Z. Anal. Anwend. 45 (2026), 485–511
DOI 10.4171/ZAA/1831

Decay character in some function spaces and applications to decay rates of generalized Navier–Stokes equations

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Abstract. The aim of this paper is twofold. First, we develop the notion of decay characters and evaluate these characters in important data spaces that often appear in problems related to fluid mechanics. Next, we apply our evaluation for the decay character of the initial datum $r^*(u_0)$ to study the decay of solution and its derivatives to n-dimensional generalized Navier–Stokes equations with fractional-order Laplace operator $(-\Delta)^\alpha$ when $0 < \alpha < \frac{n+2}{4}$. We establish the optimal uniform decay estimates for the L^2-norm of solution u, as well as decay estimates for $\|\Lambda^m u\|_{L^2}$ and $\|\partial_t^k \Lambda^m u\|_{L^2}$ for positive integer k and non-negative real number m.

1. Introduction

In this paper, we will consider the following n-dimensional incompressible generalized Navier–Stokes equations:

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u + u \cdot \nabla u + \nabla p = 0, \quad x \in \mathbb{R}^n,\ t > 0, \\ \nabla \cdot u = 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $\alpha > 0$ and $n = 2, 3, 4$. Let $u(x,t)$ and $p(x,t)$ stand for the velocity and the pressure of the fluid, respectively. The fractional Laplacian $(-\Delta)^\alpha$ is defined by Fourier transform as

$$\widehat{(-\Delta)^\alpha u}(\xi) = |\xi|^{2\alpha} \hat{u}(\xi), \quad \xi \in \mathbb{R}^n.$$

It is well known that if $(u(x,t); p(x,t))$ is a solution to the n-dimensional generalized Navier–Stokes equations, then the scaling solution

$$(u_\lambda(x,t); p_\lambda(x,t)) = (\lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t); \lambda^{4\alpha-2} p(\lambda x, \lambda^{2\alpha} t))$$

also solves (1.1). We denote the natural energy $E(u)$:

$$E(u) = \frac{1}{2} \sup_{t>0} \int_{\mathbb{R}^n} |u(x,t)|^2 dx + \int_0^\infty \int_{\mathbb{R}^n} |\Lambda^\alpha u(x,t)|^2 dx dt,$$

Mathematics Subject Classification 2020: 35Q30 (primary); 76D05, 35B65, 35B99 (secondary).
Keywords: generalized Navier–Stokes equations, decay estimates, decay characters.

where $\Lambda = (-\Delta)^{1/2}$. Hence, with $y := \lambda x, s := \lambda^{2\alpha} t$, noting that $dx = \lambda^{-n} dy, dt = \lambda^{-2\alpha} ds$, $\Lambda_x^\alpha = \lambda^\alpha \Lambda_y^\alpha$, we have

$$\begin{aligned}
E(u_\lambda) &= \frac{1}{2}\sup_{t>0} \int_{\mathbb{R}^n} |u_\lambda(x,t)|^2 dx + \int_0^\infty \int_{\mathbb{R}^n} |\Lambda^\alpha u_\lambda(x,t)|^2 dx dt \\
&= \frac{1}{2}\sup_{s>0} \int_{\mathbb{R}^n} \lambda^{2(2\alpha-1)} \lambda^{-n} |u(y,s)|^2 dy \\
&\quad + \int_0^\infty \int_{\mathbb{R}^n} \lambda^{2(2\alpha-1)} \lambda^{-n-2\alpha} \lambda^{2\alpha} |\Lambda^\alpha u(y,s)|^2 dy ds \\
&= \lambda^{4\alpha-2-n} \left(\frac{1}{2}\sup_{s>0} \int_{\mathbb{R}^n} |u_\lambda(y,s)|^2 dy + \int_0^\infty \int_{\mathbb{R}^n} |\Lambda_y^\alpha u_\lambda(y,s)|^2 dy ds \right) \\
&= \lambda^{4\alpha-2-n} E(u).
\end{aligned}$$

In the regularity theory, equation (1.1) is supercritical for $\alpha < \frac{n+2}{4}$ in the sense that $\lim_{\lambda \to 0} E(u_\lambda) = \infty$. Meanwhile, the cases $\alpha = \frac{n+2}{4}$ and $\alpha > \frac{n+2}{4}$ are called critical and subcritical, respectively. As in critical and subcritical cases, any classical solution of (1.1) is always global in time (see, e.g., [21, 37]). When $0 < \alpha < \frac{n+2}{4}$, the well-posedness as well as the global regularity of solutions still remain a challenging and open problem. The small data global (in time) existence of solutions, partial regularity and non-uniqueness have been studied extensively. Recently, Jiu [16] and Deng [9] have shown the small data partial regularity in H^s spaces, while Duan [10] has proved the regularity in homogeneous Sobolev spaces with small data also. Further research on the latest studies related to the generalized (fractional) Navier–Stokes equations can be found in [7, 19, 22, 23, 35, 38, 39, 41].

Leray in [20] published an important open problem to address the question: whether or not the global weak solution of system (1.1) when $\alpha = 1$ decays to zero in $L^2(\mathbb{R}^3)$ as time goes to infinity. After that, the temporal decay problem has gained interest in a considerable number of studies of mathematical fluid dynamics; see, e.g., Kato [18], Schonbek [29–31], Kajikiya and Miyakawa [17], Wiegner [36], Zhang [40], Schonbek and Wiegner [33], Oliver and Titi [25] and others who have proved the upper and lower bounds on the rate of decay for various norms and classes of initial data. Recently, in the 3-dimensional case, for $0 < \alpha < 5/4$, several works have been done by Jiu [16] and Melo [23] to estimate the upper bounds of the decay rates of weak solutions when u_0 belongs to L^p and $\mathcal{Y}^p$. Jiu [16] and Deng [9] established the decay estimates of higher-order derivatives of the weak solution to (1.1) in $\mathbb{R}^3$ and $\mathbb{R}^n$, respectively, with $\alpha < \frac{n+2}{4}$.

In order to study the uniform decay rates of many dissipative evolution equation with only L^2 data, Bjorland and Schonbek in [3] have introduced the concept of decay character $r^*(u_0)$ associated with $u_0 \in L^2$ that assigns an improved decay rate to the solution of the heat equation with the data u_0. It is well known that the long-time asymptotic profile of the solutions to many linear dissipative systems is determined by the behavior of the low-frequency components. Thus, the use of the decay character helps obtain better decay estimates for the solution of dissipative Cauchy problems with L^2 data without additional regularity. As far as we know, one of the first applications of the decay character in the study of fluid mechanics was published in [1]. Very recently, Brandolese, Perusato, and Zingano in [5] obtained important results to show that in the class of Leray's solutions to the unforced Navier–Stokes equations, the subclass of solutions with L^2-algebraic decay is negligible in a topological sense. Moreover, the authors in [5] proved that Leray solutions with algebraic decay generically satisfy as well two-sided bounds of the form $(1+t)^{-\sigma} \lesssim \|u(\cdot,t)\|_{L^2} \lesssim (1+t)^{-\sigma}$.

In this paper, we first analyze and evaluate the decay characters in some important function spaces that frequently play the role of data space in problems emerging from studies of mathematical fluid dynamics: the space with mixed regularity $L^2 \cap L^m$ with $m \in [1,2)$, the weighted spaces $L^{1,\gamma}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n, e^{\frac{\beta}{2}|x|^2} dx)$, the Sobolev spaces with negative order, and the special space $\mathcal{Y}^p$ that have appeared recently in the literature. After that, we will focus on setting up the uniform decay rate of $\|u\|_2$, $\|\Lambda^m u\|_2$, and $\|\partial_t^k \Lambda^m u\|_2$ using the notions of decay character. The decay rates of $\|u\|_2$ and $\|\Lambda^m u\|_2$ have been established in various papers, namely, [9,10,16,23]. However, the decay rate of $\|\partial_t^k \Lambda^m u\|_2$ has not been obtained elsewhere yet. To our knowledge, the only work involving this kind of decay estimates was carried out in the viscous Camassa–Holm equations (VCHE) framework, where the decay rate of $\|\partial_t^K \Lambda^M u\|_2$ is estimated for non-negative integers K, M; see, e.g., [1]. It is worth noting that, in the parabolic case, Guterres et al. in [13] have proved that, under some mild assumptions, the decay of the norm of the derivatives Du leads to the decay of the L^2 norm of the solution u itself. In our paper, by modifying the method proposed in [1] and utilizing the decay characters, we successfully establish the decay rate of the mixed derivatives $\|\partial_t^k \Lambda^m u\|_2$ for non-negative real number m and non-negative integer k (see Theorems 3.3, 3.10, and 3.11 for more details).

Notations

In this paper, all considered Banach or Hilbert spaces are based on the whole space $\mathbb{R}^n$, for example, $L^2 := L^2(\mathbb{R}^n)$. We denote by L^p the standard Lebesgue space with norm $\|\phi\|_p = (\int_{\mathbb{R}^n} |\phi|^p \, dx)^{1/p}$. As usual, $\langle u, v\rangle = \int_{\mathbb{R}^n} uv dx$ stands for the standard L^2 inner product. The notation $\Lambda = (-\Delta)^{1/2}$ is used throughout the paper, whereas $\widehat{\phi} = \mathcal{F}\phi$ and $\check{\phi} = \mathcal{F}^{-1}\phi$ stand for the Fourier and the inverse Fourier transform of ϕ. We denote by L^2_σ, H^m_σ and $\dot{H}^m_\sigma$ the completions of $C^\infty_{0,\sigma}$ (C^∞_0 functions with divergence free) under the usual L^2-norm, H^m-norm, and norm $\|\Lambda^m \cdot\|_2 = (\int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}(\cdot)(\xi)|^2 d\xi)^{1/2}$, respectively. Throughout the paper, for non-negative functions $f(t)$, $g(t)$ the notation $f(t) \lesssim g(t)$ is used to denote the inequalities $f(t) \le Cg(t)$ that are satisfied uniformly for all $t > 0$, with a positive constant C. Finally, the generic constants are denoted by C, and by $C(\epsilon)$ if the constant depends also on the parameter ϵ.

The paper is organized as follows: Section 2 presents the definition and computation of the decay characters in various function spaces of interest. Section 3 establishes the uniform decay rate of solution and its derivatives with respect to time and spatial variables, both in general setting and in concrete classes of initial data. Several useful inequalities and the results on the existence, regularity of solution and the generalized energy inequality are presented in Appendices A–B.

2. Decay characters in some function spaces

2.1. Definition of the decay character

Originally, Niche and Schonbek in [24] introduced the following definitions of decay indicator and decay character for a function $u_0 \in L^2(\mathbb{R}^n)$.

