

Time-periodic solutions to the Navier–Stokes equations on the whole space including the two-dimensional case

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Abstract. Let us consider the incompressible Navier–Stokes equations with the time-periodic external forces in the whole space $\mathbb{R}^n$ with $n \geq 2$ and investigate the existence and non-existence of time-periodic solutions. In the higher-dimensional case $n \geq 3$, we construct a unique small solution for given small time-periodic force in the scaling-critical spaces of Besov type and prove its stability under small perturbations. In contrast, for the two-dimensional case $n = 2$, the time-periodic solvability of the Navier–Stokes equations has long been open. It is the central work of this paper that we have now succeeded in solving this issue negatively by providing examples of small external forces such that each of them does not generate time-periodic solutions.

1. Introduction

We consider the incompressible Navier–Stokes equations with time-periodic external forces on the whole space $\mathbb{R}^n$ with $n \geq 2$:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = f, & t \in \mathbb{R}, x \in \mathbb{R}^n, \\ \operatorname{div} u = 0, & t \in \mathbb{R}, x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

where $u = u(t, x): \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $p = p(t, x): \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ represent the unknown velocity field and pressure of the fluid, respectively, whereas the given external force $f = f(t, x): \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to be T -periodic, that is, $f(t + T) = f(t)$ holds for all $t \in \mathbb{R}$ with some positive constant T . It is well known that (1.1) possesses the scaling invariant structure, that is, if u and p solve (1.1) with some external force f , then

$$u_\lambda(t, x) := \lambda u(\lambda^2 t, \lambda x), \quad p_\lambda(t, x) := \lambda^2 p(\lambda^2 t, \lambda x) \quad (1.2)$$

also satisfy (1.1) with f replaced by

$$f_\lambda(t, x) := \lambda^3 f(\lambda^2 t, \lambda x) \quad (1.3)$$

for all $\lambda > 0$. Function spaces of which the norms are invariant under the scaling transforms (1.2)–(1.3) are called the scaling-critical spaces for (1.1). The purpose of this paper

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is to consider the solvability of the time-periodic problem (1.1) in the scaling-critical Besov-type spaces framework. In the higher-dimensional case $n \geq 3$, we prove the unique existence and global-in-time stability of the T -periodic solutions to (1.1). For the two-dimensional case $n = 2$, it has been a well-known open problem whether the time-periodic solution for the two-dimensional incompressible Navier–Stokes equations (1.1) with $n = 2$ exists or not. The major outcome in this paper is to solve this question negatively and construct some arbitrarily small external forces, each of which does not produce time-periodic solutions.

1.1. Known results and the position of our study

We recall known results for the time-periodic problem of the Navier–Stokes equations on unbounded domains. It was Maremonti [13, 14] who first constructed a unique time-periodic solution to (1.1) on the three-dimensional whole space $\mathbb{R}^3$ and half-space $\mathbb{R}_+^3$. Kozono and Nakao [11] introduced the notion of integral equation (1.6) below corresponding to (1.1) and showed the existence of a unique small time-periodic mild solution to (1.1) in the Lebesgue spaces framework on the whole space $\mathbb{R}^n$, the half-space $\mathbb{R}_+^n$ with $n \geq 3$, and the exterior domains in $\mathbb{R}^n$ with $n \geq 4$. Taniuchi [15] showed the stability of the global solution to the initial value problem of Navier–Stokes equations with the time-periodic external forces around the time-periodic flow constructed in [11] in the framework of weak-mild solutions. Yamazaki [19] generalized the results in [11] to the Morrey spaces frameworks on $\mathbb{R}^n$ with $n \geq 3$. Yamazaki [18] considered the Navier–Stokes equations (1.1) with external forces that may not decay as $t \rightarrow \infty$, which is a similar situation to the time-periodic setting and proved the global existence of solutions in a scaling-critical space $u \in \text{BC}(\mathbb{R}; L^{n,\infty}(\Omega))$ for given small external force f in the scaling-critical class $(-\Delta)^{-\frac{1}{2}} f \in \text{BC}(\mathbb{R}; L^{\frac{n}{2},\infty}(\Omega))$, where Ω is the whole space $\mathbb{R}^n$, the half-space $\mathbb{R}_+^n$, or the exterior domain in $\mathbb{R}^n$ with $n \geq 3$. Geissert, Hieber, and Nguyen [9] proposed a new approach on the time-periodic problem in a general framework and applied it to several viscous incompressible fluids on $\mathbb{R}^n$ with $n \geq 3$, and constructed small time-periodic solutions in a scaling-critical space $\text{BC}(\mathbb{R}; L^{n,\infty}(\mathbb{R}^n))$ if the given time-periodic external force $f = \text{div } F$ with the scaling-critical class $F \in \text{BC}(\mathbb{R}; L^{\frac{n}{2},\infty}(\mathbb{R}^n))$ is sufficiently small.

For the two-dimensional case, the situation is completely different from that for the higher-dimensional case, and there are only a few results on the existence of time-periodic solutions in two-dimensional unbounded domains. As mentioned in [8], the solvability of time-periodic problems in two-dimensional unbounded domains has been known to be as difficult as that of stationary problems. Indeed, the proofs of all results for the higher-dimensional case mentioned above completely fails in the two-dimensional case. One of the reasons for this difficulty is that the decay rate of the heat kernels on $\mathbb{R}^2$ is so slow that it is difficult to find a function space X that establishes the key bilinear estimate

$$\left\| \int_{-\infty}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes v(\tau)) d\tau \right\|_X \leq C \|u\|_X \|v\|_X, \quad (1.4)$$

although it is known for the higher-dimensional case, such as $\text{BC}(\mathbb{R}; L^{n,\infty}(\mathbb{R}^n))$ with $n \geq 3$ by [9, 18]. On the other hand, there are some results solving the two-dimensional time-periodic problem in the whole plane by considering some special situations. Galdi [8] constructed time-periodic solutions to (1.1) around the constant flow $\hat{e}_1 = (1, 0)^\top$. Tsuda [16] proved the existence of time-periodic solutions to the compressible Navier–Stokes equations with the given small time-periodic external forces satisfying some spatial antisymmetric conditions. However, there is no previous research on time-periodic solvability in two-dimensional unbounded domains without special assumptions such as around non-zero constant equilibrium states or spatial antisymmetry. In particular, the two-dimensional analysis corresponding to the results [9, 11, 13, 14, 18, 19] for the higher-dimensional case mentioned above is completely unresolved.

In this paper, we address the solvability of the time-periodic problem (1.1) not only in the higher-dimensional case $\mathbb{R}^n$ with $n \geq 3$, but also in the two-dimensional case $\mathbb{R}^2$, and aim to reveal the existence or non-existence of time-periodic solutions in the framework of scaling-critical function spaces of Besov type. More precisely, for the higher-dimensional case $\mathbb{R}^n$ with $n \geq 3$, we prove that for $1 \leq p < n$ and $1 \leq \sigma \leq \infty$, there exists a unique small time-periodic solution $u_{\text{per}} \in \tilde{C}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))$, provided that the given time-periodic external force $f \in \tilde{C}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n))$ is sufficiently small, where $\tilde{C}(\mathbb{R}; \dot{B}_{p,\sigma}^s(\mathbb{R}^n)) := C(\mathbb{R}; \dot{B}_{p,\sigma}^s(\mathbb{R}^n)) \cap \widetilde{L^\infty}(\mathbb{R}; \dot{B}_{p,\sigma}^s(\mathbb{R}^n))$ and $\widetilde{L^r}(\mathbb{R}; \dot{B}_{p,\sigma}^s(\mathbb{R}^n))$ is the Chemin–Lerner space; the definition and basic properties of this function space are mentioned in Section 2. See Remark 1.2 for the reason why we need the Chemin–Lerner spaces. For the stability of the time-periodic solution u_{per} , we consider the initial value problem of the incompressible Navier–Stokes equations with the time-periodic external forces

$$\begin{cases} \partial_t v - \Delta v + (v \cdot \nabla)v + \nabla q = f, & t > 0, x \in \mathbb{R}^n, \\ \operatorname{div} v = 0, & t \geq 0, x \in \mathbb{R}^n, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.5)$$

and prove that if the initial disturbance $v_0(x) - u_{\text{per}}(0, x)$ is sufficiently small in $\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)$ with $1 \leq q < 2n$, then (1.5) possesses a unique mild solution *in the strong sense*¹ and it holds that

$$\lim_{t \rightarrow \infty} \|v(t) - u_{\text{per}}(t)\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)} = 0.$$

Furthermore, we consider the two-dimensional case and show that the above result on the existence of the time-periodic solution *fails*, that is, for each $1 \leq p \leq 2$ and $0 < \delta \ll 1$, there exists a time-periodic external force $f_\delta \in \tilde{C}(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2))$ with norm less than δ such that there exists no time-periodic solution to (1.1) with the force f_δ in some subset of $C(\mathbb{R}; \dot{B}_{2,1}^0(\mathbb{R}^2))$.

¹In the known result [15], the time-periodic stability is proved in the framework of the mild solutions *in the weak sense*; see Remark 1.4 below.

1.2. Main results

Now we provide the precise statements of our main theorems. To this end, we recall the notion of mild solutions to (1.1) which was proposed by [11]. By the Duhamel principle, equation (1.1) is formally equivalent to

$$u(t) = \int_{-\infty}^t e^{(t-\tau)\Delta} \mathbb{P} f(\tau) d\tau - \int_{-\infty}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes u(\tau)) d\tau, \quad (1.6)$$

where $\{e^{t\Delta}\}_{t>0}$ denotes the heat semigroup, and $\mathbb{P} := I + \nabla \operatorname{div}(-\Delta)^{-1}$ is the Helmholtz projection on $\mathbb{R}^n$. We say that u is a mild solution to (1.1) if u satisfies (1.6) for all $t \in \mathbb{R}$. See Section 2 for the definitions of function spaces appearing in the following theorems.

1.2.1. Higher-dimensional case. We first focus on the existence and stability of the time-periodic mild solutions to (1.1) in higher-dimensional whole space case $\mathbb{R}^n$ with $n \geq 3$. The first main result of this paper now reads as follows.

Theorem 1.1 (Existence of time-periodic solutions on $\mathbb{R}^n$ with $n \geq 3$). *Let $n \geq 3$ be an integer and let $1 \leq p < n$ and $1 \leq \sigma \leq \infty$. Then, there exist a positive constant $\delta_0 = \delta_0(n, p, \sigma)$ and $\varepsilon_0 = \varepsilon_0(n, p, \sigma)$ such that for any $T > 0$ and T -periodic external force $f \in \tilde{\mathcal{C}}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n))$ satisfying*

$$\|f\|_{\widetilde{L^\infty}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n))} \leq \delta_0,$$

equation (1.1) possesses a unique T -periodic mild solution $u_{\text{per}} \in U_{p,\sigma}(\mathbb{R}^n)$, where

$$U_{p,\sigma}(\mathbb{R}^n) := \left\{ u \in \tilde{\mathcal{C}}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n)); \|u\|_{\widetilde{L^\infty}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \leq \varepsilon_0 \right\}.$$

Moreover, there exists a positive constant $C = C(n, p, q)$ such that the solution u_{per} satisfies the following a priori estimate:

$$\|u_{\text{per}}\|_{\widetilde{L^\infty}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \leq C \|f\|_{\widetilde{L^\infty}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n))}. \quad (1.7)$$

Remark 1.2. We give some remarks on Theorem 1.1.

(1) It is generally acknowledged that time-periodic and stationary problems are closely related, and there is a result on the existence of the stationary solutions in the Besov-space framework that corresponds to Theorem 1.1. Kaneko, Kozono, and Shimizu [10] showed that there exists a unique small solution to the stationary Navier–Stokes equations on the whole space $\mathbb{R}^n$ with $n \geq 3$ in the scaling-critical Besov spaces $\dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n)$ for small external forces in $\dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n)$ for $1 \leq p < n$ and $1 \leq \sigma \leq \infty$.

(2) Let us explain why we use not the usual space-time Besov spaces $\text{BC}(\mathbb{R}; \dot{B}_{p,\sigma}^s(\mathbb{R}^n))$ but the Chemin–Lerner spaces $\tilde{\mathcal{C}}(\mathbb{R}; \dot{B}_{p,\sigma}^s(\mathbb{R}^n))$. Considering the bilinear estimates (1.4) with $X = \text{BC}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))$, it is difficult to show it unless $\sigma = \infty$. In contrast, if we

switch the order of the L_t^∞ -norm and the ℓ^σ -norm, then we may make use of the maximal regularity estimate Lemma 2.1 below to obtain (1.4) with $\tilde{C}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))$ for all $1 \leq \sigma \leq \infty$; see Remark 2.2 below for the detail. In particular, choosing $\sigma < \infty$ is significant in the time-periodic stability limit (1.10) in Theorem 1.3 below.

As is mentioned in the above remark, one of the advantages of Chemin–Lerner space is treating the asymptotic stability of the time-periodic solutions. For the stability of (1.5) around the time-periodic solution u_{per} constructed in Theorem 1.1 above, we set $w := v - u_{\text{per}}$ and consider the following equations which w should solve:

$$\begin{cases} \partial_t w - \Delta w + (w \cdot \nabla)w \\ \quad + (u_{\text{per}} \cdot \nabla)w + (w \cdot \nabla)u_{\text{per}} + \nabla \pi = 0, & t > 0, x \in \mathbb{R}^n, \\ \operatorname{div} w = 0, & t \geq 0, x \in \mathbb{R}^n, \\ w(0, x) = w_0(x) := v_0(x) - u_{\text{per}}(0, x), & x \in \mathbb{R}^n. \end{cases} \quad (1.8)$$

We say that w is a mild solution to (1.8) if it solves the following corresponding integral equation:

$$\begin{aligned} w(t) &= e^{t\Delta} w_0 - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_{\text{per}}(\tau) \otimes w(\tau) + w(\tau) \otimes u_{\text{per}}(\tau)) d\tau \\ &\quad - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(w(\tau) \otimes w(\tau)) d\tau. \end{aligned} \quad (1.9)$$

Our result on the time-periodic stability reads as follows.

