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# On the fractional Keller–Segel chemotaxis model

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**Abstract.** This paper studies a modified Keller–Segel chemotaxis model that uses fractional time derivatives (of order $\alpha$ and $\beta$, where $0 < \alpha, \beta < 1$) to incorporate memory effects. This approach better captures the subdiffusive behavior observed in biological systems. We perform our analysis in weak-Herz spaces by using a fixed-point argument and the decay properties of the Mittag–Leffler operators. We prove the local existence and uniqueness of mild solutions. Moreover, we show that the solutions depend continuously on the initial data, and we give a detailed description of the asymptotic behavior of solutions as $t \to 0^+$. We establish conditions for either extending solutions or encountering blow-up (chemotactic collapse).

## 1. Introduction

At the cellular level, it is known that chemotaxis plays a fundamental role in the self-organization of many biological systems. An important mathematical progress has been made in the recent decades to understand chemotaxis phenomenon. This biological topic was documented for the first time in bacterial experiments carried out by Engelmann and Pfeffer in the 1880s; see [36]. Chemotaxis can be defined as the movement of population induced by a chemical concentration gradient either toward or away from the chemical signals; such biological phenomenon is present, for example, in bacteria aggregation, immune system response or angiogenesis in embryo formation, and in tumor development. The number of mathematicians studying biological and medical processes is continuously increasing in recent years. For a broad survey with a rich selection of references on the progress of various chemotaxis models, we refer the interested reader to the papers [4, 10, 34, 35, 53]. In 1970, Keller and Segel presented a system of strongly coupled parabolic partial differential equations, which describes the aggregation of cellular slime molds like the *Dictyostelium discoideum*, see [39]. It is seen that various versions of the Keller–Segel system are available, depending on different phenomena and scales. The use of the Keller–Segel model (KS-model, for short) as a prototype in chemotaxis is due to the simplicity of the associated reaction-diffusion equations and its ability to capture natural

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experiments. The utilization of fractional differentiation can better identify the consistency between the solution and the real data, which can be referred to as an ideal tool for the mathematical modeling of real-world physical problems. In such a direction, there is a serious interest among researchers in understanding the fractional Keller–Segel model. This would imply a major breakthrough toward the mathematical development of the KS-model and its different versions. Although the implementation of the fractional approach for the KS-model is markedly different from that of the standard model, we point out that this approach cannot be done by a direct translation of the proofs from the classical time derivative to the fractional time derivative (see [1, 12, 21, 43, 44]).

In this work, we consider the time-fractional doubly parabolic Keller–Segel model for chemotaxis, which consists in a coupled system of PDEs in $\mathbb{R}^N$:

$$\begin{cases} {}^cD_t^\alpha u = \nabla\cdot(\nabla u - u\nabla v), & x\in\mathbb{R}^N,\ t>0,\\ {}^cD_t^\beta v = \Delta v - \gamma v + u, & x\in\mathbb{R}^N,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.1}$$

where ${}^cD_t^\alpha$ and ${}^cD_t^\beta$ are Caputo fractional derivatives of order $\alpha,\beta\in(0,1)$, respectively, the parameter $\gamma\geq 0$ denotes the decay rate of attractant, $u(x,t)$ represents the cell density, while $v(x,t)$ is the concentration of the chemoattractant. The terms in the first equation include the conservation of mass, the tendency of cells to diffuse under their own Brownian motion, and chemotactic drift. The terms in the second one express the diffusion and production of attractant over time. A remarkable fact is that fractional time derivative is closely related in the modeling of anomalous subdiffusion. Recent literature investigates the biological phenomenon chemotaxis with anomalous subdiffusion and memory effects, these studies are formulated with fractional temporal operators. We observe that memory effects are ubiquitous in physics and engineering. Such effects are neglected when an integer-order time derivative is assumed, see [20,23,40,60]. Fractional diffusion is referred to as subdiffusion if the fractional power is less than unity; see [30]. Note that the standard KS-model can be obtained from equation (1.1) by taking $\alpha=\beta=1$. Anomalous subdiffusion is also reported in material physics, econophysics, and biophysics (see [42]). Very recently, several papers have appeared on COVID-19 pandemic by using the Caputo fractional order derivative, see [9] and references therein. Modeling chemotactic movement can be done from two different perspectives: either from the microscopic or the macroscopic perspective (see [34]). In [41], the authors introduce mesoscopic and macroscopic model equations of chemotaxis with anomalous subdiffusion for modeling chemically directed transport of biological organisms in changing environments with diffusion hindered by traps or macromolecular crowding. These models of chemotaxis use fractional time derivative in their formulation.

### 1.1. The time-fractional Keller–Segel model

Let us recall some previous works on time-fractional KS-model. In [44], Li, Liu, and Wang have investigated a fractional time-space generalized KS-model, where Caputo derivative models memory effects in time and fractional Laplacian represents Lévy diffusion. This model takes into account the long rang interaction by using the Riesz potential with a singular kernel. The authors have provided the existence and uniqueness of the mild solutions when the initial data are in $L^p$ spaces. Furthermore, they proved the $L^1$-integrability and integral preservation when the initial data are in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. Also, some important properties such as the nonnegativity preservation, mass conservation, and blow-up behavior were established. In [43], the authors developed some compactness criteria to study the existence of weak solutions to time-fractional KS equations in $\mathbb{R}^2$. In [1], the authors established the existence of nonnegative weak solutions to a time-fractional KS-model with Dirichlet (resp., Neumann) boundary conditions in a bounded domain with smooth boundary in $\mathbb{R}^N$. In [59], the authors considered a time-fractional Keller–Segel model with Dirichlet conditions on the boundary and Caputo fractional derivative for the time. They proved the existence of solutions by using the Faedo–Galerkin method with some compactness arguments. In addition, the energy method is used to prove the stability of the weak solution and verify the uniqueness of the weak solution. The paper [29] concerns the existence, uniqueness, and $L^\infty$-boundedness for the weak solutions to a time-ractional Keller–Segel system with logistic source and Newton potential over unbounded regions. The time-fractional derivatives are taken in the Caputo sense. Assuming that the chemoattractant concentration is time independent, one obtains the parabolic-elliptic KS-model, which exhibits a very rich behavior, emphasized by the critical mass phenomenon arising in two dimension of space [12]. In the paper [38], the authors investigate the Cauchy problem of a time-space fractional parabolic-elliptic Keller–Segel equation in $\mathbb{R}^N$, $N \geq 2$, with Caputo derivative in time. By using the classical energy method combined with the generalized strong compactness criteria to time fractional partial differential equations, the global existence of weak solutions was established. Furthermore, mass conservation, the decay estimates, and hypercontractive estimates in $L^r(\mathbb{R}^N)$, $1 < r < \infty$, of weak solutions are studied. In [47], the authors have considered a fractional time-space parabolic-elliptic KS-model with Caputo derivative in time. When the spatial variable is the whole space $\mathbb{R}^N$, $N \geq 2$, the authors have established the global-in-time existence of the small-data solution in a critical Besov space. For the bounded domain case, by using a discrete spectrum of the Neumann Laplace operator, they provided the existence and uniqueness of a mild solution in Hilbert scale spaces. The authors also showed that this solution either exists globally in time or blows up in finite time. In [18], the authors have investigated the solutions and conservation laws of Keller–Segel-type time fractional diffusion equations. Lie symmetries admitted by this fractional system in Riemann–Liouville sense are derived by means of symmetry analysis. Similarity reductions are performed to construct the group invariant solution and the power series solution is deduced with the help of Erdélyi–Kober differential operator. In addition, analytical solution and numerical solution to initial value problems of time fractional KS-model in Caputo sense are established utilizing invariant subspace method and $q$-homotopy analysis method, respectively. In [26], the authors have proved the existence of a unique global classical solution that is uniform-in-time bounded for a time-space fractional parabolic-elliptic KS-model. With the aid of the Lyapunov functional and fractional Duhamel-type integral equations, exponential stabilization of the global solution toward the spatially homogeneous steady states is obtained. In [15], the geometric singular perturbation theory and Fredholm theorem are used to study persistence of traveling waves to a time fractional KS-model without and with the small parameter. Recently, many researchers are working to develop numerical and analytical techniques to determine either exact or approximate solutions for the time fractional KS-model. For instance, these techniques include the homotopy perturbation transform method [46], $q$-homotopy analysis transform method [18, 49], Daftardar–Jafari method [48], implicit-explicit time-splitting scheme [58], finite difference method [46], Faedo–Galerkin method [59], invariant subspace method [18], Laplace Adomian decomposition method [46]. Based on these tools, a wide range of time fractional Keller–Segel chemotaxis models have been analyzed.

### 1.2. The KS-model (1.1) with $\alpha = \beta \in (0, 1)$

To the best of our knowledge, there are a few results in the literature about special solutions for the time-fractional KS-model (1.1). We will comment briefly on some developments in the analysis of problem (1.1) with $\alpha = \beta \in (0, 1)$. In a recent study [22], are presented relevant results on the regularity and large time behavior in Lebesgue space. In [12], the authors have obtained some refined results on the large time behavior of solutions, which are presupposed to enjoy some uniform boundedness properties. Furthermore, the well-posedness and asymptotic stability of solutions in Marcinkiewicz spaces are studied. In [7], the authors have shown existence results with small data in a class of Besov–Morrey spaces. At the same time, self-similar solutions were obtained as well as an asymptotic stability result was presented. In [50], the authors considered the substitution of the classical diffusion in the KS-model by a fractional diffusion and they provided the global existence and self-similarity in critical homogeneous Besov spaces. In the context of Neumann boundary conditions, a recent work presented results on well-posedness and blow-up in Lebesgue and Besov spaces (see [21]). The paper [5] concerns the global well-posedness and the asymptotic behavior of solutions in critical Besov–Herz-type spaces.

Bacteria or microorganisms often live in a viscous fluid. Generally, the motion of the fluid is governed by the incompressible Navier–Stokes equations. The authors in [6] considered the KS-model coupled with the Navier–Stokes fluid and they proved the existence of global mild solutions with the small critical initial data in Besov–Morey spaces. Furthermore, the authors obtained self-similar solutions provided the initial data are homogeneous functions with small norms and considering the case of chemical attractant without degradation rate. They also have shown the asymptotic stability of solutions as the time goes to infinity and obtained a class of asymptotically self-similar ones. The first chemotaxis model under the effect of Navier–Stokes fluid was proposed by Tuval et al. [54] to describe the dynamics of swimming aerobic bacteria living in an incompressible viscous fluid, which swim toward a higher concentration of oxygen and chemical attractant. There are a large quantity of literature discussing the standard chemotaxis-fluid system, see [14,25,45,55,56] for instance. The paper [37] investigates the KS-model coupled with the Navier–Stokes fluid and Lévy diffusion in $\mathbb{R}^N$, $N \geq 2$. The local (resp., global) existence in Lebesgue space is obtained. In addition, the regularities of local and global mild solutions are improved in fractional homogeneous Sobolev spaces. Furthermore, mass conservation, decay estimates, and self-similarity are established.

### 1.3. Main results

In a very recently article by Bezerra et al. [13] are stated and proved results on local and global solutions for the fractional-in-time doubly parabolic Keller–Segel system in a bounded $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, for $\alpha \neq \beta$. More precisely, the authors obtain results on existence, uniqueness, continuous dependence on the initial data and its robustness, continuation, and blow-up alternative of solutions in Lebesgue spaces. They use those results to show the existence of global solutions to the problem, when the chemoattractant diffusion is not slower than the cell diffusion. So far, Lebesgue spaces are maximal one where one has proved the existence of mild solutions for the case $\alpha \neq \beta$. We observe that function spaces have been widely used in various areas of analysis such as harmonic analysis, and PDEs (see [17, 27, 28, 57] and references therein). In connection with this comment, the natural step is to extend the results to a wider class of solutions of (1.1).

The main target of this work is to explore Herz solvability to the time-fractional KS-model with distinct fractional diffusions of the cells and chemoattractant. To the best of our knowledge, well-posedness and blow-up results for the IVP (1.1) in the setting of Herz spaces seem not to be available in the previous literature. In many problems, one encounters quantities that arise naturally and are worth considering, but for which no exact formula is known. In such case, we can derive approximations for these quantities when the argument tends to a particular value or infinity. Asymptotic analysis is a method of describing limiting behavior and a fundamental tool in the exploration of real-world models. In this paper, we also give a detailed description of the limit behavior of the solution when the argument tends toward $0^+$. We do not treat here the long-term dynamics of system (1.1), this topic exceeds the scope of this paper. It is worth observing that in the case of Lebesgue spaces, in [13], the authors have established the global-in-time existence of the small-data solution of (1.1), when $\alpha \neq \beta$. Moreover, it satisfies $\|u(t) - E_\alpha(t^\alpha \Delta)\|_{L^\infty} \leq C t^{-\frac{\alpha N}{2p_0}}$ for all $t > 1$, with $C > 0$ and $p_0 > 0$ a suitable number. Most probably, we can obtain a similar result in the weak-Herz spaces setting. We hope to present results considering long-term dynamics of (1.1) in a future work. Our expectation is that our work will be useful to applied mathematicians who are interested in the development of the theory of the Keller–Segel models.

In [33], Herz generalized the Beurling algebras in the study of absolutely convergent Fourier transform. These generalized spaces of functions are just the prototype of Herz spaces. During the last decades, Herz spaces have attracted many researchers and have been considered in various aspects, such as in characterizing the properties of functions and multipliers on the classical Hardy spaces, in regularity theory for elliptic and parabolic equations, and in boundedness of many operators of the partial differential equations, Herz spaces also appear in some versions of classical spaces, such as Hardy–Herz, Sobolev–Herz, Besov–Herz, Fourier–Herz, and Triebel–Lizorkin–Herz spaces (see [8, 19, 31–33, 52] and references therein). Weak-Herz spaces are spaces of measurable functions that are defined using a sum of terms involving the functions' values on different scales. These spaces have interesting applications in image processing (see [16]). We note that Weak-Herz spaces contain standard spaces like Lebesgue, Marcinkiewicz, Besov–Morrey, and Herz spaces and it can be used to generalize Sobolev spaces. Weak-Herz spaces are not necessarily contained in either Morrey or Besov spaces (see Remark 2.2). In the analysis of time fractional KS-model the advantage of working in weak-Herz spaces rather than space as in the usual Besov framework lies in that they are more adapted for estimating the Mittag–Leffler operators. This work can be considered as a first approximation toward a systematic study of special solutions for (1.1). In [52], the authors use weak-Herz spaces to explore mild solutions of the incompressible Navier–Stokes equations. More precisely, the authors have constructed the solutions in the class $L^{\infty}(0,T;W\dot{K}^{\alpha}_{p,\infty})$ with initial data in $W\dot{K}^{\alpha}_{p,\infty}$.

