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Measures of noncompactness in fixed point theory for monotone operators and applications

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Abstract. We prove two counterparts of Darbo–Sadovskiĭ's theorems for monotone operators $F : X \to X$ acting on an ordered metric space X. As applications, we show the existence of solutions for some functional-integral and ordinary differential equations. The selection of a suitable measure of noncompactness is the main tool in solving the equations. Several examples are provided to illustrate our results.

1. Introduction

Fixed point theory involving measures of noncompactness is a powerful tool for studying the solutions of integral and differential equations. Darbo [8] was the first to apply the Kuratowski measure of noncompactness α to show the existence of fixed points for any continuous mapping $F : Y \to Y$ provided that $\alpha(F(\Omega)) \leq k\alpha(\Omega)$ for all $\Omega \subset Y$, where Y is a nonempty bounded closed convex subset Y of a given Banach space and $k < 1$. The theorem of Darbo generalizes Schauder's fixed point theorem and contains the existential part of Banach's fixed point theorem. Next, Sadovskiĭ [14] generalized Darbo's result for any continuous mapping $F : Y \to Y$ with assumption $\nu(F(\Omega)) < \nu(\Omega)$ for any measure of noncompactness ν on a given Banach space. In the context of applications, Szufla [15] applied Darbo's theorem to study the initial-value problem in Banach spaces. Recently, by extending the Darbo–Sadovskiĭ conditions and appropriately using measures of noncompactness, several authors have proved the existence of solutions to various types of functional equations (see [1–3, 7]). These results show that the solutions of functional equations are fixed points of a continuous map from a closed, bounded, convex subset of a function space to itself.

In this paper, we show counterparts of Darbo–Sadovskiĭ's theorem for a monotone, not necessarily continuous, mapping $F : Y \to Y$ under the assumption of Sadovskiĭ [14]. Fixed point theory for monotone operators has been investigated extensively over the past four decades. In 2018, Espínola–Wiśnicki [9] showed that, in a Hausdorff topological ordered space Y, if order intervals are compact, then every monotone mapping $F : Y \to Y$

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has a fixed point provided that $x_0 \preceq F(x_0)$ for some $x_0 \in Y$. Recently, Taoudi [16] presented some useful extensions of Espínola–Wiśnicki's theorem, and gave several examples to illustrate these extensions. Our main fixed point existence results in Section 3 exploit the arguments of Darbo, Sadovskiĭ, and Taoudi. In Section 4, we show some applications of these results to integral and ordinary differential equations.

2. Preliminaries

Let Y be a complete metric space and denote by $\mathbb{B}(Y)$ denote the family of all nonempty bounded subsets of Y, $\mathbb{CL}(Y)$ the family of all closed subsets of Y, and $\mathbb{CP}(Y)$ the family of all compact subsets of Y. The closed ball centered at $y \in Y$ with radius $r > 0$ will be denoted by $\bar{B}_r(y)$.

Definition 2.1 (Compare [4]). Let Y be a complete metric space. A measure of noncompactness (MNC for short) defined on Y is a function $\nu : \mathbb{B}(Y) \to [0, \infty)$ such that for any $\Omega_1, \Omega_2 \in \mathbb{B}(Y)$, we have the following:

(i) if $\nu(\Omega_1) = 0$, then $\bar{\Omega}_1 \in \mathbb{CP}(Y)$,

(ii) $\nu(\Omega_1) = \nu(\bar{\Omega}_1)$,

(iii) $\nu(\Omega_1 \cup \Omega_2) = \max\{\nu(\Omega_1), \nu(\Omega_2)\}$,

(iv) if $(\Omega_n)_n \subseteq \mathbb{CL}(Y) \cap \mathbb{B}(Y)$ such that $\Omega_{n+1} \subseteq \Omega_n$ for every $n \geq 1$ and $\lim_{n\to\infty} \mu(\Omega_n) = 0$, then the set

$$\Omega_\infty = \bigcap_{i=1}^{\infty} \Omega_n$$

is nonempty and compact.

In Definition 2.1, if ν satisfies (ii), (iii), and

(i′) $\nu(\Omega_1) = 0$ if and only if $\bar{\Omega}_1 \in \mathbb{CP}(Y)$,

then we say that ν is a regular measure of noncompactness (rMNC for short) on Y. From Definition 2.1, we infer the following properties:

(v) if $\Omega_1 \subseteq \Omega_2$, then $\nu(\Omega_1) \leq \nu(\Omega_2)$,

(vi) $\nu(\Omega_1 \cap \Omega_2) \leq \min\{\nu(\Omega_1), \nu(\Omega_2)\}$.

We note that if ν is a regular measure of noncompactness on Y and $\Omega = \{x_1, \dots, x_n\} \subseteq Y$, then $\nu(\Omega) = 0$, and it satisfies (iv).

Furthermore, if Y is a Banach space, the rMNC ν has the following properties:

(vii) $\nu(t\Omega_1) = |t|\nu(\Omega_1)$ for any number t,

(viii) $\nu(\Omega_1 + \Omega_2) \leq \nu(\Omega_1) + \nu(\Omega_2)$,

(ix) $\nu(x_0 + \Omega_1) = \nu(\Omega_1)$ for any $x_0 \in Y$,

(x) $\nu(\mathrm{co}(\Omega_1)) = \nu(\Omega_1)$, where $\mathrm{co}(\Omega_1)$ is the convex hull of Ω_1.

Throughout this paper, we use the following measures of noncompactness.

