

Symmetry results for a nonlocal nonlinear Poincaré–Wirtinger inequality

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Abstract. In this paper, we study the optimal constant in the nonlocal nonlinear Poincaré–Wirtinger inequality in $(a, b) \subset \mathbb{R}$:

$$\lambda_\alpha(p, q, r) \left(\int_a^b |u|^q dx \right)^{\frac{p}{q}} \leq \int_a^b |u'|^p dx + \alpha \left| \int_a^b |u|^{r-2} u dx \right|^{\frac{p}{r-1}},$$

where $\alpha \in \mathbb{R}$, $p, q, r > 1$ such that $\frac{2p}{p+2} \leq q \leq p$ and $\frac{q}{2} + 1 \leq r \leq q + \frac{q}{p}$. This problem admits a variational characterization in the nonlocal setting, as the associated Euler–Lagrange equation involves an integral term depending on the unknown function over the entire interval of definition.

We prove the existence of a critical value $\alpha_C = \alpha_C(p, q, r)$ such that the minimizers are even and have constant sign for $\alpha \leq \alpha_C$, while they are odd for $\alpha \geq \alpha_C$.

1. Introduction

Let $a, b, \alpha \in \mathbb{R}$ and $p, q, r > 1$; in this paper, we consider the following inequality:

$$\begin{aligned} & \lambda_\alpha(p, q, r) \left(\int_a^b |u|^q dx \right)^{\frac{p}{q}} \\ & \leq \int_a^b |u'|^p dx + \alpha \left| \int_a^b |u|^{r-2} u dx \right|^{\frac{p}{r-1}} \quad \forall u \in W^{1,p}(a, b) \text{ such that } u(a) = u(b) = 0. \end{aligned} \quad (1)$$

The optimal constant in (1) corresponds to the value realizing the minimum in the following eigenvalue problem:

$$\lambda_\alpha(p, q, r) = \inf \{ \mathcal{Q}_\alpha[u], u \in W_0^{1,p}(a, b), u \not\equiv 0 \}, \quad (2)$$

where

$$\mathcal{Q}_\alpha[u] := \frac{\int_a^b |u'|^p dx + \alpha \left| \int_a^b |u|^{r-2} u dx \right|^{\frac{p}{r-1}}}{\left(\int_a^b |u|^q dx \right)^{\frac{p}{q}}}. \quad (3)$$

Problems of this kind lead in general to nonstandard associated Euler–Lagrange equations, known in literature as nonlocal, because they depend on the value that the unknown function assumes on the whole domain throughout the integral over (a, b) . Specifically,

$$\begin{cases} -(|y'|^{p-2}y')' + \alpha|\gamma|^{\frac{p}{r-1}-2}\gamma|y|^{r-2} = \lambda_\alpha(p, q, r)\|y\|_q^{p-q}|y|^{q-2}y & \text{in }]a, b[, \\ y(a) = y(b) = 0, \end{cases}$$

where

$$\gamma = \int_a^b |y|^{r-2}y dx,$$

except for some trivial cases detailed in Section 3.

Nonlocal problems of this type date back at least to the 1837 papers by Duhamel [10] and Liouville [20] on thermo-elasticity. Since then, such problems have been studied in various contexts, including reaction-diffusion equations modeling chemical processes (see [13, 28]) or Brownian motion with random jumps (see [26]). They have been the object of extensive research over the last thirty years [12, 13, 15], particularly through the study of variational problems involving a nonlocal one-parameter functional, both in the n -dimensional [4] and the one-dimensional [24, 25] setting.

In higher dimensions, problem (2) has been treated [4] only in the case when $p = q = r = 2$. In that setting, the authors identify a *saturation phenomenon* under a volume constraint; they show that, among all domains with fixed measure, the optimal shape is the ball (for values of the parameter α below a critical one) and a union of two equal balls for supercritical values. A similar result holds when the Euclidean metric is replaced by a Finsler one [27]. A related but distinct line of research, not investigated in this paper, concerns the study of symmetry-breaking phenomena, where the symmetry of optimal domains is lost below or above certain threshold values; see, for instance, [3, 5, 22].

In one dimension, the analysis of the nonlocal problem (2) relies on the study of a generalized Wirtinger inequality, which was started by the pioneering work of Dacorogna, Gangbo, and Subia [8] in the case $q \leq 2p$ and $r = 2$. Subsequently, in [2, 6, 7, 11, 16, 21] various parameter ranges have been analyzed. The question of symmetry/non-symmetry of minimizers has been completely settled in [17]. Specifically, minimizers are symmetric when $q \leq (2r - 1)p$, while no odd function can be a minimizer when $q > (2r - 1)p$.

Our aim is to investigate the symmetry properties of the minimizers of (2) and, as a consequence, to provide some information on $\lambda_\alpha(p, q, r)$. To explore the full range of the exponents $p, q, r > 1$, we recall that, to the best of our knowledge, the nonlocal problem (2) has previously been studied only in the case $p = q = 2$ with $2 \leq r \leq 3$ in [24], and for $p = q \geq 2$ with $\frac{p}{2} + 1 \leq r \leq \frac{p}{2}$ in [25]. In these regimes, the minimizers of (2) are symmetric (either even or odd), and a saturation phenomenon occurs. In particular, for subcritical values of the parameter α , the minimizers are even functions with constant sign, while for supercritical values, the minimizers are odd sign-changing. It is therefore natural to ask in which regions of the parameter space symmetry breaking is expected to occur. In the present paper, we extend the range of treatable exponents and establish a new saturation phenomenon.

Throughout this paper, for the sake of simplicity, we will study the problem in the interval $(-1, 1)$ instead of (a, b) . The general case can be easily recovered since the nonlocal eigenvalue admits the following rescaling:

$$\lambda_\alpha(p, q, r; (a, b)) = \left[\left(\frac{2}{b-a} \right)^{\frac{1}{p'} + \frac{1}{q}} \right]^p \lambda_{\tilde{\alpha}}(p, q, r; (-1, 1)),$$

with

$$\tilde{\alpha} = \left(\frac{b-a}{2} \right)^{(\frac{1}{r-1} + \frac{1}{p'})p} \alpha.$$

Theorem 1.1. *Let $p, q, r > 1$ be such that $\frac{2p}{p+2} \leq q \leq p$ and $\frac{q}{2} + 1 \leq r \leq q + \frac{q}{p}$. Then, there exists a positive number $\alpha_C = \alpha_C(p, q, r)$ such that*

- (i) *if $\alpha < \alpha_C$, then $\lambda_\alpha(p, q, r) < \lambda_T(p, q, r)$,*
- (ii) *if $\alpha \geq \alpha_C$, then $\lambda_\alpha(p, q, r) = \lambda_T(p, q, r)$,*

where

$$\lambda_T(p, q, r) = \min_{\substack{u \in W_0^{1,p}(a,b) \\ \int_a^b |u|^{r-2} u \, dx = 0 \\ u \neq 0}} \frac{\int_a^b |u'|^p \, dx}{\left(\int_a^b |u|^q \, dx \right)^{\frac{p}{q}}}.$$

In addition, we prove the following symmetry results for the solutions of problem (2). We refer to Section 3 for the definition of the generalized trigonometric function $\sin_{p,q}(\cdot)$.

