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Stein–Weiss inequalities on Herz spaces with variable exponent

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Abstract. This paper extends the celebrated Stein–Weiss inequalities to the Herz spaces with variable exponents. It is an extension of the classical Stein–Weiss inequalities and the Stein–Weiss inequalities on Lebesgue spaces with variable exponents. The Stein–Weiss inequalities to the Herz spaces with variable exponents are presented as the boundedness of the fractional integral operators on the power weighted Herz spaces with variable exponents.

1. Introduction

This paper extends the Stein–Weiss inequalities to the Herz spaces with variable exponents. These inequalities are presented as the boundedness of the fractional integral operators on the power weighted Herz spaces with variable exponents.

The Stein–Weiss inequalities were established by Stein and Weiss in [40]. It gives the boundedness of the fractional integral operators on the power weighted Lebesgue spaces. The Stein–Weiss inequalities had been extended to the weighted norm inequalities for fractional integral operators in [34] and the two weight norm inequalities for potential operators in [36]. They are extended to the setting on the Carnot group in [11].

The Stein–Weiss inequalities were extended to the Lebesgue spaces with variable exponent in [5]. It gives the boundedness of the fractional integral operators on the power weighted Lebesgue spaces with variable exponents. The reader is referred to [6, 9] for the studies of the Lebesgue spaces with variable exponents. The Stein–Weiss inequalities had been further extended to the local Morrey spaces with variable exponents in [23].

The Herz space was introduced by Herz in [12] to study the Fourier series. It had been developed into one of the important function spaces in harmonic analysis. The boundedness of the singular integral operators, the fractional integral operators, and bilinear operators had been extended to the Herz spaces, see [21, 30–33]. The Herz spaces had been generalized to the Herz spaces with variable exponents in [1]. It is an extension of the Herz spaces and the Lebesgue spaces with variable exponents. For the extrapolation, the boundedness of the singular integral operators, the fractional integral operators, the

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wavelet characterizations, the spherical maximal function, the maximal Bochner-Riesz mean, the geometric maximal function and applications on partial differential equations of the classical Herz spaces, the Herz spaces with variable exponents, and the Herz type spaces, the reader is referred to [1, 3, 10, 16, 18–20, 24–33, 37, 38, 41].

The results obtained in [5, 23] motivate us to investigate the validity of the Stein–Weiss inequalities on the Herz spaces with variable exponents. We establish the Stein–Weiss inequalities on the Herz spaces with variable exponents by using a characterization of the Herz spaces with variable exponent obtained in [1] and the ideas from [22, 23].

This paper is organized as follows. The definitions and some properties of the power weighted Lebesgue spaces with variable exponents and the power weighted Herz spaces with variable exponents are presented in Section 2. The Stein–Weiss inequalities on the Herz spaces with variable exponents are established in Section 3.

2. Definitions and preliminaries

Let $\mathcal{M}$ and L^1_{loc} denote the space of Lebesgue measurable functions and the space of locally integrable functions on $\mathbb{R}^n$, respectively.

For any $x \in \mathbb{R}^n$ and $r > 0$, define

$$B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\} \quad \text{and} \quad \mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}.$$

Let $1 \le p < \infty$ and $a \in \mathbb{R}$. The power weighted Lebesgue space $L^{p,a}$ consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{L^{p,a}} = \left(\int_{\mathbb{R}^n} |x|^{ap} |f(x)|^p dx\right)^{\frac{1}{p}} < \infty.$$

Let $\alpha \in (0,n)$. For any locally integrable function f, the fractional integral operator I_α is defined as

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

The following theorem presents the celebrated Stein–Weiss inequality for the fractional integral operator.

Theorem 2.1. *Let $n \ge 1$, $\alpha \in (0,n)$, $1 < p < \infty$, $a < \frac{n}{p'}$, $b < \frac{n}{q}$, $a + b \ge 0$, and let $\frac{1}{q} = \frac{1}{p} + \frac{a+b-\alpha}{n}$. If $p \le q < \infty$, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L^{q,-b}} \le C \|f\|_{L^{p,a}} \quad \forall f \in L^{p,a}.$$

The reader is referred to [40, Theorem B*] for the proof of Theorem 2.1. The Stein–Weiss inequalities had been generalized into different settings and function spaces, see [7, 11, 13, 15, 17, 23, 34, 36].

Definition 2.1. Let $p(\cdot):\mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ so that

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f(x)/\lambda) \le 1\} < \infty,$$

where

$$\rho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx.$$

The function $p(x)$ is called the exponent function of $L^{p(\cdot)}$.

