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Characterization of BMO spaces via commutators of fractional maximal operators on the slice spaces

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Abstract. This paper mainly aims to demonstrate the boundedness for commutators of fractional maximal operator and sharp maximal operator in the slice spaces, where the symbols of the commutators belong to the BMO (bounded mean oscillation) spaces. Meanwhile, some new characterizations for BMO spaces are given.

1. Introduction and main results

Let T be the classic singular integral operator, the Coifman–Rochberg–Weiss-type commutator $[b, T]$ generated by T, and a suitable function b is defined by

$$[b, T]f = bT(f) - T(bf).$$

An important conclusion is that $b \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if $[b, T]$ is bounded on $L^s(\mathbb{R}^n)$ for $1 < s < \infty$; we refer the reader to Coifman, Rochberg, and Weiss [4] (see also [11]).

It is worth mentioning that the maximal operator is the most basic and important class of operators in harmonic analysis. Up to now, this class of operators has not only been studied by many people and extended to different fields, such as Lie group theory, number theory, etc. [6, 7, 10, 17–19], but also applied to mathematical physics, partial differential equations, and other branches [15, 16]. Moreover, we first recall some operators. (Commutators can also be understood as an operator.)

Let $0 < \alpha < n$. The fractional maximal operator can be defined as

$$M_\alpha(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. For $\alpha = 0$, $M_0 = M$ (the classic Hardy–Littlewood maximal operator).

If $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then the commutator produced by b with M_α is given by

$$M_{\alpha,b}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |b(x) - b(y)||f(y)| dy,$$

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where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. For $\alpha = 0$, $M_{0,b} = M_b$ (maximal commutator). And the nonlinear commutators of M_α can be defined by

$$[b, M_\alpha]f(x) = b(x)M_\alpha(f)(x) - M_\alpha(bf)(x).$$

For $\alpha = 0$, we write $[b, M] = [b, M_0]$.

Fefferman and Stein [5] introduced the sharp maximal function, defined as

$$M^\sharp(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, f_Q is the average of f over Q. And the nonlinear commutator produced by b with $M^\sharp$ is provided by

$$[b, M^\sharp]f(x) = b(x)M^\sharp(f)(x) - M^\sharp(bf)(x).$$

Note that $M_{\alpha,b}$ and $[b, M_\alpha]$ essentially differ from each other. For instance, $M_{\alpha,b}$ is positive and sublinear; however, $[b, M_\alpha]([b, M^\sharp])$ is neither positive nor sublinear.

When $b \in \mathrm{BMO}(\mathbb{R}^n)$, Bastero et al. [3] studied the boundedness of $[b, M]$ in $L^q(\mathbb{R}^n)$ for $1 < q < \infty$. In [21–23], Zhang and Wu further obtained the boundedness for $M_{\alpha,b}$, $[b, M_\alpha]$, $[b, M^\sharp]$ on variable Lebesgue spaces. Guliyev [9] obtained the boundedness for $M_{\alpha,b}$, $[b, M_\alpha]$ in the Orlicz space on any stratified Lie group. However, the study of operators on slice spaces seems to be scarce.

Inspired by the above literatures, the purpose of this paper is to study the boundedness for commutators of fractional maximal function and sharp maximal function in the slice spaces, where the symbols of the commutators belong to the BMO spaces; as a corollary, we also obtain the boundedness of maximal commutator in the slice spaces.

Let $0 \le \alpha$. For a fixed cube Q_*, the fractional maximal function with respect to Q_* of locally integrable function f is given by

$$M_{\alpha,Q_*}(f)(x) = \sup_{\substack{Q \ni x \\ Q \subset Q_*}} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes Q with $Q \subset Q_*$. If $\alpha = 0$, then $M_{Q_*} = M_{0,Q_*}$.

In order to introduce the following theorem, we define b^- as

$$b^-(x) = \begin{cases} 0, & \text{if } b(x) \ge 0, \\ |b(x)|, & \text{if } b(x) < 0, \end{cases} \quad \text{and} \quad b^+(x) = |b(x)| - b^-(x).$$

First of all, we consider the boundedness of nonlinear commutators generated by the fractional maximal function and BMO function on slice spaces and establish some new equivalent characterizations of BMO spaces which are different from Morrey spaces, (variable) Lebesgue spaces, and Orlicz spaces.

