(I,H^{s}_{r_i})} \le M\right\},$$
$$d(u,v) = \max_{i\in\{0,1,2\}} \|u-v\|_{L^{q_i}(I,L^{r_i})},$$

where $(q_i, r_i) \in B_0$ is given in Lemmas 4.1 and 4.2. Now, we consider the mapping

$$\mathcal{T}:\ u(t) \to S(t)u_0 - i\lambda\int_0^t S(t-\tau)|x|^{-b}|u(\tau)|^{\sigma}u(\tau)d\tau. \tag{4.11}$$

Lemma 2.9 (standard Strichartz estimates) and Corollary 3.13 (regular Strichartz estimates) yield that

$$\begin{aligned}
&\max_{i\in\{0,1,2\}} \|\mathcal{T}u\|_{L^{q_i}(I,H^{s}_{r_i,2})}\\
&\quad\le \max_{i\in\{0,1,2\}} \|\mathcal{T}u\|_{L^{q_i}(I,\dot{H}^{s}_{r_i,2})} + \max_{i\in\{0,1,2\}} \|\mathcal{T}u\|_{L^{q_i}(I,L^{r_i,2})}\\
&\quad\le \max_{i\in\{0,1,2\}} \|S(t)u_0\|_{L^{q_i}(I,H^{s}_{r_i,2})} + C\||x|^{-b}|u|^{\sigma}u\|_{L^{m_0'}(I,\dot{H}^{s-1}_{n_0',2})}\\
&\qquad + C\||x|^{-b}|u|^{\sigma}u\|_{L^{m_1'}(I,L^{n_1',2})},
\end{aligned} \tag{4.12}$$

where $(m_0, n_0) \in S$ is given in Lemma 4.1 and $(m_1, n_1) \in B_0$ is as in Lemma 4.2. By Strichartz estimate (2.1), we can see that

$$\max_{i\in\{0,1,2\}} \|S(t)u_0\|_{L^{q_i}([-T,T],H^{s}_{r_i,2})} \quad \text{as } T \to 0.$$

We take $M > 0$ satisfying $CM^\sigma \le \frac{1}{4}$ and $T > 0$ such that

$$\max_{i\in\{0,1,2\}} \|S(t)u_0\|_{L^{q_i}([-T,T],H^s_{r_i,2})} \le \frac{M}{2}. \tag{4.13}$$

Using (4.12), (4.13), Lemmas 4.1 and 4.2, we have

$$\max_{i\in\{0,1,2\}} \|\mathcal{T}u\|_{L^{q_i}(I,H^s_{r_i})} \le \frac{M}{2} + CM^{\sigma+1} \le M. \tag{4.14}$$

Similarly, it follows from Lemmas 2.9 and 4.2 that

$$\begin{aligned} d(\mathcal{T}u,\mathcal{T}v) &\lesssim \||x|^{-b}(|u|^\sigma + |v|^\sigma)|u-v|\|_{L^{m_1'}(I,L^{n_1',2})} \\ &\le 2CM^\sigma d(u,v) \le \frac{1}{2}d(u,v). \end{aligned} \tag{4.15}$$

Using (4.14), (4.15) and the standard argument(see, e.g., [39, Section 4.3]), we can get the desired result.

Let us prove the small data global well-posedness result. Let $M_1 > 0$ and $M_2 > 0$ which will be chosen later. For $X := \bigcap_{i=0}^2 L^{q_i}(\mathbb{R}, H^s_{r_i,2})$, we define the complete metric space (D_2, d) as follows:

$$\begin{aligned} D_2 &= \Big\{u \in X : \max_{i\in\{0,1,2\}} \|u\|_{L^{q_i}(\mathbb{R},H^s_{r_i})} \le M_1,\ \max_{i\in\{0,1,2\}} \|u\|_{L^{q_i}(\mathbb{R},\dot{H}^s_{r_i})} \le M_2\Big\}, \\ d(u,v) &= \max_{i\in\{0,1,2\}} \|u-v\|_{L^{q_i}(\mathbb{R},L^{r_i})}, \end{aligned}$$

where $(q_i, r_i) \in B_0$ is given in Lemmas 4.1 and 4.2. Using the similar argument to the above, we have

$$\max_{i\in\{0,1,2\}} \|\mathcal{T}u\|_{L^{q_i}(I,H^s_{r_i})} \le C\|u_0\|_{H^s} + CM_2^\sigma M_1, \tag{4.16}$$

$$\max_{i\in\{0,1,2\}} \|\mathcal{T}u\|_{L^{q_i}(I,\dot{H}^s_{r_i})} \le C\|u_0\|_{\dot{H}^s} + CM_2^{\sigma+1}, \tag{4.17}$$