Assume that $p,s,\omega>1$, $r,q\in(p,\min\{s,\omega\})$, $\nu_i,\sigma_i>0$, $i=1,2,3$. We will study well-posedness and blow-up of solutions of the IVP (1.1) with initial data $[u_0,v_0]$ belonging to the homogeneous weak-Herz space $W\dot{K}^{\nu_1}_{p,\infty}\times W\dot{K}^{\nu_2}_{q,\infty}$. Since we need to choose carefully auxiliary time-weight norm in order to control some estimates associated to the problem (1.1) on the weak-Herz spaces. We will impose the following requirement.

**Condition 1.** Suppose that parameters $p$, $q$, $r$, $s$, $\omega$, $\alpha$, $\beta$, $\sigma_i$, $\nu_i$, $i=1,2,3$, satisfy the following conditions:

(i) $\frac{N}{\omega}+\frac{\beta}{\alpha}<1$, $\frac{1}{r}+\frac{1}{\omega}<1$,

(ii) $N(\frac{1}{p}-\frac{1}{r})<\min\{1,\frac{\sigma_1}{\alpha}\}$, $\frac{N}{2}(\frac{1}{q}-\frac{1}{s})<\min\{1,\frac{\sigma_2}{\beta}\}$,

(iii) $N(\frac{1}{r}-\frac{1}{s})<1$, $N(\frac{1}{r}-\frac{1}{\omega})<1$, $N(\frac{1}{q}-\frac{1}{\omega})<1$.

Moreover, we will require that $\nu_1$, $\nu_2$, $\nu_3$, $\sigma_1$, $\sigma_2$, and $\sigma_3$ satisfy the following conditions:

(a) $\nu_2<\nu_1<\min\{N(1-\frac{1}{r})-\frac{N}{\omega},1+\nu_2\}$, $\nu_2<\min\{N(1-\frac{1}{q}),N(1-\frac{1}{s})\}$,

(b) $\nu_3<\min\{1-\frac{N}{\omega}-\frac{\beta}{\alpha},N(1-\frac{1}{r})-\frac{N}{\omega}-\nu_1,\nu_2\}$,

(c) $\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{\omega})<1$,

(d) $\sigma_1<\min\{1,\sigma_2,\sigma_3+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{\omega})))\}$,

(e) $\frac{\beta}{2}<\sigma_3<\min\{1-\sigma_1,\frac{\alpha}{2}(1-(\nu_3+\frac{N}{\omega}))\}$,

(f) $\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{\omega})<\min\{1,\frac{2\sigma_3}{\beta}-1\}$.

**Remark 1.1.** The technical conditions play a crucial role in the development of our results, defining the boundaries within which we can operate. It should be noticed that when we deal with fractional derivatives, new contraints appear that demand extra care to keep the assumptions being a nonempty set. More precisely, the presence of condition

$\alpha \neq \beta$ makes the analysis of certain terms more difficult to preserve the existence range for the parameters when compared with the case $\alpha = \beta$, where such terms would be canceled. In fact, since the equations are coupled, then sometimes implicitly, the parameter $\beta$ appears in the first equation that is governed by the $\alpha$-diffusion, and vice versa. It is worth noticing that at first glance the assumptions may appear very restrictive, e.g., either the conditions involving $\alpha$ and $\beta$ being too small or $\omega$ being very large. Note that we can choose $\omega$ so that $\omega > N(1 + (\frac{\alpha}{\beta} - 1)^{-1})$, where $\alpha > \beta$. Hence, we can choose $\alpha$ and $\beta$ in $(0, 1)$ large and the parameter $\omega$ not too big (see examples below). To the reader's convenience, in Remark 3.3, we develop an empirical algorithm to build some examples of parameters involved in Condition 1. Now, we provide some examples of physical parameters.

- $N = 3$; $\alpha = 0,98$; $\beta = 0,18$; $w = s = 4$; $p = 3,72$; $r = q = 3,8$; $\nu_1 = 0,5$; $\nu_2 = 0,02$; $\nu_3 = 0,01$; $\sigma_1 = 0,028$; $\sigma_2 = 1$; $\sigma_3 = 01023$.
- $N = 3$; $\alpha = 0,98$; $\beta = 0,75$; $w = s = 20$; $p = 5$; $r = q = 13$, $\nu_1 = 0,5$; $\nu_2 = 0,02$; $\nu_3 = 0,01$; $\sigma_1 = 0,5$; $\sigma_2 = 1$; $\sigma_3 = 04094$.

We will look at local-in-time solutions $[u, v]$ belonging to the following Fujita–Kato class:

$$\mathcal{X}^T = \big\{[u, v] : t^{\sigma_1} u \in C((0, T); W\dot{K}^{\nu_1}_{r,\infty}),\\ t^{\sigma_2} v \in C((0, T); W\dot{K}^{\nu_2}_{s,\infty}), t^{\sigma_3} \nabla v \in C((0, T); W\dot{K}^{\nu_3}_{\omega,\infty})\big\},$$

which is a Banach space endowed with the norm

$$\|[u, v]\|_{\mathcal{X}_T} = \sup_{0<t<T} t^{\sigma_1} \|u(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}} + \sup_{0<t<T} t^{\sigma_2} \|v(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}}\\ + \sup_{0<t<T} t^{\sigma_3} \|\nabla v(t)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}}. \tag{1.2}$$

We construct mild solutions for the time fractional KS-model (1.1), such solutions involve the Mainardi function and the Mittag–Leffler operators. In our approach, we use the idea that Mittag–Leffler operators are a superposition of heat kernels, and we have good properties for the Mainardi function. The strategy to construct the solutions is based on Duhamel's principle (see (2.3) and (2.4)) together with the Fujita–Kato's framework, which consists in constructing a fixed-point argument in the time-dependent space $\mathcal{X}^T$. Note that the presence of the Mittag–Leffler operators makes the analysis much more technical and the success of this implementation depends on structural bounds controlling the behavior of such operators.

The next result provides the well-posedness and asymptotic behavior of solutions to the initial value problem (1.1) in the framework of weak-Herz spaces.

**Theorem 1.2.** *Let $N \geq 2$, and let the exponents $\alpha$, $\beta$, $p$, $q$, $r$, $s$, $\omega$, $\sigma_i$, and $\nu_i$, $i = 1, 2, 3$, be as in Condition 1. Suppose that the initial data $[u_0, v_0] \in W\dot{K}^{\nu_1}_{p,\infty} \times W\dot{K}^{\nu_2}_{q,\infty}$. There exists a constant $T \in (0, 1)$ such that the problem (1.1) has a unique local mild solution*

*$[u, v] \in \mathcal{X}^T$ with the following asymptotic behavior:*

$$\lim_{t\to 0^+} t^{\sigma_1} \|u(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}} = 0, \tag{1.3}$$

$$\lim_{t\to 0^+} t^{\sigma_2} \|v(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}} = 0, \tag{1.4}$$

$$\lim_{t\to 0^+} t^{\sigma_3} \|\nabla v(t)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}} = 0. \tag{1.5}$$

*Moreover, the solutions depend continuously on the initial data.*

One of the most important and interesting aspects of the KS-model is the possibility of blow-up solutions. The literature refers to such blow-up as chemotactic collapse. In the analysis of problem (1.1) with $\alpha = \beta = 1$, extensive efforts have been undertaken to determinate conditions on the initial data which either guarantee or rule out the chemotactic collapse, see [3, 4, 11, 24]. The next result is devoted to obtain the continuation of solution and the blow-up behavior of the continued solution.

**Theorem 1.3.** *Under the hypothesis of Theorem 1.2, assume that $[u, v]$ is the weak-Herz mild solution obtained through Theorem 1.2, then $[u, v]$ can be uniquely continued up to maximal time $T_{\max} > T$. If $T_{\max} < +\infty$, then either* $\limsup_{t\to T^-_{\max}} \|u(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}} = \infty$ *or* $\limsup_{t\to T^-_{\max}} \|\nabla v(t)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}} = \infty$.

We conclude the introduction by describing the plan of this paper. The Mittag–Leffler operators associated with the problem (1.1) and the basic properties of weak-Herz spaces will be reviewed in the Section 2. In this section, we also give some key estimates of the heat semigroup in the framework of weak-Herz spaces, while Section 3 is dedicated to the proof of Theorem 1.2, which is achieved thanks to careful Herz estimates coming from the Mittag–Leffler operators. The paper finishes with Section 4, where the proof of Theorem 1.3 is given. Throughout this work, the letter $C$ will denote generic positive constants in the estimates which may vary from line to line or even within the same line. We denote by $\|T\|_{X\to Y}$ the operator norm of $T : X \to Y$.

## 2. The basic background: The ingredients

### 2.1. Weak-Herz spaces

First of all, we remind the reader about the definition and the basic properties of the weak-Herz spaces. For $1 < p_1 \le \infty$, $1 \le q_1 \le \infty$, and $\nu \in \mathbb{R}$. The homogeneous weak-Herz space $W\dot{K}^{\nu}_{p_1,q_1} = W\dot{K}^{\nu}_{p_1,q_1}(\mathbb{R}^N)$ is defined as the set of all measurable functions such that the following quantity is finite:

$$\|f\|_{W\dot{K}^{\nu}_{p_1,q_1}} := \begin{cases} \left(\sum_{k\in\mathbb{Z}} 2^{k\nu q_1} \|f\|^{q_1}_{L^{p_1,\infty}(A_k)}\right)^{\frac{1}{q_1}} & \text{if } q_1 < \infty, \\ \sup_{k\in\mathbb{Z}} 2^{k\nu} \|f\|_{L^{p_1,\infty}(A_K)} & \text{if } q_1 = \infty, \end{cases}$$

where $A_k = \{x \in \mathbb{R}^N : 2^{k-1} \le |x| < 2^k\}$ and $L^{p,\infty}$ denotes the Marcinkiewicz space. For $\nu \in \mathbb{R}$, $1 < p_1 \le \infty$, and $1 \le q_1 \le \infty$, the quantity $\|\cdot\|_{W\dot{K}^{\nu}_{p_1,q_1}}$ defines a norm in $W\dot{K}^{\nu}_{p_1,q_1}$ and the pair $(W\dot{K}^{\nu}_{p_1,q_1}, \|\cdot\|_{W\dot{K}^{\nu}_{p_1,q_1}})$ is a Banach space. Much more details on the treatment of weak-Herz spaces are given, for example, in the works of [31,32,52] and references therein.

**Remark 2.1.** Typical function belonging to $W\dot{K}^{\nu}_{p,q}$ is $f(x) = \sum_{k\in\mathbb{Z}} \frac{2^{-k\nu}}{|x-x_k|} \mathcal{X}_k(x)$, where $x_k = (\frac{3}{2}2^{k-1}, 0, \ldots, 0) \in \mathbb{R}^N$ and $\mathcal{X}_k$ is the characteristic function of $A_k$.

The next remark contains relation between weak-$L^p$, weak-Herz, Morrey, and Besov spaces, which can be found in [52]. We use the symbol $\mathcal{M}^r_q$ (resp., $\dot{B}^s_{p,r}$) for the homogeneous Morrey (resp., Besov) spaces.

**Remark 2.2.** (i) For $1 < p < \infty$, we have the continuous inclusion $L^p \subsetneq L^{p,\infty} \subsetneq W\dot{K}^0_{p,\infty}$.

(ii) If $1 \le q \le r < \infty$, $q < p$, and $\frac{N}{r} \ne \nu + \frac{N}{p}$, then $W\dot{K}^{\nu}_{p,\infty} \not\subset \mathcal{M}^r_q$.

(iii) If $1 < p < \sigma < \infty$ and $0 < \nu < N(1-\frac{1}{p})$, then $W\dot{K}^{\nu}_{p,\infty} \hookrightarrow \dot{B}^{-(\nu+N(\frac{1}{p}-\frac{1}{\sigma}))}_{\sigma,\infty}$ and $W\dot{K}^{0}_{p,\sigma} \hookrightarrow \dot{B}^{-N(\frac{1}{p}-\frac{1}{\sigma})}_{\sigma,\infty}$.

(iv) For $1 < p < \infty$ and $0 \le \nu < N(1-\frac{1}{p})$, we have $W\dot{K}^{\nu}_{p,\infty} \hookrightarrow \dot{B}^{-(\nu+\frac{N}{p})}_{\infty,\infty}$.

(v) For $1 < p \le \infty$ and $0 \le \nu \le N(1-\frac{1}{p})$, one has $W\dot{K}^{\nu}_{p,1} \hookrightarrow \dot{B}^{-(\nu+\frac{N}{p})}_{\infty,\infty}$.

(vi) For $1 < p < \sigma < \infty$ and $-N(\frac{1}{p}-\frac{1}{\sigma}) < \nu \le 0$, an embedding $W\dot{K}^{\nu}_{p,\infty} \hookrightarrow \dot{B}^{-(\nu+N(\frac{1}{p}-\frac{1}{\sigma}))}_{\sigma,\infty}$ does not hold.

(vii) For $1 < p < \infty$ and $-\frac{N}{p} < \nu < 0$, an embedding $W\dot{K}^{\nu}_{p,\infty} \hookrightarrow \dot{B}^{-(\nu+\frac{N}{p})}_{\infty,\infty}$ does not hold.

Let us recall the estimates for the norm of the product of two functions in weak-Herz spaces (see [52]).