Theorem 1.1. *Let $0 < \alpha < n$, $0 < t < \infty$, and let b be a locally integrable function on $\mathbb{R}^n$. If $1 < p < r < \infty$, $1 < q < s < \infty$, and $\alpha/n = 1/p - 1/r = 1/q - 1/s$, then the following statements are equivalent.*

(T.1) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.

(T.2) $[b, M_\alpha]$ *is bounded from* $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.

(T.3) *There exists a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{|Q|^{1/s}} \|(b - |Q|^{-\alpha/n} M_{\alpha,Q}(b))\chi_Q\|_{(E_r^s)_t(\mathbb{R}^n)} \le C.$$

(T.4) *There exists a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{|Q|} \int_Q |b(y) - M_Q(b)(y)| dy \le C.$$

When we take $\alpha = 0$ and make a slight modification of the above theorem, it is not difficult to obtain the boundedness of nonlinear commutators generated by Hardy–Littlewood maximal functions and BMO functions in slice spaces, thus establishing a new equivalent characterization different from Theorem 1.1. In fact, the proof is almost similar to the theorem mentioned above. For this, we present the conclusion below, which is new.

Corollary 1.1. *Let $0 < t < \infty$ and b be a locally integrable function on $\mathbb{R}^n$. If $1 < p < \infty$, $1 < q < \infty$, then the following statements are equivalent.*

(C.1) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.

(C.2) $[b, M]$ *is bounded on* $(E_p^q)_t(\mathbb{R}^n)$.

(C.3) *There exists a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{|Q|} \|(b - M_Q(b))\chi_Q\|^q_{(E_p^q)_t(\mathbb{R}^n)} \le C.$$

(C.4) *There exists a constant* $C > 0$ *such that*

$$\sup_Q \frac{1}{|Q|} \int_Q |b(y) - M_Q(b)(y)| dy \le C.$$

Secondly, we obtain the boundedness of commutators generated by the fractional maximal operator and BMO function on slice spaces and establish some new equivalent characterizations of BMO spaces as follows.

Theorem 1.2. *Let $0 < \alpha < n$, $0 < t < \infty$, and let b be a locally integrable function on $\mathbb{R}^n$. If $1 < p < r < \infty$, $1 < q < s < \infty$, and $\alpha/n = 1/p - 1/r = 1/q - 1/s$, then the following statements are equivalent.*

(T.1) $b \in \mathrm{BMO}(\mathbb{R}^n)$.

(T.2) $M_{\alpha,b}$ *is bounded from* $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.

(T.3) *There exists a constant* $C > 0$ *such that*

$$\sup_{Q} \frac{1}{|Q|^{1/s}} \|(b - b_Q)\chi_Q\|_{(E_r^s)_t(\mathbb{R}^n)} \le C.$$

(T.4) *There exists a constant* $C > 0$ *such that*

$$\sup_{Q} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy \le C.$$

In analogy with Corollary 1.1, for the case of $\alpha = 0$ and also making a slight modification of the above theorem, it is easy to get the boundedness of commutators generated by Hardy–Littlewood maximal functions and BMO functions in slice spaces, by which a new equivalent characterization different from Theorem 1.2 is obtained. Of course, the proof is almost similar to the theorem mentioned above. For this, we present the conclusion below, which is new.

Corollary 1.2. *Let* $0 < t < \infty$ *and* b *be a locally integrable function on* $\mathbb{R}^n$. *If* $1 < p < \infty$, $1 < q < \infty$, *then the following statements are equivalent.*

(C.1) $b \in \mathrm{BMO}(\mathbb{R}^n)$.

(C.2) M_b *is bounded on* $(E_p^q)_t(\mathbb{R}^n)$.

