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On the strongly regular locus of the inertia stack of Bun_G

Daniel R. Gulotta

Abstract. Let G be a connected reductive group over a finite extension of $\mathbb{Q}_p$. We show that for each $b \in B(G)$, the strongly regular locus of the inertia stack of Bun_G^b is open in the inertia stack of Bun_G. As a consequence, we extend the computation of Hansen–Kaletha–Weinstein of trace distributions of the cohomology of local shtuka spaces $\mathrm{Sht}_{G,b,\mu}$ to non-basic b. If b is closed in $B(G,\mu)$, or b is basic and has only one specialization in $B(G,\mu)$, then we compute the trace distribution of the entire strongly regular locus. In the process, we prove some results on the behavior of characteristic classes under cohomologically smooth pullback.

1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$, and let G be a connected reductive group over F. Scholze [15] has defined a tower of moduli spaces of mixed characteristic local shtukas

$$\mathrm{Sht}_{G,b,\mu} = \varprojlim_K \mathrm{Sht}_{G,b,\mu,K}.$$

Here, μ is a conjugacy class of cocharacters $\mu\colon \mathbb{G}_m \to G$ defined over $\overline{F}$, b is an element of the Kottwitz set $B(G,\mu)$, and K ranges over open compact subgroups of $G(F)$. Let E be the field of definition of the conjugacy class of μ. Then $\mathrm{Sht}_{G,b,\mu,K}$ is a locally spatial diamond over $\mathrm{Spd}\,\breve{E}$, where $\breve{E}$ is the completion of the maximal unramified extension of E.

The tower $\mathrm{Sht}_{G,b,\mu,K}$ has an action of $G(F) \times G_b(F)$, where $G_b(F)$ is the automorphism group of the isocrystal associated with b.

The cohomology of $\mathrm{Sht}_{G,b,\mu}$ is of interest in the local Langlands program. Let ℓ be a prime different from p. The geometric Satake equivalence attaches to each μ an object $\mathcal{S}_\mu$ in the equivariant derived category of étale $\mathbb{Z}_\ell$-sheaves on $\mathrm{Sht}_{G,b,\mu,K}$. Let C be the completion of the algebraic closure of F. For any smooth representation ρ of $G_b(F)$ with coefficients in $\overline{\mathbb{Q}}_\ell$, define

$$R\Gamma(G,b,\mu)[\rho] = \varinjlim_K \mathrm{RHom}_{G_b(F)}\big(R\Gamma_c(\mathrm{Sht}_{G,b,\mu,K,C}, \mathcal{S}_\mu), \rho\big).$$

Fargues and Scholze [2, Corollary I.7.3] have shown that if ρ is a finite length admissible representation of $G_b(F)$, then $R\Gamma(G,b,\mu)[\rho]$ is represented by a complex of $G(F) \times W_E$-representations that are finite length and admissible as representations of $G(F)$.

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One way to study $R\Gamma(G, b, \mu)[\rho]$ explicitly is via trace distributions. Recall that for any admissible smooth representation π of $G(F)$ and any compactly supported smooth function $f : G(F) \to \overline{\mathbb{Q}}_\ell$, the endomorphism of π given by $\int_{G(F)} f(g)\pi(g)\, dg$ has finite-dimensional image, so it makes sense to define its trace $\operatorname{tr}(f|\pi)$. If π also has finite length, then it has a Harish-Chandra character Θ_π, which is a smooth function from the regular semisimple locus $G(F)_{\mathrm{rs}}$ to $\overline{\mathbb{Q}}_\ell$ such that

$$\operatorname{tr}(f|\pi) = \int_{G(F)_{\mathrm{rs}}} f(g)\Theta_\pi(g) dg \quad \text{for all } f.$$

Similarly, a finite length admissible smooth representation ρ of $G_b(F)$ has a Harish-Chandra character Θ_ρ. If an object in the derived category $D(G(F), \overline{\mathbb{Q}}_\ell)$ has finite length admissible cohomology groups, then its Harish-Chandra character is defined to be the alternating sum of the Harish-Chandra characters of the cohomology groups.

For any finite length admissible smooth representation ρ of $G_b(F)$, Hansen–Kaletha–Weinstein compare the elliptic parts of the Harish-Chandra characters of ρ and $R\Gamma(G, b, \mu)[\rho]$.

