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Verdier quotients of Calabi–Yau categories from quivers with potential

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Abstract. We investigate a class of triangulated categories obtained as Verdier quotients of 3-Calabi–Yau categories combinatorially described by quivers with potential from (decorated) marked surfaces. We study their bounded t-structures and consider in particular the exchange graphs of hearts and silting objects respectively, and show that the Koszul isomorphism between these graphs is preserved under Verdier quotient.

1. Introduction

In this paper we study a class of triangulated categories which arise as Verdier quotient of Calabi–Yau categories from a differential graded algebra of a quiver with potential (Q, W), and which can be associated with some mixed-angulation of a marked Riemann surface with boundary.

For any differential graded (dg) algebra Λ, we denote by $\mathcal{D}(\Lambda)$, $\operatorname{per}(\Lambda)$, and $\operatorname{pvd}(\Lambda)$ the derived category, the perfect derived category, and the perfectly valued derived category of Λ respectively. Let e be an idempotent of a non-positive dg algebra Γ. Triangulated categories over $e\Gamma e$ appear for instance in the context of singularity categories in [16, 17], and in the context of weighted marked surfaces in [4]. Here we let $\Gamma = \Gamma(Q, W)$ be the Ginzburg dg algebra of Calabi–Yau dimension 3 from a quiver with potential (Q, W), and fix $e = \sum_{j \notin I} e_j$, the sum of trivial paths e_i associated with the complement of a subset I of the vertices Q_0 of Q. Let Γ_I be the Ginzburg dg algebra associated with the restriction of (Q, W) to I. It is known that $\mathcal{D}(e\Gamma e)$ and $\operatorname{pvd}(e\Gamma e)$ are Verdier quotients of $\mathcal{D}(\Gamma)$ and $\operatorname{pvd}(\Gamma)$ by $\mathcal{D}(\Gamma_I) = \mathcal{D}(\Gamma/\Gamma e\Gamma)$ and $\operatorname{pvd}(\Gamma_I) = \operatorname{pvd}(\Gamma/\Gamma e\Gamma)$ respectively [17],

$$0 \to \mathcal{D}(\Gamma/\Gamma e\Gamma) \to \mathcal{D}(\Gamma) \to \mathcal{D}(e\Gamma e) \to 0,$$
$$0 \to \operatorname{pvd}(\Gamma/\Gamma e\Gamma) \to \operatorname{pvd}(\Gamma) \to \operatorname{pvd}(e\Gamma e) \to 0.$$

While the first class fits into a recollement of triangulated categories induced by idempotents which is pretty well studied (see for instance [17, 29]), the latter are less known and a priori much less well-behaved triangulated categories.

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The goal of this paper is to provide a comprehensive description of the derived categories of $e\Gamma e$ and to study properties of $\mathrm{pvd}(\Gamma)$ that descend to $\mathrm{pvd}(e\Gamma e)$. We recall some preliminaries in Section 2. In Section 3 we introduce $\mathcal{D}(e\Gamma e)$, $\mathrm{per}(e\Gamma e)$, $\mathrm{pvd}(e\Gamma e)$, describe their relations, and we adapt the gluing method of bounded t-structures in a recollement, by Beilinson–Bernstein–Deligne, to the sequence of perfectly valued categories. In Sections 4, 5, and 6, we mainly restrict our attention to quivers with potential coming from triangulations of marked Riemann surfaces with boundary components. We are interested in bounded t-structures of $\mathrm{pvd}(e\Gamma e)$ regarded as a Verdier quotient and in Happel–Reiten–Smalø (HRS) tilts between them. The notions of bounded t-structures and a special case of tilting (simple tilts) are compactly encoded in a graph, and we study a HRS-tilting exchange graph of $\mathrm{pvd}(e\Gamma e)$. After recalling simple-projective duality for Γ, we state a correspondence between tilting and silting exchange graphs associated with $e\Gamma e$, obtaining a *relative* simple-projective correspondence, that is a duality between length hearts with finitely many simples and silting objects at the level of quotient triangulated categories. This kind of correspondence is also known as Koenig–Yang correspondence.

In the rest of the Introduction we summarize the main results. We refer the reader to the list of notation at the end of the Introduction for the relevant definitions.

Bounded t-structures under Verdier quotient

The exchange graph $\mathrm{EG}(\mathcal{D})$ of a triangulated category $\mathcal{D}$ is the graph whose vertices are hearts of bounded t-structures on $\mathcal{D}$ and whose directed edges correspond to simple forward HRS-tilts, i.e., tilts at a torsion-free class generate by a simple object. A complete description of this graph might be difficult to obtain, so it is standard to restrict our attention to a connected component containing a fixed distinguished heart as a vertex. In this paper we use the terminology finite heart to mean the heart of a bounded t-structure which is finite length and has finitely many simple objects, up to isomorphism.

Let $\Gamma = \Gamma(Q, W)$ be the Ginzburg algebra of a quiver with potential (Q, W) dual to a triangulation of a decorated marked surface. The exchange graph $\mathrm{EG}^{\circ}(\mathrm{pvd}(\Gamma))$ is the connected component of the exchange graph of $\mathrm{pvd}(\Gamma)$ containing the bounded t-structure with heart $\mathrm{rep}(Q, W)$ as a vertex. It was studied in [22] and is related with cluster theory and the exchange graph of triangulations of a (decorated) marked surface [30].

We consider the category $\mathrm{pvd}(e\Gamma e)$ regarded, as we have seen above, as the quotient $\mathrm{pvd}(\Gamma)/\,\mathrm{pvd}(\Gamma_I)$, and we define the subgraph $\mathrm{EG}^{\bullet}(\mathrm{pvd}(e\Gamma e))$ of $\mathrm{EG}(\mathrm{pvd}(e\Gamma e))$, whose vertices arise from localization of finite hearts of $\mathrm{pvd}(\Gamma)$ in $\mathrm{EG}^{\circ}(\mathrm{pvd}\,\Gamma)$. We show that it can be read off from $\mathrm{EG}(\mathrm{pvd}\,\Gamma)$, and that any simple forward tilt of a finite heart in $\mathrm{EG}^{\bullet}(\mathrm{pvd}(e\Gamma e))$ "lifts" to a simple forward tilt in $\mathrm{pvd}(\Gamma)$. This relies on the construction of (Q, W) as the dual quiver of the triangulation of a marked surface. Denote by $\mathrm{EG}^{\circ}(\mathrm{pvd}\,\Gamma, \mathrm{pvd}\,\Gamma_I)$ the full subgraph of $\mathrm{EG}^{\circ}(\mathrm{pvd}\,\Gamma)$ whose hearts induce a bounded t-structure on $\mathrm{pvd}(e\Gamma e)$.

Theorem 1.1 (Theorem 5.6). *The graph* $\mathrm{EG}^\bullet(\mathrm{pvd}(e\Gamma e))$ *is isomorphic to the quotient of the relative exchange graph* $\mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma, \mathrm{pvd}\,\Gamma_I)$ *obtained by contracting edges labeled by a simple object in* $\mathrm{pvd}(\Gamma_I)$

$$\mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma) \supset \mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma, \mathrm{pvd}\,\Gamma_I) \twoheadrightarrow \mathrm{EG}^\bullet\big(\mathrm{pvd}(e\Gamma e)\big).$$

It consists of connected components of $\mathrm{EG}(\mathrm{pvd}(e\Gamma e))$.

At this stage, we are not able to exclude the existence of spurious components of $\mathrm{EG}(\mathrm{pvd}(e\Gamma e))$. The vertices of $\mathrm{EG}^\bullet(\mathrm{pvd}(e\Gamma e))$ are finite hearts of bounded t-structures in $\mathrm{pvd}(e\Gamma e)$ of the form $\mathcal{H}/\mathcal{H}_I$ for some finite heart $\mathcal{H}$ of $\mathrm{pvd}(\Gamma)$ and Serre subcategory $\mathcal{H}_I = \mathcal{H} \cap \mathrm{pvd}\,\Gamma_I$. They are module categories over finite-dimensional algebras whose underlying quiver is deduced explicitly from (Q, W) and I.

A natural question is how t-structures obtained by a generic tilt at a torsion pair in a heart of $\mathrm{pvd}(e\Gamma e)$ look like. Reachable bounded t-structures of $\mathrm{pvd}(e\Gamma e)$, i.e., those that are intermediate with respect to a t-structure in $\mathrm{EG}^\bullet(\mathrm{pvd}(e\Gamma e))$, turn out to be equivalent to abelian quotients of some bounded heart in $\mathrm{pvd}(\Gamma)$.

Proposition 1.2 (Corollary 3.8). *Let* $\overline{\mathcal{H}} = \mathcal{H}/(\mathcal{H} \cap \mathrm{pvd}\,\Gamma_I) \in \mathrm{EG}^\bullet(\mathrm{pvd}(e\Gamma e))$ *and* $\mu_f^\sharp \overline{\mathcal{H}}$ *be the heart of another bounded t-structure in* $\mathrm{pvd}(e\Gamma e)$ *obtained by tilting* $\overline{\mathcal{H}}$ *at a torsion-free class* f*. Then there exists a torsion-free class* $\mathcal{F} \subset \mathcal{H}$ *and* $\mathcal{H}' = \mu_{\mathcal{F}}^\sharp \mathcal{H}$ *in* $\mathrm{pvd}\,\Gamma$ *such that* $\mu_f^\sharp \overline{\mathcal{H}} = \mathcal{H}'/(\mathcal{H}' \cap \mathrm{pvd}\,\Gamma_I)$.

This depends on the existence of recollements of abelian and triangulated categories at the level of module categories over $J := H^0\Gamma$ and derived categories over Γ

$$\begin{tikzcd}
\mathcal{D}(\Gamma/\Gamma e\Gamma) \arrow[r, "\iota"] & \mathcal{D}(\Gamma) \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "\pi"] & \mathcal{D}(e\Gamma e) \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"] \\
\cup & \cup & \cup \\
\mathrm{pvd}(\Gamma/\Gamma e\Gamma) \arrow[r, "\iota"] & \mathrm{pvd}(\Gamma) \arrow[r, "\pi"] & \mathrm{pvd}(e\Gamma e) \\
\cup & \cup & \cup \\
\mathrm{mod}(J/JeJ) \arrow[r, "\iota"] & \mathrm{mod}(J) \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "\pi"] & \mathrm{mod}(eJe) \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"]
\end{tikzcd}$$

and does not require that (Q, W) comes from a triangulation.

Koszul duality under Verdier quotient

The second main result of the paper generalizes to $e\Gamma e$ the simple-projective duality for Γ, involving a bijection between finite hearts of $\mathrm{pvd}(\Gamma)$ and silting objects of $\mathrm{per}(\Gamma)$. Silting theory is a recent area of research that is receiving a lot of attention due to its connections with other areas of representation theory. If Λ is a dg algebra, a silting object in $\mathrm{per}(\Lambda)$ is a direct sum $\mathbf{Y} = \oplus Y_i$ of non-isomorphic indecomposables that generates $\mathrm{per}(\Lambda)$. It admits a notion of silting mutation with respect to a summand, which is used to introduce the silting exchange graph $\mathrm{SEG}(\mathrm{per}(\Lambda))$. Originally, bijections between finite hearts and silting objects have been studied for example by Keller–Nicolás, King–Qiu in [22], Koenig–Yang in [23], to which we refer for contextualization. Recent works in the context of non-positive dg algebras are [32], where it is also related with a manifestation of Koszul duality, and [11], to cite some examples. In the context of 3-Calabi–Yau algebras Γ from decorated marked surfaces, the simple-projective duality, expressed as an isomorphism between (connected components of) $\mathrm{EG}(\mathrm{pvd}(\Gamma))$ and $\mathrm{SEG}(\mathrm{per}(\Gamma))$, is preserved by Verdier localization to $\mathrm{pvd}(e\Gamma e)$.

Theorem 1.3 (Theorem 6.11). *There exists an isomorphism of graphs*

$$\mathrm{EG}^{\bullet}\left(\mathrm{pvd}(e\Gamma e)\right) \simeq \mathrm{SEG}^{\bullet}\left(\mathrm{per}(e\Gamma e)\right),$$

where $\mathrm{SEG}^{\bullet}(\mathrm{per}(e\Gamma e))$ *is a full subgraph of the silting exchange graph of* $\mathrm{per}(e\Gamma e)$ *introduced in Definition* 6.9.

While the bijection between finite hearts and silting objects (Proposition 6.10) does not require that (Q, W) is dual to a triangulation, Theorem 1.3 means that under this additional assumption the correspondence is compatible with the notion of tilting and silting mutation.

Applications

We started the study of the categories associated with the dg algebra $e\Gamma e$ from the perspective of weighted decorated marked surfaces. This complementary perspective is considered in [4], where we also compute (possibly part of) the space of Bridgeland stability conditions of $\mathrm{pvd}(e\Gamma e)$. The same categories and their bounded t-structures also appear in the definition of multi-scale stability conditions from [5].

A weighted decorated marked surface (wDMS) is a bordered Riemann surface $\mathbf{S}$ with a set of marked points $\mathbb{M}$ on the boundary components and a map (weight) $\mathbf{w} : \Delta \to \mathbb{Z}_{\geq -1}$ defined on a finite set of internal points $\Delta \subset \mathring{\mathbf{S}}$. One way of constructing a weighted marked surface is by contracting sub-surfaces of a decorated marked surface $\mathbf{S_1}$. In this case we denote the resulting wDMS by $\overline{\mathbf{S}}_{\mathbf{w}}$. A mixed-angulation of a weighted marked surface is a tiling of $\overline{\mathbf{S}}_{\mathbf{w}}$ into polygons encircling exactly one point in Δ and whose number of edges is equal to the weight of such a point plus two. An edge of a mixed-angulation is a simple curve in $\mathbf{S} \setminus \Delta$ connecting two marked points, up to isotopy.

In [4], the category $\mathrm{pvd}(e\Gamma e)$ is realized as a triangulated category associated to a weighted decorated marked surface, obtained by “collapsing” subsurfaces of a marked bordered Riemann surface, together with a choice of a mixed-angulation. More precisely, Γ comes from $\mathbf{S_1}$ and the idempotent e depends on the subsurface. The exchange graph $\mathrm{EG}^{\bullet}(\mathrm{pvd}(e\Gamma e))$ is related with an exchange graph $\mathrm{EG}^{\bullet}(\overline{\mathbf{S}}_{\mathbf{w}})$ of mixed-angulations of $\overline{\mathbf{S}}_{\mathbf{w}}$. (A union of connected components of) the space of Bridgeland stability conditions on $\mathrm{pvd}(e\Gamma e)$, denoted $\mathrm{Stab}^{\bullet}(\mathrm{pvd}(e\Gamma e))$, is shown to be isomorphic to a moduli space of quadratic differentials with high order poles and possibly high order zeroes, generalizing the Bridgeland–Smith correspondence of [7] out of the usual 3-Calabi–Yau setting.

Putting together the main categorical result in [4], the Koszul duality above, and the geometric model of silting objects in $\operatorname{per}(\Gamma)$ as arcs from [31], we obtain the following theorem, which we interpret as a categorification of mixed-angulations of a weighted decorated marked surface.

Theorem 1.4. *There are isomorphisms of exchange graphs*

$$\mathrm{SEG}^{\bullet}\left(\operatorname{per}(e\Gamma e)\right) \simeq \mathrm{EG}^{\bullet}\left(\operatorname{pvd}(e\Gamma e)\right) \simeq \mathrm{EG}^{\bullet}(\overline{\mathbf{S}}_{\mathbf{w}}),$$

where $\overline{\mathbf{S}}_{\mathbf{w}}$ is a weighted decorated marked surface corresponding to $e\Gamma e$.

In terms of arcs-objects correspondence, the geometric idea behind Theorem 1.4 is the following. Mixed-angulations of $\overline{\mathbf{S}}_{\mathbf{w}}$ can be deformed to partial triangulations of $\mathbf{S}_{\mathbf{1}}$. Arcs in triangulations of $\mathbf{S}_{\mathbf{1}}$ correspond to indecomposable summands of silting objects of $\operatorname{per}(\Gamma)$. Partial triangulations therefore corresponds to those partial silting objects of Γ, which are precisely the silting objects in $\mathrm{SEG}^{\bullet}(\operatorname{per}(e\Gamma e))$.

List of notation

Quivers and algebras.

Λ	any differential graded (dg) algebra.
(Q, W)	a quiver with potential (Section 2.2).
$\Gamma = \Gamma(Q, W)$	differential non-positively graded Ginzburg algebra associated with a quiver with potential (Q, W) (Sections 2.2 and 2.5).
$J = H^0\Gamma$	Jacobian algebra.
$I \sqcup I^c = Q_0$	complementary subsets of the set Q_0 of vertices of Q.
$e = \sum_{j \notin I} e_j$	idempotent in Γ or J.
(Q_I, W_I)	full subquiver of (Q, W) with set of vertices I, not necessarily connected.
$\Gamma_I := \Gamma(Q_I, W_I)$ and $J_I := H^0\Gamma_I$.	
Q^{eJe} and $\bar{Q}^{e\Gamma e}$	quivers of the algebras eJe and $e\Gamma e$ respectively (Section 3).

Categories.

$\mathcal{D}(\Lambda)$	derived category of a dg algebra Λ.
$\operatorname{per}\Lambda$	perfect derived category of a dg algebra Λ (Section 2.5).
$\operatorname{pvd}\Lambda$	perfectly valued derived category of a dg algebra Λ (Section 2.5).
$\operatorname{mod}(eJe) \simeq \operatorname{mod} J / \operatorname{mod} J_I$	Serre quotient of abelian categories (Section 3).
$\operatorname{pvd}(e\Gamma e) \simeq \operatorname{pvd}\Gamma / \operatorname{pvd}\Gamma_I$	Verdier quotient of triangulated categories (Section 3).

A *finite heart* $\mathcal{H}$ is a heart of a bounded t-structure which is finite length and has finitely many simples up to isomorphism, $\operatorname{Sim}\mathcal{H}$ (Section 2.1).

Fix a thick subcategory $\mathcal{V}$ of a triangulated category $\mathcal{D}$ and the Verdier quotient

$$\mathcal{V} \xrightarrow{\iota} \mathcal{D} \xrightarrow{\pi} \mathcal{D}/\mathcal{V}.$$

A bounded t-structure in $\mathcal{D}$ is *compatible with* $\mathcal{V}$ if π induces a bounded t-structure on $\mathcal{D}/\mathcal{V}$ (Definition 2.9).

$\bar{\mathcal{H}} := \pi(\mathcal{H}) \simeq \mathcal{H}/(\mathcal{H} \cap \mathcal{V})$ for a $\mathcal{V}$-compatible heart $\mathcal{H}$ (Definition 2.9).

$\bar{M} := \pi(M)$ for an object $M \in \mathcal{D}$ (Notation 2.6).

In Sections 5 and 6

$$\mathcal{A} = \operatorname{mod} J, \quad \mathcal{V} = \operatorname{pvd}\Gamma_I, \quad \mathcal{K} = \operatorname{mod} J_I = \operatorname{mod} J \cap \mathcal{V}.$$

Exchange graphs.