Theorem 1.3 (Time-periodic stability on $\mathbb{R}^n$ with $n \geq 3$). *Let $n \geq 3$ be an integer and let $T > 0$. Let p, q, r , and σ satisfy*

$$\begin{aligned} 1 \leq p < n, \quad 1 \leq q < 2n, \quad \frac{1}{q} - \frac{1}{p} < \frac{1}{n}, \quad 1 \leq \sigma < \infty, \\ \max\left\{0, 1 - \frac{n}{2} \min\left\{1, \frac{2}{q}, \frac{1}{p} + \frac{1}{q}\right\}\right\} < \frac{1}{r} < \frac{1}{2} - \frac{n}{2} \max\left\{0, \frac{1}{q} - \frac{1}{p}\right\}. \end{aligned}$$

Then there exists a positive constant $\delta_0 = \delta_0(n, p, q, r, \sigma)$ such that if the T -periodic solution $u_{\text{per}} \in \tilde{C}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))$ constructed in Theorem 1.1 and the initial disturbance $w_0 \in \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)$ satisfy

$$\|u_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \leq \delta_0, \quad \|w_0\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)} \leq \delta_0,$$

then (1.8) possesses a unique mild solution w in the class

$$w \in \tilde{C}([0, \infty); \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)) \cap \widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n)).$$

Moreover, the following asymptotic limit holds:

$$\lim_{t \rightarrow \infty} \|w(t)\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)} = 0. \quad (1.10)$$

Remark 1.4. We provide some comments on Theorem 1.3.

(1) Theorem 1.3 can be compared with the results of Taniuchi [15]. In his result, the solution to the perturbed equations (1.8) should be considered in the framework that w satisfies the integral equation (1.9) in the *weak sense*. This is because if we consider the integral equation (1.9) in the strong sense by following the argument in [15], we meet a difficulty when controlling the convection terms $(u_{\text{per}} \cdot \nabla)w + (w \cdot \nabla)u_{\text{per}}$ in some time-weighted norms like $\sup_{t>0} t^{\frac{1}{2}} \|w(t)\|_{L^n(\mathbb{R}^n)}$ since u_{per} does not have any decay structure in time. In contrast, our Theorem 1.3 is able to find a solution to (1.9) in the *strong sense* thanks to the maximal regularity of the heat kernel and bilinear estimates in Chemin–Lerner spaces; see Lemmas 2.1 and 2.3 below.

(2) In Theorem 1.3, the condition $n \geq 3$ is used only for the guarantee of the existence of the time-periodic solution u_{per} . Therefore, if we obtained a two-dimensional time-periodic solution to (1.1) with some external force, then we might obtain the stability result Theorem 1.3 with $n = 2$. However, as is claimed in Theorem 1.5 below, the time-periodic problem is not solvable in the two-dimensional case.

1.2.2. Two-dimensional case. Now we introduce the central work of this paper. In the following theorem, we claim that Theorem 1.1 *fails* in the two-dimensional case.

Theorem 1.5 (Non-existence of the time-periodic solution on $\mathbb{R}^2$). *Let $n = 2$ and $1 \leq p \leq 2$. Then there exists a positive constant $\varepsilon_0 = \varepsilon_0(p)$ such that for any $0 < \delta \leq \varepsilon_0$ and $0 < T \leq 2\delta^{\frac{1}{2}}$, there exists a T -periodic external force $f_\delta \in \tilde{C}(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2))$ such that*

$$\|f_\delta\|_{\widetilde{L^\infty}(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2))} \leq \delta$$

and (1.1) possesses no T -periodic solution belonging to the class $V(\mathbb{R}^2)$, where

$$V(\mathbb{R}^2) := \{u \in \text{BC}(\mathbb{R}; \dot{B}_{2,1}^0(\mathbb{R}^2)); \|u(t_0)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \leq \varepsilon_0 \text{ for some } t_0 \in \mathbb{R}\}.$$

Remark 1.6. We make mention of some remarks.

(1) Our non-existence class $V(\mathbb{R}^2)$ may include functions with arbitrarily large $L^\infty(\mathbb{R}; \dot{B}_{2,1}^0(\mathbb{R}^2))$ -norms. From this and $\tilde{C}(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)) \hookrightarrow \text{BC}(\mathbb{R}; \dot{B}_{2,1}^0(\mathbb{R}^2))$ for $1 \leq p \leq 2$, we see that $U_{p,1}(\mathbb{R}^2) \subsetneq V(\mathbb{R}^2)$ with $1 \leq p \leq 2$, which implies Theorem 1.5 claims a stronger result than the negative proposition of Theorem 1.1 with $n = 2$, $1 \leq p \leq 2$, and $\sigma = 1$. Moreover, even if each f_δ generates a time-periodic solution $u_{\text{per},\delta}$ in a wider class than $V(\mathbb{R}^2)$, then $u_{\text{per},\delta} \notin V(\mathbb{R}^2)$ implies that $\|u_{\text{per},\delta}\|_{\widetilde{L^\infty}(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2))}$ must be bounded from below by a positive constant ε_0 independent of δ , although $\|f_\delta\|_{\widetilde{L^\infty}(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2))}$ vanishes as $\delta \downarrow 0$; this means that the a priori estimate (1.7) never holds.

(2) We compare Theorem 1.5 with the results in [16], where it was shown that the two-dimensional compressible Navier–Stokes equations with time-periodic external forces on

the whole plane possess the small time-periodic solution if the given time-periodic external force satisfies some spatial antisymmetric conditions. In contrast, our external force has an anisotropic structure due to $\cos(Mx_1)$ with some $M \gg 1$; see (4.7) below for the detail. Thus, it is crucial to impose a certain spatial symmetry for external forces in order to construct a two-dimensional time-periodic solution.

Let us explain the idea of the proof of Theorem 1.5. We use a contradiction argument. For $0 < \delta \ll 1$ and f_δ proposed in (4.7) below, there exists a T -periodic solution $u_{\text{per},\delta} \in C(\mathbb{R}; \dot{B}_{2,1}^0(\mathbb{R}^2))$ with $\|u_{\text{per},\delta}(t_0)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \leq \varepsilon_0$ for some $t_0 \in \mathbb{R}$ and $\varepsilon_0 > 0$. Then, using the method for ill-posedness (see [2, 3, 6]), Proposition 4.1 below enables us to construct external forces f_δ for each $0 < \delta \leq \varepsilon_0$ such that there exists a Navier–Stokes flow $u \in C([t_0, t_0 + kT]; \dot{B}_{2,1}^0(\mathbb{R}^2))$ starting at $t = t_0$ satisfying the initial condition $u(t_0) = u_{\text{per},\delta}(t_0)$ and the estimate

$$\|u(t_0 + kT)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \geq 2\varepsilon_0$$

for some large $k \in \mathbb{N}$. Then, since $\|u(t_0)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} = \|u_{\text{per},\delta}(t_0)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \leq \varepsilon_0$, we see that $u(t_0) \neq u(t_0 + kT)$, which means u is not T -periodic. However, since it follows from Proposition 4.5 that the uniqueness in $C([t_0, t_0 + kT]; \dot{B}_{2,1}^0(\mathbb{R}^2))$ holds for solutions to the two-dimensional Navier–Stokes equations, we see that $u = u_{\text{per},\delta}$ on $[t_0, t_0 + kT]$ and

$$u_{\text{per},\delta}(t_0) = u(t_0) \neq u(t_0 + kT) = u_{\text{per},\delta}(t_0 + kT),$$

which contradicts the assumption that $u_{\text{per},\delta}$ is T -periodic and completes the proof.

1.3. Organization and notation in this paper

This paper is organized as follows. In Section 2 we recall the definitions of Besov and Chemin–Lerner spaces and their basic properties. We focus on the higher-dimensional case in Section 3 and provide the proofs of Theorems 1.1 and 1.3. In Section 4 we prove Propositions 4.1 on the construction of non-time-periodic solutions and 4.5 for the unconditional uniqueness of two-dimensional Navier–Stokes flow to complete the proof of Theorem 1.5. In Appendix A, we note some remarks on the paradifferential calculus in Chemin–Lerner spaces.

Throughout this paper, we denote by C and c constants which may differ in each line. In particular, $C = C(*, \dots, *)$ denotes a constant which depends only on the quantities appearing in parentheses. For any $T > 0$, we say that a function $f = f(t)$ on $\mathbb{R}$ is T -periodic if $f(t + T) = f(t)$ holds for all $t \in \mathbb{R}$.

2. Preliminaries

In this section, we prepare notation used in this paper and recall the definitions of several function spaces and their basic properties which are frequently used in this paper.

We recall the definitions of Besov and Chemin–Lerner spaces. Let $\mathcal{S}(\mathbb{R}^n)$ be the set of all Schwartz functions on $\mathbb{R}^n$, and we denote by $\mathcal{S}'(\mathbb{R}^n)$ the set of all tempered distributions on $\mathbb{R}^n$. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\text{supp } \widehat{\varphi}_0 \subset \{\xi \in \mathbb{R}^n; 2^{-1} \leq |\xi| \leq 2\}, \quad 0 \leq \widehat{\varphi}_0(\xi) \leq 1,$$

and

$$\sum_{j \in \mathbb{Z}} \widehat{\varphi}_j(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\},$$

where we have set $\varphi_j(x) := 2^{nj} \varphi_0(2^j x)$. Using them, we define the dyadic frequency localized operators by

$$\Delta_j f := \mathcal{F}^{-1}[\widehat{\varphi}_j(\xi) \widehat{f}], \quad f \in \mathcal{S}'(\mathbb{R}^n), \quad j \in \mathbb{Z}.$$

For $1 \leq p, \sigma \leq \infty$, and $s \in \mathbb{R}$, the Besov space $\dot{B}_{p,\sigma}^s(\mathbb{R}^n)$ is defined as

$$\begin{aligned} \dot{B}_{p,\sigma}^s(\mathbb{R}^n) &:= \{f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n); \|f\|_{\dot{B}_{p,\sigma}^s(\mathbb{R}^n)} < \infty\}, \\ \|f\|_{\dot{B}_{p,\sigma}^s(\mathbb{R}^n)} &:= \left\| \{2^{sj} \|\Delta_j f\|_{L^p(\mathbb{R}^n)}\}_{j \in \mathbb{Z}} \right\|_{\ell^\sigma(\mathbb{Z})}, \end{aligned}$$

where $\mathcal{P}(\mathbb{R}^n)$ is the set of all polynomials on $\mathbb{R}^n$. It is well known that if $s < n/p$ or $(s, \sigma) = (n/p, 1)$, then it holds that

$$\dot{B}_{p,\sigma}^s(\mathbb{R}^n) \sim \{f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{\dot{B}_{p,\sigma}^s(\mathbb{R}^n)} < \infty, f = \sum_{j \in \mathbb{Z}} \Delta_j f \text{ in } \mathcal{S}'(\mathbb{R}^n)\}.$$

For $1 \leq p, r, \sigma \leq \infty$, $s \in \mathbb{R}$, and an interval $I \subset \mathbb{R}$, we define the Chemin–Lerner space $\widetilde{L}^r(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n))$ by

$$\begin{aligned} \widetilde{L}^r(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n)) &:= \{F: I \rightarrow \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n); \|F\|_{\widetilde{L}^r(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n))} < \infty\}, \\ \|F\|_{\widetilde{L}^r(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n))} &:= \left\| \{2^{sj} \|\Delta_j F\|_{L^p(I; L^p(\mathbb{R}^n))}\}_{j \in \mathbb{Z}} \right\|_{\ell^\sigma(\mathbb{Z})}. \end{aligned}$$

Since $\dot{H}^s(\mathbb{R}^n) = \dot{B}_{2,2}^s(\mathbb{R}^n)$, we write

$$\widetilde{L}^r(I; \dot{H}^s(\mathbb{R}^n)) := \widetilde{L}^r(I; \dot{B}_{2,2}^s(\mathbb{R}^n)).$$

We also use the notation

$$\widetilde{C}(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n)) := C(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n)) \cap \widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n)).$$

The Chemin–Lerner spaces were first introduced by [4] and continue to be frequently used for the analysis of compressible viscous fluids in critical Besov spaces. The Chemin–Lerner spaces possess similar embedding properties to the usual Besov spaces:

$$\begin{aligned} \widetilde{L}^r(I; \dot{B}_{p_1,\sigma_1}^s(\mathbb{R}^n)) &\hookrightarrow \widetilde{L}^r(I; \dot{B}_{p_2,\sigma_2}^s(\mathbb{R}^n)) \quad \text{for } 1 \leq \sigma_1 \leq \sigma_2 \leq \infty, \\ \widetilde{L}^r(I; \dot{B}_{p_1,\sigma_1}^{s+\frac{n}{p_1}}(\mathbb{R}^n)) &\hookrightarrow \widetilde{L}^r(I; \dot{B}_{p_2,\sigma_2}^{s+\frac{n}{p_2}}(\mathbb{R}^n)) \quad \text{for } 1 \leq p_1 \leq p_2 \leq \infty. \end{aligned}$$

It also holds by the Hausdorff–Young inequality that

$$\begin{aligned}\widetilde{L}^r(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n)) &\hookrightarrow L^r(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n)) \quad \text{for } 1 \leq \sigma \leq r \leq \infty, \\ L^r(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n)) &\hookrightarrow \widetilde{L}^r(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n)) \quad \text{for } 1 \leq r \leq \sigma \leq \infty.\end{aligned}$$

See [1] for more precise information on the Chemin–Lerner spaces. One advantage of using the Chemin–Lerner spaces is that there holds the following maximal regularity estimates for the heat kernel $e^{t\Delta}$.

Lemma 2.1. *Let $n \in \mathbb{N}$. Then there exists a positive constant $C = C(n)$ such that*

$$\|e^{(t-t_0)\Delta}a\|_{\widetilde{L}^r(I; \dot{B}_{p,\sigma}^{s+\frac{2}{r}}(\mathbb{R}^n))} \leq C\|a\|_{\dot{B}_{p,\sigma}^s(\mathbb{R}^n)}, \quad (2.1)$$

$$\left\| \int_{t_0}^t e^{(t-\tau)\Delta} f(\tau) d\tau \right\|_{\widetilde{L}^r(I; \dot{B}_{p,\sigma}^{s+\frac{2}{r}}(\mathbb{R}^n))} \leq C\|f\|_{\widetilde{L}^{r_1}(I; \dot{B}_{p,\sigma}^{s-2+\frac{2}{r_1}}(\mathbb{R}^n))} \quad (2.2)$$

for all $I = (t_0, t_1) \subset \mathbb{R}$ (additionally $t_0 > -\infty$ for (2.1)), $1 \leq p, \sigma \leq \infty$, $1 \leq r_1 \leq r \leq \infty$, $s \in \mathbb{R}$, $a \in \dot{B}_{p,\sigma}^s(\mathbb{R}^n)$, and $f \in \widetilde{L}^{r_1}(I; \dot{B}_{p,\sigma}^{s+\frac{2}{r_1}}(\mathbb{R}^n))$.