Theorem 1.1 ([8, Theorem 6.5.2]). *Assume b is basic. Let ρ be a finite length admissible smooth representation of $G_b(F)$, and let $\pi = R\Gamma(G, b, \mu)[\rho]$.*

Let $g \in G(F)$ be an elliptic element. Then

$$\Theta_\pi(g) = (-1)^{\langle \mu, 2\rho_G\rangle} \sum_{(g,g',\lambda)\in \mathrm{Rel}_b} \dim r_\mu[\lambda]\Theta_\rho(g').$$

Here, ρ_G is the sum of the positive roots of G, Rel_b is a set parametrizing pairs of stably conjugate elements of $G(F)$ and $G_b(F)$, plus some additional data, and $r_\mu[\lambda]$ is the λ-isotypic subspace of the representation of the Langlands dual group $\widehat{G}$ with highest weight μ. If ρ belongs to a supercuspidal L-packet, then this identity confirms a prediction of the Kottwitz and refined local Langlands conjectures.

Recall that an element of a reductive group is *strongly regular* if its centralizer is a torus. It would be useful to generalize Theorem 1.1 to strongly regular g, since the strongly regular locus is dense in $G(F)$, while the elliptic locus may not be dense. We make some progress toward this goal.

Theorem 1.2 (Theorem 5.3.4). *Let ρ be a finite length admissible representation of $G_b(F)$, and let $\pi = R\Gamma(G, b, \mu)[\rho]$. Let $g \in G(F)$ be a strongly regular element. Suppose that for all specializations b' of b in $B(G, \mu)$, g is not stably conjugate to any element of $G_{b'}(F)$. Then*

$$\Theta_\pi(g) = (-1)^{\langle \mu, 2\rho_G\rangle} \sum_{(g,g',\lambda)\in \mathrm{Rel}_b} \dim r_\mu[\lambda]\Theta_\rho(g') \prod_{\alpha\in\Phi^+} \left|1 - \alpha(\lambda)\right|^{-1}.$$

Here, Φ^+ is the set of roots of G that are positive with respect to the parabolic subgroup of G associated with b, and the norm is defined so that $|p| = p^{-[F:\mathbb{Q}_p]}$.

Our definition of Rel_b is slightly different from the one in [8], but the elliptic locus of our Rel_b is the same as the elliptic locus of theirs.

Example 1.3. Let M be a Levi subgroup of G, let μ be a cocharacter of G whose centralizer is M, and let $b = \mu(p)$. Then $G_b = M$. After accounting for normalization, Theorem 1.2 recovers van Dijk's formula for the Harish-Chandra character of a parabolically induced representation [18, Theorem 3].

A key step in the proof of Theorem 1.2 is the following result.

Theorem 1.4 (Theorem 3.2.4). *For each $b \in B(G)$, the strongly regular locus of the inertia stack of* Bun_G^b *is open in the inertia stack of* Bun_G.

Here, Bun_G is the stack classifying G-bundles on the Fargues–Fontaine curve. This theorem also holds when $F = \mathbb{F}_q((T))$.

The reason that Theorem 1.2 does not cover the entire strongly regular locus is the following. An admissible representation ρ of $G_b(F)$ can be considered as a constructible sheaf on Bun_G^b, and the trace distribution of ρ can be regarded as the characteristic class of this sheaf. One can also consider the characteristic class of the sheaf $i_{b*}\rho$ on Bun_G. Theorem 1.4 implies that this characteristic class determines a trace distribution on the strongly regular locus of $G_{b'}(F)$ for each $b' \in B(G)$. The b-component is the usual trace distribution of ρ. If b' is a specialization of b, then the b'-component may be nonzero. These components show up in the computation of Θ_π.

The problem of computing these b'-components seems interesting in its own right. We compute some of these components in Examples 4.5.20–4.5.21, but we do not have a general formula. In the case where $\mathrm{Sht}_{G,b,\mu}$ is the Drinfeld upper half plane, Example 5.3.5 uses the result of Example 4.5.20 to compute Θ_π over the entire strongly regular locus.

In order to determine the factor of $\prod_{\alpha\in\Phi^+}|1-\alpha(\lambda)|^{-1}$ appearing in Theorem 1.2, and to work out Examples 4.5.20–4.5.21, we establish a method for computing the behavior of characteristic classes under cohomologically smooth pullback. This method is developed in Section 4.5.

Outline of the paper

In Section 2, we prove some general results about endomorphisms of G-bundles. In Section 3, we use these results to prove Theorem 1.4. In Section 4, we prove some results on the behavior of characteristic classes under pullback, and use these to compute characteristic classes on Bun_G. In Section 5, we explain how to modify the argument of [8] to prove Theorem 1.2.

