Z. Anal. Anwend. 45 (2026), 465–484
DOI 10.4171/ZAA/1818

Bounds for generalizations of Steffensen's inequality using Taylor's formula

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Abstract. In this paper, we generalize the identities related to Steffensen's inequality using the Taylor formula by introducing a positive weight. For these generalized identities, we obtain bounds by applying the weighted Hermite–Hadamard inequality for a convex or concave function of the form $|f^{(n)}|^q$. Furthermore, we generalize known weighted Hermite–Hadamard-type inequalities by using new identities associated with Steffensen's inequality.

1. Introduction

In recent decades, considerable attention has been devoted to the theory of convexity due to its practical applications across various fields in both pure and applied sciences. The theory of convexity is described in detail in many books such as [6, 12, 20, 23]. The theory of convex functions is closely related to the theory of inequalities, and one of the most well-known inequalities showing this connection is the Hermite–Hadamard inequality. It is noteworthy that this inequality was independently discovered by Hermite [7] and Hadamard [5]. For a long time, this inequality was attributed only to Hadamard because Hermite's result was not acknowledged in the history and theory of convex functions. Even Fejér [3] mentioned only Hadamard when obtaining the generalization of this inequality. Today, the inequality is commonly known as the Hermite–Hadamard inequality, and for a convex function $f : [a,b] \to \mathbb{R}$, it states

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

A generalization due to Fejér [3] for a convex function f and a positive, integrable function g, symmetric to $\frac{a+b}{2}$ in the interval (a,b), states

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx.$$

In this paper, we use the weighted Hermite–Hadamard inequality for convex functions stated in the following theorem (see [15, 20, 28]).

Mathematics Subject Classification 2020: 26D15 (primary); 26A51 (secondary).
Keywords: Steffensen's inequality, Taylor's formula, convex function, the weighted Hermite–Hadamard inequality.

Theorem 1.1. *Let $w : [a,b] \to \mathbb{R}$ be a non-negative function. If $f : [a,b] \to \mathbb{R}$ is a convex function, then the following inequality holds:*

$$W(b)f(m) \leq \int_a^b w(x)f(x)dx \leq W(b)\left[\frac{b-m}{b-a}f(a) + \frac{m-a}{b-a}f(b)\right], \tag{1.1}$$

where

$$W(t) = \int_a^t w(x)dx \quad \text{and} \quad m = \frac{1}{W(b)}\int_a^b w(x)x\,dx.$$

If $f : [a,b] \to \mathbb{R}$ is a concave function, the inequalities in (1.1) are reversed.

In some sources, inequality (1.1) is referred to as Hermite–Hadamard–Fejér inequality, due to its historical development and weighted nature.

Steffensen's inequality has intrigued many mathematicians who have generalized and refined it in numerous papers, not only using the classical Riemann integral but also fractional integrals [10, 14, 27], time scales [1, 2, 13, 25] and quantum calculus [4, 22, 24]. Furthermore, by using various interpolation polynomials many identities associated with Steffensen's inequality have been obtained in the last decade; for example, see [16, 17]. A detailed overview can be found in [8]. Most of the advances on Steffensen's inequality since its appearance in [26] have been collected in [19]. For a non-increasing function f on $[a,b]$, a function g satisfying $0 \leq g \leq 1$ on $[a,b]$ and $\lambda = \int_a^b g(t)dt$, the Steffensen inequality states

$$\int_{b-\lambda}^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt.$$

The aim of this paper is to generalize the identities related to Steffensen's inequality via Taylor's formula as obtained in [17], by adding a positive weight. For the resulting identities, we establish bounds using the weighted Hermite–Hadamard inequality for convex or concave functions of the form $|f^{(n)}|^q$. We conclude the paper with weighted generalizations of results from [18]. In the final section, we present results concerning higher-order convex functions. For the sake of clarity and readability, we begin by recalling their definition and main properties.

The definition of n-convex functions is characterized by nth order divided differences. The nth order divided difference of a function $f : [a,b] \to \mathbb{R}$ at mutually distinct points $x_0, \ldots, x_n \in [a,b]$ is defined recursively by

$$f[x_i] = f(x_i), \quad (i = 0, \ldots, n)$$

and

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0, \ldots, x_n]$ is independent of the order of the points $x_0, \ldots, x_n$.

The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x,\ldots,x}_{j\text{-times}}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \tag{1.2}$$

The notion of n-convexity goes back to Popoviciu [21]. We follow the definition given by Karlin [9].

Definition 1.1. A function $f : [a,b] \to \mathbb{R}$ is said to be n-convex on $[a,b]$, $n \geq 0$, if for all choices of $(n+1)$ distinct points in $[a,b]$, nth order divided difference of f satisfies

$$f[x_0,\ldots,x_n] \geq 0.$$

Note that 1-convex functions are non-decreasing functions. Certain differentiability properties hold for n-convex functions. If f is n-convex on $[a,b]$ for $n \geq 2$ then f^k exists and is $(n-k)$-convex function for $1 \leq k \leq n-2$. Moreover, the right-hand derivative f_+^{n-1} exists, is right-continuous and increasing on $[a,b)$. An n-convex function f need not be n-times differentiable; however, if $f^{(n)}$ exists, then f is n-convex iff $f^{(n)} \geq 0$.