Remark 2.2. If we attempt to show (2.1) with the norm on the left-hand side replaced by that for the usual time-space Besov norm $L^r(I; \dot{B}_{p,\sigma}^{s+\frac{2}{r}}(\mathbb{R}^n))$, we fail by the following argument:

$$\begin{aligned}\|e^{t\Delta}a\|_{L^r(I; \dot{B}_{p,\sigma}^{s+\frac{2}{r}}(\mathbb{R}^n))} &= \left(\int_{t_0}^{t_1} \|e^{(t-t_0)\Delta}a\|_{\dot{B}_{p,\sigma}^{s+\frac{2}{r}}(\mathbb{R}^n)}^r dt \right)^{\frac{1}{r}} \\ &\leq C \left[\int_{t_0}^{t_1} [(t-t_0)^{-\frac{1}{r}} \|a\|_{\dot{B}_{p,\sigma}^s(\mathbb{R}^n)}]^r dt \right]^{\frac{1}{r}} = \infty\end{aligned}$$

for non-zero a , where we have used the smoothing estimate for the heat kernel (see [12, Lemma 2.2]). In contrast, we succeed in obtaining the maximal regularity estimate by changing the order of the L_t^r -norm and the ℓ^σ -norm in the Besov norm. A similar situation to above holds for (2.2).

Proof of Lemma 2.1. Although the proof is immediately obtained by [1, Corollary 2.5], we shall give the outline of the proof for the readers' convenience. It follows from [1, Lemma 2.4] that there exists an absolute positive constant C_* such that

$$2^{\frac{2}{r}j} \|\Delta_j e^{(t-t_0)\Delta}a\|_{L^p(\mathbb{R}^n)} \leq C_* e^{-C_*^{-1}(t-t_0)2^{2j}} \|\Delta_j a\|_{L^p(\mathbb{R}^n)} \quad (2.3)$$

for all $j \in \mathbb{Z}$. Taking the $L^r(I)$ norm of (2.3), we see that

$$\begin{aligned}\|\Delta_j e^{(t-t_0)\Delta}a\|_{L^r(I; L^p(\mathbb{R}^n))} &\leq C_* \|e^{-C_*^{-1}(t-t_0)2^{2j}}\|_{L^r(t_0, \infty)} \|\Delta_j F\|_{L^p(\mathbb{R}^n)} \\ &= C_*(C_*^{-1}r)^{-\frac{1}{r}} 2^{-\frac{2}{r}j} \|\Delta_j F\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

As $(C_*^{-1}r)^{-\frac{1}{r}}$ is bounded with respect to r , we obtain (2.1) by multiplying (2.3) by 2^{sj} and taking the ℓ^σ -norm. Thus, we complete the proof of (2.1).

We next prove (2.2). Let $1 \leq r_2 \leq \infty$ satisfy $1 + 1/r = 1/r_2 + 1/r_1$. It follows from (2.3) and the Hausdorff–Young inequality for the time convolution that

$$\begin{aligned} \left\| \Delta_j \int_{t_0}^t e^{(t-\tau)\Delta} f(\tau) d\tau \right\|_{L^r(I; L^p(\mathbb{R}^n))} &\leq C_* \left\| \int_{t_0}^t e^{-C_*^{-1}(t-\tau)2^{2j}} \|\Delta_j f(\tau)\|_{L^p(\mathbb{R}^n)} d\tau \right\|_{L^r(I)} \\ &\leq C_* \|e^{-C_*^{-1}t2^{2j}}\|_{L^{r_2}(0,\infty)} \|\Delta_j f\|_{L^{r_1}(I; L^p(\mathbb{R}^n))} \\ &= C_*(C_*^{-1}r_2)^{-\frac{1}{r_2}} 2^{(-2+\frac{2}{r_1}-\frac{2}{r})j} \|\Delta_j f\|_{L^r(I; L^p(\mathbb{R}^n))}. \end{aligned}$$

Note that $(C_*^{-1}r_2)^{-\frac{1}{r_2}}$ is bounded with respect to $1 \leq r_2 \leq \infty$. Thus, we obtain (2.2) by multiplying (2.3) by $2^{(s+\frac{2}{r})j}$ and taking the ℓ^σ -norm. Thus, we complete the proof of (2.1). $\blacksquare$

Combining Lemma 2.1 and the paradifferential calculus in Appendix A, we obtain the following bilinear estimate.

Lemma 2.3. *Let $n \geq 2$ be an integer. Let $1 \leq p, q, \sigma \leq \infty$ and $2 \leq r \leq r_1 \leq \infty$ satisfy*

$$1 \leq p, q, r, \sigma \leq \infty, \quad r \leq r_1 \leq \infty, \\ \max\left\{0, n\left(\frac{1}{q} - \frac{1}{p}\right)\right\} < 1 - \frac{2}{r}, \quad \min\left\{n, n\left(\frac{1}{p} + \frac{1}{q}\right)\right\} - 2 + \frac{2}{r} > 0. \quad (2.4)$$

Then there exists a positive constant $C = C(n, p, q, r, \sigma)$ such that

$$\begin{aligned} &\left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes v(\tau)) d\tau \right\|_{\widetilde{L}^{r_1}(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r_1}}(\mathbb{R}^n))} \\ &\leq C \|u\|_{\widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|v\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \end{aligned}$$

for all $I = (t_0, t_1) \subset \mathbb{R}$, $u \in \widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))$, and $v \in \widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))$.

Remark 2.4. In the proof of Theorem 1.1, we use the case $p = q$ and $r = \infty$:

$$\begin{aligned} &\left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes v(\tau)) d\tau \right\|_{\widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \\ &\leq C_1 \|u\|_{\widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|v\|_{\widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))}, \end{aligned}$$

where we need to assume $n \geq 3$ and $1 \leq p < n$ due to the conditions (2.4).

Proof of Lemma 2.3. It follows from Lemma 2.1 that

$$\begin{aligned} &\left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes v(\tau)) d\tau \right\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\ &\leq C \|u \otimes v\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-2+\frac{2}{r}}(\mathbb{R}^n))} \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{k,\ell=1}^n \|T_{u_k} v_\ell\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-2+\frac{2}{r}}(\mathbb{R}^n))} + C \sum_{k,\ell=1}^n \|T_{v_\ell} u_k\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-2+\frac{2}{r}}(\mathbb{R}^n))} \\
&\quad + C \sum_{k,\ell=1}^n \|R(u_k, v_\ell)\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-2+\frac{2}{r}}(\mathbb{R}^n))} \\
&=: A_1[u, v] + A_2[u, v] + A_3[u, v].
\end{aligned}$$

See Appendix A for the definitions of the Bony decompositions $T_f g$ and $R(f, g)$. First, we consider the estimate of $A_1[u, v]$. By Lemma A.1 (1), we have

$$\begin{aligned}
A_1[u, v] &\leq C \|u\|_{\widetilde{L}^\infty(I; \dot{B}_{\infty,\sigma}^{-1}(\mathbb{R}^n))} \|v\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\
&\leq C \|u\|_{\widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|v\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))}.
\end{aligned}$$

Next we focus on the estimate of $A_2[u, v]$. For the case of $p \leq q$, it holds by Lemma A.1 and $-1 + 2/r < 0$ that

$$\begin{aligned}
A_2[u, v] &\leq C \|v\|_{\widetilde{L}^r(I; \dot{B}_{\infty,\sigma}^{-1+\frac{2}{r}}(\mathbb{R}^n))} \|u\|_{\widetilde{L}^\infty(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n))} \\
&\leq C \|v\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \|u\|_{\widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))}.
\end{aligned}$$

For the case of $q \leq p$, we define $1 \leq \theta \leq \infty$ by $1/q = 1/\theta + 1/p$. Then, using $q \leq \theta$, $1/q = 1/\theta + 1/p$, and $n/\theta - 1 + 2/r = n(1/q - 1/p) - 1 + 2/r < 0$, we see that

$$\begin{aligned}
A_2[u, v] &\leq C \|v\|_{\widetilde{L}^r(I; \dot{B}_{\theta,\sigma}^{\frac{n}{\theta}-1+\frac{2}{r}}(\mathbb{R}^n))} \|u\|_{\widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \\
&\leq C \|v\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \|u\|_{\widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))}.
\end{aligned}$$

Finally, we consider the estimate of $A_3[u, v]$. For the case of $1/p + 1/q \geq 1$, it holds by Lemma A.1 (2) and $n - 2 + 2/r > 0$ that

$$\begin{aligned}
A_3[u, v] &\leq C \sum_{k,\ell=1}^n \|R(u_k, v_\ell)\|_{\widetilde{L}^r(I; \dot{B}_{1,\sigma}^{n-2+\frac{2}{r}}(\mathbb{R}^n))} \\
&\leq C \|u\|_{\widetilde{L}^\infty(I; \dot{B}_{q',\sigma}^{\frac{n}{q'}-1}(\mathbb{R}^n))} \|v\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\
&\leq C \|u\|_{\widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|v\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))},
\end{aligned}$$

where we have used $p \leq q'$. For the case of $1/p + 1/q \leq 1$, we define $1 \leq \zeta \leq \infty$ by $1/\zeta = 1/p + 1/q$. Then we have by $n/\zeta - 2 + 2/r = n(1/p + 1/q) - 2 + 2/r > 0$ that

$$\begin{aligned}
A_3[u, v] &\leq C \sum_{k,\ell=1}^n \|R(u_k, v_\ell)\|_{\widetilde{L}^r(I; \dot{B}_{\zeta,\sigma}^{\frac{n}{\zeta}-2+\frac{2}{r}}(\mathbb{R}^n))} \\
&\leq C \|u\|_{\widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|v\|_{\widetilde{L}^r(I; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))}.
\end{aligned}$$

Hence, we complete the proof. ■

3. Higher dimensional analysis: Proofs of Theorems 1.1 and 1.3

In this section we provide the proofs of our main theorems in the higher-dimensional case. We are ready to prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1. By Lemmas 2.1 and 2.3 with $p = q$ and $r = \infty$ (see also Remark 2.4), there exists a positive constant $C_1 = C_1(n, p, \sigma)$ such that

$$\begin{aligned} & \left\| \int_{-\infty}^t e^{(t-\tau)\Delta} \mathbb{P} f(\tau) d\tau \right\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \leq C_1 \|f\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n))}, \\ & \left\| \int_{-\infty}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes v(\tau)) d\tau \right\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \\ & \leq C_1 \|u\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|v\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \end{aligned}$$

for all $f \in \widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n))$ and $u, v \in \widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))$. Now let f be a T -periodic external force satisfying

$$f \in \widetilde{C}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n)), \quad \|f\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n))} \leq \frac{1}{16C_1^2}.$$

We consider the map

$$\Phi[u](t) := \int_{-\infty}^t e^{(t-\tau)\Delta} \mathbb{P} f(\tau) d\tau - \int_{-\infty}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes u(\tau)) d\tau$$

on the complete metric space $(S_{p,\sigma}, d_{S_{p,\sigma}})$, which is defined by

$$S_{p,\sigma} := \left\{ u \in \widetilde{C}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n)); u(t+T) = u(t) \text{ for all } t \in \mathbb{R}, \right. \\ \left. \|u\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \leq 2C_1 \|f\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n))} \right\},$$

$$d_{S_{p,\sigma}}(u, \tilde{u}) := \|u - \tilde{u}\|_{\widetilde{L}^\infty(0,\infty; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))}.$$

Then, for any $u \in S_{p,\sigma}$, since $\Phi[u]$ is T -periodic and satisfies

$$\begin{aligned} \|\Phi[u]\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} & \leq C_1 \|f\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n))} + C_1 \|u\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))}^2 \\ & \leq C_1 \|f\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n))} + 4C_1^3 \|f\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n))}^2 \\ & \leq 2C_1 \|f\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-3}(\mathbb{R}^n))}, \end{aligned}$$

we see that $\Phi[u] \in S_{p,\sigma}$. For $u, v \in S_{p,\sigma}$, there holds by Lemma 2.3 with $p = q$ and $r = \infty$ that

$$\begin{aligned} \Phi[u](t) - \Phi[v](t) & = - \int_{-\infty}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes (u(\tau) - v(\tau))) d\tau \\ & \quad - \int_{-\infty}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}((u(\tau) - v(\tau)) \otimes v(\tau)) d\tau, \end{aligned} \quad (3.1)$$

which with Lemma 2.3 with $p = q$ and $r = \infty$ implies

$$\begin{aligned} & \|\Phi[u] - \Phi[v]\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \\ & \leq C_1 \left(\|u\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} + \|v\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \right) \|u - v\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \\ & \leq 4C_1^2 \|f\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|u - v\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \\ & \leq \frac{1}{4} \|u - v\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))}. \end{aligned}$$

Hence, $\Phi[\cdot]$ is a contraction map on $S_{p,\sigma}$ and the Banach fixed point theorem implies there exists a unique $u_{\text{per}} \in S_{p,\sigma}$ such that $u_{\text{per}} = \Phi[u_{\text{per}}]$, which yields a T -periodic mild solution satisfying (1.7).