2. Automorphisms of G-bundles in general

Let F be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, and let G be a connected reductive group over F. We will sometimes abuse notation and write G for the diamond $(G^{\mathrm{an}})^\diamond$, and do similarly with other schemes.

2.1. Centralizers and bundles

Since we are interested in G-bundles on the Fargues–Fontaine curve, it is natural to work in the category of sousperfectoid spaces defined in [9] (see also [15, Section 6.3]). There are several equivalent ways of defining G-bundles on a sousperfectoid space.

Definition/Proposition 2.1.1 ([15, Theorem 19.5.2], [2, Definition/Proposition III.1.1]). Let Y be a sousperfectoid space. The following categories are naturally equivalent:

(1) The category of adic spaces $T \to Y$ with a G-action such that étale locally on Y, there is a G-equivariant isomorphism $T \cong G \times Y$.

(2) The category of étale sheaves $\mathcal{E}$ on Y with a G-action such that étale locally, $\mathcal{E} \cong G$.

(3) The category of exact $\otimes$-functors from the category $\operatorname{Rep}_F G$ of algebraic F-vector space representations of G to the category $\operatorname{Bun}(Y)$ of vector bundles on Y.

We define a *G-bundle* on Y to be an object in the category (3). We will sometimes also view it as an object in one of the first two categories.

Definition 2.1.2 ([15, Definition 5.3.2]). Let Y be a sousperfectoid space. A *Cartier divisor* on Y is an ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$ that is locally free of rank 1.

Definition 2.1.3. Let Y be a sousperfectoid space, let $\mathcal{I}$ be a Cartier divisor on Y, and let U be the complement of the support of $\mathcal{O}_Y/\mathcal{I}$.

Let $\mathcal{V}$, $\mathcal{V}'$ be vector bundles on Y. A *modification* $\mathcal{E} \dashrightarrow \mathcal{E}'$ along $\mathcal{I}$ is an isomorphism of vector bundles $\mathcal{V}|_U \xrightarrow{\sim} \mathcal{V}'|_U$ that extends to a map of sheaves of $\mathcal{O}_Y$-modules $\mathcal{V} \to \varinjlim_n \mathcal{V}' \otimes \mathcal{I}^{\otimes -n}$.

Let $\mathcal{E}$, $\mathcal{E}'$ be G-bundles on Y. A *modification* $\mathcal{E} \dashrightarrow \mathcal{E}'$ along $\mathcal{I}$ is an isomorphism of G-bundles $\mathcal{E}|_U \xrightarrow{\sim} \mathcal{E}'|_U$ such that for any object of $\operatorname{Rep}_F G$, the associated map of vector bundles is a modification along $\mathcal{I}$.

Proposition 2.1.4. *Let Y be a sousperfectoid space over F, and let $\mathcal{E}, \mathcal{E}'$ be G-bundles on Y. Let $g \in \operatorname{Aut} \mathcal{E}$, $g' \in \operatorname{Aut} \mathcal{E}'$. Let $\widetilde{\mathcal{E}} \subset \underline{\operatorname{Aut}}\mathcal{E}$ denote the centralizer of g, and let $\widetilde{\mathcal{E}}'$ denote the equalizer of the maps $\underline{\operatorname{Isom}}(\mathcal{E}, \mathcal{E}') \to \underline{\operatorname{Isom}}(\mathcal{E}, \mathcal{E}')$ given by $x \mapsto gx$ and $x \mapsto xg'$.*

Suppose that étale locally on Y, g and g' are conjugate. Also suppose that we are given a connected reductive group H over F and an isomorphism $h\colon H \times_F Y \xrightarrow{\sim} \widetilde{\mathcal{E}}$ of groups over Y. Then:

(1) *$\widetilde{\mathcal{E}}'$ is an H-bundle.*

(2) *There is an isomorphism of G-bundles $\mathcal{E}' \cong (\widetilde{\mathcal{E}}' \times_Y \mathcal{E})/H$.*

Further suppose that there exists a finite extension K of F and a homomorphism $h'\colon H_K \to G_K$ such that the induced map $H_K \times_F Y \to G_K \times_F Y$ is étale locally conjugate to h_K. Then:

(3) *For any Cartier divisor $\mathcal{I}$ on Y, the isomorphism (2) induces a bijection between modifications of G-bundles $\mathcal{E} \dashrightarrow \mathcal{E}'$ along $\mathcal{I}$ intertwining g and g' and modifications of H-bundles $\widetilde{\mathcal{E}} \dashrightarrow \widetilde{\mathcal{E}}'$ along $\mathcal{I}$.*

Proof. Items (1) and (2) are clear from Definition/Proposition 2.1.1 (2).