Definition 2.1 (Decay indicator). For $u_0 \in L^2(\mathbb{R}^n)$, $s > 0$, and $r \in (-n/2 + s, \infty)$, the decay indicator $P_r^s(u_0)$ corresponding to $\Lambda^s u_0$ is defined by

$$P_r^s(u_0) = \lim_{\rho \to 0} \rho^{-2r-n} \int_{B(\rho)} |\xi|^{2s} |\widehat{u_0}(\xi)|^2 d\xi,$$

if this limit exists, where $B(\rho)$ is the ball centered at origin with radius ρ.

Definition 2.2 (Decay character). The decay character of $\Lambda^s u_0$, denoted by $r_s^* = r_s^*(u_0)$ is the unique $r \in (-n/2 + s, \infty)$ such that $0 < P_r^s(u_0) < \infty$, if this number $P_r^s(u_0)$ exists. If such $P_r^s(u_0)$ does not exist, we set $r_s^* = -n/2 + s$ when $P_r^s(u_0) = +\infty$ for all $r \in (-n/2 + s, \infty)$, or $r_s^* = +\infty$ when $P_r^s(u_0) = 0$ for all $r \in (-n/2 + s, \infty)$. For $s = 0$, we denote $P_r^0(u_0) = P_r(u_0)$ and $r_0^* = r^*$.

However, the decay character does not always exist for functions in $L^2(\mathbb{R}^n)$. In order to overcome this situation, Brandolese in [4] has proposed the following improved definitions of these notions.

Definition 2.3. Let $u_0 \in L^2(\mathbb{R}^n)$, $r \in (-n/2 + s, \infty)$. We define the *lower decay indicator* and *upper decay indicator* by

$$P_r(u_0)_- = \liminf_{\rho \to 0^+} \rho^{-2r-n} \int_{B(\rho)} |\widehat{u_0}(\xi)|^2 d\xi,$$
$$P_r(u_0)_+ = \limsup_{\rho \to 0^+} \rho^{-2r-n} \int_{B(\rho)} |\widehat{u_0}(\xi)|^2 d\xi.$$

We note that the lower and upper decay indicators of $u_0 \in L^2$ always exist in $\overline{\mathbb{R}}$. Next, we define the *lower decay character* and *upper decay character* as follows.

Definition 2.4. We have

$$r(u_0)_- = \inf \{ r \in \mathbb{R} : P_r(u_0)_- > 0 \},$$
$$r(u_0)_+ = \sup \{ r \in \mathbb{R} : P_r(u_0)_+ < \infty \}.$$

Finally, the decay character is defined, according to Brandolese's approach, in the following manner.

Definition 2.5. If there exists $r^* \in (-\frac{n}{2}, \infty)$ such that

$$r^* = \min\left\{r \in \mathbb{R} : P_r(u_0)_- > 0\right\} = \max\left\{r \in \mathbb{R} : P_r(u_0)_+ < \infty\right\},$$

then we say $r^* = r^*(u_0)$ is the decay character of u_0.

An advantage of introducing separately the upper and lower decay indicators and decay characters of u_0 lies in obtaining the upper and lower decay bounds for the solution of the evolution problem even in the case when the data may not have the classical decay character (by the approach of Niche and Schonbek). We recall the following result in [4].

Proposition 2.6. *Consider the Cauchy problem*

$$\begin{cases} u_t + (-\Delta)^\alpha u = 0, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,\cdot) = u_0(\cdot) \in L^2(\mathbb{R}^n). \end{cases} \tag{2.1}$$

Then, the following statements hold.

(a) *If $P_r(u_0)_+ < \infty$, then the solution $u(t,x)$ to* (2.1) *satisfies*

$$\|u(t,\cdot)\|_2^2 \lesssim (1+t)^{-\frac{1}{\alpha}(r+\frac{n}{2})}.$$

(b) *If $P_r(u_0)_- > 0$, then the solution $u(t,x)$ to* (2.1) *satisfies*

$$\|u(t,\cdot)\|_2^2 \gtrsim (1+t)^{-\frac{1}{\alpha}(r+\frac{n}{2})}.$$

Remark 2.7. The notions of $P_r^s(u_0)_\pm$ and $r_s(u_0)_\pm$, for $s > 0$, can be defined in a similar manner by following Brandolese's approach, so we may skip the details.

Niche and Schonbek in [24] obtained the following result on the decay characters $r_s^*(u_0)$.

Proposition 2.8 ([24, Theorem 2.11]). *Let $u_0 \in H^s(\mathbb{R}^n)$, $s > 0$. If $-n/2 < r^*(u_0) < \infty$, then $-n/2 + s < r_s^*(u_0) < \infty$ and $r_s^*(u_0) = s + r^*(u_0)$.*

Remark 2.9. Implicitly, the authors in [24] require the existence of both $r^* := r^*(u_0)$ and $r_s^*(u_0)$ for the validity of Proposition 2.8. In that case, it follows that $r^*(\Lambda^s u_0) = r_s^*(u_0) = r^*(u_0) + s \geq -n(1 - 1/m) + s$. If, however, $r_s^*(u_0)$ does not exist, we can assert only that $(P_{r^*+s}(u_0))_+ < +\infty$.

Definition 2.10 (Decay indicators and decay characters with negative orders). For $v \in L^2(\mathbb{R}^n)$, $s > 0$ and $r \in (-n/2, \infty)$, the decay indicator $P_r^{-s}(v)$ corresponding to $\Lambda^{-s} v$ is defined by

$$P_r^{-s}(v) = \lim_{\rho \to 0} \rho^{-2r-n} \int_{B(\rho)} |\xi|^{-2s} |\hat{v}(\xi)|^2 \, d\xi.$$

If $u := \mathcal{F}^{-1}(|\xi|^{-s}\hat{v}) \in L^2$ for $s > 0$, then we denote $r_{-s}^*(v) := r^*(u)$.

Our recent result in [8, Theorem A.4] regarding the relation between $r_{-s}^*(v)$ and $r^*(v)$ for a function $v \in L^2$ is stated as follows.

Proposition 2.11. *Let $v \in L^2(\mathbb{R}^n)$ such that there exists $r^*(v)$ satisfying $r^*(v) - s > -\frac{n}{2}$ for some $s \geq 0$. Then, the function $u := \mathcal{F}^{-1}(|\xi|^{-s}\hat{v}) \in L^2(\mathbb{R}^n)$ and there exists $r^*_{-s}(v) = r^*(u) = r^*(v) - s$.*

2.2. Decay characters in $L^2 \cap L^m$ for $m \in [1, 2)$

The following result has been discussed in [12] and is presented here with minor adjustment and correction.

Proposition 2.12. *Let $u_0 \in L^2 \cap L^m$ for $m \in [1, 2)$. Then, the following statements hold:*

(i) $P_r(u_0)_+ < +\infty$;

(ii) $r^*(u_0)_+ \geq -n(1 - \frac{1}{m})$.

Moreover, if the decay character $r^*(u_0)$ exists, then $r^*(u_0) \geq -n(1 - \frac{1}{m})$.

Proof. Following arguments in [12], with $\frac{1}{m} + \frac{1}{m'} = 1$, we have

$$\begin{aligned}\int_{B_\rho} |\widehat{u_0}(\xi)|^2 d\xi &\leq \left(\int |\widehat{u_0}(\xi)|^{m'} d\xi\right)^{2/m'} \cdot \left(\int_{B_\rho} 1 d\xi\right)^{1-2/m'} \\ &\leq C\rho^{n(1-\frac{2}{m'})} = C\rho^{n(\frac{2}{m}-1)}.\end{aligned} \tag{2.2}$$

Therefore,

$$\rho^{-n-2r} \int_{B_\rho} |\widehat{u_0}(\xi)|^2 d\xi \leq C \quad \text{with } r := -n\Big(1 - \frac{1}{m}\Big). \tag{2.3}$$

The assertion of Proposition 2.12 then follows from the definitions of $P_r(u_0)_+, r^*(u_0)_+$ and $r^*(u_0)$. ∎

In [12], the authors have mistakenly concluded, based on (2.2) and (2.3), that $r^*(u_0) = -n(1 - \frac{1}{m})$ for $u_0 \in L^2 \cap L^m, m \in [1, 2)$. It is not always the case, since amongst functions $u_0 \in L^2 \cap L^m, m \in [1, 2)$ admitting decay characters, the decay character of u_0 can take arbitrarily large values greater than $-n(1 - \frac{1}{m})$, as the following proposition shows.

Proposition 2.13. *For any $s > 0$, there exists $v_0 \in L^2 \cap L^m$ such that $r^*(v_0) \geq -n(1 - 1/m) + s$. Moreover, for any $r \in [r^*(v_0) - s, r^*(v_0)]$, there exists a function $v \in L^2 \cap L^m$ with $r^*(v) = r$.*

Proof. For any $s > 0$, let us choose $u_0 \in H^s_m(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$. Recall that the Bessel potential space $H^s_m = F^s_{m,2}$ (the Sobolev space of fractional order s) is defined as

$$H^s_m = \left\{f \in \mathcal{S}' \mid \|f\|_{H^s_m} = \|\mathcal{F}^{-1}((1 + |\xi|^2)^{\frac{s}{2}}\mathcal{F}(f))\|_{L^m}\right\} < \infty$$

and $H^s = H^s_2$.

Then, $u_0 \in L^2 \cap L^m$ and $\Lambda^s u_0 = (-\Delta)^{s/2} u_0 \in L^2 \cap L^m$, for $m \in [1, 2)$. Let $v_0 := \Lambda^s u_0 \in L^2 \cap L^m$. Hence, by definition, $r^*(v_0) = r^*(u_0) + s \geq -n(1 - 1/m) + s$.

Moreover, thanks to the embeddings in the Triebel–Lizorkin spaces (see [27, Proposition in Section 2.1, page 32])

$$F^s_{m,2} \hookrightarrow F^s_{m,\infty} \hookrightarrow F^{s_1}_{m,2}$$

for any s, s_1 such that $0 < s_1 < s$, by the above construction, we are able to pick $v := \Lambda^{s_1} u_0 \in L^m \cap L^2$ to conclude that $r^*(v) = r^*(\Lambda^{s_1} u_0) = r^*(u_0) + s_1$. Hence, the decay characters of functions from $L^m \cap L^2$ can take any value in the range $[r^*(u_0), r^*(u_0)+s]$ for $s > 0$. ■

Now, we will construct a function $u_0 \in L^m \cap L^2$ that $r^*(u_0)$ is exactly $-n(1-1/m)$. First, let us recall the definition of the Hardy space $\mathcal{H}^p$.

Definition 2.14. The Hardy space $\mathcal{H}^p$ for $0 < p < \infty$ consists of tempered distributions f such that for some $\Phi \in \mathcal{S}$ with $\int_{\mathbb{R}^n} \Phi(x)dx = 1$, the maximal function

$$M_\Phi f := \sup_{t>0} |(f * \Phi_t)(x)|$$

satisfies $M_\Phi f \in L^p(\mathbb{R}^n)$, where $\Phi_t := t^{-n}\Phi(\frac{x}{t})$.