**Lemma 2.3.** *If $1 < p, p_1, p_2 \le \infty$, $1 \le q, q_1, q_2 \le \infty$, and $\nu, \beta_1, \beta_2 \in \mathbb{R}$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and $\nu = \beta_1 + \beta_2$, then*

$$\|fg\|_{W\dot{K}^{\nu}_{p,q}} \le C\|f\|_{W\dot{K}^{\beta_1}_{p_1,q_1}} \|g\|_{W\dot{K}^{\beta_2}_{p_2,q_2}},$$

*where $C > 0$.*

The following two lemmata give the behavior of the heat semigroup $G(t)$ in homogeneous weak-Herz spaces (see [52]).

**Lemma 2.4.** *For $1 < p < \infty$, $-\frac{N}{p} \le \mu \le \nu < N(1-\frac{1}{p})$, the following inequalities hold:*

$$\|G(t)\|_{W\dot{K}^{\nu}_{p,\infty} \to W\dot{K}^{\mu}_{p,\infty}} \le Ct^{-\frac{1}{2}(\nu-\mu)},$$

$$\|\nabla G(t)\|_{W\dot{K}^{\nu}_{p,\infty} \to W\dot{K}^{\mu}_{p,\infty}} \le Ct^{-\frac{1}{2}(1+\nu-\mu)}.$$

**Lemma 2.5.** *For $1 < p < \sigma \le \infty$, $-\frac{N}{\sigma} \le \mu \le \nu < N(1-\frac{1}{p})$, the following inequalities hold:*

$$\|G(t)\|_{W\dot{K}^{\nu}_{p,\infty}\to W\dot{K}^{\mu}_{\sigma,\infty}} \le Ct^{-\frac{1}{2}(\nu-\mu+N(\frac{1}{p}-\frac{1}{\sigma}))}, \tag{2.1}$$

$$\|\nabla G(t)\|_{W\dot{K}^{\nu}_{p,\infty}\to W\dot{K}^{\mu}_{\sigma,\infty}} \le Ct^{-\frac{1}{2}(1+\nu-\mu+N(\frac{1}{p}-\frac{1}{\sigma}))}. \tag{2.2}$$

## 2.2. Mittag–Leffler operators

In accordance with Duhamel's principle, we have the following representation:

$$u(t) = E_\alpha(t^\alpha\Delta)u_0 - \int_0^t (t-s)^{\alpha-1}\nabla E_{\alpha,\alpha}((t-s)^\alpha\Delta)(u\nabla v)(s)ds, \tag{2.3}$$

$$v(t) = E_\beta(t^\beta(\Delta-\gamma))v_0 + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}((t-s)^\beta(\Delta-\gamma))u(s)ds. \tag{2.4}$$

The families $\{E_\alpha(t^\alpha\Delta)\}_{t\ge0}$, $\{E_\beta(t^\beta(\Delta-\gamma))\}_{t\ge0}$, $\{E_{\alpha,\alpha}(t^\alpha\Delta)\}_{t\ge0}$, and $\{E_{\beta,\beta}(t^\beta(\Delta-\gamma))\}_{t\ge0}$ are called the Mittag–Leffler operators. One of the crucial observations in the proof of our results is the following one: we have a mutual connection between Mittag–Leffler operators and the Mainardi function:

$$\begin{aligned} E_\alpha(t^\alpha\Delta) &= \int_0^\infty M_\alpha(s)G(st^\alpha)ds, \\ E_{\alpha,\alpha}(t^\alpha\Delta) &= \int_0^\infty \alpha s M_\alpha(s)G(st^\alpha)ds, \\ E_\beta(t^\beta(\Delta-\gamma)) &= \int_0^\infty M_\beta(s)G_\gamma(st^\beta)ds, \\ E_{\beta,\beta}(t^\beta(\Delta-\gamma)) &= \int_0^\infty \beta s M_\beta(s)G_\gamma(st^\beta)ds, \end{aligned}$$

where $M_{\cdot}(\cdot)$ is the Mainardi function (see [12]) and $G_\gamma(t) = e^{-t\gamma}G(t)$ is the damped heat semigroup (see [51]). The following classical result plays an important role in obtaining structural estimates for the Mittag–Leffler operators (see Lemma 3.2).

**Proposition 2.6.** *For $-1 < r < \infty$, the following properties hold:*

$$M_\alpha(t) \ge 0 \quad \text{for all } t \ge 0 \text{ and } \int_0^\infty t^r M_\alpha(t)dt = \frac{\Gamma(r+1)}{\Gamma(1+\alpha r)},$$

*where $\Gamma(\cdot)$ denotes the Gamma function.*

**Definition 2.7.** Let $u_0 \in W\dot{K}^{\nu_1}_{p,\infty}$, $v_0 \in W\dot{K}^{\nu_2}_{q,\infty}$, $T > 0$ and $p, s, w > 1$ with $r, q \in (p, \min\{s, w\})$. A continuous function $[u, v] : (0, T) \to W\dot{K}^{\nu_1}_{r,\infty} \times W\dot{K}^{\nu_2}_{s,\infty}$ that satisfies the Duhamel-type integral equations (2.3) and (2.4) is called mild solution to the time-fractional Keller–Segel model (1.1).

## 3. Proof of Theorem 1.2

The first thing we need is the following lemma.

**Lemma 3.1.** *Let $1 < \rho < \infty$ and $0 \le \mu < N(1 - \frac{1}{\rho})$; then, the families $(E_\alpha(t^\alpha \Delta))_{t \ge 0}$, $(E_{\alpha,\alpha}(t^\alpha \Delta))_{t \ge 0}$, $(\nabla E_{\alpha,\alpha}(t^\alpha \Delta))_{t \ge 0}$, $(E_\beta(t^\beta(\Delta - \gamma)))_{t \ge 0}$, $(\nabla E_\beta(t^\beta(\Delta - \gamma)))_{t \ge 0}$, and $(E_{\beta,\beta}(t^\beta(\Delta - \gamma)))_{t \ge 0}$ are strongly continuous in $W\dot{K}^{\mu}_{\rho,\infty}$.*

*Proof.* Let $\varphi$ be in $W\dot{K}^{\mu}_{\rho,\infty}$ taking advantage of the Lemma 2.4, we get that

$$\sup_{t \ge 0} \|G(t)\|_{W\dot{K}^{\mu}_{\rho,\infty} \to W\dot{K}^{\mu}_{\rho,\infty}} < +\infty.$$

For $0 \le t_1 < t_2$, one sees that

$$\begin{aligned} &\|E_\alpha(t_2^\alpha \Delta)\varphi - E_\alpha(t_1^\alpha \Delta)\varphi\|_{W\dot{K}^{\mu}_{\rho,\infty}} \\ &\quad \le \sup_{t \ge 0} \|G(t)\|_{W\dot{K}^{\mu}_{\rho,\infty} \to W\dot{K}^{\mu}_{\rho,\infty}} \int_0^\infty M_\alpha(s) \|G(s(t_2^\alpha - t_1^\alpha))\varphi - \varphi\|_{W\dot{K}^{\mu}_{\rho,\infty}} \, ds. \end{aligned}$$

Since the heat semigroup is strongly continuous in $W\dot{K}^{\mu}_{\rho,\infty}$, the integrand goes to zero as $t_2 \to t_1^+$. So, applying the Lebesgue's dominated convergence theorem, the continuity of map $t \to E_\alpha(t^\alpha \Delta)\varphi$ is guaranteed. Proceeding in an analogous way, the reader can check the details for the other families. This finishes the proof of Lemma 3.1. ■

We are interested in getting a priori estimates for the Mittag–Leffler operators in weak-Herz spaces.

**Lemma 3.2.** *The following upper bounds hold:*

$$\begin{aligned} &\|E_\alpha(t^\alpha \Delta)\|_{W\dot{K}^{\nu_1}_{p,\infty} \to W\dot{K}^{\nu_1}_{r,\infty}} \\ &\quad \le C t^{-\frac{\alpha N}{2}(\frac{1}{p} - \frac{1}{r})} \frac{\Gamma(1 - \frac{N}{2}(\frac{1}{p} - \frac{1}{r}))}{\Gamma(1 - \frac{\alpha N}{2}(\frac{1}{p} - \frac{1}{r}))}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\|E_\beta(t^\beta(\Delta - \gamma))\|_{W\dot{K}^{\nu_2}_{q,\infty} \to W\dot{K}^{\nu_2}_{s,\infty}} \\ &\quad \le C t^{-\frac{\beta N}{2}(\frac{1}{q} - \frac{1}{s})} \frac{\Gamma(1 - \frac{N}{2}(\frac{1}{q} - \frac{1}{s}))}{\Gamma(1 - \frac{\beta N}{2}(\frac{1}{q} - \frac{1}{s}))}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\|\nabla E_{\alpha,\alpha}(t^\alpha \Delta)\|_{W\dot{K}^{\nu_1 + \nu_3}_{\frac{rw}{r+w},\infty} \to W\dot{K}^{\nu_1}_{r,\infty}} \\ &\quad \le C t^{-\frac{\alpha}{2}(1 + \nu_3 + \frac{N}{w})} \frac{\Gamma(2 - \frac{1}{2}(1 + \nu_3 + \frac{N}{w}))}{\Gamma(1 + \alpha - \frac{\alpha}{2}(1 + \nu_3 + \frac{N}{w}))}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\|E_{\beta,\beta}(t^{\beta}(\Delta-\gamma))\|_{W\dot{K}^{\nu_1}_{r,\infty}\to W\dot{K}^{\nu_2}_{s,\infty}} \\ &\quad \le Ct^{-\frac{\beta}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))}\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}, \qquad (3.4) \\ &\|\nabla E_{\beta,\beta}(t^{\beta}(\Delta-\gamma))\|_{W\dot{K}^{\nu_1}_{r,\infty}\to W\dot{K}^{\nu_3}_{w,\infty}} \\ &\quad \le Ct^{-\frac{\beta}{2}(1+\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))}\frac{\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}, \qquad (3.5) \\ &\|\nabla E_{\beta}(t^{\beta}(\Delta-\gamma))\|_{W\dot{K}^{\nu_2}_{q,\infty}\to W\dot{K}^{\nu_3}_{w,\infty}} \\ &\quad \le Ct^{-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w}))}\frac{\Gamma(1-\frac{1}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))}{\Gamma(1-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))}. \qquad (3.6) \end{aligned}$$

*Proof.* In estimating the $W\dot{K}^{\nu_1}_{r,\infty}$-norm of $E_\alpha(t^\alpha\Delta)f$, it is convenient to keep in mind that $r > p$, $\frac{N}{2}(\frac{1}{p}-\frac{1}{r}) < 1$; from (2.1) and Proposition 2.6, we get the upper bound

$$\begin{aligned} \|E_\alpha(t^\alpha\Delta)f\|_{W\dot{K}^{\nu_1}_{r,\infty}} &\le \int_0^\infty M_\alpha(s)\|G(st^\alpha)f\|_{W\dot{K}^{\nu_1}_{r,\infty}}ds \\ &\le Ct^{-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})}\int_0^\infty M_\alpha(s)s^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{r})}ds\|f\|_{W\dot{K}^{\nu_1}_{p,\infty}} \\ &= Ct^{-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{p}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r}))}\|f\|_{W\dot{K}^{\nu_1}_{p,\infty}}. \end{aligned}$$

This allows us to conclude the proof of (3.1).

To prove (3.2), we apply Lemma 2.5 to obtain

$$\begin{aligned} \|E_\beta(t^\beta(\Delta-\gamma))f\|_{W\dot{K}^{\nu_2}_{s,\infty}} &\le \int_0^\infty M_\beta(s)e^{-st^\beta\gamma}\|G(st^\beta)f\|_{W\dot{K}^{\nu_2}_{s,\infty}}ds \\ &= Ct^{-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{q}-\frac{1}{s}))}{\Gamma(1-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s}))}\|f\|_{W\dot{K}^{\nu_2}_{q,\infty}}. \end{aligned}$$

Since $\nu_1+\nu_3 < N(1+(\frac{1}{\omega}+\frac{1}{r}))$, using again Lemma 2.5, we have that

$$\begin{aligned} &\|\nabla E_{\alpha,\alpha}(t^\alpha\Delta)f\|_{W\dot{K}^{\nu_1}_{r,\infty}} \\ &\quad \le Ct^{-\frac{\alpha}{2}(1+\nu_3+\frac{N}{\omega})}\int_0^\infty \alpha sM_\alpha(s)s^{-\frac{1}{2}(1+\nu_3+\frac{N}{\omega})}ds\|f\|_{W\dot{K}^{\nu_1+\nu_3}_{\frac{r\omega}{r+\omega},\infty}} \\ &\quad = Ct^{-\frac{\alpha}{2}(1+\nu_3+\frac{N}{\omega})}\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{\omega}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{\omega}))}\|f\|_{W\dot{K}^{\nu_1+\nu_3}_{\frac{r\omega}{r+\omega},\infty}}, \end{aligned}$$

which yields (3.3).

In order to prove (3.4), we should stress that $\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s}) < 1$, and thereupon using (2.1), we get

$$\begin{aligned}
&\|E_{\beta,\beta}(t^\beta(\Delta-\gamma))f\|_{W\dot{K}^{\nu_2}_{s,\infty}} \\
&\quad \le C\int_0^\infty \beta s M_\beta(s)(st^\beta)^{-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))}ds\|f\|_{W\dot{K}^{\nu_1}_{r,\infty}} \\
&\quad = C\beta t^{-\frac{\beta}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))}\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}\|f\|_{W\dot{K}^{\nu_1}_{r,\infty}}.
\end{aligned}$$

By using (2.2) and $\nu_1 - \nu_3 + N(\frac{1}{r} - \frac{1}{w}) < 1$, we have that

$$\begin{aligned}
&\|\nabla E_{\beta,\beta}(t^\beta(\Delta-\gamma))f\|_{W\dot{K}^{\nu_3}_{\omega,\infty}} \\
&\quad \le Ct^{-\frac{\beta}{2}(1+\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{\omega}))}\int_0^\infty M_\beta(s)s^{\frac{1}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{\omega})))}ds\|f\|_{W\dot{K}^{\nu_1}_{r,\infty}} \\
&\quad = Ct^{-\frac{\beta}{2}(1+\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{\omega}))}\frac{\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{\omega}))))}{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{\omega}))))}\|f\|_{W\dot{K}^{\nu_1}_{r,\infty}};
\end{aligned}$$

hence, we get (3.5).