Now we will prove item (3). Let U be the complement of the support of $\mathcal{O}_Y/\mathcal{I}$. It is clear that (2) induces a bijection between isomorphisms $\mathcal{E}|_U \cong \mathcal{E}'|_U$ intertwining g and g' and isomorphisms $\widetilde{\mathcal{E}}|_U \cong \widetilde{\mathcal{E}}'|_U$. It remains to check that this bijection preserves the property of being meromorphic along $\mathcal{I}$.

By étale descent for vector bundles [12, Theorem 8.2.22 (d)], we can reduce to the case where $\mathcal{E}$, $\mathcal{E}'$ are trivial and $K = F$. Let U be the complement of the support of $\mathcal{O}_Y/\mathcal{I}$. Modifications $\mathcal{E} \dashrightarrow \mathcal{E}'$ along $\mathcal{I}$ are then identified with maps $\phi\colon U \to G$ such that for every finite-dimensional algebraic representation $r\colon G \to GL(V)$ (equivalently, for some faithful r), $r \circ \phi$ induces an automorphism of $V \otimes_F \mathcal{O}_U$ that is meromorphic along $\mathcal{I}$. If r is faithful, then $r \circ h'$ is also faithful. Conjugating an automorphism of $V \otimes_F \mathcal{O}_U$ by an automorphism of $V \otimes_F \mathcal{O}_Y$ will not change whether the former is meromorphic along $\mathcal{I}$. ■

2.2. The coarse quotient $G/\!/G$, and the strongly regular locus

Consider the action of G on itself by conjugation, and let $G/\!/G = \mathrm{Spec}(\Gamma(G, \mathcal{O}_G)^G)$ denote the coarse quotient.

Any morphism $H \to G$ of connected reductive groups over F induces a morphism $H/\!/H \to G/\!/G$.

Let Y be a scheme or sousperfectoid space over F, and let $\mathcal{E}$ be a G-bundle over Y. Suppose we are given an étale covering $Z \to Y$ along with a trivialization

$$\tau\colon \mathcal{E} \times_Y Z \xrightarrow{\sim} G \times_F Z.$$

The action $\underline{\mathrm{Aut}}\,\mathcal{E} \times_Y (G \times_F Z) \to (G \times_F Z)$ is of the form $(h, g, z) \mapsto (g\gamma_\tau(h, z), z)$ for some $\gamma_\tau\colon \underline{\mathrm{Aut}}\,\mathcal{E} \times_Y Z \to G$. The map γ_τ depends τ only up to conjugation. Moreover, the composite

$$\underline{\mathrm{Aut}}\,\mathcal{E} \times_Y Z \xrightarrow{\gamma_\tau} G \to G/\!/G$$

descends to a map $\chi_{\mathcal{E}}\colon \underline{\mathrm{Aut}}\,\mathcal{E} \to G/\!/G$.

Example 2.2.1. If G' is an inner form of G, then $\underline{\mathrm{Isom}}(G, G')$ is a G-bundle over F, with automorphism group G'. There is an induced map $G' \to G/\!/G$.

Definition 2.2.2. Let K be an extension of F. An element of $G(K)$ is *strongly regular* if its centralizer is a torus.

Lemma 2.2.3. *The locus of strongly regular points of G is an open subscheme G_{sr} of G. This open subscheme is the pullback of an open subscheme $(G/\!/G)_{\mathrm{sr}}$ of the coarse quotient $G/\!/G$.*

Proof. Since the property of being strongly regular is insensitive to field extensions, there is no harm in assuming that G is split. Let T be a split maximal torus of G, and let W be the Weyl group of G.

Let g be a geometric point of G. If g is strongly regular, then it is regular semisimple, i.e. the multiplicity of the eigenvalue 1 in the adjoint action of g on $\mathfrak{g}$ is as small as possible. It is clear that the regular semisimple locus is the pullback of an open subscheme of $G/\!/G$.

If g is regular semisimple, then it is conjugate to a geometric point of T. Using the Bruhat decomposition, we see that a regular semisimple geometric point of T is strongly regular iff its stabilizer under the action of the Weyl group W is trivial. The locus of points of T with trivial stabilizer is a W-stable open subspace of T. By [4, Proposition V.1.1 (ii)], this locus is the pullback of an open subspace of $\mathrm{Spec}(\Gamma(T, \mathcal{O}_T)^W)$.