We fix the following function $v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap \mathcal{H}^1(\mathbb{R}^n)$ such that its decay character $r^*(v) = 0$. We know that such function exists, for example, it is sufficient to choose $v \in L^1 \cap L^2 \cap \mathcal{H}^1$ with non-zero mean, i.e., $\int_{\mathbb{R}^n} v(x)dx \neq 0$.

For $m \in (1,2)$, put $s := n - \frac{n}{m}$. Then, $0 < s < \frac{n}{2}$. Let $u := \Lambda^{-s} v = I_s v$, where the Riesz potential I_β is defined by the Fourier transform as follows:

$$\widehat{I_\beta f}(\xi) := \frac{\hat{f}(\xi)}{|\xi|^\beta}.$$

Proposition 2.15. *For $s = n - \frac{n}{m}, m \in (1,2)$, and $v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap \mathcal{H}^1(\mathbb{R}^n)$, it holds that $u := \Lambda^{-s} v \in L^2 \cap L^m$ with $r^*(u) = -n(1-\frac{1}{m})$.*

Proof of Proposition 2.15. We verify first that $u \in L^2(\mathbb{R}^n)$. It is, in fact, a direct consequence of Lemma A.5 (the Sobolev lemma).

By choosing $q = 2$, relation (A.4) becomes $\frac{1}{2} = \frac{1}{p} + \frac{1}{m} - 1$, which leads to $p := \frac{2m}{3m-2}$. It is easy to check that $\frac{2m}{3m-2} \in (1,2)$ for $m \in (1,2)$. Thus, $\|v\|_{L^p}$ is bounded for our choice $v \in L^1 \cap L^2$. Therefore, by (A.5), $\|u\|_{L^2} < +\infty$.

Let us now verify that $u \in L^m(\mathbb{R}^n)$. We cannot apply the Sobolev lemma A.5, since relation (A.4) with $q = m, s = n - \frac{n}{m}$ implies that $p = 1$, which violates condition $1 < p < q < \infty$. Instead, we will employ a version of the Hardy–Littlewood–Sobolev inequality, stated in Lemma A.6, for the critical $p = 1$ obtained recently in [28].

With $s := n - \frac{n}{m}$, it is clear that condition (A.6) is satisfied. Moreover, by the classical Stein–Fefferman result in [11] on the characterization of spaces $\mathcal{H}^p$, it is known that for function $f \in L^1(\mathbb{R}^n)$ the condition $f \in \mathcal{H}^1(\mathbb{R}^n)$ is equivalent to $Rf \in L^1(\mathbb{R}^n)$. Recall that the Riesz transform R_j of function f is given by the Fourier transform as

$$R_j f(x) = \mathcal{F}^{-1}\Big(- i \frac{\xi_j}{|\xi|} \mathcal{F} f(\xi)\Big).$$

We denote the vector $R := (R_1, R_2, \ldots, R_n)$. Therefore, our condition $v \in L^1 \cap L^2 \cap \mathcal{H}^1$ implies that $\|Rv\|_{L^1} < +\infty$. Thus, we have verified that $u = I_s v \in L^m$. Summarizing, we have shown that $u \in L^2 \cap L^m$.

Next, we will show that the above constructed function u satisfies $r^*(u) = -s = -n(1 - \frac{1}{m})$. Indeed, we will apply Proposition 2.11 on the decay character with negative order. In our case, we have that $r^*(v) = 0$ and $s = n - \frac{n}{m}$, therefore, the condition $r^*(v) - s > -\frac{n}{2}$ is clearly satisfied. Proposition 2.11 implies that

$$r^*(u) = r^*(v) - s = -s = -n\Big(1 - \frac{1}{m}\Big). \qquad \blacksquare$$

Thus, we have constructed the function $u_0 \in L^2 \cap L^m$ with $r^*(u_0)$ being exactly equal to $-n(1 - \frac{1}{m})$.

2.3. The decay characters of function in the weighted spaces $L^{1,\gamma}(\mathbb{R}^n)$

Let the weighted space $L^{1,\gamma}(\mathbb{R}^n)$ be defined as $L^{1,\gamma}(\mathbb{R}^n) := L^1(\mathbb{R}^n, (1 + |x|)^\gamma dx)$ for $\gamma \in [0, 1]$. The following analysis was carried out in [6] and is stated below with slight modification and correction.

Proposition 2.16 ([6]). *Let $u_0 \in L^{1,\gamma}(\mathbb{R}^n) \cap L^2_\sigma$. Then, $P_r(u_0) = 0$ for $r < \gamma$, and let $(P_\gamma(u_0))_+ < +\infty$. Moreover, if there exists the decay character $r^*(u_0) =: r^*$, then $r^* \geq \gamma$.*

Proof. We follow the analysis from [6]. For a general function $u_0 \in L^{1,\gamma}(\mathbb{R}^n)$, Ikehata in [15, Lemma 3.1] has established the following estimate:

$$|\widehat{u_0}(\xi)| \leq C_\gamma |\xi|^\gamma \|u_0\|_{1,\gamma} + \Big| \int_{\mathbb{R}^n} u_0(x) dx \Big|, \tag{2.4}$$

with some constant $C_\gamma > 0$, which depends only on γ. However, since $u_0 \in L^{1,\gamma}(\mathbb{R}^n) \cap L^2_\sigma$, by Borchers' lemma (see [32]) the function u_0 has zero mean, i.e.,

$$\widehat{u_0}(0) = \int_{\mathbb{R}^n} u_0(x) dx = 0.$$

Now, from (2.4), it follows that $|\widehat{u_0}(\xi)| \leq C_\gamma |\xi|^\gamma$, which implies the following:

(i) $P_r(u_0) = 0$ for $r < \gamma$;

(ii) $(P_\gamma(u_0))_+ < +\infty$.

If the decay character $r^*(u_0) =: r^*$ exists, then the above shows that $r^*(v) \geq \gamma$. $\blacksquare$

2.4. Decay characters in weighted space $L^2(\mathbb{R}^n, e^{\frac{\beta}{2}|x|^2}dx)$

The weighted space $L^2(\mathbb{R}^n, e^{\frac{\beta}{2}|x|^2}dx)$ appears in many problems in order to overcome the compact support condition required for the initial data. Thanks to the inclusion of the weighted space $L^2(\mathbb{R}^n, e^{\frac{\beta}{2}|x|^2}dx) \subset L^{1,\gamma}(\mathbb{R}^n)$ for any $\beta, \gamma > 0$, we see that $r^*(u_0) \geq 1$ for the divergence-free $u_0 \in L^2(\mathbb{R}^n, e^{\frac{\beta}{2}|x|^2}dx)$.

2.5. Decay characters in Sobolev spaces with negative order

Proposition 2.17. *Let $u_0 \in (L^2 \cap \dot{H}^{-\gamma})$ with $\gamma \in (0, \frac{n}{2})$. Then, $P_r(u_0)_+ < +\infty$ with $r := \gamma - \frac{n}{2}$ and $r^*(u_0)_+ \geq \gamma - \frac{n}{2}$.*

Proof. Since $|\xi|^{-\gamma}\hat{u}_0 \in L^2(\mathbb{R}^n)$, we obtain

$$\rho^{-2r-n}\int_{B_\rho}|\hat{u}_0(\xi)|^2 d\xi \leq \rho^{2\gamma-2r-n}\int_{B_\rho}|\xi|^{-2\gamma}|\hat{u}_0(\xi)|^2 d\xi \leq C\rho^{2\gamma-2r-n}.$$

From the definition of upper and lower decay indicators, with $r := \gamma - \frac{n}{2}$, we obtain

$$P_r(u_0)_+ < +\infty.$$

By Definition 2.5, we conclude immediately that $r^*(u_0)_+ \geq \gamma - \frac{n}{2}$. ∎

2.6. Decay character in $\mathcal{Y}^p$

We define (see Melo [23])

$$\mathcal{Y}^p(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; \hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } \sup_{\xi\in\mathbb{R}^n}\{|\xi|^p|\hat{f}(\xi)|\} < \infty \right\},$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions and $\mathcal{Y}^p(\mathbb{R}^n)$-norm is given by

$$\|f\|_{\mathcal{Y}^p} := \sup_{\xi\in\mathbb{R}^n}\{|\xi|^p|\hat{f}(\xi)|\}.$$

Proposition 2.18. *Let $p \in [-1, n/2)$. If $u_0 \in \mathcal{Y}^p(\mathbb{R}^n)$, then $r^*(u_0)_+ \geq -p$.*

Proof. If $u_0 \in L^2 \cap \mathcal{Y}(\mathbb{R}^n)$, then we have

$$\begin{aligned}\rho^{-2r-n}\int_{B(\rho)}|\hat{u}_0(\xi)|^2 d\xi &\leq \rho^{-2r-n}\int_{B(\rho)} C|\xi|^{-2p}d\xi \\ &\leq C\rho^{-2r-n}\int_0^\rho r^{n-1-2p}dr \leq C\rho^{-2p-2r}.\end{aligned}$$

Therefore, it is deduced from the definition of decay character that $r^*(u_0) \geq -p$ (if u_0 has decay character), and $r^*(u_0)_+ \geq -p$ (in general). ∎

3. The decay of solution and its derivatives

3.1. Non-uniform decay

We first prove that the L^2-norm of solution u to (1.1) tends to zero when $t \to \infty$. After that, we show that there is no uniform decay in L^2 for solutions of (1.1) in the following sense: for each sphere with radius γ in L^2, there is an initial datum on the sphere for which the corresponding solution decays arbitrarily slowly.

The next theorem has already been obtained in [9]. It can be proved also by the generalized energy inequality method; see, e.g., [2].

Theorem 3.1. *Assume that $0 < \alpha < \frac{n+2}{4}$, $u_0 \in L^2_\sigma$. Let u be the solution to (1.1) constructed in Theorem B.2. Then,*

$$\lim_{t\to\infty} \|u(t)\|_2 = 0. \tag{3.1}$$

The next theorem shows a lack of uniformity of decay rates of solutions u to (1.1).

Theorem 3.2. *Let the assumptions in Theorem 3.1 hold. Then, there is no function $G(t,\gamma) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that the following statements hold:*

- $\|u\|_2^2 \le G(t, \|u_0\|_2)$;
- $\lim_{t\to\infty} G(t,\gamma) = 0$ *for all* $\gamma > 0$.