We now proceed with the proof of (3.6), if we take $\nu_2 - \nu_3 + N(\frac{1}{q} - \frac{1}{w}) < 1$, $0 \le \nu_3 < \nu_2 < N(1 - \frac{1}{q})$ and $q < w$ into account and use Lemma 2.5, we get the upper bound

$$\begin{aligned}
&\|\nabla E_\beta(t^\beta(\Delta-\gamma))f\|_{W\dot{K}^{\nu_3}_{w,\infty}} \\
&\quad \le Ct^{-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w}))}\left(\int_0^\infty s^{-\frac{1}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w}))}M_\beta(s)ds\right)\|f\|_{W\dot{K}^{\nu_2}_{q,\infty}} \\
&\quad = Ct^{-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w}))}\frac{\Gamma(1-\frac{1}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))}{\Gamma(1-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))}\|f\|_{W\dot{K}^{\nu_2}_{q,\infty}}.
\end{aligned}$$

This completes the proof. ∎

*Proof of Theorem* 1.2. The proof of Theorem 1.2 subdivides into several parts. Let $R$ be a positive number and $T \in (0, 1)$ small enough to be specified below. Define $B_T[0, R]$ as the closed ball in $\mathcal{X}^T$ centered in the origin with radius $R$. Let $[u, v]$ be in $B_T[0, R]$, define $\Phi([u, v])$ by

$$\Phi([u,v]) = [\varphi_1([u,v]), \varphi_2([u,v])], \tag{3.7}$$

where

$$\varphi_1([u,v])(t) = E_\alpha(t^\alpha\Delta)u_0 - \int_0^t (t-s)^{\alpha-1}\nabla E_{\alpha,\alpha}((t-s)^\alpha\Delta)(u\nabla v)(s)ds,$$

whereas

$$\varphi_2([u,v])(t) = E_\beta(t^\beta(\Delta-\gamma))v_0 + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}((t-s)^\beta(\Delta-\gamma))u(s)ds.$$

The next step in our analysis will be to prove that $\Phi$ is well defined.

• Let $[u,v]$ be in $B_T[0,R]$ and $t\in(0,T)$. Using (3.3), Lemma 2.3, and the fact that $0\le \nu_3<1-\frac{N}{w}$ and that $\sigma_1+\sigma_3<1$, one might obtain the following estimate:

$$\begin{aligned}
&\left\|\int_0^t (t-s)^{\alpha-1}\nabla E_{\alpha,\alpha}((t-s)^\alpha\Delta)(u\nabla v)(s)ds\right\|_{W\dot{K}^{\nu_1}_{r,\infty}}\\
&\le \int_0^t (t-s)^{\alpha-1}\|\nabla E_{\alpha,\alpha}((t-s)^\alpha\Delta)(u\nabla v)(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}ds\\
&\le C\frac{\Gamma(\frac12(3-(\nu_3+\frac{N}{w})))}{\Gamma(1+\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w})))}\int_0^t (t-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1}\|(u\nabla v)(s)\|_{W\dot{K}^{\nu_1+\nu_3}_{\frac{rw}{r+w},\infty}}ds\\
&\le C\frac{\Gamma(\frac12(3-(\nu_3+\frac{N}{w})))}{\Gamma(1+\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w})))}\int_0^t (t-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1}\|u(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\|\nabla v(s)\|_{W\dot{K}^{\nu_3}_{w,\infty}}ds\\
&\le C\frac{\Gamma(\frac12(3-(\nu_3+\frac{N}{w})))}{\Gamma(1+\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w})))}\left(\int_0^t (t-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1}s^{-(\sigma_1+\sigma_3)}ds\right)\\
&\quad\times\sup_{0<t<T} t^{\sigma_1}\|u(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\sup_{0<t<T} t^{\sigma_3}\|\nabla v(t)\|_{W\dot{K}^{\nu_3}_{w,\infty}}\\
&\le C\frac{\Gamma(\frac12(3-(\nu_3+\frac{N}{w})))}{\Gamma(1+\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w})))}t^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-(\sigma_1+\sigma_3)}\\
&\quad\times \mathrm{B}\left(1-(\sigma_1+\sigma_3),\frac{\alpha}{2}\left(1-\left(\nu_3+\frac{N}{w}\right)\right)\right)\|[u,v]\|^2_{\mathcal{X}^T},
\end{aligned}\tag{3.8}$$

where B is the Euler-Beta function. If we combine inequalities (3.1) and (3.8), we see that $t^{\sigma_1}\|\varphi_1([u,v])(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}}$, $t\in(0,T)$, is bounded by

$$\begin{aligned}
&CT^{\sigma_1-\frac{\alpha N}{2}(\frac1p-\frac1r)}\frac{\Gamma(1-\frac{N}{2}(\frac1p-\frac1r))}{\Gamma(1-\frac{\alpha N}{2}(\frac1p-\frac1r))}\|u_0\|_{W\dot{K}^{\nu_1}_{p,\infty}}\\
&+CR^2T^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-\sigma_3}\frac{\Gamma(\frac12(3-(\nu_3+\frac{N}{w})))}{\Gamma(1+\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w})))}\\
&\times \mathrm{B}\left(1-(\sigma_1+\sigma_3),\frac{\alpha}{2}\left(1-\left(\nu_3+\frac{N}{w}\right)\right)\right).
\end{aligned}$$

We will choose $T$ so small to guarantee that

$$CT^{\sigma_1-\frac{\alpha N}{2}(\frac1p-\frac1r)}\frac{\Gamma(1-\frac{N}{2}(\frac1p-\frac1r))}{\Gamma(1-\frac{\alpha N}{2}(\frac1p-\frac1r))}\|u_0\|_{W\dot{K}^{\nu_1}_{p,\infty}}<\frac{R}{6}$$

and

$$\begin{aligned}
&CT^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-\sigma_3}\frac{\Gamma(\frac13(3-(\nu_3+\frac{N}{w})))}{\Gamma(1+\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w})))}\\
&\times \mathrm{B}\left(1-(\sigma_1+\sigma_3),\frac{\alpha}{2}\left(1-\left(\nu_3+\frac{N}{w}\right)\right)\right)<\frac{1}{6R}.
\end{aligned}$$

From all this, we obtain at once

$$\sup_{0<t<T} t^{\sigma_1}\|\varphi_1([u,v])(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}} < \frac{R}{3}. \tag{3.9}$$

Let $t$ be in $[0,T)$, using (3.4), and the fact that $\sigma_1<1$ and $\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))<1$, we can infer that

$$\begin{aligned}
&\left\|\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}((t-s)^\beta(\Delta-\gamma))u(s)ds\right\|_{W\dot{K}^{\nu_2}_{s,\infty}}\\
&\le \int_0^t (t-s)^{\beta-1}\|E_{\beta,\beta}((t-s)^\beta(\Delta-\gamma))u(s)\|_{W\dot{K}^{\nu_2}_{s,\infty}}ds\\
&\le C\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}\\
&\quad\times\int_0^t (t-s)^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))-1}\|u(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}ds\\
&\le C\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}t^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))-\sigma_1}\\
&\quad\times \mathrm{B}\left(1-\sigma_1,\beta\left(1-\frac{1}{2}\left(\nu_1-\nu_2+N\left(\frac{1}{r}-\frac{1}{s}\right)\right)\right)\right)\|[u,v]\|_{\mathcal{X}^T}.
\end{aligned}\tag{3.10}$$

In view of (3.2), (3.10) and recalling that $\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s})<\sigma_2<\sigma_1$, we can achieve that

$$\begin{aligned}
&\sup_{0<t<T} t^{\sigma_2}\|\varphi_2([u,v])(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}}\\
&\le CT^{\sigma_2-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{q}-\frac{1}{s}))}{\Gamma(1-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s}))}\|v_0\|_{W\dot{K}^{\nu_2}_{q,\infty}}\\
&\quad+CRT^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))+\sigma_2-\sigma_1}\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}\\
&\quad\times \mathrm{B}\left(1-\sigma_1,\beta\left(1-\frac{1}{2}\left(\nu_1-\nu_2+N\left(\frac{1}{r}-\frac{1}{s}\right)\right)\right)\right).
\end{aligned}$$

Next, we may even choose $T>0$ small enough so that

$$CT^{\sigma_1-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{q}-\frac{1}{s}))}{\Gamma(1-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s}))}\|v_0\|_{W\dot{K}^{\nu_2}_{q,\infty}} < \frac{R}{6}$$

and

$$\begin{aligned}
&CT^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))+\sigma_2-\sigma_1}\\
&\quad+\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}\\
&\quad\times \mathrm{B}\left(1-\sigma_1,\beta\left(1-\frac{1}{2}\left(\nu_1-\nu_2+N\left(\frac{1}{r}-\frac{1}{s}\right)\right)\right)\right) < \frac{1}{6};
\end{aligned}$$

this enables us to infer that

$$\sup_{0<t<T} t^{\sigma_2} |\|\varphi_2([u,v])(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}} < \frac{R}{3}. \tag{3.11}$$

Let $t$ be in $(0,T)$, using (3.5), the fact that $\sigma_1 < 1$ and $\nu_1 - \nu_3 + N(\frac{1}{r} - \frac{1}{w}) < 1$, we can infer the following estimate:

$$\begin{aligned}
&\left\| \int_0^t (t-s)^{\beta-1} \nabla E_{\beta,\beta}((t-s)^\beta(\Delta-\gamma))u(s)ds \right\|_{W\dot{K}^{\nu_3}_{w,\infty}} \\
&\le \int_0^t (t-s)^{\beta-1} \|\nabla E_{\beta,\beta}((t-s)^\beta(\Delta-\gamma))u(s)\|_{W\dot{K}^{\nu_3}_{w,\infty}} ds \\
&\le C \frac{\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))} \|[u,v]\|_{\mathcal{X}^T} \\
&\quad \times \int_0^t (t-s)^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))-1} s^{-\sigma_1} ds \\
&\le C t^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))-\sigma_1} \frac{\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))} \\
&\quad \times \mathrm{B}\left(1-\sigma_1, \frac{\beta}{2}\left(1-\left(\nu_1-\nu_3+N\left(\frac{1}{r}-\frac{1}{w}\right)\right)\right)\right) \|[u,v]\|_{\mathcal{X}^T}.
\end{aligned} \tag{3.12}$$

By applying (3.6), (3.12), (c) and (f) of Condition 1, we, therefore, get the estimate

$$\begin{aligned}
&\sup_{0<t<T} t^{\sigma_3} \|\nabla\varphi_2([u,v])(t)\|_{W\dot{K}^{\nu_3}_{w,\infty}} \\
&\le C T^{\sigma_3-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w}))} \frac{\Gamma(1-\frac{1}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))}{\Gamma(1-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))} \|v_0\|_{W\dot{K}^{\nu_2}_{q,\infty}} \\
&\quad + CR \frac{\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))} \\
&\quad \times \mathrm{B}\left(1-\sigma_1, \frac{\beta}{2}\left(1-\left(\nu_1-\nu_3+N\left(\frac{1}{r}-\frac{1}{w}\right)\right)\right)\right) T^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))+\sigma_3-\sigma_1}.
\end{aligned}$$

Once again, it is convenient to choose $T > 0$ so small that

$$C T^{\sigma_3-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w}))} \frac{\Gamma(1-\frac{1}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))}{\Gamma(1-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))} \|v_0\|_{W\dot{K}^{\nu_2}_{q,\infty}} < \frac{R}{6}$$

and

$$\begin{aligned}
&C T^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))+\sigma_3-\sigma_1} \frac{\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))} \\
&\quad \times \mathrm{B}\left(1-\sigma_1, \frac{\beta}{2}\left(1-\left(\nu_1-\nu_3+N\left(\frac{1}{r}-\frac{1}{w}\right)\right)\right)\right) < \frac{1}{6}.
\end{aligned}$$

All this enables us to conclude that

$$\sup_{0<t<T} t^{\sigma_3} \|\nabla\varphi_2([u,v])(t)\|_{W\dot{K}^{\nu_3}_{w,\infty}} < \frac{R}{3}. \tag{3.13}$$

It follows at once from (3.9), (3.11), and (3.13) that

$$\|\Phi([u,v])\|_{\mathcal{X}^T} \le R.$$

• Continuing our analysis, we will prove that $t^{\sigma_1}\varphi_1([u,v]) \in C((0,T); W\dot{K}^{\nu_1}_{r,\infty})$, $t^{\sigma_2}\varphi_2([u,v]) \in C((0,T); W\dot{K}^{\nu_2}_{s,\infty})$, and $t^{\sigma_3}\nabla\varphi_2([u,v]) \in C((0,T); W\dot{K}^{\nu_3}_{w,\infty})$. Let $0 < t_1 < t_2$, the next inequality will be our starting point for the argument which follows

$$\begin{aligned}
&\|t_2^{\sigma_1}\varphi_1([u,v])(t_2) - t_1^{\sigma_1}\varphi_1([u,v])(t_1)\|_{W\dot{K}^{\nu_1}_{r,\infty}} \\
&\le \frac{R}{3} t_2^{-\sigma_1}|t_2^{\sigma_1} - t_1^{\sigma_1}| + t_1^{\sigma_1}\|E_\alpha(t_2^\alpha\Delta)u_0 - E_\alpha(t_1^\alpha\Delta)u_0\|_{W\dot{K}^{\nu_1}_{r,\infty}} \\
&\quad + t_1^{\sigma_1}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|\nabla E_{\alpha,\alpha}((t_2-s)^\alpha\Delta)(u\nabla v)(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}ds \\
&\quad + t_1^{\sigma_1}\int_0^{t_1}\big(\|(t_1-s)^{\alpha-1}\nabla E_{\alpha,\alpha}((t_1-s)^\alpha\Delta)(u\nabla v)(s) \\
&\quad - (t_2-s)^{\alpha-1}\nabla E_{\alpha,\alpha}((t_2-s)^\alpha\Delta)(u\nabla v)(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\big)ds.
\end{aligned} \tag{3.14}$$