2. Identities obtained by Taylor's formula

Let n be a positive integer and $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then for $c, x \in [a,b]$, the following Taylor formula around the point c holds:

$$f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(c)}{i!}(x-c)^i + \frac{1}{(n-1)!}\int_c^x f^{(n)}(t)(x-t)^{n-1}dt.$$

In this paper, we use Taylor's formula around a point a given by

$$f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i + \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}dt \tag{2.1}$$

and Taylor's formula around a point b given by

$$f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(b)}{i!}(x-b)^i - \frac{1}{(n-1)!}\int_x^b f^{(n)}(t)(x-t)^{n-1}dt. \tag{2.2}$$

By applying Taylor's formulas around the points a and b, we obtain the following identities related to weighted generalizations of Steffensen's inequality.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_1 be defined by*

$$G_1(x) = \begin{cases} \int_a^x (1-g(t))p(t)dt, & x \in [a, a+\lambda], \\ \int_x^b g(t)p(t)dt, & x \in [a+\lambda, b]. \end{cases} \tag{2.3}$$

Then, the following identities hold:

$$\begin{aligned} &\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_1(x)(x-a)^i dx \\ &\quad = -\frac{1}{(n-2)!} \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) f^{(n)}(t)dt \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i dx \\ &\quad = \frac{1}{(n-2)!} \int_a^b \left(\int_a^t G_1(x)(x-t)^{n-2} dx \right) f^{(n)}(t)dt. \end{aligned} \tag{2.5}$$

Proof. Similarly, as in the proof of [17, Theorem 2.1] using the Mitrinović identity given in [11] with the positive weight p and applying the integration by parts, we obtain

$$\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt = -\int_a^b G_1(x)f'(x)dx.$$

Further, we apply Taylor's formula at the point a, given by (2.1), to the function f', replacing n with $n-1$, to obtain

$$\begin{aligned} \int_a^b G_1(x)f'(x)dx &= \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_1(x)(x-a)^i dx \\ &\quad + \frac{1}{(n-2)!} \int_a^b G_1(x) \left(\int_a^x f^{(n)}(t)(x-t)^{n-2} dt \right) dx. \end{aligned} \tag{2.6}$$

To obtain the desired identity (2.4), we apply Fubini's theorem to the last integral in (2.6). In a similar way, we obtain the identity (2.5). ■

Remark 2.1. Taking $p \equiv 1$ in Theorem 2.1, we obtain the result given in [17, Theorem 2.1].

Theorem 2.2. *Let us assume that $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that*

p is positive and $0 \le g \le 1$. Let $\int_{b-\lambda}^{b} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_2 be defined by

$$G_2(x) = \begin{cases} \int_a^x g(t)p(t)dt, & x \in [a, b-\lambda], \\ \int_x^b (1-g(t))p(t)dt, & x \in [b-\lambda, b]. \end{cases} \tag{2.7}$$

Then, the following identities hold:

$$\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_2(x)(x-a)^i dx$$
$$= -\frac{1}{(n-2)!} \int_a^b \left(\int_t^b G_2(x)(x-t)^{n-2} dx \right) f^{(n)}(t)dt \tag{2.8}$$

and

$$\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_2(x)(x-b)^i dx$$
$$= \frac{1}{(n-2)!} \int_a^b \left(\int_a^t G_2(x)(x-t)^{n-2} dx \right) f^{(n)}(t)dt. \tag{2.9}$$

Proof. The proof is similar to the proof of [17, Theorem 2.2] except that we use the Mitrinović identity [11] and the function G_2 with the positive weight p. ∎

Remark 2.2. Taking $p \equiv 1$ in Theorem 2.2, we obtain the result given in [17, Theorem 2.2].

3. Main results

In this section, we establish Hermite–Hadamard-type bounds for the identities obtained in Theorems 2.1 and 2.2. These identities were derived by applying Taylor's formula at the left endpoint a and the right endpoint b, respectively. Therefore, we divide the results into two subsections, depending on which boundary point the results pertain to. The results are established for a class of convex or concave functions of the form $|f^{(n)}|^q$.

3.1. Results for Taylor's formula at the left endpoint *a*

Our first result provides a weighted Hermite–Hadamard-type bound for identity (2.4) obtained by using a convex function of the form $|f^{(n)}|^q$. After each inequality, a numerical example is provided to illustrate and confirm the validity of the result.

Theorem 3.1. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n \ge 2$. Let $g, p : [a, b] \to \mathbb{R}$ be integrable functions such that p is positive* *and* $0 \leq g \leq 1$. *Let* $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ *and let the function* G_1 *be defined by* (2.3). *If* $|f^{(n)}|^q$ *is convex for* $n \geq 2$ *and* $q \geq 1$, *the following inequality holds:*

$$(n-2)!\left|\int_a^b f(t)g(t)p(t)dt - \int_a^{a+\lambda} f(t)p(t)dt - \sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i dx\right|$$

$$\leq W_1(b)\left(\frac{b-m_1}{b-a}|f^{(n)}(a)|^q + \frac{m_1-a}{b-a}|f^{(n)}(b)|^q\right)^{\frac{1}{q}}, \tag{3.1}$$

where

$$W_1(b) = \frac{1}{n-1}\int_a^b G_1(x)(x-a)^{n-1}dx \tag{3.2}$$

and

$$m_1 = \frac{1}{(n-1)W_1(b)}\int_a^b G_1(x)\left(a\cdot(x-a)^{n-1} + \frac{(x-a)^n}{n}\right)dx. \tag{3.3}$$

Proof. Let us begin the proof by defining the function w_1 on $[a,b]$ as follows:

$$w_1(t) = \int_t^b G_1(x)(x-t)^{n-2}dx, \tag{3.4}$$

where the function G_1 is defined by (2.3). Since the condition $0 \leq g \leq 1$ is satisfied and the function p is positive, the function G_1 is non-negative. Therefore, the function w_1 is non-negative on $[a,b]$ for every $n \geq 2$.