$\lvert G \rvert$	set of vertices of a graph G.
$\mathrm{EG}(\mathrm{pvd}\,\Lambda)$	exchange graph of bounded hearts in $\mathrm{pvd}\,\Lambda$ (Definition 2.3).
$\mathrm{EG}^{\circ}(\mathrm{pvd}\,\Gamma)$	the principal part of $\mathrm{EG}(\mathrm{pvd}\,\Gamma)$, i.e., the connected component containing $\mathrm{mod}\,J$ as a vertex. It only contains finite hearts (Definition 2.3 and Notation 3.5).
$\mathrm{EG}^{\circ}(\mathrm{pvd}\,\Gamma, \mathrm{pvd}\,\Gamma_I)$	the full subgraph of $\mathrm{EG}^{\circ}(\mathrm{pvd}\,\Gamma)$ whose vertices are hearts compatible with $\mathrm{pvd}\,\Gamma_I$ (Definition 2.10).
$\mathrm{EG}^{\bullet}(\mathrm{pvd}(e\Gamma e))$	the full subgraph of $\mathrm{EG}(\mathrm{pvd}(e\Gamma e))$ whose vertices are hearts of type $\bar{\mathcal{H}}$ for some $\mathcal{H} \in \mathrm{EG}^{\circ}(\mathrm{pvd}\,\Gamma)$ (Definition 2.10 and Notation 3.5).
$\lvert \mathrm{EG}^{\bullet}_{\mathcal{K}}(\mathrm{pvd}\,\Gamma) \rvert$	set of hearts $\mathcal{H}$ in $\mathrm{EG}^{\circ}(\mathrm{pvd}\,\Gamma, \mathrm{pvd}\,\Gamma_I)$ satisfying $\bar{\mathcal{H}} \in \mathrm{EG}^{\bullet}(\mathrm{pvd}(e\Gamma e))$ and $\mathcal{H} \supset \mathcal{K}$ (Equation 6.5).
$\mathrm{SEG}(\mathrm{per}\,\Lambda)$	silting exchange graph of $\mathrm{per}\,\Lambda$ (Definition 6.2).
$\mathrm{SEG}^{\circ}(\mathrm{per}\,\Gamma)$	principal part of $\mathrm{SEG}(\mathrm{per}\,\Gamma)$ containing Γ as a vertex, it is isomorphic to $\mathrm{EG}^{\circ}(\mathrm{pvd}\,\Gamma)$ under simple-projective duality for Γ (Definition 6.2).
$\mathrm{pSEG}_{\perp\mathcal{V}}(\mathrm{per}\,\Gamma)$	exchange graph of $\mathcal{V}$-perpendicular partial silting objects in $\mathrm{per}\,\Gamma$ (Definition 6.7).
$\mathrm{pSEG}^{\bullet}_{\perp\mathcal{V}}(\mathrm{per}\,\Gamma)$	principal part of $\mathrm{pSEG}_{\perp\mathcal{V}}(\mathrm{per}\,\Gamma)$ consisting on partial silting objects that can be completed to silting objects in $\mathrm{SEG}^{\circ}(\mathrm{per}\,\Gamma)$ (Definition 6.9).
$\mathrm{SEG}^{\bullet}(\mathrm{per}(e\Gamma e))$	silting exchange graph of $\mathrm{per}(e\Gamma e)$ of silting objects corresponding to partial silting objects in $\mathrm{pSEG}^{\bullet}_{\perp\mathcal{V}}(\mathrm{per}\,\Gamma)$ (Definition 6.9).

2. Preliminaries on categories and notation

2.1. Categories and t-structures: notation

We fix the notations and some basic facts regarding abelian and triangulated categories, their bounded t-structures, and the tilting procedure.

Through the paper, $\mathbf{k}$ is an algebraically closed field, all categories are $\mathbf{k}$-linear, and all subcategories are full. The shift functor of a triangulated category is denoted by [1], and distinguished triangles are represented as $X \to E \to Y$, likewise short exact sequences. For an additive (either abelian or triangulated) category $\mathcal{C}$ with subcategories $\mathcal{H}_1, \mathcal{H}_2$ and for a set of objects $\mathcal{B}$, we define

$$\langle \mathcal{B} \rangle := \{M \in \mathcal{C} \mid \exists\, T, F \in \mathrm{Add}\,\mathcal{B} \text{ s.t. } T \to M \to F\},$$
$$\mathcal{H}_1 * \mathcal{H}_2 := \{M \in \mathcal{C} \mid \exists\, T \in \mathcal{H}_1,\ F \in \mathcal{H}_2 \text{ s.t. } T \to M \to F\}.$$

If $\mathcal{H}_1, \mathcal{H}_2$ satisfy $\mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2) = 0$, then we write $\mathcal{H}_1 \perp \mathcal{H}_2$ for $\mathcal{H}_1 * \mathcal{H}_2$. If $\mathcal{H}_1 \perp \mathcal{H}_2 = \mathcal{C}$, the pair $(\mathcal{H}_1, \mathcal{H}_2)$ is called a torsion pair. We denote moreover by $\mathcal{B}^{\perp_{\mathcal{C}}}$ and ${}^{\perp_{\mathcal{C}}}\mathcal{B}$ the right and left orthogonal to $\mathcal{B}$ in $\mathcal{C}$, omitting the subscript when there is no confusion. More precisely,

$$\mathcal{B}^{\perp_{\mathcal{C}}} := \big\{C \in \mathcal{C} : \mathrm{Hom}_{\mathcal{C}}(B, C) = 0,\ \forall B \in \mathcal{B}\big\},$$

and ${}^{\perp_{\mathcal{C}}}\mathcal{B}$ is defined similarly. If $\mathcal{C}$ is triangulated, we denote by $\mathrm{thick}(\mathcal{B})$ the smallest thick additive full subcategory in $\mathcal{C}$ containing $\mathcal{B}$. It is a triangulated subcategory.

Notation 2.1. A finite length abelian category will be said to be *finite* if it has finitely many simple objects, up to isomorphism. The set of isomorphism classes of simples in an abelian category $\mathcal{A}$ will be denoted $\operatorname{Sim}\mathcal{A}$.

A subcategory $\mathcal{S}$ of an abelian category $\mathcal{A}$ is called a *Serre subcategory* if it is abelian and for any short exact sequence

$$0 \to A_1 \to E \to A_2 \to 0$$

in $\mathcal{A}$, we have that $E \in \mathcal{S}$ if and only if $A_1, A_2 \in \mathcal{S}$.

Serre subcategories of a finite length abelian category $\mathcal{A}$ are in bijection with subsets of the set $\operatorname{Sim}\mathcal{A}$.

A *t-structure* on a triangulated category $\mathcal{D}$ is the torsion part $\mathcal{P}$ of a torsion pair (so that $\mathcal{D} = \mathcal{P} \perp \mathcal{P}^{\perp}$) satisfying $\mathcal{P}[1] \subset \mathcal{P}$. The t-structure is said to be *bounded* if

$$\mathcal{D} = \bigcup_{m \in \mathbb{Z}} \mathcal{P}[m] \cap \mathcal{P}^{\perp}[-m].$$

The *heart* of a t-structure $\mathcal{P} \subset \mathcal{D}$ is the full subcategory $\mathcal{H} = \mathcal{P} \cap \mathcal{P}^{\perp}[1]$, which is abelian. If $\mathcal{H} \subset \mathcal{D}$ is the heart of a bounded t-structure, we call it a *(bounded) heart*. The bounded t-structure and its heart determine each other uniquely and hence we will use them interchangeably. The cohomology functor $\mathcal{D} \to \mathcal{H}$ is denoted by $H^0_{\mathcal{H}}$ and the subscript is omitted if there is no possibility of confusion. As standard notation, we write $H^n_{\mathcal{H}}$ for $H^0_{\mathcal{H}} \circ [n]$.

If the heart of a bounded t-structure is a finite length abelian category with finitely many simples, we call it a *finite heart*.

Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in the abelian heart of a bounded t-structure $\mathcal{H} = \mathcal{T} \perp \mathcal{F}$, there is a new heart $\mu^{\sharp}_{\mathcal{F}}\mathcal{H} := \mathcal{F}[1] \perp_{\mathcal{D}} \mathcal{T}$. The operation of creating the new heart is known as Happel–Reiten–Smalø (HRS) tilt or *forward tilt with respect to* $(\mathcal{T}, \mathcal{F})$ of $\mathcal{H}$, see e.g. [14]. The inverse operation is the backward tilt with respect to the torsion class; the new heart is denoted by $\mu^{\flat}_{\mathcal{T}}\mathcal{H} := \mathcal{F} \perp_{\mathcal{D}} \mathcal{T}[-1]$.

A forward tilt $\mathcal{H} \to \mu^{\sharp}_{\mathcal{F}}\mathcal{H}$ (resp., backward tilt $\mathcal{H} \to \mu^{\flat}_{\mathcal{T}}\mathcal{H}$) is *simple* if the corresponding torsion-free class $\mathcal{F}$ (resp., torsion class $\mathcal{T}$) is generated by a simple S of $\mathcal{H}$, i.e., $\mathcal{F} = \langle S \rangle$ (resp., $\mathcal{T} = \langle S \rangle$). For a finite heart $\mathcal{H}$ with a rigid simple S, i.e., such that $\operatorname{Ext}^1(S, S) = 0$, the simple forward tilt with respect to S exists and is denoted by $\mu^{\sharp}_S\mathcal{H}$. It is finite too. While in general it is not true that the forward tilt of a finite heart is finite, if a torsion-free class contains finitely many indecomposables, then tilting at $\mathcal{F}$ is equivalent to perform a finite sequence of simple tilts and the new heart is finite.

We recall general definitions about graphs and we define two graphs encoding information about t-structures of a triangulated category.

Definition 2.2. Let $\mathrm{G} = (V, E)$ be a (oriented) graph with set of vertices V and (directed) edges E. A *subgraph* of G consists of (V', E') where $V' \subseteq V$ and $E' \subseteq E$ such that the endpoints of edges in E' connect vertices in V'. Given $V' \subset V$, the subgraph (V', E') is *full* if $E' \subset E$ is the maximal subset of edges connecting vertices in V'; it is *connected* if, given any two vertices, there is a path connecting them.

Definition 2.3. Let $\mathcal{D}$ be a triangulated category and $\mathcal{A}$ a fixed distinguished finite bounded heart of $\mathcal{D}$.

- The *exchange graph* $\mathrm{EG}(\mathcal{D})$ is the oriented graph whose vertices are all hearts in $\mathcal{D}$ and whose directed edges correspond to simple forward tilt between them.
- The *principal component* $\mathrm{EG}^\circ(\mathcal{D})$ is the *connected component* of the exchange graph $\mathrm{EG}(\mathcal{D})$ containing the distinguished heart $\mathcal{A}$.

2.2. Quivers with potential

Recall that a quiver with potential (Q, W) is defined by a set of vertices Q_0, a set of arrows $\alpha : i \to j \in Q_1$, the source and target functions $s(\alpha) = i$, $t(\alpha) = j$, and a formal sum of cycles W in the completion $\widehat{\mathbf{k}Q}$ of the path algebra of Q with respect to the bilateral ideal generated by the arrows. We refer to [9] for basic notions. We assume that a quiver Q has no loops nor 2-cycles. For any $k \in Q_0$, we denote by μ_k the operation of mutation of (Q, W) at the vertex k, producing another quiver with potential (Q', W') with no loops and no 2-cycles in the following way. The set of vertices Q'_0 is the same as Q_0, arrows α incoming from or outgoing to the vertex k are replaced by their opposite α^*, and for any pair of consecutive arrows $\alpha : i \to k$ and $\beta : k \to j$ passing through k, an arrow $[\alpha\beta] : i \to j$ is added. The potential W' is defined as $\widetilde{W} + \sum_{\alpha,\beta}[\alpha\beta]\beta^*\alpha^*$, where $\widetilde{W}$ is obtained from W by replacing any composition $\alpha\beta : i \to k \to j$ by $[\alpha\beta]$. We say that (Q, W) is non-degenerate if moreover any finite sequence of mutations produces a quiver with no loops nor 2-cycles.

We say that a quiver is *finite* if it has finitely many vertices and finitely many arrows.

The Jacobian algebra of (Q, W) is the quotient $J(Q, W)$ of the complete path algebra $\widehat{\mathbf{k}Q}$ by the bilateral ideal $\partial W = \langle \partial_\alpha W \mid \alpha \in Q_1 \rangle$, where $\partial_\alpha W$ is obtained by cyclically permuting W in a way that α is in the first position in each word, and then deleting α and all words not containing α.

We make the assumption on (Q, W) that it is non-degenerate and that $J(Q, W)$ is finite dimensional.

For a fixed finite non-degenerate quiver with potential (Q, W), we denote by $\Gamma(Q, W)$ the complete Ginzburg differential graded (dg) algebra of Calabi–Yau dimension 3. It is the dg path algebra of a graded quiver $\overline{Q}$ obtained from (Q, W) in the following way

- $\overline{Q}_0 = Q_0$,
- if $\alpha \in Q_1$, then $\alpha \in \overline{Q}_1$ in degree 0,
- for any $\alpha \in Q_1$, there is $\alpha^\vee \in \overline{Q}_1$ in degree -1,
- for any $k \in \overline{Q}_0$, there is a loop $\ell_k : k \to k \in \overline{Q}_1$ in degree -2.

The differential d is the unique continuous linear endomorphism, homogeneous of degree 1, which satisfies the Leibniz rule and takes values $d\alpha^\vee = \partial_\alpha W$, $d\ell_k = \sum_{\alpha \in Q_1} \ell_k[\alpha, \alpha^\vee]\ell_k$. We refer to [19, 21] for more details. The zero-th homology of this dg algebra is the Jacobian algebra of the original quiver with potential: $H^0\Gamma(Q, W) \simeq J(Q, W)$.

2.3. Localization of categories and hearts

We recall the notions of quotient of an abelian category by a Serre subcategory and of the Verdier quotient of a triangulated category, and give some definitions concerning localization of bounded t-structures. References for these notions are Neeman's book [27], in particular, Appendix A and Chapter 2, and [3].

Given a Serre subcategory $\mathcal{S}$ of an abelian category $\mathcal{H}$, one can construct the quotient $\mathcal{H}/\mathcal{S}$, together with a projection functor $\pi : \mathcal{H} \to \mathcal{H}/\mathcal{S}$ defined in the following way.

Definition 2.4. If $\mathcal{S}$ is a Serre subcategory of an abelian category $\mathcal{H}$, $\mathcal{H}/\mathcal{S}$ is a category with the same set of objects as $\mathcal{H}$ and where, for all $X, Y \in \mathrm{Obj}(\mathcal{H}/\mathcal{S})$, a morphism in $\mathrm{Hom}_{\mathcal{H}/\mathcal{S}}(X, Y)$ is an equivalence class of roofs $(\widetilde{\eta}, \eta)$ of the form

$$X \xleftarrow{\widetilde{\eta}} Z \xrightarrow{\eta} Y$$

for some $Z \in \mathrm{Obj}(\mathcal{H}/\mathcal{S})$, $\widetilde{\eta} \in \mathrm{Hom}_{\mathcal{H}}(Z, X)$, $\eta \in \mathrm{Hom}_{\mathcal{H}}(Z, Y)$, with $\ker \widetilde{\eta} \in \mathcal{S} \ni \operatorname{coker} \widetilde{\eta}$, and with obvious composition law

$$\begin{tikzcd}[column sep=small] & & & T \arrow[dl] \arrow[dr] & & & \\ & & Z \arrow[dll] \arrow[dr] & & U \arrow[dl] \arrow[drr] & & \\ X & & & Y & & & W. \end{tikzcd}$$

(Here we say that two roofs are equivalent, if they are dominated by a third roof.) The projection $\pi : \mathcal{H} \to \mathcal{H}/\mathcal{S}$ is a functor whose kernel is $\mathcal{S}$, sending objects identically in themselves and morphisms

$$\mathrm{Hom}_{\mathcal{H}}(A, B) \ni f \mapsto (\mathrm{id}, f) \in \mathrm{Hom}_{\mathcal{H}/\mathcal{S}}(A, B).$$

Lemma 2.5 ([27, Lemma A.2.3], [12]). *The category $\mathcal{H}/\mathcal{S}$ is an abelian category. The functor π is exact, and takes the objects of $\mathcal{S}$ to objects in $\mathcal{H}/\mathcal{S}$ isomorphic to zero. Furthermore, π is universal with this property, and the subcategory $\mathcal{S}$ is the full subcategory of objects of $\mathcal{H}$ whose images under π are isomorphic to zero.*

Roughly speaking, this means that all morphisms in $\mathcal{S}$ have become invertible, and objects in $\mathcal{S}$ can be treated as zero objects, and that we have a short exact sequence

$$0 \to \mathcal{S} \xrightarrow{\iota} \mathcal{H} \xrightarrow{\pi} \mathcal{H}/\mathcal{S} \to 0,$$

with ι and π exact functors of abelian categories.

Verdier (triangulated) localization. Given a triangulated category $\mathcal{D}$ and a triangulated subcategory $\mathcal{V} \hookrightarrow \mathcal{D}$, we can construct the so-called *Verdier quotient* $\mathcal{D}/\mathcal{V}$ with a procedure similar to that discussed for abelian categories. The category $\mathcal{D}/\mathcal{V}$ has the same objects as $\mathcal{D}$ and a morphism between two objects R and S in $\mathcal{D}/\mathcal{V}$ is an equivalence class of roofs

$$R \xleftarrow{\tilde{f}} T \xrightarrow{f} S$$

with $\tilde{f}$ a morphism in $\mathcal{D}$ fitting in a distinguished triangle $T \xrightarrow{\tilde{f}} R \to V$, with V in $\mathcal{V}$. There is a natural triangulated functor $\pi : \mathcal{D} \to \mathcal{D}/\mathcal{V}$, called the *Verdier localization*. If the triangulated subcategory $\mathcal{V}$ is thick, that is it contains all direct summands of its objects, then

$$0 \to \mathcal{V} \xrightarrow{\iota} \mathcal{D} \xrightarrow{\pi} \mathcal{D}/\mathcal{V} \to 0$$

is a short exact sequence of categories with exact functors [33, Proposition 2.3.1].

Notation 2.6. Whenever $\pi : \mathcal{D} \to \mathcal{D}/\mathcal{V}$ is a quotient functor of triangulated categories, and $\mathcal{B}$ is a subcategory of $\mathcal{D}$, by $\pi(\mathcal{B})$ we will mean the *essential image* of $\mathcal{B}$ through π. This will apply, in particular, to the image of an abelian heart $\mathcal{H} \subset \mathcal{D}$.

Notation 2.7. To avoid confusion, if $M \in \mathcal{D}$ is an object, we will denote by $\bar{M}$ the same object regarded in the quotient category $\mathcal{D}/\mathcal{V}$.

We recall a general fact about bounded t-structures under localization.