Theorem 1.2. *Let $p, q, r > 1$ be such that $\frac{2p}{p+2} \leq q \leq p$ and $\frac{q}{2} + 1 \leq r \leq q + \frac{q}{p}$. Then, the following statements hold.*

- (i) *If $\alpha < \alpha_C$, then any minimizer y of $\lambda_\alpha(p, q, r)$ is an even function with constant sign in $(-1, 1)$.*
- (ii) *If $\alpha > \alpha_C$, the function $y(x) = \sin_{p,q}(\lambda_T(p, q, r)x)$, $x \in (-1, 1)$, is the unique minimizer, up to a multiplicative constant, of $\lambda_\alpha(p, q, r)$. Hence, it is an odd function, $\int_{-1}^1 |y|^{r-2} y \, dx = 0$, and $\bar{x} = 0$ is the only point in $(-1, 1)$ such that $y(\bar{x}) = 0$.*
- (iii) *If $\alpha = \alpha_C$, then $\lambda_{\alpha_C}(p, q, r)$ admits both a positive minimizer and the minimizer $y(x) = \sin_{p,q}(\pi_{p,q}x)$, up to a multiplicative constant. Moreover, if $r > \frac{1}{2} + \frac{q}{2} + \frac{q}{p}$ any minimizer has a constant sign or it is odd.*

The outline of the paper follows. In Section 2, we show how the treated nonlocal inequalities generalize classical Poincaré, Wirtinger, and twisted inequalities; in Section 3, we provide some recalls on the nonlocal eigenvalue problem we are dealing with; in Section 4, we study the properties of an auxiliary function useful to give some representation formulas of the eigenvalue and the eigenfunctions of problem (2); in Section 5, we give the proof of the main theorems.

2. A unified treatment of Poincaré–Wirtinger and twisted inequalities

One of the main aims of this paper is to unify and extend the study of Poincaré–Wirtinger, and twisted inequalities, through the introduction of a penalization term. In this section, we describe each of these classical inequalities in detail, explaining their connections with the nonlocal inequality under investigation.

Let $\lambda \in \mathbb{R}$; inequality (1) is a nonlinear generalization of the celebrated one-dimensional inequality

$$\lambda \int_a^b |u|^2 dx \leq \int_a^b |u'|^2 dx \quad \forall u \in C^1(a, b), \quad (4)$$

that is, the Poincaré inequality when

$$u(a) = u(b) = 0, \quad (5)$$

and that is, the Wirtinger inequality when

$$\int_a^b u dx = 0. \quad (6)$$

The best constant λ in both Poincaré (4)–(5) and Wirtinger inequalities (4)–(6) is obtained for

$$\lambda = \left(\frac{\pi}{b-a} \right)^2.$$

The generalized Poincaré inequality (see, e.g., [17] and references therein) states that there exists a constant $\lambda_P(p, q)$ such that

$$\lambda_P(p, q) \left(\int_a^b |u|^q dx \right)^{\frac{p}{q}} \leq \int_a^b |u'|^p dx \quad \forall u \in W^{1,p}(a, b) \text{ s.t. } u(a) = u(b) = 0. \quad (7)$$

It is easily seen that, when $\alpha = 0$, the nonlocal inequality (1) is the Poincaré inequality (7). When $p = q = 2$, we come back to the classical Poincaré inequality (4)–(5). Moreover, the optimal constant in (7) is also the minimum for the variational problem

$$\lambda_P(p, q) = \min_{\substack{u \in W_0^{1,p}(a,b) \\ u \neq 0}} \frac{\int_a^b |u'|^p dx}{\left(\int_a^b |u|^q dx \right)^{\frac{p}{q}}},$$

and the minimizing functions are even functions with constant sign. It is easily seen that $\lambda_P(p, q)$ is a homogeneous (p, q) -Dirichlet–Laplacian eigenvalue (see, e.g., [18, Theorem 3.3]).

On the other hand, the generalized Wirtinger inequality states that there exists a constant $\lambda_W(p, q, r)$ such that

$$\lambda_W(p, q, r) \left(\int_a^b |u|^q dx \right)^{\frac{p}{q}} \leq \int_a^b |u'|^p dx \quad \forall u \in W^{1,p}(a, b) \text{ s.t. } \int_a^b |u|^{r-2} u dx = 0. \quad (8)$$

When $p = q = r = 2$, we come back to the classical Wirtinger inequality (4)–(6). Moreover, the optimal constant in (8) is also the minimum for the variational problem

$$\lambda_W(p, q, r) = \min_{\substack{u \in W^{1,p}(a,b) \\ \int_a^b |u|^{r-2} u \, dx = 0 \\ u \neq 0}} \frac{\int_a^b |u'|^p \, dx}{\left(\int_a^b |u|^q \, dx \right)^{\frac{p}{q}}},$$

and the minimizing functions are odd functions. For the exact value of $\lambda_W(p, q, r)$ see [16]. It is easily seen that $\lambda_W(p, q, r)$ is a Neumann Laplacian eigenvalue (see, e.g., [18, Theorem 3.4]).

When both Dirichlet (5) and Neumann (6) boundary conditions hold, we speak of twisted boundary conditions [1, 14]. The best constant in the twisted inequality (4)–(6) is equal to

$$\lambda_T = \left(\frac{2\pi}{b-a} \right)^2.$$

The generalized twisted inequality states that there exists a constant $\lambda_T(p, q, r)$ such that

$$\begin{aligned} \lambda_T(p, q, r) \left(\int_a^b |u|^q \, dx \right)^{\frac{p}{q}} &\leq \int_a^b |u'|^p \, dx \quad \forall u \in W^{1,p}(a, b) \\ \text{s.t. } u(a) = u(b) = 0 \quad &\text{and} \quad \int_a^b |u|^{r-2} u \, dx = 0. \end{aligned} \quad (9)$$

It is easily seen that, meanwhile when $\alpha \rightarrow +\infty$, inequality (1) tends to the twisted inequality (9). When $p = q = r = 2$, we come back to the classical twisted inequality (4)–(6).

Moreover, the optimal constant in (9) is also the minimum for the variational problem

$$\lambda_T(p, q, r) = \min_{\substack{u \in W_0^{1,p}(a,b) \\ \int_a^b |u|^{r-2} u \, dx = 0 \\ u \neq 0}} \frac{\int_a^b |u'|^p \, dx}{\left(\int_a^b |u|^q \, dx \right)^{\frac{p}{q}}}, \quad (10)$$

and the minimizing functions are odd functions. See [7, Theorem 1.1] for the exact value of $\lambda_T(p, q, r)$, and note that there is no dependence on the parameter r .

3. The eigenvalue problem

In this section, we firstly recall some results on the generalized trigonometric functions and then some properties of the eigenvalue problem (2).

3.1. The (p, q) -circular functions

We briefly summarize some properties of the p -trigonometric functions for any fixed $1 < p < +\infty$ (see, e.g., [18, 19, 23]). These functions generalize the familiar trigonometric functions and coincide with them when $p = 2$.

Let us consider the function $F_p : [0, 1] \rightarrow \mathbb{R}$ defined as

$$F_p(x) = \int_0^x \frac{dt}{(1-t^p)^{\frac{1}{p}}}.$$

Denote by $z(s)$ the inverse function of F which is defined on the interval $[0, \frac{\pi_p}{2}]$, where

$$\pi_p = 2 \int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}.$$

Therefore, the p -sine function $\sin_p$ is defined as the following periodic extension of $z(t)$:

$$\sin_p(t) = \begin{cases} z(t) & \text{if } t \in [0, \frac{\pi_p}{2}], \\ z(\pi_p - t) & \text{if } t \in [\frac{\pi_p}{2}, \pi_p], \\ -\sin_p(-t) & \text{if } t \in [-\pi_p, 0]. \end{cases}$$

It is extended periodically to all $\mathbb{R}$, with period $2\pi_p$. Furthermore, the p -cosine function is defined by

$$\cos_p(t) = \frac{d}{dt} \sin_p(t)$$

and is a $2\pi_p$ -periodic and odd function.