Multiplying identity (2.4) by -1, taking the absolute value, and then applying Hölder inequality with $q \geq 1$ and using the fact that the function w_1 is non-negative, we arrive at

$$\left|\int_a^b f(t)g(t)p(t)dt - \int_a^{a+\lambda} f(t)p(t)dt - \sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i dx\right|$$

$$= \left|\frac{1}{(n-2)!}\int_a^b\left(\int_t^b G_1(x)(x-t)^{n-2}dx\right)f^{(n)}(t)dt\right|$$

$$= \left|\frac{1}{(n-2)!}\int_a^b w_1(t)f^{(n)}(t)dt\right|$$

$$\leq \frac{1}{(n-2)!}\int_a^b w_1(t)|f^{(n)}(t)|dt$$

$$\leq \frac{1}{(n-2)!}\left(\int_a^b w_1(t)dt\right)^{1-\frac{1}{q}}\left(\int_a^b w_1(t)|f^{(n)}(t)|^q dt\right)^{\frac{1}{q}}. \tag{3.5}$$

Since the function $|f^{(n)}|^q$ is convex and w_1 is a non-negative function, we can apply the right-hand side weighted Hermite–Hadamard inequality given by (1.1), which yields the inequality

$$\int_a^b w_1(t)|f^{(n)}(t)|^q dt \leq W_1(b)\cdot\left(\frac{b-m_1}{b-a}|f^{(n)}(a)|^q + \frac{m_1-a}{b-a}|f^{(n)}(b)|^q\right), \tag{3.6}$$

where

$$W_1(b) = \int_a^b w_1(t)dt = \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2}dx\right)dt$$
$$= \int_a^b G_1(x)\left(\int_a^x (x-t)^{n-2}dt\right)dx = \int_a^b G_1(x)\frac{(x-a)^{n-1}}{n-1}dx$$

and

$$m_1 = \frac{1}{W_1(b)}\int_a^b w_1(t)t\,dt = \frac{1}{W_1(b)}\int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2}dx\right)t\,dt$$
$$= \frac{1}{W_1(b)}\int_a^b G_1(x)\left(\int_a^x (x-t)^{n-2}\cdot t\,dt\right)dx$$
$$= \frac{1}{W_1(b)}\int_a^b G_1(x)\left(t\cdot\frac{-(x-t)^{n-1}}{n-1}\Big|_a^x + \int_a^x \frac{(x-t)^{n-1}}{n-1}dt\right)dx$$
$$= \frac{1}{W_1(b)}\int_a^b G_1(x)\left(\frac{a\cdot(x-a)^{n-1}}{n-1} + \frac{(x-a)^n}{(n-1)\cdot n}\right)dx.$$

Clearly, by combining relations (3.5) and (3.6), we obtain

$$\left|\int_a^b f(t)g(t)p(t)dt - \int_a^{a+\lambda} f(t)p(t)dt - \sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i dx\right|$$
$$\le \frac{1}{(n-2)!}\left(\int_a^b w_1(t)dt\right)^{1-\frac{1}{q}}\left(\int_a^b w_1(t)|f^{(n)}(t)|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{1}{(n-2)!}\big(W_1(b)\big)^{1-\frac{1}{q}}\left(\int_a^b w_1(t)|f^{(n)}(t)|^q dt\right)^{\frac{1}{q}}$$
$$\le \frac{1}{(n-2)!}\big(W_1(b)\big)^{1-\frac{1}{q}}\left(W_1(b)\cdot\left(\frac{b-m_1}{b-a}|f^{(n)}(a)|^q + \frac{m_1-a}{b-a}|f^{(n)}(b)|^q\right)\right)^{\frac{1}{q}}$$
$$= \frac{1}{(n-2)!}W_1(b)\left(\frac{b-m_1}{b-a}|f^{(n)}(a)|^q + \frac{m_1-a}{b-a}|f^{(n)}(b)|^q\right)^{\frac{1}{q}},$$

which proves our assertion. The proof is complete. ■

Example 3.1. Let us set $a = 0$, $b = 1$, $f(t) = e^t$, $p(t) = 1+t$, $g(t) = t$, $n = 3$ and $q = 1$. Then, the conditions of Theorem 3.1 are satisfied, and we have

$$\left|\int_0^1 e^t(t+t^2)dt - \int_0^\lambda te^t dt - \int_0^1 G_1(x)dx - \int_0^1 G_1(x)xdx\right| = 0.0650933 \tag{3.7}$$

and

$$W_1(b)\big(1 - m_1 + m_1 e\big) = 0.0709719. \tag{3.8}$$

It is evident that $0.0650933 \le 0.0709719$, which confirms the inequality stated in Theorem 3.1.