Proof. Fix $u_0 \in C_c^\infty$, and let $u_0^\lambda(x) = \lambda^{n/2} u_0(\lambda x)$. Hence, $\|u_0^\lambda\|_2 = \|u_0\|_2$ and $\|\Lambda^s u_0^\lambda\|_2 = \lambda^s \|\Lambda^s u_0\|_2$ for all $s > 0$. Therefore, there exists a constant $\lambda_0 > 0$ small enough such that for all $0 < \lambda < \lambda_0$,

$$\|\Lambda^{\frac{n+2-4\alpha}{2}} u_0^\lambda\|_2 = \lambda^{\frac{n+2-4\alpha}{2}} \|\Lambda^{\frac{n+2-4\alpha}{2}} u_0\|_2 < \varepsilon \quad \text{for the } \varepsilon \text{ mentioned in Theorem B.2.}$$

We denote by u^λ the weak solution to (1.1) corresponding to u_0^λ. Taking inner product (1.1) with $\Lambda^{2\alpha} u$ yields

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\alpha u^\lambda\|_2^2 + \|\Lambda^{2\alpha} u^\lambda\|_2^2 \le |\langle [\Lambda^\alpha, u^\lambda \cdot \nabla] u^\lambda, \Lambda^\alpha u^\lambda\rangle|.$$

By the fractional Gagliardo–Nirenberg inequality, Young's inequality and Lemma A.2, we have

$$\begin{aligned}
|\langle [\Lambda^\alpha, u^\lambda \cdot \nabla] u^\lambda, \Lambda^\alpha u^\lambda\rangle| &\le C\|[\Lambda^\alpha, u^\lambda \cdot \nabla] u^\lambda\|_2 \|\Lambda^\alpha u^\lambda\|_2 \\
&\le C\|\Lambda^\alpha u^\lambda\|_{\frac{2n}{n-2\alpha}} \|\nabla u^\lambda\|_{\frac{n}{\alpha}} \|\Lambda^\alpha u^\lambda\|_2 \\
&\le C\|\Lambda^{2\alpha} u^\lambda\|_2 \|\Lambda^{\frac{n+2-2\alpha}{2}} u^\lambda\|_2 \|\Lambda^\alpha u^\lambda\|_2 \\
&\le C\|\Lambda^{\frac{n+2-2\alpha}{2}} u^\lambda\|_2^2 \|\Lambda^\alpha u^\lambda\|_2^2 + \|\Lambda^{2\alpha} u^\lambda\|_2^2.
\end{aligned}$$

Hence,

$$\frac{d}{dt}\|\Lambda^\alpha u^\lambda\|_2^2 \le C\|\Lambda^\alpha u^\lambda\|_2^2 \|\Lambda^{\frac{n+2-2\alpha}{2}} u^\lambda\|_2^2.$$

By Gronwall's inequality and Theorem B.2, we obtain

$$\|\Lambda^\alpha u^\lambda(t)\|_2^2 \leq \lambda^{2\alpha}\|\Lambda^\alpha u_0\|_2^2 e^C \leq C\lambda^{2\alpha}.$$

From there, multiplying (1.1) with u^λ gives

$$\frac{1}{2}\frac{d}{dt}\|u^\lambda\|_2^2 \geq -C\lambda^{2\alpha},$$

so that

$$\|u^\lambda(t)\|_2^2 \geq \|u_0\|_2^2 - C\lambda^{2\alpha}t.$$

Hence, we can deduce that there is no function $G(t,\gamma)$ approaching zero in t for each fixed γ such that $\|u(t)\|_2^2 \leq G(t, \|u_0\|_2)$. Indeed, if there were such a function, then at some t_0 we would have $G(t_0, \|u\|_2) \leq \frac{\|u_0\|_2^2}{2}$. Then, by choosing λ small enough, i.e., $\lambda^\alpha < \frac{\|u_0\|_2^2}{4Ct_0}$, it would lead to $\frac{\|u_0\|_2^2}{2} \geq \frac{3\|u_0\|_2^2}{4}$, which is a contradiction. ∎

3.2. Uniform decay

In this section, we will apply the notion of decay characters to generalize the uniform decay rates in L^2 of u and its derivatives.

3.2.1. Decay in L^2. We first establish the decay of solution u in L^2.

Theorem 3.3. *Suppose that $0 < \alpha < \frac{n+2}{4}$. Let $u_0 \in L^2_\sigma$ and u be the corresponding solution to system (1.1) constructed in Theorem B.2. Assume further that u_0 has decay character $r^*(u_0) = r^* \in (-\frac{n}{2}, \infty)$. Then, for all $t > 0$, we have*

$$\|u(t)\|_2^2 \leq C(1+t)^{-\min\{\frac{r^*}{\alpha}+\frac{n}{2\alpha};\frac{n+2}{2\alpha}\}}. \tag{3.2}$$

Proof. To begin, applying the Fourier transform to (1.1) yields

$$\begin{aligned} \hat{u}_t(\xi,t) + |\xi|^{2\alpha}\hat{u}(\xi,t) &= -\widehat{H}(\xi,t), \\ \hat{u}(\xi,0) &= \hat{u}_0(\xi), \end{aligned}$$

where $\widehat{H}(\xi,t) = \widehat{u\cdot\nabla u}(\xi,t) + \widehat{\nabla p}(\xi,t)$. Hence,

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)e^{-|\xi|^{2\alpha}t} - \int_0^t e^{-|\xi|^{2\alpha}(t-\tau)}\widehat{H}(\xi,\tau)d\tau.$$

Therefore,

$$|\hat{u}(\xi,t)|^2 \leq C|\hat{u}_0(\xi)|^2 e^{-2|\xi|^{2\alpha}t} + C|\xi|^2\left(\int_0^t e^{-|\xi|^{2\alpha}(t-\tau)}\|u(\tau)\|_2^2 d\tau\right)^2. \tag{3.3}$$

Meanwhile, taking the L^2-scalar product of (1.1) with u, by Plancherel's theorem, we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}|\hat{u}(t,\xi)|^2 d\xi + 2\int_{\mathbb{R}^n}|\xi|^{2\alpha}|\hat{u}(t,\xi)|^2 d\xi = 0.$$

Let $B(\rho(t)) := \{\xi \in \mathbb{R}^n \mid |\xi| \le \rho(t)\}$. By the Fourier splitting argument, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^n} |\hat{u}(t,\xi)|^2 d\xi + 2\rho^{2\alpha}(t)\int_{\mathbb{R}^n} |\hat{u}(t,\xi)|^2 d\xi \le 2p^{2\alpha}(t)\int_{B(\rho(t))} |\hat{u}(t,\xi)|^2 d\xi.$$

Let d be a large constant. By choosing $\rho^{2\alpha}(t) = \frac{d}{2(1+t)}$, from (3.3) and by the Fourier splitting method, we see that

$$\begin{aligned}&\frac{d}{dt}\left((1+t)^d \int_{\mathbb{R}^n} |\hat{u}(t,\xi)|^2 d\xi\right)\\ &\le C(1+t)^{d-1}\int_{B(\rho(t))} |\hat{u}_0(\xi)|^2 d\xi + C(1+t)^{d-1}\left(\int_0^t \|u(s)\|_2^2 ds\right)^2 \int_{B(\rho(t))} |\xi|^2 d\xi.\end{aligned} \tag{3.4}$$

Therefore, since the decay indicator $P_{r^*}(u_0) < C$ for some constant $C > 0$, we have

$$\frac{d}{dt}\left((1+t)^d \int_{\mathbb{R}^n} |\hat{u}(t,\xi)|^2 d\xi\right) \le C(1+t)^{d-1-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{d-\frac{n+2}{2\alpha}}\int_0^t \|u(\tau)\|_2^4 d\tau.$$

Integrating from 0 to t yields

$$\|u(t)\|_2^2 \le C(1+t)^{-d} + C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{-\frac{n+2}{2\alpha}+1}\int_0^t \|u(\tau)\|_2^4 d\tau.$$

Since d is large enough, we get

$$\|u(t)\|_2^2 \le C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{-\frac{n+2}{2\alpha}+1}\int_0^t \|u(\tau)\|_2^4 d\tau.$$

Hence, thanks to $\|u(\tau)\|_2^2 \le \|u_0\|_2^2$, it follows that

$$\begin{aligned}&(1+t)^{\frac{n+2}{2\alpha}-1}\|u(t)\|_2^2\\ &\quad\le C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}+\frac{n+2}{2\alpha}-1} + C\int_0^t (1+\tau)^{-\frac{n+2}{2\alpha}+1}(1+\tau)^{\frac{n+2}{2\alpha}-1}\|u(\tau)\|_2^2 d\tau.\end{aligned}$$

By Gronwall's inequality, we have

$$\begin{aligned}&(1+t)^{\frac{n+2}{2\alpha}-1}\|u(t)\|_2^2\\ &\quad\le C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}+\frac{n+2}{2\alpha}-1} + C\int_0^t (1+\tau)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}}\exp\left(\int_s^t (1+r)^{-\frac{n+2}{2\alpha}+1} dr\right) d\tau.\end{aligned}$$

Since $-\frac{n+2}{2\alpha}+1 < -1$, $\exp(\int_s^t (1+r)^{-\frac{n+2}{2\alpha}+1} dr) \le C$. Therefore,

$$\|u(t)\|_2^2 \le C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{-\frac{n+2}{2\alpha}+1}\int_0^t (1+\tau)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} d\tau.$$

It implies that

$$\|u(t)\|_2^2 \le \begin{cases} C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{-\frac{n+2}{2\alpha}+1} & \text{if } 0<\alpha<r^*+\frac{n}{2}, \\ C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{-\frac{n+2}{2\alpha}+1}\ln(1+t) & \text{if } \alpha=r^*+\frac{n}{2}, \\ C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{-\frac{n+2}{2\alpha}+2-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} & \text{if } r^*+\frac{n}{2}<\alpha<\frac{n+2}{4}, \end{cases}$$

$$\le \begin{cases} C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{-\frac{n+2}{2\alpha}+1} & \text{if } 0<\alpha<r^*+\frac{n}{2}, \\ C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{-\frac{n+2}{2\alpha}+1}\ln(1+t) & \text{if } \alpha=r^*+\frac{n}{2}, \\ C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} & \text{if } r^*+\frac{n}{2}<\alpha<\frac{n+2}{4}. \end{cases}$$

Due to the fact that $\ln(1+t)\le C(1+t)^{\gamma}$ for $0<\gamma<\frac{n+2}{2r^*+n}$, we get

$$\int_0^t \|u(s)\|_2^2 ds \le \begin{cases} C & \text{if } 0<\alpha<r^*+\frac{n}{2}, \\ C\ln(1+t) & \text{if } \alpha=r^*+\frac{n}{2}, \\ C(1+t)^{1-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} & \text{if } r^*+\frac{n}{2}<\alpha<\frac{n+2}{4}. \end{cases} \tag{3.5}$$

Plugging (3.5) into (3.4) yields

$$\frac{d}{dt}((1+t)^d\|u(t)\|_2^2)$$
$$\le \begin{cases} C(1+t)^{d-1-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{d-1-\frac{n+2}{2\alpha}} & \text{if } 0<\alpha<r^*+\frac{n}{2}, \\ C(1+t)^{d-1-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{d-1-\frac{n+2}{2\alpha}}\ln^2(1+t) & \text{if } \alpha=r^*+\frac{n}{2}, \\ C(1+t)^{d-1-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{d+1-\frac{n+2}{2\alpha}-\frac{2r^*}{\alpha}-\frac{n}{\alpha}} & \text{if } r^*+\frac{n}{2}<\alpha<\frac{n+2}{4}. \end{cases}$$