Proposition 2.8 ([3, Proposition 2.20]). *Let $\iota : \mathcal{C} \to \mathcal{D}$ be a t-exact fully faithful functor of triangulated categories equipped with bounded t-structures, with a well-defined quotient functor $\pi : \mathcal{D} \to \mathcal{D}/\mathcal{C}$. Let $\mathcal{H}_\mathcal{D}$ and $\mathcal{H}_\mathcal{C}$ be the two hearts in $\mathcal{D}$ and $\mathcal{C}$ respectively. Then the following are equivalent*

(a) *the essential image $\iota(\mathcal{H}_\mathcal{C}) \subset \mathcal{H}_\mathcal{D}$ is a Serre subcategory, and*

(b) *the quotient $\mathcal{D}/\mathcal{C}$ has a bounded t-structure such that π is t-exact, whose heart is equivalent to $\mathcal{H}_\mathcal{D}/\mathcal{H}_\mathcal{C}$.*

The (bounded) t-structure corresponding to $\mathcal{H}_\mathcal{D}/\mathcal{H}_\mathcal{C}$ in $\mathcal{D}/\mathcal{C}$ of point (b) is described in [3, Proposition 2.20].

Suppose $\mathcal{V}$ is a full thick triangulated subcategory of a triangulated category $\mathcal{D}$, and let $\mathcal{H}$ be a bounded heart of $\mathcal{D}$.

Definition 2.9 ([5, Definition 2.3]). We say that $\mathcal{H}$ is *compatible* with $\mathcal{V}$ if $\mathcal{H} \cap \mathcal{V}$ is a bounded heart of $\mathcal{V}$ making the inclusion of $\mathcal{V}$ in $\mathcal{D}$ t-exact, and it is a Serre full subcategory of $\mathcal{H}$.

If a heart in $\mathcal{D}/\mathcal{V}$ arises as in Proposition 2.8, we say that it is *induced* by the heart $\mathcal{H}_\mathcal{D}$, and we call it a heart *of quotient type*. It will be denoted by $\overline{\mathcal{H}}$ whenever it is equivalent to $\mathcal{H}/(\mathcal{H} \cap \mathcal{V})$.

We introduce moreover the following notation, based on Definition 2.3:

Definition 2.10. Fix a distinguished bounded t-structure of $\mathcal{D}$ with finite heart $\mathcal{A}$ compatible with $\mathcal{V}$.

- $\mathrm{EG}^\circ(\mathcal{D}, \mathcal{V})$ is the full subgraph of $\mathrm{EG}^\circ(\mathcal{D})$ whose vertices are finite hearts of bounded t-structures in $\mathcal{D}$ that are compatible with $\mathcal{V}$;
- $\mathrm{EG}^\bullet(\mathcal{D}/\mathcal{V})$ is the full subgraph of $\mathrm{EG}(\mathcal{D}/\mathcal{V})$ whose vertices can be realized as quotients of hearts $\mathcal{H} \in \mathrm{EG}^\circ(\mathcal{D})$.

The next lemma is easy to understand, and is proven in [4, Proposition 6.7] for the dual notion of HRS backward tilt (at a torsion class).

Lemma 2.11. *Suppose that $\mathcal{H}, \mathcal{H}'$ are $\mathcal{V}$-compatible hearts of $\mathcal{D}$. Then*

$$\mathcal{H}/(\mathcal{H} \cap \mathcal{V}) \simeq \mathcal{H}'/(\mathcal{H}' \cap \mathcal{V}) \subset \mathcal{D}/\mathcal{V},$$

if $\mathcal{H}' = \mu^{\sharp}_{\mathcal{F}} \mathcal{H}$ at some torsion-free class $\mathcal{F} \subset \mathcal{H}$ such that $\mathcal{F} \subset \mathcal{V}$.

We end by noting that if $\mathcal{H}_{\mathcal{C}}$ and $\mathcal{H}_{\mathcal{D}}$ are finite hearts as in Proposition 2.8, then $\iota\mathcal{F}$ is a torsion-free class in $\mathcal{H}_{\mathcal{D}}$ if and only if $\mathcal{F}$ is a torsion-free class in $\mathcal{H}_{\mathcal{C}}$.

2.4. Recollements of triangulated categories and hearts

If the inclusion and quotient functor ι and π are exact and admit left and right adjoints, we are in a *recollement* situation. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be either abelian or triangulated categories.

Definition 2.12. A recollement $R(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r, "\iota"] & \mathcal{B} \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "\pi"] & \mathcal{C} \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"] \arrow[r] & 0 \end{tikzcd} \tag{2.1}$$

such that (ℓ, π, r) and (q, ι, p) are adjoint triples, and, moreover, the functors ι, ℓ, r are fully faithful with $\operatorname{Im}\iota = \ker\pi$.

We write $R_{ab}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ for a recollement of abelian categories and $R_{tr}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ for a recollement of triangulated categories.

A recollement $R_{ab}(\mathcal{H}', \mathcal{H}, \mathcal{H}'')$ is a recollement of hearts $R_{\heartsuit}(\mathcal{H}', \mathcal{H}, \mathcal{H}'')$ if it is induced by a t-exact recollement of triangulated categories $R_{tr}(\mathcal{D}', \mathcal{D}, \mathcal{D}'')$ by composing with the cohomology functors and the inclusions, in the sense that it fits into a commutative diagram

$$\begin{tikzcd}[row sep=huge, column sep=huge] \mathcal{D}' \arrow[r, "\iota"] \arrow[d, "H^0_{\mathcal{H}'}"] & \mathcal{D} \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "\pi"] \arrow[d, "H^0_{\mathcal{H}}"] & \mathcal{D}'' \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"] \arrow[d, "H^0_{\mathcal{H}''}"] \\ \mathcal{H}' \arrow[r, "\iota'"] \arrow[u, bend left, "\varepsilon'"] & \mathcal{H} \arrow[l, bend right, "q'"'] \arrow[l, bend left, "p'"] \arrow[r, "\pi'"] \arrow[u, bend left, "\varepsilon"] & \mathcal{H}'', \arrow[l, bend right, "\ell'"'] \arrow[l, bend left, "r'"] \arrow[u, bend left, "\varepsilon''"] \end{tikzcd} \tag{2.2}$$

where the ε's map denote the inclusions and the H^0's maps denote the cohomology functors defined by the bounded t-structures. In a recollement of hearts,

$$\begin{aligned} q' &= H^0_{\mathcal{H}'} \circ q \circ \varepsilon, & \ell' &= H^0_{\mathcal{H}} \circ \ell \circ \varepsilon'', \\ \iota' &= H^0_{\mathcal{H}} \circ \iota \circ \varepsilon', & \pi' &= H^0_{\mathcal{H}''} \circ \pi \circ \varepsilon, \\ p' &= H^0_{\mathcal{H}'} \circ p \circ \varepsilon, & r' &= H^0_{\mathcal{H}} \circ r \circ \varepsilon''. \end{aligned}$$

The notions of compatible and glued torsion pairs defined below are taken from [29]. They ensure that a recollement of hearts from a recollement of triangulated categories induces another recollement of hearts by tilting. Proposition 2.14 characterizes torsion pairs whose images under the localization map π are still torsion pairs.

Definition 2.13. Let $R_{ab}(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ be the recollement of abelian categories

$$\begin{tikzcd} \mathcal{A}' \arrow[r, "\iota"] & \mathcal{A} \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "\pi"] & \mathcal{A}''. \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"] \end{tikzcd} \tag{2.3}$$

A torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{A}$ is said to be *compatible* with $R_{ab}(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ if $\ell\pi(\mathcal{X}) \subset \mathcal{X}$ or $r\pi(\mathcal{Y}) \subset \mathcal{Y}$.

Proposition 2.14 ([29, Proposition 9.1]). *Let $R_{ab}(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ be a recollement of abelian categories.*

(i) *Let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in $\mathcal{B}$. Then*
- *the pair $(q(\mathcal{X}), p(\mathcal{Y}))$ is a torsion pair in $\mathcal{A}'$, and*
- *the pair $(\pi(\mathcal{X}), \pi(\mathcal{Y}))$ is a torsion pair in $\mathcal{A}''$ if and only if $\ell\pi(\mathcal{X}) \subset \mathcal{X}$ if and only if $r\pi(\mathcal{Y}) \subset \mathcal{Y}$.*

(ii) *Let $(\mathcal{X}', \mathcal{Y}')$ and $(\mathcal{X}'', \mathcal{Y}'')$ be torsion pairs in $\mathcal{A}'$ and $\mathcal{A}''$ respectively. Then*

$$\mathcal{X} := \{A \in \mathcal{A} \mid q(A) \in \mathcal{X}',\ \pi(A) \in \mathcal{X}''\},$$
$$\mathcal{Y} := \{A \in \mathcal{A} \mid p(A) \in \mathcal{Y}',\ \pi(A) \in \mathcal{Y}''\}$$

is a torsion pair in $\mathcal{A}$ compatible with $R_{ab}(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$.

Definition 2.15. The torsion pair $(\mathcal{X}, \mathcal{Y})$ in Proposition 2.14 (ii), is said to be *glued* from $(\mathcal{X}', \mathcal{Y}')$ and $(\mathcal{X}'', \mathcal{Y}'')$. Similarly, if $\mathcal{A}', \mathcal{A}, \mathcal{A}''$ are hearts of bounded t-structures, the t-structure obtained by (forward or backward) tilting $\mathcal{A}$ at the glued torsion pair will be said to be *glued* from bounded t-structures with hearts $\mathcal{A}', \mathcal{A}''$.

Let $R_{tr}(\mathcal{D}', \mathcal{D}, \mathcal{D}'')$ be a recollement of triangulated categories. It is shown in [29, Section 9] that, if $R_{\heartsuit}(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ is a recollement of hearts and the torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{A}$ is glued from $(\mathcal{X}', \mathcal{Y}')$ and $(\mathcal{X}'', \mathcal{Y}'')$, then

$$R_{\heartsuit}\big(\mu^{\sharp}_{\mathcal{Y}'}\mathcal{A}', \mu^{\sharp}_{\mathcal{Y}}\mathcal{A}, \mu^{\sharp}_{\mathcal{Y}''}\mathcal{A}''\big) \quad \text{and} \quad R_{\heartsuit}\big(\mu^{\flat}_{\mathcal{X}'}\mathcal{A}', \mu^{\flat}_{\mathcal{X}}\mathcal{A}, \mu^{\flat}_{\mathcal{X}''}\mathcal{A}''\big)$$

are other recollements of hearts. It is however not true in general that adjoint triples restrict to subcategories.

On the other hand, whether a recollement of abelian hearts lifts to a recollement of triangulated categories is also an interesting question considered for instance in [2].

In Section 3 we will be interested in an intermediate situation where a localization of triangulated categories

$$0 \to \mathcal{C}' \xrightarrow{\iota} \mathcal{C} \xrightarrow{\pi} \mathcal{C}'' \to 0$$

does not admit adjoints, but some bounded hearts fit into a recollement, and we would like to understand when tilting bounded t-structures in $\mathcal{C}''$ arise from tilting a bounded t-structure in $\mathcal{C}$. We will show in Theorem 3.7 that under suitable conditions, we can *"glue"* bounded t-structures even in the absence of a recollement of triangulated categories.

2.5. Categories from quivers with potential

In this paper we work with the following triangulated categories associated to a differential graded (dg) algebra Λ:

- the derived category $\mathcal{D}(\Lambda)$,
- the perfect derived category $\mathrm{per}(\Lambda) \subset \mathcal{D}(\Lambda)$, and
- the perfectly valued derived category $\mathrm{pvd}(\Lambda) \subset \mathcal{D}(\Lambda)$.

The latter consists of objects M in $\mathcal{D}(\Lambda)$ such that $\mathrm{Hom}_{\mathcal{D}(\Lambda)}(P, M[i])$ has finite dimension for any $P \in \mathrm{per}(\Lambda)$ and for any $i \in \mathbb{Z}$.

Here and through the rest of the paper, we fix a non-degenerate finite quiver with potential (Q, W) whose Jacobian algebra is finite dimensional. We denote the 3-Calabi–Yau Ginzburg dg algebra $\Gamma(Q, W)$ simply by Γ and its Jacobian algebra $J(Q, W)$ by J. We recall that $J = H^0\Gamma$. The main features of $\mathrm{pvd}(\Gamma)$ that will be used in the next sections are recalled here. We refer the reader to [21] for details.

The category $\mathrm{pvd}(\Gamma)$ coincides with the subcategory of $\mathcal{D}(\Gamma)$ consisting of dg modules of finite-dimensional total co-homology. It is generated by the simple dg modules S_i associated with the vertices of Q. It is Hom-finite and of Calabi–Yau dimension 3, meaning that there is a natural isomorphism of $\mathbf{k}$-vector spaces

$$\nu : \mathrm{Hom}_{\mathrm{pvd}\,\Gamma}(E, F) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{pvd}\,\Gamma}\left(F, E[3]\right)^{\vee} \tag{2.4}$$

for any pair of objects E, F. Indeed, for any pair of simple dg modules S_i, S_j, $i, j \in Q_0$, a basis of $\mathrm{Hom}^{d+1}(S_i, S_j)$ is given by the (finitely many) degree $(-d)$ arrows $i \xrightarrow{-d} j$ in $\bar{Q}$. The Calabi–Yau property (2.4) can be written as

$$\mathrm{Hom}^n(S_i, S_j) \simeq \mathrm{Hom}^{3-n}(S_j, S_i)^{\vee},$$

and is combinatorically expressed by the existence of pairs of opposite arrows in $\bar{Q}$ in degree $-d, -d^*$ such that $(d+1) + (d^*+1) = 3$, with the convention that $\mathrm{Hom}^0(S_i, S_i) = \mathbf{k}$ is implicitly assumed and not represented by arrows.

Since $\Gamma = \Gamma(Q, W)$ is homologically smooth (defined and proven in [21, Section 7]), we have the following sequence of inclusions

$$\mathrm{pvd}(\Gamma) \subset \mathrm{per}(\Gamma) \subset \mathcal{D}(\Gamma),$$

where the category $\mathrm{per}\,\Gamma$ is generated by the indecomposable projective dg modules $P_k = e_k\Gamma$, where e_k is the idempotent element in Γ associated with the vertex $k \in Q_0$. As $\Gamma = \bigoplus_{k=1}^{n} P_k$, we have

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(P_k, S_i) = \delta_{ki}\mathbf{k}, \tag{2.5}$$

for any $k, i \in Q_0$.

We recall that $\operatorname{pvd}(\Gamma), \operatorname{per}(\Gamma), \mathcal{D}(\Gamma)$ are equivalent to $\operatorname{pvd}(\Gamma'), \operatorname{per}(\Gamma'), \mathcal{D}(\Gamma')$, if $\Gamma' = \Gamma(Q', W')$ is associated with a quiver (Q', W') obtained from (Q, W) by mutations at its vertices.

The standard t-structure with heart $\operatorname{Mod}(J)$ on $\mathcal{D}(\Gamma)$ restricts to bounded t-structures with heart $\operatorname{rep}(Q, W) \simeq \operatorname{mod} J$ on $\operatorname{pvd}\Gamma$ and $\operatorname{per}\Gamma$, that we still call standard. The abelian category $\operatorname{mod} J$ is finite-length with simple objects in bijection with the vertices of Q. The correspondence between HRS simple tilts in $\operatorname{Mod} J \subset \mathcal{D}(\Gamma)$ or $\operatorname{mod} J \subset \operatorname{pvd}(\Gamma)$ and mutations at a vertex $i \in Q_0$ realizes $\operatorname{rep}(Q', W') = \operatorname{mod} J'$ as another heart in $\operatorname{pvd}(\Gamma)$, when (Q', W') is obtained from (Q, W) by a sequence of mutations. In fact, if $\operatorname{EG}^{\circ}(\operatorname{pvd}\Gamma)$ is the connected component of the exchange graph containing $\operatorname{mod} J$ as a vertex, any other vertex corresponds to a finite heart equivalent to $\operatorname{mod} J'$ for the Jacobian algebra J' of a quiver with potential (Q', W') related to (Q, W) by a sequence of mutations.

For later purposes, we recall here the relations between simples of $\operatorname{mod} J$ and simples of $\mu_S^{\sharp/\flat} \operatorname{mod} J$ inside $\operatorname{pvd}(\Gamma)$, assuming that (Q, W) has no loops nor 2-cycles. Let $\mathcal{H} = \operatorname{mod} J$. Let $\operatorname{Sim}\mathcal{H} = \{[S_j] \mid j = 1, \dots, n\}$ and $S = S_i$ for some i.

Then we have that

$$\operatorname{Sim}\mu_S^{\sharp}\mathcal{H} = \big\{[S[1]], [F_j] \mid j \neq i\big\} \quad \text{and} \quad \operatorname{Sim}\mu_S^{\flat}\mathcal{H} = \big\{[S[-1]], [E_j] \mid j \neq i\big\}$$

where F_j and E_j are defined as cones of universal extensions in $\operatorname{pvd}(\Gamma)$ and fit into the following distinguished triangles

$$S \otimes \operatorname{Ext}^1_{\mathcal{H}}(S_j, S) \to F_j \to S_j \quad \text{and} \quad S_j \to E_j \to S \otimes \operatorname{Ext}^1_{\mathcal{H}}(S, S_j).$$

In particular, $F_j = S_j$ if $\operatorname{Ext}^1_{\mathcal{H}}(S_j, S) = 0$, and $E_j = S_j$ if $\operatorname{Ext}^1_{\mathcal{H}}(S, S_j) = 0$.

3. Localization of Calabi–Yau categories from quivers

We use the hypotheses and the notation set in Section 2.5. Moreover, we fix once for all a proper subset $I \subset Q_0$ of the set of vertices of Q. We denote by $I^c = Q_0 \setminus I$ its complement. The quiver with potential (Q_I, W_I) is the full subquiver of (Q, W) whose set of vertices is I, and with potential W_I obtained from W by deleting all cycles passing through vertices of I^c. We write Γ_I for $\Gamma(Q_I, W_I)$ and $J_I = H^0\Gamma_I = \mathbf{k}Q_I/\partial W_I$. The quiver (Q_I, W_I) and the associated algebras Γ_I, J_I satisfy the same properties as (Q, W), Γ and J. The following lemma is clear and will be useful later.

Lemma 3.1. *Let k be a vertex of Q in $I \subset Q_0$, or $k \notin I$ be a vertex of Q such that there are no arrows from i to k or from k to i for all $i \in I$. Then $(\mu_k(Q, W))_I = (Q_I, W_I)$.*

We introduce here the Verdier quotients $\mathcal{D}(\Gamma)/\mathcal{D}(\Gamma_I)$ and $\operatorname{pvd}(\Gamma)/\operatorname{pvd}(\Gamma_I)$ and their underlying algebra $e\Gamma e$ for an idempotent $e \in \Gamma$. Under our assumptions, the Verdier quotient is a dg quotient in the sense of [10, 18], so the two perspectives are available. The goal of this Section is to present and study some homological properties of $\operatorname{pvd}(e\Gamma e) = \operatorname{pvd}(\Gamma)/\operatorname{pvd}(\Gamma_I)$.