By Chevalley restriction, the map

$$\Gamma(G, \mathcal{O}_G)^G \to \Gamma(T, \mathcal{O}_T)^W$$

is an isomorphism. So the strongly regular locus of G is the pullback of an open subscheme of $G/\!/G$. ∎

Definition 2.2.4. Let X be a scheme, sousperfectoid space, or v-stack over $G/\!/G$. The *strongly regular locus* of X, denoted X_{sr}, is the pullback $X \times_{G/\!/G} (G/\!/G)_{\mathrm{sr}}$.

Two strongly regular elements of G are stably conjugate if and only if they have the same image in $G/\!/G$. The following lemma generalizes this fact.

Lemma 2.2.5. *The map $G \times_F G_{\mathrm{sr}} \to G_{\mathrm{sr}} \times_{G/\!/G} G_{\mathrm{sr}}$ given by $(g, h) \mapsto (h, ghg^{-1})$ admits an étale local section.*

Proof. We claim the map is smooth and surjective. Indeed, it is straightforward to check smoothness using Chevalley restriction, and [17, Corollary 6.6] implies surjectivity. Then an étale local section exists by [3, Corollaire 17.16.3 (ii)].

Alternatively, one can use the Bruhat decomposition to give an explicit étale local section. Choose a finite extension K/F such that G_K is split, and let B and $\bar{B}$ be opposite Borel subgroups of G_K. Let $T = B \cap \bar{B}$, and let N and $\bar{N}$ be the unipotent radicals of B and $\bar{B}$, respectively. Let W be the Weyl group of G_K. Recall that $G_K = \bigsqcup_{w \in W} Bw\bar{B}$, and that $Bw\bar{B} \subseteq B\bar{B}w = N\bar{N}wT$. Therefore, $G/T = \bigcup_{w \in W} N\bar{N}wT/T$. Choose a representative in the normalizer of T for each element of W. Then there is an étale covering

$$T_{\mathrm{sr}} \times_K N \times_K \bar{N} \times_K N \times_K \bar{N} \times W \to G_{\mathrm{sr}} \times_{G/\!/G} G_{\mathrm{sr}}$$

given by

$$(t, n_1, \bar{n}_1, n_2, \bar{n}_2, w) \mapsto (n_1\bar{n}_1 t \bar{n}_1^{-1} n_1^{-1}, n_2\bar{n}_2 w t w^{-1} \bar{n}_2^{-1} n_2^{-1}),$$

and the map to $G \times_F G_{\mathrm{sr}}$ given by

$$(t, n_1, \bar{n}_1, n_2, \bar{n}_2, w) \mapsto (n_2\bar{n}_2 w \bar{n}_1^{-1} n_1^{-1}, n_1\bar{n}_1 t \bar{n}_1^{-1} n_1^{-1})$$

is the desired étale local section. ∎

Corollary 2.2.6. *Let Y be a sousperfectoid space over F, let $\mathcal{E}$, $\mathcal{E}'$ be G-bundles over Y, and let $g \in \operatorname{Aut}\mathcal{E}$, $g' \in \operatorname{Aut}\mathcal{E}'$ be strongly regular automorphisms. Then g and g' are étale locally conjugate iff the diagram*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "g'"] & \underline{\operatorname{Aut}}\,\mathcal{E} \arrow[d, "\chi_{\mathcal{E}}"] \\ \underline{\operatorname{Aut}}\,\mathcal{E}' \arrow[r, "\chi_{\mathcal{E}'}"] & G/\!/G \end{tikzcd}$$

commutes.

3. Automorphisms of G-bundles on the Fargues–Fontaine curve

3.1. G-bundles on the Fargues–Fontaine curve

As in the previous section, F denotes a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, and G denotes a connected reductive group over F. Let $\breve{F}$ denote the completion of the maximal unramified extension of F, and let $\sigma\colon \breve{F} \to \breve{F}$ denote the Frobenius map.

Given $b \in G(\breve{F})$, let G_b denote the algebraic group over F whose functor of points is given by

$$G_b(R) = \{g \in G(R \otimes_F \breve{F}) \mid b\sigma(g) = gb\}.$$

The group G_b is an inner form of a Levi subgroup of the quasisplit inner form of G.