For the uniqueness of T -periodic solutions, let us assume a T -periodic external force generates two T -periodic mild solutions u_{per} and v_{per} to (1.1) in the following class:

$$\left\{ u \in \widetilde{C}(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n)); \|u\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \leq \frac{1}{4C_1} \right\}.$$

Then, using a similar observation to (3.1), we have

$$\begin{aligned} \|u_{\text{per}} - v_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} & \leq C_1 \|u_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|u_{\text{per}} - v_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \\ & \quad + C_1 \|v_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|u_{\text{per}} - v_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \\ & \leq \frac{1}{2} \|u_{\text{per}} - v_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))}, \end{aligned}$$

which implies $u_{\text{per}} = v_{\text{per}}$. Thus, we complete the proof. $\blacksquare$

Proof of Theorem 1.3. From Lemmas 2.1 and 2.3 there exists a positive constant $C_1 = C_1(n, p, q, r, \sigma)$ such that

$$\begin{aligned} & \|e^{t\Delta} w_0\|_{\widetilde{L}^\infty(0,\infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)) \cap \widetilde{L}^r(0,\infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \leq C_1 \|w_0\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)}, \\ & \left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes v(\tau)) d\tau \right\|_{\widetilde{L}^\infty(0,\infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)) \cap \widetilde{L}^r(0,\infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\ & \leq C_1 \|u\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|v\|_{\widetilde{L}^r(0,\infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))}, \\ & \left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(v(\tau) \otimes w(\tau)) d\tau \right\|_{\widetilde{L}^\infty(0,\infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)) \cap \widetilde{L}^r(0,\infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\ & \leq C_2 \|v\|_{\widetilde{L}^\infty(0,\infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n))} \|w\|_{\widetilde{L}^r(0,\infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \end{aligned}$$

for all $w_0 \in \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)$, $u \in \widetilde{L}^\infty(I; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))$, and $v, w \in \widetilde{L}^\infty(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)) \cap \widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))$. We assume that the time-periodic solution u_{per} and the initial

disturbance $w_0 \in \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)$ satisfy

$$\|u_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \leq \frac{1}{8C_1^2}, \quad \|w_0\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)} \leq \frac{1}{8C_1^2}.$$

To construct a mild solution to (1.8), we consider a map

$$\begin{aligned} \Psi[w](t) &:= e^{t\Delta} w_0 - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_{\text{per}}(\tau) \otimes w(\tau) + w(\tau) \otimes u_{\text{per}}(\tau)) d\tau \\ &\quad - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(w(\tau) \otimes w(\tau)) d\tau. \end{aligned}$$

We define the complete metric space $(S_{q,\sigma}^r, d_{S_{q,\sigma}^r})$ by

$$\begin{aligned} S_{q,\sigma}^r &:= \left\{ w \in \widetilde{C}([0, \infty); \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)) \cap \widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n)); \right. \\ &\quad \left. \|w\|_{\widetilde{L}^\infty(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)) \cap \widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \leq 2C_1 \|w_0\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)} \right\}, \\ d_{S_{q,\sigma}^r}(w, \tilde{w}) &:= \|w - \tilde{w}\|_{\widetilde{L}^\infty(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)) \cap \widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))}. \end{aligned}$$

Then, for any $w, \tilde{w} \in S_{q,\sigma}^r$, there holds

$$\begin{aligned} &\|\Psi[w]\|_{\widetilde{L}^\infty(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)) \cap \widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\ &\leq C_1 \|w_0\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)} + 2C_1 \|u_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|w\|_{\widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\ &\quad + C_1 \|w\|_{\widetilde{L}^\infty(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n))} \|w\|_{\widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\ &\leq C_1 \|w_0\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)} + 4C_1^3 \|u_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|w_0\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)} \\ &\quad + 4C_1^3 \left(\|w_0\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)} \right)^2 \\ &\leq 2C_1 \|w_0\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)} \end{aligned}$$

and

$$\begin{aligned} &\|\Psi[w] - \Psi[\tilde{w}]\|_{\widetilde{L}^\infty(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)) \cap \widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\ &\leq 2C_1 \|u_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|w - \tilde{w}\|_{\widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\ &\quad + C_1 \left(\|w\|_{\widetilde{L}^\infty(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n))} + \|\tilde{w}\|_{\widetilde{L}^\infty(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n))} \right) \|w - \tilde{w}\|_{\widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\ &\leq 2C_1^2 \left(\|u_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} + \|w_0\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)} \right) \|w - \tilde{w}\|_{\widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\ &\leq \frac{1}{2} \|w - \tilde{w}\|_{\widetilde{L}^\infty(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)) \cap \widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))}, \end{aligned}$$

which implies that $\Psi[\cdot]$ is a contraction map on $(S_{q,\sigma}^r, d_{S_{q,\sigma}^r})$. Hence, it follows from the Banach fixed point theorem that there exists a unique $w \in S_{q,\sigma}^r$ such that $w = \Psi[w]$, which yields a mild solution to (1.8). The uniqueness in $\widetilde{C}([0, \infty); \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)) \cap \widetilde{L}^r(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))$ is a straightforward argument.

Finally, we show (1.10). Let $T' > T > 0$. It holds that

$$\begin{aligned} w(t) &= e^{(t-T)\Delta} w(T) - \int_T^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_{\text{per}}(\tau) \otimes w(\tau) + w(\tau) \otimes u_{\text{per}}(\tau)) d\tau \\ &\quad - \int_T^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(w(\tau) \otimes w(\tau)) d\tau \end{aligned}$$

for $t > T$. Then, similarly to above, we have

$$\begin{aligned} \|w\|_{\widetilde{L}^\infty(T', \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n))} &\leq \|e^{(t-T)\Delta} w(T)\|_{\widetilde{L}^\infty(T', \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n))} \\ &\quad + 2C_1 \|u_{\text{per}}\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,\sigma}^{\frac{n}{p}-1}(\mathbb{R}^n))} \|w\|_{\widetilde{L}^r(T, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))} \\ &\quad + C_1 \|w\|_{\widetilde{L}^\infty(0, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n))} \|w\|_{\widetilde{L}^r(T, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))}, \end{aligned}$$

which implies

$$\begin{aligned} \|w\|_{\widetilde{L}^\infty(T', \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n))} &\leq C \|e^{(t-T)\Delta} w(T)\|_{\widetilde{L}^\infty(T', \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n))} \\ &\quad + C \|w\|_{\widetilde{L}^r(T, \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1+\frac{2}{r}}(\mathbb{R}^n))}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|w(T')\|_{\dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n)} &\leq \|w\|_{\widetilde{L}^\infty(T', \infty; \dot{B}_{q,\sigma}^{\frac{n}{q}-1}(\mathbb{R}^n))} \\ &\leq C \left\{ \sum_{j \in \mathbb{Z}} (e^{-c2^{2j}(T'-T)} 2^{(\frac{n}{q}-1)j} \|\Delta_j w(T)\|_{L^q})^\sigma \right\}^{\frac{1}{\sigma}} \\ &\quad + C \left\{ \sum_{j \in \mathbb{Z}} (2^{(\frac{n}{q}-1+\frac{2}{r})j} \|\Delta_j w\|_{L^r(T, \infty; L^q)})^\sigma \right\}^{\frac{1}{\sigma}}. \end{aligned}$$

Hence, letting $T' \rightarrow \infty$ and then letting $T \rightarrow \infty$, we complete the proof. $\blacksquare$

4. Two-dimensional analysis: Proof of Theorem 1.5

The goal of this section is to prove Theorem 1.5. To this end, we investigate some properties of the following initial value problem of the Navier–Stokes equations:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = f, & t > t_0, x \in \mathbb{R}^2, \\ \operatorname{div} u = 0, & t \geq t_0, x \in \mathbb{R}^2, \\ u(t_0, x) = a(x), & t = t_0, x \in \mathbb{R}^2, \end{cases} \quad (4.1)$$

where $t_0 \in \mathbb{R}$ is a given initial time and $a = a(x)$ is a given initial data. We say that v is a mild solution to (4.1) if it satisfies

$$u(t) = e^{(t-t_0)\Delta} a + \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} f(\tau) d\tau - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes u(\tau)) d\tau. \quad (4.2)$$

4.1. Construction of non-periodic-in-time mild solutions

The aim of this subsection is to show that for any initial data, there exists a mild solution to (4.1) that is not T -periodic if we choose an appropriate time-periodic external force. More precisely, we prove the following proposition.

Proposition 4.1. *Let $1 \leq p \leq 2$. Then there exists a positive constant $\varepsilon_0 = \varepsilon_0(p)$ such that the following statement holds. For any $0 < \delta \leq \varepsilon_0$, and $0 < T \leq 2^{\frac{1}{\delta^2}}$, there exist a T -periodic external force $f_\delta \in \tilde{C}(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2))$ with*

$$\|f_\delta\|_{\widetilde{L^\infty}(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2))} \leq \delta$$

and a $k_{\delta,T} \in \mathbb{N}$ such that for any initial time $t_0 \in \mathbb{R}$ and initial data $a \in \dot{B}_{2,1}^0(\mathbb{R}^2)$ with

$$\operatorname{div} a = 0, \quad \|a\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \leq \varepsilon_0, \quad (4.3)$$

(4.1) possesses a mild solution $u_\delta[a] \in \tilde{C}([t_0, t_0 + k_{\delta,T}T]; \dot{B}_{2,1}^0(\mathbb{R}^2))$ satisfying

$$\|u_\delta[a](t_0 + k_{\delta,T}T)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \geq 2\varepsilon_0. \quad (4.4)$$

Remark 4.2. By (4.3) and (4.4) we see that

$$u_\delta[a](t_0) \neq u_\delta[a](t_0 + k_{\delta,T}T),$$

which implies the solution $u_\delta[a]$ is never T -periodic, regardless of the choice of small initial data $a \in \dot{B}_{2,1}^0(\mathbb{R}^2)$.

Before starting the proof of Proposition 4.1, we mention the idea and outline of it. The idea is based on [6, Theorem 3.1]. In order to obtain (4.4), we shall follow the method used in the context of ill-posedness [2, 3, 6, 17, 20] and we deduce the lower bound estimate by the second iteration of the standard successive approximation system. To this end, we decompose a solution u of (4.1) into the first iteration, the second iteration, and the remainder part:

$$u(t) = u^{(1)}(t) + u^{(2)}(t) + \tilde{u}(t),$$

where $u^{(1)}$ and $u^{(2)}$ solve the first iterative system

$$\begin{cases} \partial_t u^{(1)} - \Delta u^{(1)} + \nabla p^{(1)} = f_\delta, & t > t_0, x \in \mathbb{R}^2, \\ \operatorname{div} u^{(1)} = 0, & t \geq t_0, x \in \mathbb{R}^2, \\ u^{(1)}(t_0, x) = 0, & x \in \mathbb{R}^2, \end{cases}$$

and the second iterative system

$$\begin{cases} \partial_t u^{(2)} - \Delta u^{(2)} + (u^{(1)} \cdot \nabla) u^{(1)} + \nabla p^{(2)} = 0, & t > t_0, x \in \mathbb{R}^2, \\ \operatorname{div} u^{(2)} = 0, & t \geq t_0, x \in \mathbb{R}^2, \\ u^{(2)}(t_0, x) = 0, & x \in \mathbb{R}^2, \end{cases}$$

respectively, and the remainder $\tilde{u}$ should be a solution to

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} \\ + (u^{(1)} \cdot \nabla) u^{(2)} + (u^{(2)} \cdot \nabla) u^{(1)} + (u^{(2)} \cdot \nabla) u^{(2)} \\ + (u^{(1)} \cdot \nabla) \tilde{u} + (u^{(2)} \cdot \nabla) \tilde{u} + (\tilde{u} \cdot \nabla) u^{(1)} + (\tilde{u} \cdot \nabla) u^{(2)} \\ + (\tilde{u} \cdot \nabla) \tilde{u} + \nabla \tilde{p} = 0, & t > t_0, x \in \mathbb{R}^2, \\ \operatorname{div} \tilde{u} = 0, & t \geq t_0, x \in \mathbb{R}^2, \\ \tilde{u}(t_0, x) = a(x), & x \in \mathbb{R}^2. \end{cases}$$

Note that we regard the linear term $e^{(t-t_0)\Delta}a$ as not part of the first iteration $u^{(1)}$ but a piece of the remainder $\tilde{u}$; this allows us to obtain the lower-bound estimate for the second iteration $u^{(2)}$ with arbitrariness in the choice of the initial data a since $u^{(2)}$ is independent of a . For sufficiently small $0 < \delta \ll 1$, choosing a suitable interval $I_\delta = [t_0, t_0 + k_{\delta,T}T]$ with some large $k_{\delta,T} \in \mathbb{N}$ and a suitable external force f_δ with $\|f_\delta\|_{\widetilde{L^\infty}(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2))} \leq \delta$,

we have

$$\|u^{(1)}\|_{\widetilde{L^\infty}(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} \leq C\delta, \quad \|u^{(2)}(t_0 + k_{\delta,T}T)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \geq cM^2$$

for some positive constant $M \gg 1$ independent of δ . Hence, the essential part of the proof is to construct the remainder $\tilde{u}$. However, since the estimate

$$\left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(v(\tau) \otimes w(\tau)) d\tau \right\|_X \leq C \|v\|_X \|w\|_X \quad (4.5)$$

fails with $X = \widetilde{L^\infty}(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))$, ingenuity is needed to achieve the objectives. To overcome this, we define a norm

$$\|v\|_{\delta,I} := \|v\|_{\widetilde{L^\infty}(I; \dot{B}_{2,1}^0(\mathbb{R}^2))} + \frac{1}{\delta} \|v\|_{\widetilde{L^{\frac{2}{\delta^2}}}(I; \dot{H}^{\delta^2}(\mathbb{R}^2))}$$

for all $0 < \delta \leq 1/4$, intervals $I \subset \mathbb{R}$, and $v \in \widetilde{L^\infty}(I; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L^{\frac{2}{\delta^2}}}(I; \dot{H}^{\delta^2}(\mathbb{R}^2))$. Then we obtain the estimate (4.5) with the norm replaced by $\|\cdot\|_{\delta,I}$ and the constant C independent of δ . See Lemma 4.3 below. Then, choosing the initial data a so small that $\|a\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \leq \varepsilon_0$ with sufficiently small $0 < \varepsilon_0 \ll 1$ independent of δ and using the contraction mapping principle via the norm $\|\cdot\|_{\delta,I_\delta}$, we may construct the remainder part $\tilde{u}$ satisfying

$$\|\tilde{u}\|_{\widetilde{L^\infty}(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} \leq C\varepsilon_0.$$

Hence, collecting the above estimates, and for $0 < \delta \leq \varepsilon_0$ and sufficiently large $M \gg 1$, we have

$$\begin{aligned}
 & \|u(t_0 + k_{\delta,T}T)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \\
 & \geq \|u^{(2)}(t_0 + k_{\delta,T}T)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} - \|u^{(1)}\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} - \|\tilde{u}\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} \\
 & \geq cM^2 - C\delta - C\varepsilon_0 \\
 & \geq 2\varepsilon_0,
 \end{aligned}$$

which completes the outline of the proof.

The following lemma provides non-linear estimates for the norm $\|\cdot\|_{\delta,I}$.

Lemma 4.3. *For an interval $I = [t_0, t_1) \subset \mathbb{R}$, the following statements hold:*

(1) *There exists an absolute positive constant C such that*

$$\left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes v(\tau)) d\tau \right\|_{\delta,I} \leq C \|u\|_{\delta,I} \|v\|_{\delta,I} \quad (4.6)$$

for all $0 \leq \delta \leq 1/4$ and $u, v \in \widetilde{L}^\infty(I; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^{\frac{2}{\delta^2}}(I; \dot{H}^{\delta^2}(\mathbb{R}^2))$.