To further extend the definitions of trigonometric functions, let us consider $p, q > 1$ and set

$$\pi_{p,q} := 2 \int_0^1 \frac{1}{(1-t^q)^{\frac{1}{p}}} dt = \frac{2}{q} B\left(\frac{1}{p'}, \frac{1}{q}\right) = \frac{2}{q} \frac{\Gamma(\frac{1}{p'})\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{p'} + \frac{1}{q})},$$

where B and Γ are the beta and gamma functions, respectively.

This definition coincides with π_p when $p = q$. Therefore, the function $\sin_{p,q}$ is defined on the interval $[0, \frac{\pi_{p,q}}{2}]$ as the inverse of $F_{p,q} : [0, 1] \rightarrow \mathbb{R}$ given by

$$F_{p,q}(x) = \int_0^x \frac{1}{(1-t^q)^{\frac{1}{p}}} dx$$

and extended to the real line by the usual process involving the symmetry and the $2\pi_{p,q}$ periodicity.

Finally, we recall from [18, Theorem 3.3] that any eigenvalue and eigenfunction of the 1-dimensional Dirichlet (p, q) -Laplacian eigenvalue problem

$$\begin{cases} -(|y'|^{p-2}y')' = \lambda|y|^{q-2}y & \text{in }]-1, 1[, \\ y(-1) = y(1) = 0 \end{cases} \quad (11)$$

are of the form

$$\lambda_n = c_1 \frac{q}{p'} \left(\frac{n\pi_{p,q}}{2} \right) \quad \text{and} \quad y_n(x) = c_2 \sin_{p,q} \left(\frac{n\pi_{p,q}}{2} (x+1) \right) \quad \forall n \in \mathbb{N},$$

respectively, for $c_1, c_2 \in \mathbb{R}$. Clearly, when $p = q$, we fall in the case of the p -Laplacian problem.

3.2. The eigenvalue problem

We firstly show some properties of the solution of the eigenvalue problem (2).

Proposition 3.1. *Let $\alpha \in \mathbb{R}$, $p, q, r > 1$ be such that $r \leq p + 1$. Then, problem (2) admits a solution in $W_0^{1,p}(-1, 1)$ and any minimizer y of (2) is a solution of the following Dirichlet homogeneous problem:*

$$\begin{cases} -(|y'|^{p-2}y')' + \alpha|\gamma|^{\frac{p}{r-1}-2}\gamma|y|^{r-2} = \lambda_\alpha(p, q, r) \|y\|_q^{p-q} |y|^{q-2}y & \text{in }]-1, 1[, \\ y(-1) = y(1) = 0, \end{cases} \quad (12)$$

where

$$\gamma = \begin{cases} 0 & \text{if both } r = p + 1 \text{ and } \int_{-1}^1 |y|^{r-2}y dx = 0, \\ \int_{-1}^1 |y|^{r-2}y dx & \text{otherwise.} \end{cases}$$

Moreover, $y, y'|y|^{p-2} \in C^1[-1, 1]$.

Proof. Standard methods of Calculus of Variations prove the existence of a minimizer. If $p > r - 1$, the functional $\mathcal{Q}_\alpha[\cdot]$ in (3) is differentiable and hence the associated Euler–Lagrange equation leads to (12); meanwhile, when $p = r - 1$, the problem (2) coincides with problem (10); hence, $\gamma = 0$ and we get the conclusion.

Finally, the fact that $y, y'|y|^{p-2} \in C^1[-1, 1]$ is easily seen from (12). ■

At this stage, we analyze the monotonicity and asymptotic properties of the eigenvalue (2) with respect to the parameter α .

Proposition 3.2. *For any fixed $p, q, r > 1$ such that $r \leq p + 1$, the function $\alpha \in \mathbb{R} \mapsto \lambda_\alpha(p, q, r)$ is Lipschitz continuous, nondecreasing with respect to $\alpha \in \mathbb{R}$ and*

$$\lim_{\alpha \rightarrow -\infty} \lambda_\alpha(p, q, r) = -\infty, \quad \lim_{\alpha \rightarrow +\infty} \lambda_\alpha(p, q, r) = \lambda_T(p, q, r).$$

Proof. Let us fix $\varepsilon > 0$; then, by using the Hölder inequality with conjugate exponents $\frac{q}{r-1}$ and $\frac{q}{q-r+1}$, we have

$$\mathcal{Q}_{\alpha+\varepsilon}[u] \leq \mathcal{Q}_\alpha[u] + \varepsilon \frac{(\int_{-1}^1 |u|^{r-1} dx)^{\frac{p}{r-1}}}{(\int_{-1}^1 |u|^q dx)^{\frac{p}{q}}} \leq \mathcal{Q}_\alpha[u] + 2^{\frac{p(q-r+1)}{q(r-1)}} \varepsilon.$$

Therefore, we gain the following chain of inequalities:

$$\mathcal{Q}_\alpha[u] \leq \mathcal{Q}_{\alpha+\varepsilon}[u] \leq \mathcal{Q}_\alpha[u] + 2^{\frac{p(q-r+1)}{q(r-1)}} \varepsilon \quad \forall \varepsilon > 0.$$

By taking the minimum for any $u \in W_0^{1,p}(-1, 1)$, we have

$$\lambda_\alpha(p, q, r) \leq \lambda_{\alpha+\varepsilon}(p, q, r) \leq \lambda_\alpha(p, q, r) + 2^{\frac{p(q-r+1)}{q(r-1)}} \varepsilon \quad \forall \varepsilon > 0,$$

which implies the desired Lipschitz continuity and monotonicity.

Now, let us fix a positive admissible function $\varphi \in W_0^{1,p}(-1, 1)$. Then, we have that $\mathcal{Q}_\alpha[\varphi] \rightarrow -\infty$ as $\alpha \rightarrow -\infty$ and since $\lambda_\alpha(p, q, r) \leq \mathcal{Q}_\alpha[\varphi]$, we have that

$$\lim_{\alpha \rightarrow -\infty} \lambda_\alpha(p, q, r) = -\infty.$$

Finally, let us consider a sequence $\{\alpha_n\}_{n \in \mathbb{N}} \rightarrow +\infty$. Since $\lambda_\alpha(p, q, r)$ is nondecreasing with respect to α , we have that $\lambda_\alpha(p, q, r) \leq \lambda_T(p, q, r)$ for any $\alpha \in \mathbb{R}$. Let us denote by $u_n = u_{\alpha_n}$ the normalized ($\|u_n\|_{L^q} = 1$) minimizer in $W_0^{1,p}$ of (2) when the value of the parameter is α_n ; we have that

$$\lambda_{\alpha_n}(p, q, r) = \int_{-1}^1 |u'_n|^p dx + \alpha_n \left(\int_{-1}^1 |u_n|^{r-2} u_n dx \right)^{\frac{p}{r-1}} \leq \lambda_T(p, q, r).$$