$$d(\mathcal{T}u,\mathcal{T}v) \le 2CM_2^\sigma d(u,v). \tag{4.18}$$

Putting $M_1 = 2C\|u_0\|_{H^s}$, $M_2 = 2C\|u_0\|_{\dot{H}^s}$ and $\delta = 2(4C)^{-\frac{\sigma+1}{\sigma}}$. If $\|u_0\|_{\dot{H}^s} \le \delta$, i.e., $CM_2^\sigma < \frac{1}{4}$, it follows from (4.16)–(4.18) that $\mathcal{T} : (D_2, d) \to (D_2, d)$ is a contraction mapping. Hence, there exists a unique global solution of (1.1) in D_2. The rest part is proved by using the standard argument (see, e.g., [3]). This completes the proof. ■

5. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. To this end, we establish the following nonlinear estimates.

Lemma 5.1. *Let $s > 0$, $\sigma > \lceil s \rceil$ and $1 < p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{r} + \frac{\sigma}{q}$. Assume that $f \in C^{\lceil s \rceil}(\mathbb{C} \to \mathbb{C})$ satisfies*

$$|f^{(k)}(u)| \lesssim |u|^{\sigma+1-k} \tag{5.1}$$

for any $0 \le k \le 2$ and $u \in \mathbb{C}$. Then, we have

$$\begin{aligned}\|f(u) - f(v)\|_{\dot{H}^s_{p,2}} &\lesssim (\|u\|^{\sigma}_{L^{q,2}} + \|v\|^{\sigma}_{L^{q,2}})\|u - v\|_{\dot{H}^s_{r,2}} \\ &\quad + (\|u\|^{\sigma-1}_{L^{q,2}} + \|v\|^{\sigma-1}_{L^{q,2}})(\|u\|_{\dot{H}^s_{r,2}} + \|v\|_{\dot{H}^s_{r,2}})\|u - v\|_{L^{q,2}}.\end{aligned}$$

Proof. Without loss of generality and for simplicity, we assume that f is a function of a real variable. We have

$$f(u) - f(v) = (u - v)\int_0^1 f'(v + t(u - v))dt.$$

Putting

$$\frac{1}{p_1} = \frac{\sigma}{q}, \quad \frac{1}{p_2} = \frac{\sigma - 1}{q} + \frac{1}{r}, \tag{5.2}$$

it follows from Lemmas 2.2 and 2.5 that

$$\|f(u) - f(v)\|_{\dot{H}^s_{p,2}} = \left\|(u - v)\int_0^1 f'(v + t(u - v))dt\right\|_{\dot{H}^s_{p,2}} \lesssim I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \left\|\int_0^1 f'(v + t(u - v))dt\|_{L^{p_1,2}}\right\|u - v\|_{\dot{H}^s_{r,2}}, \\ I_2 &= \left\|\int_0^1 f'(v + t(u - v))dt\|_{\dot{H}^s_{p_2,2}}\right\|u - v\|_{L^{q,2}}.\end{aligned}$$

First, we estimate I_1. It follows from (5.1) that

$$|f'(v + t(u - v))| \lesssim \max_{t\in[0,1]} |v + t(u - v)|^{\sigma} \lesssim |u|^{\sigma} + |v|^{\sigma} \tag{5.3}$$

for any $0 \le t \le 1$. Using (5.2), (5.3), Lemmas 2.1 and 2.2, we have

$$\begin{aligned} I_1 &\le \|u - v\|_{\dot{H}^s_{r,2}} \int_0^1 \|f'(v + t(u - v))\|_{L^{p_1,2}}dt \\ &\lesssim \||u|^{\sigma} + |v|^{\sigma}\|_{L^{p_1,2}}\|u - v\|_{\dot{H}^s_r} \le (\|u\|^{\sigma}_{L^{q,2}} + \|v\|^{\sigma}_{L^{q,2}})\|u - v\|_{\dot{H}^s_{r,2}}. \end{aligned} \tag{5.4}$$

Next, we estimate I_2. We have

$$\left\|\int_0^1 f'(v + t(u - v))dt\right\|_{\dot{H}^s_{p_2,2}} \le \int_0^1 \|f'(v + t(u - v))\|_{\dot{H}^s_{p_2,2}}dt. \tag{5.5}$$