For any Lebesgue measurable function $p(x):\mathbb{R}^n \to (0,\infty)$, define $p_- = \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$ and $p_+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$.

The associate space of $L^{p(\cdot)}$ is given in [9, Theorem 3.2.13]. For the definition of associate space of Banach function space, see [2, Chapter 1, Definitions 2.1 and 2.3].

Theorem 2.2. *Let $p(\cdot)$ be a Lebesgue measurable function on $\mathbb{R}^n$. If $1 < p(x) < \infty$, then the associate space of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$, where p' satisfies $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.*

We call $p'(x)$ the conjugate function of $p(x)$. Whenever $\sup_{x\in\mathbb{R}^n} p(x) < \infty$, the dual space of $L^{p(\cdot)}$ is equal to the associate space of $L^{p(\cdot)}$, see [9, Theorem 3.4.6].

Definition 2.2. A continuous function g is belonging to $g \in C_0^{\log}(\mathbb{R}^n)$ if

$$|g(x) - g(0)| \le \frac{c_{\log}}{\log(e + 1/|x|)} \quad \forall x \in \mathbb{R}^n \tag{2.1}$$

for $c_{\log} > 0$. We say that g is log-Hölder continuous at the origin if $g \in C_0^{\log}(\mathbb{R}^n)$.

A continuous function $g : \mathbb{R}^n \to (0,\infty)$ is locally log-Hölder continuous if there exists $c_{\log} > 0$ such that

$$|g(x) - g(y)| \le \frac{c_{\log}}{\log(e + 1/|x-y|)} \quad \forall x, y \in \mathbb{R}^n. \tag{2.2}$$

We denote the class of locally log-Hölder continuous function by $C_{\mathrm{loc}}^{\log}(\mathbb{R}^n)$. Furthermore, a continuous function is globally log-Hölder continuous if $g \in C_{\mathrm{loc}}^{\log}(\mathbb{R}^n)$, and there exists $g_\infty \in \mathbb{R}$ so that

$$|g(x) - g_\infty| \le \frac{c_{\log}}{\log(e + |x|)} \quad \forall x \in \mathbb{R}^n. \tag{2.3}$$

The class of globally log-Hölder continuous function is denoted by $C^{\log}(\mathbb{R}^n)$.

When $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \le p_+ < \infty$, then $p'(\cdot) \in C^{\log}(\mathbb{R}^n)$. Moreover, we have $(p')_\infty = (p_\infty)'$. Therefore, we write $p'_\infty = (p')_\infty$.

Whenever $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \le p_+ < \infty$, the Hardy–Littlewood maximal operator

$$(\mathrm{M}f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \quad f \in L^1_{\mathrm{loc}},$$

is bounded on $L^{p(\cdot)}$.

Theorem 2.3. *If $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $1 < p_- \le p_+ < \infty$, then the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}$.*

For the proof of the above theorem, the reader is referred to [4, Theorem 1.5], [8, Theorem 3.5], [9, Theorem 4.3.8], and [35].

We now define the power weighted Lebesgue spaces with variable exponent.

Definition 2.3. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $b \in \mathbb{R}$, the power weighted Lebesgue space with variable exponent consists of all $f \in \mathcal{M}$ such that

$$\|f\|_{L^{p(\cdot),b}} = \|v_b f\|_{L^{p(\cdot)}} < \infty,$$

where $v_b(x) = |x|^b$, $x \in \mathbb{R}^n$.

The associate space of $L^{p(\cdot),b}$, denoted by $(L^{p(\cdot),b})'$, consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{(L^{p(\cdot),b})'} = \sup\left\{\int_{\mathbb{R}^n} |f(x)g(x)|dx : \|g\|_{L^{p(\cdot),b}} \le 1\right\} < \infty.$$

The associate space of the power weighted Lebesgue spaces with variable exponents is given in [23, Lemma 2.4].

Lemma 2.4. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $b \in \mathbb{R}$. If $1 < p_- \le p_+ < \infty$, then*

$$(L^{p(\cdot),b})' = L^{p'(\cdot),-b}.$$

Consequently, we have the Hölder inequality for $L^{p(\cdot),b}$:

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \le C\|f\|_{L^{p(\cdot),b}}\|g\|_{L^{p'(\cdot),-b}} \tag{2.4}$$

for some $C > 0$.

Furthermore, we have the following estimate from [23, Lemma 2.6].