Since $\ln^2(1+t)\le C(1+t)^{\eta}$ for some sufficiently small $0<\eta<\frac{2-2r^*}{2r^*+n}$ (noting that if $r^*\ge 1$, then the second case will not happen) and d is large enough, we deduce that

$$\|u(t)\|_2^2 \le \begin{cases} C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{-\frac{n+2}{2\alpha}} & \text{if } 0<\alpha<r^*+\frac{n}{2}, \\ C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{-\frac{n+2}{2\alpha}+\eta} & \text{if } \alpha=r^*+\frac{n}{2}, \\ C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{2-\frac{n+2}{2\alpha}-\frac{2r^*}{\alpha}-\frac{n}{\alpha}} & \text{if } r^*+\frac{n}{2}<\alpha<\frac{n+2}{4}, \end{cases}$$

$$\le \begin{cases} C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{-\frac{n+2}{2\alpha}} & \text{if } 0<\alpha<r^*+\frac{n}{2}, \\ C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} & \text{if } \alpha=r^*+\frac{n}{2}, \\ C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} & \text{if } r^*+\frac{n}{2}<\alpha<\frac{n+2}{4}. \end{cases} \qquad \blacksquare$$

Remark 3.4. The decay rate described in (3.2) is optimal in the sense that it coincides with the ones for the solution of the generalized heat equation $v_t+(-\Delta)^{\alpha}v=0$ with the same initial data u_0. Niche and Schonbek [24, Theorem 2.10] proved that if $\mathcal{L}$ is the pseudo-differential operator with homogeneous symbol of degree 2α, then for the solution to the linear problem

$$\begin{cases} u_t=\mathcal{L}u, & t>0,\ x\in\mathbb{R}^n, \\ u|_{t=0}=u_0, \end{cases} \tag{3.6}$$

the decay rate of the form

$$\|u(t)\|_2^2 \le C(1+t)^{-\frac{r^*}{\alpha}+\frac{n}{2\alpha}}$$

is sharp. It means that the solution to (3.6) satisfies both upper and lower bound estimates for the L^2-norm:

$$C_1(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} \le \|u(t)\|_2^2 \le C_2(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}}.$$

In [24], it has been proved that the decay rate $(1+t)^{-\frac{r^*}{\alpha}+\frac{n}{2\alpha}}$ is also sharp for the solution θ to the dissipative quasi-geostrophic equations

$$\theta_t + u\cdot\nabla\theta + \kappa(-\delta)^\alpha\theta = 0, \quad 0<\alpha\le 1,\ \kappa>0,$$

as well as for the solution u^ϵ to the compressible approximation to Navier–Stokes equations (corresponding to $\alpha = 1$) in $\mathbb{R}^3$ with $-\frac{3}{2} < r^* \le 1$.

Regarding the solution to the Navier–Stokes equations in $\mathbb{R}^3$ (i.e., when $\alpha = 1$), our estimate coincides with the decay estimate obtained by Bjorland and Schonbek in [3]. For this classical model with $n = 3, \alpha = 1$, Brandolese in [4, Theorem 5.1] has proved that under condition $r^* = r^*(u_0) \in (-\frac{3}{2}, 1]$ the solution u to the Navier–Stokes equations in $\mathbb{R}^3$ with initial datum u_0 satisfies the following two-sided sharp decay estimates:

$$C_1(1+t)^{-r^*-\frac{3}{2}} \le \|u(t)\|_2^2 \le C_2(1+t)^{-r^*-\frac{3}{2}}.$$

From Theorem 3.3, we deduce the following consequences related to the initial data chosen from special spaces.

Corollary 3.5. *Let $\alpha = 1$ and $u_0 \in L^2_\sigma \cap L^p$ with $p \in [1,2)$. Assume further that u_0 has decay character. Then, the solution u to system* (1.1) *constructed in Theorem* B.2 *satisfies for all $t > 0$ the following estimate:*

$$\|u(t)\|_2 \le C(1+t)^{-\frac{n}{4}(\frac{2}{p}-1)}.$$

Proof. For $\alpha = 1$, if $u_0 \in L^2 \cap L^p$ with $p \in [1,2)$, then, by Proposition 2.12, $r^*(u_0)_+ \ge -n(1-\frac{1}{p})$. Therefore, thanks to Theorem 3.3, we have

$$\|u(t)\|_2 \le C(1+t)^{-\min\{\frac{n}{4}(\frac{2}{p}-1);\frac{n+2}{4\alpha}\}} \le C(1+t)^{-\frac{n}{4}(\frac{2}{p}-1)}.$$

One could see that it is the same decay rate obtained in [30]. ∎

Corollary 3.6. *Let $n = 3$, $0 < \alpha < \frac{5}{4}$. Assume that $u_0 \in L^2_\sigma \cap L^p$ for $p \in [1,2)$. Assume further that u_0 has decay character. Then, the solution u to system* (1.1) *constructed in Theorem* B.2 *satisfies for all $t > 0$ the following estimate:*

$$\|u(t)\|_2^2 \le C(1+t)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}.$$

Proof. By Theorem 3.3 and Proposition 2.12, we have

$$\|u(t)\|_2^2 \leq C(1+t)^{-\min\{\frac{3}{2\alpha}(\frac{2}{p}-1);\frac{n+2}{2\alpha}\}} \leq C(1+t)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)}. \qquad \blacksquare$$

Remark 3.7. It is clear that the decay rate stated in Corollary 3.6 is exactly the same as that one obtained by Jiu [16]. However, in our proof we do not need to use the technical assumption $\max\{\frac{1}{3-2\alpha};1\} \leq p < 2$ introduced by Jiu.

Corollary 3.8. *Let $n=3$ and the initial datum $u_0 \in L^2_\sigma \cap \mathcal{Y}^p$. Assume further that u_0 has decay character. Then, the solution u to system (1.1) constructed in Theorem B.2 satisfies for all $t>0$ the following estimate:*

$$\|u(t)\|_2 \leq C(1+t)^{-\frac{3-2p}{4\alpha}}.$$

Proof. From Proposition 2.18, we have $r^*(u_0) \geq -p$. Hence, by Theorem 3.3, we have

$$\|u(t)\|_2 \leq C(1+t)^{-\min\{\frac{3-2p}{4\alpha};\frac{n+2}{4\alpha}\}} \leq C(1+t)^{-\frac{3-2p}{4\alpha}}.$$

One could realize that the same decay rate appears in [23]. $\blacksquare$

3.2.2. Decays of derivatives with respect to time and spatial variables.

Proposition 3.9. *Suppose that $0<\alpha<\frac{n+2}{4}$. Let $u_0 \in H^s_\sigma$ ($s>\frac{n+2-2\alpha}{2}$) satisfy $\|u_0\|_{H^s} \leq \epsilon$ mentioned in Theorem B.3. Assume further that u_0 has decay character $r^*(u_0)=r^* \in (-\frac{n}{2},\infty)$. Then, for all $t>0$, the following estimate holds:*

$$\|u(t)\|_{H^s}^2 \leq C(1+t)^{-\min\{\frac{r^*}{\alpha}+\frac{n}{2\alpha};\frac{n+2}{2\alpha}\}}. \tag{3.7}$$

Proof. We use the technique that has been used in [16, Proposition 3.3]. By applying the Fourier splitting method to (B.1), we have

$$\begin{aligned}\frac{d}{dt}\|u\|_{H^s}^2 + C\rho^{2\alpha}(t)\|u\|_{H^s}^2 &\leq \rho^{2\alpha}(t)\int_{B(\rho(t))}(|\hat{u}(\xi)|^2 + |\widehat{\Lambda^s u}(\xi)|^2)d\xi \\ &\leq \rho^{2\alpha}(t)\int_{B(\rho(t))}|\hat{u}(\xi)|^2 d\xi + \rho^{2\alpha+2s}(t)\int_{B(\rho(t))}|\hat{u}(\xi)|^2 d\xi.\end{aligned}$$

Choosing $\rho^{2\alpha}(t) = \frac{d}{C(1+t)}$ with d large enough, we get from (3.2) that

$$\frac{d}{dt}((1+t)^d\|u\|_{H^s}^2) \leq (1+t)^{d-1-\min\{\frac{r^*}{\alpha}+\frac{n}{2\alpha};\frac{n+2}{2\alpha}\}}.$$

Therefore,

$$\|u(t)\|_{H^s}^2 \leq C(1+t)^{-\min\{\frac{r^*}{\alpha}+\frac{n}{2\alpha};\frac{n+2}{2\alpha}\}},$$

which finishes the proof. $\blacksquare$

The next result establishes the decay rates of x-derivatives of u in L^2.

Theorem 3.10. *Let $0 < \alpha < \frac{n+2}{4}$ and $u_0 \in H^s_\sigma$ ($s > \frac{n+2-2\alpha}{2}$) satisfying $\|u_0\|_{H^s} \le \epsilon$ mentioned in Theorem B.3. Assume that u_0 has decay character $r^*(u_0) = r^* \in (-\frac{n}{2}, \infty)$. Then, for all $t > T_0$, with some $T_0 > 0$ sufficiently large, and any real number $0 \le m \le s$, we have*

$$\|\Lambda^m u(t)\|_2^2 \le C(1+t)^{-\min\{\frac{r^*+m}{\alpha}+\frac{n}{2\alpha};\frac{n+2}{2\alpha}+\frac{m}{\alpha}\}}.$$

Proof. In the following proof, we denote $f(t) := C(1+t)^{-\min\{\frac{r^*}{\alpha}+\frac{n}{2\alpha};\frac{n+2}{2\alpha}\}}$. By taking an inner product in L^2 of (1.1) with $\Lambda^{2m}u$ with $0 \le m \le s$, since u is divergence-free, we get

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^m u\|_2^2 + \|\Lambda^{m+\alpha}u\|_2^2 \le |\langle[\Lambda^m, u\cdot\nabla]u, \Lambda^m u\rangle| := I.$$

To estimate the right-hand side, we follow the idea from [9] and consider the following three cases.

Case 1: $0 < \alpha \le \frac{n}{4}$. Using (A.1) and Hölder's inequality yields

$$I \le \|\Lambda^m u\|^2_{\frac{2n}{n-2\alpha}}\|\nabla u\|_{\frac{n}{2\alpha}} \le C\|\Lambda^{m+\alpha}u\|_2^2\|\Lambda^{\frac{n+2-4\alpha}{2}}u\|_2.$$

Here, for the term $\|\Lambda^m u\|^2_{\frac{2n}{n-2\alpha}}$, we have applied (A.3) with $\theta = 1, s = m, p = \frac{2n}{n-2\alpha}$ and $\sigma = m+\alpha$. The term $\|\nabla u\|_{\frac{n}{2\alpha}}$ is estimated by using (A.3) with $\theta = 1, s = 1, p = \frac{n}{2\alpha}$ and $\sigma = \frac{n+2-4\alpha}{2}$.