Example 2.2 (See [11]). Let Y be a complete metric space. The Kuratowski rMNC is defined by

$$\alpha(\Omega) = \inf\left\{\varepsilon > 0 : \Omega \text{ can be covered by finitely many sets of diameter } \leq \varepsilon\right\}$$

for every $\Omega \in \mathbb{B}(Y)$.

Example 2.3. Put $I := [0,1] \subset \mathbb{R}$. Let $\mathcal{C}(I,\mathbb{R})$ represent the space of all continuous real-valued functions defined on I. With the maximum norm

$$\|f\|_\infty = \max_{x\in I} |f(x)|, \quad f \in \mathcal{C}(I,\mathbb{R}),$$

$\mathcal{C}(I,\mathbb{R})$ is a Banach space. Take $\Omega \in \mathbb{B}(\mathcal{C}(I,\mathbb{R}))$, $f \in \Omega$, $\delta > 0$ and define

$$\begin{aligned}
\Psi(f,\delta) &= \sup\left\{|f(x) - f(y)| : x, y \in I, |x-y| \leq \delta\right\},\\
\Psi(\Omega,\delta) &= \sup_{f\in\Omega} \Psi(f,\delta),\\
\Psi_0(\Omega) &= \lim_{\delta\to 0} \Psi(\Omega,\delta).
\end{aligned}$$

The function Ψ_0 is an rMNC on $\mathcal{C}(I;\mathbb{R})$ (see [6]).

Example 2.4. Consider the space $\mathcal{BC}(\mathbb{R}_+)$ of all continuous and bounded real functions on $\mathbb{R}_+ := [0,\infty)$ with the standard sup-norm

$$\|f\|_{\mathcal{BC}} = \sup_{x\geq 0} |f(x)| \quad \text{for every } f \in \mathcal{BC}(\mathbb{R}_+).$$

Clearly, $(\mathcal{BC}(\mathbb{R}_+), \|\cdot\|_{\mathcal{BC}})$ is a Banach space. Take $\Omega \in \mathbb{B}(\mathcal{BC}(\mathbb{R}_+))$, $f \in \Omega$, $L > 0$, $\varepsilon > 0$ and define

$$\begin{aligned}
\Psi^L(f,\varepsilon) &= \sup\left\{|f(x) - f(y)| : x, y \in [0,L], |x-y| \leq \varepsilon\right\},\\
\Psi^L(\Omega,\varepsilon) &= \sup_{f\in\Omega} \Psi^L(f,\varepsilon),\\
\Psi_0^L(\Omega) &= \lim_{\varepsilon\to 0} \Psi^L(\Omega,\varepsilon),\\
\Psi_0(\Omega) &= \lim_{L\to\infty} \Psi_0^L(\Omega).
\end{aligned}$$

Consider the function

$$b(\Omega) = \lim_{L\to\infty} \left\{ \sup_{f\in\Omega} \{\sup\{|f(x) - f(y)| : x, y \geq L\}\}\right\}$$

and define

$$\Psi_1(\Omega) := \Psi_0(\Omega) + b(\Omega).$$

In [5], Banaś showed that Ψ_1 is an MNC on $\mathcal{BC}(\mathbb{R}_+)$.

Next, we need to recall some basic definitions in ordered metric spaces. Let Y be a metric space and $\preceq$ a partial order on Y. The order intervals have the form

$$[a, \to) := \{y \in Y : a \preceq y\},$$
$$(\leftarrow, b] := \{y \in Y : y \preceq b\},$$
$$[a, b] := [a, \to) \cap (\leftarrow, b]$$

for every $a, b \in Y$.

Definition 2.5 ([10]). An ordered metric space is a triple $(Y, d, \preceq)$ such that the order intervals $[a, \to)$, $(\leftarrow, a]$ are closed for every $a \in Y$.

Definition 2.6 ([10]). A mapping $F : Y \to Y$ is called monotone (or non-decreasing) if, for every $x, y \in Y$ such that $x \preceq y$, it satisfies $F(x) \preceq F(y)$.

A point x in Y is called a fixed point of F if $F(x) = x$.

Definition 2.7 ([10]). A sequence $(x_n)_n$ in the partially ordered set $(Y, \preceq)$ is called non-decreasing (respectively, non-increasing) if $x_n \preceq x_{n+1}$ $(x_{n+1} \preceq x_n)$ for any $n \geq 1$. A sequence $(x_n)_n$ is called monotone if it is non-decreasing or non-increasing.

Proposition 2.8 ([10]). *If a non-decreasing (respectively, non-increasing) sequence $(x_n)_n$ in an ordered metric space $(X, d, \preceq)$ has a limit point y_0, then $y_0 = \sup_n x_n$ (respectively, $y_0 = \inf_n x_n$).*

Proposition 2.9 ([10]). *If C is a chain in a metric ordered space, then each monotone sequence in C converges if and only if $\bar{C}$ is compact.*

Definition 2.10 ([16]). A mapping $F : Y \to Y$ is said to be monotone sequentially continuous on Y if, for every monotone sequence $(x_n)_n$ of Y which converges to some $x \in Y$, the sequence $(F(x_n))_n$ converges to $F(x)$.

Obviously, continuous maps are monotone sequentially continuous.

Example 2.11. In $(\mathbb{R}^2, d)$ with the Euclidean metric d, we consider the partial order

$$(u, v) \preceq (t, w) \Leftrightarrow u = v \leq t = w \text{ or } (u = t \text{ and } v = w).$$

The function g is defined as follows:

$$g(u, v) = \begin{cases} (1, 1) & \text{if } u \leq v \\ (0, 0) & \text{if } u > v \end{cases} \quad \text{for every } (u, v) \in \mathbb{R}^2.$$

Then, g is monotone sequentially continuous on $\mathbb{R}^2$, but it is not continuous at any point $(u, u) \in \mathbb{R}^2$.