3.1. Idempotents

For any $k \in Q_0$, we let e_k be the idempotent element in Γ associated with the vertex k of the quiver Q. We define

$$e := \sum_{j \in I^c} e_j$$

to be the idempotent associated to the complement $I^c = Q_0 \setminus I$. We use the same notation, e_k and e, for the same elements regarded as idempotents in J. As explained in [17], the Jacobian algebra J_I and the Ginzburg algebra Γ_I of the subquiver (Q_I, W_I) are isomorphic to the quotients by the bilateral ideals generated by e:

$$J_I \simeq J/JeJ, \quad \text{and} \quad \Gamma_I \simeq \Gamma/\Gamma e \Gamma, \quad \text{respectively,}$$

and we have exact inclusions of categories

$$\mathrm{Mod}(J_I) \hookrightarrow \mathrm{Mod}(J), \quad \text{and}\, \mathrm{mod}(J) \hookrightarrow \mathrm{mod}(J),$$

and

$$\mathcal{D}(\Gamma_I) \hookrightarrow \mathcal{D}(\Gamma).$$

At the abelian level, mod J_I is a Serre subcategory of mod J generated by the simple objects labeled by indexes in $I \subset Q_0$. In fact, Serre subcategories of mod J are in bijection with full subquivers (Q_I, W_I) of (Q, W), or equivalently with subsets $I \subset Q_0$.

The other algebras we are interested in here are eJe and $e\Gamma e$, that we describe briefly. Let A denote either $J = J(Q, W)$ or $\Gamma = \Gamma(Q, W)$, then eAe consists of all (graded) paths in A that start and end at some vertex in I^c. For $A = J$, this is still a quiver algebra for a finite quiver with potential (Q^{eJe}, W^{eJe}) with

- set of vertices $Q_0^{eJe} = I^c$, and
- set of arrows Q_1^{eJe}, consisting of arrows in Q_1 connecting two vertices in I^c, and a basis of arrows $j_1 \to j_2$ for distinct paths connecting $j_1, j_2 \in I^c$ in $\mathbf{k}Q$, not in the bilateral ideal ∂W_I.

If J is a finite-dimensional algebra, eJe is still a finite-dimensional algebra, and the quiver with potential (Q^{eJe}, W^{eJe}) has finitely many vertices and finitely many arrows. See Figure 1 for examples.

$$A_3 = \bullet_1 \xrightarrow{\alpha} \bullet_2 \xrightarrow{\beta} \bullet_3, \qquad \mu_2 A_3 = \begin{tikzcd} \bullet_1 \arrow[rr, "{[\alpha\beta]}"] & & \bullet_3 \arrow[dl, "\beta^*"] \\ & \bullet_2 \arrow[ul, "\alpha^*"] & \end{tikzcd}, \qquad W = [\alpha\beta]\beta^*\alpha^*,$$

$$Q^{eJe} = \bullet_{\bar{1}} \xrightarrow{\alpha\beta} \bullet_{\bar{3}}.$$

Figure 1. The choice of $I = \{2\}$ for the quivers with potential $(A_3, 0)$ and $(\mu_2 A_3, W)$ produces the same quiver Q^{eJe} with trivial potential. The abelian quotient category does not change.

Figure 2. The graded quiver $\bar{Q}^{e\Gamma e}$ obtained from $\Gamma = \Gamma(A_2, 0)$ for $e = e_2$ has infinitely many arrows: one in degree -2 and one for any odd negative integer, obtained by concatenating α, $(\ell_2)^k$, and $\alpha^\vee$ in $\bar{Q} = \overline{A_2}$. The differential is the same of Γ.

A similar argument shows that $e\Gamma e$ is the dg path algebra of a graded quiver $\bar{Q}^{e\Gamma e}$ with finitely many vertices I^c, but possibly infinitely many arrows: for any pair of vertices $j_1, j_2 \in I^c$ and for any degree $-d \in \mathbb{Z}_{\leq 0}$, an arrow $j_1 \xrightarrow{-d} j_2$ appears whenever there is a path of degree $-d$ in $\Gamma(Q, W)$. See Figure 2 for an example. The differential is the same of Γ (defined in Section 2.2), inducing an isomorphism $H^k(e\Gamma e) \simeq eH^k(\Gamma)e$.

Writing $e\Gamma e = e\Gamma \otimes \Gamma e$ identifies the dg algebra $e\Gamma e$ with the endomorphism dg algebra of the projective module $\Gamma e = \sum_{j \in I^c} \Gamma e_j$ in $\mathcal{D}(\Gamma)$.

3.2. Recollements induced by idempotents

As explained in [17], the Ginzburg algebra Γ_I is isomorphic to $\Gamma / \Gamma e \Gamma$, for $e = \sum_{j \in I^c} e_j$ defined above. On the other hand, the Verdier quotient $\mathcal{D}(\Gamma)/\mathcal{D}(\Gamma_I)$ coincides with $\mathcal{D}(e\Gamma e)$. The derived category associated to Γ and its restriction to I fits in a recollement of triangulated categories induced by an idempotent,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{D}(\Gamma_I) \arrow[r, "\iota"] & \mathcal{D}(\Gamma) \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "\pi"] & \mathcal{D}(e\Gamma e) \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"] \arrow[r] & 0 \end{tikzcd} \tag{3.1}$$

where

$$\begin{aligned} q(-) &= - \overset{\mathbb{L}}{\otimes}_\Gamma \Gamma/\Gamma e\Gamma, & \ell(-) &= - \overset{\mathbb{L}}{\otimes}_{e\Gamma e} e\Gamma, \\ \iota(-) &= \mathbb{R}\operatorname{Hom}_{\Gamma/\Gamma e \Gamma}(\Gamma/\Gamma e\Gamma, -), & \pi(-) &= \mathbb{R}\operatorname{Hom}_\Gamma(e\Gamma, -), \\ p(-) &= \mathbb{R}\operatorname{Hom}_\Gamma(\Gamma/\Gamma e\Gamma, -), & r(-) &= \mathbb{R}\operatorname{Hom}_{e\Gamma e}(\Gamma e, -), \end{aligned}$$

are adjoint pairs of fully faithful functors. They induce a recollement of hearts

$$\begin{tikzcd} \mathcal{D}(\Gamma_I) \arrow[r, "\iota"] \arrow[d, "H^0_{\operatorname{Mod} J_I}"] & \mathcal{D}(\Gamma) \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "\pi"] \arrow[d, "H^0_{\operatorname{Mod} J}"] & \mathcal{D}(e\Gamma e) \arrow[l, bend right, "l"'] \arrow[l, bend left, "r"] \arrow[d, "H^0_{\operatorname{Mod} eJe}"] \\ \operatorname{Mod} J_I \arrow[r, "\iota"] \arrow[u, bend left, "\varepsilon_I"] & \operatorname{Mod} J \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "\pi"] \arrow[u, bend left, "\varepsilon"] & \operatorname{Mod} eJe, \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"] \arrow[u, bend left, "\varepsilon_e"] \end{tikzcd} \tag{3.2}$$

where the ε's maps are inclusions, and the functors between hearts are denoted exactly as the triangulated functors with abuse of notation. We refer to [29] for examples of recollements induced by idempotents and for the general theory. See also [17, Section 7] for the special example involving Calabi–Yau algebras from quivers with potential.

At the level of finitely generated modules over the Jacobian algebras, $\operatorname{mod} J_I \subset \operatorname{mod} J$ is localizing and co-localizing, and we have a similar recollement of abelian categories

$$\begin{tikzcd} \operatorname{mod} J_I \arrow[r, "\iota"] & \operatorname{mod} J \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "\pi"] & \operatorname{mod} eJe, \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"] \end{tikzcd} \tag{3.3}$$

where we use the same notation as in (3.2) for obvious reasons:

$$\begin{aligned} \iota(-) &= \operatorname{Hom}_{J/JeJ}(J/JeJ, -), & \pi(-) &= \operatorname{Hom}_J(eJ, -),\\ q(-) &= - \otimes_J J/JeJ, & \ell(-) &= - \otimes_{eJe} eJ,\\ p(-) &= \operatorname{Hom}_J(J/JeJ, -), & r(-) &= \operatorname{Hom}_{eJe}(Je, -). \end{aligned}$$

The article [13] by Geigle and Lenzing is a reference text for the description of $\operatorname{mod} J_I$ and $\operatorname{mod} eJe$ inside $\operatorname{mod} J$.

A trivial corollary of Proposition 2.14 is the following.

Corollary 3.2. *Any torsion pair in* $\operatorname{mod}(eJe)$ *lifts to a torsion pair in* $\operatorname{mod} J$ *(in non-unique way).*

3.3. The quotient category $\mathbf{pvd}(e\Gamma e)$

By definition, the maps ι and π appearing in (3.1) restrict to the perfectly valued derived categories of the same dg algebras, i.e., to modules with finite-dimensional total homology. Therefore the Verdier quotient $\operatorname{pvd}(\Gamma)/\operatorname{pvd}(\Gamma_I)$ is equivalent to $\operatorname{pvd}(e\Gamma e)$

$$0 \longrightarrow \operatorname{pvd}(\Gamma_I) \xrightarrow{\iota} \operatorname{pvd}(\Gamma) \xrightarrow{\pi} \operatorname{pvd}(e\Gamma e) \longrightarrow 0.$$

The functors ℓ and q restrict to the perfect categories

$$0 \longleftarrow \operatorname{per}(\Gamma_I) \xleftarrow{q} \operatorname{per}(\Gamma) \xleftarrow{\ell} \operatorname{per}(e\Gamma e) \longleftarrow 0.$$

Remark 3.3. The left adjoints q, ℓ do not descend to the pvd level.

We learned this argument from Dong Yang; we include a proof for completeness. The crucial feature is Calabi–Yau dimensional finiteness.

Proof. Suppose that $\pi : \operatorname{pvd}(\Gamma) \to \operatorname{pvd}(e\Gamma e)$ had a left adjoint $\pi_!$. Then the pair

$$(\mathcal{X}, \mathcal{Y}) := \big(\pi_!\big(\operatorname{pvd}(e\Gamma e)\big), \iota\big(\operatorname{pvd}(\Gamma/\Gamma e\Gamma)\big)\big)$$

would be a stable t-structure (i.e., the aisle and the co-aisle are closed under arbitrary shifts) on $\operatorname{pvd}(\Gamma)$, hence a split t-structure since $\operatorname{pvd}(\Gamma)$ is Calabi–Yau. Indeed, the Calabi–Yau condition applied to $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$,

$$\operatorname{Hom}(Y, X)^\vee \simeq \operatorname{Hom}\big(X, Y[3]\big) = 0,$$

implies that, for any $E \in \mathrm{pvd}(\Gamma)$,

$$x(E) \to E \to y(E) \to x(E)[1], \quad x(E) \in \mathcal{X},\ y(E) \in \mathcal{Y}$$

is a splitting sequence with $y(E) \xrightarrow{0} x(E)[1]$ and $E \simeq x(E) \oplus y(E)$.

Similarly, one proves that π does not admit a right adjoint, and that there are no adjoints in $\mathrm{pvd}(\Gamma)$ to the inclusion ι. ∎

Theorem 3.4. *The cohomology theories in the derived categories of* (3.2) *associated with* $\mathrm{Mod}\, J_I$, $\mathrm{Mod}\, J$, *and* $\mathrm{Mod}\, eJe$ *restrict to bounded t-structures and cohomology theories on* $\mathrm{pvd}(\Gamma_I)$, $\mathrm{pvd}(\Gamma)$, *and* $\mathrm{pvd}(e\Gamma e)$, *with hearts* $\mathrm{mod}\, J_I$, $\mathrm{mod}\, J$, *and* $\mathrm{mod}\, eJe$, *respectively.*

Proof. The statement follows from [17, Proposition 3.2], being J, J_I, eJe Noetherian rings, and Γ, Γ_I, $e\Gamma e$ non-positive dg algebras. ∎

In particular,

$$\mathrm{mod}\, J_I = \mathrm{pvd}\, \Gamma_I \cap \mathrm{mod}\, J \subset \mathrm{mod}\, J$$

as a Serre subcategory, and $\mathrm{mod}\, eJe \simeq \mathrm{mod}\, J / \mathrm{mod}\, J_I$.

Notation 3.5. We fix the standard heart $\mathcal{A} := \mathrm{mod}\, J$ as a distinguished heart for $\mathrm{pvd}(\Gamma)$, so that $\mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma)$ is the connected component of $\mathrm{EG}(\mathrm{pvd}\,\Gamma)$ containing $\mathrm{mod}\, J$ as a vertex.

We recall that the graph $\mathrm{EG}^\bullet(\mathrm{pvd}(e\Gamma e))$ is the full subgraph of $\mathrm{EG}(\mathrm{pvd}(e\Gamma e))$ whose vertices correspond to hearts $\overline{\mathcal{H}}$ for some $\mathcal{H}$ in $\mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma)$. Then $\mathrm{EG}^\bullet(\mathrm{pvd}(e\Gamma e))$ contains $\mathrm{mod}(eJe)$ as a vertex.

While homological smoothness guarantees the inclusion $\mathrm{pvd}(\Lambda) \subset \mathrm{per}(\Lambda) \subset \mathcal{D}(\Lambda)$, for $\Lambda = \Gamma, \Gamma_I$, we have no reasons to expect the same chain of inclusions for $e\Gamma e$, and the relation between $\mathrm{pvd}(e\Gamma e)$ and $\mathrm{per}(e\Gamma e)$ remains unknown to us,

$$\begin{tikzcd}
\mathcal{D}(\Gamma_I) \arrow[r, "\iota"] & \mathcal{D}(\Gamma) \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "\pi"] & \mathcal{D}(e\Gamma e) \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"] \\
\cup & \cup & \\
\mathrm{per}(\Gamma_I) & \mathrm{per}(\Gamma) \arrow[l, "q"'] & \mathrm{per}(e\Gamma e) \arrow[l, "\ell"'] \\
\cup & \cup & \\
\mathrm{pvd}(\Gamma_I) \arrow[r, "\iota"] & \mathrm{pvd}(\Gamma) \arrow[r, "\pi"] & \mathrm{pvd}(e\Gamma e).
\end{tikzcd}$$

On the other hand, we can compare $\mathrm{per}(e\Gamma e)$ and $\mathrm{pvd}(\Gamma_I)$, as they can be realized inside $\mathrm{per}\,\Gamma$, by composing ι with the inclusion $\mathrm{pvd}\,\Gamma \subset \mathrm{per}\,\Gamma$

$$\mathrm{pvd}(\Gamma_I) \xrightarrow{\ \iota\ } \mathrm{per}\,\Gamma \xleftarrow{\ \ell\ } \mathrm{per}(e\Gamma e). \tag{3.4}$$

Lemma 3.6. *The orthogonality relation* $\ell(\mathrm{per}(e\Gamma e)) = {}^{\perp_{\mathrm{per}\,\Gamma}}\iota(\mathrm{pvd}\,\Gamma_I)$ *holds in* $\mathrm{per}\,\Gamma$.

Proof. By equation (3.1), $\ell(\mathcal{D}(e\Gamma e)) = {}^{\perp_{\mathcal{D}(\Gamma)}}\iota(\mathcal{D}(\Gamma_I))$ (see [6, Section 1.4.3.3]), and this, together with (3.4), implies that $\ell(\mathrm{per}(e\Gamma e)) \subseteq {}^{\perp_{\mathrm{per}\,\Gamma}}\iota(\mathrm{pvd}\,\Gamma_I)$. To prove the statement, we need to show that this inclusion is indeed an equality. For simplicity, we identify the categories associated with Γ_I and with $e\Gamma e$ with their images under ι and ℓ respectively. We show that, for any $X \in \mathrm{per}\,\Gamma$ not in $\mathrm{per}(e\Gamma e)$, then $X \notin {}^{\perp_{\mathrm{per}\,\Gamma}}\,\mathrm{pvd}\,\Gamma_I$.

By [28, Lemma 2.14], any object X in $\mathrm{per}\,\Gamma$ (resp. in $\mathrm{per}(e\Gamma e)$) admits a minimal perfect presentation, i.e., it is quasi-isomorphic to a finite direct sum of shifts of projective objects $\bigoplus_{k\in K\subset Q_0} P_k[d_k]$ (resp. of $\bigoplus_{k\in K\subset I^c} P_k[d_k]$), and its differential, as a degree 1 map, is a strictly upper triangular matrix with entries in the ideal of Γ generated by the arrows of the double quiver $\bar{Q}$. Note that for any arrow $\alpha\colon i \to j$ in $\bar{Q}$ that induces some irreducible morphism $P_j \xrightarrow{f_\alpha} P_i[\deg\alpha]$, we have $\mathrm{Hom}(f_\alpha, S) \equiv 0$ for any simple $S \in \mathrm{mod}\,J$ (since either $\mathrm{Hom}(P_j, S) = 0$ or $\mathrm{Hom}(P_i[d_i], S) = 0$). This implies, in particular, that for a simple $S \in \mathrm{mod}\,J$,

$$\mathrm{Hom}_{\mathrm{per}\,\Gamma}(X, S) = \bigoplus_k \mathrm{Hom}_{\mathrm{per}\,\Gamma}\big(P_k[d_k], S\big). \tag{3.5}$$

Therefore, for any X in $\mathrm{per}\,\Gamma$ not in $\mathrm{per}(e\Gamma e)$, its perfect presentation has a summand $P_i[d_i]$ for some $i \in I$, hence by (2.5) $\mathrm{Hom}^0(X, S_i[-d_i]) \neq 0$ with $S_i[-d_i] \in \mathrm{pvd}\,\Gamma_I$. ∎

Lemma 3.6 will be used for the proof of the simple-projective duality for $\mathrm{pvd}\,e\Gamma e$ and $\mathrm{per}\,e\Gamma e$, in Section 6.

3.4. Gluing torsion pairs, and tilting in pvd$(e\Gamma e)$

The perfectly valued categories

$$\mathrm{pvd}(\Gamma_I) \to \mathrm{pvd}(\Gamma) \to \mathrm{pvd}(e\Gamma e),$$

despite admitting no adjoints to the quotient functor, are sandwiched between recollements $R_{tr}(\mathcal{D}(\Gamma_I), \mathcal{D}(\Gamma), \mathcal{D}(e\Gamma e))$ and $R_{ab}(\mathrm{mod}(J_I), \mathrm{mod}(J), \mathrm{mod}(eJe))$ of triangulated and abelian categories, involving the derived categories of the Ginzburg algebras and the small module categories of the Jacobian algebras, respectively. See equations (3.1) and (3.3). We exploit this feature to show that tilting the standard heart $\mathrm{mod}(eJe)$ of $\mathrm{pvd}(e\Gamma e)$ at any torsion pair results in another heart of quotient type. To make the presentation lighter, in the following we write denote $\mathcal{H}_{\mathcal{F}}$ for $\mu^{\sharp}_{\mathcal{F}}\mathcal{H}$ and $\mathcal{H}_{\mathcal{T}}$ for $\mu^{\flat}_{\mathcal{T}}\mathcal{H}$, whenever $(\mathcal{T}, \mathcal{F})$ is a torsion pair in a bounded heart $\mathcal{H}$.