The following result is, in some sense, complementary to Theorem 3.1, as it concerns a class of concave functions of the form $|f^{(n)}|^q$.

Theorem 3.2. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$. Let $g,p:[a,b]\to\mathbb{R}$ be integrable functions such that p is positive and $0\leq g\leq 1$. Let $\int_a^{a+\lambda}p(t)dt=\int_a^b g(t)p(t)dt$ and let the function G_1 be defined by (2.3). If $|f^{(n)}|^q$ is concave for $n\geq 2$ and $q\geq 1$, then the following inequality holds:*

$$(n-2)!\left|\int_a^b f(t)g(t)p(t)dt-\int_a^{a+\lambda}f(t)p(t)dt-\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i dx\right|$$
$$\leq W_1(b)|f^{(n)}(m_1)|, \tag{3.9}$$

where $W_1(b)$ and m_1 are given by (3.2) and (3.3), respectively.

Proof. Let us define the function w_1 by (3.4). As in the proof of Theorem 3.1, we have that w_1 is non-negative and that the inequalities in (3.5) hold.

Since the function $|f^{(n)}|^q$ is concave and w_1 is non-negative, we can apply the reverse left-hand side weighted Hermite–Hadamard inequality in (1.1) and obtain

$$W_1(b)|f^{(n)}(m_1)|^q\geq\int_a^b w_1(t)|f^{(n)}(t)|^q dt, \tag{3.10}$$

where $W_1(b)$ and m_1 are given by (3.2) and (3.3), respectively.

Combining the relations (3.5) and (3.10), we arrive at

$$\left|\int_a^b f(t)g(t)p(t)dt-\int_a^{a+\lambda}f(t)p(t)dt-\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i dx\right|$$
$$\leq\frac{1}{(n-2)!}\left(\int_a^b w_1(t)dt\right)^{1-\frac{1}{q}}\left(\int_a^b w_1(t)|f^{(n)}(t)|^q dt\right)^{\frac{1}{q}}$$
$$\leq\frac{1}{(n-2)!}\left(W_1(b)\right)^{1-\frac{1}{q}}\left(W_1(b)|f^{(n)}(m_1)|^q\right)^{\frac{1}{q}}$$
$$=\frac{1}{(n-2)!}W_1(b)|f^{(n)}(m_1)|,$$

which proves our assertion. The proof is complete. ∎

Example 3.2. Let us set $a=1$, $b=2$, $f(t)=t^{\frac{7}{2}}$, $p(t)=1+t$ and $g(t)=\frac{t}{2}$, $n=3$ and $q=1$. Then, the conditions of Theorem 3.2 are satisfied, and we have

$$\left|\int_1^2(t^{\frac{11}{2}}+t^{\frac{9}{2}})dt-\int_1^{1+\lambda}t^{\frac{7}{2}}(1+t)dt-\sum_{i=0}^{1}\frac{f^{(i+1)}(1)}{i!}\int_1^2 G_1(x)(x-1)^i dx\right|$$
$$=0.885589$$

and

$$W_1(b)\frac{105\sqrt{m_1}}{8} = 0.88786.$$

It is evident that $0.885589 \leq 0.88786$, which confirms the inequality stated in Theorem 3.2.

Now, our aim is to establish a Hermite–Hadamard-type bound for the identity (2.8) by using a convex function of the form $|f^{(n)}|^q$.

Theorem 3.3. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$ Let $g,p:[a,b]\to\mathbb{R}$ be integrable functions such that p is positive and $0\leq g\leq 1$. Let $\int_{b-\lambda}^{b}p(t)dt=\int_a^b g(t)p(t)dt$ and let the function G_2 be defined by (2.7). If $|f^{(n)}|^q$ is convex for $n\geq 2$ and $q\geq 1$ then the following inequality holds:*

$$(n-2)!\left|\int_{b-\lambda}^{b} f(t)p(t)dt-\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_2(x)(x-a)^i dx-\int_a^b f(t)g(t)p(t)dt\right|$$

$$\leq W_2(b)\left(\frac{b-m_2}{b-a}|f^{(n)}(a)|^q+\frac{m_2-a}{b-a}|f^{(n)}(b)|^q\right)^{\frac{1}{q}}, \tag{3.11}$$

where

$$W_2(b)=\frac{1}{n-1}\int_a^b G_2(x)(x-a)^{n-1}dx \tag{3.12}$$

and

$$m_2=\frac{1}{(n-1)W_2(b)}\int_a^b G_2(x)\left(a\cdot(x-a)^{n-1}+\frac{(x-a)^n}{n}\right)dx. \tag{3.13}$$

Proof. We follow the lines of the proof of Theorem 3.1, except that we use the identity (2.8). ∎

Example 3.3. Let $f(t)=e^t$ on $[0,1]$, $n=3$, and $q=1$. By applying (3.11) with $p(t)=1+t$ and $g(t)=t$, where $t\in[0,1]$, we obtain the estimate

$$0.0115411 \leq 0.0125495.$$

Next, we establish a result complementary to Theorem 3.3.