We derive from the strong continuity of the Mittag–Leffler family $(E_\alpha(t^\alpha\Delta))_{t\ge 0}$ (see Lemma 3.1) that the second term on the right-hand side of (3.14) goes to zero $t_2 \to t_1^+$. The third term on the right-hand side of (3.14) is estimated as follows:

$$\begin{aligned}
&\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|\nabla E_{\alpha,\alpha}((t_2-s)^\alpha\Delta)(u\nabla v)(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}ds \\
&\le C\frac{\Gamma(\frac{1}{2}(3-(\nu_3+\frac{N}{w})))}{\Gamma(1+\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w})))} \\
&\quad \times \int_{t_1}^{t_2}(t_2-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1}\|u(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\|\nabla v(s)\|_{W\dot{K}^{\nu_3}_{w,\infty}}ds \\
&\le C\frac{\Gamma(\frac{1}{2}(3-(\nu_3+\frac{N}{w})))}{\Gamma(1+\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w})))}\|[u,v]\|^2_{\mathcal{X}^T} t_2^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-\sigma_1-\sigma_3} \\
&\quad \times \int_{\frac{t_1}{t_2}}^{1}(1-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1}s^{-\sigma_1-\sigma_3}ds,
\end{aligned}$$

which in turn implies that the third term on the right-hand side of (3.14) converges to zero as $t_2 \to t_1^+$. We seek now an upper bound for the fourth term on the right-hand side of (3.14) that converges to zero as $t_2 \to t_1^+$:

$$\begin{aligned}
&\int_0^{t_1} \|(t_1-s)^{\alpha-1}\nabla E_{\alpha,\alpha}((t_1-s)^\alpha\Delta)(u\nabla v)(s) \\
&\qquad -(t_2-s)^{\alpha-1}\nabla E_{\alpha,\alpha}((t_2-s)^\alpha\Delta)(u\nabla v)(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\,ds \\
&\quad\le \int_0^{t_1}(t_1-s)^{\alpha-1}\|(\nabla E_{\alpha,\alpha}((t_1-s)^\alpha\Delta)-\nabla E_{\alpha,\alpha}((t_2-s)^\alpha\Delta))(u\nabla v)(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\,ds \\
&\qquad + C\|[u,v]\|^2_{\mathcal{X}^T}\int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|(t_1-s)^{-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w})}s^{-\sigma_1-\sigma_3}\,ds.
\end{aligned} \tag{3.15}$$

Recalling $\frac{\alpha}{2}(1+\nu_3+\frac{N}{w})<1$, $\nu_3+\frac{N}{w}<1$ and that $\sigma_1+\sigma_3<1$. We reach the conclusion that the integrand of the first (resp., second) term on the right-hand side of (3.15) is dominated by the integrable function $2C\|[u,v]\|^2_{\mathcal{X}^T}(t_1-s)^{-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w})}s^{-\sigma_1-\sigma_3}$ (resp., $2C(t_1-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1}s^{-\sigma_1-\sigma_3}$). Taking advantage of the $W\dot{K}^{\nu_1}_{r,\infty}$-strong continuity of $(\nabla E_{\alpha,\alpha}(t^\alpha\Delta))_{t\ge 0}$ and the Lebesgue dominated convergence theorem, we deduce that the first (resp., second) term on the right-hand side of (3.15) goes to zero as $t_2 \to t_1^+$. Hence, we have reached that the fourth term on the right-hand side of (3.14) goes to zero as $t_2\to t_1^+$, from which the continuity of $t^{\sigma_1}\varphi_1(u,v)$ follows.

We will see that
$$t^{\sigma_2}\varphi_2([u,v]) \in C((0,T);W\dot{K}^{\nu_2}_{s,\infty}).$$

Let $0<t_1<t_2$, we have that

$$\begin{aligned}
&\|t_2^{\sigma_2}\varphi_2[(u,v)](t_2)-t_1^{\sigma_2}\varphi_2[(u,v)](t_1)\|_{W\dot{K}^{\nu_2}_{s,\infty}} \\
&\quad\le \frac{R}{3}t_2^{-\sigma_2}|t_2^{\sigma_2}-t_1^{\sigma_2}| + t_1^{\sigma_1}\|E_\beta(t_2^\beta(\Delta-\gamma))v_0 - E_\beta(t_1^\beta(\Delta-\gamma))v_0\|_{W\dot{K}^{\nu_2}_{s,\infty}} \\
&\qquad + t_1^{\sigma_2}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\|E_{\beta,\beta}((t_2-s)^\beta(\Delta-\gamma))u(s)\|_{W\dot{K}^{\nu_2}_{s,\infty}}\,ds \\
&\qquad + t_1^{\sigma_2}\int_0^{t_1}(t_2-s)^{\beta-1}\|E_{\beta,\beta}((t_2-s)^\beta(\Delta-\gamma))u(s) \\
&\qquad - E_{\beta,\beta}((t_1-s)^\beta(\Delta-\gamma))u(s)\|_{W\dot{K}^{\nu_2}_{s,\infty}}\,ds \\
&\qquad + t_1^{\sigma_2}\int_0^{t_1}|(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}|\|E_{\beta,\beta}((t_1-s)^\beta(\Delta-\gamma))u(s)\|_{W\dot{K}^{\nu_2}_{s,\infty}}\,ds.
\end{aligned} \tag{3.16}$$

From the strong continuity of the Mittag–Leffler family $(E_\beta(t^\beta(\Delta-\gamma)))_{t\ge 0}$ (see Lemma 3.1), the second term, on the right-hand side of (3.16), goes to zero as $t_2\to t_1^+$. Next, we determine an upper bound for the third term on the right-hand side of (3.16):

$$\begin{aligned}
&\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\|E_{\beta,\beta}((t_2-s)^\beta(\Delta-\gamma))u(s)\|_{W\dot{K}^{\nu_2}_{s,\infty}}ds\\
&\quad\le C\|[u,v]\|_{\mathcal{X}^T}\frac{\Gamma(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}t_2^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))-\sigma_1}\\
&\qquad\times\int_{\frac{t_1}{t_2}}^{1}(1-s)^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))-1}s^{-\sigma_1}ds. \qquad (3.17)
\end{aligned}$$

Therefore, by virtue of $\sigma_1<1$ and $\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))<1$, one deduces that the integrand on the right-hand side of (3.17) is integrable. Consequently, we reach the conclusion that (3.17) goes to zero as $t_2\to t_1^+$, thus, we obtain that the third term, on the right-hand side of (3.16), goes to zero as $t_2\to t_1^+$. Since $(E_{\beta,\beta}(t^\beta(\Delta-\gamma)))_{t\ge0}$ is strongly continuous and the integrand of the fourth term on the right-hand side of (3.16) is dominated by the integrable function $C(t_1-s)^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))-1}s^{-\sigma_1}$, where $C>0$ is an appropriate constant. We therefore get that the fourth term on the right-hand side of (3.16) goes to zero as $t_2\to t_1^+$. The integrand of the fifth term on the right-hand side of (3.16) goes to zero as $t_2\to t_1^+$ and also notice that this term is dominated by the integrable function $C(t_1-s)^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))-1}s^{-\sigma_1}$, one then concludes that the fifth term on the right-hand side of (3.16) goes to zero as $t_2\to t_1^+$. Next, we will see that $t^{\sigma_3}\nabla\varphi_2(u,v)$ is a continuous function. Let $0<t_1<t_2$, we have that

$$\begin{aligned}
&\|t_2^{\sigma_3}\nabla\varphi_2([u,v])(t_2)-t_1^{\sigma_3}\nabla\varphi_2([u,v])(t_1)\|_{W\dot{K}^{\nu_3}_{w,\infty}}\\
&\quad\le\frac{R}{3}|t_2^{\sigma_3}-t_1^{\sigma_3}|t_2^{-\sigma_3}+t_1^{\sigma_3}\|\nabla E_\beta(t_2^\beta(\Delta-\gamma))v_0-\nabla E_\beta(t_1^\beta(\Delta-\gamma))v_0\|_{W\dot{K}^{\nu_3}_{w,\infty}}\\
&\qquad+t_1^{\sigma_3}\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\|\nabla E_{\beta,\beta}((t_2-s)^\beta(\Delta-\gamma))u(s)\|_{W\dot{K}^{\nu_3}_{w,\infty}}ds\\
&\qquad+t_1^{\sigma_3}\int_0^{t_1}(t_2-s)^{\beta-1}\|\nabla E_{\beta,\beta}((t_2-s)^\beta(\Delta-\gamma))u(s)\\
&\qquad-\nabla E_{\beta,\beta}((t_1-s)^\beta(\Delta-\gamma))u(s)\|_{W\dot{K}^{\nu_3}_{w,\infty}}ds\\
&\qquad+t_1^{\sigma_3}\int_0^{t_1}|(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}|\|\nabla E_{\beta,\beta}((t_1-s)^\beta(\Delta-\gamma))u(s)\|_{W\dot{K}^{\nu_3}_{w,\infty}}ds.
\end{aligned} \qquad (3.18)$$

The second and fourth terms on the right-hand side of (3.18) go to zero as $t_2\to t_1^+$ as a consequence of the strong continuity of the families $(\nabla E_\beta(t^\beta(\Delta-\gamma)))_{t\ge0}$ and $(\nabla E_{\beta,\beta}(t^\beta(\Delta-\gamma)))_{t\ge0}$, respectively. We have that the third term on the right-hand side of (3.18) is dominated by

$$Ct_1^{\sigma_3}t_2^{\frac{\beta}{2}(1-(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{w})))-\sigma_1}\int_{\frac{t_1}{t_2}}^{1}(1-s)^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))-1}s^{\sigma_1}ds\to0\quad\text{as }t_2\to t_1^+.$$

The fifth term on the right-hand side of (3.18) goes to zero as $t_2 \to t_1^+$ because the integrand of such term goes to zero and it is dominated by the integrable function $C(t_1 - s)^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))-1}$. Therefore, the function $t^{\sigma_3}\nabla\varphi_2([u,v]) : (0,T) \to W\dot{K}^{\nu_3}_{w,\infty}$ is continuous. It is not difficult to see, from the proof of the previous step, that $\Phi([u,v]) \in C((0,T); W\dot{K}^{\nu_1}_{r,\infty}) \times C((0,T); W\dot{K}^{\nu_2}_{s,\infty})$.

• Next, we will show that $\Phi$ is a contraction in $B_{\mathcal{X}^T}[0,R]$. Let $[u_1,v_1]$ and $[u_2,v_2]$ be in $B_{\mathcal{X}^T}[0,R]$, let us observe now that using (3.3), we get the following estimate:

$$
\begin{aligned}
&t^{\sigma_1}\|\varphi_1([u_1,v_1])(t) - \varphi_1([u_2,v_2])(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}} \\
&\quad\le Ct^{\sigma_1}\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))} \\
&\qquad\times\int_0^t (t-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1}\big(\|u_1(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\|\nabla(v_1-v_2)(s)\|_{W\dot{K}^{\nu_3}_{w,\infty}} \\
&\qquad+\|(u_1-u_2)(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\|(\nabla v_2)(s)\|_{W\dot{K}^{\nu_3}_{w,\infty}}\big)ds \\
&\quad\le 2CRt^{\sigma_1}\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))} \\
&\qquad\times\left(\int_0^t (t-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1}s^{-(\sigma_1+\sigma_3)}ds\right)\|[u_1-u_2,v_1-v_2]\|_{\mathcal{X}^T} \\
&\quad\le 2CR\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))}\mathrm{B}\left(1-(\sigma_1+\sigma_3),\frac{\alpha}{2}\left(1-\left(\nu_3+\frac{N}{w}\right)\right)\right) \\
&\qquad\times t^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-\sigma_3}\|[u_1-u_2,v_1-v_2]\|_{\mathcal{X}^T},
\end{aligned}
\tag{3.19}
$$

according to $\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-\sigma_3>0$ and choosing $T\in\mathbb{R}^+$ so that

$$
\begin{aligned}
&0 < T^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-\sigma_3} \\
&\quad< \frac{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))}{8CR\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))\mathrm{B}(1-(\sigma_1+\sigma_3),\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w})))}.
\end{aligned}
$$

Because of a priori bound (3.19), we get the following inequality:

$$
\sup_{0<t<T} t^{\sigma_1}\|\varphi_1([u_1,v_1])(t)-\varphi_1([u_2,v_2])(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}} < \frac{1}{4}\|[u_1-u_2,v_1-v_2]\|_{\mathcal{X}^T}. \tag{3.20}
$$

On the other hand, we have the following string of inequalities:

$$
\begin{aligned}
&t^{\sigma_2}\|\varphi_2([u_1,v_1])(t)-\varphi_2([u_2,v_2])(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}} \\
&\quad\le t^{\sigma_2}\int_0^t (t-s)^{\beta-1}\|E_{\beta,\beta}((t-s)^\beta(\Delta-\gamma))(u_1(s)-u_2(s))\|_{W\dot{K}^{\nu_2}_{s,\infty}}ds \\
&\quad\le C\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(t^{\sigma_2}\int_0^t (t-s)^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))-1} s^{-\sigma_1}\,ds\right)\\
&\times \|[u_1-u_2, v_1-v_2]\|_{\mathcal{X}^T}\\
&\le C\,\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}\\
&\times \mathrm{B}\left(1-\sigma_1, \beta\left(1-\frac{1}{2}\left(\nu_1-\nu_2+N\left(\frac{1}{r}-\frac{1}{s}\right)\right)\right)\right)\\
&\times t^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))+\sigma_2-\sigma_1}\|[u_1-u_2, v_1-v_2]\|_{\mathcal{X}^T}.
\end{aligned}
\tag{3.21}
$$

Taking advantage of $\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))<1$, $\sigma_2>\sigma_1$ and choosing $T>0$ so that

$$
\begin{aligned}
&T^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))+\sigma_2-\sigma_1}\\
&\quad<\frac{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}{4C\,\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))\mathrm{B}(1-\sigma_1,\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))},
\end{aligned}
$$

we obtain, directly from (3.21), the following estimate:

$$
\sup_{0<t<T} t^{\sigma_2}\|\varphi_2([u_1,v_1])(t)-\varphi([u_2,v_2])(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}}<\frac{1}{4}\|[u_1-u_2, v_1-v_2]\|_{\mathcal{X}^T}. \tag{3.22}
$$

Finally, we can prove that

$$
\begin{aligned}
&t^{\sigma_3}\|\nabla\varphi_2([u_1,v_1])(t)-\nabla\varphi_2([u_2,v_2])(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}}\\
&\le C\,\frac{\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}\\
&\times \mathrm{B}\left(1-\sigma_1, \frac{\beta}{2}\left(1-\left(\nu_1-\nu_3+N\left(\frac{1}{r}-\frac{1}{w}\right)\right)\right)\right)\\
&\times t^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))+\sigma_3-\sigma_1}\|[u_1-u_2, v_1-v_2]\|_{\mathcal{X}^T}.
\end{aligned}
\tag{3.23}
$$

Its passage can be obtained in quite an analogous way to the preceding argument. Combining the conditions $\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})<1$, $\sigma_1<\sigma_3+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))$ and choosing $T>0$, we get

$$
\begin{aligned}
&T^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))+\sigma_3-\sigma_1}\\
&\quad<\frac{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}{4C\,\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))\mathrm{B}(1-\sigma_1,\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}
\end{aligned}
$$

We derive, directly from (3.23), that

$$
\sup_{0<t<T} t^{\sigma_3}\|\nabla\varphi_2([u_1,v_1])(t)-\nabla\varphi_2([u_2,v_2])(t)\|_{W\dot{K}^{\nu_3}_{w,\infty}}<\frac{1}{4}\|[u_1-u_2, v_1-v_2]\|_{\mathcal{X}^T}. \tag{3.24}
$$

We can therefore assert from (3.20), (3.22), and (3.24) the following estimate:

$$\|\Phi([u_1,v_1])-\Phi([u_2,v_2])\|_{\mathcal{X}^T}\le\frac{3}{4}\|[u_1-u_2,v_1-v_2]\|_{\mathcal{X}^T},\tag{3.25}$$

for every $[u_i,v_i]\in B_{\mathcal{X}^T}[0,R]$, $i=1,2$. Notice that the contraction principle implies the existence of a unique fixed-point $[u,v]$ in $B_{\mathcal{X}^T}[0,R]$. Moreover, we have that $[u,v]\in C((0,T);W\dot{K}^{\nu_1}_{r,\infty})\times C((0,T);W\dot{K}^{\nu_2}_{s,\infty})$.