Proposition 3.7. *Let* $(\mathcal{U}, \mathcal{V})$ *and* $(\mathcal{T}, \mathcal{F})$ *be torsion pairs in the standard hearts* $\mathrm{mod}\,J_I$ *and* $\mathrm{mod}\,eJe$ *of* $\mathrm{pvd}\,\Gamma_I$ *and* $\mathrm{pvd}\,e\Gamma e$ *respectively. Then* $(\mathcal{U}, \mathcal{V})$ *and* $(\mathcal{T}, \mathcal{F})$ glue *to a torsion pair* $(\mathcal{X}, \mathcal{Y})$ *in* $\mathrm{mod}\,J$ *which is compatible with the recollement of abelian categories. The (forward or backward) tilts at* $(\mathcal{U}, \mathcal{V})$, $(\mathcal{X}, \mathcal{Y})$, $(\mathcal{T}, \mathcal{F})$ *produces new short exact sequences*

$$0 \to (\mathrm{mod}\,J_I)^{\sharp}_{\mathcal{V}} \xrightarrow{\iota} (\mathrm{mod}\,J)^{\sharp}_{\mathcal{Y}} \xrightarrow{\pi} (\mathrm{mod}\,eJe)^{\sharp}_{\mathcal{F}} \to 0$$

and

$$0 \to (\operatorname{mod} J_I)^{\flat}_{\mathcal{U}} \xrightarrow{\iota} (\operatorname{mod} J)^{\flat}_{\mathcal{X}} \xrightarrow{\pi} (\operatorname{mod} eJe)^{\flat}_{\mathcal{T}} \to 0$$

of hearts of bounded t-structures in $\operatorname{pvd}\Gamma_I \xrightarrow{\iota} \operatorname{pvd}\Gamma \xrightarrow{\pi} \operatorname{pvd}(e\Gamma e)$.

Proof. We prove the proposition for backward tilt, the forward tilt being completely analogous. The closure by limits of $\mathcal{T} \subset \operatorname{mod} eJe$ (resp. $\mathcal{U} \subset \operatorname{mod} J_I$) is a torsion class $\widetilde{\mathcal{T}}$ in $\operatorname{Mod} eJe$ (resp. $\widetilde{\mathcal{U}}$ in $\operatorname{Mod} J_I$) such that $\widetilde{\mathcal{T}} \cap \operatorname{mod} eJe = \mathcal{T}$ (resp. $\widetilde{\mathcal{U}} \cap \operatorname{mod} J_I = \mathcal{U}$), see [8]. We consider the recollement of abelian hearts (3.2). The gluing method by Beilinson–Bernstein–Deligne, together with [29, Propositions 9.1 (ii) and 10.2] implies that the torsion classes $\widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{U}}$ induce a torsion class $\widetilde{\mathcal{X}}$ in $\operatorname{Mod} J$ such that $\pi(\widetilde{\mathcal{X}}) = \widetilde{\mathcal{T}}$, $q(\widetilde{\mathcal{X}}) = \widetilde{\mathcal{U}}$, and a new recollement of abelian hearts

$$\begin{tikzcd}[column sep=large]
(\operatorname{Mod} J_I)^{\flat}_{\widetilde{\mathcal{U}}} \arrow[r, "\iota''"] & (\operatorname{Mod} J)^{\flat}_{\widetilde{\mathcal{X}}} \arrow[l, bend right, "q''"'] \arrow[l, bend left, "p''"] \arrow[r, "\pi''"] & (\operatorname{Mod} eJe)^{\flat}_{\widetilde{\mathcal{T}}}. \arrow[l, bend right, "\ell''"'] \arrow[l, bend left, "r''"]
\end{tikzcd}$$

According to [29] (cf. proof of Proposition 10.2 and equations therein), we write

$$\begin{aligned}
(\operatorname{Mod} J_I)^{\flat}_{\widetilde{\mathcal{U}}} = \big\{X \in \mathcal{D}(\Gamma_I) \mid\ & H^0_{\operatorname{Mod} J_I}(X) \in \widetilde{\mathcal{U}},\ H^{-1}_{\operatorname{Mod} J_I}(X) \in \widetilde{\mathcal{U}}^{\perp}, \\
& H^i_{\operatorname{Mod} J_I}(X) = 0,\ i \neq 0, -1\big\}, \qquad (3.6)\\
(\operatorname{Mod} J)^{\flat}_{\widetilde{\mathcal{X}}} = \big\{X \in \mathcal{D}(\Gamma) \mid\ & H^0_{\operatorname{Mod} J}(X) \in \mathcal{X},\ H^{-1}_{\operatorname{Mod} J}(X) \in \widetilde{\mathcal{X}}^{\perp}, \\
& H^i_{\operatorname{Mod} eJe}(X) = 0,\ i \neq 0, -1\big\}, \qquad (3.7)\\
(\operatorname{Mod} eJe)^{\flat}_{\widetilde{\mathcal{T}}} = \big\{X \in \mathcal{D}(\Gamma) \mid\ & H^0_{\operatorname{Mod} eJe}(X) \in \widetilde{\mathcal{T}},\ H^{-1}_{\operatorname{Mod} eJe}(X) \in \widetilde{\mathcal{T}}^{\perp}, \\
& H^i_{\operatorname{Mod} eJe}(X) = 0,\ i \neq 0, -1\big\}. \qquad (3.8)
\end{aligned}$$

Since the cohomology functors appearing in (3.6), (3.7), and (3.8) restrict to the perfectly valued categories by Theorem 3.4, then

$$\begin{aligned}
(\operatorname{Mod} J_I)^{\flat}_{\widetilde{\mathcal{U}}} \cap \operatorname{pvd}\Gamma_I &= (\operatorname{mod} J_I)^{\flat}_{\mathcal{U}},\\
(\operatorname{Mod} J)^{\flat}_{\widetilde{\mathcal{X}}} \cap \operatorname{pvd}\Gamma &= (\operatorname{mod} J)^{\flat}_{\mathcal{X}},\\
(\operatorname{Mod} eJe)^{\flat}_{\widetilde{\mathcal{T}}} \cap \operatorname{pvd} e\Gamma e &= (\operatorname{mod} eJe)^{\flat}_{\mathcal{T}},
\end{aligned}$$

where $\mathcal{X} = \widetilde{\mathcal{X}} \cap \operatorname{mod} J$. For the thesis to follow, we need to know that ι'' and π'' descend to the intersections. This is true because they are the restrictions of ι and π to $\operatorname{pvd}\Gamma_I$ and $\operatorname{pvd}\Gamma$ respectively, see (3.2). Boundedness of the resulting t-structures follows from HRS tilting theory and [3, Proposition 2.20]. ■

It follows that tilting at a torsion pair a heart $\overline{\mathcal{H}} = \mathcal{H}/\mathcal{H} \cap \mathcal{V}$ of $\operatorname{pvd}(e\Gamma e)$ from $\mathcal{H} \in \operatorname{EG}^{\circ}(\operatorname{pvd}\Gamma)$, always produces another heart of quotient type.

Corollary 3.8. *Let* $\overline{\mathcal{H}} \in \mathrm{EG}^{\bullet}(\mathrm{pvd}(e\Gamma e))$ *and* $\overline{\mathcal{H}}^{\sharp}_{f}$ *be the heart of another bounded t-structure in* $\mathrm{pvd}(e\Gamma e)$ *obtained by tilting* $\overline{\mathcal{H}}$ *at a torsion-free class* f. *Then* $\overline{\mathcal{H}}^{\sharp}_{f}$ *is of quotient type.*

More precisely, if $\overline{\mathcal{H}} = \mathcal{H}/(\mathcal{H} \cap \mathrm{pvd}\,\Gamma_I)$ *for some* $\mathcal{H} \in |\,\mathrm{EG}^{\circ}(\mathrm{pvd}\,\Gamma)|$, *then there exists* $\mathcal{F} \subset \mathcal{H}$ *such that* $\overline{\mathcal{H}}^{\sharp}_{f} = \mathcal{H}'/(\mathcal{H}' \cap \mathrm{pvd}\,\Gamma_I)$ *for* $\mathcal{H}' = \mu^{\sharp}_{\mathcal{F}}\mathcal{H}$.

Proof. The statement follows from Proposition 3.7, using the fact that any bounded heart in $\mathrm{EG}^{\bullet}(\mathrm{pvd}(e\Gamma e))$ is $\mathrm{mod}(e'J'e') \subset \mathrm{pvd}(e'\Gamma'e')$ with $\mathrm{pvd}(e'\Gamma'e') \simeq \mathrm{pvd}(e\Gamma e)$, for $\Gamma' = \Gamma(Q', W')$, $J' = H^0\Gamma'$, and the quivers (Q', W') and (Q, W) related by mutations. ∎

4. Decorated marked surfaces and collapsing surfaces

We recall some definitions related with decorated marked surfaces and the construction of the dual quiver of a triangulation, and with the operation of collapsing subsurfaces. In Sections 5 and 6 we will restrict to a Ginzburg algebra Γ of a quiver with potential from a decorated marked surface.

4.1. Decorated marked surfaces and exchange graphs

A *decorated marked surface (DMS)* $\mathbf{S}_\Delta$ *(without punctures)* is determined by the following data:

- a compact genus $g \geq 0$ surface $\mathbf{S}$,
- a set $\mathbf{M} \neq \emptyset$ of *marked points* on the boundary $\partial\mathbf{S}$ of $\mathbf{S}$, and
- a set Δ of *decorations*, i.e., points in the interior $\mathbf{S}^{\circ}$ of $\mathbf{S}$,

such that each boundary component of $\partial\mathbf{S}$ contains at least one marked point, and

$$|\Delta| = 4g + 2|\partial\mathbf{S}_\Delta| + |\mathbf{M}| - 4 \geq 1, \tag{4.1}$$

where $|\partial\mathbf{S}_\Delta|$ is the number of boundary components.

An open arc on $\mathbf{S}_\Delta$ is an isotopy class of a curve whose interior is in $\mathbf{S}^{\circ} \setminus \Delta$ and whose endpoints are marked points.

A *(decorated) triangulation* $\mathbb{T}$ of $\mathbf{S}_\Delta$ is a collection of open arcs that, together with boundary segments defined by consecutive marked points, divides $\mathbf{S}_\Delta$ into once-decorated triangles, i.e., triangles encircling exactly one decoration. The open arcs of $\mathbb{T}$ are usually called *internal edges* of the triangulation, and once-decorated triangles are said *internal triangles*. The assumptions on $\mathbf{S}_\Delta$ ensure that no decorated triangulation admits *self-folded triangles*. There is an operation of forward flip on triangulations, which changes an arc of a triangulation by moving its endpoints along the boundary of the quadrilateral containing it, without crossing Δ. We refer to [25, 26] for the classical notion of flip of edges of a triangulation and to [30] for the decorated version. We denote by $\mathrm{EG}(\mathbf{S}_\Delta)$ the exchange graph of triangulations of $\mathbf{S}_\Delta$, whose vertices are triangulations and whose directed edges are forward flips.

Given a triangulation $\mathbb{T}$ of $\mathbf{S}_\Delta$, there is an associated quiver with potential $(Q_\mathbb{T}, W_\mathbb{T})$, where

- the vertices of $Q_\mathbb{T}$ are the open arcs in $\mathbb{T}$,
- the arrows of $Q_\mathbb{T}$ are the (anticlockwise) angle between open arcs in $\mathbb{T}$, and
- $W_\mathbb{T}$ is the sum of all 3-cycles $W_\mathbb{T}$, that are the compositions of three open arcs for any triangle of $\mathbb{T}$ whose edges are all internal edges.

Definition 4.1. We say that a quiver with potential is of *geometric type*, precisely when it can be defined by this procedure.

Fixing an initial triangulation $\mathbb{T}_0$, we let $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$ be the principal component of $\mathrm{EG}(\mathbf{S}_\Delta)$ containing $\mathbb{T}_0$. We let moreover $(Q_{\mathbb{T}_0}, W_{\mathbb{T}_0})$ be the quiver associated with $\mathbb{T}_0$, and $\Gamma := \Gamma_{\mathbb{T}_0}$ be its Ginzburg algebra. Key features of $\mathrm{pvd}(\Gamma_{\mathbb{T}_0})$ are the correspondences between

- simple objects of a heart and internal edges of a triangulation of $\mathbf{S}_\Delta$, and
- simple (forward) tilts and (forward) flips.

This correspondence was first studied by Labardini Fragoso in [25] for the unpunctured case and in [26] in the presence of *self-folded* triangles, and can be expressed more precisely in the following theorem including results contained in [30, 31].

Theorem 4.2. *There is an isomorphism of oriented graphs* $\mathrm{EG}^\circ(\mathbf{S}_\Delta) \cong \mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma)$*, sending a triangulation* $\mathbb{T}$ *to a heart* $\mathcal{H}_\mathbb{T}$*, such that simples of* $\mathcal{H}_\mathbb{T}$ *are naturally parameterized by open arcs in* $\mathbb{T}$ *or vertices of the associated quiver* $Q_\mathbb{T}$*. Each directed edge of* $\mathrm{EG}^\circ(\mathbf{S}_\Delta)$ *corresponding to flipping an internal edge of the triangulation matches with a directed edge of* $\mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma)$ *corresponding to a simple forward tilt of a heart of a bounded t-structure in* $\mathrm{pvd}(\Gamma)$*. Moreover, the dimension of* Ext^1 *between simples of* $\mathcal{H}_\mathbb{T}$ *equals the number of arrows between the corresponding vertices in* $Q_\mathbb{T}$*.*

In the setup of Theorem 4.2, if R and S are the simples corresponding to open arcs β and γ in a triangulation $\mathbb{T}$, we have the following easy lemma.

Lemma 4.3 (No-arrow/Ext^1-vanishing condition). *If* β, γ *are not counter-clockwise consecutive internal edges of an internal triangle of* $\mathbb{T}$*, i.e., if there is no arrow in* $(Q_\mathbb{T}, W_\mathbb{T})$ *from* β *to* γ*, then* $\mathrm{Ext}^1(R, S) = 0$*.*

4.2. Collapsing subsurfaces

Let $\Gamma = \Gamma(Q_{\mathbb{T}_0}, W_{\mathbb{T}_0})$ be the Ginzburg algebra of the dual quiver of a triangulation $\mathbb{T}_0$ of a decorated marked surface (without punctures) $\mathbf{S}_\Delta$ and $e = \sum_{j \in I^c} e_j$ an idempotent as in the previous sections. The category $\mathrm{pvd}(e\Gamma e)$ admits a combinatorial description in terms of a *weighted decorated marked surface* obtained by *collapsing a subsurface of* $\mathbf{S}_\Delta$ roughly associated with the subset I. This perspective is considered in [4] and briefly summarized here. See Figure 3 for an example.

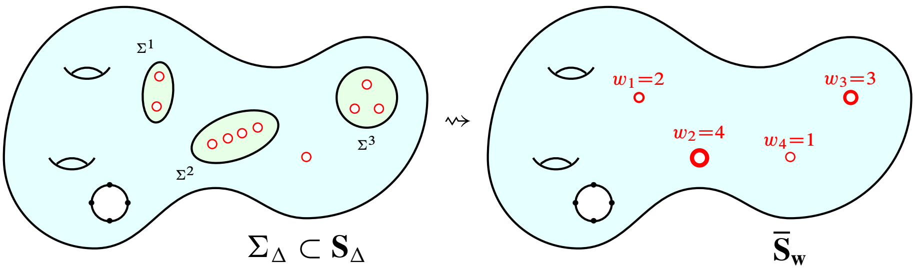

Figure 3. Example: collapse of a subsurface Σ with three connected components of genus 0 from a surface $\mathbf{S}_\Delta$ of genus 2 with 10 decorations and four marked points on the boundary component.

Let Σ be a subsurface cut out of $\mathbf{S}$ along simple closed curves of $\mathbf{S} \setminus \Delta$. It can be realized as a decorated marked surface Σ_Δ by taking its closure $\overline{\Sigma}$ and adding marked points on its (new) boundary components. For a fixed triangulation $\mathbb{T}$ of $\mathbf{S}_\Delta$, assume that we cut Σ_Δ in a way that the restriction of $\mathbb{T}$ to Σ_Δ identifies a subquiver (Q_I, W_I) of $(Q_\mathbb{T}, W_\mathbb{T})$. The details of this procedure can be found in [4, Section 5.1]. Then one can *collapse* Σ_Δ *in* $\mathbf{S}_\Delta$ to get a *weighted decorated marked surface (wDMS)* $\overline{\mathbf{S}}_\mathbf{w}$, defined as follows and diagrammatically represented by

$$\Sigma_\Delta \hookrightarrow \mathbf{S}_\Delta \rightsquigarrow \overline{\mathbf{S}}_\mathbf{w}. \tag{4.2}$$

The underlying surface of $\overline{\mathbf{S}}_\mathbf{w}$ is the compactification of $\mathbf{S} \setminus \overline{\Sigma}$ by points. The new points, together with the decorations in $\mathbf{S}_\Delta \setminus \Sigma_\Delta$, are taken as the decorations of $\overline{\mathbf{S}}_\mathbf{w}$, denoted again Δ with abuse of notation. The marked points of $\overline{\mathbf{S}}_\mathbf{w}$ are inherited from $\mathbf{S}_\Delta$ and coincide with those in $\mathbf{M}$. The resulting DMS is enriched with a map $\mathbf{w}\colon \Delta \to \mathbb{Z}_{\geq -1}$, called a *weight function*, such that the condition (4.1) is replaced by

$$\sum_{Z \in \Delta} \mathbf{w}(Z) = 4g + 2|\partial \mathbf{S}_\Delta| + |\mathbf{M}| - 4 \geq 1. \tag{4.3}$$

Under suitable conditions on $\mathbf{S}_\Delta$ and Σ_Δ, the weight function on $\overline{\mathbf{S}}_\mathbf{w}$ takes values in $\mathrm{Z}_{\geq 1}$. Very roughly speaking, $\mathbb{T}$ induces a (decorated) "partial triangulation" and a "mixed-angulation" of $\overline{\mathbf{S}}_\mathbf{w}$, that is a tiling of $\overline{\mathbf{S}}_\mathbf{w}$ into once-decorated polygons. A notion of forward flip for a mixed-angulation is available, that is used to introduce [4, Section 5] an exchange graph $\mathrm{EG}^\bullet(\overline{\mathbf{S}}_\mathbf{w})$ of mixed-angulations. We refer to [4] for more details, and in particular, for the description of $\operatorname{pvd}(e\Gamma e) \simeq \operatorname{pvd}(\Gamma)/\operatorname{pvd}(\Gamma_I)$ as a triangulated category associated with $\overline{\mathbf{S}}_\mathbf{w}$.

Recall the definition of $\mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$ in Notation 3.5. Similarly to Theorem 4.2 we have the following Theorem 4.4 (1), depending on the choice of the initial partial triangulation and its refinement to a triangulation $\mathbb{T}_0$ of $\mathbf{S}_\Delta$. For part (2), we let m be the cardinality of $I^c = Q_0 \setminus I$.