Example 2.12. Monotone nonexpansive mappings in metric spaces are natural examples of monotone sequentially continuous mappings (see [13]).

3. Fixed point theorems

Let us begin with the first result under Darbo's assumption for monotone maps.

Theorem 3.1. *Let Y be a nonempty closed bounded subset in a complete ordered metric space $(X, d, \preceq)$, and let ν be an MNC on X. Let $T : Y \to Y$ be a monotone mapping satisfying*

$$\nu(T(\Omega)) \leq k\nu(\Omega)$$

for any $\Omega \subseteq Y$ with $k \in [0, 1)$. Assume that the set $\{y \in Y : y \preceq T(y)\}$ is nonempty. Then, T has a fixed point.

Proof. Define $Y_0 = Y, \ldots, Y_{n+1} = \overline{T(Y_n)}$ for any $n \geq 0$. We have

$$\begin{aligned}\nu(Y_{n+1}) &= \nu(\overline{T(Y_n)}) = \nu(T(Y_n)) \leq k\nu(Y_n) \\ &\leq k^2\nu(Y_{n-1}) \leq \cdots \leq k^{n+1}\nu(Y_0).\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \nu(Y_{n+1}) = 0.$$

We have

$$\begin{aligned}Y_1 &= \overline{T(Y_0)} = \overline{T(Y)} \subseteq \overline{Y} = Y = Y_0, \\ Y_2 &= \overline{T(Y_1)} \subseteq \overline{T(Y_0)} = Y_1.\end{aligned}$$

By induction, we have that $Y_{n+1} \subseteq Y_n$ for any $n \geq 0$. By Definition 2.1 (iv), we infer that the set $A = \bigcap_{n=0}^{\infty} Y_n$ is a nonempty compact subset of Y.

For each $n \geq 0$, we have

$$T(Y_{n+1}) \subseteq T(Y_n) \subseteq \overline{T(Y_n)} = Y_{n+1}.$$

Therefore, we deduce that $T(A) \subseteq A$.

Take $x_0 \in \{y \in Y : y \preceq T(y)\}$. Put

$$A_n = \{T^n(x_0), T^{n+1}(x_0), \ldots\} \quad \text{for any } n \geq 0,$$

where $T^0(x_0) = x_0$. Note that for any $n \geq 0$, $A_n \subseteq Y_n$, and $T^n(x_0) \preceq T^{n+1}(x_0)$ due to the monotonicity of T. Clearly, $T(A_n) = A_{n+1}$ for any $n \geq 0$. By an argument analogous to the one above, we get

$$\bigcap_{n=0}^{\infty} \overline{A_n} \neq \emptyset.$$

Take $c \in \bigcap_{n=0}^{\infty} \overline{A_n} \subset A$. Then, $c \in \overline{A_n}$ for any $n \geq 0$. If any subsequence of $(T^n(x_0))_n$ does not converge to c, then $c \in A_n$ for any $n \geq 0$. It implies that $c \preceq T(c)$.

Assume now that there exists a subsequence $(T^{n_k}(x_0))_k$ of $(T^n(x_0))_n$ such that

$$\lim_{k\to\infty} T^{n_k}(x_0) = c.$$

Since $(T^{n_k}(x_0))_k$ is non-decreasing, it follows from Proposition 2.8 that

$$c = \sup_n T^n(x_0).$$

By monotonicity of T, we get

$$T^n(x_0) \preceq T(c) \quad \text{for each } n \geq 0,$$

and thus, $c \preceq T(c)$.

Let $U = \{x \in A : x \preceq T(x)\}$. Since $c \in U$, $U \neq \emptyset$. Obviously, $T(x) \in U$ whenever $x \in U$. Suppose that Z is a chain in U. For each $z \in Z$, we set

$$V_z = [z, \to) \cap \overline{Z}.$$

Clearly, V_z is a nonempty closed subset of A for any $z \in Z$. Let $z_1, \ldots, z_n \in Z$. Since Z is a chain, there exists $i_0 \in \{1, \ldots, n\}$ with $z_{i_0} = \max\{z_1, \ldots, z_n\}$. It shows that $z_{i_0} \in V_{z_i}$ for all $i = 1, \ldots, n$. Thus,

$$\bigcap_{i=1}^{n} V_{z_i} \neq \emptyset.$$

It means that the family $(V_z)_{z\in Z}$ has the finite intersection property. Hence,

$$Z_0 = \bigcap_{z\in Z} V_z \neq \emptyset.$$

Take $v \in Z_0$. We have $z \preceq v$, so that $T(z) \preceq T(v)$ for all $z \in Z$. Hence, $Z \subseteq (\leftarrow, T(v)]$. By the closedness of A and $(\leftarrow, T(v)]$, we deduce that $v \in \overline{Z} \subseteq (\leftarrow, T(v)] \cap A$. Thus, $v \in U$ is an upper bound of Z in U. By Kuratowski–Zorn lemma, there is a maximal element u^* in U. Furthermore, $u^* \preceq T(u^*) \in U$ from the monotonicity of T. By maximality of u^* in U, we have

$$u^* = T(u^*).$$

■

With Sadovskiĭ's assumption, we have the following theorem.