(2) *There exists an absolute positive constant C such that*

$$\begin{aligned}
 & \left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes v(\tau)) d\tau \right\|_{\widetilde{L}^\infty(I; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^4(I; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} \\
 & \leq C \|u\|_{\widetilde{L}^4(I; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} \|v\|_{\widetilde{L}^\infty(I; \dot{B}_{2,1}^0(\mathbb{R}^2))}
 \end{aligned}$$

for all $u \in \widetilde{L}^4(I; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))$ and $v \in \widetilde{L}^\infty(I; \dot{B}_{2,1}^0(\mathbb{R}^2))$.

Remark 4.4. We should emphasize that the positive constant C appearing in (4.6) is independent of δ .

Proof of Lemma 4.3. As (2) is obtained by Lemma 2.3, we only prove (1). This is inspired by [6, Lemma 2.5]. We decompose the left-hand side by the Bony decomposition as follows:

$$\begin{aligned}
 & \left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes v(\tau)) d\tau \right\|_{\delta,I} \\
 & \leq \left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}\{T_{u_k(\tau)}v_\ell(\tau) + T_{v_\ell(\tau)}u_k(\tau)\}_{1 \leq k, \ell \leq 3} d\tau \right\|_{\delta,I} \\
 & \quad + \left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}\{R(u_k(\tau), v_\ell(\tau))\}_{1 \leq k, \ell \leq 3} d\tau \right\|_{\delta,I}.
 \end{aligned}$$

Here, see Appendix A for the definition of $T_f g$ and $R(f, g)$. It follows from Lemmas 2.1 and A.1 that

$$\begin{aligned}
 & \left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div} \{T_{u_k}(\tau) v_\ell(\tau) + T_{v_\ell}(\tau) u_k(\tau)\}_{1 \leq k, \ell \leq 3} d\tau \right\|_{\delta, I} \\
 & \leq C \sum_{1 \leq k, \ell \leq 3} \|T_{u_k} v_\ell + T_{v_\ell} u_k\|_{\widetilde{L}^\infty(I; \dot{B}_{2,1}^{-1}(\mathbb{R}^2))} \\
 & \quad + \frac{C}{\delta} \sum_{1 \leq k, \ell \leq 3} \|T_{u_k} v_\ell + T_{v_\ell} u_k\|_{\widetilde{L}^{\frac{2}{\delta^2}}(I; \dot{H}^{\delta^2-1}(\mathbb{R}^2))} \\
 & \leq C \|u\|_{\widetilde{L}^\infty(I; \dot{B}_{2,1}^0(\mathbb{R}^2))} \|v\|_{\widetilde{L}^\infty(I; \dot{B}_{2,1}^0(\mathbb{R}^2))} + \frac{C}{\delta} \|u\|_{\widetilde{L}^{\frac{2}{\delta^2}}(I; \dot{H}^{\delta^2}(\mathbb{R}^2))} \|v\|_{\widetilde{L}^\infty(I; \dot{B}_{2,1}^0(\mathbb{R}^2))} \\
 & \leq C \|u\|_{\delta, I} \|v\|_{\delta, I}.
 \end{aligned}$$

Using Lemma A.2, we have

$$\begin{aligned}
 & \left\| \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div} \{R(u_k(\tau), v_\ell(\tau))\}_{1 \leq k, \ell \leq 3} d\tau \right\|_{\delta, I} \\
 & \leq C \sum_{1 \leq k, \ell \leq 3} \left(\|R(u_k, v_\ell)\|_{\widetilde{L}^{\frac{1}{\delta^2}}(I; \dot{B}_{2,1}^{2\delta^2-1}(\mathbb{R}^2))} + \frac{1}{\delta} \|R(u_k, v_\ell)\|_{\widetilde{L}^{\frac{1}{\delta^2}}(I; \dot{H}^{2\delta^2-1}(\mathbb{R}^2))} \right) \\
 & \leq C \sum_{1 \leq k, \ell \leq 3} \left(\|R(u_k, v_\ell)\|_{\widetilde{L}^{\frac{1}{\delta^2}}(I; \dot{B}_{1,1}^{2\delta^2}(\mathbb{R}^2))} + \frac{1}{\delta} \|R(u_k, v_\ell)\|_{\widetilde{L}^{\frac{1}{\delta^2}}(I; \dot{B}_{1,2}^{2\delta^2}(\mathbb{R}^2))} \right) \\
 & \leq \frac{C}{\delta^2} \|u\|_{\widetilde{L}^{\frac{2}{\delta^2}}(I; \dot{H}^{\delta^2}(\mathbb{R}^2))} \|v\|_{\widetilde{L}^{\frac{2}{\delta^2}}(I; \dot{H}^{\delta^2}(\mathbb{R}^2))} \\
 & \leq C \|u\|_{\delta, I} \|v\|_{\delta, I},
 \end{aligned}$$

which completes the proof. $\blacksquare$

Now we provide the rigorous proof of Proposition 4.1.

Proof of Proposition 4.1. Let the initial data $a \in \dot{B}_{2,1}^0(\mathbb{R}^2)$ satisfy

$$\operatorname{div} a = 0, \quad \|a\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \leq \varepsilon.$$

Let $M \geq 10$ and $0 < \varepsilon, \eta \leq 1/2$ be positive constants to be determined later. Let $0 < \delta \leq \eta$ and $0 < T \leq 2^{\frac{1}{\delta^2}}$. For any $\mathbb{R}^2$ -valued divergence-free T -periodic function $h \in \widetilde{C}(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2)) \cap \widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,2}^{\frac{2}{p}-3+\delta^2}(\mathbb{R}^2))$ with

$$\|h\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2))} + \|h\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3+\delta^2}(\mathbb{R}^2))} \leq 1,$$

we define the external force as

$$\begin{aligned}
 f_\delta(t, x) &:= \eta \delta \Delta g(x) + \eta^2 \delta h(t, x), \\
 g(x) &:= \nabla^\perp(\psi(x) \cos(Mx_1)),
 \end{aligned} \tag{4.7}$$

where the function $\psi \in \mathcal{S}(\mathbb{R}^2)$ satisfies that $\hat{\psi}$ is radial symmetric and

$$0 \leq \hat{\psi}(\xi) \leq 1, \quad \hat{\psi}(\xi) = \begin{cases} 1 & (|\xi| \leq 1), \\ 0 & (|\xi| \geq 2). \end{cases}$$

We note that f_δ is $\mathbb{R}^3$ -valued, T -periodic, and divergence-free. Using Lemma 2.1 and $\text{supp } \hat{g} \subset \{\xi \in \mathbb{R}^2; M - 2 \leq |\xi| \leq M + 2\}$, we have

$$\begin{aligned} \|e^{t\Delta}a\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} &\leq C_0 \|a\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)}, \\ \|f_\delta\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2))} &\leq C\eta\delta \|g\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)} + \eta^2\delta \|h\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2))} \\ &\leq CM^{\frac{2}{p}} \|\psi\|_{L^p(\mathbb{R}^2)} \eta\delta + \eta^2\delta \\ &\leq C_0 M^{\frac{2}{p}} \eta\delta \end{aligned} \quad (4.8)$$

for some positive constant $C_0 = C_0(p, \|\psi\|_{L^p(\mathbb{R}^2)})$.

We set $k_{\delta,T} \in \mathbb{N}$ and $I_\delta \subset \mathbb{R}$ as

$$\frac{2^{\frac{1}{\delta^2}}}{T} \leq k_{\delta,T} < \frac{2^{\frac{1}{\delta^2}}}{T} + 1, \quad I_\delta := [t_0, t_0 + k_{\delta,T}T].$$

Here, we note that

$$2 \leq (k_{\delta,T}T)^{\delta^2} < 4. \quad (4.9)$$

We define

$$\begin{aligned} u_\delta^{(1)}(t) &:= \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} f_\delta(\tau) d\tau = \int_{t_0}^t e^{(t-\tau)\Delta} f_\delta(\tau) d\tau, \\ u_\delta^{(2)}(t) &:= - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_\delta^{(1)}(\tau) \otimes u_\delta^{(1)}(\tau)) d\tau, \\ u_\delta^{(3)}(t) &:= - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_\delta^{(1)}(\tau) \otimes u_\delta^{(2)}(\tau) + u_\delta^{(2)}(\tau) \otimes u_\delta^{(1)}(\tau) \\ &\quad + u_\delta^{(2)}(\tau) \otimes u_\delta^{(2)}(\tau)) d\tau. \end{aligned}$$

and consider the following integral equation:

$$\begin{aligned} \tilde{u}(t) &= e^{t\Delta}a + u_\delta^{(3)}(t) \\ &\quad - \sum_{m=1}^2 \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_\delta^{(m)}(\tau) \otimes \tilde{u}(\tau) + \tilde{u}(\tau) \otimes u_\delta^{(m)}(\tau)) d\tau \\ &\quad - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(\tilde{u}(\tau) \otimes \tilde{u}(\tau)) d\tau. \end{aligned} \quad (4.10)$$

We note that once we establish a solution $\tilde{u}_\delta[a]$ to (4.10), we obtain the mild solution to (4.1) by $u_\delta[a](t) := u_\delta^{(1)}(t) + u_\delta^{(2)}(t) + \tilde{u}_\delta[a](t)$.

For the estimates of $u_\delta^{(1)}$, we decompose it as

$$\begin{aligned} u_\delta^{(1)}(t) &= \eta\delta\Delta \int_{t_0}^t e^{(t-\tau)\Delta} g \, d\tau + \int_{t_0}^t e^{(t-\tau)\Delta} h(\tau) \, d\tau \\ &= -\eta\delta g + \eta\delta e^{(t-t_0)\Delta} g + \eta^2\delta \int_{t_0}^t e^{(t-\tau)\Delta} h(\tau) \, d\tau \\ &=: u_\delta^{(1;1)} + u_\delta^{(1;2)}(t) + u_\delta^{(1;3)}(t). \end{aligned}$$

Then it follows from Lemma 2.1 and the Bernstein inequality that

$$\begin{aligned} \|u_\delta^{(1;1)}\|_{\widetilde{L^\infty}(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} &= \eta\delta \|g\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \leq CM \|\psi\|_{L^2(\mathbb{R}^2)} \eta\delta \leq C_1 M \eta\delta, \\ \|u_\delta^{(1;2)}\|_{\widetilde{L^\infty}(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L^{\frac{2}{\delta^2}}}(I_\delta; \dot{H}^{\delta^2}(\mathbb{R}^2))} &\leq C \eta\delta \|g\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \leq C_1 M \eta\delta, \\ \|u_\delta^{(1;3)}\|_{\widetilde{L^\infty}(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} &\leq C \eta^2\delta \|h\|_{\widetilde{L^\infty}(I_\delta; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2))} \leq C_1 \eta^2\delta \end{aligned}$$

and

$$\begin{aligned} \|u_\delta^{(1;1)}\|_{\widetilde{L^{\frac{2}{\delta^2}}}(I_\delta; \dot{H}^{\delta^2}(\mathbb{R}^2))} &\leq C \eta\delta (k_\delta T)^{\frac{\delta^2}{2}} \|g\|_{\dot{H}^{\delta^2}(\mathbb{R}^2)} \leq C_1 M^2 \eta\delta, \\ \|u_\delta^{(1;3)}\|_{\widetilde{L^{\frac{2}{\delta^2}}}(I_\delta; \dot{H}^{\delta^2}(\mathbb{R}^2))} &\leq C \eta^2\delta \|h\|_{\widetilde{L^{\frac{2}{\delta^2}}}(I_\delta; \dot{H}^{-2+\delta^2}(\mathbb{R}^2))} \\ &\leq C \eta^2\delta (k_\delta T)^{\frac{\delta^2}{2}} \|h\|_{\widetilde{L^\infty}(I_\delta; \dot{B}_{p,2}^{\frac{2}{p}-3+\delta^2}(\mathbb{R}^2))} \leq C_1 \eta^2\delta \end{aligned}$$

for some positive constant C_1 . Thus, we have

$$\|u_\delta^{(1)}\|_{\widetilde{L^\infty}(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} \leq 3C_1 M^2 \eta\delta, \quad \|u_\delta^{(1)}\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \leq 6C_1 M^2 \eta. \quad (4.11)$$

For the estimates for $u_\delta^{(2)}$, we decompose it as

$$\begin{aligned} u_\delta^{(2)}(t) &= \sum_{k,\ell=1}^3 u_\delta^{(2;k,\ell)}(t) \\ &= u_\delta^{(2;1,1)}(t) + \sum_{(k,\ell) \in \{1,3\}^2 \setminus \{(1,1)\}} u_\delta^{(2;k,\ell)}(t) + \sum_{(k,\ell) \in \{1,2,3\}^2 \setminus \{1,3\}^2} u_\delta^{(2;k,\ell)}(t) \\ &=: u_\delta^{(2;1)}(t) + u_\delta^{(2;2)}(t) + u_\delta^{(2;3)}(t), \end{aligned}$$

where we have set

$$u_\delta^{(2;k,\ell)}(t) := - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_\delta^{(1;k)}(\tau) \otimes u_\delta^{(1;\ell)}(\tau)) \, d\tau.$$