This implies that, up to a subsequence, u_n strongly converges in $L^p(-1, 1)$ and weakly in $W_0^{1,p}(-1, 1)$ to a function $u \in W_0^{1,p}(-1, 1)$ such that $\|u\|_{L^q} = 1$. Furthermore, we have that

$$\left(\int_{-1}^1 |u_n|^{r-2} u_n dx \right)^{\frac{p}{r-1}} \leq \frac{\lambda_T(p, q, r)}{\alpha_n} \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

which means that $\int_{-1}^1 |u|^{r-2} u dx = 0$. On the other hand, since u is an admissible function for (10), by using the lower semicontinuity of the integral, we have that

$$\begin{aligned} \lambda_T(p, q, r) &\leq \int_{-1}^1 |u'|^p dx \leq \liminf_{n \rightarrow +\infty} \left[\int_{-1}^1 |u'_n|^p dx + \alpha_n \left(\int_{-1}^1 |u_n|^{r-2} u_n dx \right)^{\frac{p}{r-1}} \right] \\ &= \lim_{n \rightarrow +\infty} \lambda_{\alpha_n}(p, q, r) \leq \lambda_T(p, q, r), \end{aligned}$$

and hence, the conclusion follows. ■

4. The auxiliary function H

In this section, we study the behavior of an auxiliary function on which the proof of the main results is based (Theorems 1.1 and 1.2). We consider the following integral function:

$$H(m, p, q, r) : (m, p, q, r) \in [0, 1] \times [1, +\infty[\times [1, +\infty[\times [1, +\infty[\mapsto \mathbb{R},$$

defined as

$$\begin{aligned} H(m, p, q, r) &:= \int_{-m}^1 \frac{dy}{[1 - R(m, q, r)(1 - |y|^{r-2}y) - |y|^q]^{\frac{1}{p}}} \\ &= \int_0^1 \frac{dy}{[1 - R(m, q, r)(1 - y^{r-1}) - y^q]^{\frac{1}{p}}} \\ &\quad + \int_0^1 \frac{mdy}{[1 - R(m, q, r)(1 + m^{r-1}y^{r-1}) - m^q y^q]^{\frac{1}{p}}}, \end{aligned} \quad (13)$$

where

$$R(m, q, r) = \frac{1 - m^q}{1 + m^{r-1}}. \quad (14)$$

It will also be very useful in the sequel to consider h , the integrand function of H , that is defined as

$$\begin{aligned} h(m, p, q, r, y) &:= \frac{1}{[1 - R(m, q, r)(1 - y^{r-1}) - y^q]^{\frac{1}{p}}} \\ &\quad + \frac{m}{[1 - R(m, q, r)(1 + m^{r-1}y^{r-1}) - m^q y^q]^{\frac{1}{p}}} \end{aligned}$$

for any $y \in [0, 1[$, except when $m = y = 0$.

We will prove the monotonicity of the auxiliary function with respect to r (more precisely, the function h) in Lemma 4.1 and then with respect to m in Lemma 4.2. Finally, in Proposition 4.3, we provide some useful estimates for the function H .

Lemma 4.1. *Let $p, q, r > 1$ such that $\frac{1}{2} + \frac{q}{2} + \frac{q}{2p} \leq r \leq q + \frac{q}{p}$. For any fixed $y \in [0, 1[$ and*

- *for any fixed $m \in [0, 1[$, the function $h(m, p, q, r, y)$ is strictly increasing with respect to r ;*
- *for $m = 1$, the function $h(1, p, q, r, y)$ is constant with respect to r .*

Proof. We divide the proof into three steps: in the first step, we compute the expression of the derivative of h with respect to r for any $m \in]0, 1[$ and $y \in]0, 1[$; in the second step, we study the sign of the aforementioned derivative; in the third step, we analyze the cases excluded by the previous steps. From now on, for the sake of simplicity, we will set $R = R(m, q, r)$.

Step 1 (The derivative of h). Let us start by considering the case when $m \in]0, 1[$ and $y \in]0, 1[$. Differentiating h with respect to r , we have

$$\begin{aligned} \partial_r h(m, p, q, r, y) &= -\frac{1}{p} \frac{(1 - y^{r-1})\partial_r R + R y^{r-1} \log y}{[1 - R(1 - y^{r-1}) - y^q]^{\frac{p+1}{p}}} \\ &\quad + -\frac{m}{p} \frac{[-(1 + m^{r-1}y^{r-1})\partial_r R - R m^{r-1}y^{r-1}(\log m + \log y)]}{[1 - R(1 + m^{r-1}y^{r-1}) - m^q y^q]^{\frac{p+1}{p}}}. \end{aligned}$$

Therefore, in order to compute the derivative of h with respect to r , we need to differentiate R (defined in (14)) with respect to r . We have

$$\partial_r R = -\frac{1-m^q}{(1+m^{r-1})^2} m^{r-1} \log m,$$

and hence,

$$\begin{aligned} \partial_r h(m, p, q, r, y) &= -\frac{1}{p} \frac{1-m^q}{(1+m^{r-1})^2} \left\{ \frac{(1-y^{r-1})m^{r-1} \log m + y^{r-1}(1+m^{r-1}) \log y}{[1-R(1-y^{r-1})-y^q]^{\frac{p+1}{p}}} \right. \\ &\quad \left. + m^r \frac{(1+m^{r-1}y^{r-1}) \log m - (1+m^{r-1})y^{r-1}(\log m + \log y)}{[1-R(1+m^{r-1}y^{r-1})-m^q y^q]^{\frac{p+1}{p}}} \right\}. \end{aligned} \quad (15)$$

Step 2 (The monotonicity of h). It is easily seen that the numerator of the first ratio, in the curly brackets of (15), is negative. If the numerator of the second ratio is also negative, we get the desired monotonicity. Otherwise, if this second numerator is positive, let us observe that

$$m^q(1-R(1-y^{r-1})-y^q) \leq [1-R(1+m^{r-1}y^{r-1})-m^q y^q],$$

which implies

$$\begin{aligned} \partial_r h(m, p, q, r, y) &\geq -\frac{1}{p} \frac{1-m^q}{(1+m^{r-1})^2} \left\{ \frac{(1-y^{r-1})m^{r-1} \log m + y^{r-1}(1+m^{r-1}) \log y}{m^{-\frac{q(p+1)}{p}} [1-R(1+m^{r-1}y^{r-1})-m^q y^q]^{\frac{p+1}{p}}} \right. \\ &\quad \left. + m^r \frac{(1+m^{r-1}y^{r-1}) \log m - (1+m^{r-1})y^{r-1}(\log m + \log y)}{[1-R(1+m^{r-1}y^{r-1})-m^q y^q]^{\frac{p+1}{p}}} \right\}. \end{aligned} \quad (16)$$

Hence, by setting

$$\begin{aligned} g(m, p, q, r, y) &:= [-(1-y^{r-1})m^{r-1} \log m - y^{r-1}(1+m^{r-1}) \log y] \\ &\quad + [(y^{r-1}-1) \log m + (1+m^{r-1})y^{r-1} \log y] m^{r-\frac{q(p+1)}{p}}, \end{aligned}$$

we have that inequality (16) can be written as

$$\partial_r h(m, p, q, r, y) \geq \frac{1}{p} \frac{1-m^q}{(1+m^{r-1})^2 m^{\frac{q(p+1)}{p}}} g(m, p, q, r, y). \quad (17)$$

To prove the positivity of $\partial_r h$, we will show that

$$g(m, p, q, r, y) > 0, \quad (18)$$

by proving that g is decreasing with respect to y in $]0, 1[$, since $g(m, p, q, r, 1) = 0$. By differentiating g with respect to y , we obtain

$$\begin{aligned} \partial_y g(m, p, q, r, y) &= [(r-1)y^{r-2}m^{r-1}\log m - (r-1)y^{r-2}(1+m^{r-1})\log y - y^{r-2}(1+m^r)] \\ &\quad + [(r-1)y^{r-2}\log m + (1+m^{r-1})((r-1)y^{r-2}\log y + y^{r-2})]m^{r-\frac{q(p+1)}{p}} \\ &= y^{r-2}[(r-1)(m^{r-1}+m^{r-\frac{q(p+1)}{p}})\log m + (r-1)(1+m^{r-1})(m^{r-\frac{q(p+1)}{p}}-1)\log y \\ &\quad + (1+m^{r-1})(m^{r-\frac{q(p+1)}{p}}-1)]. \end{aligned}$$