Theorem 4.4 ([4, Theorem 6.9 and Proposition 5.13]). *Set $\Gamma = \Gamma(Q_{\mathbb{T}_0}, W_{\mathbb{T}_0})$ as above and $J = J(Q_{\mathbb{T}_0}, W_{\mathbb{T}_0})$, for a fixed refining triangulation $\mathbb{T}_0$, and let $\mathrm{EG}^\circ(\operatorname{pvd}\Gamma)$ be the principal component of $\mathrm{EG}(\operatorname{pvd}\Gamma)$ containing $\operatorname{mod} J$. Then*

(1) *There is an isomorphism of graphs*

$$\mathrm{EG}^\bullet(\overline{\mathbf{S}}_{\mathrm{w}}) \simeq \mathrm{EG}^\bullet\big(\operatorname{pvd}(e\Gamma e)\big),$$

sending a partial triangulation $\mathbb{A}$ of $\overline{\mathbf{S}}_{\mathrm{w}}$ to a finite heart of a bounded t-structure of $\operatorname{pvd}(e\Gamma e)$ whose simples (up to isomorphism) correspond to internal edges of $\mathbb{A}$, and associating to any (forward) flip of edges the tilting of the corresponding heart at a simple object.

(2) *The graph $\mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$ is (m, m)-regular, i.e., each vertex has exactly m outgoing and m incoming directed edges, and consists of connected components of $\mathrm{EG}(\operatorname{pvd} e\Gamma e)$.*

5. Hearts of $\operatorname{pvd}(e\Gamma e)$

We continue to use the terminology from the previous sections: (Q, W) is a finite nondegenerate quiver with potential, with set of vertices $Q^0 = I \sqcup I^c$, $\Gamma = \Gamma(Q, W)$ is its Ginzburg algebra, $e = \sum_{j\in I^c} e_j$ is a fixed idempotent associated with the complement of the subquiver (Q_I, W_I). The exchange graphs $\mathrm{EG}^\circ(\operatorname{pvd}\Gamma)$ and $\mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$ are defined in Notation 3.5. Corollary 3.8 tells us that all hearts obtained by titling at *any* torsion pair a finite heart in $\mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$ are of quotient type, i.e., they are induced by bounded t-structures in $\operatorname{pvd}\Gamma$, not necessarily finite.

In this section, we assume that (Q, W) is a quiver with potential *of geometric type*, and we study $\mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$ and *simple* tilts, i.e., tilts at simple objects. The main result is Theorem 5.6, that shows that $\mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$ can be read off from $\mathrm{EG}^\bullet(\operatorname{pvd}\Gamma)$. It relies in the procedure described in the proof of Proposition 5.2 for which the assumption that (Q, W) is of geometric type is crucial. The content of this section is complementary to the study of the exchange graph $\mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$ in relation with the exchange graph of $\overline{\mathbf{S}}_{\mathrm{w}}$ and $\mathbf{S}_\Delta$, recalled in Theorem 4.4.

5.1. Lifting simple tilts

The goal of this subsection is to give a constructive explanation of the fact that we can "lift" any simple tilt of $\overline{\mathcal{H}}$ in $\mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$ to a simple tilt in $\mathrm{EG}^\circ(\operatorname{pvd}\Gamma)$ (Proposition 5.2), thanks to the geometric description. We do this by showing that, for any simple $\overline{S} \in \overline{\mathcal{H}}$, we can find a "convenient" heart $\mathcal{H}$ in $\mathrm{EG}^\circ(\operatorname{pvd}\Gamma)$ whose quotient is $\overline{\mathcal{H}}$. Being "convenient" with respect to S for a heart in fact means to satisfy the hypotheses of Proposition 5.1 below. A similar argument appeared in [4], using refinements of more general "mixed-angulations" of a wDMS. Proposition 5.3 can later be compared with Lemma 6.6.

We denote

$$\mathcal{V} := \operatorname{pvd}\Gamma_I,$$

and we fix a finite heart $\mathcal{H} \in \mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma)$ compatible with $\mathcal{V}$, i.e., such that

$$\mathcal{K}^{\mathcal{H}} := \mathcal{H} \cap \mathcal{V}$$

is a Serre subcategory of $\mathcal{H}$. Then the following proposition holds.

Proposition 5.1. *Suppose that $S \in \mathcal{H} \setminus (\mathcal{H} \cap \mathcal{V})$ is a simple object such that*

$$\dim \mathrm{Ext}^1_{\mathcal{H}}(T, S) = 0 \quad \textit{for any } T \in \mathcal{H} \cap \mathcal{V}. \tag{5.1}$$

Then $\mu^\sharp_S \mathcal{H}$ is also compatible with $\mathcal{V}$ and $\mu^\sharp_S \overline{\mathcal{H}} \simeq \overline{\mu^\sharp_S \mathcal{H}}$.

Proof. It is an application of [4, Proposition 6.8], since any simple object in a finite heart $\mathcal{H}$ of $\mathrm{pvd}\,\Gamma$ is rigid. ∎

We assume now that (Q, W) is of geometric type for some decorated marked surface $\mathbf{S}_\Delta$, and that the finite heart $\mathcal{H}$ is associated with a triangulation $\mathbb{T}$ of $\mathbf{S}_\Delta$

$$\mathcal{H} = \mathrm{mod}\, J(Q_{\mathbb{T}}, W_{\mathbb{T}}).$$

Recall that its simple objects can be identified with a subset of the edges of the triangulation $\mathbb{T}$. The combinatorial counterpart of equation (5.1) above is given in Lemma 4.3. Moreover, we can use the geometric model to prove the following two propositions.

Proposition 5.2. *Let $S \in \mathcal{H} \setminus \mathcal{K}^{\mathcal{H}}$ be a simple object. Then there exists a finite sequence of simple tilts at objects in $\mathcal{V} = \mathrm{pvd}\,\Gamma_I$ such that the resulting heart is $\mu^\sharp_{\mathcal{F}} \mathcal{H}$, for some torsion-free class $\mathcal{F} \subset \mathcal{K}^{\mathcal{H}}$, and satisfies*

(i) *$S \in \mu^\sharp_{\mathcal{F}} \mathcal{H}$ and it is simple,*

(ii) *$\dim \mathrm{Ext}^1(T, S) = 0$ for any $T \in \mu^\sharp_{\mathcal{F}} \mathcal{H} \cap \mathcal{V}$, and*

(iii) *the quotient abelian categories $\mu^\sharp_{\mathcal{F}} \mathcal{H} / (\mu^\sharp_{\mathcal{F}} \mathcal{H} \cap \mathcal{V})$ and $\mathcal{H}/\mathcal{K}^{\mathcal{H}}$ coincide:*

$$\overline{\mu^\sharp_{\mathcal{F}} \mathcal{H}} = \overline{\mathcal{H}}.$$

Proof. We consider the triangulation $\mathbb{T}$ of $\mathbf{S}_\Delta$ induced by $\mathcal{H}$. We call S the edge corresponding to the simple object S in $\mathcal{H} \setminus \mathcal{K}^{\mathcal{H}}$, and color in blue (resp. black) the edges corresponding to the simples of $\mathcal{H}$ in $\mathcal{H} \cap \mathcal{V}$ (resp. $\mathcal{H} \setminus \mathcal{K}^{\mathcal{H}}$). Black edges define a tiling of $\mathbf{S}_\Delta$ into polygons, and blue edges refine this tiling to the triangulation $\mathbb{T}$. Without loss of generality, we may restrict our attention to two adjacent polygons P_k and P_ℓ, with k and ℓ vertices respectively, sharing S as common edge. We denote by d_k (resp. d_ℓ) the diagonal of P_k (resp. P_ℓ) which, together with S and its counter-clockwise neighboring edge of the polygon, form a triangle. Examples of such a configuration are displayed in Figure 4. The common edge S is displayed in red, and the diagonals d_k, d_ℓ are displayed as black dotted lines.

In order to get a torsion-free class $\mathcal{F}$ as required in the lemma, we construct a suitable triangulation of $P_k \cup P_\ell$ by iteratively flipping some blue edges inside $P_k \cup P_\ell$. Using the

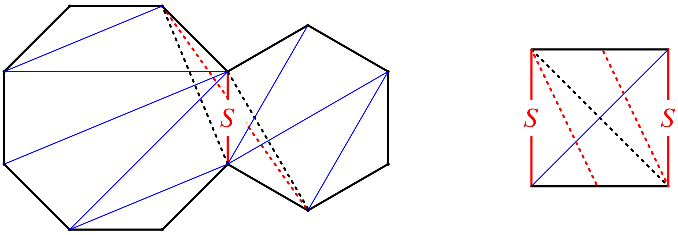

Figure 4. On the left, an example of $P_k \cup P_\ell$ with $k = 8$ and $\ell = 6$. On the right, a polygon with two edges identified.

correspondence between flips and simple tilts, this guaranties that the resulting quotient category does not change, i.e., condition (iii). Since flips will occur at most once at each edge, the procedure will define a torsion-free class $\mathcal{F} \subset \mathcal{K}^{\mathcal{H}}$.

We note that the heart corresponding to a triangulation of $\mathbf{S}_\Delta$ satisfies condition (ii) if and only if it contains d_k and d_ℓ (this is the no-arrow condition of Lemma 4.3 and the construction of the quiver from the triangulation $\mathbb{T}$).

Consider the finitely many edges e_i that intersect, out of the extremes, the diagonals d_k and d_ℓ, and label them counter-clockwise. We mutate at these edges in lexicographic order. The diagonal d_k subdivides the k-gon into a triangle and a $(k-1)$-gon sharing d_k as an edge and its extremes as vertices. Any e_i intersecting d_k has the characterizing vertex of the triangle as an endpoint. Flipping e_i gives us a new edge, with two different endpoints, that lies completely in the $(k-1)$-gon. Once all crossing edges have been flipped exactly once, the resulting triangulation has d_k among its edges. The same procedure will be applied to the other polygon P_ℓ. Note that the procedure works also when two black edges are identified, or, e.g., in the example displayed on the right of Figure 4, where the black dashed edge is the resulting edge after flipping the blue edge.

Let S_i be the simple object of $\mathcal{H}$ corresponding to e_i. Saying $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{H}_i = \mu^\sharp_{S_i}\mathcal{H}_{i-1}$ the result of a single simple tilt, we note that at each step, S stays simple in $\mathcal{H}_i$, hence in $\mu^\sharp_{\mathcal{F}}\mathcal{H}$, which is condition (ii). ∎

Proposition 5.3. *Let $\mathcal{H} \in \mathrm{EG}^\circ(\operatorname{pvd}\Gamma)$ and $S \in \mathcal{H} \setminus \mathcal{K}^{\mathcal{H}}$ as in Proposition 5.1. Then, there exists $\mathcal{G} \in \mathrm{EG}^\circ(\operatorname{pvd}\Gamma)$ compatible with $\operatorname{pvd}\Gamma_I$ such that*

(i) $\mathcal{G} \supset \mathcal{K}^{\mathcal{H}}$, *and*

(ii) $\overline{\mathcal{G}} = \mu^\sharp_{\overline{S}}\overline{\mathcal{H}}$.

Proof. Let $\mathcal{F} \subset \mathcal{K}^{\mathcal{H}}$ be the torsion-free class constructed in the proof of Proposition 5.2 and $\mathcal{T}$ its corresponding torsion class in $\mathcal{H}$. The heart $\mathcal{H}$ is compatible with $\operatorname{pvd}\Gamma_I$ by hypothesis, while the hearts

$$\mathcal{G}' := \mu^\sharp_{\mathcal{F}}\mathcal{H} = \mathcal{F}[1] * \mathcal{T}, \quad \text{and} \quad \mathcal{G}'' := \mu^\sharp_S\mathcal{G}'$$

are compatible with $\operatorname{pvd}\Gamma_I$, by Lemma 2.11 because $\mathcal{F} \subset \operatorname{pvd}\Gamma_I$, and by Proposition 5.1, respectively. This means that the corresponding bounded t-structures in $\operatorname{pvd}\Gamma$ induce bounded t-structures in $\operatorname{pvd}\Gamma(e\Gamma e)$, and, in particular, that $\mathcal{K}' := \mathcal{G}' \cap \operatorname{pvd}\Gamma_I \subset \mathcal{G}'$ and $\mathcal{K}'' := \operatorname{pvd}\Gamma_I \cap \mathcal{G}'' \subset \mathcal{G}''$ are Serre subcategories of the hearts. We can therefore consider the restricted bounded t-structures in $\operatorname{pvd}\Gamma_I$ as well as the restricted torsion pairs, and trace them. We have that

- the torsion pair $(\mathcal{F}, \mathcal{T})$ in $\mathcal{H}$ induces the torsion pair $(\mathcal{F}, \mathcal{T} \cap \mathcal{K}^{\mathcal{H}})$ in $\mathcal{K}$, with

$$\mu^\sharp_{\mathcal{F}} \mathcal{K}^{\mathcal{H}} = \mathcal{F}[1] * (\mathcal{T} \cap \mathcal{K}^{\mathcal{H}}) = \mathcal{K}'.$$

Since $S \in \mathcal{G}'$ is simple and not contained in $\mathcal{K}'$, then $\mathcal{K}' \subset {}^{\perp}S$. Therefore

- the torsion pair $(\operatorname{filt}(S), {}^{\perp}S)$ in $\mathcal{H}'$ induces the torsion pair $(\mathcal{K}', 0)$ in $\mathcal{K}'$, so that

$$\mathcal{K}'' = \mu^\sharp_S \mathcal{K}' = \mathcal{K}'.$$

Last, the torsion pair $(\mathcal{F}[1], \mathcal{F}[1]^{\perp_{\mathcal{G}''}})$ restricts to $(\mathcal{F}[1], \mathcal{T} \cap \mathcal{K}^{\mathcal{H}})$ in $\mathcal{K}'' \subset \mathcal{G}''$. We can therefore perform a *backward* tilt at $\mathcal{F}[1]$ of $\mathcal{G}''$ to get

$$\mathcal{G} := \mu^\flat_{\mathcal{F}[1]} \mathcal{G}'',$$

compatible with $\operatorname{pvd}\Gamma_I$, and containing $\mu^\flat_{\mathcal{F}[1]} \mathcal{K}'' = \mu^\flat_{\mathcal{F}[1]} \mu^\sharp_{\mathcal{F}} \mathcal{K}^{\mathcal{H}} = \mathcal{K}^{\mathcal{H}}$. ∎

Remark 5.4. The dual and the corresponding analogues of Propositions 5.1, 5.2, 5.3 hold for backward tilting.

5.2. The exchange graph $\mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$

We first recall the general definition of quotient of a graph, and then show that the graph $\mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$ is isomorphic to a quotient of $\mathrm{EG}^\circ(\operatorname{pvd}\Gamma, \operatorname{pvd}\Gamma_I)$.

Definition 5.5. Let $\mathrm{G} = (V, E)$ be a (oriented) graph with set of vertices V and (directed) edges E. Let $E' \subset E$. We say that E' induces the partition $P = \{P_1, \ldots, P_t\}$ of V, where, for any pair $i \neq j$, the vertices V_i, V_j are in the same subset P_r if and only if there is a path of (directed) edges in E' connecting them.

The *quotient of* G *with respect to* E' is defined as the (oriented) graph G/E' whose set of vertices coincides with P, the partition of V induced by E', and such that there is a (directed) edge connecting P_r and (to) P_s for any (directed) edge in $E \setminus E'$ connecting a vertex of V in P_r and one in P_s.

Theorem 5.6. *The graph* $\mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$ *is isomorphic to the quotient*

$$\mathrm{EG}^\bullet\left(\operatorname{pvd}(e\Gamma e)\right) \simeq \mathrm{EG}^\circ(\operatorname{pvd}\Gamma, \operatorname{pvd}\Gamma_I)/E_I,$$

of $\mathrm{EG}^\circ(\operatorname{pvd}\Gamma, \operatorname{pvd}\Gamma_I)$ *with respect to the set* E_I *of directed edges labeled by an object in* $\operatorname{pvd}\Gamma_I$.

It consists of connected components of $\mathrm{EG}(\operatorname{pvd}(e\Gamma e))$.

Proof. We use the "lifting simple tilts" procedure of the previous subsection. First we note that there is a bijection between vertices of $\mathrm{EG}^\bullet(\mathrm{pvd}(e\Gamma e))$ and vertices of

$$\mathrm{EG}^\bullet(\mathrm{pvd}\,\Gamma, \mathrm{pvd}\,\Gamma_I)/E_I.$$

Indeed, any simple tilt at an object in $\mathcal{V} := \mathrm{pvd}\,\Gamma_I$ does not change the induced abelian heart in $\mathrm{pvd}(e\Gamma e)$. Proposition 5.2 together with Proposition 5.1 above show that any simple tilt in $\mathrm{EG}^\bullet(\mathrm{pvd}(e\Gamma e))$ lifts to a simple tilt in $\mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma)$ and in $\mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma, \mathrm{pvd}\,\Gamma_I)/E_I$. On the other hand, having an edge in $\mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma, \mathrm{pvd}\,\Gamma_I)/E_I$ connecting P_i and P_j means to have a simple tilt $\mu_S^\sharp$ relating two $\mathcal{V}$-compatible hearts $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathrm{pvd}(\Gamma)$ for some $S \in \mathcal{H}_1$ not in $\mathcal{V}$. This is possible if and only if S satisfies $\dim \mathrm{Ext}^1_{\mathcal{H}_1}(T, S) = 0$ for any T. Indeed, if $\mathrm{Sim}(\mathcal{H}_1) = \{[S], [S_2], \dots, [S_n]\}$ and $\mathrm{Sim}(\mathcal{H}_1 \cap \mathcal{V}) = \{[S_k], \dots, [S_n]\}$, the new simples in $\mathcal{H}_2$ are $S[1]$, S_i for any i such that

$$\dim \mathrm{Ext}^1_{\mathcal{H}_1}(S_i, S) = 0, \quad \text{and} \quad F_i = \mathrm{Cone}\big(S_i \to S \otimes \mathrm{Ext}^1(S_i, S)^\vee[1]\big)[-1]$$

for any other i. Suppose that S_k is replaced by F_k. Then S_k in $\mathcal{H}_2$ is an extension of $S[1]$ and F_k contradicting the hypothesis that $\mathcal{H}_2 \cap \mathcal{V}$ is Serre in $\mathcal{H}_2$.

The second part of the statement was proven in [4, Theorem 6.9] and also follows from Proposition 3.8. ■

Example 5.7 (Figure 5). As an example of Theorem 5.6, we consider the linear quiver $Q = A_3 = \bullet \to \bullet \to \bullet$ with straight orientation and trivial potential $W = 0$. Let $\mathcal{D} = \mathrm{pvd}(Q, W)$ and call X, Y, Z the simple dg modules generating the standard heart $\mathcal{H} = \mathrm{rep}\, A_3$. Figure 5 displays in gray part of the exchange graph of $\mathcal{D}$, which is taken from [22], and in blue and red part of the exchange graph of $\mathcal{D}/\mathcal{V}$, as a quotient graph, for two different choices of thick subcategories $\mathcal{V} \subset \mathcal{D}$. We explain here the notation. We call X, Y, Z the simples of $\mathcal{H}$ and U, V, W the new simples obtained by forward tilting $\mathcal{H}$ at $\langle X \rangle$, $\langle Y \rangle$, $\langle X, U \rangle$ respectively. For any object $M \in \mathcal{D}$, denote by $M_i = M[i-1]$, so that $M_1 = M$ and $M_2 = M[1]$. $\bar{M}$ denotes the image of M under the appropriate quotient functor. Vertices of the graphs are labeled with the simple generators of the corresponding bounded heart and edges are labeled by the simple object the tilting is performed at. Colored dotted lines emanate from those hearts that are compatible with the subcategory $\mathcal{V}$, and connect them with their abelian quotient.