It follows from Lemma 4.3 that

$$\begin{aligned} \|u_\delta^{(2;1)}\|_{\delta, I_\delta} &\leq C \|u_\delta^{(1;1)}\|_{\delta, I_\delta}^2 \leq C_2 M^4 \eta^2, \\ \|u_\delta^{(2;2)}\|_{\delta, I_\delta} &\leq C \sum_{(k,\ell) \in \{1,3\}^2 \setminus \{(1,1)\}} \|u_\delta^{(1;k)}\|_{\delta, I_\delta} \|u_\delta^{(1;\ell)}\|_{\delta, I_\delta} \leq C_2 M^2 \eta^3 \end{aligned} \quad (4.12)$$

$$\begin{aligned} \|u_\delta^{(2;3)}\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^{\frac{2}{\delta^2}}(I_\delta; \dot{H}^{\delta^2}(\mathbb{R}^2))} \\ \leq C \|u_\delta^{(1;2)}\|_{\widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} \sum_{\ell=1}^3 \|u_\delta^{(1;\ell)}\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} \leq C_2 M^2 \eta^2 \delta^2 \end{aligned} \quad (4.13)$$

for some positive constant C_2 . Thus, we have²

$$\|u_\delta^{(2)}\|_{\delta, I_\delta} \leq 3C_2 M^4 \eta^2. \quad (4.14)$$

For the estimates of $u_\delta^{(3)}$, we rewrite it as

$$u_\delta^{(3)}(t) = - \sum_{(i,j) \in \{1,2\}^2 \setminus \{(1,1)\}} \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_\delta^{(i)}(\tau) \otimes u_\delta^{(j)}(\tau)) d\tau,$$

and by Lemma 4.3, there holds

$$\|u_\delta^{(3)}\|_{\delta, I_\delta} \leq C \sum_{(i,j) \in \{1,2\}^2 \setminus \{(1,1)\}} \|u_\delta^{(i)}\|_{\delta, I_\delta} \|u_\delta^{(j)}\|_{\delta, I_\delta} \leq C_3 M^8 \eta^3. \quad (4.15)$$

for some positive constant C_3 . Now we construct a solution $\tilde{u}_\delta$ to the equation (4.10). To this end, we define a complete metric space (S_δ, d_{S_δ}) by

$$\begin{aligned} S_\delta &:= \{u = u' + u'' \in \widetilde{C}(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)); u' \in S'_\delta, u'' \in S''_\delta\}, \\ d_{S_\delta}(u, v) &:= \inf_{\substack{u=u'+u'' \\ v=v'+v'' \\ u', v' \in S'_\delta \\ u'', v'' \in S''_\delta}} \left(\|u' - v'\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} + \|u'' - v''\|_{\delta, I_\delta} \right), \end{aligned}$$

where

$$\begin{aligned} S'_\delta &:= \left\{ u' \in \widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2)); \right. \\ &\quad \left. \|u'\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} \leq 4C_0 \varepsilon \right\}, \\ S''_\delta &:= \left\{ u'' \in \widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^{\frac{2}{\delta^2}}(I_\delta; \dot{H}^{\delta^2}(\mathbb{R}^2)); \|u''\|_{\delta, I_\delta} \leq 4C_3 M^8 \eta^3 \right\}, \end{aligned}$$

²To obtain the estimate (4.14), we do not need the decomposition and it suffices to apply Lemma 4.3 to the definition of $u_\delta^{(2)}$. However, constructing the estimate below in (4.20), we use these decompositions $u^{(2,k)}$; therefore we prepare them here.

and let us consider a map

$$\begin{aligned}\Phi_\delta[u](t) &= e^{t\Delta}a + u_\delta^{(3)}(t) \\ &\quad - \sum_{m=1}^2 \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_\delta^{(m)}(\tau) \otimes u(\tau) + u(\tau) \otimes u_\delta^{(m)}(\tau)) d\tau \\ &\quad - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u(\tau) \otimes u(\tau)) d\tau, \quad u \in S_\delta.\end{aligned}$$

For any $u = u' + u'' \in S_\delta$ with $u' \in S'_\delta$ and $u'' \in S''_\delta$, we decompose $\Phi_\delta[u]$ as

$$\Phi_\delta[u](t) = \Phi'_\delta[u', u''](t) + \Phi''_\delta[u''](t),$$

where we have set

$$\begin{aligned}\Phi'_\delta[u', u''](t) &:= e^{t\Delta}a - \sum_{m=1}^2 \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_\delta^{(m)}(\tau) \otimes u'(\tau) + u'(\tau) \otimes u_\delta^{(m)}(\tau)) d\tau \\ &\quad - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u'(\tau) \otimes u''(\tau) + u''(\tau) \otimes u'(\tau)) d\tau \\ &\quad - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u'(\tau) \otimes u'(\tau)) d\tau, \\ \Phi''_\delta[u''](t) &:= u_\delta^{(3)}(t) - \sum_{m=1}^2 \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_\delta^{(m)}(\tau) \otimes u''(\tau) + u''(\tau) \otimes u_\delta^{(m)}(\tau)) d\tau \\ &\quad - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u''(\tau) \otimes u''(\tau)) d\tau.\end{aligned}$$

It follows from Lemma 4.3, (4.8), (4.11), (4.14), and (4.15) that

$$\begin{aligned}\|\Phi'_\delta[u', u'']\| &\widetilde{L^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} \cap \widetilde{L^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} \\ &\leq C_0 \|a\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} + C \sum_{m=1}^2 \|u_\delta^{(m)}\|_{\widetilde{L^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))}} \|u'\|_{\widetilde{L^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))}} \\ &\quad + C \|u''\|_{\widetilde{L^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))}} \|u'\|_{\widetilde{L^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))}} + C \|u'\|_{\widetilde{L^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))}}^2 \\ &\leq C_0 \varepsilon + C_4 M^4 \eta^2 \varepsilon + C_4 M^8 \eta^3 \varepsilon + C_4 \varepsilon^2\end{aligned}$$

and

$$\begin{aligned}\|\Phi''_\delta[u'']\|_{\delta, I_\delta} &\leq \|v_\delta^{(3)}\|_{\delta, I_\delta} + C \sum_{m=1}^2 \|u_\delta^{(m)}\|_{\delta, I_\delta} \|u''\|_{\delta, I_\delta} + C \|u''\|_{\delta, I_\delta}^2 \\ &\leq C_3 M^8 \eta^3 + C_4 (M^4 \eta^2) M^8 \eta^3 + C_4 (M^8 \eta^3)^2\end{aligned}$$

for some positive constant C_4 . Here, we have used

$$\sum_{m=1}^2 \|u_\delta^{(m)}\|_{\delta, I_\delta} \leq CM^2\eta\delta + CM^4\eta^2 \leq CM^4\eta^2,$$

which is implied by (4.11), (4.14), and $\delta \leq \eta$. Let $0 < \eta_1, \varepsilon_1 \leq 1/2$ satisfy

$$\max\{M^4\eta_1^2, \varepsilon_1, M^8\eta_1^3\} \leq \min\left\{\frac{C_0}{3C_4}, \frac{C_3}{3C_4}\right\}$$

and assume $0 < \eta \leq \eta_1$ and $0 < \varepsilon \leq \varepsilon_1$ in the remainder of this proof. Then we see that for every $u \in S_\delta$,

$$\begin{aligned} \|\Phi'_\delta[u', u'']\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} &\leq 2C_0\varepsilon, \\ \|\Phi''_\delta[u'']\|_{\delta, I_\delta} &\leq 2C_3M^8\eta^3, \end{aligned} \quad (4.16)$$

which implies $\Phi'_\delta[u', u''] \in S'_\delta$ and $\Phi''_\delta[u''] \in S''_\delta$. Thus, we have $\Phi_\delta[u] \in S_\delta$ for all $u \in S_\delta$. For $u = u' + u'', v = v' + v'' \in S_\delta$ with $u', v' \in S'_\delta$ and $u'', v'' \in S''_\delta$, since it holds that

$$\begin{aligned} &\Phi'_\delta[u', u''](t) - \Phi'_\delta[v', v''](t) \\ &= - \sum_{m=1}^2 \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_\delta^{(m)}(\tau) \otimes (u'(\tau) - v'(\tau))) d\tau \\ &\quad - \sum_{m=1}^2 \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}((u'(\tau) - v'(\tau)) \otimes u_\delta^{(m)}(\tau)) d\tau \\ &\quad - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}((u'(\tau) - v'(\tau)) \otimes u''(\tau) + v'(\tau) \otimes (u''(\tau) - v''(\tau))) d\tau \\ &\quad - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}((u''(\tau) - v''(\tau)) \otimes u'(\tau) + v''(\tau) \otimes (u'(\tau) - v'(\tau))) d\tau \\ &\quad - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}((u'(\tau) - v'(\tau)) \otimes u'(\tau) + v'(\tau) \otimes (u'(\tau) - v'(\tau))) d\tau \end{aligned}$$

and

$$\begin{aligned} &\Phi''_\delta[u''](t) - \Phi''_\delta[v''](t) \\ &= - \sum_{m=1}^2 \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u_\delta^{(m)}(\tau) \otimes (u''(\tau) - v''(\tau))) d\tau \\ &\quad - \sum_{m=1}^2 \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}((u''(\tau) - v''(\tau)) \otimes u_\delta^{(m)}(\tau)) d\tau \\ &\quad - \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}((u''(\tau) - v''(\tau)) \otimes u''(\tau) + v''(\tau) \otimes (u''(\tau) - v''(\tau))) d\tau, \end{aligned}$$

we have

$$\begin{aligned}
& \|\Phi'_\delta[u', u''] - \Phi'_\delta[v', v'']\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} \\
& \leq C \sum_{m=1}^2 \|u_\delta^{(m)}\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} \|u' - v'\|_{\widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} \\
& \quad + C \left(\|u'\|_{\widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} + \|v'\|_{\widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} \right) \|u'' - v''\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} \\
& \quad + C \left(\|u''\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} + \|v''\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} \right) \|u' - v'\|_{\widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} \\
& \quad + C \left(\|u'\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} + \|v'\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} \right) \|u' - v'\|_{\widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} \\
& \leq C_5(\varepsilon + M^8\eta) \left(\|u' - v'\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} + \|u'' - v''\|_{\delta, I_\delta} \right)
\end{aligned}$$

and

$$\begin{aligned}
& \| \Phi''_\delta[u''] - \Phi''_\delta[v''] \|_{\delta, I_\delta} \\
& \leq C \sum_{m=1}^2 \|u_\delta^{(m)}\|_{\delta, I_\delta} \|u'' - v''\|_{\delta, I_\delta} \\
& \quad + C (\|u''\|_{\delta, I_\delta} + \|v''\|_{\delta, I_\delta}) \|u'' - v''\|_{\delta, I_\delta} \\
& \leq C_5 M^8 \eta \left(\|u' - v'\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} + \|u'' - v''\|_{\delta, I_\delta} \right)
\end{aligned}$$

for some positive constant C_5 . Thus, it holds that

$$\begin{aligned}
& d_{S_\delta}(\Phi_\delta[u], \Phi_\delta[v]) \\
& \leq \inf_{\substack{u=u'+u'' \\ v=v'+v'' \\ u', v' \in S'_\delta \\ u'', v'' \in S''_\delta}} \left(\|\Phi'_\delta[u', u''] - \Phi'_\delta[v', v'']\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2)) \cap \widetilde{L}^4(I_\delta; \dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R}^2))} \right. \\
& \quad \left. + \| \Phi''_\delta[u''] - \Phi''_\delta[v''] \|_{\delta, I_\delta} \right) \\
& \leq C_5(\varepsilon + 2M^8\eta) d_{S_\delta}(u, v),
\end{aligned}$$

where the first estimate above is ensured by the fact $\Phi'_\delta[u', u''] - \Phi'_\delta[v', v''] \in S'_\delta$ and $\Phi''_\delta[u''] - \Phi''_\delta[v''] \in S''_\delta$, which are implied by (4.16). Let $0 < \varepsilon_2, \eta_2 \leq 1/2$ satisfy

$$C_5(\varepsilon_2 + 2M^8\eta_2) \leq \frac{1}{2}.$$

In the remainder of this proof, we assume $0 < \varepsilon \leq \min\{\varepsilon_1, \varepsilon_2\}$ and $0 < \eta \leq \min\{\eta_1, \eta_2\}$. Then we see that

$$d_{S_\delta}(\Phi_\delta[u], \Phi_\delta[v]) \leq \frac{1}{2} d_{S_\delta}(u, v)$$

for all $u, v \in S_\delta$. Hence, the Banach fixed point theorem yields the unique existence of $\tilde{u}_\delta[a] \in S_\delta$ satisfying $\tilde{u}_\delta[a] = \Phi_\delta[\tilde{u}_\delta[a]]$, which implies $\tilde{u}_\delta[a]$ is a solution to (4.10). We note that $\tilde{u}_\delta[a] \in S_\delta$ implies

$$\|\tilde{u}_\delta[a]\|_{\widetilde{L^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))}} \leq 4C_0\varepsilon + 4C_3M^8\eta^3. \quad (4.17)$$

Moreover, $u_\delta[a] := u_\delta^{(1)} + u_\delta^{(2)} + \tilde{u}_\delta[a]$ is a solution to (4.2).

Finally, we establish the second estimate in (4.4). To this end, we focus on $u_\delta^{(2;1)}$. As $g = g(x)$ is time independent, there holds

$$\begin{aligned} u_\delta^{(2;1)}(t_0 + k_{\delta,T}T) &= -\eta^2\delta^2 \int_{t_0}^{t_0+k_{\delta,T}T} e^{(t_0+k_{\delta,T}T-\tau)\Delta} \mathbb{P} \operatorname{div}(g \otimes g) d\tau \\ &= \eta^2\delta^2(1 - e^{k_{\delta,T}T\Delta})(-\Delta)^{-1} \mathbb{P} \operatorname{div}(g \otimes g). \end{aligned}$$