Since $r - \frac{q(p+1)}{p} \leq 0$ by assumption, this derivative is negative if and only if

$$\log y < -\frac{(m^{r-1}+m^{r-\frac{q(p+1)}{p}})\log m}{(1+m^{r-1})(m^{r-\frac{q(p+1)}{p}}-1)} - \frac{1}{r-1}. \quad (19)$$

Since the left-hand term is negative, then if the right-hand side of (19) is nonnegative, then the inequality (19) holds. To this aim, we will equivalently show that

$$f(m, p, q, r) := -(m^{r-1}+m^{r-\frac{q(p+1)}{p}})\log m - \frac{1}{r-1}(1+m^{r-1})(m^{r-\frac{q(p+1)}{p}}-1) > 0. \quad (20)$$

We have

$$\begin{aligned} f(m, p, q, r) &= (m^{r-1}+m^{r-\frac{q(p+1)}{p}})\log \frac{1}{m} \\ &\quad + \frac{1}{r-1}(1+m^{r-1}-m^{r-\frac{q(p+1)}{p}}-m^{2r-1-\frac{q(p+1)}{p}}) \\ &= m^{r-1}\left(\log \frac{1}{m} + \frac{1}{r-1}\right) + m^{r-\frac{q(p+1)}{p}}\left(\log \frac{1}{m} - \frac{1}{r-1}\right) \\ &\quad + \frac{1}{r-1}(1-m^{2r-1-\frac{q(p+1)}{p}}) \\ &\geq m^{r-1}\left(\log \frac{1}{m} + \frac{1}{r-1}\right) + m^{r-\frac{q(p+1)}{p}}\left(\log \frac{1}{m} - \frac{1}{r-1}\right) \\ &= m^{r-\frac{q(p+1)}{p}}\left(m^{\frac{q(p+1)}{p}-1}\left(\log \frac{1}{m} + \frac{1}{r-1}\right) + \log \frac{1}{m} - \frac{1}{r-1}\right) \end{aligned}$$

because $2r-1-\frac{q(p+1)}{p} \geq 0$. Hence, the positivity of f follows if

$$e(m, p, q, r) := m^{\frac{q(p+1)}{p}-1}\left(\log \frac{1}{m} + \frac{1}{r-1}\right) + \log \frac{1}{m} - \frac{1}{r-1} > 0. \quad (21)$$

Since $e(1, p, q, r) = 0$, to prove (21), we show that e is decreasing with respect to m ; indeed, we have

$$\partial_m e(n, p, q, r) = m^{\frac{q(p+1)}{p}-2}\left(\log \left(\frac{1}{m^{\frac{q(p+1)}{p}-1}}\right) + \frac{\frac{q(p+1)}{p}-1}{r-1} - 1 - \frac{1}{m^{\frac{q(p+1)}{p}-1}}\right),$$

which is negative since $\log z < z - 1$ when $z > 1$, $m^{\frac{q(p+1)}{p}-1} < 1$ and $r \geq \frac{1}{2} + \frac{q}{2} + \frac{q}{2p}$, by assumption.

Hence (21), (20), (19), and (18) are satisfied, and recalling the behavior of h from (17), this implies that

$$\partial_r h(m, p, q, r, y) \geq \frac{1}{p} \frac{1 - m^q}{(1 + m^{r-1})^2} g(m, p, q, r, y) > \frac{1}{p} \frac{1 - m^q}{(1 + m^{r-1})^2} g(m, p, q, r, 1) = 0.$$

when $m \in]0, 1[$ and $y \in]0, 1[$.

Step 3 (The trivial cases). We observe that if $m = 0$, then $R = 1$ and

$$h(0, p, q, r, y) = \frac{1}{(y^{r-1} - y^q)^{\frac{1}{p}}},$$

which is strictly increasing with respect to r .

Meanwhile, if $m = 1$, then $R = 0$ and

$$h(1, p, q, r, y) = \frac{2}{(1 - y^q)^{\frac{1}{p}}},$$

which is constant with respect to r .

Finally, when $y = 0$, we have

$$h(m, p, q, r, 0) = \frac{1 + m}{1 - R},$$

which is strictly increasing with respect to r . ■

At this stage, to prove the monotonicity of H with respect to m , we argue using a change of variables similarly as in [17].

Lemma 4.2. *Let $p, q > 1$ be such that $q \leq p$; then, $\partial_m H(m, p, q, \frac{q}{2} + 1) \geq 0$ for any $m \in]0, 1[$.*

Proof. Let us explicitly observe that when $r = \frac{q}{2} + 1$, then

$$R\left(m, q, \frac{q}{2} + 1\right) = 1 - m^{\frac{q}{2}} \quad \forall m \in [0, 1].$$

Hence, for any fixed p and q in the assumed ranges, we can write

$$K(m) := H\left(m, p, q, \frac{q}{2} + 1\right) = \int_0^1 (A(m, y)^{-\frac{1}{p}} + m B(m, y)^{-\frac{1}{p}}) dy, \quad (22)$$

where

$$A(m, y) := m^{\frac{q}{2}} + (1 - m^{\frac{q}{2}}) y^{\frac{q}{2}} - y^q \quad \forall (m, y) \in [0, 1]^2,$$

$$B(m, y) := m^{\frac{q}{2}} - (1 - m^{\frac{q}{2}}) m^{\frac{q}{2}} y^{\frac{q}{2}} - m^q y^q \quad \forall (m, y) \in [0, 1]^2.$$

We get

$$K'(m) = -\frac{1}{p} \int_0^1 \left(A(m, y)^{-\frac{1}{p}-1} \partial_m A(m, y) + B(m, y)^{-\frac{1}{p}-1} (-pB(m, y) + m \partial_m B(m, y)) \right) dy.$$

Differentiating $A(m, y)$ and $B(m, y)$ with respect to m , we obtain

$$\begin{aligned} \partial_m A(m, y) &= \frac{q}{2} m^{\frac{q}{2}-1} (1 - y^{\frac{q}{2}}), \\ -pB(m, y) + m \partial_m B(m, y) &= \left(\frac{q}{2} - p \right) m^{\frac{q}{2}} (1 - y^{\frac{q}{2}}) + (q - p) m^q (y^{\frac{q}{2}} - y^q). \end{aligned}$$

Hence, we have

$$\begin{aligned} K'(m) &= m^{\frac{q}{2}-1} \int_0^1 -\frac{q}{2p} \frac{1 - y^{\frac{q}{2}}}{A(m, y)^{\frac{1}{p}+1}} + \left(1 - \frac{q}{2p} \right) \frac{m(1 - y^{\frac{q}{2}})}{B(m, y)^{\frac{1}{p}+1}} \\ &\quad + \left(1 - \frac{q}{p} \right) \frac{m^{\frac{q}{2}+1} (y^{\frac{q}{2}} - y^q)}{B(m, y)^{\frac{1}{p}+1}} dy. \end{aligned}$$

To prove the nonnegativity of the integral, we have to show that

$$\begin{aligned} &-\frac{q}{2p} \int_0^1 \frac{1 - y^{\frac{q}{2}}}{A(m, y)^{\frac{p+1}{p}}} dy \\ &+ \int_0^1 \left[\left(1 - \frac{q}{2p} \right) \frac{m(1 - y^{\frac{q}{2}})}{B(m, y)^{\frac{1}{p}+1}} + \left(1 - \frac{q}{p} \right) \frac{m^{\frac{q}{2}+1} (y^{\frac{q}{2}} - y^q)}{B(m, y)^{\frac{1}{p}+1}} \right] dy \geq 0. \end{aligned} \quad (23)$$