(C.3) *There exists a constant* $C > 0$ *such that*

$$\sup_{Q} \frac{1}{|Q|} \|(b - b_Q)\chi_Q\|^q_{(E_p^q)_t(\mathbb{R}^n)} \le C.$$

(C.4) *There exists a constant* $C > 0$ *such that*

$$\sup_{Q} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy \le C.$$

Finally, we study the boundedness of nonlinear commutators generated by the sharp maximal operator and BMO function on slice spaces and establish some new equivalent characterizations of BMO spaces.

Theorem 1.3. *Let* $0 < t < \infty$ *and* b *be a locally integrable function on* $\mathbb{R}^n$. *If* $1 < p < \infty$, $1 < q < \infty$, *then the following statements are equivalent.*

(T.1) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.

(T.2) $[b, M^\sharp]$ *is bounded on* $(E_p^q)_t(\mathbb{R}^n)$.

(T.3) *There exists a constant* $C > 0$ *such that*

$$\sup_{Q} \frac{1}{|Q|} \|(b - 2M^\sharp(b\chi_Q))\chi_Q\|^q_{(E_p^q)_t(\mathbb{R}^n)} \le C.$$

(T.4) *There exists a constant* $C > 0$ *such that*

$$\sup_{Q} \frac{1}{|Q|} \int_Q |b(y) - 2M^\sharp(b\chi_Q)(y)| dy \le C.$$

Throughout this paper, the letter C always takes place of a constant independent of the primary parameters involved and whose value may differ from line to line. In addition, we give some notations. Here and hereafter, $|E|$ will always denote the Lebesgue measure of a measurable set E on $\mathbb{R}^n$ and χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^n$. $X \approx Y$ means there is a constant $C > 1$ such that $C^{-1}Y \leq X \leq CY$.

2. Preliminaries

2.1. Some function spaces

This subsection introduces some main function spaces, for example, slice and BMO spaces.

In the last years, the research on slice spaces has attracted considerable attention. In 2019, the earliest lesson on slice spaces $(E_2^p)_t(\mathbb{R}^n)(0 < t < \infty,\ 1 < p < \infty)$ can be traced back to the work of Auscher and Mourgoglou in [1], for the purpose of the weak solutions of boundary value problems with a t-independent elliptic systems in the upper half-plane. Recently, Auscher and Prisuelos-Arribas [2] considered the boundedness of some classical operators on the slice space $(E_r^p)_t(\mathbb{R}^n)(0 < t < \infty$ and $1 < r, p < \infty)$. Of course, for more research on slice spaces, we refer the reader to [14, 20].

Definition 2.1. Let $0 < t < \infty$ and $1 < r,\ p < \infty$. The slice space $(E_r^p)_t(\mathbb{R}^n)$ is defined as the set of all locally r-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{(E_r^p)_t(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n}\left(\frac{1}{|Q(x,t)|}\int_{Q(x,t)}|f(y)|^r dy\right)^{p/r} dx\right)^{1/p} < \infty,$$

where $Q(x,t)$ is a cube centered at x and with edge length t.

John and Nirenberg [12] introduced the following space for the purpose of studying the partial differential equations problem.

Definition 2.2. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then, the BMO space can be defined as

$$\mathrm{BMO}(\mathbb{R}^n) = \left\{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \sup_Q \frac{1}{|Q|}\int_Q |f(y) - f_Q| dy < \infty\right\},$$

where f_Q is the average of f over Q.

2.2. Auxiliary propositions and lemmas

In this part, we state some auxiliary propositions and lemmas, which will be needed for proving our main theorems. And we only describe the partial results we need.

First, we introduce the well-known Hölder inequality, which plays an extensive role in the proof of this paper.

Lemma 2.1. *Let $1 < p < \infty$ and p' is the conjugate exponent of p. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$, Then*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq C\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)}.$$

Lemmas 2.2 and 2.3 below can be obtained in [14], where Lemma 2.2 reveals the boundedness of the maximal function in the slice space, and Lemma 2.3 gives the norm of the characteristic function on the slice space, which is different from the norm of the characteristic function on the (variable) Lebesgue and Morrey spaces.