Lemma 2.5. *Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$. If $b \in (-\frac{n}{p_+}, \infty)$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $1 < p_- \le p_+ < \infty$, then there exist constants $C_0, C_1 > 0$ such that, for any $R > 0$, we have*

$$C_0 R^b \|\chi_{B(0,R)}\|_{L^{p(\cdot)}} \le \|\chi_{B(0,R)}\|_{L^{p(\cdot),b}} \le C_1 R^b \|\chi_{B(0,R)}\|_{L^{p(\cdot)}}. \tag{2.5}$$

In view of Lemmas 2.4 and 2.5, the power weighted Lebesgue spaces with variable exponents is a ball quasi-Banach function space. For simplicity, we refer the readers to [39] for the definition of the ball quasi-Banach function space.

We recall the Stein–Weiss inequalities for the Lebesgue spaces with variable exponents from [5, Theorem 3.1].

Theorem 2.6. *Let $\alpha \in (0,n)$, $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p(x) \le q(x)$ $\forall x \in \mathbb{R}^n$, $1 < p_- \le p_+ < \infty$, and $1 < q_- \le q_+ < \infty$. If $a, b, p(\cdot), q(\cdot)$ satisfy*

$$-\frac{n}{q_+} < a \le b < \frac{n}{(p_-)'}, \tag{2.6}$$

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n} + \frac{a-b}{n}, \tag{2.7}$$

then there is a constant $C > 0$ *such that, for any* $f \in L^{p(\cdot),b}$*, we have*

$$\|I_\alpha f\|_{L^{q(\cdot),a}} \leq C\|f\|_{L^{p(\cdot),b}}.$$

For the proof of the above theorem, the reader is referred to [5, Section 4].

We have the following estimate from [23, Lemma 2.8].

Lemma 2.7. *Let* $\alpha \in (0, n)$*,* $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n)$ *with* $p(x) \leq q(x)\ \forall x \in \mathbb{R}^n$*,* $1 < p_- \leq p_+ < \infty$*, and* $1 < q_- \leq q_+ < \infty$*. If* $a, b, p(\cdot), q(\cdot)$ *satisfy* (2.6)-(2.7)*, then there is a constant* $C > 0$ *such that, for any* $B \in \mathbb{B}$*, we have*

$$\|\chi_B\|_{L^{p'(\cdot),-b}}\|\chi_B\|_{L^{q(\cdot),a}} \leq C|B|^{1-\frac{\alpha}{n}}. \tag{2.8}$$

In fact, the above estimate is a consequence of the Stein–Weiss inequalities in Theorem 2.6, see the proof of [23, Lemma 2.8].

We now give the definition of the Herz spaces with variable exponents [1]. Let $R_0 = B(0, 1)$ and $\chi_0 = \chi_{R_0}$; then,

$$B_k = B(0, 2^k), \quad R_k = B_k \backslash B_{k-1}, \quad \chi_k = \chi_{R_k}, \quad k \in \mathbb{N}\backslash\{0\}.$$

Definition 2.4. Let $\theta \in (0, \infty)$, $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The Herz space with variable exponent $K^{\alpha(\cdot)}_{p(\cdot),\theta}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),\theta}} = \left(\sum_{k=0}^{\infty} \|2^{k\alpha(\cdot)} f\chi_k\|^\theta_{L^{p(\cdot)}}\right)^{1/\theta} < \infty.$$

When $p(\cdot) = p$ and $\alpha(\cdot) = \alpha$ are constant functions, then the Herz space with variable exponent becomes the Herz space $K^\alpha_{p,\theta}$.

We have the following characterization of the Herz spaces with variable exponents from [1, Proposition 3.8].

Proposition 2.8. *Let* $\theta \in (0, \infty)$*,* $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ *be a Lebesgue measurable function and* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$*. If* α *satisfies* (2.3)*, then*

$$K^{\alpha(\cdot)}_{p(\cdot),\theta} = K^{\alpha_\infty}_{p(\cdot),\theta}.$$

The reader is referred to [1,16,19,25,26] for the boundedness of singular integral operators, potential operators, and some sublinear operators and inequalities on Herz spaces with variable exponents.

Definition 2.5. Let $b \in \mathbb{R}$, $\theta \in (0, \infty)$, $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The power weighted Herz space with variable exponent $K^{\alpha(\cdot),b}_{p(\cdot),\theta}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{K^{\alpha(\cdot),b}_{p(\cdot),\theta}} = \|v_b f\|_{K^{\alpha(\cdot)}_{p(\cdot),\theta}} < \infty,$$

where $v_b(x) = |x|^b$, $x \in \mathbb{R}^n$.