Theorem 3.4. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$. Let $g,p:[a,b]\to\mathbb{R}$ be integrable functions such that p is positive and $0\leq g\leq 1$. Let $\int_{b-\lambda}^{b}p(t)dt=\int_a^b g(t)p(t)dt$ and let the function G_2 be defined by (2.7). If $|f^{(n)}|^q$ is concave for $n\geq 2$ and $q\geq 1$ then the following inequality holds:*

$$(n-2)!\left|\int_{b-\lambda}^{b} f(t)p(t)dt-\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_2(x)(x-a)^i dx-\int_a^b f(t)g(t)p(t)dt\right|$$

$$\leq W_2(b)|f^{(n)}(m_2)|, \tag{3.14}$$

where $W_2(b)$ and m_2 are given by (3.12) and (3.13), respectively.

Proof. We follow the lines of the proof of Theorem 3.2, except that we use the identity (2.8). ∎

Example 3.4. Let us consider again our constructed example, $a = 1$, $b = 2$, $f(t) = t^{\frac{7}{2}}$, $p(t) = 1 + t$, $g(t) = \frac{t}{2}$, $n = 3$, and $q = 1$. Then, the conditions of Theorem 3.4 are satisfied, and we obtain the estimate

$$0.150519 \leq 0.150785,$$

which confirms inequality (3.14).

3.2. Results for Taylor's formula at the right endpoint *b*

In this subsection, we aim to establish the Hermite–Hadamard-type bound for identities (2.5) and (2.9) by utilizing convex or concave functions of the form $|f^{(n)}|^q$. We begin with results related to identity (2.5).

Theorem 3.5. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$.*

Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_1 be defined by (2.3). If $|f^{(n)}|^q$ is convex for $n \geq 2$, $q \geq 1$ and if

$$\int_a^t G_1(x)(x-t)^{n-2}dx \leq 0, \quad t \in [a,b], \tag{3.15}$$

then the following inequality holds:

$$\begin{aligned} &(n-2)!\left|\int_a^b f(t)g(t)p(t)dt - \int_a^{a+\lambda} f(t)p(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!}\int_a^b G_1(x)(x-b)^i dx\right| \\ &\leq W_3(b)\left(\frac{b-m_3}{b-a}|f^{(n)}(a)|^q + \frac{m_3-a}{b-a}|f^{(n)}(b)|^q\right)^{\frac{1}{q}}, \end{aligned} \tag{3.16}$$

where

$$W_3(b) = \frac{1}{n-1}\int_a^b G_1(x)(x-b)^{n-1}dx \tag{3.17}$$

and

$$m_3 = \frac{1}{(n-1)W_3(b)}\int_a^b G_1(x)\left(b\cdot(x-b)^{n-1} + \frac{(x-b)^n}{n}\right)dx. \tag{3.18}$$

Proof. The assumption (3.15) implies

$$-\int_a^t G_1(x)(x-t)^{n-2}dx \geq 0, \quad t \in [a,b].$$

Define a non-negative function

$$w_3(t) = -\int_a^t G_1(x)(x-t)^{n-2}dx, \tag{3.19}$$

where the function G_1 is defined by (2.3). Multiplying the identity (2.5) by -1 and taking the absolute value, and then using Hölder's inequality with $q \geq 1$ and the fact that the function w_3 is non-negative, we obtain the inequality

$$\begin{aligned}
&\left|\int_a^b f(t)g(t)p(t)dt - \int_a^{a+\lambda} f(t)p(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i dx\right| \\
&= \left|\frac{-1}{(n-2)!} \int_a^b \left(\int_a^t G_1(x)(x-t)^{n-2}dx\right) f^{(n)}(t)dt\right| \\
&= \left|\frac{1}{(n-2)!} \int_a^b w_3(t) f^{(n)}(t)dt\right| \leq \frac{1}{(n-2)!} \int_a^b w_3(t)|f^{(n)}(t)|dt \\
&\leq \frac{1}{(n-2)!} \left(\int_a^b w_3(t)dt\right)^{1-\frac{1}{q}} \left(\int_a^b w_3(t)|f^{(n)}(t)|^q dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.20}$$

Since the function $|f^{(n)}|^q$ is convex and w_3 is non-negative, we can apply the right-hand side weighted Hermite–Hadamard inequality in (1.1), which yields

$$\int_a^b w_3(t)|f^{(n)}(t)|^q dt \leq W_3(b) \cdot \left(\frac{b-m_3}{b-a}|f^{(n)}(a)|^q + \frac{m_3-a}{b-a}|f^{(n)}(b)|^q\right), \tag{3.21}$$

where

$$\begin{aligned}
W_3(b) &= \int_a^b w_3(t)dt = -\int_a^b \left(\int_a^t G_1(x)(x-t)^{n-2}dx\right)dt \\
&= -\int_a^b G_1(x)\left(\int_x^b (x-t)^{n-2}dt\right)dx = \int_a^b G_1(x)\frac{(x-b)^{n-1}}{n-1}dx
\end{aligned}$$

and

$$\begin{aligned}
m_3 &= \frac{1}{W_3(b)} \int_a^b w_3(t)t\,dt = -\frac{1}{W_3(b)} \int_a^b \left(\int_a^t G_1(x)(x-t)^{n-2}dx\right)t\,dt \\
&= -\frac{1}{W_3(b)} \int_a^b G_1(x)\left(\int_x^b (x-t)^{n-2} \cdot t\,dt\right)dx \\
&= -\frac{1}{W_3(b)} \int_a^b G_1(x)\left(t \cdot \frac{-(x-t)^{n-1}}{n-1}\Big|_x^b + \int_x^b \frac{(x-t)^{n-1}}{n-1}dt\right)dx \\
&= -\frac{1}{W_3(b)} \int_a^b G_1(x)\left(-b \cdot \frac{(x-b)^{n-1}}{n-1} - \frac{(x-b)^n}{(n-1)\cdot n}\right)dx.
\end{aligned}$$