To construct (part of) the exchange graphs of the corresponding weighted decorated marked surfaces, that are discs with three decorations of weights $\mathbf{w} = (1, 1, 2)$, see [4, Section 3.3]. Mixed-angulations for the two cases considered here will not look the same, despite giving rise to similar pentagon-shaped partial exchange graphs. Compare it also with [24, Figure 3].

The graph $\mathrm{EG}(\mathrm{pvd}(e\Gamma e))$ is not necessarily connected. An example of disconnected graph is provided in [4, Section 5.5] in terms of graphs of an associated weighted decorated marked surface.

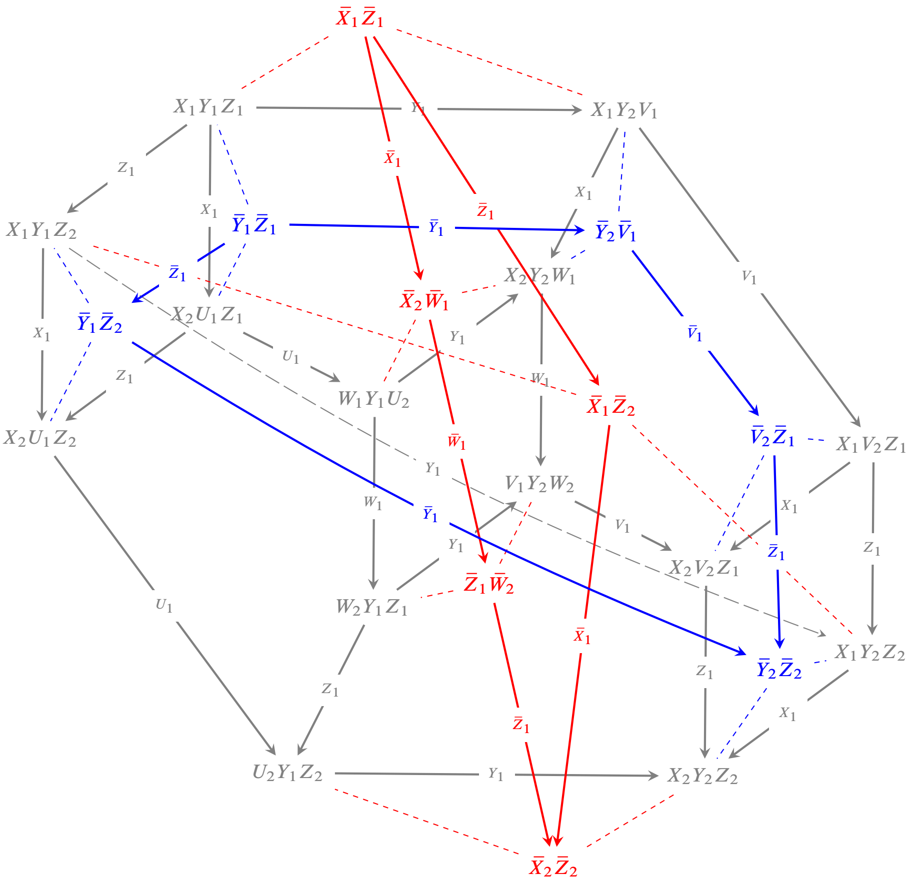

Figure 5. (Part of) the exchange graphs $\mathrm{EG}^\bullet(\mathcal{D})$ and $\mathrm{EG}^\bullet(\mathcal{D}/\mathcal{V})$ for $\mathcal{D} = \mathrm{pvd}(A_3) = \langle X_1, Y_1, Z_1\rangle$ and $\mathcal{V} = \langle X_1\rangle$ (blue) or $\mathcal{V} = \langle Y_1\rangle$ (red). The dotted lines stand for taking the quotient of hearts $\mathcal{A}/(\mathcal{A} \cap \mathcal{V})$ when the t-structure $\mathcal{A}$ is compatible with $\mathcal{V}$.

6. Simple-projective duality

In this section, we relate simple tilts in $\mathrm{pvd}(e\Gamma e)$ with *silting mutations* (Definition 6.1 below) in the perfect derived category $\mathrm{per}(e\Gamma e)$, and give an isomorphism between the exchange graph $\mathrm{EG}^\bullet(\mathrm{pvd}(e\Gamma e))$ and the silting exchange graph of $\mathrm{per}(e\Gamma e)$, see Theorem 6.11. For, we first construct a bijection between sets of silting objects and finite hearts of $\mathrm{pvd}(e\Gamma e)$ (Proposition 6.10), and then we show that the bijection is compatible with silting and tilting mutations when $\Gamma = \Gamma(Q, W)$ for a quiver of geometric type. In fact, we use simple-projective duality for $\mathrm{pvd}\,\Gamma$ and $\mathrm{per}\,\Gamma$ and show that it descends to $e\Gamma e$.

We start by recalling some definitions and simple-projective duality for Γ.

6.1. Simple-projective duality for pvd Γ and per Γ

Let $\mathcal{C}$ be an additive category with a full subcategory $\mathcal{X}$. A *left $\mathcal{X}$-approximation* of an object C in $\mathcal{C}$ is a morphism $f : C \to X$ with $X \in \mathcal{X}$, such that $\operatorname{Hom}(f, X')$ is an epimorphism for any $X' \in \mathcal{X}$. A *minimal left $\mathcal{X}$-approximation* is a left approximation f that is moreover left-minimal, i.e., for any $g : X \to X$ such that $g \circ f = f$, the morphism g is an isomorphism.

Definition 6.1 ([1, 20]). Let $\mathcal{C}$ be a triangulated category.

(1) A full subcategory $\mathcal{Y}$ of $\mathcal{C}$ is called a *partial silting subcategory* of $\mathcal{C}$ if

$$\operatorname{Hom}^{>0}(\mathcal{Y}, \mathcal{Y}) = 0.$$

It is said *silting* if, furthermore, $\operatorname{thick}(\mathcal{Y}) = \mathcal{C}$.[1]

(2) A *(partial) silting object* $\mathbf{Y}$ is a direct sum of non-isomorphic indecomposable objects Y_i of $\mathcal{C}$ such that $\operatorname{Add}\mathbf{Y}$ is a (partial) silting subcategory.[2]

(3) The *forward mutation* $\mu^\sharp_{Y_k}$ of a partial silting object $\mathbf{Y} = \bigoplus_i Y_i$ with respect to a summand Y_k is an operation that produces another partial silting object $\mu^\sharp_{Y_k}\mathbf{Y} := Y_k^\sharp \oplus \bigoplus_{i\neq k} Y_i$, defined by

$$Y_k^\sharp := \operatorname{Cone}\left(Y_k \xrightarrow{f} \bigoplus_{i\neq k} \operatorname{Irr}(Y_k, Y_i)^\vee \otimes Y_i\right), \tag{6.1}$$

where $\operatorname{Irr}(X, Y)$ is the space of irreducible maps $X \to Y$ in $\operatorname{Add}(\mathbf{Y})$, and f is the left minimal $\operatorname{Add}(\bigoplus_{i\neq k} Y_i)$-approximation of Y_k.

If the Grothendieck group of $\mathcal{C}$ is finite (e.g., when $\mathcal{C} = \operatorname{per}\Gamma$), any partial silting object has finitely many summands.

Definition 6.2. For the perfect derived category $\mathcal{C} = \operatorname{per}\Lambda$ of a non-positive dg algebra Λ, we denote by $\operatorname{SEG}\mathcal{C}$ the *silting exchange graph*, i.e., the graph whose vertices are silting objects and whose directed edges are forward mutations between them, and by SEG° the connected component containing Λ as a vertex (same notation as for the exchange graphs EGs).

The following theorem, which we refer to as "simple-projective duality", is due to Keller–Nicolàs in our setting. The original proof has never been published, but it is very similar to the proofs of analogous results in slightly different contexts that can be found for instance in [20, 22, 23]. It holds for the non-positive dg Ginzburg algebras $\Gamma(Q, W)$ of finite quivers with potential. As a bijection of sets, i.e., irrespectively of compatibility with mutations, it is proved in the more general setting of locally finite non-positive dg algebras by [11, Theorem 3.5], which also covers the case of $e\Gamma e$.

Recall that $\operatorname{pvd}\Gamma \subset \operatorname{per}\Gamma$.

[1] These are often called *pre*-silting subcategories.

[2] Note that in the literature this definition usually corresponds to a *basic* (partial/pre) silting object.

Theorem 6.3 (Simple-projective duality for $\operatorname{pvd}(\Gamma)$ and $\operatorname{per}(\Gamma)$). *There is an isomorphism between oriented graphs*

$$\iota_\Gamma \colon \mathrm{EG}^\circ\big(\operatorname{pvd}(\Gamma)\big) \to \mathrm{SEG}^\circ\big(\operatorname{per}(\Gamma)\big), \tag{6.2}$$

sending a finite heart $\mathcal{H}$ with iso-classes of simples $\operatorname{Sim}\mathcal{H} = \{[S_1], \dots, [S_n]\}$ to a silting object $\mathbf{Y}_{\mathcal{H}} = \bigoplus_{i=1}^n Y_i$, satisfying

$$\operatorname{Hom}_{\operatorname{per}\Gamma}(Y_i, S_j) = \delta_{ij}\mathbf{k} \tag{6.3}$$

and

$$\operatorname{Irr}_{\operatorname{Add}(\mathbf{Y})}(Y_j, Y_i) \simeq \operatorname{Ext}^1(S_i, S_j)^\vee. \tag{6.4}$$

In particular, the silting mutation corresponds to simple tilting.

The key idea of the proof of Theorem 6.3 is that the category $\operatorname{per}\Gamma$ can be realized as $\operatorname{per}\Gamma_{\mathcal{H}}$, where $\Gamma_{\mathcal{H}}$ is the dg endomorphism algebra of the silting object $\mathbf{Y}_{\mathcal{H}}$ for any $\mathcal{H} \in \mathrm{EG}^\circ(\operatorname{pvd}(\Gamma))$. Then the summands Y_i of $\mathbf{Y}_{\mathcal{H}}$ are projective $\Gamma_{\mathcal{H}}$-modules and the corresponding simples S_i in $\mathcal{H}$ are simple $\Gamma_{\mathcal{H}}$-modules, for which (6.3) and (6.4) hold.

6.2. Simple-projective duality for $\operatorname{pvd}(e\Gamma e)$ and $\operatorname{per}(e\Gamma e)$

We show that Theorem 6.3 is preserved under localization. Proposition 6.10 below states a correspondence between finite hearts of bounded t-structures and silting objects. Theorem 6.11 promotes this bijection to an isomorphism of graphs, in the case that $\Gamma = \Gamma(Q, W)$ comes from a quiver with potential with geometric description.

Notation 6.4. Let $\mathcal{K} := \operatorname{mod} J_I = \langle S_i \mid i \in Q_I \rangle$ be the Serre subcategory of $\mathcal{A} := \operatorname{mod} J$ associated with $I \subset Q_0$, and $\mathcal{V} := \operatorname{pvd}(\Gamma_I) = \operatorname{thick}\mathcal{K} \subset \operatorname{pvd}\Gamma$.

Notation 6.5. If G is a graph, $|\,\mathrm{G}\,|$ denotes its set of vertices.

Recall that the principal part $\mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$ of $\mathrm{EG}(\operatorname{pvd}(e\Gamma e))$ consists of the quotients of $\mathcal{V}$-compatible hearts in $\mathrm{EG}^\circ(\operatorname{pvd}(\Gamma))$, the principal part of the exchange graph of $\operatorname{pvd}\Gamma$ containing $\mathcal{A} = \operatorname{mod} J$.

Before proving the main results, we rephrase a result by H. Jin, originally expressed in terms of *simple minded collections*, corresponding to collections of simple objects (up to isomorphism) in finite hearts of bounded t-structures.

Lemma 6.6. *Each finite heart $\overline{\mathcal{H}} \in \mathrm{EG}^\bullet(\operatorname{pvd}(e\Gamma e))$ can be lifted uniquely to a finite heart $\mathcal{H} \in \mathrm{EG}^\circ(\operatorname{pvd}\Gamma)$ such that $\operatorname{Sim}\mathcal{H}$ contains $\operatorname{Sim}\mathcal{K}$.*

Proof. It is an application of [15, Theorem 3.1], using the correspondence between finite hearts and simple minded collections in $\operatorname{pvd}\Gamma$ (see e.g. [23]). ∎

We denote by $|\,\mathrm{EG}^\bullet_{\mathcal{K}}(\operatorname{pvd}(\Gamma))|$ the sub-set of all such lifts:

$$\begin{tikzcd} \left|\mathrm{EG}^\bullet_{\mathcal{K}}\operatorname{pvd}(\Gamma)\right| \arrow[r, hook] & \left|\mathrm{EG}^\circ\operatorname{pvd}\Gamma\right| \\ \left|\mathrm{EG}^\bullet\operatorname{pvd}(e\Gamma e)\right| \arrow[u, "1:1"', "{[15]}"] & \end{tikzcd} \tag{6.5}$$

We also need some preliminary results and to set some notation. Recall that $\operatorname{per}(e\Gamma e)$ is orthogonal to $\mathcal{V} = \operatorname{pvd}\Gamma_I$ inside $\operatorname{per}\Gamma$, by Lemma 3.6.

Definition 6.7. A partial silting object $\mathbf{X}$ in $\operatorname{per}(\Gamma)$ is called *$\mathcal{V}$-perpendicular* if

$$\operatorname{Hom}_{\operatorname{per}\Gamma}(\mathbf{X}, \mathcal{V}) = 0 \quad \text{and} \quad \operatorname{thick}(\mathbf{X}) = {}^{\perp_{\operatorname{per}(\Gamma)}}\mathcal{V}.$$

We define

$$\operatorname{pSEG}_{\perp\mathcal{V}}(\operatorname{per}\Gamma)$$

as the graph whose vertices are $\mathcal{V}$-perpendicular *partial* silting objects $\mathbf{X}$ and whose directed edges are forward mutations. We call it the *$\mathcal{V}$-perpendicular silting exchange graph*.

Lemma 6.8. *There is a canonical bijection*

$$\operatorname{SEG}\left(\operatorname{per}(e\Gamma e)\right) \simeq \operatorname{pSEG}_{\perp\mathcal{V}}(\operatorname{per}\Gamma).$$

Proof. Any silting object $\mathbf{Y}$ in $\operatorname{per}(e\Gamma e)$ is naturally a partial silting object in $\operatorname{per}\Gamma$, because ℓ is fully faithful. Moreover, $\mathbf{Y} \in \operatorname{per}(e\Gamma e)$ is $\mathcal{V}$-perpendicular because $\operatorname{thick}\mathbf{Y} = \operatorname{per}(e\Gamma e) = {}^{\perp_{\operatorname{per}\Gamma}}\mathcal{V}$ (Lemma 3.6). Hence we have an inclusion

$$\left|\operatorname{SEG}\left(\operatorname{per}(e\Gamma e)\right)\right| \hookrightarrow \left|\operatorname{pSEG}_{\perp\mathcal{V}}(\operatorname{per}\Gamma)\right|.$$

It is straightforward to see that this map is a bijection.

Since the definition of partial silting mutation of a (partial) silting object $\mathbf{Y}$ only depends on the summands of $\mathbf{Y}$ and maps between them, and since ℓ is fully faithful, it is in fact an isomorphism of graphs. ■

Definition 6.9. Fix the connected component $\operatorname{SEG}^\circ(\operatorname{per}(\Gamma))$ containing Γ.

- We let $\operatorname{pSEG}^\bullet_{\perp\mathcal{V}}(\operatorname{per}(\Gamma))$ be the principal part of $\operatorname{pSEG}_{\perp\mathcal{V}}(\operatorname{per}(\Gamma))$ consisting of partial silting objects that can be completed into silting objects in $\operatorname{SEG}^\circ(\operatorname{per}(\Gamma))$.
- We denote by $\operatorname{SEG}^\bullet(\operatorname{per}(e\Gamma e))$ the full subgraph of $\operatorname{SEG}(\operatorname{per}(e\Gamma e))$ which is the preimage of $\operatorname{pSEG}^\bullet_{\perp\mathcal{V}}(\operatorname{per}(\Gamma))$ under the bijection in Lemma 6.8, so that we have a resulting bijection

$$\text{i}\colon \operatorname{SEG}^\bullet\left(\operatorname{per}(e\Gamma e)\right) \simeq \operatorname{pSEG}^\bullet_{\perp\mathcal{V}}\left(\operatorname{per}(\Gamma)\right). \tag{6.6}$$

The graph $\operatorname{SEG}^\bullet(\operatorname{per}(e\Gamma e))$ is the relevant graph appearing in simple-projective duality for $e\Gamma e$.

Proposition 6.10. *There is a bijection*

$$\iota_s\colon \left|\operatorname{EG}^\bullet\left(\operatorname{pvd}(e\Gamma e)\right)\right| \simeq \left|\operatorname{SEG}^\bullet\left(\operatorname{per}(e\Gamma e)\right)\right|. \tag{6.7}$$

Proof. In order to prove the bijection, we construct explicitly a map

$$\iota_v : \left|\operatorname{EG}^\bullet\left(\operatorname{pvd}(e\Gamma e)\right)\right| \to \left|\operatorname{pSEG}^\bullet_{\perp\mathcal{V}}(\operatorname{per}\Gamma)\right|$$

and compose it with the bijection (6.6). The construction of ι_v involves three steps: a heart $\bar{\mathcal{H}} \in |\,\mathrm{EG}^\bullet(\mathrm{pvd}(e\Gamma e))|$ is first lifted to a heart $\mathcal{H}$ of $\mathrm{pvd}\,\Gamma$ containing $\mathcal{K}$, then the silting-projective duality for Γ is applied to get a silting object in $\mathrm{per}\,\Gamma$, which is restricted to a $\mathcal{V}$-perpendicular silting object (diagram (6.9) below). To conclude the proof, the map is shown to be injective (*step 4*), and finally bijective (*step 5*).

Steps 1 and 2. Let $\bar{\mathcal{H}}$ be a finite heart in $\mathrm{EG}^\bullet(\mathrm{pvd}(e\Gamma e))$. By Lemma 6.6 [15, Theorem 3.1], it can be lifted uniquely to a finite heart $\mathcal{H} \in \mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma)$ such that $\mathrm{Sim}\,\mathcal{H}$ contains $\mathrm{Sim}\,\mathcal{K}$. We identify I with the set of labels of $\mathrm{Sim}\,\mathcal{K}$ and I^c with the set of labels of $\mathrm{Sim}\,\mathcal{H} \setminus \mathrm{Sim}\,\mathcal{K}$ for any $\mathcal{H} \supset \mathcal{K}$. Secondly, we apply the simple-projective duality

$$\iota_\Gamma : \mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma) \xrightarrow{1:1} \mathrm{SEG}^\circ(\mathrm{per}\,\Gamma)$$

of Theorem 6.3, to get a silting object $\mathbf{Y}_{\mathcal{H}} := \iota_\Gamma(\mathcal{H})$.