Theorem 3.2. *Let Y be a nonempty closed bounded subset in a complete ordered metric space $(X, d, \preceq)$, and let ν be an rMNC on X. Let $F : Y \to Y$ be a monotone mapping satisfying*

$$\nu(F(\Omega)) < \nu(\Omega)$$

for any closed subset $\Omega \in \mathbb{B}(X)$ with $\nu(\Omega) > 0$. Assume the set $\{z \in Y : F(z) \in [z, \to)\}$ is nonempty. Then, F has a fixed point.

Proof. Take $x_0 \in \{z \in Y : F(z) \in [z, \rightarrow)\}$ and define

$$\mathcal{M} = \big\{M \subseteq Y : M \in \mathbb{CL}(X), F(M) \subseteq M \text{ and } x_0 \in M\big\}.$$

Clearly, $\mathcal{M} \neq \emptyset$ since $Y \in \mathcal{M}$. Set

$$A = \bigcap_{M \in \mathcal{M}} M \quad \text{and} \quad B = \{x_0\} \cup \overline{F(A)}.$$

It is not difficult to show that A belongs to $\mathcal{M}$, and so, we get $F : A \to A$. Since $x_0 \in A$ and $F(A) \subseteq A$, we deduce that $B \subseteq A$. Thus,

$$F(B) \subseteq F(A) \subseteq B.$$

It implies that $B \in \mathcal{M}$; hence, $A \subseteq B$. Therefore, $B = A$. By the properties of ν, we get

$$\nu(A) = \nu(B) = \nu(\overline{F(A)} \cup \{x_0\}) = \nu(\overline{F(A)}) = \nu(F(A)).$$

It follows that $\nu(A) = 0$. Hence, A is compact.

Let $U = \{x \in A : x \preceq F(x)\}$. Since $x_0 \in U$, $U \neq \emptyset$. Obviously, $F(x) \in U$ whenever $x \in U$. Suppose that Z is a chain in U. For each $z \in Z$, we set

$$V_z = [z, \rightarrow) \cap \bar{Z}.$$

Clearly, V_z is a nonempty, closed subset of C for any $z \in Z$. Let $z_1, \dots, z_n \in Z$. Since Z is a chain, there exists $i_0 \in \{1, \dots, n\}$ with $z_{i_0} = \max\{z_1, \dots, z_n\}$. It shows that $z_{i_0} \in V_{z_i}$ for all $i = 1, \dots, n$. Thus,

$$\bigcap_{i=1}^{n} V_{z_i} \neq \emptyset.$$

Since $\bar{Z}$ is compact, we get

$$Z_0 = \bigcap_{z \in Z} V_z \neq \emptyset.$$

Take $v \in Z_0$. We have $z \preceq v$, so that $F(z) \preceq F(v)$ for all $z \in Z$. Hence, $Z \subseteq (\leftarrow, F(v)]$. By the closedness of C and $(\leftarrow, F(v)]$, we deduce that $v \in \bar{Z} \subseteq (\leftarrow, F(v)] \cap C$. Thus, $v \in U$ is an upper bound of Z in U. By Kuratowski–Zorn lemma, it follows that there is a maximal element u^* in U. Hence, $x \preceq F(x) \preceq F(u^*)$ for every $x \in U$, and thus, $F(u^*)$ is an upper bound of U. Therefore, $u^* \preceq F(u^*)$. Furthermore, $F(u^*) \in U$. By maximality of u^*, we have $u^* = F(u^*)$. ∎

Theorem 3.3. *Let $\nu : \mathbb{B}(X) \to [0, \infty)$ be an rMNC on a complete ordered metric space $(X, d, \preceq)$ and let Y be a nonempty closed bounded subset of X. Suppose that $F : Y \to Y$ is a monotone mapping satisfying the following:*

(i) $\nu(F(\Omega)) < \nu(\Omega)$ *for any chain $\Omega \subset Y$ with $\nu(\Omega) > 0$,*

(ii) *F is monotone sequentially continuous.*

If $x_0 \in \{z \in Y : z \preceq F(z)\}$, then the iteration $\{F^n(x_0)\}$ tends to a fixed point of F.

Proof. Take $x_0 \in Y$ with $x_0 \preceq F(x_0)$, and let

$$\Omega = \{x_0, F(x_0), F^2(x_0), \ldots\}.$$

It is not difficult to prove that Ω is a chain. We get

$$\nu(\Omega) = \nu(F(\Omega) \cup \{x_0\}) = \nu(F(\Omega)).$$

It follows that $\nu(\Omega) = 0$, i.e., $\bar{\Omega}$ is compact. By Proposition 2.9, it implies that $\{F^n(x_0)\}$ converges to $y \in Y$. Since F is monotone sequentially continuous, we have

$$\lim_{n\to\infty} F(F^n(x_0)) = F(y).$$

Note that $\{F(F^n(x_0))\}$ is a subsequence of $\{F^n(x_0)\}$. By the uniqueness of the limit point, we get $F(y) = y$. ■

4. Applications

4.1. An integral equation of Hammerstein type

Let us study the existence of solutions in $\mathcal{C}(I, \mathbb{R})$ of integral equations with the following form:

$$f(x) = F(x, f(x)) + \int_0^1 H(x,s)U(s, f(s))ds \quad \text{for every } x \in I, \tag{4.1}$$

where $F : I \times \mathbb{R} \to \mathbb{R}$, $H : I \times I \to [0, \infty)$ are continuous and $U(\cdot, f(\cdot))$ is Lebesgue measurable on I for each $f \in \mathcal{C}(I, \mathbb{R})$.