Combining the relations (3.20) and (3.21), we obtain

$$\begin{aligned}
&\left|\int_a^b f(t)g(t)p(t)dt - \int_a^{a+\lambda} f(t)p(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i dx\right| \\
&\quad \leq \frac{1}{(n-2)!}\left(\int_a^b w_3(t)dt\right)^{1-\frac{1}{q}}\left(\int_a^b w_3(t)|f^{(n)}(t)|^q dt\right)^{\frac{1}{q}} \\
&\quad \leq \frac{1}{(n-2)!}\left(W_3(b)\right)^{1-\frac{1}{q}}\left(W_3(b)\cdot\left(\frac{b-m_3}{b-a}|f^{(n)}(a)|^q + \frac{m_3-a}{b-a}|f^{(n)}(b)|^q\right)\right)^{\frac{1}{q}} \\
&\quad = \frac{1}{(n-2)!}W_3(b)\left(\frac{b-m_3}{b-a}|f^{(n)}(a)|^q + \frac{m_3-a}{b-a}|f^{(n)}(b)|^q\right)^{\frac{1}{q}},
\end{aligned}$$

which proves the assertion. ■

Remark 3.1. For an odd number $n \geq 2$, condition (3.15) is always satisfied. For an even number $n \geq 2$, condition (3.15) is never satisfied, but we can use the function $-w_2$ in the proof of Theorem 3.5. Since $-w_2$ is a non-negative function for even $n \geq 2$, the inequality

$$\begin{aligned}
&(n-2)!\left|\int_a^{a+\lambda} f(t)p(t)dt - \int_a^b f(t)g(t)p(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i dx\right| \\
&\quad \leq W_3(b)\left(\frac{b-m_3}{b-a}|f^{(n)}(a)|^q + \frac{m_3-a}{b-a}|f^{(n)}(b)|^q\right)^{\frac{1}{q}}
\end{aligned}$$

holds with

$$W_3(b) = \frac{-1}{n-1}\int_a^b G_1(x)(x-b)^{n-1} dx$$

and

$$m_3 = \frac{-1}{(n-1)W_3(b)}\int_a^b G_1(x)\left(b\cdot(x-b)^{n-1} + \frac{(x-b)^n}{n}\right)dx.$$

The following result is, in some way, complementary to Theorem 3.5, as it refers to a class of concave functions of the form $|f^{(n)}|^q$.

Theorem 3.6. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_1 be defined by (2.3). If $|f^{(n)}|^q$ is concave for $n \geq 2$, $q \geq 1$ and if (3.15) holds, then the following inequality holds:*

$$\begin{aligned}
&(n-2)!\left|\int_a^b f(t)g(t)p(t)dt - \int_a^{a+\lambda} f(t)p(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i dx\right| \\
&\quad \leq W_3(b)|f^{(n)}(m_3)|, \qquad (3.22)
\end{aligned}$$

where $W_3(b)$ and m_3 are given by (3.17) and (3.18), respectively.

Proof. Let us define the function w_3 by (3.19). As in the proof of Theorem 3.5, we have that the function w_3 is non-negative and that the inequalities in (3.20) hold.

Since the function $|f^{(n)}|^q$ is concave and w_3 is non-negative, we can apply the reverse left-hand side weighted Hermite–Hadamard inequality in (1.1) and obtain

$$W_3(b)|f^{(n)}(m_3)|^q \geq \int_a^b w_3(t)|f^{(n)}(t)|^q dt, \tag{3.23}$$

where $W_3(b)$ and m_3 are given by (3.17) and (3.18), respectively.

Now, combining (3.20) and (3.23), we obtain

$$\begin{aligned}
&\left|\int_a^b f(t)g(t)p(t)dt - \int_a^{a+\lambda} f(t)p(t)dt - \sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{i!}\int_a^b G_1(x)(x-b)^i dx\right| \\
&\quad \leq \frac{1}{(n-2)!}\left(W_3(b)\right)^{1-\frac{1}{q}}\left(\int_a^b w_3(t)|f^{(n)}(t)|^q dt\right)^{\frac{1}{q}} \\
&\quad \leq \frac{1}{(n-2)!}W_3(b)|f^{(n)}(m_3)|.
\end{aligned}$$

Therefore, we proved the desired inequality (3.22). ∎

In the following theorem, we establish a Hermite–Hadamard-type bound for identity (2.9) which corresponds to a class of convex functions.