Lemma 2.2. *Let $0 < t < \infty$ and $1 < r,\ p < \infty$. If $f \in (E_r^p)_t(\mathbb{R}^n)$, then*

$$\|Mf\|_{(E_r^p)_t(\mathbb{R}^n)} \leq C\|f\|_{(E_r^p)_t(\mathbb{R}^n)}.$$

Lemma 2.3. *If $0 < t < \infty$ and $1 < r,\ p < \infty$, then, for any cube $Q \subset \mathbb{R}^n$, we have*

$$\|\chi_Q\|_{(E_r^p)_t(\mathbb{R}^n)} \approx |Q|^{1/p}.$$

The following result shows the boundedness of the fractional maximal operator in slice spaces, which plays an important role in the estimates of commutators (see [13] for more details).

Lemma 2.4. *Let $0 < \alpha < n$, $0 < t < \infty$, $1 < r,\ p < \infty$, and $1 < q < s < \infty$ with $\alpha/n = 1/p - 1/r = 1/q - 1/s$. If $f \in (E_p^q)_t(\mathbb{R}^n)$, then*

$$\|M_\alpha f\|_{(E_r^s)_t(\mathbb{R}^n)} \leq C\|f\|_{(E_p^q)_t(\mathbb{R}^n)}.$$

The following estimate can be obtained by similar derivation and calculation in [8].

Lemma 2.5. *Let $0 < \alpha < n$. If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that, for almost every $x \in \mathbb{R}^n$ and for all functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we have*

$$M_{\alpha,b}(f)(x) \leq C\|b\|_*(M(M_\alpha f)(x) + M_\alpha(Mf)(x)).$$

Lemma 2.6 ([23]). *Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For any fixed cube $Q \subset \mathbb{R}^n$, the following statements hold..*

(1) *If $0 \leq \alpha < n$, then, for all $y \in Q$, we have*

$$M_\alpha(b\chi_Q)(y) = M_{\alpha,Q}(b)(y)$$

and

$$M_\alpha(\chi_Q)(y) = M_{\alpha,Q}(\chi_Q)(y) = |Q|^{\frac{\alpha}{n}}.$$

(2) *Then, for any $y \in Q$, we have*

$$|b_Q| \leq |Q|^{-\frac{\alpha}{n}} M_{\alpha,Q}(b)(y).$$

(3) *Let $E = \{y \in Q : b(y) \leq b_Q\}$ and $F = Q \setminus E = \{y \in Q : b(y) > b_Q\}$. Then, the following equality is trivially true:*

$$\int_E |b(y) - b_Q| dy = \int_F |b(y) - b_Q| dy.$$

Bastero et al. [3] obtained the following equivalent relations for BMO space.

Lemma 2.7. *Let b be a locally integrable function on $\mathbb{R}^n$. Then, the following assertions are equivalent.*

(1) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.

(2) *There exists a positive constant C such that*

$$\sup_Q \frac{1}{|Q|} \int_Q |b(y) - M_Q(b)(y)| dy \leq C.$$

(3) *There exists a positive constant C such that*

$$\sup_Q \frac{1}{|Q|} \int_Q |b(y) - 2M^\sharp(b\chi_Q)(y)| dy \leq C.$$

3. Proof of the principal results

Lemma 3.1. *Let $0 < \alpha < n$, and let b be a locally integrable function on $\mathbb{R}^n$. If there exists a constant $C > 0$ such that*

$$\sup_Q \frac{1}{|Q|^{1/s}} \|(b - |Q|^{-\alpha/n} M_{\alpha,Q}(b))\chi_Q\|_{(E_r^s)_t(\mathbb{R}^n)} \leq C$$

for $0 < t < \infty$ and $1 < r, s < \infty$, then $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Proof. For any fixed cube $Q \subset \mathbb{R}^n$, according to Lemma 2.6 (2), if $y \in Q$, then

$$|b_Q| \leq |Q|^{-\frac{\alpha}{n}} M_{\alpha,Q}(b)(y).$$

Let $E = \{y \in Q : b(y) \leq b_Q\}$ and $F = Q \setminus E = \{y \in Q : b(y) > b_Q\}$. Then, for any $y \in E$, we have $b(y) \leq b_Q \leq |b_Q| \leq |Q|^{-\frac{\alpha}{n}} M_{\alpha,Q}(b)(y)$, which implies

$$|b - b_Q| \leq |b - |Q|^{-\alpha/n} M_{\alpha,Q}(b)(y)|.$$

Thus, applying Lemma 2.6 (3), we obtain

$$\begin{aligned} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy &= \frac{2}{|Q|} \int_E |b(y) - b_Q| dy \\ &\leq \frac{2}{|Q|} \int_Q |b(y) - |Q|^{-\alpha/n} M_{\alpha,Q}(b)(y)| dy. \end{aligned}$$