Theorem 4.1. *Assume that the functions in (4.1) satisfy the following:*

(i) *$F(\cdot,\cdot)$ is continuous on $I \times \mathbb{R}$, and $F(x,\cdot)$ is non-decreasing on $\mathbb{R}$ for every $x \in I$;*

(ii) *there exists $k \in [0, 1)$ such that*

$$|F(x, y) - F(x, z)| \le k|y - z| \quad \textit{for any } x \in I \textit{ and } y, z \in \mathbb{R};$$

(iii) *$H(\cdot,\cdot)$ is continuous on $I \times I$;*

(iv) *$U(s,\cdot)$ is non-decreasing on $\mathbb{R}$ for every $s \in I$, and $U(\cdot, f(\cdot))$ is Lebesgue measurable on I for each $f \in \mathcal{C}(I, \mathbb{R})$;*

(v) *there exists a function $g : I \to [0, \infty)$ such that*

$$\sup_{y\in\mathbb{R}} |U(s, y)| \le g(s) \quad \textit{for a.e. } s \in I$$

and

$$\int_0^1 g(s)ds < \infty;$$

(vi) *there exists a function* $f_0 \in \mathcal{C}(I, \mathbb{R})$ *such that*

$$f_0(x) \leq F(x, f_0(x)) + \int_0^1 H(x,s)U(s, f_0(s))ds \quad \textit{for every } x \in I.$$

Then, equation (4.1) *has a solution* $f = f(x)$ *in* $\mathcal{C}(I; \mathbb{R})$.

Proof. Consider a partial order $\preceq$ in $\mathcal{C}(I, \mathbb{R})$ defined by

$$f \preceq g \Leftrightarrow f(x) \leq g(x) \quad \text{for all } x \in I$$

for every $f, g \in \mathcal{C}(I, \mathbb{R})$. It is not difficult to show that order intervals are closed.

We can assume that $\gamma := \int_0^1 g(s)ds > 0$. Take $f \in \mathcal{C}(I, \mathbb{R})$ and define

$$v(x) := \int_0^1 H(x,s)U(s, f(s))ds \quad \text{for all } x \in I.$$

Fix $s \in I$. Since $H(\cdot, s)$ is continuous on a compact set I, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $t_1, t_2 \in I$, $|t_1 - t_2| \leq \delta$, we get

$$|H(t_1, s) - H(t_2, s)| \leq \frac{\varepsilon}{\gamma}.$$

Hence,

$$\begin{aligned}|v(t_1) - v(t_2)| &= \left| \int_0^1 (H(t_1, s) - H(t_2, s))U(s, f(s))ds \right| \\ &\leq \int_0^1 |H(t_1, s) - H(t_2, s)||U(s, f(s))|ds \leq \frac{\varepsilon}{\gamma} \int_0^1 g(s)ds = \varepsilon.\end{aligned}$$

Therefore, v is uniformly continuous on I. Now, fix a function $f \in \mathcal{C}(I, \mathbb{R})$ and define

$$T(f)(x) = F(x, f(x)) + \int_0^1 H(x,s)U(s, f(s))ds \quad \text{for all } x \in I.$$

Obviously, $T(f)$ is continuous on I. It implies that $T(\mathcal{C}(I, \mathbb{R})) \subseteq \mathcal{C}(I, \mathbb{R})$. Furthermore,

$$\begin{aligned}|T(f)(x)| &\leq |F(x, f(x)) - F(x, 0)| + |F(x, 0)| + \int_0^1 |H(x,s)||U(s, f(s))|ds \\ &\leq k|f(x)| + \|F(\cdot, 0)\|_\infty + \gamma_1 \int_0^1 g(s)ds\end{aligned}$$

for all $x \in I$, where $\gamma_1 = \max\{H(x,s) : (x,s) \in I \times I\}$. Hence,

$$\|T(f)\|_\infty \leq k\|f\|_\infty + M,$$

where $M = \|F(\cdot, 0)\|_\infty + \gamma_1 \gamma$. We deduce that $T(\overline{B}_r(0)) \subseteq \overline{B}_r(0)$, where $r = M/(1-k)$.

Now, take a nonempty subset Ω of $\overline{B}_r(0)$, $f \in \Omega$, and $\varepsilon > 0$. Choosing $x, y \in I$ such that $|x - y| \le \varepsilon$, we get

$$
\begin{aligned}
&|T(f)(x) - T(f)(y)| \\
&\quad \le |F(x, f(x)) - F(y, f(y))| + \int_0^1 |H(x,s) - H(y,s)||U(s, f(s))|ds \\
&\quad \le |F(x, f(x)) - F(x, f(y))| + |F(x, f(y)) - F(y, f(y))| \\
&\qquad + \int_0^1 |H(x,s) - H(y,s)||U(s, f(s))|ds \\
&\quad \le k|f(x) - f(y)| + |F(x, f(y)) - F(y, f(y))| \\
&\qquad + \int_0^1 |H(x,s) - H(y,s)||U(s, f(s))|ds \\
&\quad \le k\Psi(f,\varepsilon) + \overline{\Psi}_1(F,\varepsilon) + \gamma\overline{\Psi}_2(H,\varepsilon),
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi(f,\varepsilon) &= \sup\left\{|f(x) - f(y)| : x, y \in I, |x - y| \le \varepsilon\right\}, \\
\overline{\Psi}_1(F,\varepsilon) &= \sup\left\{|F(z,x) - F(w,x)| : z, w \in I, |z - w| \le \varepsilon, |x| \le r\right\}, \\
\overline{\Psi}_2(h,\varepsilon) &= \sup\left\{\sup\{|H(x,s) - H(y,s)| : s \in I\} : x, y \in I, |x - y| \le \varepsilon\right\}.
\end{aligned}
$$

Since F, H are continuous, respectively, on compact sets $I \times [-r, r]$ and $I \times I$, then $\overline{\Psi}_1(F,\varepsilon) \to 0$ and $\overline{\Psi}_2(H,\varepsilon) \to 0$ as $\varepsilon \to 0$.