• Next, we look at the asymptotic profiles of the solution $[u,v]$ which are given by (1.3), (1.4), and (1.5). We have the following estimate:

$$\begin{aligned}t^{\sigma_1}\|u(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}}&\le Ct^{\sigma_1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{p}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r}))}\|u_0\|_{W\dot{K}^{\nu_1}_{p,\infty}}\\&+Ct^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-\sigma_3}\frac{\Gamma(\frac{1}{3}(3-(\nu_3+\frac{N}{w})))}{\Gamma(1+\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w})))}\\&\times\mathrm{B}\left(1-(\sigma_1+\sigma_3),\frac{\alpha}{2}\left(1-\left(\nu_3+\frac{N}{w}\right)\right)\right)\|[u,v]\|^2_{\mathcal{X}^T}.\end{aligned}$$

If we use $\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})<\sigma_1$ and the fact that $\sigma_3<\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))$, we have that (1.3) holds. On the other hand, we get

$$\begin{aligned}&t^{\sigma_2}\|v(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}}\\&\le Ct^{\sigma_2-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{q}-\frac{1}{s}))}{\Gamma(1-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s}))}\|v_0\|_{W\dot{K}^{\nu_2}_{q,\infty}}\\&\quad+Ct^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))+\sigma_2-\sigma_1}\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}\\&\quad\times\mathrm{B}\left(1-\sigma_1,\beta-\frac{\beta}{2}\left(\nu_1-\nu_2+N\left(\frac{1}{r}-\frac{1}{s}\right)\right)\right)\|[u,v]\|_{\mathcal{X}^T}.\end{aligned}$$

It is important to observe that $\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s})<\sigma_2$, $\sigma_1<\sigma_2$ and $\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))<1$, thus, we conclude that (1.4) holds.

Note that (1.5) follows from the following upper bound:

$$\begin{aligned}&t^{\sigma_3}\|\nabla v(t)\|_{W\dot{K}^{\nu_3}_{w,\infty}}\\&\le Ct^{\sigma_3-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w}))}\frac{\Gamma(1-\frac{1}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))}{\Gamma(1-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))}\|v_0\|_{W\dot{K}^{\nu_2}_{q,\infty}}\\&+Ct^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))+\sigma_3-\sigma_1}\frac{\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}\\&\times\mathrm{B}\left(1-\sigma_1,\frac{\beta}{2}\left(1-\left(\nu_1-\nu_3+N\left(\frac{1}{r}-\frac{1}{w}\right)\right)\right)\right)\|[u,v]\|_{\mathcal{X}^T}.\end{aligned}$$

The next step in our analysis will be to prove that the mild solution of (1.1) is unique in $\mathcal{X}^T$. In fact, if $[\bar{u}, \bar{v}] \in \mathcal{X}^T$ is a mild solution of (1.1), we may take $0 < \overline{T} \leq T$ such that

$$\|[\bar{u}, \bar{v}]\|_{\mathcal{X}^T} \leq R.$$

Then, by uniqueness in $B_{\mathcal{X}^{\overline{T}}}[0, R]$, we see that

$$[u(t), v(t)] = [\bar{u}(t), \bar{v}(t)]$$

for all $t \in [0, \overline{T}]$. Let us set

$$\overline{R} := \max\Big\{ \sup_{0<t<T} t^{\sigma_1} \|\bar{u}(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}}, \sup_{0<t<T} t^{\sigma_2} \|\bar{v}(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}}, \sup_{0<t<T} t^{\sigma_3} \|\nabla\bar{v}(t)\|_{W\dot{K}^{\nu_3}_{w,\infty}} \Big\}.$$

We have the following estimates:

$$\begin{aligned}
&t^{\sigma_1}\|u(t)-\bar{u}(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}} \\
&\quad \leq t^{\sigma_1}\int_T^t (t-s)^{\alpha-1}\|\nabla E_{\alpha,\alpha}((t-s)^\alpha\Delta)((u-\bar{u})\nabla v+\bar{u}\nabla(v-\bar{v}))(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}ds \\
&\quad \leq Ct^{\sigma_1}\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))}\int_T^t (t-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1} \\
&\qquad \times\big(\|u(s)-\bar{u}(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\|(\nabla v)(s)\|_{W\dot{K}^{\nu_3}_{w,\infty}}+\|\bar{u}(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\|\nabla(v-\bar{v})(s)\|_{W\dot{K}^{\nu_3}_{w,\infty}}\big)ds \\
&\quad \leq Ct^{\sigma_1}(R+\overline{R})\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))}\int_T^t (t-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1}s^{-\sigma_1-\sigma_3} \\
&\qquad \times\big(s^{\sigma_1}\|u(s)-\bar{u}(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}+s^{\sigma_3}\|\nabla(v-\bar{v})(s)\|_{W\dot{K}^{\nu_3}_{w,\infty}}\big)ds \\
&\quad \leq C(R+\overline{R})T^{\sigma_1}\overline{T}^{-\sigma_1-\sigma_3}\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))} \\
&\qquad \times\int_{\overline{T}}^t (t-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1}\big(s^{\sigma_1}\|u(s)-\bar{u}(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}+s^{\sigma_3}\|\nabla(v-\bar{v})(s)\|_{W\dot{K}^{\nu_3}_{w,\infty}}\big)ds.
\end{aligned} \tag{3.26}$$

Furthermore, we have that

$$\begin{aligned}
&t^{\sigma_3}\|\nabla v(t)-\nabla\bar{v}(t)\|_{W\dot{K}^{\nu_3}_{w,\infty}} \\
&\quad \leq t^{\sigma_3}\int_{\overline{T}}^t (t-s)^{\beta-1}\|\nabla E_{\beta,\beta}((t-s)^\beta(\Delta-\gamma))(u(s)-\bar{u}(s))\|_{W\dot{K}^{\nu_3}_{w,\infty}}ds \\
&\quad \leq CT^{\sigma_3}\overline{T}^{-\sigma_1}\frac{\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))} \\
&\qquad \times\int_{\overline{T}}^t (t-s)^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))-1}s^{\sigma_1}\|u(s)-\bar{u}(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}ds.
\end{aligned} \tag{3.27}$$

From (3.26) and (3.27), we infer that

$$\xi(t) \le \int_{\bar{T}}^{t} b(t-s)\xi(s)ds, \quad \bar{T} \le t \le T,$$

where

$$\xi(t) = t^{\sigma_1}\|u(t)-\bar{u}(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}} + t^{\sigma_3}\|\nabla(v-\bar{v})(t)\|_{W\dot{K}^{\nu_3}_{w,\infty}},$$

$$b(t) = \bar{C}t^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1} + \bar{C}t^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))-1},$$

$$\bar{C} = C(R+\bar{R})T^{\sigma_1}\bar{T}^{-\sigma_1-\sigma_3}\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))}$$

$$+ CT^{\sigma_3}\bar{T}^{-\sigma_1}\frac{\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}.$$

With the knowledge of a generalization of Gronwall's lemma (see [2]), we deduce that $u = \bar{u}$ in $(0,T)$. Consequently, we easily reach the conclusion that $v = \bar{v}$ in $(0,T)$. This completes the uniqueness of the solution of (1.1).

• In order to obtain the continuous dependence, we can proceed similarly to the contraction step. We commence by assuming that $[u_0, v_0]$ (respectively, $[\bar{u}_0, \bar{v}_0]$) be in $W\dot{K}^{\nu_1}_{p,\infty} \times W\dot{K}^{\nu_2}_{q,\infty}$; then, there is a time $T_1 > 0$ (respectively, $T_2 > 0$) such that the problem (1.1) has a unique mild solution $[u,v]$ in $\mathcal{X}^{T_1}$ (respectively, $[\bar{u},\bar{v}]$ in $\mathcal{X}^{T_2}$) with initial data $[u_0, v_0]$ (respectively, $[\bar{u}_0, \bar{v}_0]$). Next, we will take $0 < T < \min\{T_1, T_2\}$. We have the following a priori estimates:

$$t^{\sigma_1}\|E_\alpha(t^\alpha\Delta)(u_0-\bar{u}_0)\|_{W\dot{K}^{\nu_1}_{r,\infty}} \le Ct^{\sigma_1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{p}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r}))}\|u_0-\bar{u}_0\|_{W\dot{K}^{\nu_1}_{p,\infty}}, \tag{3.28}$$

$$\begin{aligned}\|\nabla E_{\alpha,\alpha}&((t-s)^\alpha\Delta)((u-\bar{u})\nabla v + \bar{u}\nabla(v-\bar{v}))(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}} \\ &\le C(t-s)^{-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w})}\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))} \\ &\quad\times\left(\|u(s)-\bar{u}(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\|(\nabla v)(s)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}} + \|\bar{u}(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\|\nabla(v-\bar{v})(s)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}}\right) \\ &\le CR(t-s)^{-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w})}s^{-\sigma_1-\sigma_3}\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))}\|[u-\bar{u}, v-\bar{v}]\|_{\mathcal{X}^T}.\end{aligned} \tag{3.29}$$

Let $t$ be in $(0,T)$, and let us now turn our attention toward estimating the $W\dot{K}^{\nu_1}_{r,\infty}$-norm of $u(t)-\bar{u}(t)$. From the a priori bounds (3.28) and (3.29) and with the knowledge that $\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})<\sigma_1$ and $\sigma_3<\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))$, we infer that

$$\begin{aligned}
&t^{\sigma_1}\|u(t)-\bar{u}(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\\
&\quad\le Ct^{\sigma_1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{p}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r}))}\|u_0-\bar{u}_0\|_{W\dot{K}^{\nu_1}_{p,\infty}}\\
&\qquad+Ct^{\sigma_1}R\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))}\\
&\qquad\times\int_0^t(t-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1}s^{-\sigma_1-\sigma_3}\,ds\,\|[u-\bar{u},v-\bar{v}]\|_{\mathcal{X}^T}\\
&\quad\le Ct^{\sigma_1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{p}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r}))}\|u_0-\bar{u}_0\|_{W\dot{K}^{\nu_1}_{p,\infty}}\\
&\qquad+CRt^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-\sigma_3}\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))}\\
&\qquad\times\mathrm{B}\left(1-(\sigma_1+\sigma_3),\frac{\alpha}{2}\left(1-\left(\nu_3+\frac{N}{w}\right)\right)\right)\|[u-\bar{u},v-\bar{v}]\|_{\mathcal{X}^T}\\
&\quad\le CT^{\sigma_1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{p}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r}))}\|u_0-\bar{u}_0\|_{W\dot{K}^{\nu_1}_{p,\infty}}\\
&\qquad+CRT^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-\sigma_3}\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))}\\
&\qquad\times\mathrm{B}\left(1-(\sigma_1+\sigma_3),\frac{\alpha}{2}\left(1-\left(\nu_3+\frac{N}{w}\right)\right)\right)\|[u-\bar{u},v-\bar{v}]\|_{\mathcal{X}^T}.
\end{aligned}$$

Taking $T>0$ so that

$$T^{\sigma_1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{p}-\frac{1}{r}))}{\Gamma(1-\frac{\alpha N}{2}(\frac{1}{p}-\frac{1}{r}))}<1$$

and

$$\begin{aligned}
&CRT^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-\sigma_3}\frac{\Gamma(2-\frac{1}{2}(1+\nu_3+\frac{N}{w}))}{\Gamma(1+\alpha-\frac{\alpha}{2}(1+\nu_3+\frac{N}{w}))}\\
&\qquad\times\mathrm{B}\left(1-(\sigma_1+\sigma_3),\frac{\alpha}{2}\left(1-\left(\nu_3+\frac{N}{w}\right)\right)\right)<\frac{1}{4},
\end{aligned}$$

we derive that

$$\sup_{0<t<T}t^{\sigma_1}\|u(t)-\bar{u}(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\le C\|u_0-\bar{u}_0\|_{W\dot{K}^{\nu_1}_{p,\infty}}+\frac{1}{4}\|[u-\bar{u},v-\bar{v}]\|_{\mathcal{X}^T}.\tag{3.30}$$