Case 2: $\frac{n}{4} < \alpha \le 1$. By (A.2) and Hölder's inequality, we have

$$\begin{aligned} I &= |\langle\Lambda^{m-\alpha}(u\cdot\nabla u), \Lambda^{m+\alpha}u\rangle| \le \|\Lambda^{m+1-\alpha}(u\otimes u)\|_2\|\Lambda^{m+\alpha}u\|_2 \\ &\le \|u\|_{\frac{n}{2\alpha-1}}\|\Lambda^{m+1-\alpha}u\|_{\frac{2n}{n+2-4\alpha}}\|\Lambda^{m+\alpha}u\|_2 \le C\|\Lambda^{m+\alpha}u\|_2^2\|\Lambda^{\frac{n+2-4\alpha}{2}}u\|_2. \end{aligned}$$

In the above, we estimate $\|u\|_{\frac{n}{2\alpha-1}}$ by (A.3) with $\theta = 1, s = 0, p = \frac{n}{2\alpha-1}$ and $\sigma = \frac{n+2-4\alpha}{2}$. The term $\|\Lambda^{m+1-\alpha}u\|_{\frac{2n}{n+2-4\alpha}}$ is estimated by (A.3) with $\theta = 1, s = m+1-\alpha, p = \frac{2n}{n+2-4\alpha}$ and $\sigma = m+\alpha$.

Case 3: $1 < \alpha < \frac{n+2}{4}$. We use (A.2) and Hölder's inequality to deduce that

$$\begin{aligned} I &= |\langle\Lambda^{m-\alpha}(u\cdot\nabla u), \Lambda^{m+\alpha}u\rangle| \le \|\Lambda^{m+1}(u\otimes u)\|_{\frac{2n}{n+2\alpha}}\|\Lambda^m u\|_{\frac{2n}{n-2\alpha}} \\ &\le \|u\|_{\frac{n}{2\alpha-1}}\|\Lambda^{m+1}u\|_{\frac{2n}{n+2-2\alpha}}\|\Lambda^m u\|_{\frac{2n}{n-2\alpha}} \le C\|\Lambda^{m+\alpha}u\|_2^2\|\Lambda^{\frac{n+2-4\alpha}{2}}u\|_2. \end{aligned}$$

In the last deduction, the term $\|u\|_{\frac{n}{2\alpha-1}}$ is estimated by using (A.3) with $\theta = 1, s = 0, p = \frac{n}{2\alpha-1}$ and $\sigma = \frac{n+2-4\alpha}{2}$; meanwhile, the term $\|\Lambda^{m+1}u\|_{\frac{2n}{n+2-2\alpha}}$ is estimated by (A.3) with $\theta = 1, s = m+1, p = \frac{2n}{n+2-2\alpha}$ and $\sigma = m+\alpha$. Finally, (A.3) with $\theta = 1, s = m, p = \frac{2n}{n-2\alpha}$ and $\sigma = m+\alpha$ gives the estimation for $\|\Lambda^m u\|_{\frac{2n}{n-2\alpha}}$.

Therefore, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Lambda^m u\|_2^2 + \|\Lambda^{m+\alpha}u\|_2^2 &\le C\|\Lambda^{m+\alpha}u\|_2^2\|\Lambda^{\frac{n+2-4\alpha}{2}}u\|_2 \\ &\le C\|\Lambda^{m+\alpha}u\|_2^2\|\Lambda^{\frac{n+2-4\alpha}{2}}u\|_2^2 + \frac{1}{4}\|\Lambda^{m+\alpha}u\|_2^2. \end{aligned}$$

Then, by (3.7), we deduce that

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^m u\|_2^2 + \frac{3}{4}\|\Lambda^{m+\alpha}u\|_2^2 \le Cf(t)\|\Lambda^{m+\alpha}u\|_2^2.$$

Let $T_0 > 0$ be such that $f(t) \le \frac{1}{4C}$ for all $t > T_0$. We have

$$\frac{d}{dt}\|\Lambda^m u\|_2^2 + \|\Lambda^{m+\alpha}u\|_2^2 \le 0.$$

By the Fourier splitting method and choosing $\rho^{2\alpha}(t) = \frac{2d}{(1+t)}$, with $d > 0$ large enough, we obtain

$$\begin{aligned}
&\frac{d}{dt}((1+t)^d\|\Lambda^m u\|_2^2) \\
&\quad\le C(1+t)^{d-\frac{m}{\alpha}-1}\int_{B(\rho(t))}|\hat{u}(\xi,t)|^2 d\xi \\
&\quad\le C(1+t)^{d-\frac{m}{\alpha}-1-\frac{r^*}{\alpha}-\frac{n}{2\alpha}} + C(1+t)^{d-\frac{m}{\alpha}-1-\frac{n+2}{2\alpha}}\left(\int_0^t \|u(\tau)\|_2^2 d\tau\right)^2.
\end{aligned}$$

By (3.5), using arguments similar to the proof of Theorem 3.3, we conclude that

$$\|\Lambda^m u\|_2^2 \le \begin{cases} C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}-\frac{m}{\alpha}} + C(1+t)^{-\frac{n+2}{2\alpha}-\frac{m}{\alpha}} & \text{if } 0 < \alpha < r^* + \frac{n}{2}, \\ C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}-\frac{m}{\alpha}} & \text{if } \alpha = r^* + \frac{n}{2}, \\ C(1+t)^{-\frac{r^*}{\alpha}-\frac{n}{2\alpha}-\frac{m}{\alpha}} & \text{if } r^* + \frac{n}{2} < \alpha < \frac{n+2}{4}. \end{cases}$$ ∎

Now, we can formulate the main result of our paper.

Theorem 3.11. *Assume that* $0 < \alpha < \frac{n+2}{4}$. *Let* $u_0 \in H^s_\sigma$, *where*

$$s \ge \frac{n+2-2\alpha}{2} + K\max\{2\alpha, \alpha+1\},$$

satisfying $\|u_0\|_{H^s} \le \epsilon$ *with the numbers* ϵ *and* K *mentioned in Theorem B.3. Assume further that* u_0 *has decay character* $r^*(u_0) = r^* \in (-\frac{n}{2}, \infty)$. *Then, there exists* $T_0 > 0$ *such that for all* $t \ge T_0$, *the corresponding weak solution* u *to (1.1) constructed in Theorem B.3 satisfies the following estimate:*

$$\|\partial_t^k \Lambda^m u(t)\|_2^2 \le C(1+t)^{-\min\{\frac{r^*+m}{\alpha}+\frac{n}{2\alpha}+2k;\frac{n+2}{2\alpha}+\frac{m}{\alpha}+2k\}} \tag{3.8}$$

for all $m \in \mathbb{R}_+$ *and* $k \in \mathbb{N}$ *satisfying* $k \le K+1$ *and* $m + k\max\{2\alpha, \alpha+1\} \le s$.

Proof. The proof is by induction on k. Without loss of generality, we assume for the moment that $K \ge 1$. Consider the case $k = 1$, and let $m \in \mathbb{R}_+$ be arbitrary such that $m + \max\{2\alpha, \alpha+1\} \le s$. From the proof of Theorem B.3, we have the following estimate:

$$\begin{aligned}
\|\partial_t\Lambda^m u\|_2^2 &\le C\|\Lambda^{m+2\alpha}u\|_2^2 + C\|\Lambda^m(u\cdot\nabla u)\|_2^2 \\
&\le C\|\Lambda^{m+2\alpha}u\|_2^2 + C\|\Lambda^{m+\alpha}u\|_2^2\|\Lambda^{\frac{n+2-2\alpha}{2}}u\|_2^2 + C\|\Lambda^{\frac{n-2\alpha}{2}}u\|_2^2\|\Lambda^{m+\alpha+1}u\|_2^2.
\end{aligned}$$

Since

$$\begin{aligned} m+2\alpha &\le m+\max\{2\alpha,\alpha+1\}\le s,\\ m+\alpha &\le m+\max\{2\alpha,\alpha+1\}\le s,\\ m+\alpha+1 &\le m+\max\{2\alpha,\alpha+1\}\le s, \end{aligned}$$

from Theorem 3.10, it follows that

$$\begin{aligned} \|\Lambda^{m+2\alpha}u(t)\|_2^2 &\le C(1+t)^{-\min\{\frac{r^*+m}{\alpha}+\frac{n}{2\alpha}+2;\frac{n+2}{2\alpha}+\frac{m}{\alpha}+2\}},\\ \|\Lambda^{m+\alpha}u\|_2^2\|\Lambda^{\frac{n+2-2\alpha}{2}}u\|_2^2 &\le C(1+t)^{-\min\{\frac{2r^*+n}{\alpha}+\frac{m}{\alpha}+\frac{n+2-4\alpha}{2\alpha}+2;\frac{n+2}{\alpha}+\frac{m}{\alpha}+\frac{n+2-4\alpha}{2\alpha}+2\}}, \end{aligned}$$

and

$$\|\Lambda^{\frac{n-2\alpha}{2}}u\|_2^2\|\Lambda^{m+\alpha+1}u\|_2^2 \le C(1+t)^{-\min\{\frac{2r^*+n}{\alpha}+\frac{m}{\alpha}+\frac{n+2-4\alpha}{2\alpha}+2;\frac{n+2}{\alpha}+\frac{m}{\alpha}+\frac{n+2-4\alpha}{2\alpha}+2\}}.$$

Combining the above estimates, we conclude that

$$\|\Lambda^{m}u\|_2^2 \le C(1+t)^{-\min\{\frac{r^*+m}{\alpha}+\frac{n}{2\alpha}+2;\frac{n+2}{2\alpha}+\frac{m}{\alpha}+2\}}.$$

Now, let us assume that the statement of Theorem 3.11 holds for every $j=1,2,\dots,k$, with $k\le K$. We will prove the theorem for $j=k+1$. Let m be non-negative such that $m+(k+1)\max\{2\alpha,\alpha+1\}\le s$. Recall the following estimate from the proof of Theorem B.3:

$$\begin{aligned} \|\partial_t^{k+1}\Lambda^m u\|_2^2 &\le C\|\partial_t^k\Lambda^{m+2\alpha}u\|_2^2 + C\|\partial_t^k\Lambda^m(u\cdot\nabla u)\|_2^2\\ &\le C\|\partial_t^k\Lambda^{m+2\alpha}u\|_2^2 + C\sum_{i=0}^{k} 2\|\partial_t^{k-i}\Lambda^{m+\alpha}u\|_2^2\|\partial_t^i\Lambda^{\frac{n+2-2\alpha}{2}}u\|_2^2\\ &\quad + \|\partial_t^{k-i}\Lambda^{\frac{n-2\alpha}{2}}u\|_2^2\|\partial_t^i\Lambda^{m+\alpha+1}u\|_2^2. \end{aligned}$$