When $p(\cdot) = p$ and $\alpha(\cdot) = \alpha$ are constant functions, then the power weighted Herz space with variable exponent becomes the power weighted Herz space $K^{\alpha,b}_{p,\theta}$.

Lemma 2.9. *Let $b \in \mathbb{R}$, $\theta \in (0,\infty)$, $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$; then, the following statements hold.*

(1) *If $b \geq 0$ and α satisfies (2.3), then, for any $B \in \mathbb{B}$, $\chi_B \in K^{\alpha(\cdot),b}_{p(\cdot),\theta}$.*

(2) *If $b > -\frac{n}{p_+}$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, and α satisfies (2.3), then, for any $B \in \mathbb{B}$, $\chi_B \in K^{\alpha(\cdot),b}_{p(\cdot),\theta}$.*

Proof. It suffices to prove that for any $j \in \mathbb{N}$, $\chi_{B(0,2^j)} \in K^{\alpha(\cdot),b}_{p(\cdot),\theta}$. When $b \geq 0$, we see that

$$\begin{aligned}
\sum_{k=0}^{\infty} \|2^{k\alpha_\infty}\chi_{B(0,2^j)}\chi_k v_b\|^\theta_{L^{p(\cdot)}} &= \sum_{k=0}^{j} \|2^{k\alpha_\infty}\chi_k v_b\|^\theta_{L^{p(\cdot)}} \\
&\leq \sum_{k=0}^{j} 2^{k\alpha_\infty\theta}2^{kb\theta}\|\chi_k\|^\theta_{L^{p(\cdot)}} \\
&\leq C(j+1)\max(1, 2^{j\alpha_\infty\theta})2^{jb\theta}\|\chi_{B(0,2^j)}\|^\theta_{L^{p(\cdot)}} < \infty.
\end{aligned}$$

Therefore, $\chi_{B(0,2^j)} \in K^{\alpha(\cdot),b}_{p(\cdot),\theta}$.

When $b \in (-\frac{n}{p_+}, 0)$, Lemma 2.5 yields

$$\begin{aligned}
\sum_{k=0}^{\infty} \|2^{k\alpha_\infty}\chi_{B(0,2^j)}\chi_k v_b\|^\theta_{L^{p(\cdot)}} &= \sum_{k=0}^{j} 2^{k\alpha_\infty\theta}\|\chi_k\|^\theta_{L^{p(\cdot),b}} \\
&\leq \sum_{k=0}^{j} 2^{k\alpha_\infty\theta}2^{(k-1)b\theta}\|\chi_{B(0,2^k)}\|^\theta_{L^{p(\cdot)}} \\
&\leq C(j+1)\max(1, 2^{j\alpha_\infty\theta}2^{(j-1)b\theta})\|\chi_{B(0,2^j)}\|^\theta_{L^{p(\cdot)}} \\
&< \infty.
\end{aligned}$$

Therefore, $\chi_{B(0,2^j)} \in K^{\alpha(\cdot),b}_{p(\cdot),\theta}$. ∎

The above result shows that the power weighted Herz spaces with variable exponent is non-trivial.

In view of [16, Corollaries 2.7 and 2.8] and Definition 2.5, we obtain the following Hölder inequalities and characterization of $K^{\alpha(\cdot),b}_{p(\cdot),\theta}$.

Proposition 2.10. *Let $b \in \mathbb{R}$, $\theta \in (1,\infty)$ and let $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ be a Lebesgue measurable function and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. If α satisfies (2.3), then*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq C\|f\|_{K^{\alpha(\cdot),b}_{p(\cdot),\theta}}\|g\|_{K^{-\alpha(\cdot),-b}_{p'(\cdot),\theta'}}$$

for some $C > 0$.