The Kottwitz set $B(G)$ is defined to be the set of σ-conjugacy classes of $G(\breve{F})$. In other words, two elements $b, b' \in G(\breve{F})$ are considered to be equivalent if there exists $\gamma \in G(\breve{F})$ such that $b' = \sigma(\gamma)b\gamma^{-1}$. The isomorphism class of the group G_b depends only on the image of b in $B(G)$.

Let $\mathrm{Perf}_{\overline{\mathbb{F}}_p}$ denote the category of perfectoid spaces over $\overline{\mathbb{F}}_p$. For any affinoid perfectoid $S = \mathrm{Spa}(R, R^+)$ in $\mathrm{Perf}_{\overline{\mathbb{F}}_p}$ with pseudouniformizer ϖ, define

$$\begin{aligned} Y_S &= \left(\mathrm{Spa}\, W_{\mathcal{O}_F}(R^+)\right) \setminus \{p[\varpi] = 0\}, \\ X_S &= Y_S / \mathrm{Frob}^{\mathbb{Z}}. \end{aligned}$$

The space X_S is the adic Fargues–Fontaine curve corresponding to the pair (F, S). This definition can be glued, so it makes sense to define Y_S and X_S for any S in $\mathrm{Perf}_{\overline{\mathbb{F}}_p}$.

For any $b \in B(G)$ and any $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$, there is a corresponding vector bundle $\mathcal{E}_{b,S}$ on X_S.

Definition 3.1.1 ([2, Definition III.0.1, Theorem III.0.2]). Let Bun_G be the v-stack sending $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$ to the groupoid of G-bundles on X_S.

For any $b \in \mathrm{Bun}_G$, let Bun_G^b be the substack classifying G-bundles that are isomorphic to $\mathcal{E}_{b,x}$ at every geometric point x.

Let $\widetilde{G}_b$ denote the functor $\mathrm{Perf}_{\overline{\mathbb{F}}_p} \to \mathrm{Set}$ defined by $\widetilde{G}_b(S) = \operatorname{Aut}\mathcal{E}_{b,S}$. It is representable by a locally spatial diamond. There are maps $\underline{G_b(F)} \to \widetilde{G}_b \to \underline{G_b(F)}$, whose composition is the identity.

3.2. The strongly regular locus of $\mathrm{In}(\mathrm{Bun}_G)$

Lemma 3.2.1. *Let S be a perfectoid space over $\overline{\mathbb{F}}_p$. For any map $X_S \to (G/\!/G)^\diamond$, there is a unique continuous map $|S| \to (G/\!/G)(F)$ that makes the following diagram commute.*

$$\begin{tikzcd} X_S \arrow[r] \arrow[d] & (G/\!/G)^\diamond \\ \underline{|S|} \arrow[r] & \underline{(G/\!/G)(F)} \arrow[u] \end{tikzcd}$$

Proof. By [2, Proposition II.2.5 (ii)], maps $X_S \to (\mathbb{A}^1_F)^\diamond$ over $\mathrm{Spd}\, F$ are in bijection with maps $|S| \to F$. Since $G/\!/G$ is an affine variety, the lemma follows. ∎

Definition 3.2.2. For any map of v-stacks $X \to S$, the *inertia stack* $\mathrm{In}_S(X)$ is the fiber product $X \times_{X \times_S X} X$.

In the case $S = \mathrm{Spd}\, \overline{\mathbb{F}}_p$, we will shorten the notation to $\mathrm{In}(X)$.

The stack $\mathrm{In}(\mathrm{Bun}_G)$ classifies G-bundles equipped with an automorphism.

Corollary 3.2.3. *There is a map $\chi\colon \mathrm{In}(\mathrm{Bun}_G) \to \underline{(G/\!/G)(F)}$ such that for any $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$, and any morphism $S \to \mathrm{In}(\mathrm{Bun}_G)$ inducing a G-bundle $\mathcal{E}$ over X_S equipped with an automorphism $g \in \mathrm{Aut}\,\mathcal{E}$, the diagram*

$$\begin{tikzcd} \underline{\mathrm{Aut}\,\mathcal{E}} \arrow[drr, "\chi_{\mathcal{E}}"] & & \\ X_S \arrow[u, "g"] \arrow[rr] \arrow[d] & & G/\!/G \\ \underline{|S|} \arrow[rr] \arrow[d] & & \underline{(G/\!/G)(F)} \arrow[u] \\ \underline{|\mathrm{In}(\mathrm{Bun}_G)|} \arrow[urr, "|\chi|"] & & \end{tikzcd}$$

commutes.