Thanks to (B.2) and

$$\begin{aligned} m+2\alpha+k\max\{2\alpha,\alpha+1\} &\le m+(k+1)\max\{2\alpha,\alpha+1\}\le s,\\ m+\alpha+k\max\{2\alpha,\alpha+1\} &\le m+(k+1)\max\{2\alpha,\alpha+1\}\le s,\\ m+\alpha+1+k\max\{2\alpha,\alpha+1\} &\le m+(k+1)\max\{2\alpha,\alpha+1\}\le s, \end{aligned}$$

from the inductive assumption, we have

$$\begin{aligned} &\|\partial_t^k\Lambda^{m+2\alpha}u(t)\|_2^2 \le C(1+t)^{-\min\{\frac{r^*+m}{\alpha}+\frac{n}{2\alpha}+2(k+1);\frac{n+2}{2\alpha}+\frac{m}{\alpha}+2(k+1)\}},\\ &\|\partial_t^{k-i}\Lambda^{m+\alpha}u\|_2^2\|\partial_t^i\Lambda^{\frac{n+2-2\alpha}{2}}u\|_2^2\\ &\qquad \le C(1+t)^{-\min\{\frac{2r^*+n}{\alpha}+\frac{m}{\alpha}+\frac{n+2-4\alpha}{2\alpha}+2(k+1);\frac{n+2}{\alpha}+\frac{m}{\alpha}+\frac{n+2-4\alpha}{2\alpha}+2(k+1)\}}, \end{aligned}$$

and

$$\begin{aligned}&\|\partial_t^{k-i}\Lambda^{\frac{n-2\alpha}{2}}u\|_2^2\|\partial_t^i\Lambda^{m+\alpha+1}u\|_2^2\\&\quad\le C(1+t)^{-\min\{\frac{2r^*+n}{\alpha}+\frac{m}{\alpha}+\frac{n+2-4\alpha}{2\alpha}+2(k+1);\frac{n+2}{\alpha}+\frac{m}{\alpha}+\frac{n+2-4\alpha}{2\alpha}+2(k+1)\}}.\end{aligned}$$

Combining the above estimates, we conclude that

$$\|\partial_t^{k+1}\Lambda^m u\|_2^2\le C(1+t)^{-\min\{\frac{r^*+m}{\alpha}+\frac{n}{2\alpha}+2(k+1);\frac{n+2}{2\alpha}+\frac{m}{\alpha}+2(k+1)\}}. \qquad \blacksquare$$

From the above theorem, we deduce the following consequences related to the estimates of the x-derivatives of the solution when the initial data are chosen from special spaces.

Corollary 3.12. *Let $n=3$ and $0<\alpha<5/4$. Assume that $u_0\in H^s_\sigma\cap L^p$ with $s>\frac{5}{2}-\alpha$ and $0\le m\le s$, $p\in[1,2)$. Assume further that u_0 has decay character. Then, the solution u to system (1.1) constructed in Theorem B.2 satisfies for all $t>0$ the following estimate for its derivatives:*

$$\|D^m u(t)\|_2^2\le C(1+t)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)+\frac{m}{\alpha}}.$$

Proof. Let $k=0$ in (3.8). From Theorem 3.11 with $k=0$ in (3.8) and Proposition 2.18 with $p\in[1,2)$, it follows that

$$\|D^m u(t)\|_2^2\le C(1+t)^{-\min\{\frac{3}{2\alpha}(\frac{2}{p}-1)+\frac{m}{\alpha};\frac{n+2}{2\alpha}+\frac{m}{\alpha}\}}\le C(1+t)^{-\frac{3}{2\alpha}(\frac{2}{p}-1)+\frac{m}{\alpha}}. \qquad \blacksquare$$

Remark 3.13. One could see that the decay rate obtained in Corollary 3.12 is exactly the same as in [16] without any further assumptions on p as Jiu required in [16].

Corollary 3.14. *Let $n=2,3,4$, and $0<\alpha<\frac{n+2}{4}$. Assume that $u_0\in H^s_\sigma\cap\dot H^{-l}$ $(l>0)$, and $0\le m\le s$. Assume further that u_0 has decay character. Then, the solution u to system (1.1) constructed in Theorem B.2 satisfies for all $t>0$ the following estimate for its x-derivatives:*

$$\|D^m u(t)\|_2\le C(1+t)^{-\frac{m+l}{2\alpha}}.$$

Proof. If $u_0\in L^2\cap\dot H^{-l}$, $(l>0)$, then, by Proposition 2.17, we have $r^*(u_0)=l-\frac{n}{2}$. Hence, Theorem 3.11 implies that

$$\|D^m u(t)\|_2\le C(1+t)^{-\min\{\frac{m+l}{2\alpha};\frac{n+2}{4\alpha}+\frac{m}{2\alpha}\}}\le C(1+t)^{-\frac{m+l}{2\alpha}}.$$

The same decay rate was also obtained in [9]. $\blacksquare$

A. Some auxiliary inequalities

In this appendix, we recall some auxiliary inequalities.

Lemma A.1 ([26]). *Assume that $a, b \geq 0$. Then, the following estimate is valid:*

$$\int_{\mathbb{R}} e^{c|\xi|^a t} |\xi|^b d\xi \leq C t^{-\frac{n+b}{a}}.$$

Lemma A.2 (The Kato–Ponce inequality [9]). *Let $s > 0$. Assume that $1 < r < \infty$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$, with $q_1, p_2 \in (1, \infty)$ and $p_1, q_2 \in [1, \infty]$. Then,*

$$\|[\Lambda^s, f \cdot \nabla]g\|_r \leq C(\|\nabla f\|_{p_1} \|\Lambda^s g\|_{q_1} + \|\Lambda^s f\|_{p_2} \|\nabla g\|_{q_2}) \tag{A.1}$$

and

$$\|\Lambda^s(fg)\|_r \leq C(\|g\|_{p_1} \|\Lambda^s f\|_{q_1} + \|\Lambda^s g\|_{p_2} \|f\|_{q_2}), \tag{A.2}$$

where $[\Lambda^s, f \cdot \nabla]g := \Lambda^s(f \cdot \nabla g) - f \cdot \nabla \Lambda^s g$, and C's are positive constants depending on the indices s, r, p_1, q_1, p_2, and q_2.

Lemma A.3 (Gronwall's inequality in integral form). *Let I be an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $0 \leq a < b$. Let f, g, and u be real-valued functions defined on I. Assume further that g and u are continuous and the negative part of f is integrable on every closed and bounded subinterval of I. If g is non-negative and if u satisfies the integral inequality*

$$u(t) \leq f(t) + \int_a^t g(s)u(s)ds \quad \forall t \in I,$$

then

$$u(t) \leq f(t) + \int_a^t f(s)g(s) \exp\left\{\int_s^t g(r)dr\right\} ds, \quad t \in I.$$

Lemma A.4 (Fractional Gagliardo–Nirenberg inequality [14]). *Let $1 < r, r_0, r_1 < \infty, \sigma > 0$ and $s \in [0, \sigma)$. Then, the following inequality holds:*

$$\|\Lambda^s v\|_r \leq C \|v\|_{r_0}^{1-\theta} \|\Lambda^\sigma v\|_{r_1}^\theta, \tag{A.3}$$

where $\theta = \dfrac{\frac{1}{r_0} - \frac{1}{r} + \frac{s}{n}}{\frac{1}{r_0} - \frac{1}{r_1} + \frac{\sigma}{n}}$ and $\frac{s}{\sigma} \leq \theta \leq 1$.

Let the Riesz potential I_β be defined as follows: $\widehat{I_\beta f}(\xi) := \frac{\hat{f}(\xi)}{|\xi|^\beta}$. We formulate below the Sobolev lemma, known also as the Hardy–Littlewood–Sobolev fractional integration theorem.

Lemma A.5 (Aubin and Stein [34]). *If $1 < p < q < \infty$ satisfying*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{n}, \tag{A.4}$$

then

$$\|I_s v\|_{L^q} \leq C \|v\|_{L^p}, \tag{A.5}$$

with a positive constant $C = C(n, p, q, s)$.

For $p = 1$, we have the following alternative.

Lemma A.6 (Schikorra, Spector, and Van Schaftingen [28]). *For $m, s > 0$ such that*

$$\frac{1}{m} = 1 - \frac{s}{n}, \tag{A.6}$$

the estimate

$$\|I_s v\|_{L^m} \leq C\|Rv\|_{L^1}$$

holds, with a positive constant $C = C(n, m, s)$.

B. Well-posedness, regularity, and generalized energy inequality

In this appendix, we recall the proofs of the existence and regularity of the solution u to (1.1) from [9, 10, 16] with some necessary modifications. Moreover, we also estimate the higher order t-derivatives of u in L^2. Next, the generalized energy inequality is formulated. To begin, let us define the weak solution to problem (1.1).

Definition B.1. For $T > 0$, the function u is called a *weak solution* to system (1.1) on the interval $(0, T)$ if it satisfies the following:

- $u \in L^\infty([0, T]; L^2_\sigma) \cap L^2([0, T]; H^\alpha_\sigma)$;
- for any $0 \leq s < t \leq T$, the identity

$$\begin{aligned}\int_s^t \big\{ -\langle u(\tau), \phi_t(\tau)\rangle + \langle u(\tau)\cdot\nabla u(\tau), \phi(\tau)\rangle + \langle \Lambda^\alpha u(\tau), \Lambda^\alpha \phi(\tau)\rangle\big\} d\tau \\ = -\langle u(t), \phi(t)\rangle + \langle u(s), \phi(s)\rangle\end{aligned}$$

  is valid for all test function $\phi \in C^1([0, T]; C^\infty_{0,\sigma})$,
- the following strong energy inequality:

$$\|u(t)\|_2^2 + \int_s^t \|\Lambda^\alpha u(\tau)\|_2^2 d\tau \leq \|u(s)\|_2^2$$

  is valid for almost all $s \in [0, T]$, and for all $t \in [s, T]$.

Theorem B.2. *Assume that $0 < \alpha < \frac{n+2}{4}$, and let $u_0 \in L^2_\sigma$. Then, there exists a weak solution $u \in L^\infty([0, T]; L^2_\sigma) \cap L^2([0, T]; H^\alpha_\sigma)$ to system (1.1) for all $T > 0$. Moreover, if $u_0 \in \dot H_\sigma^{\frac{n+2-4\alpha}{2}}$ satisfies $\|u_0\|_{\dot H_\sigma^{\frac{n+2-4\alpha}{2}}} \leq \varepsilon$ for some $\varepsilon > 0$ small enough and independent of u_0, then the solution u is unique for all $t > 0$ and belongs to $L^\infty([0, T]; \dot H_\sigma^{\frac{n+2-4\alpha}{2}}) \cap L^2([0, T]; \dot H_\sigma^{\frac{n+2-2\alpha}{2}})$ for all $T > 0$.*

Proof. Following the same strategy as in [16, Theorem A.1] and [10, Theorem 1], one can obtain a similar result in the n-dimensional case. ∎

The following theorem is devoted to the regularity of solution u in inhomogeneous Sobolev spaces and also to the bounding of higher order derivatives with respect to t in L^2 space.