Theorem 3.7. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$. Let $g,p:[a,b]\to\mathbb{R}$ be integrable functions such that p is positive and $0\leq g\leq 1$. Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_2 be defined by (2.7). If $|f^{(n)}|^q$ is convex for $n\geq 2$, $q\geq 1$ and if*

$$\int_a^t G_2(x)(x-t)^{n-2}dx \leq 0, \tag{3.24}$$

then the following inequality holds:

$$\begin{aligned}
&(n-2)!\left|\int_{b-\lambda}^b f(t)p(t)dt - \sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{i!}\int_a^b G_2(x)(x-b)^i dx - \int_a^b f(t)g(t)p(t)dt\right| \\
&\quad \leq W_4(b)\left(\frac{b-m_4}{b-a}|f^{(n)}(a)|^q + \frac{m_4-a}{b-a}|f^{(n)}(b)|^q\right)^{\frac{1}{q}},
\end{aligned} \tag{3.25}$$

where

$$W_4(b) = \frac{1}{n-1}\int_a^b G_2(x)(x-b)^{n-1}dx \tag{3.26}$$

and

$$m_4 = \frac{1}{(n-1)W_4(b)}\int_a^b G_2(x)\left(b\cdot(x-b)^{n-1} + \frac{(x-b)^n}{n}\right)dx. \tag{3.27}$$

Proof. We follow the lines of the proof of Theorem 3.5, except that we use the identity (2.9). ∎

Remark 3.2. For an odd number $n \geq 2$, condition (3.24) is always satisfied. For an even number $n \geq 2$, condition (3.24) is never satisfied, but as in Remark 3.1, we can obtain that the inequality

$$(n-2)!\left|\int_a^b f(t)g(t)p(t)dt - \int_{b-\lambda}^b f(t)p(t)dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!}\int_a^b G_2(x)(x-b)^i dx\right| \leq W_4(b)\left(\frac{b-m_4}{b-a}|f^{(n)}(a)|^q + \frac{m_4-a}{b-a}|f^{(n)}(b)|^q\right)^{\frac{1}{q}} \tag{3.28}$$

holds with

$$W_4(b) = \frac{-1}{n-1}\int_a^b G_2(x)(x-b)^{n-1}dx$$

and

$$m_4 = \frac{-1}{(n-1)W_4(b)}\int_a^b G_2(x)\left(b\cdot(x-b)^{n-1} + \frac{(x-b)^n}{n}\right)dx.$$

To conclude the main results, it remains to establish the Hermite–Hadamard-type bound for identity (2.9) that corresponds to a class of concave functions.

Theorem 3.8. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_2 be defined by (2.7). If $|f^{(n)}|^q$ is concave for $n \geq 2$, $q \geq 1$ and if (3.24) holds, then the following inequality holds:*

$$(n-2)!\left|\int_{b-\lambda}^b f(t)p(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!}\int_a^b G_2(x)(x-b)^i dx - \int_a^b f(t)g(t)p(t)dt\right| \leq W_4(b)|f^{(n)}(m_4)|, \tag{3.29}$$

where $W_4(b)$ and m_4 are given by (3.26) and (3.27), respectively.

Proof. We follow the lines of the proof of Theorem 3.6, except that we use the identity (2.9). ∎

Remark 3.3. Taking $p \equiv 1$ in Theorems 3.1–3.8, we obtain special cases for the Hermite–Hadamard-type bounds related to the identities proved in [17].

Example 3.5. For numerical example in Theorems 3.5 and 3.7, we take $a = 0$, $b = 1$, $f(t) = e^t$, $p(t) = 1 + t$, $g(t) = t$, $n = 3$, and $q = 1$. Using the inequality (3.16), we obtain the estimate

$$0.0840471 \leq 0.0887289.$$

Applying inequality (3.25) yields the estimate

$$0.0175881 \leq 0.0185214.$$

These estimates demonstrate the validity of our inequalities.

Example 3.6. For numerical example in Theorems 3.6 and 3.8, we take $a = 1$, $b = 2$, $f(t) = t^{\frac{7}{2}}$, $p(t) = 1 + t$, $g(t) = \frac{t}{2}$, $n = 3$, and $q = 1$. Applying inequality (3.22) gives the estimate

$$0.630532 \leq 0.631249,$$

while inequality (3.29) yields

$$0.476723 \leq 0.477369.$$

These results clearly confirm the correctness of the inequalities.

4. Generalizations of the weighted Hermite–Hadamard-type inequalities by identities related to generalizations of Steffensen's inequality

In [18], Pečarić et al. proved weighted Hermite–Hadamard-type inequalities using identities related to generalizations of Steffensen's inequality via Taylor's formula. In this section, we generalize these results using the identities obtained in Section 2. By applying the weighted Hermite–Hadamard inequality to identity (2.4), we obtain the following result.

Theorem 4.1. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Let $g, p : [a, b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_1 be defined by (2.3). If f is $(n + 2)$-convex, then the following inequality holds:*

$$\begin{aligned} W_1(b) \cdot f^{(n)}(m_1) &\leq (n-2)!\Bigg[\int_a^b f(t)g(t)p(t)dt - \int_a^{a+\lambda} f(t)p(t)dt \\ &\qquad - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_1(x)(x-a)^i dx\Bigg] \\ &\leq W_1(b) \cdot \left[\frac{b-m_1}{b-a} f^{(n)}(a) + \frac{m_1-a}{b-a} f^{(n)}(b)\right], \end{aligned} \tag{4.1}$$

where $W_1(b)$ and m_1 are given by (3.2) and (3.3), respectively.