By using the assumption, Lemmas 2.1 and 2.3, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |b(y)-b_Q|dy \\
&\quad\le 2\left(\frac{1}{|Q|}\int_Q |b(y)-|Q|^{-\alpha/n}M_{\alpha,Q}(b)(y)|^r dy\right)^{1/r}\left(\frac{1}{|Q|}\int_Q |\chi_Q|^{r'}dy\right)^{1/r'} \\
&\quad\le \frac{C}{|Q|}\|(b-|Q|^{-\alpha/n}M_{\alpha,Q}(b))\chi_Q\|_{(E_r^s)_t(\mathbb{R}^n)}\|\chi_Q\|_{(E_{r'}^{s'})_t(\mathbb{R}^n)} \\
&\quad\le C,
\end{aligned}$$

which implies $b \in \mathrm{BMO}(\mathbb{R}^n)$. Thus, we finish the proof of Lemma 3.1. ■

Proof of Theorem 1.1. Since (T.4)⇒(T.1) follow readily from Lemma 2.7, we only need to prove (T.1)⇒(T.2), (T.2)⇒(T.3), and (T.3)⇒(T.4).

(T.1)⇒(T.2): Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$. We need to prove that $[b, M_\alpha]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$. It follows from Lemma 2.4 that $M_\alpha f(x) < \infty$(when $f \in (E_p^q)_t(\mathbb{R}^n)$) for almost $x \in \mathbb{R}^n$; we have (for example, see [22])

$$|[b, M_\alpha]f(x)| \le M_{\alpha,b}(f)(x) + 2b^- M_\alpha(f)(x). \tag{3.1}$$

Using Lemma 2.5, we further obtain

$$|[b, M_\alpha]f(x)| \le C\|b\|_*(M(M_\alpha f)(x) + M_\alpha(Mf)(x)) + 2b^- M_\alpha(f)(x).$$

With the help of Lemma 2.2 and Lemma 2.4, we get

$$\begin{aligned}
&\|[b, M_\alpha]f\|_{(E_r^s)_t(\mathbb{R}^n)} \\
&\quad\le C\|b\|_*\|M(M_\alpha f) + M_\alpha(Mf)\|_{(E_r^s)_t(\mathbb{R}^n)} + 2\|b^-\|_{L^\infty(\mathbb{R}^n)}\|M_\alpha(f)\|_{(E_r^s)_t(\mathbb{R}^n)} \\
&\quad\le C(\|M_\alpha f\|_{(E_r^s)_t(\mathbb{R}^n)} + \|Mf\|_{(E_p^q)_t(\mathbb{R}^n)}) \\
&\quad\le C\|f\|_{(E_p^q)_t(\mathbb{R}^n)}.
\end{aligned}$$

Thus, we obtain that $[b, M_\alpha]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.

(T.2)⇒(T.3): For any fixed cube $Q \subset \mathbb{R}^n$ and $y \in Q$, it follows from Lemma 2.6 (1) that

$$M_\alpha(b\chi_Q)(y) = M_{\alpha,Q}(b)(y) \quad \text{and} \quad M_\alpha(\chi_Q)(y) = M_{\alpha,Q}(\chi_Q)(y) = |Q|^{\frac{\alpha}{n}}.$$

Then, for any $y \in Q$, we have

$$\begin{aligned}
b(y) - |Q|^{-\alpha/n}M_{\alpha,Q}(b)(y) &= |Q|^{-\alpha/n}(b(y)|Q|^{\alpha/n} - M_{\alpha,Q}(\chi_Q)(y)) \\
&= |Q|^{-\alpha/n}(b(y)M_\alpha(\chi_Q)(y) - M_\alpha(b\chi_Q)(y)) \\
&= |Q|^{-\alpha/n}[b, M_\alpha](\chi_Q)(y).
\end{aligned}$$