Proposition 2.11. *Let $b \in \mathbb{R}$, $\theta \in (1,\infty)$, $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ be a Lebesgue measurable function and let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. If α satisfies (2.3), then there exist constants $C_1, C_2 > 0$ such that, for any $f \in K^{\alpha(\cdot),b}_{p(\cdot),\theta}$,*

$$C_1\|f\|_{K^{\alpha(\cdot),b}_{p(\cdot),\theta}} \le \sup_{g\in K^{-\alpha(\cdot),-b}_{p'(\cdot),\theta'}} \int_{\mathbb{R}^n} |f(x)g(x)|dx \le C_2\|f\|_{K^{\alpha(\cdot),b}_{p(\cdot),\theta}}. \tag{2.9}$$

Proposition 2.12. *Let $b \in \mathbb{R}$, $\theta \in (1,\infty)$, $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function and let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. If $b < n - \frac{n}{p_-}$ and α satisfies (2.3), then $K^{\alpha(\cdot),b}_{p(\cdot),\theta} \subset L^1_{\mathrm{loc}}$.*

Proof. It suffices to show that for any $f \in K^{\alpha(\cdot),b}_{p(\cdot),\theta}$ and $k \in \mathbb{N}$, we have

$$\int_{B(0,2^k)} |f(x)|dx < \infty.$$

As $b < n - \frac{n}{p_-}$ yields $-b > -\frac{n}{p'_+}$, Lemma 2.9 guarantees that $\chi_{B(0,2^k)} \in K^{-\alpha(\cdot),-b}_{p'(\cdot),\theta'}$. Proposition 2.10 asserts that for any $f \in K^{\alpha(\cdot),b}_{p(\cdot),\theta}$,

$$\int_{B(0,2^k)} |f(x)|dx \le C\|f\|_{K^{\alpha(\cdot),b}_{p(\cdot),\theta}} \|\chi_{B(0,2^k)}\|_{K^{-\alpha(\cdot),-b}_{p'(\cdot),\theta'}} \le C\|f\|_{K^{\alpha(\cdot),b}_{p(\cdot),\theta}}.$$

Therefore, f is locally integrable. ∎

In addition, Lemma 2.9 and Proposition 2.12 show that whenever $1 \le p_- \le p_+ < \infty$, $\theta \in (1,\infty)$, $-\frac{n}{p_+} < b < n - \frac{n}{p_-}$ and $\alpha \in L^\infty(\mathbb{R}^n)$ satisfies (2.3), then the weighted Herz space with variable exponent $K^{\alpha(\cdot),b}_{p(\cdot),\theta}$ is a ball Banach function space. For simplicity, we refer the reader to [39] for the definition and properties of ball Banach function spaces.

3. Main result

We establish the Stein–Weiss inequalities on the Herz spaces with variable exponents in this section.

Theorem 3.1. *Let $\alpha \in (0,n)$, $\theta \in (0,\infty)$, $\beta(\cdot) \in L^\infty(\mathbb{R}^n)$, $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p(x) \le q(x)$ $\forall x \in \mathbb{R}^n$, $1 < p_- \le p_+ < \infty$, and $1 < q_- \le q_+ < \infty$. If $a, b, p(\cdot), q(\cdot)$ satisfy (2.6), (2.7), α satisfies (2.3), and*

$$-\frac{n}{p_+} + \alpha - b < \beta_\infty < n - \frac{n}{p_-} - b, \tag{3.1}$$

then, there is a constant $C > 0$ such that, for any $f \in K^{\beta(\cdot),b}_{p(\cdot),\theta}$,

$$\|I_\alpha f\|_{K^{\beta(\cdot),a}_{q(\cdot),\theta}} \le C\|f\|_{K^{\beta(\cdot),b}_{p(\cdot),\theta}}.$$

Proof. For any $f \in K^{\beta(\cdot),b}_{p(\cdot),\theta}$, Proposition 2.8 gives

$$\begin{aligned}
\|I_\alpha f\|^\theta_{K^{\beta(\cdot),a}_{q(\cdot),\theta}} &= \|v_a I_\alpha f\|^\theta_{K^{\beta(\cdot)}_{q(\cdot),\theta}} \\
&\le C\sum_{k=0}^{\infty}\|2^{k\beta_\infty} v_a \chi_k I_\alpha f\|^\theta_{L^{q(\cdot)}} \\
&= C\sum_{k=0}^{\infty}\|2^{k\beta_\infty} \chi_k I_\alpha f\|^\theta_{L^{q(\cdot),a}} \\
&\le C\Bigg(\sum_{k=0}^{\infty}\Bigg\|\sum_{j=0}^{\min(k-2,0)} 2^{k\beta_\infty}\chi_k I_\alpha(f\chi_j)\Bigg\|^\theta_{L^{q(\cdot),a}} \\
&\quad + \sum_{k=0}^{\infty}\Bigg\|\sum_{j=\min(k-1,0)}^{k+1} 2^{k\beta_\infty}\chi_k I_\alpha(f\chi_j)\Bigg\|^\theta_{L^{q(\cdot),a}} \\
&\quad + \sum_{k=0}^{\infty}\Bigg\|\sum_{j=k+2}^{\infty} 2^{k\beta_\infty}\chi_k I_\alpha(f\chi_j)\Bigg\|^\theta_{L^{q(\cdot),a}}\Bigg) = I + II + III. \qquad (3.2)
\end{aligned}$$