Thus,

$$\Psi(T(f),\varepsilon) \le k\Psi(f,\varepsilon) + \overline{\Psi}_1(F,\varepsilon) + \gamma\overline{\Psi}_2(H,\varepsilon).$$

Hence,

$$\Psi_0(T(\Omega)) \le k\Psi_0(\Omega) < \Psi_0(\Omega),$$

where Ψ_0 is the MNC on $\mathcal{C}(I, \mathbb{R})$ defined in Example 2.3. Clearly, T is monotone on $\overline{B}_r(0)$. Using Theorem 3.2, equation (4.1) has a solution f in $\mathcal{C}(I, \mathbb{R})$. ∎

The following is an example satisfying the assumptions of Theorem 4.1.

Example 4.2. Assume that the function $H : I \times I \to [0, \infty)$ is continuous. Then, the following equation has a solution f in $\mathcal{C}(I, \mathbb{R})$:

$$f(x) = \frac{x(f(x) + 1)}{1 + x^2} + \int_0^1 H(x,s)\frac{f(s)}{\sqrt{s}(1 + s)(1 + |f(s)|)}ds. \tag{4.2}$$

Proof. We consider the functions

$$F(x,y) := \frac{x(y+1)}{1+x^2}, \quad U(x,y) := \frac{y}{\sqrt{x}(1+x)(1+y)}, \quad g(x) := \frac{1}{\sqrt{x}(1+x)},$$

and choose $f_0(x) = 0$ for any $x \in [0, 1]$. Hence, the conditions of Theorem 4.1 are satisfied. Hence, equation (4.2) has a solution. ∎

4.2. An integral equation of Volterra type

In what follows, we study the functional-integral equation of the form

$$f(x) = F(f(x)) + \int_0^x U(s, f(s))ds \quad \text{for every } x \geq 0, \tag{4.3}$$

where $F : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$, and for each $f \in \mathcal{BC}(\mathbb{R}_+)$, $x \geq 0$, the function $U(\cdot, f(\cdot))$ is Lebesgue measurable on $[0, x]$.

Theorem 4.3. *Assume the following:*

(i) $F(\cdot)$ *is continuous and non-decreasing on* $\mathbb{R}$,

(ii) *there exists* $k \in [0, 1)$ *such that*

$$|F(x) - F(y)| \leq k|x - y| \quad \textit{for all } x, y \in \mathbb{R},$$

(iii) $U(s, \cdot)$ *is non-decreasing on* $\mathbb{R}$ *for every* $s \geq 0$, *and for each* $f \in \mathcal{BC}(\mathbb{R}_+)$, $x \geq 0$ *the function* $U(\cdot, f(\cdot))$ *is Lebesgue measurable on* $[0, x]$,

(iv) *there exists a function* $g : [0, \infty) \to [0, \infty)$ *such that*

$$\sup_{x \in \mathbb{R}} |U(s, x)| \leq g(s) \quad \textit{for a.e. } [0, +\infty)$$

and

$$\int_0^\infty g(s)ds < \infty,$$

(v) *there exists* $f_0 \in \mathcal{BC}(\mathbb{R}_+)$ *such that*

$$f_0(x) \leq F(f_0(x)) + \int_0^x U(s, f_0(s))ds \quad \textit{for every } x \geq 0. \tag{4.4}$$

Then, equation (4.3) *has a solution* f *that belongs to* $\mathcal{BC}(\mathbb{R}_+)$.

Proof. Consider a partial order $\preceq$ in $\mathcal{BC}(\mathbb{R}_+)$ given by

$$f \preceq g \Leftrightarrow f(x) \leq g(x) \quad \text{for all } x \in \mathbb{R}_+$$

for every $f, g \in \mathcal{BC}(\mathbb{R}_+)$. It is not difficult to show that the order intervals are closed.

Fix a function $f \in \mathcal{BC}(\mathbb{R}_+)$ and define

$$T(f)(x) = F(f(x)) + \int_0^x U(s, f(s))ds \quad \text{for all } x \geq 0.$$

We note that the function $u(x) := \int_0^x U(s, f(s))ds$ is continuous on $[0, \infty)$. Then, the function $F(f)$ is continuous on $[0, \infty)$. Furthermore,

$$\begin{aligned} |T(f)(x)| &\leq |F(f(x)) - F(0)| + |F(0)| + \int_0^x |U(s, f(s))|ds \\ &\leq k|f(x) - 0| + |F(0)| + \int_0^x g(s)ds \leq k|f(x)| + |F(0)| + \int_0^\infty g(s)ds \end{aligned}$$

for every $x \in \mathbb{R}_+$. Hence, $T(f) \in \mathcal{BC}(\mathbb{R}_+)$. It shows that $T(\mathcal{BC}(\mathbb{R}_+)) \subseteq \mathcal{BC}(\mathbb{R}_+)$.

Now, we get

$$\|T(f)\|_{\mathcal{BC}} \le k\|f\|_{\mathcal{BC}} + |F(0)| + \int_0^\infty g(s)ds.$$

It follows that if $\|f\|_{\mathcal{BC}} \le r$, where $r = \frac{R}{1-k}$, $R = |F(0)| + \int_0^\infty g(s)ds$, then

$$\|T(f)\|_{\mathcal{BC}} \le r.$$

Thus, $T(\overline{B}_r(0)) \subseteq \overline{B}_r(0)$. Furthermore, T is obviously monotone on $\overline{B}_r(0)$.