Following the ideas of [17], for all $m \in (0, 1)$ and $z \in (0, 1)$, we set

$$\delta(z) := [1 - (1 - m^{\frac{q}{2}})z^{\frac{q}{2}}]^{\frac{2}{q}}$$

and

$$\ell(z) := \frac{mz}{\delta(z)}.$$

It holds that $\ell(0) = 0$, $\ell(1) = 1$, and

$$\ell'(z) := \frac{m}{\delta(z)^{\frac{q}{2}+1}}.$$

Since the function ℓ is strictly increasing, keeping the change of variables $y = \ell(z)$ into account, inequality (23) follows if we prove that

$$\begin{aligned} &-\frac{q}{2p} \int_0^1 \frac{1 - m^{\frac{q}{2}} z^{\frac{q}{2}} \delta(z)^{-\frac{q}{2}}}{(m^{\frac{q}{2}} + (1 - m^{\frac{q}{2}})m^{\frac{q}{2}} z^{\frac{q}{2}} \delta(z)^{-\frac{q}{2}} - m^q z^q \delta(z)^{-q})^{\frac{p+1}{p}}} \cdot \frac{m}{\delta(z)^{\frac{q}{2}+1}} dz \\ &+ \int_0^1 \frac{(1 - \frac{q}{2p})m(1 - z^{\frac{q}{2}}) + (1 - \frac{q}{p})m^{\frac{q}{2}+1}(z^{\frac{q}{2}} - z^q)}{(m^{\frac{q}{2}} - (1 - m^{\frac{q}{2}})m^{\frac{q}{2}} z^{\frac{q}{2}} - m^q z^q)^{\frac{p+1}{p}}} dz \geq 0, \end{aligned}$$

Since it is easily checked that $1 - z^{\frac{q}{2}} = \delta(z)^{\frac{q}{2}} - m^{\frac{q}{2}} z^{\frac{q}{2}}$ and

$$m^{\frac{q}{2}} - (1 - m^{\frac{q}{2}}) m^{\frac{q}{2}} z^{\frac{q}{2}} - m^q z^q = \delta(z)^q \left(m^{\frac{q}{2}} + (1 - m^{\frac{q}{2}}) \frac{m^{\frac{q}{2}} z^{\frac{q}{2}}}{\delta(z)^{\frac{q}{2}}} - \frac{m^q z^q}{\delta(z)^q} \right),$$

the conclusion follows. $\blacksquare$

The two previous lemmas lead to the following estimates for the function H .

Proposition 4.3. *Let $p, q, r > 1$ be such that $\frac{2p}{p+2} \leq q \leq p$.*

(i) *If $\frac{q}{2} + 1 \leq r \leq q + \frac{q}{p}$, then*

$$H(m, p, q, r) \geq \pi_{p,q}$$

for any $m \in [0, 1]$.

(ii) *If $\frac{q}{2} + 1 < r \leq q + \frac{q}{p}$, then*

$$H(m, p, q, r) = \pi_{p,q}$$

if and only if $m = 1$.

Proof. Case (i). If $m = 1$, by the definition (13) of H , we have that

$$H(1, p, q, r) = 2 \int_0^1 \frac{dy}{(1 - y^q)^{\frac{1}{p}}} = \pi_{p,q}. \quad (24)$$

If $m = 0$, it is easily seen that

$$H(0, p, q, r) = \int_0^1 \frac{dy}{(y^{r-1} - y^q)^{\frac{1}{p}}} \geq \int_0^1 \frac{dy}{(1 - y^q)^{\frac{1}{p}}} = \pi_{p,q}. \quad (25)$$

When $0 < m < 1$, since $\bar{r} := \frac{q}{2} + 1 \geq \frac{1}{2} + \frac{q}{2} + \frac{q}{2p}$, exploiting first the monotonicity with respect to r (Lemma 4.1) and then the monotonicity with respect to m (Lemma 4.2), we obtain

$$H(m, p, q, r) \geq K(m) \geq K(0) = \int_0^1 \frac{dy}{(y^{\frac{q}{2}} - y^q)^{\frac{1}{p}}} \geq \pi_{p,q}, \quad (26)$$

where

$$K(m) = H(m, p, q, \bar{r})$$

is the function defined in (22).

Case (ii). The sufficient condition follows by (24), while the necessary condition follows by observing that both the inequality in (25) when $m = 0$ and the first inequality in (26) when $m \in (0, 1)$ are strict. $\blacksquare$

5. Proof of the main theorems

A key role in the proof of the main results is played by the sign-changing minimizers. When such solutions occur, both the eigenvalue and the corresponding eigenfunctions can be represented in terms of the function H introduced in the previous section.

Proposition 5.1. *Let $p, q, r > 1$ be such that $\frac{2p}{p+2} \leq q \leq p$ and $\frac{q}{2} + 1 \leq r \leq q + \frac{q}{p}$ and suppose that there exists $\alpha > 0$ such that $\lambda_\alpha(p, q, r)$ admits a minimizer y that changes sign in $[-1, 1]$. Then, the following properties hold.*

- (i) *The minimizer y has exactly one maximum point η_M in $[-1, 1]$, has exactly one minimum point η_m in $[-1, 1]$, and, up to a multiplicative constant, satisfies*

$$y(\eta_M) = 1 = \max_{[-1,1]} y(x), \quad y(\eta_m) = -m = \min_{[-1,1]} y(x) \quad \text{with } m \in]0, 1]. \quad (27)$$

- (ii) *If $y_+ \geq 0$ and $y_- \leq 0$ are, respectively, the positive and negative parts of y , then y_+ and y_- are, respectively, symmetric about $x = \eta_M$ and $x = \eta_m$.*
- (iii) *There exists a unique zero of y in $] -1, 1[$.*
- (iv) *The following representations hold:*

$$\lambda_\alpha(p, q, r) = \frac{q}{p'} \|y\|_q^{q-p} H^p(m, p, q, r),$$

$$\|y\|_q = \left[\frac{r-1+p'}{q+p'} \gamma + (1-R(m, q, r)) \frac{2p'}{p'+q} \right]^{\frac{1}{q}},$$

where λ , H , and R have been defined in (2), (13), and (14), respectively.

- (v) $\lambda_\alpha(p, q, r) = \lambda_T(p, q, r)$.

Proof. For the sake of simplicity, throughout the proof, we will write $\lambda = \lambda_\alpha(p, q, r)$.

Multiplying the equation in (12) by y' and integrating in $] -1, 1[$, we get

$$\frac{1}{p'} |y'|^p + \frac{\lambda \|y\|_q^{p-q}}{q} |y|^q = \frac{\alpha |\gamma|^{\frac{p}{r-1}-2} \gamma}{r-1} |y|^{r-2} y + c \quad (28)$$

for a suitable constant c , where $\frac{1}{p} + \frac{1}{p'} = 1$.