It also follows from Lemmas 2.2 and 2.6 that

$$\|f'(v+t(u-v))\|_{\dot{H}^s_{p_2,2}} \lesssim \|f''(v+t(u-v))\|_{L^{p_3,2}} \|v+t(u-v)\|_{\dot{H}^s_{r,2}}, \tag{5.6}$$

where

$$\frac{1}{p_3} = \frac{\sigma-1}{q}.$$

We can also see that

$$|f''(v+t(u-v))| \lesssim \max_{t\in[0,1]} |v+t(u-v)|^{\sigma-1} \lesssim |u|^{\sigma-1} + |v|^{\sigma-1} \tag{5.7}$$

and

$$\begin{aligned} \|v+t(u-v)\|_{\dot{H}^s_{r,2}} &\le \max_{t\in[0,1]} \|(-\Delta)^{s/2}v + t[(-\Delta)^{s/2}u - (-\Delta)^{s/2}v]\|_{L^{r,2}} \\ &\lesssim \|u\|_{\dot{H}^s_{r,2}} + \|v\|_{\dot{H}^s_{r,2}} \end{aligned} \tag{5.8}$$

for any $0 \le t \le 1$. Using (5.2), (5.5)–(5.8), Lemmas 2.1 and 2.2, we have

$$\begin{aligned} I_2 &= \left\| \int_0^1 f'(v+t(u-v))dt \right\|_{\dot{H}^s_{p_2,2}} \|u-v\|_{L^{q,2}} \\ &\lesssim (\|u\|^{\sigma-1}_{L^{q,2}} + \|v\|^{\sigma-1}_{L^{q,2}})(\|u\|_{\dot{H}^s_{r,2}} + \|v\|_{\dot{H}^s_{r,2}})\|u-v\|_{L^{q,2}}. \end{aligned} \tag{5.9}$$

In view of (5.4) and (5.9), we get the desired result. This completes the proof. ∎

Lemma 5.2. *Let $d \ge 3$, $1 \le s < \frac{d}{2}$, $0 < b < \min\{4, 2+\frac{d}{2}-s\}$ and $\sigma = \frac{8-2b}{d-2s}$. If σ is not an even integer, assume that $\sigma > \lceil s \rceil - 1$. Then, for any interval $I(\subset \mathbb{R})$, we have*

$$\begin{aligned} &\||x|^{-b}(|u|^\sigma u - |v|^\sigma v)\|_{L^{m_0'}(I,\dot{H}^{s-1}_{n_0',2})} \\ &\quad\lesssim (\|u\|^\sigma_{L^{q_0}(I,\dot{H}^s_{r_0,2})} + \|u\|^\sigma_{L^{q_0}(I,\dot{H}^s_{r_0,2})})\|u-v\|_{L^{q_0}(I,\dot{H}^s_{r_0,2})}, \end{aligned}$$

where $(m_0, n_0) \in S$ and $(q_0, r_0) \in B_0$ are as in Lemma 4.1.

Proof. The result follows directly from Lemma 5.1 and the same argument as in the proof of Lemma 4.1. ∎

We are ready to prove Theorem 1.5.

Proof of Theorem 1.5. For interval $I \subset \mathbb{R}$, we define the complete metric space (D_3, d_3) as follows:

$$\begin{aligned} D_3 &= \{u \in L^{q_0}(I, \dot{H}^s_{r_0,2}) : \|u\|_{L^{q_0}(I,\dot{H}^s_{r_0})} \le M_3\}, \\ d_3(u,v) &= \|u-v\|_{L^{q_0}(I,\dot{H}^s_{r_0})}, \end{aligned}$$

where $(q_0, r_0) \in B_0$ is given in Lemma 4.1 and $M_3 > 0$ is determined later. Considering the mapping $\mathcal{T}$ defined by (4.11), it follows from Corollary 3.13 (regular Strichartz estimates), Lemmas 4.1 and 5.2 that

$$\begin{aligned} \|\mathcal{T}u\|_{L^{q_0}(I,\dot{H}^s_{r_0,2})} &\leq \|S(t)u_0\|_{L^{q_0}(I,\dot{H}^s_{r_0,2})} + C\||x|^{-b}|u|^\sigma u\|_{L^{m_0'}(I,\dot{H}^{s-1}_{n_0',2})} \\ &\leq \|S(t)u_0\|_{L^{q_0}(I,\dot{H}^s_{r_0,2})} + C\|u\|^{\sigma+1}_{L^{q_0}(I,\dot{H}^s_{r_0,2})} \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} d_3(\mathcal{T}u, \mathcal{T}v) &\lesssim \||x|^{-b}(|u|^\sigma u - |v|^\sigma v)\|_{L^{m_0'}(I,\dot{H}^{s-1}_{n_0',2})} \\ &\lesssim (\|u\|^\sigma_{L^{q_0}(I,\dot{H}^s_{r_0,2})} + \|u\|^\sigma_{L^{q_0}(I,\dot{H}^s_{r_0,2})})\|u-v\|_{L^{q_0}(I,\dot{H}^s_{r_0,2})}, \end{aligned} \tag{5.11}$$

where $(m_0, n_0) \in S$ is given in Lemma 4.1. Using (5.10), (5.11) and repeating the argument similar to that in the proof of Theorem 1.1, we can get the desired result. ■

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