Now, take a nonempty subset Ω of $\overline{B}_r(0)$, a function $f \in \Omega$, and $L > 0$, $\varepsilon > 0$. If $z, w \in [0, L]$ and $|z - w| \le \varepsilon$, then

$$\begin{aligned}
|T(f)(z) - T(f)(w)| &\le |F(f(z)) - F(f(w))| + \left|\int_z^w U(s, f(s))ds\right| \\
&\le k|f(z) - f(w)| + \left|\int_z^w |U(s, f(s))|ds\right| \\
&\le k|f(z) - f(w)| + \left|\int_z^w g(s)ds\right| \\
&\le k\Psi^L(f, \varepsilon) + \sup\left\{\left|\int_z^w g(s)ds\right| : z, w \in [0, L], |z - w| \le \varepsilon\right\}.
\end{aligned}$$

Thus,

$$\Psi^L(T(f), \varepsilon) \le k\Psi^L(f, \varepsilon) + \sup\left\{\left|\int_z^w g(s)ds\right| : z, w \in [0, L], |z - w| \le \varepsilon\right\}.$$

It yields

$$\Psi^L(T(\Omega), \varepsilon) \le k\Psi^L(\Omega, \varepsilon) + \sup\left\{\left|\int_z^w g(s)ds\right| : z, w \in [0, L], |z - w| \le \varepsilon\right\}.$$

Using our assumptions, it is not difficult to prove that

$$h(t) = \int_0^t g(s)ds$$

is uniformly continuous on the compact set $[0, L]$. It follows that

$$\lim_{\varepsilon\to 0} \sup\left\{\left|\int_z^w g(s)ds\right| : z, w \in [0, L], |z - w| \le \varepsilon\right\} = 0.$$

Hence,

$$\Psi_0^L(T(\Omega)) \le k\Psi_0^L(\Omega). \tag{4.5}$$

Now, choosing $z, w \in [L, \infty)$, we get

$$|T(f)(z) - T(f)(w)| \le k|f(z) - f(w)| + \left|\int_z^w g(s)ds\right|.$$

Thus,

$$\Phi(T(f), L) \leq k\Phi(f, L) + \overline{\Phi}(g, L),$$

where

$$\begin{aligned}\Phi(T(f), L) &= \sup\left\{|T(f)(z) - T(f)(w)| : z, w \in [L, \infty)\right\},\\ \Phi(f, L) &= \sup\left\{|f(z) - f(w)| : z, w \in [L, \infty)\right\},\\ \overline{\Phi}(g, L) &= \sup\left\{\left|\int_z^w g(s)ds\right| : z, w \in [L, \infty)\right\}.\end{aligned}$$

It implies that

$$\sup_{f\in\Omega} \Phi(T(f), L) \leq k \sup_{f\in\Omega} \Phi(f, L) + \overline{\Phi}(g, L).$$

Note that the function

$$\eta(U) = \int_U g(s)ds$$

for $U \in \mathcal{B}(\mathbb{R}_+)$ is a measure on the family $\mathcal{B}(\mathbb{R}_+)$ of Borel subsets of $\mathbb{R}_+$. Take $z, w \in [L, \infty)$. Since $g(s) \geq 0$ for almost all $s \geq 0$, we get

$$\left|\int_z^w g(s)ds\right| \leq \int_T^\infty g(s)ds = \eta([L, \infty)).$$

Hence,

$$\sup\left\{\left|\int_z^w g(s)ds\right| : z, w \in [L, \infty)\right\} \leq \eta([L, \infty)),$$

and thus,

$$\begin{aligned}\lim_{L\to\infty} \sup\left\{\left|\int_z^w g(s)ds\right| : z, w \in [L, \infty)\right\} &\leq \lim_{L\to\infty} \eta([L, \infty))\\ &\leq \eta\left(\bigcap_{a\geq L}[a, \infty)\right) = \eta(\emptyset) = 0.\end{aligned}$$

Therefore,

$$b(T(\Omega)) \leq kb(\Omega). \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\Psi_1(T(\Omega)) < \Psi_1(\Omega),$$

where

$$\Psi_1(\Omega) = \Psi_0(\Omega) + b(\Omega)$$

is the MNC on $\mathcal{BC}(\mathbb{R}_+)$ defined in Example 2.4. By Theorem 3.2, there exists a solution f of (4.3) in $\mathcal{BC}(\mathbb{R}_+)$. ■

The following are examples satisfying the assumptions of Theorem 4.3.

Example 4.4. We have

$$f(x) = \arctan \frac{f(x)+2}{2} + \frac{1}{2}\int_0^x \frac{f(s)\sin^2 s}{s^2\sqrt{f^2(s)+1}}ds, \tag{4.7}$$

$$2f(x) = \sqrt{9+f^2(x)} + \frac{3}{\pi^2 \log 2}\int_0^x \frac{s}{\sqrt{e^s-1}}\arctan f(s)ds. \tag{4.8}$$

Proof. To prove (4.7), we consider the functions

$$F(x) := \arctan \frac{x+2}{2}, \quad U(x,y) := \frac{y\sin^2 x}{x^2\sqrt{y^2+1}}, \quad g(x) := \frac{\sin^2 x}{x^2}.$$