On the other hand, we have that

$$
\begin{aligned}
&t^{\sigma_2}\|v(t)-\bar{v}(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}}\\
&\le Ct^{\sigma_2-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{q}-\frac{1}{s}))}{\Gamma(1-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s}))}\|v_0-\bar{v}_0\|_{W\dot{K}^{\nu_2}_{q,\infty}}\\
&\quad + Ct^{\sigma_2}\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}\\
&\quad \times\int_0^t (t-s)^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))-1}\|u(s)-\bar{u}(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}}ds\\
&\le Ct^{\sigma_2-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{q}-\frac{1}{s}))}{\Gamma(1-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s}))}\|v_0-\bar{v}_0\|_{W\dot{K}^{\nu_2}_{q,\infty}}\\
&\quad + Ct^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))+\sigma_2-\sigma_1}\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}\\
&\quad \times \mathrm{B}\left(1-\sigma_1,\beta\left(1-\frac{1}{2}\left(\nu_1-\nu_2+N\left(\frac{1}{r}-\frac{1}{s}\right)\right)\right)\right)\|[u-\bar{u},v-\bar{v}]\|_{\mathcal{X}^T}\\
&\le CT^{\sigma_2-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{q}-\frac{1}{s}))}{\Gamma(1-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s}))}\|v_0-\bar{v}_0\|_{W\dot{K}^{\nu_2}_{q,\infty}}\\
&\quad + CT^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))+\sigma_2-\sigma_1}\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}\\
&\quad \times \mathrm{B}\left(1-\sigma_1,\beta\left(1-\frac{1}{2}\left(\nu_1-\nu_2+N\left(\frac{1}{r}-\frac{1}{s}\right)\right)\right)\right)\|[u-\bar{u},v-\bar{v}]\|_{\mathcal{X}^T}.
\end{aligned}
$$

Choosing $T>0$ so that

$$
T^{\sigma_2-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s})}\frac{\Gamma(1-\frac{N}{2}(\frac{1}{q}-\frac{1}{s}))}{\Gamma(1-\frac{\beta N}{2}(\frac{1}{q}-\frac{1}{s}))}\|v_0-\bar{v}_0\|_{W\dot{K}^{\nu_2}_{q,\infty}}<1
$$

and

$$
\begin{aligned}
&CT^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))+\sigma_2-\sigma_1}\frac{\Gamma(2-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}{\Gamma(1+\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s}))))}\\
&\quad\times \mathrm{B}\left(1-\sigma_1,\beta\left(1-\frac{1}{2}\left(\nu_1-\nu_2+N\left(\frac{1}{r}-\frac{1}{s}\right)\right)\right)\right)<\frac{1}{4}.
\end{aligned}
$$

The last inequalities imply that

$$
\sup_{0<t<T} t^{\sigma_2}\|v(t)-\bar{v}(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}}\le C\|v_0-\bar{v}_0\|_{W\dot{K}^{\nu_2}_{q,\infty}}+\frac{1}{4}\|[u-\bar{u},v-\bar{v}]\|_{\mathcal{X}^T}. \tag{3.31}
$$

We can prove that

$$
\begin{aligned}
&t^{\sigma_3}\|\nabla(v-\bar{v})(t)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}} \\
&\quad\le CT^{\sigma_3-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w}))}\frac{\Gamma(1-\frac{1}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))}{\Gamma(1-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))}\|v_0-\bar{v}_0\|_{W\dot{K}^{\nu_2}_{q,\infty}} \\
&\qquad+CT^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))+\sigma_3-\sigma_1}\frac{\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))} \\
&\qquad\times \mathrm{B}\left(1-\sigma_1,\frac{\beta}{2}\left(1-\left(\nu_1-\nu_3+N\left(\frac{1}{r}-\frac{1}{w}\right)\right)\right)\right)\|[u-\bar{u},v-\bar{v}]\|_{\mathcal{X}^T}.
\end{aligned}
$$

Choosing $T>0$ so that

$$
T^{\sigma_3-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w}))}\frac{\Gamma(1-\frac{1}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))}{\Gamma(1-\frac{\beta}{2}(1+\nu_2-\nu_3+N(\frac{1}{q}-\frac{1}{w})))}<1
$$

and

$$
\begin{aligned}
&CT^{\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w})))+\sigma_3-\sigma_1}\frac{\Gamma(\frac{1}{2}(3-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))}{\Gamma(1+\frac{\beta}{2}(1-(\nu_1-\nu_3+N(\frac{1}{r}-\frac{1}{w}))))} \\
&\quad\times \mathrm{B}\left(1-\sigma_1,\frac{\beta}{2}\left(1-\left(\nu_1-\nu_3+N\left(\frac{1}{r}-\frac{1}{w}\right)\right)\right)\right)<\frac{1}{4},
\end{aligned}
$$

gives the upper bound

$$
\sup_{0<t<T} t^{\sigma_3}\|\nabla(v-\bar{v})(t)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}}\le C\|v_0-\bar{v}_0\|_{W\dot{K}^{\nu_2}_{q,\infty}}+\frac{1}{4}\|[u-\bar{u},v-\bar{v}]\|_{\mathcal{X}^T}. \tag{3.32}
$$

We then get from (3.30), (3.31) together with (3.32) that

$$
\|[u-\bar{u},v-\bar{v}]\|_{\mathcal{X}^T}\le C\left(\|u-\bar{u}_0\|_{W\dot{K}^{\nu_1}_{p,\infty}}+\|v_0-\bar{v}_0\|_{W\dot{K}^{\nu_2}_{s,\infty}}\right),
$$

that is, the solution depends continuously on the initial data. The proof of Theorem 1.2 is concluded. ∎

**Remark 3.3** (Parameter generation information). We show an empirical algorithm to generate some examples of parameters involved in Condition 1. Let $\alpha$ and $\beta$ be in $(0,1)$ so that $\alpha>\beta$. Take $\omega=s$ so that $\omega>N(1+(\frac{\alpha}{\beta}-1)^{-1})$. We can choose $r=q$ so that $((\frac{\alpha}{\beta}-1)(\frac{1}{N}-\frac{1}{\omega}))^{-1}<q<\omega$, then $\beta(1+N(\frac{1}{q}-\frac{1}{\omega}))<\alpha(1-\frac{N}{\omega})$. Among the possible candidates for parameters $\nu_i's$, $i=1,2,3$, we first look for those that satisfy the following property:

$$
\frac{\beta}{2}\nu_2+\frac{\alpha-\beta}{2}\nu_3<\frac{\alpha}{2}\left(1-\frac{N}{\omega}\right)-\frac{\beta}{2}\left(1+N\left(\frac{1}{q}-\frac{1}{\omega}\right)\right).
$$

Then, we select those $\nu_i$'s that satisfy (a), (b), and (c) of Condition 1. Now, we choose $\sigma_3 \in (0, \beta)$ so that

$$\frac{\beta}{2}\left(\nu_1 - \nu_3 + 1 + N\left(\frac{1}{q} - \frac{1}{\omega}\right)\right) < \sigma_3 < \frac{\alpha}{2}\left(1 - \left(\nu_3 + \frac{N}{\omega}\right)\right).$$

Next, we consider $\sigma_1, \sigma_2$ and $p$ such that $0 < \sigma_1 < \min\{\alpha, 1 - \frac{\beta}{2}, 1 - \sigma_3, \sigma_2\}, \sigma_2 > \beta$ and $p > (\frac{1}{r} + \frac{\sigma_1}{N\alpha})^{-1}$. Using our algorithm, we obtain the following set of parameters.

• $N = 4$; $\alpha = 0,9$; $\beta = 0,8$; $p = 8$; $r = q = 36$; $w = s = 37$; $\nu_1 = 1$, $\nu_2 = 0,006$; $\nu_3 = 0,00059$; $\sigma_1 = 0,4$; $\sigma_2 = 1$; $\sigma_3 = 0,401125$.

## 4. Proof of Theorem 1.3

Given a mild solution $[u, v] : [0, T] \to W\dot{K}^{\nu_1}_{r,\infty} \times W\dot{K}^{\nu_2}_{s,\infty}$ of (1.1), let us consider $\overline{T} > T$. We say that $[\bar{u}, \bar{v}] : [0, \overline{T}] \to W\dot{K}^{\nu_1}_{r,\infty} \times W\dot{K}^{\nu_2}_{s,\infty}$ is a continuation of $[u, v]$ if $[\bar{u}, \bar{v}]$ is a mild solution and $u(t) = \bar{u}(t)$, $v(t) = \bar{v}(t)$ whenever $t \in [0, T]$.

*Proof of Theorem* 1.3. (Continuation of the solution). Let us assume that $[u, v] : [0, T] \to W\dot{K}^{\nu_1}_{r,\infty} \times W\dot{K}^{\nu_2}_{s,\infty}$ is the solution given by Theorem 1.2. Consider now $\overline{T} > T$ and we define the following complete metric space:

$$\mathcal{G} = \Big\{[\bar{u}, \bar{v}] \in \mathcal{X}^{\overline{T}} : [\bar{u}, \bar{v}] \equiv [u, v] \text{ in } [0, T], \sup_{T<t<\overline{T}} \|\bar{u}(t) - u(T)\|_{W\dot{K}^{\nu_1}_{r,\infty}} < \overline{R},$$
$$\sup_{T<t<\overline{T}} \|\bar{v}(t) - v(T)\|_{W\dot{K}^{\nu_2}_{s,\infty}} < \overline{R}, \sup_{T<t<\overline{T}} \|\nabla\bar{v}(t) - \nabla v(T)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}} < \overline{R}\Big\},$$

equipped by $\|\cdot\|_{\mathcal{X}^{\overline{T}}}$ (see (1.2)). We define $\Phi = (\varphi_1, \varphi_2) : \mathcal{G} \to \mathcal{G}$ by (3.7).

• We will prove that $\Phi$ is well defined. Given $[\bar{u}, \bar{v}] \in \mathcal{G}$, by arguing exactly as in the proof of Theorem 1.2, we can prove that $t^{\sigma_1}\varphi_1([\bar{u}, \bar{v}]) \in C((0, \overline{T}); W\dot{K}^{\nu_1}_{r,\infty})$, $t^{\sigma_2}\varphi_2([\bar{u}, \bar{v}]) \in C((0, \overline{T}); W\dot{K}^{\nu_2}_{s,\infty})$, $t^{\sigma_3}\nabla\varphi_2([\bar{u}, \bar{v}]) \in C((0, \overline{T}); W\dot{K}^{\nu_3}_{\omega,\infty})$. Moreover, it is easily checked that

$$\Phi([\bar{u}, \bar{v}])(t) = [u, v](t) \quad \text{for every } t \in [0, T].$$

For $T < t < \overline{T}$, the difference $\varphi_1([\bar{u}, \bar{v}])(t) - u(T)$ (respectively, $\varphi_2([\bar{u}, \bar{v}])(t) - v(T)$) can be decomposed as follows:

$$\begin{aligned}\varphi_1([\bar{u}, \bar{v}])(t) - u(T) = {} & E_\alpha(t^\alpha\Delta)u_0 - E_\alpha(T^\alpha\Delta)u_0 \\ & - \int_0^T \big((t - s)^{\alpha-1}\nabla E_{\alpha,\alpha}((t - s)^\alpha\Delta)(u\nabla v)(s) \\ & - (T - s)^{\alpha-1}\nabla E_{\alpha,\alpha}((T - s)^\alpha\Delta)(u\nabla v)(s)\big)ds \\ & - \int_T^t (t - s)^{\alpha-1}\nabla E_{\alpha,\alpha}((t - s)^\alpha\Delta)(\bar{u}\nabla\bar{v})(s)ds,\end{aligned} \tag{4.1}$$

respectively,

$$\begin{aligned}\varphi_2([\bar u,\bar v])(t)-v(T)&=E_\beta(t^\beta(\Delta-\gamma))v_0-E_\beta(T^\beta(\Delta-\gamma))v_0\\&\quad+\int_0^T\big((t-s)^{\beta-1}E_{\beta,\beta}((t-s)^\beta(\Delta-\gamma))u(s)\\&\quad-(T-s)^{\beta-1}E_{\beta,\beta}((T-s)^\beta(\Delta-\gamma))u(s)\big)ds\\&\quad+\int_T^t(t-s)^{\beta-1}E_{\beta,\beta}((t-s)^\beta(\Delta-\gamma))\bar u(s)ds.\end{aligned}\tag{4.2}$$

Let us keep in mind that $\nu_1<N(1-\frac{1}{r})$ (respectively, $\nu_2<N(1-\frac{1}{s})$). By Lemma 3.1, we note that the first term on the right-hand side of representation (4.1) (respectively, (4.2)) goes to zero in the $W\dot K^{\nu_1}_{r,\infty}$-norm (respectively, $W\dot K^{\nu_2}_{s,\infty}$-norm) as $t\to T^+$. On the other hand, by the Lebesgue dominated convergence theorem, one deduces that the second term on the right-hand side of (4.1) (respectively, (4.2)) goes to zero as $t\to T^+$. Furthermore, we have the following upper estimates:

$$\begin{aligned}&\int_T^t(t-s)^{\alpha-1}\|\nabla E_{\alpha,\alpha}((t-s)^\alpha\Delta)(\bar u\nabla\bar v)(s)\|_{W\dot K^{\nu_1}_{r,\infty}}ds\\&\quad\le\frac{C\Gamma(2-\frac12(1+\nu_3+\frac{N}{w}))(\bar R+\|u(T)\|_{W\dot K^{\nu_1}_{r,\infty}})(\bar R+\|\nabla v(T)\|_{W\dot K^{\nu_3}_{\omega,\infty}})}{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))\Gamma(1+\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w})))}\\&\qquad\times\left(1-\frac{T}{t}\right)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))}\to0,\quad\text{as } t\to T^+,\end{aligned}$$

respectively,

$$\begin{aligned}&\int_T^t(t-s)^{\beta-1}\|E_{\beta,\beta}((t-s)^\beta(\Delta-\gamma))\bar u(s)\|_{W\dot K^{\nu_2}_{s,\infty}}ds\\&\quad\le\frac{C\Gamma(1-\frac12(\nu_1-\nu_2+N(\frac1r-\frac1s)))(\bar R+\|u(T)\|_{W\dot K^{\nu_1}_{r,\infty}})}{\Gamma(1+\beta(1-\frac12(\nu_1-\nu_2+N(\frac1r-\frac1s))))\beta(1-\frac12(\nu_1-\nu_2+N(\frac1r-\frac1s)))}\\&\qquad\times(t-T)^{\beta(1-\frac12(\nu_1-\nu_2+N(\frac1r-\frac1s)))}\to0,\quad\text{as } t\to T^+.\end{aligned}$$

Therefore, we can take $\overline{T}$ so close to $T$ such that

$$\sup_{T<t<\overline{T}}\|\varphi_1([\bar u,\bar v])(t)-u(T)\|_{W\dot K^{\nu_1}_{r,\infty}}<\bar R,$$

respectively,

$$\sup_{T<t<\overline{T}}\|\varphi_2([\bar u,\bar v])(t)-v(T)\|_{W\dot K^{\nu_2}_{s,\infty}}<\bar R.$$

In a similar fashion, we get the upper bound

$$\sup_{T<t<\overline{T}}\|\nabla\varphi_2([\bar u,\bar v])(t)-\nabla v(T)\|_{W\dot K^{\nu_3}_{\omega,\infty}}<\bar R.$$

Notice that this implies that $\Phi$ is well defined.