Therefore, since $y'(\eta_M) = 0$ and $y(\eta_M) = 1$, $y'(\eta_m) = 0$ and $y(\eta_m) = -m$, we have

$$c = \frac{\lambda \|y\|_q^{p-q}}{q} - \frac{\alpha |\gamma|^{\frac{p}{r-1}-2} \gamma}{r-1} = \frac{\lambda \|y\|_q^{p-q}}{q} m^q + \frac{\alpha |\gamma|^{\frac{p}{r-1}-2} \gamma}{r-1} m^{r-1}.$$

Hence, we obtain

$$\begin{cases} \frac{\alpha |\gamma|^{\frac{p}{r-1}-2} \gamma}{r-1} = \frac{\lambda \|y\|_q^{p-q}}{q} R(m, q, r), \\ c = \frac{\lambda \|y\|_q^{p-q}}{q} (1 - R(m, q, r)). \end{cases} \quad (29)$$

So, equation (28) can be written as

$$\frac{1}{p'}|y'|^p + \frac{\lambda\|y\|_q^{p-q}}{q}|y|^q = \frac{\lambda\|y\|_q^{p-q}}{q}R(m, q, r)|y|^{r-2}y + \frac{\lambda\|y\|_q^{p-q}}{q}(1 - R(m, q, r)), \quad (30)$$

that is,

$$|y'|^p = \frac{p'\lambda\|y\|_q^{p-q}}{q}(1 - R(m, q, r)(1 - |y|^{r-2}y) - |y|^q). \quad (31)$$

It is easy to see that the number of zeros of y has to be finite; let us denote them by

$$-1 = \zeta_1 < \dots < \zeta_j < \zeta_{j+1} < \dots < \zeta_n = 1,$$

which are the zeros of y . Moreover, as in [7], we observe that

$$y'(x) = 0 \iff y(x) = -m \text{ or } y(x) = 1. \quad (32)$$

Indeed, if we set

$$\mu(y) := 1 - R(m, q, r)(1 - |y|^{r-2}y) - |y|^q, \quad y \in [-m, 1],$$

then (31) gives

$$|y'|^p = \frac{p'\lambda\|y\|_q^{p-q}}{q}\mu(y). \quad (33)$$

Let us observe that $\mu(-m) = \mu(1) = 0$. Since $q \geq r - 1$ by assumption, it is easily seen that, for any $\bar{y}$ such that $\mu'(\bar{y}) = 0$ then $\mu(\bar{y}) > 0$. Hence, μ does not vanish in $] - m, 1[$ and, therefore, by (33), $y'(x) \neq 0$ if $y(x) \neq 1$ and $y(x) \neq -m$, which proves (32).

This implies that y has no other local minima or maxima in $] - 1, 1[$; that is, in any interval $]\zeta_j, \zeta_{j+1}[$ where $y > 0$ there is a unique maximum point and that in any interval $]\zeta_j, \zeta_{j+1}[$ where $y < 0$, there is a unique minimum point.

Then, the properties (i), (ii) and (iii) follow by adapting the argument of [8, Lemma 2.6], see also [24] for the case $p = 2$. We remark that they can also be proved by using a symmetrization argument, rearranging the functions y^+ and y^- , and using the Pólya–Szegő inequality and the properties of rearrangements (see also, for example, [4, 9]). Specifically, one can prove the following:

- any interval $]\zeta_j, \zeta_{j+1}[$ given by two subsequent zeros of y , and in which $y = y^+ > 0$, has the same length; for any of such interval, y^+ is symmetric about $x = \frac{\zeta_j + \zeta_{j+1}}{2}$;
- any interval $]\zeta_j, \zeta_{j+1}[$ given by two subsequent zeros of y , and in which $y = y^- < 0$, has the same length; for any of such interval, y^- is symmetric about $x = \frac{\zeta_j + \zeta_{j+1}}{2}$;
- there is a unique zero of y in $] - 1, 1[$.

In order to show (iv), it is not restrictive to suppose the order relation $\eta_M < \eta_m$ between the unique maximum and the unique minimum point of y . It is easily seen [8, Lemma 2.6] that $\eta_M - \eta_m = 1$, with $y' < 0$ in $]\eta_M, \eta_m[$. Then, from (31), we have

$$\frac{-y'}{[1 - R(m, q, r)(1 - |y|^{r-2}y) - |y|^q]^{\frac{1}{p}}} = \left(\frac{p'\lambda\|y\|_q^{p-q}}{q} \right)^{\frac{1}{p}} \quad \text{in }]\eta_M, \eta_m[.$$

Then, integrating between η_M and η_m , we have

$$\begin{aligned}\lambda &= \frac{q}{p'} \|y\|_q^{q-p} \left[\int_{-m}^1 \frac{dy}{[1 - R(m, q, r)(1 - |y|^{r-2}y) - |y|q]^{\frac{1}{p}}} \right]^p \\ &= \frac{q}{p'} \|y\|_q^{q-p} H^p(m, p, q, r),\end{aligned}\quad (34)$$

which is the first part of (iv). The second part follows by integrating (30) over $(-1, 1)$ and recalling that $\|y'\|_p^p + \alpha|y|^{\frac{p}{r-1}} = \lambda\|y\|_q^p$.

Finally, since by Proposition 3.2 we know that $\lim_{\alpha \rightarrow +\infty} \lambda_\alpha(p, q, r) = \lambda_T(p, q, r)$ and since the relation (34) does not depend on α , we have

$$\frac{q}{p'} \|y\|_q^{q-p} H^p(m, p, q, r) = \lim_{\alpha \rightarrow +\infty} \lambda_\alpha(p, q, r) = \lambda_T(p, q, r),$$

which gives (v). ■

At this stage, we are in a position to state that each sign-changing minimizer of problem (2) is a symmetric and zero-average function.

Proposition 5.2. *Let $p, q, r > 1$ be such that $\frac{2p}{p+2} \leq q \leq p$ and suppose that there exists $\alpha > 0$ such that $\lambda_\alpha(p, q, r)$ admits a minimizer y that changes sign in $[-1, 1]$ and satisfies the conditions in (27). The following statements hold.*

- (i) *If $\frac{q}{2} + 1 < r \leq q + \frac{q}{p}$, then*
 - (a) $\int_{-1}^1 |y|^{r-2} y dx = 0$;
 - (b) $y(x) = C \sin_{p,q}(\lambda_T(p, q, r)x)$, with $C \in \mathbb{R} \setminus \{0\}$;
 - (c) *the only point $\bar{x} \in]-1, 1[$ where y vanishes is $\bar{x} = 0$.*
- (ii) *If $r = \frac{q}{2} + 1$ and $\int_{-1}^1 |y|^{r-2} y dx = 0$, then $y(x) = C \sin_{p,q}(\lambda_T(p, q, r)x)$, with $C \in \mathbb{R} \setminus \{0\}$, and the only point in $\bar{x} \in]-1, 1[$ where y vanishes is $\bar{x} = 0$.*

Proof. Since the exact value of the best constant in the twisted inequality is known from [7, Theorem 1.1] and, notably, it does not depend on the parameter r , Proposition 5.1 (iv) and (v) yield

$$\begin{aligned}\lambda_T(p, q, r) &= \left[\left(\frac{1}{p'} \right)^{\frac{1}{q}} \left(\frac{1}{q} \right)^{\frac{1}{p'}} \left(\frac{2}{p' + q} \right)^{\frac{1}{p} - \frac{1}{q}} q \right]^p \pi_{p,q}^p = \frac{q}{p'} \left(\frac{2p'}{p' + q} \right)^{1 - \frac{p}{q}} \pi_{p,q}^p \\ &\leq \frac{q}{p'} \left[\frac{r-1+p'}{q+p'} \gamma + (1 - R(m, q, r)) \frac{2p'}{p' + q} \right]^{1 - \frac{p}{q}} H^p(m, p, q, r) \\ &= \lambda_\alpha(p, q, r) = \lambda_T(p, q, r).\end{aligned}\quad (35)$$

Hence, since by Proposition 4.3 (ii) we know that $H(m, p, q, r) = \pi_{p,q}$ if and only if $m = 1$, the strict decrease of R with respect to m and the first identity in (29) give that

$$\int_{-1}^1 |y|^{r-2} y dx = 0, \quad (36)$$

which is (a). To prove (b) and (c), let us explicitly observe that, when (36) holds, then y solves problem (11) with $\lambda = \lambda_T(p, q, r) \|y\|_q^{p-q}$. Hence, $y(x) = C \sin_{p,q}(\pi_{p,q}x)$, with $C \in \mathbb{R} \setminus \{0\}$.