Then, for any $y \in \mathbb{R}^n$, we obtain

$$b(y) - |Q|^{-\alpha/n} M_{\alpha,Q}(b)(y)\chi_Q(y) = |Q|^{-\alpha/n}[b, M_\alpha](\chi_Q)(y).$$

By applying Lemma 2.3, statement (T.2), and the condition $\alpha/n = 1/q - 1/s$, we get

$$\begin{aligned}
\|(b - |Q|^{-\alpha/n} M_{\alpha,Q}(b))\chi_Q\|_{(E_r^s)_t(\mathbb{R}^n)} &= |Q|^{-\alpha/n}\|[b, M_\alpha](\chi_Q)\|_{(E_r^s)_t(\mathbb{R}^n)} \\
&\le C|Q|^{-\alpha/n}\|\chi_Q\|_{(E_p^q)_t(\mathbb{R}^n)} \\
&\le C|Q|^{-\alpha/n+1/q} \\
&= C|Q|^{1/s},
\end{aligned}$$

where the arbitrary constant C is independent of Q. Thus, we deduce (T.3).

(T.3)⇒(T.4): For any fixed cube $Q \subset \mathbb{R}^n$, we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |b(y) - M_Q(b)(y)|dy &\le \frac{1}{|Q|}\int_Q |b(y) - |Q|^{-\alpha/n} M_{\alpha,Q}(b)(y)|dy \\
&\quad + \frac{1}{|Q|}\int_Q ||Q|^{-\alpha/n} M_{\alpha,Q}(b)(y) - M_Q(b)(y)|dy \\
&= I_1 + I_2.
\end{aligned}$$

For the first term I_1, by using Lemma 2.3, statement (T.3), and Lemma 2.1, we obtain

$$\begin{aligned}
I_1 &\le \left(\frac{1}{|Q|}\int_Q |b(y) - |Q|^{-\alpha/n} M_{\alpha,Q}(b)(y)|^r dy\right)^{1/r} \left(\frac{1}{|Q|}\int_Q |\chi_Q|^{r'} dy\right)^{1/r'} \\
&\le \frac{C}{|Q|}\|(b - |Q|^{-\alpha/n} M_{\alpha,Q}(b))\chi_Q\|_{(E_r^s)_t(\mathbb{R}^n)}\|\chi_Q\|_{(E_{r'}^{s'})_{t}(\mathbb{R}^n)} \\
&\le C.
\end{aligned}$$

Next, for all $y \in Q$, it follows from Lemma 2.6 (1) that

$$M_\alpha(b\chi_Q)(y) = M_{\alpha,Q}(b)(y) \quad \text{and} \quad M_\alpha(\chi_Q)(y) = |Q|^{\frac{\alpha}{n}}$$

and

$$M(b\chi_Q)(y) = M_Q(b)(y) \quad \text{and} \quad M(\chi_Q)(y) = 1.$$

Then, for all $y \in Q$,

$$\begin{aligned}
&||Q|^{-\alpha/n} M_{\alpha,Q}(b)(y) - M_Q(b)(y)| \\
&\quad \le |Q|^{-\alpha/n}|M_{\alpha,Q}(b)(y) - |Q|^{\alpha/n}|b(y)|| + ||b(y)| - M_Q(b)(y)| \\
&\quad \le |Q|^{-\alpha/n}|M_\alpha(b\chi_Q)(y) - M_\alpha(\chi_Q)(y)|b(y)|| + ||b(y)|M(\chi_Q)(y) - M(b\chi_Q)(y)| \\
&\quad \le |Q|^{-\alpha/n}|[|b|, M_\alpha](\chi_Q)(y)| + |[|b|, M](\chi_Q)(y)|.
\end{aligned}$$