Define $\mathrm{In}(\mathrm{Bun}_G)_{\mathrm{sr}} = \chi^{-1}(\underline{(G/\!/G)(F)_{\mathrm{sr}}})$.

Theorem 3.2.4. *For each $b \in B(G)$, $\mathrm{In}(\mathrm{Bun}^b_G)_{\mathrm{sr}}$ is open in $\mathrm{In}(\mathrm{Bun}_G)$. In particular, $\mathrm{In}(\mathrm{Bun}_G)_{\mathrm{sr}} = \bigsqcup_{b \in B(G)} \mathrm{In}(\mathrm{Bun}^b_G)_{\mathrm{sr}}$.*

The strategy of the proof will be to show that if $\mathcal{E}$ is a G-bundle on X_S admitting a strongly regular automorphism, then locally on $|S|$, $\mathcal{E}$ can be obtained from the construction of Proposition 2.1.4, with H a torus. If H is a torus, then $|\mathrm{Bun}_H|$ is discrete.

Any $b \in B(G)$ induces a morphism $\mathrm{Spd}\, \overline{\mathbb{F}}_p \to \mathrm{Bun}_G$. The group $\widetilde{G}_b$ can be identified with the fiber product $\mathrm{Spd}\, \overline{\mathbb{F}}_p \times_{\mathrm{Bun}_G} \mathrm{In}(\mathrm{Bun}_G)$. Let $\chi_{\widetilde{G}_b}$ denote the composite

$$\widetilde{G}_b \to \mathrm{In}(\mathrm{Bun}_G) \to \underline{(G/\!/G)(F)}.$$

One can also define a map $\chi_{G_b}\colon G_b \to (G/\!/G)$ as in Example 2.2.1.

Lemma 3.2.5 ([2, Proposition III.5.1]). *The group $\widetilde{G}_b$ has a slope filtration $\{\widetilde{G}_b^{\geq\lambda}\}$ such that*

$$\widetilde{G}_b \cong \underline{G_b(F)} \ltimes \widetilde{G}_b^{>0},$$

and for each $\lambda > 0$, $\widetilde{G}_b^{\geq\lambda}/\widetilde{G}_b^{>\lambda}$ is isomorphic to the Banach–Colmez space associated with the slope $-\lambda$ part of the isocrystal $(\mathrm{Lie}\, G \otimes_F \breve{F}, \mathrm{Ad}(b)\sigma)$.

Lemma 3.2.6. *The diagram*

$$\begin{tikzcd}
\widetilde{G}_b \arrow[r] \arrow[dr, "\chi_{\widetilde{G}_b}"'] & \underline{G_b(F)} \arrow[r] \arrow[d, "\chi_{G_b}"] & \widetilde{G}_b \arrow[dl, "\chi_{\widetilde{G}_b}"] \\
& \underline{(G/\!/G)(F)} &
\end{tikzcd}$$

commutes.

Proof. It is clear from the definitions that the triangle on the right commutes. To see that the large triangle also commutes, observe that the composite $\widetilde{G}_b \to \underline{G_b(F)} \to \widetilde{G}_b$ is a limit of conjugation maps, and $\chi_{\widetilde{G}_b}$ is conjugation invariant. ∎

Lemma 3.2.7. *Let T be a maximal torus of G_b, and let T_{sr} denote the preimage of $(G/\!/G)_{\mathrm{sr}}$ under the map $T \to G/\!/G$ induced by χ_{G_b}. Then the induced map*

$$T(F)_{\mathrm{sr}} \to (G/\!/G)(F)_{\mathrm{sr}}$$

is a local homeomorphism.

Proof. By Chevalley restriction, the map $T_{\mathrm{sr}} \to (G/\!/G)_{\mathrm{sr}}$ is étale. Then the lemma follows from the inverse function theorem for locally F-analytic manifolds [16, Section II.III.9]. ∎

Proof of Theorem 3.2.4. Let S be a perfectoid space, and let $\mathcal{E}$ be a G-bundle over X_S equipped with a strongly regular endomorphism g. It induces a map $S \to \mathrm{In}(\mathrm{Bun}_G)$. Let $b \in B(G)$, and let $z = \mathrm{Spa}(K, K^+)$ be a point of S whose image in Bun_G is contained in Bun_G^b. We need to show that some open neighborhood of z also maps to Bun_G^b.