The case (ii) easily follows using the same arguments. ■

At this stage, we are in a position to prove the main theorems of the paper.

Proof of Theorem 1.1. When $\alpha \leq 0$, the minimizers of (2) have constant sign; indeed,

$$\mathcal{Q}_\alpha[u] \geq \mathcal{Q}_\alpha[|u|],$$

with equality if and only if $u \geq 0$ or $u \leq 0$.

In order to prove the main result, we will show that there exists $\alpha > 0$ for which problem (2) admits a minimizer y that changes sign. Suppose, by contradiction, that for any $k \in \mathbb{N}$, there exists a divergent sequence α_k , with a corresponding sequence of nonnegative eigenfunctions $\{y_k\}_{k \in \mathbb{N}}$ associated to $\lambda_{\alpha_k}(p, q, r)$ such that $\|y_k\|_q = 1$.

By Proposition 3.2, we have that $\lambda_{\alpha_k}(p, q, r) \leq \lambda_T(p, q, r)$ and hence, it holds that

$$\int_{-1}^1 |y'_k|^p dx + \alpha_k \left(\int_{-1}^1 y_k^{r-1} dx \right)^{\frac{p}{r-1}} \leq \lambda_T(p, q, r). \quad (37)$$

Therefore, y_k converges (up to a subsequence) to a function $y \in W_0^{1,p}(-1, 1)$, strongly in $L^p(-1, 1)$ and weakly in $W_0^{1,p}(-1, 1)$. Moreover, $\|y\|_p = 1$ and y is not identically zero. As a consequence, $\|y\|_{r-1} > 0$ and, letting $\alpha_k \rightarrow +\infty$ in (37), we have a contradiction. We have thus proved that there exists a positive value of α such that the minimization problem (2) admits an eigenfunction y satisfying $\int_{-1}^1 |y|^{r-2} y dx = 0$. In this case, $\lambda_\alpha(p, q, r) = \lambda_T(p, q, r)$ and, up to a multiplicative constant, $y = \sin_{p,q}(\pi_{p,q}x)$.

Since, by Proposition 3.1, $\lambda_\alpha(p, q, r)$ is nondecreasing and Lipschitz continuous with respect to α , we can define

$$\alpha_C = \min\{\alpha \in \mathbb{R} : \lambda_\alpha(p, q, r) = \lambda_T(p, q, r)\} = \sup\{\alpha \in \mathbb{R} : \lambda_\alpha(p, q, r) < \lambda_T(p, q, r)\}.$$

It is easily seen that this critical value of the parameter is strictly positive. ■

Proof of Theorem 1.2. If $\alpha < \alpha_C$, then all minimizers corresponding to $\lambda_\alpha(p, q, r)$ have constant sign; otherwise, when $\alpha < \alpha_C$, we have $\lambda_\alpha(p, q, r) = \lambda_T(p, q, r)$.

Let us now consider the case $\alpha > \alpha_C$, and suppose by contradiction that there exist $\bar{\alpha} > \alpha_C$ and a minimizer $\bar{y}$ such that

$$\int_{-1}^1 |\bar{y}|^{r-2} \bar{y} dx > 0, \quad \|\bar{y}\|_p = 1, \quad \mathcal{Q}_{\bar{\alpha}}[\bar{y}] = \lambda_{\bar{\alpha}}(p, q, r).$$

Then, for any $\varepsilon > 0$ sufficiently small, we compute

$$\begin{aligned} \mathcal{Q}_{\bar{\alpha}-\varepsilon}[\bar{y}] &= \mathcal{Q}_{\bar{\alpha}}[\bar{y}] - \varepsilon \left(\int_{-1}^1 |\bar{y}|^{r-2} \bar{y} dx \right)^{\frac{p}{r-1}} \\ &= \lambda_{\bar{\alpha}}(p, q, r) - \varepsilon \left(\int_{-1}^1 |\bar{y}|^{r-2} \bar{y} dx \right)^{\frac{p}{r-1}} < \lambda_{\bar{\alpha}}(p, q, r). \end{aligned}$$

This implies that

$$\lambda_T(p, q, r) = \lambda_{\alpha_C}(p, q, r) \leq \lambda_{\bar{\alpha}-\varepsilon}(p, q, r) < \lambda_{\bar{\alpha}}(p, q, r),$$

which contradicts the definition of α_C . Hence, for all $\alpha > \alpha_C$, any minimizer y satisfies

$$\int_{-1}^1 |y|^{r-2} y \, dx = 0.$$

Hence, by Proposition 5.2, the claims (i) and (ii) follow. Regarding (iii), it is not difficult to show, using suitable approximating sequences, that $\lambda_{\alpha_C}(p, q, r)$ admits both a nonnegative minimizer and a minimizer with vanishing r -average. ■

Remark 5.3. To analyze the behavior of the solutions when $r = p + 1$, we consider the case $\alpha = \alpha_C(p, q, p + 1)$. In this setting, the corresponding positive minimizer y is a solution of

$$\begin{cases} (|y'|^{p-2} y')' + \lambda_T(p, q, p + 1) \|y\|_q^{p-q} y^{q-1} = \alpha_C(p, q, p + 1) y^{q-1} & \text{in }]-1, 1[, \\ y(-1) = y(1) = 0. \end{cases}$$

The positivity of the eigenfunction ensures (see also (35)) that

$$\begin{aligned} \lambda_T(p, q, p + 1) \|y\|_q^{p-q} - \alpha_C(p, q, p + 1) &= \lambda_0(p, q, p + 1) \|y\|_q^{p-q} \\ &= \frac{q}{p'} \left(\frac{2p'}{p' + q} \right)^{1-\frac{p}{q}} \left(\frac{\pi_{p,q}}{2} \right)^p. \end{aligned}$$

As a consequence, we deduce the explicit expression

$$\alpha_C(p, q, p + 1) = \frac{2^p - 1}{2^p} \frac{q}{p'} \left(\frac{2p'}{p' + q} \right)^{1-\frac{p}{q}} \pi_{p,q}^p.$$

Remark 5.4. When the exponents p, q, r satisfy the assumptions of the main theorems, we obtain the following lower bound for the critical value $\alpha_C(p, q, r)$:

$$\alpha_C(p, q, r) \geq \frac{2^p - 1}{2^{\frac{p}{r-1} + p - 1}} \frac{q}{p'} \left(\frac{2p'}{p' + q} \right)^{1-\frac{p}{q}} \pi_{p,q}^p. \quad (38)$$

To derive estimate (38), we exploit the monotonicity of $\lambda_\alpha(p, q, r)$ with respect to α and test the Rayleigh quotient with the function $u(x) = \sin_{p,q}(\frac{\pi_{p,q}}{2}(x + 1))$. Hence,

$$\begin{aligned} \lambda_T(p, q, r) &= \lambda_{\alpha_C}(p, q, r) \leq \mathcal{Q}[u, \alpha_C] \\ &= \frac{q}{p'} \left(\frac{2p'}{p' + q} \right)^{1-\frac{p}{q}} \left(\frac{\pi_{p,q}}{2} \right)^p + \alpha_C \left(\int_{-1}^1 u^{r-1} \, dx \right)^{\frac{p}{r-1}} \\ &\leq \frac{q}{p'} \left(\frac{2p'}{p' + q} \right)^{1-\frac{p}{q}} \left(\frac{\pi_{p,q}}{2} \right)^p + \alpha_C 2^{\frac{p}{r-1} - 1}. \end{aligned}$$

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