In view of Theorem 2.6, we have

$$\begin{aligned}
II &\le C\sum_{k=0}^{\infty}\sum_{j=\min(k-1,0)}^{k+1}\|2^{k\beta_\infty} I_\alpha(f\chi_j)\chi_k\|^\theta_{L^{q(\cdot),a}} \\
&\le C\sum_{k=0}^{\infty}\sum_{j=\min(k-1,0)}^{k+1}\|2^{k\beta_\infty} f\chi_j\|^\theta_{L^{p(\cdot),b}} \\
&\le C\sum_{k=0}^{\infty}\|2^{k\beta_\infty} f\chi_k\|^\theta_{L^{p(\cdot),b}} \le C\|f\|^\theta_{K^{\beta(\cdot),b}_{p(\cdot),\theta}} \qquad (3.3)
\end{aligned}$$

for some $C > 0$.

Next, we consider I for the case $\theta \in (0,1]$. Since $j \le k-2$, we have a constant $C > 0$ such that

$$\chi_k(x)I_\alpha(f\chi_j)(x) \le C2^{-k(n-\alpha)}\int_{R_j}|f(y)|dy.$$

The Hölder inequality (2.4) yields

$$\|\chi_k I_\alpha(f\chi_j)\|_{L^{q(\cdot),a}} \le C2^{-k(n-\alpha)}\|\chi_k\|_{L^{q(\cdot),a}}\|f\chi_j\|_{L^{p(\cdot),b}}\|\chi_j\|_{L^{p'(\cdot),-b}}.$$

Lemma 2.7 gives

$$\begin{aligned}
\|\chi_k I_\alpha(f\chi_j)\|_{L^{q(\cdot),a}} &\le C2^{-(k-j)(n-\alpha)}\frac{\|\chi_k\|_{L^{q(\cdot),a}}}{\|\chi_j\|_{L^{q(\cdot),a}}}\|f\chi_j\|_{L^{p(\cdot),b}} \\
&\le C2^{-(k-j)(n-\alpha)}2^{(k-j)a}\frac{\|\chi_k\|_{L^{q(\cdot)}}}{\|\chi_j\|_{L^{q(\cdot)}}}\|f\chi_j\|_{L^{p(\cdot),b}}.
\end{aligned}$$

According to (2.7), we have

$$\frac{1}{p_-} = \frac{1}{q_-} + \frac{\alpha + a - b}{n},$$

and hence, (3.1) gives

$$\beta_\infty < n - \frac{n}{p_-} - b = n - \frac{n}{q_-} - \alpha - a.$$

Thus, there is a $\gamma \in (1, q_-)$ such that

$$\beta_\infty < n - \frac{n}{\gamma} - \alpha - a. \tag{3.4}$$

According to [14, Proposition 2.5 and Lemma 6.3], for any $\gamma \in (1, q_-)$, there is a constant $C > 0$ such that

$$\left\|\chi_k I_\alpha(f\chi_j)\right\|_{L^{q(\cdot),a}} \le C 2^{-(k-j)(n-\alpha-a)} 2^{(k-j)\frac{n}{\gamma}} \|f\chi_j\|_{L^{p(\cdot),b}}. \tag{3.5}$$

When $\theta \in (0,1)$, by using the θ-inequality, (3.4), and (3.5), we obtain

$$\begin{aligned} I &\le C \sum_{k=0}^{\infty} \sum_{j=0}^{\min(k-2,0)} \left\|2^{k\beta_\infty} \chi_k I_\alpha(f\chi_j)\right\|_{L^{q(\cdot),a}}^{\theta} \\ &\le C \sum_{k=0}^{\infty} \sum_{j=0}^{\min(k-2,0)} 2^{-(k-j)(n-\frac{n}{\gamma}-\alpha-a)\theta} \left\|2^{k\beta_\infty} f\chi_j\right\|_{L^{p(\cdot),b}}^{\theta} \\ &\le C \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} 2^{-(k-j)(n-\frac{n}{\gamma}-\alpha-a-\beta_\infty)\theta} \left\|2^{j\beta_\infty} f\chi_j\right\|_{L^{p(\cdot),b}}^{\theta} \\ &\le C \sum_{j=0}^{\infty} \left\|2^{j\beta_\infty} f\chi_j\right\|_{L^{p(\cdot),b}}^{\theta} \le C\|f\|_{K_{p(\cdot),\theta}^{\beta(\cdot),b}}^{\theta} \end{aligned} \tag{3.6}$$

for some $C > 0$.