Proof. The proof is similar to the proof of [18, Theorem 5], so we will only give a brief sketch of the proof. The convexity of the function $f^{(n)}$ follows from the $(n + 2)$-convexity of the function f (see [20]). Define the non-negative function w_1 by (3.4). Apply the weighted Hermite–Hadamard inequality with non-negative function w_1 and convex function $f^{(n)}$ to arrive at

$$W_1(b) \cdot f^{(n)}(m_1) \leq \int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2} dx \right) f^{(n)}(t) dt \\ \leq W_1(b) \cdot \left[\frac{b-m_1}{b-a} f^{(n)}(a) + \frac{m_1-a}{b-a} f^{(n)}(b) \right], \tag{4.2}$$

where $W_1(b)$ and m_1 are given by (3.2) and (3.3). Inserting the identity (2.4) into the inequality (4.2), we get the desired inequality (4.1). ∎

Remark 4.1. Taking $p \equiv 1$ in Theorem 4.1, we obtain the result given by Pečarić et al. [18, Theorem 5].

We continue with the result related to identity (2.8) obtained using the weighted Hermite–Hadamard inequality.

Theorem 4.2. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_2 be defined by (2.7). If f is $(n+2)$-convex, then the following inequality holds:*

$$W_2(b) \cdot f^{(n)}(m_2) \\ \leq (n-2)! \left[\int_{b-\lambda}^b f(t)p(t)dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_2(x)(x-a)^i dx \right. \\ \left. - \int_a^b f(t)g(t)p(t)dt \right] \\ \leq W_2(b) \cdot \left[\frac{b-m_2}{b-a} f^{(n)}(a) + \frac{m_2-a}{b-a} f^{(n)}(b) \right],$$

where $W_2(b)$ and m_2 are given by (3.12) and (3.13), respectively.

Proof. We follow similar arguments as in the proof of Theorem 4.1. ∎

Remark 4.2. Taking $p \equiv 1$ in Theorem 4.2, we obtain the result given by Pečarić et al. [18, Theorem 7].

Using the identities (2.5) and (2.9), we obtain the following results.

Theorem 4.3. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_a^{a+\lambda} p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_1 be defined by (2.3).*

If f is $(n+2)$-convex and (3.15) holds, then the following inequality holds:

$$
\begin{aligned}
&W_3(b) \cdot f^{(n)}(m_3) \\
&\leq (n-2)!\bigg[\int_a^b f(t)g(t)p(t)dt - \int_a^{a+\lambda} f(t)p(t)dt \\
&\qquad\qquad - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i dx\bigg] \\
&\leq W_3(b) \cdot \left[\frac{b-m_3}{b-a} f^{(n)}(a) + \frac{m_3-a}{b-a} f^{(n)}(b)\right],
\end{aligned} \tag{4.3}
$$

where $W_3(b)$ and m_3 are given by (3.17) and (3.18), respectively.

Proof. We follow similar arguments as in the proof of Theorem 4.1. ■

Remark 4.3. Taking $p \equiv 1$ in Theorem 4.3, we obtain the result given by Pečarić et al. [18, Theorem 6].

Theorem 4.4. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$. Let $g, p : [a,b] \to \mathbb{R}$ be integrable functions such that p is positive and $0 \leq g \leq 1$. Let $\int_{b-\lambda}^b p(t)dt = \int_a^b g(t)p(t)dt$ and let the function G_2 be defined by (2.7). If f is $(n+2)$-convex and (3.24) holds, then the following inequality holds:*

$$
\begin{aligned}
&W_4(b) \cdot f^{(n)}(m_4) \\
&\leq (n-2)!\bigg[\int_{b-\lambda}^b f(t)p(t)dt \\
&\qquad\qquad - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_2(x)(x-b)^i dx - \int_a^b f(t)g(t)p(t)dt\bigg] \\
&\leq W_4(b) \cdot \left[\frac{b-m_4}{b-a} f^{(n)}(a) + \frac{m_4-a}{b-a} f^{(n)}(b)\right],
\end{aligned}
$$

where $W_4(b)$ and m_4 are given by (3.26) and (3.27), respectively.

Proof. We follow similar arguments as in the proof of Theorem 4.1. ■

Remark 4.4. Taking $p \equiv 1$ in Theorem 4.4, we obtain the result given in Pečarić et al. [18, Theorem 8].

Remark 4.5. If f is an $(n+2)$-concave function, the inequalities in Theorems 4.1–4.4 are reversed. In this case, we apply the weighted Hermite–Hadamard inequality for a convex function $-f^{(n)}$.

5. Conclusion

This paper makes a significant contribution to the theory of mathematical inequalities by introducing a general method that connects classical inequalities, such as Steffensen's and Hermite–Hadamard's, with Taylor's formula. By extending known identities from [17] with a positive weight function, the authors have developed a new type of inequalities that provides sharper bounds for expressions involving the absolute values of higher-order derivatives, specifically those of the form $|f^{(n)}|^q$, where $n \geq 2$ and $q \geq 1$. The importance of this work lies in its adaptability. The approach is not limited to a specific inequality; it can be applied to other classical inequalities as well. Furthermore, there are numerous possible applications for these results in numerical analysis, optimization, error estimation, and other areas where it is important that the integral approximation is both accurate and efficient.

Acknowledgments. The authors would like to express their gratitude to the anonymous referees for their careful reading and helpful suggestions, which improved the presentation and quality of their paper.

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Received 21 November 2024; revised 5 February 2026.

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