The statement (T.3) along with Lemma 3.1 gives $b \in \mathrm{BMO}(\mathbb{R}^n)$, which implies $|b| \in \mathrm{BMO}(\mathbb{R}^n)$. It follows from (3.1), Lemma 2.5, and Lemma 2.6 (1) that

$$\begin{aligned}|[|b|, M_\alpha](\chi_Q)(y)| &\le M_{\alpha,b}(\chi_Q)(y)\\ &\le C\|b\|_*(M(M_\alpha(\chi_Q))(y) + M_\alpha(M(\chi_Q))(y))\\ &\le C\|b\|_*|Q|^{\alpha/n}\end{aligned}$$

and

$$|[|b|, M](\chi_Q)(y)| \le M_{|b|}(\chi_Q)(y) \le C\|b\|_* M(M_\alpha(\chi_Q))(y) \le C\|b\|_*.$$

Thus, we obtain

$$I_2 \le \frac{1}{|Q|}\int_Q |Q|^{-\alpha/n}|[|b|, M_\alpha](\chi_Q)(y)| + |[|b|, M](\chi_Q)(y)|dy \le C\|b\|_*.$$

Combining with I_1 and I_2, we have

$$\sup_Q \frac{1}{|Q|}\int_Q |b(y) - M_Q(b)(y)|dy \le C.$$

Therefore, we finish the proof of Theorem 1.1. ∎

Proof of Theorem 1.2. Since the statements (T.1)⇔(T.4) directly are obtained by Definition 2.2, we only need to prove (T.1)⇒(T.2), (T.2)⇒(T.3), and (T.3)⇒(T.4).

(T.1)⇒(T.2). Since $b \in \mathrm{BMO}(\mathbb{R}^n)$, by using Lemma 2.5, we obtain

$$M_{\alpha,b}(f)(x) \le C\|b\|_*(M(M_\alpha f)(x) + M_\alpha(Mf)(x)).$$

Similar to (T.1)⇒(T.2) of Theorem 1.1, it follows from Lemma 2.2 and Lemma 2.4 that $M_{\alpha,b}$ is bounded from $(E^q_p)_t(\mathbb{R}^n)$ to $(E^s_r)_t(\mathbb{R}^n)$.

(T.2)⇒(T.3). For any fixed cube $Q \subset \mathbb{R}^n$ and $y \in Q$, we have

$$\begin{aligned}|b(y) - b_Q| &= \frac{1}{|Q|}\int_Q |b(y) - b(z)|\chi_Q(z)dz\\ &= \frac{1}{|Q|^{\alpha/n}} M_{\alpha,b}(\chi_Q)(y).\end{aligned}$$

Then, for any $y \in Q$, we obtain

$$|(b(y) - b_Q)\chi_Q(y)| = |Q|^{-\alpha/n} M_{\alpha,b}(\chi_Q)(y).$$

By applying Lemma 2.3, statement (T.2), and the condition $\alpha/n = 1/q - 1/s$, we get

$$\begin{aligned}\|(b - b_Q)\chi_Q\|_{(E^s_r)_t(\mathbb{R}^n)} &= |Q|^{-\alpha/n}\|M_{\alpha,b}(\chi_Q)\|_{(E^s_r)_t(\mathbb{R}^n)}\\ &\le C|Q|^{-\alpha/n}\|\chi_Q\|_{(E^q_p)_t(\mathbb{R}^n)}\\ &\le C|Q|^{-\alpha/n+1/q} = C|Q|^{1/s},\end{aligned}$$

where the arbitrary constant C is independent of Q. Thus, we deduce (T.3).

(T.3)⇒(T.4). For any fixed cube $Q \subset \mathbb{R}^n$, using assertion (T.3) and Lemmas 2.1 and 2.3, we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q |b(y)-b_Q|dy &\le 2\left(\frac{1}{|Q|}\int_Q |b(y)-b_Q|^r dy\right)^{1/r}\left(\frac{1}{|Q|}\int_Q |\chi_Q|^{r'} dy\right)^{1/r'}\\ &\le \frac{C}{|Q|}\|(b-b_Q)\chi_Q\|_{(E_r^s)_t(\mathbb{R}^n)}\|\chi_Q\|_{(E_{r'}^{s'})_t(\mathbb{R}^n)}\\ &\le C.\end{aligned}$$

Thus, the proof of Theorem 1.2 is completed. ■

Proof of Theorem 1.3. Since the implications (T.1)⇔(T.4) follow readily from Lemma 2.7, we only need to prove (T.1)⇒(T.2), (T.2)⇒(T.3), and (T.3)⇒(T.4).