Theorem B.3. *Let $0 < \alpha < \frac{n+2}{4}$. Suppose that $u_0 \in H^s_\sigma$ with $s > \frac{n+2-2\alpha}{2}$. Then, there exists a positive constant C_0 such that for all $\epsilon \in (0, C_0)$, for all data u_0 with $\|u_0\|_{H^s} < \epsilon$, system (1.1) has a unique global solution $u \in L^\infty([0,T]; H^s_\sigma) \cap L^2([0,T]; H^{s+\alpha}_\sigma)$ for all $T > 0$. In addition, the following inequality is valid:*

$$\frac{d}{dt}\|u(t)\|^2_{H^s} \le -C\|\Lambda^\alpha u(t)\|^2_{H^s}. \tag{B.1}$$

Moreover, let

$$K := \Big[\frac{2s-(n+2-2\alpha)}{2\max\{2\alpha, 1+\alpha\}}\Big]. \tag{B.2}$$

Then, for every non-negative integer $k \le K+1$ and non-negative m satisfying $m + k\max\{2\alpha, \alpha+1\} \le s$, we have

$$\|\partial_t^k \Lambda^m u\|_2^2 \le C(n,\epsilon) \quad \text{and} \quad \int_0^t \|\partial_t^k \Lambda^{m+\alpha} u(\tau)\|_2^2 d\tau \le C(n,\epsilon) \quad \text{for all } t \ge 0. \tag{B.3}$$

Proof. The first two statements have been proved in [9, Theorem 1.6]. It remains to show inequality (B.3).

By the same strategy as in [2], we prove (B.3) by induction in k. Note that the following estimates are carried out for the approximated solutions obtained, let us say, by Galerkin's method. By the limiting process and by the uniqueness, we obtain the same bounds for the exact solution. Let $k = 1$ and m be arbitrarily non-negative such that $m + \max\{2\alpha, \alpha+1\} \le s$. Applying Λ^m to (1.1) yields

$$\begin{aligned}
\|\partial_t \Lambda^m u\|_2^2 &\le C\|\Lambda^{m+2\alpha} u\|_2^2 + C\|\Lambda^m(u\cdot\nabla u)\|_2^2 \\
&\le C\|\Lambda^{m+2\alpha} u\|_2^2 + C\|[\Lambda^m, u\cdot\nabla]u\|_2^2 + C\|u\cdot\nabla\Lambda^m u\|_2^2 \\
&\le C\|\Lambda^{m+2\alpha} u\|_2^2 + C\|\Lambda^m u\|^2_{\frac{2n}{n-2\alpha}}\|\nabla u\|^2_{\frac{n}{\alpha}} + C\|u\|^2_{\frac{n}{\alpha}}\|\nabla\Lambda^m u\|^2_{\frac{n}{n-2\alpha}} \\
&\le C\|\Lambda^{m+2\alpha} u\|_2^2 + C\|\Lambda^{m+\alpha} u\|_2^2\|\Lambda^{\frac{n+2-2\alpha}{2}} u\|_2^2 + C\|\Lambda^{\frac{n-2\alpha}{2}} u\|_2^2\|\Lambda^{m+\alpha+1} u\|_2^2 \\
&\le C\|u\|^2_{H^s} + C\|u\|^4_{H^s}.
\end{aligned}$$

By (B.1), we deduce that

$$\|u(t)\|_{H^s} \le \|u_0\|_{H^s} \le \epsilon.$$

Therefore,

$$\|\partial_t \Lambda^m u\|_2^2 \le C(n,\epsilon).$$

Assume that the following estimates hold for all $j = 1, \dots, k$ with $k \le K$:

$$\|\partial_t^j \Lambda^m u\|_2^2 \le C(n,\epsilon) \quad \text{for any non-negative } m \text{ such that } m + j\max\{2\alpha, \alpha+1\} \le s. \tag{B.4}$$

We show that (B.4) is also true for $k+1$. Indeed, let m be any non-negative real number satisfying $m+(k+1)\max\{2\alpha,\alpha+1\}\le s$. Applying $\partial_t^k\Lambda^m$ to solution u of (1.1), we have

$$\begin{aligned}\|\partial_t^{k+1}\Lambda^m u\|_2^2 &\le C\|\partial_t^k\Lambda^{m+2\alpha}u\|_2^2 + C\|\partial_t^k\Lambda^m(u\cdot\nabla u)\|_2^2\\ &\le C(n,\epsilon)+C\sum_{i=0}^{k}\|[\Lambda^m,\partial_t^{k-i}u\cdot\nabla]\partial_t^i u\|_2^2+\|(\partial_t^{k-i}u\cdot\nabla\partial_t^i\Lambda^m u)\|_2^2.\end{aligned}$$

We estimate the sum in the above inequality

$$\begin{aligned}&C\sum_{i=0}^{k}\|[\Lambda^m,\partial_t^{k-i}u\cdot\nabla]\partial_t^i u\|_2^2+\|(\partial_t^{k-i}u\cdot\nabla\partial_t^i\Lambda^m u)\|_2^2\\ &\le C\sum_{i=0}^{k}\|\partial_t^{k-i}\Lambda^m u\|_{\frac{2n}{n-2\alpha}}^2\|\partial_t^i\nabla u\|_{\frac{n}{\alpha}}^2\\ &\quad+\|\partial_t^i\Lambda^m u\|_{\frac{2n}{n-2\alpha}}^2\|\partial_t^{k-i}\nabla u\|_{\frac{n}{\alpha}}^2+\|\partial_t^{k-i}u\|_{\frac{n}{\alpha}}^2\|\nabla\partial_t^i\Lambda^m u\|_{\frac{n}{n-2\alpha}}^2\\ &\le C\sum_{i=0}^{k}2\|\partial_t^{k-i}\Lambda^{m+\alpha}u\|_2^2\|\partial_t^i\Lambda^{\frac{n+2-2\alpha}{2}}u\|_2^2+\|\partial_t^{k-i}\Lambda^{\frac{n-2\alpha}{2}}u\|_2^2\|\partial_t^i\Lambda^{m+\alpha+1}u\|_2^2.\end{aligned}$$

From (B.2) and (B.4), we obtain $\|\partial_t^{k+1}\Lambda^m u\|_2^2\le C(n,\epsilon)$.

On the other hand, by a similar argument, for $m+k\max\{2\alpha,\alpha+1\}<s$, with non-negative m and non-negative integer $k\le K+1$, we obtain

$$\|\partial_t^k\Lambda^{m+\alpha}u\|_2^2\le C\|\Lambda^\alpha u\|_{H^s}^2+C\|u\|_{H^s}^2\|\Lambda^\alpha u\|_{H^s}^2.$$

Also, from (B.1), we deduce that $\int_0^t\|\Lambda^\alpha u(\tau)\|_{H^s}^2d\tau<\epsilon^2$. Therefore,

$$\int_0^t\|\partial_t^k\Lambda^{m+\alpha}u(\tau)\|_2^2d\tau\le C(n,\epsilon)\quad\text{for all } t\ge 0,$$

which finishes the proof. ∎

The generalized energy inequalities presented in the following proposition play a crucial role in proving the non-uniform decay of solution u to (1.1).

Proposition B.4. *Let u be the solution to (1.1) obtained from Theorem B.2. Let $E\in C^1([0,\infty))$ be non-negative. Then, for $s=0$ and almost $s>0$ for all $t\ge s$, we have*

$$\begin{aligned}&E(t)\|\psi(t)*u(t)\|_2^2\\ &=E(s)\|\psi(s)*u(s)\|_2^2+\int_s^t E'(\tau)\|\psi(\tau)*u(\tau)\|_2^2d\tau\\ &\quad+2\int_s^t E(\tau)\langle\psi'(\tau)*u(\tau),\psi(\tau)*u(\tau)\rangle d\tau-2\int_s^t E(\tau)\|\Lambda^\alpha\psi(\tau)*u(\tau)\|_2^2d\tau\\ &\quad-2\int_s^t E(\tau)\langle u(\tau)\cdot\nabla u(\tau),\psi(\tau)*\psi(\tau)*u(\tau)\rangle d\tau\end{aligned}$$

for all $\psi \in C^1([0,t]; C^1 \cap L^2)$*, and*

$$\begin{aligned}
&E(t)\|\widetilde{\psi}(t)\hat{u}(t)\|_2^2 \\
&\quad = E(s)\|\widetilde{\psi}(s)\hat{u}(s)\|_2^2 + \int_s^t E'(\tau)\|\widetilde{\psi}(\tau)\hat{u}(\tau)\|_2^2 d\tau \\
&\qquad + 2\int_s^t E(\tau)\langle \widetilde{\psi}'(\tau)\hat{u}(\tau), \widetilde{\psi}(\tau)\overline{\hat{u}(\tau)}\rangle d\tau - 2\int_s^t E(\tau)\||\xi|^\alpha \widetilde{\psi}(\tau)\hat{u}(\tau)\|_2^2 d\tau \\
&\qquad - 2\int_s^t E(\tau)\langle \mathcal{F}[u(\tau)\cdot\nabla u(\tau)], \widetilde{\psi}^2(\tau)\overline{\hat{u}(\tau)}\rangle d\tau
\end{aligned}$$

for all $\widetilde{\psi} \in C^1([0,\infty); L^\infty)$.

Proof. The proof follows exactly the same arguments as in [2]. Note that the only differences are

$$\int_{\mathbb{R}^n} (-\Delta)^\alpha u E(t)\psi * \psi * u dx = E(t)\int_{\mathbb{R}^n} |\Lambda^\alpha(\psi * u)|^2 dx = E(t)\|\Lambda^\alpha \psi * u\|_2^2$$

and

$$\int_{\mathbb{R}^n} |\xi|^{2\alpha}\hat{u}(\xi)E(t)\widetilde{\psi}^2\bar{\hat{u}}(\xi)d\xi = E(t) = E(t)\||\xi|^\alpha \widetilde{\psi}\hat{u}\|_2^2. \qquad \blacksquare$$

Acknowledgments. The authors wish to thank Vietnam Institute for Advanced Study in Mathematics (VIASM) for the invitation and hospitality during their stay at the Institute in October–November 2024. The authors also would like to thank Hanoi National University of Education for providing a fruitful working environment. The authors express their gratitude to the anonymous referees for their helpful and valuable comments that improved the quality of the paper.

Funding. This research was funded by the Ministry of Education and Training of Vietnam (Project no. B2025-SPH-01).

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Received 26 November 2024; revised 10 February 2026.

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