For the case $\theta \in [1, \infty)$, (3.5) yields

$$\begin{aligned} I &\le C \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\min(k-2,0)} \left\|2^{k\beta_\infty} \chi_k I_\alpha(f\chi_j)\right\|_{L^{q(\cdot),a}} \right)^{\theta} \\ &\le C \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\min(k-2,0)} 2^{-(k-j)(n-\frac{n}{\gamma}-\alpha-a-\beta_\infty)} \left\|2^{j\beta_\infty} f\chi_j\right\|_{L^{p(\cdot),b}} \right)^{\theta}. \end{aligned}$$

The Hölder inequality and (3.4) assert that

$$I \leq C \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\min(k-2,0)} 2^{-\theta \frac{(k-j)}{2}(n-\frac{n}{\gamma}-\alpha-a-\beta_\infty)} \|2^{j\beta_\infty} f\chi_j\|^{\theta}_{L^{p(\cdot),b}} \right)$$
$$\times \left(\sum_{j=0}^{\min(k-2,0)} 2^{-\theta' \frac{(k-j)}{2}(n-\frac{n}{\gamma}-\alpha-a-\beta_\infty)} \right)^{\frac{\theta}{\theta'}}$$
$$\leq C \sum_{k=0}^{\infty} \sum_{j=0}^{\min(k-2,0)} 2^{-\frac{(k-j)}{2}(n-\frac{n}{\gamma}-\alpha-a-\beta_\infty)\theta} \|2^{j\beta_\infty} f\chi_j\|^{\theta}_{L^{p(\cdot),b}}$$
$$\leq C \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} 2^{-\frac{(k-j)}{2}(n-\frac{n}{\gamma}-\alpha-a-\beta_\infty)\theta} \|2^{j\beta_\infty} f\chi_j\|^{\theta}_{L^{p(\cdot),b}}$$
$$\leq C \sum_{j=0}^{\infty} \|2^{j\beta_\infty} f\chi_j\|^{\theta}_{L^{p(\cdot),b}} \leq C\|f\|^{\theta}_{K^{\beta(\cdot),b}_{p(\cdot),\theta}} \tag{3.7}$$

for some $C > 0$.

We now consider III. As $j \geq k+2$, we have

$$\chi_k(x) I_\alpha(f\chi_j)(x) \leq C 2^{-j(n-\alpha)} \int_{R_j} |f(y)| dy.$$

Thus, Lemmas 2.5 and 2.7 guarantee that

$$\|\chi_k I_\alpha(f\chi_j)\|_{L^{q(\cdot),a}} \leq C 2^{-j(n-\alpha)} \|\chi_k\|_{L^{q(\cdot),a}} \|f\chi_j\|_{L^{p(\cdot),b}} \|\chi_j\|_{L^{p'(\cdot),-b}}$$
$$\leq C 2^{(k-j)a} \frac{\|\chi_k\|_{L^{q(\cdot)}}}{\|\chi_j\|_{L^{q(\cdot)}}} \|f\chi_j\|_{L^{p(\cdot),b}}$$

for some $C > 0$.

In view of (2.7), we have

$$\frac{1}{p_+} = \frac{1}{q_+} + \frac{\alpha + a - b}{n}.$$

Hence, (3.1) gives

$$-\frac{n}{q_+} - a = -\frac{n}{p_+} + \alpha - b < \beta_\infty.$$

There is a $\gamma \in (q_+, \infty)$ such that

$$-\frac{n}{\gamma} - a < \beta_\infty. \tag{3.8}$$

Consequently, [14, Proposition 2.5 and Lemma 6.3] assure that, for any $\gamma \in (q_+, \infty)$, there exists a constant $C > 0$ such that

$$\|2^{k\beta_\infty} \chi_k I_\alpha(f\chi_j)\|_{L^{q(\cdot),a}} \leq C 2^{(k-j)(a+\frac{n}{\gamma}+\beta_\infty)} \|2^{j\beta_\infty} f\chi_j\|_{L^{p(\cdot),b}}. \tag{3.9}$$