• We will prove that $\Phi$ is a contraction. We proceed as follows: let $[\bar{u}_i, \bar{u}_i]$ be in $\mathcal{G}$, $i = 1, 2$, for $0 < t < \overline{T}$, we have the following string of inequalities:

$$\begin{aligned}
&\|\varphi_1([\bar{u}_1, \bar{v}_1])(t) - \varphi_1([\bar{u}_2, \bar{v}_2])(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}} \\
&\quad \le C \frac{\Gamma(2 - \frac{1}{2}(1 + \nu_3 + \frac{N}{w}))}{\Gamma(1 + \alpha - \frac{\alpha}{2}(1 + \nu_3 + \frac{N}{w}))} \int_0^t (t-s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1} s^{-\sigma_1-\sigma_3} \\
&\qquad \times \big(s^{\sigma_1}\|\bar{u}_1(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}} s^{\sigma_3}\|\nabla(\bar{v}_1 - \bar{v}_2)(s)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}} \\
&\qquad + s^{\sigma_1}\|(\bar{u}_1 - \bar{u}_2)(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}} s^{\sigma_3}\|(\nabla\bar{v}_2)(s)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}}\big) ds \\
&\quad \le C t^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-\sigma_1-\sigma_3} \frac{\Gamma(2 - \frac{1}{2}(1 + \nu_3 + \frac{N}{w}))\mathrm{B}(1 - (\sigma_1 + \sigma_3), \frac{\alpha}{2}(1 - (\nu_3 + \frac{N}{w})))}{\Gamma(1 + \alpha - \frac{\alpha}{2}(1 + \nu_3 + \frac{N}{w}))} \\
&\qquad \times \big(\overline{T}^{\sigma_1}(\overline{R} + \|u(T)\|_{W\dot{K}^{\nu_1}_{r,\infty}}) + \overline{T}^{\sigma_3}(\overline{R} + \|(\nabla v)(T)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}})\big)\|[\bar{u}_1 - \bar{u}_2, \bar{v}_1 - \bar{v}_2]\|_{\mathcal{X}^{\overline{T}}}.
\end{aligned}$$

Choosing $\overline{T} > 0$ so that

$$\begin{aligned}
&C\overline{T}^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-\sigma_3} \frac{\Gamma(2 - \frac{1}{2}(1 + \nu_3 + \frac{N}{w}))\mathrm{B}(1 - (\sigma_1 + \sigma_3), \frac{\alpha}{2}(1 - (\nu_3 + \frac{N}{w})))}{\Gamma(1 + \alpha - \frac{\alpha}{2}(1 + \nu_3 + \frac{N}{w}))} \\
&\quad \times \big(\overline{T}^{\sigma_1}(\overline{R} + \|u(T)\|_{W\dot{K}^{\nu_1}_{r,\infty}}) + \overline{T}^{\sigma_3}(\overline{R} + \|(\nabla v)(T)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}})\big) < \frac{1}{4},
\end{aligned}$$

we, therefore, have

$$\sup_{0<t<\overline{T}} t^{\sigma_1}\|\varphi_1([\bar{u}_1, \bar{v}_1])(t) - \varphi_1([\bar{u}_2, \bar{v}_2])(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}} < \frac{1}{4}\|[\bar{u}_1 - \bar{u}_2, \bar{v}_1 - \bar{v}_2]\|_{\mathcal{X}^{\overline{T}}}. \tag{4.3}$$

Continuing our analysis likewise, one may prove that

$$\begin{aligned}
&t^{\sigma_2}\|\varphi_2([\bar{u}_1, \bar{v}_1])(t) - \varphi_2([\bar{u}_2, \bar{v}_2])(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}} \\
&\quad \le C t^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))+\sigma_2-\sigma_1} \\
&\qquad \times \frac{\Gamma(2 - \frac{1}{2}(\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s})))\mathrm{B}(1 - \sigma_1, \beta(1 - \frac{1}{2}(\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s}))))}{\Gamma(1 + \beta(1 - \frac{1}{2}(\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s}))))} \\
&\qquad \times \|[\bar{u}_1 - \bar{u}_2, \bar{v}_1 - \bar{v}_2]\|_{\mathcal{X}^T}.
\end{aligned}$$

By invoking $\frac{1}{2}(\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s})) < 1, \sigma_2 > \sigma_1$ and choosing $\overline{T} > 0$ so that

$$\begin{aligned}
&\overline{T}^{\beta(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))+\sigma_2-\sigma_1} \\
&\quad < \frac{\Gamma(1 + \beta(1 - \frac{1}{2}(\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s}))))}{4C\Gamma(2 - \frac{1}{2}(\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s})))\mathrm{B}(1 - \sigma_1, \beta(1 - \frac{1}{2}(\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s}))))},
\end{aligned}$$

we immediately get

$$\sup_{0<t<\overline{T}} t^{\sigma_2}\|\varphi_2([\bar{u}_1, \bar{v}_1])(t) - \varphi_2([\bar{u}_2, \bar{v}_2])(t)\|_{W\dot{K}^{\nu_2}_{s,\infty}} \le \frac{1}{4}\|[\bar{u}_1 - \bar{u}_2, \bar{v}_1 - \bar{v}_2]\|_{\mathcal{X}^{\overline{T}}}. \tag{4.4}$$

We can prove also that

$$\sup_{0<t<\bar{T}} t^{\sigma_3}\|\nabla\varphi_2([\bar{u}_1,\bar{v}_1])(t)-\nabla\varphi_2([\bar{u}_2,\bar{v}_2])(t)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}} < \frac{1}{4}\|[\bar{u}_1-\bar{u}_2,\bar{v}_1-\bar{v}_2]\|_{\mathcal{X}^{\bar{T}}}. \tag{4.5}$$

With the help of bounds (4.3), (4.4), and (4.5), we have that $\Phi$ is a $\frac{3}{4}$-contraction. Hence, it has a fixed point $[\bar{u},\bar{v}]$ in $\mathcal{G}$, which is a continuation of $[u,v]$. By using the same argument as in Theorem 1.2, we can establish the uniqueness of continuation.

• (Blow-up) We next concentrate our attention on proving the blow-up. We will reason by contradiction. Suppose that $T_{\max}<+\infty$ and that there exists a constant $K>0$ such that $\|u(t)\|_{W\dot{K}^{\nu_1}_{r,\infty}}\le K$, $\|\nabla v(t)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}}\le K$ for all $t\in[a,T_{\max})$ with $a>0$.

We consider a sequence of positive real numbers $t_n\to T^-_{\max}$. Let us show that $(u(t_n))_n$ and $(\nabla v(t_n))_n$ are Cauchy sequences in $W\dot{K}^{\nu_1}_{r,\infty}$, and $W\dot{K}^{\nu_3}_{\omega,\infty}$, respectively. Indeed, for $0<t_m<t_n<T_{\max}$, we see that the differences $u(t_n)-u(t_m)$ and $v(t_n)-v(t_m)$ can be written as follows:

$$\begin{aligned} u(t_n)-u(t_m) &= E_\alpha(t_n^\alpha\Delta)u_0-E_\alpha(t_m^\alpha\Delta)u_0\\ &\quad-\int_0^{t_m}\big((t_n-s)^{\alpha-1}\nabla E_{\alpha,\alpha}((t_n-s)^\alpha\Delta)\\ &\quad-(T_{\max}-s)^{\alpha-1}\nabla E_{\alpha,\alpha}((T_{\max}-s)^\alpha\Delta)\big)(u\nabla v)(s)ds\\ &\quad+\int_0^{t_m}\big((t_m-s)^{\alpha-1}\nabla E_{\alpha,\alpha}((t_m-s)^\alpha\Delta)\\ &\quad-(T_{\max}-s)^{\alpha-1}\nabla E_{\alpha,\alpha}((T_{\max}-s)^\alpha\Delta)\big)(u\nabla v)(s)ds\\ &\quad-\int_{t_m}^{t_n}(t_n-s)^{\alpha-1}\nabla E_{\alpha,\alpha}((t_n-s)^\alpha\Delta)(u\nabla v)(s)ds \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} v(t_n)-v(t_m) &= E_\beta(t_n^\beta(\Delta-\gamma))v_0-E_\beta(t_m^\beta(\Delta-\gamma))v_0\\ &\quad-\int_0^{t_m}\big((t_m-s)^{\beta-1}E_{\beta,\beta}((t_m-s)^\beta(\Delta-\gamma))\\ &\quad-(T_{\max}-s)^{\beta-1}E_{\beta,\beta}((T_{\max}-s)^\beta(\Delta-\gamma))\big)u(s)ds\\ &\quad+\int_0^{t_m}\big((t_n-s)^{\beta-1}E_{\beta,\beta}((t_n-s)^\beta(\Delta-\gamma))\\ &\quad-(T_{\max}-s)^{\beta-1}E_{\beta,\beta}((T_{\max}-s)^\beta(\Delta-\gamma))\big)u(s)ds\\ &\quad+\int_{t_m}^{t_n}(t_n-s)^{\beta-1}E_{\beta,\beta}((t_n-s)^\beta(\Delta-\gamma))u(s)ds. \end{aligned} \tag{4.7}$$

Employing the strong continuity of Mittag–Leffler families $(E_\alpha(t^\alpha\Delta))_{t\ge0}$, $(E_\beta(t^\beta(\Delta-\gamma)))_{t\ge0}$, $(E_{\beta,\beta}(t^\beta(\Delta-\gamma)))_{t\ge0}$ and $(\nabla E_{\alpha,\alpha}(t^\alpha\Delta))_{t\ge0}$ and the Lebesgue dominated convergence theorem, we infer that the first three terms on the right-hand side of (4.6) and (4.7) go to zero. On the other hand, we have that

$$\begin{aligned}
&\left\| \int_{t_m}^{t_n} (t_n - s)^{\alpha-1} \nabla E_{\alpha,\alpha}((t_n - s)^{\alpha} \Delta)(u \nabla v)(s) ds \right\|_{W\dot{K}^{\nu_1}_{r,\infty}} \\
&\quad \le C \frac{\Gamma(2 - \frac{1}{2}(1 + \nu_3 + \frac{N}{w}))}{\Gamma(1 + \frac{\alpha}{2}(1 - (\nu_3 + \frac{N}{w})))} \\
&\qquad \times \int_{t_m}^{t_n} (t_n - s)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))-1} \|u(s)\|_{W\dot{K}^{\nu_1}_{r,\infty}} \|(\nabla v)(s)\|_{W\dot{K}^{\nu_3}_{\omega,\infty}} ds \\
&\quad \le CK^2 \frac{\Gamma(2 - \frac{1}{2}(1 + \nu_3 + \frac{N}{w}))}{\Gamma(1 + \frac{\alpha}{2}(1 - (\nu_3 + \frac{N}{w})))} \frac{(t_n - t_m)^{\frac{\alpha}{2}(1-(\nu_3+\frac{N}{w}))}}{\frac{\alpha}{2}(1 - (\nu_3 + \frac{N}{w}))}
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
&\left\| \int_{t_m}^{t_n} (t_n - s)^{\beta-1} E_{\beta,\beta}((t_n - s)^{\beta} (\Delta - \gamma)) u(s) ds \right\|_{W\dot{K}^{\nu_3}_{\omega,\infty}} \\
&\quad \le CK \frac{\Gamma(2 - \frac{1}{2}(\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s})))}{\Gamma(1 + \beta(1 - \frac{1}{2}(\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s}))))} \\
&\qquad \times \int_{t_m}^{t_n} (t_n - s)^{\frac{\beta}{2}(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))-1} ds \\
&\quad \le CK \frac{\Gamma(2 - \frac{1}{2}(\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s})))}{\Gamma(1 + \beta(1 - \frac{1}{2}(\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s}))))} \\
&\qquad \times \frac{1}{\frac{\beta}{2}(1 - \frac{1}{2}(\nu_1 - \nu_2 + N(\frac{1}{r} - \frac{1}{s})))} (t_n - t_m)^{\frac{\beta}{2}(1-\frac{1}{2}(\nu_1-\nu_2+N(\frac{1}{r}-\frac{1}{s})))}.
\end{aligned} \tag{4.9}$$

From (4.6) and (4.8) (respectively, (4.7) and (4.9)), we get

$$\lim_{n,m\to\infty} \|u(t_n) - u(t_m)\|_{W\dot{K}^{\nu_1}_{r,\infty}} = 0,$$

respectively,

$$\lim_{n,m\to\infty} \|v(t_n) - v(t_m)\|_{W\dot{K}^{\nu_2}_{s,\infty}} = 0.$$

One might therefore conclude that the limits $\lim_{n\to\infty} u(t_n)$ and $\lim_{n\to\infty} v(t_n)$ exist. Consequently, $u(T_{\max})$ and $v(T_{\max})$ exist in $W\dot{K}^{\nu_1}_{r,\infty}$ and $W\dot{K}^{\nu_2}_{s,\infty}$, respectively. Then, $[u, v]$ can be continued, which yields a contradiction. The proof of Theorem 1.3 is concluded. ∎

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