Then we see that

$$\begin{aligned} &\|u_\delta^{(2;1)}(t_0 + k_{\delta,T}T)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \\ &\geq \eta^2\delta^2 \sum_{-\frac{1}{2\delta^2} \leq j \leq -2} \|\Delta_j(1 - e^{k_{\delta,T}T\Delta})(-\Delta)^{-1} \mathbb{P} \operatorname{div}(g \otimes g)\|_{L^2(\mathbb{R}^2)} \\ &\geq c\eta^2\delta^2 \sum_{-\frac{1}{2\delta^2} \leq j \leq -2} (1 - e^{-\frac{1}{4}2^{2j}k_{\delta,T}T}) \|\Delta_j(-\Delta)^{-1} \mathbb{P} \operatorname{div}(g \otimes g)\|_{L^2(\mathbb{R}^2)} \\ &\geq c\eta^2\delta^2 \sum_{-\frac{1}{2\delta^2} \leq j \leq -2} \|\Delta_j(-\Delta)^{-1} \mathbb{P} \operatorname{div}(g \otimes g)\|_{L^2(\mathbb{R}^2)}, \end{aligned}$$

where we have used the estimate

$$1 - e^{-\frac{1}{4}2^{2j}k_{\delta,T}T} \geq 1 - e^{-\frac{1}{4}2^{-\frac{1}{\delta^2}}k_{\delta,T}T} \geq 1 - e^{-\frac{1}{4}2^{-\frac{1}{\delta^2}}2^{\frac{1}{\delta^2}}} = 1 - e^{-\frac{1}{4}},$$

which is implied by $-1/(2\delta^2) \leq j \leq -2$ and (4.9). We consider the estimate for $\Delta_j(g \otimes g)$. It follows from [5, Lemmas 2.1 and 2.4] that for $-1/(2\delta^2) \leq j \leq -2$,

$$\Delta_j \operatorname{div}(g \otimes g) = \frac{M^2}{2} \Delta_j \begin{pmatrix} 0 \\ \partial_{x_2}(\psi^2) \end{pmatrix} + \frac{1}{2} \Delta_j \operatorname{div}(\nabla^\perp \psi \otimes \nabla^\perp \psi) \quad (4.18)$$

and

$$\left\| \Delta_j(-\Delta)^{-1} \mathbb{P} \begin{pmatrix} 0 \\ \partial_{x_2}(\psi^2) \end{pmatrix} \right\|_{L^2(\mathbb{R}^2)} \geq c \quad (4.19)$$

for some positive constant c independent of j . Thus, by (4.18) and (4.19), we have

$$\begin{aligned}
 & \eta^2 \delta^2 \sum_{-\frac{1}{2\delta^2} \leq j \leq -2} \|\Delta_j(-\Delta)^{-1} \mathbb{P} \operatorname{div}(g \otimes g)\|_{L^2(\mathbb{R}^2)} \\
 & \geq c M^2 \eta^2 \delta^2 \sum_{-\frac{1}{2\delta^2} \leq j \leq -2} \left\| \Delta_j(-\Delta)^{-1} \mathbb{P} \begin{pmatrix} 0 \\ \partial_{x_2}(\psi^2) \end{pmatrix} \right\|_{L^2(\mathbb{R}^2)} \\
 & \quad - C \eta^2 \delta^2 \sum_{-\frac{1}{2\delta^2} \leq j \leq -2} \|\Delta_j(-\Delta)^{-1} \mathbb{P} \operatorname{div}(\nabla^\perp \psi \otimes \nabla^\perp \psi)\|_{L^2(\mathbb{R}^2)} \\
 & \geq c M^2 \eta^2 - C \eta^2 \|\nabla^\perp \psi \otimes \nabla^\perp \psi\|_{\dot{B}_{2,\infty}^{-1}(\mathbb{R}^2)} \\
 & \geq c M^2 \eta^2 - C \eta^2 \|\nabla^\perp \psi\|_{L^2(\mathbb{R}^2)}^2,
 \end{aligned}$$

which implies

$$\|u_\delta^{(2;1)}(t_0 + k_\delta, T)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \geq c_1 M^2 \eta^2 - C_6 \eta^2$$

for some positive constant c_1 and C_6 . Here, we have used the estimate

$$\|\nabla^\perp \psi \otimes \nabla^\perp \psi\|_{\dot{B}_{2,\infty}^{-1}(\mathbb{R}^2)} \leq C \|\nabla^\perp \psi \otimes \nabla^\perp \psi\|_{L^1(\mathbb{R}^2)} \leq C \|\nabla^\perp \psi\|_{L^2(\mathbb{R}^2)}^2.$$

Using (4.11), (4.12), (4.13), (4.15), and (4.17), we obtain

$$\begin{aligned}
 & \|u_\delta[a](t_0 + k_\delta, T)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \\
 & \geq \|u_\delta^{(2;1)}(t_0 + k_\delta, T)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} - \|u_\delta^{(1)}\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} \\
 & \quad - \sum_{k=2}^3 \|u_\delta^{(2;k)}\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} - \|\tilde{u}_\delta[a]\|_{\widetilde{L}^\infty(I_\delta; \dot{B}_{2,1}^0(\mathbb{R}^2))} \\
 & \geq c_1 M^2 \eta^2 - C_6 \eta^2 \\
 & \quad - 3C_1 M \eta \delta - C_2 M^2 \eta^3 - C_2 M^2 \eta^2 \delta^2 - 4C_0 \varepsilon - 4C_3 M^8 \eta^3 \quad (4.20) \\
 & \geq c_1 M^2 \eta^2 - C_6 \eta^2 - 3C_1 M \eta^2 - C_2 M^2 \eta^3 - C_2 M^2 \eta^4 - 4C_0 \varepsilon - 4C_3 M^8 \eta^3 \\
 & = (c_1 M^2 - C_6 - 3C_1 M) \eta^2 - (C_2 M^2 \eta + C_2 M^2 \eta^2 + 4C_3 M^8 \eta) \eta^2 - 4C_0 \varepsilon.
 \end{aligned}$$

We now choose $M = M_0 \geq 10$ and $0 < \eta = \eta_0 \leq \min\{\eta_1, \eta_2\}$, so that

$$\begin{aligned}
 & c_1 M_0^2 - C_6 - 3C_1 M_0 \geq 3, \\
 & C_2 M_0^2 \eta_0 + C_2 M_0^2 \eta_0^2 + 4C_3 M_0^8 \eta_0 \leq 1, \quad C_0 M_0 \eta_0 \leq 1,
 \end{aligned}$$

and let

$$\varepsilon_0 := \min\left\{\varepsilon_1, \varepsilon_2, \eta_0, \frac{\eta_0^2}{4C_0}, \frac{\eta_0^2}{2}\right\}.$$

Then, for any $0 < \delta \leq \varepsilon_0$, there holds

$$\|a\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \leq \varepsilon_0, \quad \|f_\delta\|_{\widetilde{L}^\infty(\mathbb{R}; \dot{B}_{p,1}^{\frac{2}{p}-3}(\mathbb{R}^2))} \leq \delta, \quad \|u_\delta[a](t_0 + k_\delta, T)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \geq 2\varepsilon_0,$$

and we complete the proof. $\blacksquare$

4.2. Unconditional uniqueness

To complete the proof of Theorem 1.5, we need the following unconditional uniqueness for the initial value problem (4.1).

Proposition 4.5. *Let $I = [t_0, T_0) \subset \mathbb{R}$ and $a \in \dot{B}_{2,1}^0(\mathbb{R}^2)$. If two vector fields $u_1, u_2 \in C(I; \dot{B}_{2,1}^0(\mathbb{R}^2))$ are mild solutions to (4.1) with $u_1(t_0) = u_2(t_0) = a$ and the same external force, then $u_1 = u_2$.*

Remark 4.6. For the unconditional uniqueness of the incompressible Navier–Stokes equations, Furioli, Lemarié-Rieusset, and Terraneo [7] considered the three-dimensional case and showed the uniqueness in the class $C(I; L^3(\mathbb{R}^3))$. Their method is directly applicable to the general higher-dimensional case $C(I; L^n(\mathbb{R}^n))$ with $n \geq 3$, whereas its proof fails in the two-dimensional case since the key embedding $L^{\frac{n}{2}}(\mathbb{R}^n) \hookrightarrow \dot{W}^{-1,n}(\mathbb{R}^n)$ is not valid when $n = 2$. In Proposition 4.5, we find that the unconditional uniqueness holds in the class $C(I; \dot{B}_{2,1}^0(\mathbb{R}^2))$, which is slightly narrower than $C(I; L^2(\mathbb{R}^2))$, by following the idea in [7].

To show Proposition 4.5, we first establish some bilinear estimates.

Lemma 4.7. *Let $I = [t_0, T_0) \subset \mathbb{R}$ be an interval. Then there exists a positive constant C such that*

$$\begin{aligned} \sup_{t \in I} \left\| \int_{t_0}^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(u(s) \otimes v(s)) \, ds \right\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \\ \leq C \sup_{t \in I} \|u(t)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \sup_{t \in I} \|v(t)\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \end{aligned}$$

for all $u \in C(I; \dot{B}_{2,1}^0(\mathbb{R}^2))$ and $v \in C(I; \dot{B}_{2,\infty}^0(\mathbb{R}^2))$.

Proof. As there holds

$$\begin{aligned} \left\| \Delta_j \int_{t_0}^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(u(s) \otimes v(s)) \, ds \right\|_{L^2(\mathbb{R}^2)} \\ \leq C \int_{t_0}^t e^{-\frac{1}{4}2^{2j}(t-s)} 2^j \|\Delta_j(u(s) \otimes v(s))\|_{L^2(\mathbb{R}^2)} \, ds \\ \leq C \int_{t_0}^t e^{-\frac{1}{4}2^{2j}(t-s)} \, ds \, 2^j \sup_{t_0 \leq s \leq t} \|\Delta_j(u(s) \otimes v(s))\|_{L^2(\mathbb{R}^2)} \\ = C 2^{-j} \sup_{s \in I} \|\Delta_j(u(s) \otimes v(s))\|_{L^2(\mathbb{R}^2)}, \end{aligned}$$

we have

$$\sup_{t \in I} \left\| \int_{t_0}^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(u(s) \otimes v(s)) \, ds \right\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \leq C \sup_{t \in I} \|u(t) \otimes v(t)\|_{\dot{B}_{2,\infty}^{-1}(\mathbb{R}^2)}.$$

Hence, it suffices to show

$$\|fg\|_{\dot{B}_{2,\infty}^{-1}(\mathbb{R}^2)} \leq C \|f\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \|g\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)}$$

for all $f \in \dot{B}_{2,1}^0(\mathbb{R}^2)$ and $g \in \dot{B}_{2,\infty}^0(\mathbb{R}^2)$. To prove this, we use the Bony paraproduct decomposition:

$$fg = T_f g + R(f, g) + T_g f.$$

See Appendix A for the definitions of $T_f g$ and $R(f, g)$. It follows from Lemma A.1 and the continuous embeddings $\dot{B}_{2,\infty}^0(\mathbb{R}^2) \hookrightarrow \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^2)$ and $\dot{B}_{2,1}^0(\mathbb{R}^2) \hookrightarrow \dot{B}_{2,\infty}^0(\mathbb{R}^2)$ that

$$\begin{aligned} \|T_f g\|_{\dot{B}_{2,\infty}^{-1}(\mathbb{R}^2)} &\leq C \|f\|_{\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^2)} \|g\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \\ &\leq C \|f\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \|g\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \\ &\leq C \|f\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \|g\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \end{aligned}$$

and similarly

$$\begin{aligned} \|T_g f\|_{\dot{B}_{2,\infty}^{-1}(\mathbb{R}^2)} &\leq C \|g\|_{\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^2)} \|f\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \\ &\leq C \|g\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \|f\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \\ &\leq C \|f\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \|g\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)}. \end{aligned}$$

By the continuous embedding $\dot{B}_{1,\infty}^0(\mathbb{R}^2) \hookrightarrow \dot{B}_{2,\infty}^{-1}(\mathbb{R}^2)$ and Lemma A.1, we see that

$$\|R(f, g)\|_{\dot{B}_{2,\infty}^{-1}(\mathbb{R}^2)} \leq C \|R(f, g)\|_{\dot{B}_{1,\infty}^0(\mathbb{R}^2)} \leq C \|f\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \|g\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)}.$$

Combining the above three estimates, we complete the proof. $\blacksquare$

Lemma 4.8. *Let $I = [t_0, T_0) \subset \mathbb{R}$ be an interval. Then there exists a positive constant C such that*

$$\begin{aligned} \sup_{t \in I} \left\| \int_{t_0}^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(u(s) \otimes v(s)) ds \right\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \\ \leq C \sup_{t \in I} (t - t_0)^{\frac{1}{4}} \|u(t)\|_{\dot{B}_{4,1}^0(\mathbb{R}^2)} \sup_{t \in I} \|v(t)\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \end{aligned}$$

for all $u \in C((t_0, T_0); \dot{B}_{4,1}^0(\mathbb{R}^2))$ and $v \in C(I; \dot{B}_{2,\infty}^0(\mathbb{R}^2))$.

Proof. By the smoothing estimate for the kernel of $e^{(t-s)\Delta} \mathbb{P} \operatorname{div}$, we see that

$$\begin{aligned} \left\| \int_{t_0}^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(u(s) \otimes v(s)) ds \right\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \\ \leq C \int_{t_0}^t \|e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(u(s) \otimes v(s))\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} ds \\ \leq C \int_{t_0}^t (t-s)^{-\frac{3}{4}} \|u(s) \otimes v(s)\|_{\dot{B}_{2,\infty}^{-\frac{1}{2}}(\mathbb{R}^2)} ds \end{aligned}$$

$$\begin{aligned}
&\leq C \int_{t_0}^t (t-s)^{-\frac{3}{4}} (s-t_0)^{-\frac{1}{4}} ds \sup_{t_0 < s \leq t} (s-t_0)^{\frac{1}{4}} \|u(s) \otimes v(s)\|_{\dot{B}_{2,\infty}^{-\frac{1}{2}}(\mathbb{R}^2)} \\
&\leq C \sup_{t_0 < s \leq t} (s-t_0)^{\frac{1}{4}} \|u(s) \otimes v(s)\|_{\dot{B}_{2,\infty}^{-\frac{1}{2}}(\mathbb{R}^2)}.
\end{aligned}$$

Hence, it suffices to show

$$\|fg\|_{\dot{B}_{2,\infty}^{-\frac{1}{2}}(\mathbb{R}^2)} \leq C \|f\|_{\dot{B}_{4,1}^0(\mathbb{R}^2)} \|g\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)}.$$

Similarly to the argument in the proof of Lemma 4.7, we use the paraproduct decomposition. It follows from Lemma A.1 and the continuous embeddings $\dot{B}_{4,1}^0(\mathbb{R}^2) \hookrightarrow \dot{B}_{4,\infty}^0(\mathbb{R}^2) \hookrightarrow \dot{B}_{\infty,\infty}^{-\frac{1}{2}}(\mathbb{R}^2)$ and $\dot{B}_{2,\infty}^0(\mathbb{R}^2) \hookrightarrow \dot{B}_{4,\infty}^{-\frac{1}{2}}(\mathbb{R}^2)$ that

$$\|T_f g\|_{\dot{B}_{2,\infty}^{-\frac{1}{2}}(\mathbb{R}^2)} \leq C \|f\|_{\dot{B}_{\infty,\infty}^{-\frac{1}{2}}(\mathbb{R}^2)} \|g\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \leq C \|f\|_{\dot{B}_{4,1}^0(\mathbb{R}^2)} \|g\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)},$$

$$\|T_g f\|_{\dot{B}_{2,\infty}^{-\frac{1}{2}}(\mathbb{R}^2)} \leq C \|g\|_{\dot{B}_{4,\infty}^{-\frac{1}{2}}(\mathbb{R}^2)} \|f\|_{\dot{B}_{4,\infty}^0(\mathbb{R}^2)} \leq C \|g\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \|f\|_{\dot{B}_{4,1}^0(\mathbb{R}^2)}.$$

By the continuous embedding $\dot{B}_{\frac{4}{3},\infty}^0(\mathbb{R}^2) \hookrightarrow \dot{B}_{2,\infty}^{-\frac{1}{2}}(\mathbb{R}^2)$ and Lemma A.1, we see that

$$\|R(f, g)\|_{\dot{B}_{2,\infty}^{-\frac{1}{2}}(\mathbb{R}^2)} \leq C \|R(f, g)\|_{\dot{B}_{\frac{4}{3},\infty}^0(\mathbb{R}^2)} \leq C \|f\|_{\dot{B}_{4,1}^0(\mathbb{R}^2)} \|g\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)}.$$

Combining the above three estimates, we complete the proof. $\blacksquare$

Now we present the proof of Proposition 4.5.