For (4.8), we consider the functions

$$F(x) := \frac{1}{2}\sqrt{9+x^2}, \quad U(x,y) := \frac{3}{\pi^2\log 2}\cdot\frac{x\arctan y}{\sqrt{e^x-1}}, \quad g(x) := \frac{1}{4\pi\log 2}\cdot\frac{x}{\sqrt{e^x-1}}.$$

We choose the function $f_0(x) = 0$ for any $x \geq 0$. Then, the conditions of Theorem 4.3 are satisfied. Therefore, the above equations always have solutions. ∎

4.3. A first order initial value problem with discontinuities

Let $(E, \|\cdot\|, \preceq)$ be an ordered Banach space with the partial order $\preceq$. Take $r > 0$ and $\overline{B}_r = \overline{B}_r(0) \subset E$. Let $\mathcal{C}(I,E)$ denote the Banach space of all mappings $x : I \to E$ that are continuous on I with the supremum norm

$$\|x\|_{\mathcal{C}} = \sup\{\|x(t)\| : t \in I\}.$$

Define a partial order in $\mathcal{C}(I,E)$ by

$$x \preceq_1 y \Leftrightarrow x(t) \preceq y(t) \quad \text{for every } t \in I$$

for every $x, y \in \mathcal{C}(I,E)$, and note that the order intervals in $\mathcal{C}(I,E)$ are closed.

Let α denote the Kuratowski MNCs on $\mathcal{C}(I,E)$. The following result is proved in [4, Theorem II.2.11].

Theorem 4.5. *Suppose that Ω is an equicontinuous bounded subset in $\mathcal{C}(I,E)$. Then,*

$$\alpha(\Omega) = \sup_{t\in I}\alpha(\{x(t) : x \in \Omega\}).$$

Consider the following first-order initial value problem

$$x'(t) = f(t, x(t)) \quad \text{for } t \in I \text{ a.e.}, \ x(0) = 0, \tag{4.9}$$

where $f : I \times \overline{B}_r(0) \to \overline{B}_r(0)$.

Before going to our next theorem, recall the following result (see [12, Lemma II.1.3]).

Lemma 4.6. *If $f : I \to E$ is integrable, then*

$$\int_0^t f(s)ds \in t\overline{\mathrm{co}}(\{f(s) : s \in [0,t]\})$$

for each $t \in I$.

Theorem 4.7. *Assume that $f : I \times \overline{B}_r \to \overline{B}_r$ satisfies the following conditions:*

(i) *$f(t,\cdot)$ is increasing on $\overline{B}_r$ for almost all $t \in I$ and*

$$M = \sup\{\|f(t,x)\| : (t,x) \in I \times \overline{B}_r\};$$

(ii) *$f(\cdot, x(\cdot))$ is integrable on I for each $x \in \mathcal{C}(I, \overline{B}_r)$;*

(iii) *there exists $k \geq 0$ such that*

$$\alpha(f(I \times \Omega)) \leq k\alpha(\Omega) \quad \textit{for any } \Omega \subseteq \overline{B}_r;$$

(iv) *there exists a function $x_0 \in \overline{B}_r$ such that*

$$x_0(t) \preceq \int_0^t f(s, x_0(s))ds \quad \textit{for every } t \in I.$$

Then, equation (4.9) has a solution $x \in \mathcal{C}(J, \overline{B}_r)$, where

$$J = [0, \beta], \quad 0 < \beta < \min\{1, r/M, 1/k\}.$$

Proof. Consider the following integral equation:

$$x(t) = \int_0^t f(s, x(s))ds, \tag{4.10}$$

which is equivalent to (4.9). Fix $x \in \mathcal{C}(J, \overline{B}_r)$ and define

$$T(x)(t) = \int_0^t f(s, x(s))ds.$$

Since $f(\cdot, x(\cdot))$ is integrable, then $T(x)$ is continuous on J. For $z, w \in J$, we get

$$\|T(x)(z) - T(x)(w)\| = \left\| \int_z^w f(s, x(s))ds \right\| \leq M|z - w|$$

and

$$\|T(x)(z)\| \leq \beta M.$$

It means that $T(\mathcal{C}(J, \overline{B}_r)) \subseteq \mathcal{C}(J, \overline{B}_r)$, and $T(\mathcal{C}(J, \overline{B}_r))$ is equicontinuous and bounded. Take $\Omega \subseteq \mathcal{C}(J, \overline{B}_r)$ such that $\alpha(\Omega) > 0$. It follows from Theorem 4.5 that

$$\begin{aligned}\alpha(T(\Omega)) &= \sup\left\{\alpha(\{(Tx)(t) : x \in \Omega\}) : t \in J\right\} \\ &= \sup\left\{\alpha\Big(\Big\{\int_0^t f(s, x(s))ds : x \in \Omega\Big\}\Big) : t \in J\right\}.\end{aligned}$$

From Lemma 4.6, we deduce that

$$\begin{aligned}\alpha(T(\Omega)) &\leq \sup_{t\in J}\{\alpha(\{t\overline{\mathrm{co}}(\{f(s,x(s)) : s \in [0,t], x \in \Omega\})\})\} \\ &\leq \sup_{t\in J}\{t\alpha(\overline{\mathrm{co}}(f(J \times \Omega)))\} \leq \beta\alpha(f(J \times \Omega)) \leq \beta k\alpha(\Omega) < \alpha(\Omega).\end{aligned}$$

Obviously, T is a monotone operator on $\mathcal{C}(J, \overline{B}_r)$. Note that $x_0 \preceq_1 Tx_0$. Hence, Theorem 3.2 yields that T has a fixed point $x \in \mathcal{C}(J, \overline{B}_r)$ that is a solution of equation (4.9). ■

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