Step 3. The silting object $\mathbf{Y}_{\mathcal{H}} := \iota_\Gamma(\mathcal{H}) \in |\,\mathrm{SEG}^\circ(\mathrm{pvd}(\Gamma))|$ corresponding to $\mathcal{H}$ decomposes as

$$\mathbf{Y}_{\mathcal{H}} =: \bigoplus_{j \in I^c} Y_j \oplus \bigoplus_{i \in I} Y_i =: \mathbf{Y}_{\mathcal{R}} \oplus \mathbf{Y}_{\mathcal{K}}. \tag{6.8}$$

Similarly, there is a decomposition

$$\mathcal{H} = \langle \mathcal{R}, \mathcal{K} \rangle$$

with $\mathcal{R} = \langle S \mid [S] \in \mathrm{Sim}\,\mathcal{H} \setminus \mathrm{Sim}\,\mathcal{K} \rangle = \langle S_j \mid j \in I^c \rangle$, and $\mathcal{K} = \langle S_i \mid i \in I \rangle$. The summands Y_ℓ of $\mathbf{Y}_{\mathcal{H}}$, and consequently of $\mathbf{Y}_{\mathcal{K}}$ and $\mathbf{Y}_{\mathcal{R}}$, satisfy $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(Y_\ell, S_t) = \delta_{\ell t}\mathbf{k}$ (equation (6.3)), for any $[S_t] \in \mathrm{Sim}\,\mathcal{H}$. Therefore

$$\mathrm{Hom}_{\mathrm{per}\,\Gamma}(\mathbf{Y}_{\mathcal{R}}, \mathcal{K}) = 0 = \mathrm{Hom}_{\mathrm{per}\,\Gamma}(\mathbf{Y}_{\mathcal{K}}, \mathcal{R}),$$

and $\mathbf{Y}_{\mathcal{K}}$ and $\mathbf{Y}_{\mathcal{R}}$ are partial silting objects in $\mathrm{per}\,\Gamma$. Clearly $\mathrm{thick}(\mathbf{Y}_{\mathcal{R}}) \subseteq {}^{\perp_{\mathrm{per}\,\Gamma}}\mathcal{V}$, because $\mathcal{V} = \mathrm{thick}\,\mathcal{K}$, and the equality can be proven arguing as in Lemma 3.6: for any $X \in {}^{\perp_{\mathrm{per}\,\Gamma}}\mathcal{V} \subset \mathrm{per}\,\Gamma$ and for any $S_i \in \mathcal{K}$

$$0 = \mathrm{Hom}_{\mathrm{per}\,\Gamma}(X, S_i) = \oplus \mathrm{Hom}_{\mathrm{per}\,\Gamma}\big(Y_\ell[d_\ell], S_i\big),$$

where the Y_ℓ's are, up to shifts, the summands of the minimal perfect presentation of X, and must be in $\mathrm{thick}\,\mathbf{Y}_{\mathcal{R}}$.

This construction defines a map of sets

$$\iota_v : \big|\,\mathrm{EG}^\bullet\big(\mathrm{pvd}(e\Gamma e)\big)\big| \to \big|\,\mathrm{pSEG}^\bullet_{\perp\mathcal{V}}(\mathrm{per}\,\Gamma)\big|,$$

diagrammatically represented by

$$\begin{tikzcd}
\big|\,\mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma)\big| & \ni & \mathcal{H} \arrow[r, mapsto, "\iota_\Gamma"] \arrow[d, equal] & \mathbf{Y}_{\mathcal{H}} = \mathbf{Y}_{\mathcal{R}} \oplus \mathbf{Y}_{\mathcal{K}} \arrow[dd, mapsto] & \in & \big|\,\mathrm{SEG}^\circ(\mathrm{per}\,\Gamma)\big| \\
\big|\,\mathrm{EG}^\bullet_{\mathcal{K}}(\mathrm{pvd}\,\Gamma)\big| \arrow[u, hook] & \ni & \mathcal{H} & & & \\
\big|\,\mathrm{EG}^\bullet\big(\mathrm{pvd}(e\Gamma e)\big)\big| \arrow[u, "1:1"] & \ni & \bar{\mathcal{H}} \arrow[u, mapsto] \arrow[r, mapsto, "\iota_v"] & \mathbf{Y}_{\mathcal{R}} & \in & \big|\,\mathrm{pSEG}^\bullet_{\perp\mathcal{V}}(\mathrm{per}\,\Gamma)\big|
\end{tikzcd} \tag{6.9}$$

Step 4. The map ι_v is injective. Suppose

$$\mathbf{Y}_{\mathcal{R}} = \iota_v(\overline{\mathcal{H}}) = \iota_v(\overline{\mathcal{H}'}). \tag{6.10}$$

Then the finite hearts $\mathcal{H}$ and $\mathcal{H}'$ of pvd Γ are generated by simples $\{S_i, S_j \mid i \in I,\ j \in I^c\}$ and $\{S_i', S_j \mid i \in I,\ j \in I^c\}$, corresponding by hypothesis (6.10) to the same silting object $\mathbf{Y}_{\mathcal{H}} = \mathbf{Y}_{\mathcal{R}} \oplus \mathbf{Y}_{\mathcal{K}} = \mathbf{Y}_{\mathcal{H}'}$. Therefore $S_i \simeq S_i'$ and $\mathcal{H} = \mathcal{H}'$.

Step 5. We now prove surjectivity, and conclude that ι_v is a bijection.

We take any $\mathbf{Y}_{\mathcal{R}'} = \bigoplus_{j\in I^c} Y_j'$ in $|\,\mathrm{pSEG}^{\bullet}_{\perp\mathcal{V}}(\mathrm{per}\,\Gamma)|$, a completable partial silting object such that $\mathrm{thick}(\mathbf{Y}_{\mathcal{R}'}) = {}^{\perp_{\mathrm{per}\,\Gamma}}\mathcal{V}$, and complete it to a silting object $\mathbf{Y}' = \mathbf{Y}_{\mathcal{R}'} \oplus \mathbf{Y}_{\mathcal{K}'}$, with corresponding finite heart $\mathcal{H}' = \langle \mathcal{R}', \mathcal{K}' \rangle \in |\,\mathrm{EG}^{\circ}(\mathrm{pvd}\,\Gamma)|$. The decompositions correspond again to the decomposition of the set $Q_0 = I \sqcup I^c$. Simple-projective duality for Γ (Theorem 6.3), implies that if $M \in \mathcal{H}'$, then $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(\mathbf{Y}_{\mathcal{R}'}, M) = 0$ if and only if $M \in \mathcal{K}'$. This, together with $\mathrm{thick}(\mathbf{Y}_{\mathcal{R}'}) = {}^{\perp_{\mathrm{per}\,\Gamma}}\mathcal{V} = \mathrm{per}(e\Gamma e)$ (Lemma 3.6) implies that $\mathcal{H}' \cap \mathcal{V} = \mathcal{K}'$, which is a Serre subcategory of $\mathcal{H}'$. By Proposition 2.8, this means that the heart $\mathcal{H}'$ induces a quotient heart $\overline{\mathcal{H}}$ in $\mathrm{EG}^{\bullet}(\mathrm{pvd}(e\Gamma e))$, and we want to show that $\mathbf{Y}_{\mathcal{R}'} = \iota_v \overline{\mathcal{H}}$. For, we apply the construction of ι_v to $\overline{\mathcal{H}}$ to get a heart in $\mathcal{H} = \langle \mathcal{R}, \mathcal{K} \rangle$ in $\mathrm{EG}^{\bullet}_{\mathcal{K}}(\mathrm{pvd}\,\Gamma)$ (*step 1*), with dual silting object $\mathbf{Y} = \mathbf{Y}_{\mathcal{R}} \oplus \mathbf{Y}_{\mathcal{K}}$ as in (6.8) (*step 2*), and image $\iota_v \overline{\mathcal{H}} = \mathbf{Y}_{\mathcal{R}}$ (*step 3*). Let

$$\mathrm{Sim}\,\mathcal{H} = \mathrm{Sim}\,\mathcal{R} \cup \mathrm{Sim}\,\mathcal{K} = \big\{[S_j]\big\}_{j\in I^c} \cup \big\{[S_i]\big\}_{i\in I}$$

and

$$\mathrm{Sim}\,\mathcal{H}' = \mathrm{Sim}\,\mathcal{R}' \cup \mathrm{Sim}\,\mathcal{K}' = \big\{[S_j']\big\}_{j\in I^c} \cup \big\{[S_i']\big\}_{i\in I}.$$

Since S_j and S_j' are mapped to the same simples in $\overline{\mathcal{H}}$, we have $S_j \in \mathcal{V} * S_j' * \mathcal{V}$. And since the summands Y_k of $\mathbf{Y}_{\mathcal{K}}$ are orthogonal to $\mathcal{V}$, we conclude that $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(Y_k, S_j') = \mathrm{Hom}_{\mathrm{per}\,\Gamma}(Y_k, S_j) = \delta_{kj}\mathbf{k}$. When we regard $\mathrm{per}(e\Gamma e)$ as $\mathrm{per}\,\mathbf{Y}_{\mathcal{R}'}$, the object Y_k admits a minimal perfect presentation with summands the shifts of Y_j' for $j \in I^c$ [28, Lemma 2.14]. Then the Hom-condition above implies by (3.5) that $Y_j = Y_j'$ for any $j \in I^c$ as required, i.e., $\mathbf{Y}_{\mathcal{R}} = \mathbf{Y}_{\mathcal{R}'}$. This concludes the proof of surjectivity, and hence of bijectivity. ■

We now assume that (Q, W) is a quiver with potential arising from a triangulation of a (unpunctured) decorated marked surface $\mathbf{S}_\Delta$ defined in Section 4.1.

Theorem 6.11. *If $\Gamma = \Gamma(Q, W)$ for some quiver with potential with geometric description, then the map*

$$\iota_s \colon \mathrm{EG}^{\bullet}\left(\mathrm{pvd}(e\Gamma e)\right) \to \mathrm{SEG}^{\bullet}\left(\mathrm{per}(e\Gamma e)\right) \tag{6.11}$$

from Lemma 6.10 is an isomorphism between oriented graphs, and the analogous of (6.3) holds.

Proof. The map ι_s is constructed in the previous proof at the level of sets as the composition

$$\iota_s : \left|\mathrm{EG}^{\bullet}\left(\mathrm{pvd}(e\Gamma e)\right)\right| \xrightarrow{\iota_v} \left|\mathrm{pSEG}^{\bullet}_{\perp\mathcal{V}}(\mathrm{per}\,\Gamma)\right| \simeq \left|\mathrm{SEG}^{\bullet}\left(\mathrm{per}(e\Gamma e)\right)\right|.$$

We recall that the second bijection is an isomorphism of graphs by definition.

We identify any silting object in $\mathrm{SEG}^\bullet(\mathrm{per}(e\Gamma e))$ with the corresponding partial silting object in $\mathrm{pSEG}^\bullet_{\perp\mathcal{V}}(\mathrm{per}\,\Gamma)$ (Lemma 6.8), and we consider a partial silting mutation

$$\mathbf{X}_1 \xrightarrow{Y} (\mathbf{X}_1)^\sharp_Y =: \mathbf{X}_2 \quad \text{in } \mathrm{pSEG}^\bullet_{\perp\mathcal{V}}(\mathrm{per}\,\Gamma)$$

with respect to a summand Y of $\mathbf{X}_1$. Arguing as in *Step 5* of the previous proof, we complete $\mathbf{X}_1$ and $\mathbf{X}_2$ to silting objects $\mathbf{Y}_1 = \mathbf{X}_1 \oplus \mathbf{Y}_\mathcal{K}$ and $\mathbf{Y}_2 = \mathbf{X}_2 \oplus \mathbf{Y}_\mathcal{K}$ in $|\,\mathrm{SEG}^\circ(\mathrm{per}\,\Gamma)|$ and we let $\mathcal{H}_1 := \langle \mathcal{R}_1, \mathcal{K}\rangle$ and $\mathcal{H}_2 := \langle \mathcal{R}_2, \mathcal{K}\rangle$ be the corresponding hearts in $|\,\mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma)|$ under the simple projective duality for Γ (top left corner of (6.9)).

The mutation formula (6.1), together with the condition $\mathrm{Hom}_{\mathrm{per}\,\Gamma}(\mathbf{X}_1, \mathcal{V}) = 0$ from Definition 6.7, implies that the silting mutation of $\mathbf{Y}_1 = \mathbf{X}_1 \oplus \mathbf{Y}_\mathcal{K}$ at Y does not depend on the summands in $\mathbf{Y}_\mathcal{K}$, which implies that the operations of silting mutation at the levels of $\mathrm{pSEG}^\bullet_{\perp\mathcal{V}}(\mathrm{per}\,\Gamma)$ and of $\mathrm{SEG}^\circ(\mathrm{per}\,\Gamma)$ commute with the restriction map (the right vertical arrow in diagram (6.9)):

$$\begin{tikzcd}
\mathbf{Y}_1 = \mathbf{X}_1 \oplus \mathbf{Y}_\mathcal{K} \arrow[r, "Y"] \arrow[d, mapsto] \arrow[dr, phantom, "\square"] & \mu^\sharp_Y \mathbf{Y}_1 = \mathbf{X}_2 \oplus \mathbf{Y}_\mathcal{K} \arrow[d, mapsto] \\
\mathbf{X}_1 \arrow[r, "Y"] & \mathbf{X}_2
\end{tikzcd}$$

We now apply the isomorphism of oriented graphs

$$\iota_\Gamma^{-1} : \mathrm{SEG}^\circ(\mathrm{per}\,\Gamma) \to \mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma)$$

associating the silting mutation

$$\mathbf{Y}_1 \xrightarrow{Y} \mathbf{Y}_2$$

to the tilting mutation

$$\mathcal{H}_1 \xrightarrow{S_Y} \mathcal{H}_2$$

for a simple object $S_Y \in \mathcal{R}_1 \subset \mathcal{H}_1$.

The fact that $\mathcal{H}_1 = \langle \mathcal{R}_1, \mathcal{K}\rangle \xrightarrow{S_Y} \mu^\sharp_{S_Y}\mathcal{H}_1 = \mathcal{H}_2 = \langle \mathcal{R}_2, \mathcal{K}\rangle$ does not change the simples in $\mathcal{K}$ means that $\mathrm{Ext}^1(\mathcal{K}, S_Y) = 0$ and the hypotheses of Proposition 5.1 are satisfied. Therefore $\overline{\mathcal{H}_2} = \overline{\mu^\sharp_{S_Y}\mathcal{H}_1}$ equals $\mu^\sharp_{\overline{S_Y}}\overline{\mathcal{H}_1}$, which shows that for any silting mutation there is a uniquely defined corresponding simple tilt.

Now let $\overline{\mathcal{H}} \in |\,\mathrm{EG}^\bullet(\mathrm{pvd}(e\Gamma e))|$ and assume $\mathcal{H} \supset \mathcal{K}$ as in Jin's correspondence (6.5). We consider a simple tilt

$$\overline{\mathcal{H}} \xrightarrow{\overline{S}} \mu^\sharp_{\overline{S}}\overline{\mathcal{H}} \quad \text{in } \mathrm{EG}^\bullet\big(\mathrm{pvd}(e\Gamma e)\big). \tag{6.12}$$

Combining Propositions 5.2 and 5.1 as in Theorem 5.6, the tilt (6.12) can be lifted to a simple tilt

$$\mathcal{H}_1 \xrightarrow{S} \mathcal{H}_2 \quad \text{in } \mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma) \tag{6.13}$$

such that $\overline{\mathcal{H}_1} = \overline{\mathcal{H}}$, $\overline{\mathcal{H}_2} = \mu^\sharp_{\overline{S}}\overline{\mathcal{H}}$, and $\mathrm{Ext}^1(\mathcal{V} \cap \mathcal{H}_1, S) = 0$. Under simple-projective duality for Γ (Theorem 6.3), the simple tilt (6.13) corresponds to a silting mutation

$$\mathbf{Y}_1 = \mathbf{Y}_{\mathcal{R}_1} \oplus \mathbf{Y}_{\mathcal{V}\cap\mathcal{H}_1} \xrightarrow{Y_S} \mathbf{Y}_2 = \mathbf{Y}_{\mathcal{R}_2} \oplus \mathbf{Y}_{\mathcal{V}\cap\mathcal{H}_2}, \tag{6.14}$$

with Y_S a summand of $\mathbf{Y}_{\mathcal{R}_1}$. The condition $\mathrm{Ext}^1(\mathcal{V} \cap \mathcal{H}_1, S) = 0$ implies that $(Y_S)^\sharp$ does not depend on summands of $\mathbf{Y}_{\mathcal{V}\cap\mathcal{H}_1}$, therefore the partial silting mutation

$$\mathbf{Y}_{\mathcal{R}_1} \xrightarrow{Y_S} \mathbf{Y}_{\mathcal{R}_2}$$

is well defined.

Note that $\mathcal{H}_1$ and $\mathcal{H}_2$ need not to contain $\mathcal{K}$. However, by construction (proof of Proposition 5.2) and thanks to the geometric assumption, $\mathcal{H}_1$ is obtained from $\mathcal{H}$ by finitely many simple tilts at objects in $\mathcal{K}$. Therefore, comparing

$$\mathcal{H} \overset{\iota_\Gamma}{\longleftrightarrow} \mathbf{Y} = \mathbf{Y}_{\mathcal{R}} \oplus \mathbf{Y}_{\mathcal{K}},$$
$$\mathcal{H}_1 \overset{\iota_\Gamma}{\longleftrightarrow} \mathbf{Y}_1 = \mathbf{Y}_{\mathcal{R}_1} \oplus \mathbf{Y}_{\mathcal{V}\cap\mathcal{H}_1},$$

we have that $\mathbf{Y}_{\mathcal{R}_1} = \mathbf{Y}_{\mathcal{R}} = \iota_v(\overline{\mathcal{H}})$. Similarly, it follows from Proposition 5.3, that there exists $\mathcal{G} \in |\mathrm{EG}^\circ(\mathrm{pvd}\,\Gamma)|$ such that $\mathcal{G} \supset \mathcal{K}$, $\overline{\mathcal{G}} = \mu^\sharp_{\overline{S}}\overline{\mathcal{H}} = \overline{\mathcal{H}_2}$, and with $\iota_\Gamma(\mathcal{G}) = \mathbf{Y}_{\mathcal{R}_2} \oplus \mathbf{Y}_{\mathcal{K}}$. Therefore (6.14) restricts to

$$\iota_v(\overline{\mathcal{H}}) \xrightarrow{Y_S} \iota_v\big(\mu^\sharp_{\overline{S}}\overline{\mathcal{H}}\big). \qquad \blacksquare$$

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