(T.1)⇒(T.2). Assume $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b \in L^\infty(\mathbb{R}^n)$, for any fixed cube $Q \subset \mathbb{R}^n$, the following estimate was obtained in [21]:

$$|[|b|, M^\sharp]f(x)| \le 2M_{|b|}f(x).$$

Since $M^\sharp(f) \le 2M(f)$, for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned}|[b, M^\sharp](f)(x)| &\le 2((b^-(x))M^\sharp(f)(x) + M_p^\sharp(b^- f)(x)) + |[|b|, M^\sharp](f)(x)|\\ &\le 4((b^-(x))M(f)(x) + M(b^- f)(x)) + 2M_{|b|}f(x).\end{aligned}$$

Since $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $|b| \in \mathrm{BMO}(\mathbb{R}^n)$. It follows from Corollary 1.2 and Lemma 2.2 that

$$\|[b, M^\sharp](f)\|_{(E_p^q)_t(\mathbb{R}^n)} \le C\|b\|_*\|f\|_{(E_p^q)_t(\mathbb{R}^n)},$$

which implies that $[b, M^\sharp]$ is bounded on $(E_p^q)_t(\mathbb{R}^n)$.

(T.2)⇒(T.3). Assume that $[b, M^\sharp]$ is bounded on $(E_p^q)_t(\mathbb{R}^n)$. For any fixed cube Q, we have (see also [21])

$$M^\sharp(\chi_Q)(y) = \frac{1}{2}, \quad y \in Q.$$

Then, for all $y \in Q$, we obtain

$$\begin{aligned}b(y) - 2M^\sharp(b\chi_Q)(y) &= 2\left(\frac{1}{2}b(y) - M^\sharp(b\chi_Q)(y)\right)\\ &= 2(b(y)M^\sharp(\chi_Q)(y) - M^\sharp(b\chi_Q)(y))\\ &= 2[b, M^\sharp](\chi_Q)(y).\end{aligned}$$

Then, for any $y \in \mathbb{R}^n$, we obtain

$$(b(y) - 2M^\sharp(b\chi_Q)(y))\chi_Q = 2[b, M^\sharp](\chi_Q)(y).$$

Since $[b, M^\sharp]$ is bounded on $(E_p^q)_t(\mathbb{R}^n)$, then, by applying Lemma 2.3, we get

$$\begin{aligned}\|(b - 2M^\sharp(b\chi_Q))\chi_Q\|^q_{(E_p^q)_t(\mathbb{R}^n)} &= 2^q \|[b, M^\sharp](\chi_Q)\|^q_{(E_p^q)_t(\mathbb{R}^n)} \\ &\le C\|\chi_Q\|^q_{(E_p^q)_t(\mathbb{R}^n)} \\ &\le C|Q|.\end{aligned}$$

Then, we obtain (T.3).

(T.3)$\Rightarrow$(T.4). For any fixed cube $Q \subset \mathbb{R}^n$, using assertion (T.3) and Lemmas 2.1 and 2.3, we have

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |b(y) - 2M^\sharp(b\chi_Q)(y)|dy \\ &\le \Big(\frac{1}{|Q|}\int_Q |b(y) - 2M^\sharp(b\chi_Q)(y)|^p dy\Big)^{1/p}\Big(\frac{1}{|Q|}\int_Q |\chi_Q|^{p'} dy\Big)^{1/p'} \\ &\le \frac{C}{|Q|}\|b - 2M^\sharp(b\chi_Q)\|_{(E_p^q)_t(\mathbb{R}^n)}\|\chi_Q\|_{(E_{p'}^{q'})_t(\mathbb{R}^n)} \\ &\le C,\end{aligned}$$

which implies statement (T.4) since the constant C is independent of Q. Therefore, we finish the proof of Theorem 1.3. ■

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