When $\theta \in (0,1]$, (3.8), (3.9) and the θ-inequality give

$$\begin{aligned}
III &\le C \sum_{k=0}^{\infty} \sum_{j=\min(k-1,0)}^{\infty} \|2^{k\beta_\infty} \chi_k I_\alpha(f\chi_j)\|_{L^{q(\cdot),a}}^{\theta} \\
&\le C \sum_{k=0}^{\infty} \sum_{j=\min(k-1,0)}^{\infty} 2^{\theta(k-j)(a+\frac{n}{\gamma}+\beta_\infty)} \|2^{j\beta_\infty} f\chi_j\|_{L^{p(\cdot),b}}^{\theta} \\
&\le C \sum_{j=0}^{\infty} \sum_{k=0}^{j} 2^{\theta(k-j)(a+\frac{n}{\gamma}+\beta_\infty)} \|2^{j\beta_\infty} f\chi_j\|_{L^{p(\cdot),b}}^{\theta} \\
&\le C \sum_{j=0}^{\infty} \|2^{j\beta_\infty} f\chi_j\|_{L^{p(\cdot),b}}^{\theta} = C\|f\|_{K_{p(\cdot),\theta}^{\beta(\cdot),b}}^{\theta}.
\end{aligned} \tag{3.10}$$

When $\theta \in (1,\infty)$, (3.8), (3.9), and the Hölder inequality yield

$$\begin{aligned}
III &\le C \sum_{k=0}^{\infty} \left(\sum_{j=\min(k-2,0)}^{\infty} 2^{(k-j)(a+\frac{n}{\gamma}+\beta_\infty)} \|2^{j\beta_\infty} f\chi_j\|_{L^{p(\cdot),b}} \right)^{\theta} \\
&\le C \sum_{k=0}^{\infty} \left(\sum_{j=\min(k-2,0)}^{\infty} 2^{\theta\frac{(k-j)}{2}(a+\frac{n}{\gamma}+\beta_\infty)} \|2^{j\beta_\infty} f\chi_j\|_{L^{p(\cdot),b}}^{\theta} \right) \\
&\quad \times \left(\sum_{j=\min(k-2,0)}^{\infty} 2^{\theta'\frac{(k-j)}{2}(a+\frac{n}{\gamma}+\beta_\infty)} \right)^{\frac{\theta}{\theta'}} \\
&\le C \sum_{k=0}^{\infty} \sum_{j=\min(k-2,0)}^{\infty} 2^{\theta\frac{(k-j)}{2}(a+\frac{n}{\gamma}+\beta_\infty)} \|2^{j\beta_\infty} f\chi_j\|_{L^{p(\cdot),b}}^{\theta} \\
&\le C \sum_{j=0}^{\infty} \sum_{k=0}^{j} 2^{\theta\frac{(k-j)}{2}(a+\frac{n}{\gamma}+\beta_\infty)} \|2^{j\beta_\infty} f\chi_j\|_{L^{p(\cdot),b}}^{\theta} \\
&\le C \sum_{j=0}^{\infty} \|2^{j\beta_\infty} f\chi_j\|_{L^{p(\cdot),b}}^{\theta} = C\|f\|_{K_{p(\cdot),\theta}^{\beta(\cdot),b}}^{\theta}
\end{aligned} \tag{3.11}$$

for some $C > 0$. Therefore, (3.2), (3.3), (3.6), (3.7), (3.10), and (3.11) establish the boundedness of $I_\alpha : K_{p(\cdot),\theta}^{\beta(\cdot),b} \to K_{q(\cdot),\theta}^{\beta(\cdot),a}$. ∎

As a special case of the above main result, we have the Stein–Weiss inequalities on the power weighted Herz spaces.

Corollary 3.2. *Let $\alpha \in (0,n)$, $\theta,\beta \in (0,\infty)$, $a,b \in \mathbb{R}$, and $p,q \in (1,\infty)$. If a,b,p,q satisfy $p \le q$,*

$$-\frac{n}{q} < a \le b < \frac{n}{p'},$$
$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n} + \frac{a-b}{n},$$
$$-\frac{n}{p} + \alpha - b < \beta < n - \frac{n}{p} - b,$$

then there is a constant $C > 0$ such that, for any $f \in K^{\beta,b}_{p,\theta}$, we have

$$\|I_\alpha f\|_{K^{\beta,a}_{q,\theta}} \le C\|f\|_{K^{\beta,b}_{p,\theta}}.$$

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