Proof of Proposition 4.5. Let u_1 and u_2 be a solution to (4.1) with the same initial data $a \in \dot{B}_{2,1}^0(\mathbb{R}^2)$. Let $v_m(t) := u_m(t) - e^{(t-t_0)\Delta} a$ ($m = 1, 2$) and $w(t) := v_1(t) - v_2(t)$. Then it holds that

$$\begin{aligned}
w(t) &= - \int_{t_0}^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(e^{(s-t_0)\Delta} a \otimes w(s)) ds \\
&\quad - \int_{t_0}^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(w(s) \otimes e^{(s-t_0)\Delta} a) ds \\
&\quad - \int_{t_0}^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(v_1(s) \otimes w(s)) ds \\
&\quad - \int_{t_0}^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(w(s) \otimes v_2(s)) ds.
\end{aligned}$$

It follows from Lemmas 4.7 and 4.8 that for any $T \in (t_0, T_0)$,

$$\begin{aligned}
&\sup_{t_0 < t < T} \|w(t)\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \\
&\leq C_0 \sup_{t_0 < t < T} (t-t_0)^{\frac{1}{4}} \|e^{(t-t_0)\Delta} a\|_{\dot{B}_{4,1}^0(\mathbb{R}^2)} \sup_{t_0 < t < T} \|w(t)\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \\
&\quad + C_0 \sum_{m=1}^2 \sup_{t_0 < t < T} \|v_m(t)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \sup_{t_0 < t < T} \|w(t)\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \quad (4.21)
\end{aligned}$$

with some positive constant C_0 independent of T . As the density argument and $v_m(t_0) = 0$ yields

$$\lim_{T \downarrow t_0} \left(\sup_{t_0 < t < T} (t - t_0)^{\frac{1}{4}} \|e^{(t-t_0)\Delta} a\|_{\dot{B}_{4,1}^0(\mathbb{R}^2)} + \sum_{m=1}^2 \sup_{t_0 < t < T} \|v_m(t)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \right) = 0,$$

there exists a time $t_0 < T_1 \leq T_0$ such that

$$\sup_{t_0 < t < T_1} (t - t_0)^{\frac{1}{4}} \|e^{(t-t_0)\Delta} a\|_{\dot{B}_{4,1}^0(\mathbb{R}^2)} + \sum_{m=1}^2 \sup_{t_0 < t < T_1} \|v_m(t)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \leq \frac{1}{4C_0}. \quad (4.22)$$

Then we see by (4.21) and (4.22) that

$$\sup_{t_0 < t < T_1} \|w(t)\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)} \leq \frac{1}{2} \sup_{t_0 < t < T_1} \|w(t)\|_{\dot{B}_{2,\infty}^0(\mathbb{R}^2)},$$

which implies $w(t) = 0$ for all $t_0 \leq t \leq T_1$. If $T_1 < T_0$, then we repeat the same procedure many times to obtain $w(t) = 0$ for all $t \in I$. Thus, we complete the proof. ■

4.3. Proof of Theorem 1.5

Now we are in a position to present the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $1 \leq p \leq 2$ and let ε_0 be a positive constant determined in Proposition 4.1. Let $0 < \delta \leq \varepsilon_0$ and $0 < T \leq 2^{\frac{1}{\delta^2}}$. Suppose by contradiction that the external force f_δ appearing in Proposition 4.1 generates a T -periodic mild solution $u_{\text{per},\delta} \in C(\mathbb{R}; \dot{B}_{2,1}^0(\mathbb{R}^2))$ to (1.1) satisfying

$$\|u_{\text{per},\delta}(t_0)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \leq \varepsilon_0 \quad (4.23)$$

for some $t_0 \in \mathbb{R}$. It is easy to see that $u_{\text{per},\delta}$ is also a mild solution to (4.1) with the initial data $a = u_{\text{per},\delta}(t_0)$. On the other hand, by Proposition 4.1, there exists a mild solution $u_\delta[u_{\text{per},\delta}(t_0)] \in \tilde{C}([t_0, t_0 + k_{\delta,T}T]; \dot{B}_{2,1}^0(\mathbb{R}^2))$ to (4.1) with $a = u_{\text{per},\delta}(t_0)$ satisfying

$$\|u_\delta[u_{\text{per},\delta}(t_0)](t_0 + k_{\delta,T}T)\|_{\dot{B}_{2,1}^0(\mathbb{R}^2)} \geq 2\varepsilon_0,$$

which with (4.23) implies

$$u_\delta[u_{\text{per},\delta}(t_0)](t_0) \neq u_\delta[u_{\text{per},\delta}(t_0)](t_0 + k_{\delta,T}T).$$

Using Proposition 4.5, we see by $u_\delta[u_{\text{per},\delta}(t_0)](t_0) = u_{\text{per},\delta}(t_0)$ that

$$u_\delta[u_{\text{per},\delta}(t_0)](t) = u_{\text{per},\delta}(t) \quad (4.24)$$

holds for all $t_0 \leq t \leq t_0 + k_{\delta,T}T$. From (4.23) and (4.24) with $t = t_0 + k_{\delta,T}T$, we have

$$u_{\text{per},\delta}(t_0) = u_\delta[u_{\text{per},\delta}(t_0)](t_0) \neq u_\delta[u_{\text{per},\delta}(t_0)](t_0 + k_{\delta,T}T) = u_{\text{per},\delta}(t_0 + k_{\delta,T}T)$$

which yields a contradiction to the periodicity of $u_{\text{per},\delta}$. Thus, we complete the proof. ■

A. Remarks on the paradifferential calculus

In this appendix we consider the Bony decomposition of the product fg for two functions f and g :

$$fg = T_f g + R(f, g) + T_g f,$$

where we have set

$$T_f g := \sum_{k \in \mathbb{Z}} \left(\sum_{\ell \leq k-3} \Delta_\ell f \right) \Delta_k g, \quad R(f, g) := \sum_{k \in \mathbb{Z}} \sum_{|k-\ell| \leq 2} \Delta_k f \Delta_\ell g.$$

We first recall the basic estimates for $T_f g$ and $R(f, g)$ in Besov and Chemin–Lerner spaces.

Lemma A.1. *Let $n \in \mathbb{N}$ and let $I \subset \mathbb{R}$ be an interval. Then the following statements hold:*

- (1) *Let $1 \leq p, p_1, p_2, r, r_1, r_2, \sigma, \sigma_1 \leq \infty$ and $s, s_1, s_2 \in \mathbb{R}$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad \frac{1}{\sigma} \leq \frac{1}{\sigma_1} + \frac{1}{\sigma_2}, \quad s = s_1 + s_2, \quad s_1 < 0.$$

Then there exists a positive constant $K_1 = K_1(\sigma_1, s_1, s_2)$ such that

$$\|T_f g\|_{\dot{B}_{p,\sigma}^s(\mathbb{R}^n)} \leq K_1 \|f\|_{\dot{B}_{p_1,\sigma_1}^{s_1}(\mathbb{R}^n)} \|g\|_{\dot{B}_{p_2,\sigma}^{s_2}(\mathbb{R}^n)} \quad (\text{A.1})$$

for all $f \in \dot{B}_{p_1,\sigma_1}^{s_1}(\mathbb{R}^n)$ and $g \in \dot{B}_{p_2,\sigma}^{s_2}(\mathbb{R}^n)$, as well as

$$\|T_F G\|_{\widetilde{L}^r(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n))} \leq K_1 \|F\|_{\widetilde{L}^{r_1}(I; \dot{B}_{p_1,\sigma_1}^{s_1}(\mathbb{R}^n))} \|G\|_{\widetilde{L}^{r_2}(I; \dot{B}_{p_2,\sigma}^{s_2}(\mathbb{R}^n))} \quad (\text{A.2})$$

for all $F \in \widetilde{L}^{r_1}(I; \dot{B}_{p_1,\sigma_1}^{s_1}(\mathbb{R}^n))$ and $G \in \widetilde{L}^{r_2}(I; \dot{B}_{p_2,\sigma}^{s_2}(\mathbb{R}^n))$.

- (2) *Let $1 \leq p, p_1, p_2, r, r_1, r_2, \sigma, \sigma_1, \sigma_2 \leq \infty$ and $s, s_1, s_2 \in \mathbb{R}$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad \frac{1}{\sigma} \leq \frac{1}{\sigma_1} + \frac{1}{\sigma_2}, \quad s = s_1 + s_2 > 0.$$

Then there exists a positive constant $K_2 = K_2(s, s_1, s_2)$ such that

$$\|R(f, g)\|_{\dot{B}_{p,\sigma}^s(\mathbb{R}^n)} \leq K_2 \|f\|_{\dot{B}_{p_1,\sigma_1}^{s_1}(\mathbb{R}^n)} \|g\|_{\dot{B}_{p_2,\sigma_2}^{s_2}(\mathbb{R}^n)} \quad (\text{A.3})$$

for all $f \in \dot{B}_{p_1,\sigma_1}^{s_1}(\mathbb{R}^n)$ and $g \in \dot{B}_{p_2,\sigma_2}^{s_2}(\mathbb{R}^n)$, as well as

$$\|R(F, G)\|_{\widetilde{L}^r(I; \dot{B}_{p,\sigma}^s(\mathbb{R}^n))} \leq K_2 \|F\|_{\widetilde{L}^{r_1}(I; \dot{B}_{p_1,\sigma_1}^{s_1}(\mathbb{R}^n))} \|G\|_{\widetilde{L}^{r_2}(I; \dot{B}_{p_2,\sigma_2}^{s_2}(\mathbb{R}^n))} \quad (\text{A.4})$$

for all $F \in \widetilde{L}^{r_1}(I; \dot{B}_{p_1,\sigma_1}^{s_1}(\mathbb{R}^n))$ and $G \in \widetilde{L}^{r_2}(I; \dot{B}_{p_2,\sigma_2}^{s_2}(\mathbb{R}^n))$.

One may prove Lemma A.1 using the arguments in [1, Theorems 2.47 and 2.52]. It follows from the proof of estimates (A.1), (A.2) and (A.3), (A.4) that there exist absolute

positive constants C_1 and C_2 such that we may choose K_1 and K_2 as

$$K_1(\sigma_1, s_1, s_2) = \frac{C_1}{s_1^{1-\frac{1}{\sigma_1}}} 2^{3|s_2|}, \quad K_2(s, s_1, s_2) = \frac{C_2^{|s_1|+|s_2|}}{s}. \quad (\text{A.5})$$

While we see from (A.5) that $K_2 = O(s^{-1})$ as $s \downarrow 0$, we may relax the singularity s^{-1} to $s^{-\frac{1}{\sigma}}$ if σ_1 and σ_2 satisfy a strict condition in the following lemma, which is crucial in the proof of Lemma 4.3.

Lemma A.2. *Let $1 \leq p, p_1, p_2, r, r_1, r_2, \sigma, \sigma_1, \sigma_2 \leq \infty$ and $s, s_1, s_2 \in \mathbb{R}$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}, \quad 1 \leq \frac{1}{\sigma_1} + \frac{1}{\sigma_2}, \quad s = s_1 + s_2 > 0.$$

Then there holds

$$\|R(f, g)\|_{\widetilde{L}^r(I; \dot{B}_{p, \sigma}^s(\mathbb{R}^n))} \leq K_3 \|f\|_{\widetilde{L}^{r_1}(I; \dot{B}_{p_1, \sigma_1}^{s_1}(\mathbb{R}^n))} \|g\|_{\widetilde{L}^{r_2}(I; \dot{B}_{p_2, \sigma_2}^{s_2}(\mathbb{R}^n))}$$

for all $f \in \widetilde{L}^{r_1}(I; \dot{B}_{p_1, \sigma_1}^{s_1}(\mathbb{R}^n))$ and $g \in \widetilde{L}^{r_2}(I; \dot{B}_{p_2, \sigma_2}^{s_2}(\mathbb{R}^n))$, where the positive constant $K_3 = K_3(\sigma, s, s_1, s_2)$ is given by

$$K_3(\sigma, s, s_1, s_2) = \frac{C_3^{|s_1|+|s_2|}}{s^{\frac{1}{\sigma}}}$$

for some absolute positive constant C_3 .

Proof. By $R(f, g) = R(g, f)$, Applying Δ_j to $R(f, g)$, we see that

$$\Delta_j R(f, g) = \Delta_j \sum_{k \geq j-4} \sum_{|k-\ell| \leq 2} \Delta_k f \Delta_\ell g,$$

which implies

$$\begin{aligned} 2^{sj} \|\Delta_j R(f, g)\|_{L^r(I; L^p(\mathbb{R}^n))} &\leq C 2^{2|s_2|} \sum_{k \geq j-4} 2^{s(j-k)} 2^{s_1 k} \|\Delta_k f\|_{L^{r_1}(I; L^{p_1}(\mathbb{R}^n))} \\ &\quad \times \sum_{|k-\ell| \leq 2} 2^{s_2 \ell} \|\Delta_\ell g\|_{L^{r_2}(I; L^{p_2}(\mathbb{R}^n))}. \end{aligned}$$

Taking the $\ell^\sigma(\mathbb{Z})$ -norm of this and using the Hausdorff–Young inequality for the discrete convolution we obtain

$$\begin{aligned} &\|R(f, g)\|_{\widetilde{L}^r(I; \dot{B}_{p, \sigma}^s(\mathbb{R}^n))} \\ &\leq C 2^{2|s_2|} \left(\sum_{j \leq 4} 2^{s \sigma j} \right)^{\frac{1}{\sigma}} \\ &\quad \times \left\| \left\{ 2^{s_1 k} \|\Delta_k f\|_{L^{r_1}(I; L^{p_1}(\mathbb{R}^n))} \sum_{|k-\ell| \leq 2} 2^{s_2 \ell} \|\Delta_\ell g\|_{L^{r_2}(I; L^{p_2}(\mathbb{R}^n))} \right\}_{k \in \mathbb{Z}} \right\|_{\ell^1(\mathbb{Z})} \\ &\leq C 2^{2|s_2|} 2^{4s} s^{-\frac{1}{\sigma}} \|f\|_{\widetilde{L}^{r_1}(I; \dot{B}_{p_1, \sigma_1}^{s_1}(\mathbb{R}^n))} \|g\|_{\widetilde{L}^{r_2}(I; \dot{B}_{p_2, \sigma_2}^{s_2}(\mathbb{R}^n))}, \end{aligned}$$

which completes the proof. ■

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