Let t be the image of g_z under the map $\widetilde{G}_b(K) \to G_b(F)$. By Lemma 3.2.6, g_z and t determine the same element of $(G/\!/G)(F)$. Let T be the centralizer of t in G_b. By Lemma 3.2.7, after replacing S with an open neighborhood of z, we can find a map $|S| \to T(F)_{\mathrm{sr}}$ such that the composite

$$|S| \to T(F)_{\mathrm{sr}} \xrightarrow{\chi_{G_b}} (G/\!/G)(F)_{\mathrm{sr}}$$

agrees with the map $|S| \to (G/\!/G)(F)_{\mathrm{sr}}$ induced by g. Let g' the endomorphism of the bundle $\mathcal{E}_{b,S}$ induced by this map. By Corollary 2.2.6, g and g' are étale locally conjugate. By Proposition 2.1.4, we can find a T-bundle $\widetilde{\mathcal{E}}$ such that $\mathcal{E} = (\widetilde{\mathcal{E}} \times \mathcal{E}_{b,S})/T$. Then the map $S \to \mathrm{Bun}_G$ factors through Bun_T. Since $|\mathrm{Bun}_T|$ is discrete, the map $|S| \to |\mathrm{Bun}_G|$ must be locally constant. ∎

4. Characteristic classes

In this section, we consider the theory of characteristic classes, both in general and on Bun_G. We are interested in characteristic classes because of their connection to trace distributions of representations of $G(F)$. This connection will be recalled in Section 4.2. In Section 4.3, we will relate characteristic classes on Bun_G to distributions on the groups $G_b(F)$ for $b \in B(G)$ (including non-basic b). Sections 4.4 and 4.5 develop some techniques for computing characteristic classes on Bun_G, and more generally for computing the behavior of characteristic classes under cohomologically smooth pullback.

From now on, we assume that F is a finite extension of $\mathbb{Q}_p$.

4.1. Definition of characteristic classes

First, we recall the definition of characteristic classes from [8, Section 4.3]. In the interest of space, we avoid introducing categories of cohomological correspondences, and give all definitions directly in terms of the category $D_{\text{ét}}(X, \Lambda)$ defined in [14, Definition 14.13].

Let $f: X \to S$ be a fine map of decent v-stacks, as defined in [5, Definitions 1.1 and 1.3]. Let ℓ be a prime different from p, and let Λ be a $\mathbb{Z}_\ell$-algebra that is killed by a power of ℓ. Let $K_{X/S}$ denote the dualizing complex of f.

For any object A of $D_{\text{ét}}(X, \Lambda)$, let $A^\vee = \underline{\mathrm{RHom}}(A, K_{X/S})$ denote its dual. There is a natural morphism

$$\mathrm{ev}: A^\vee \otimes A \to K_{X/S},$$

called the *evaluation map*.

Let $p_1, p_2: X \times_S X \to X$ denote the projection maps, and let $\Delta: X \to X \times_S X$ denote the diagonal. There is a natural morphism

$$A^\vee \boxtimes A \to \underline{\mathrm{RHom}}(p_1^* A, p_2^! A), \tag{4.1.1}$$

defined as follows. By the adjunction between $\otimes$ and $\underline{\mathrm{RHom}}$, specifying such a morphism is equivalent to specifying a morphism $(A^\vee \otimes A) \boxtimes A \to p_2^! A$, which is in turn equivalent to specifying a morphism

$$p_1^*(A^\vee \otimes A) \to \underline{\mathrm{RHom}}(p_2^* A, p_2^! A).$$

We take the composite

$$p_1^*(A^\vee \otimes A) \xrightarrow{p_1^* \mathrm{ev}} p_1^* K_{X/S} \to p_2^! \Lambda \to p_2^! \underline{\mathrm{RHom}}(A, A) \xrightarrow{\sim} \underline{\mathrm{RHom}}(p_2^* A, p_2^! A).$$

Here, the second arrow is base change.

Definition 4.1.2. We say that A is *dualizable* over S if (4.1.1) is an isomorphism (cf. [8, Proposition 4.3.5]).

If A is dualizable, we define the *coevaluation map* $\mathrm{coev}: \Lambda \to \Delta^!(A^\vee \boxtimes A)$ as the composite

$$\Lambda \to \underline{\mathrm{RHom}}(A, A) \xrightarrow{\sim} \underline{\mathrm{RHom}}(\Delta^* p_1^* A, \Delta^! p_2^! A) \xrightarrow{\sim} \Delta^! \underline{\mathrm{RHom}}(p_1^* A, p_2^! A) \xrightarrow{\sim} \Delta^!(A^\vee \otimes A),$$

where the last isomorphism is induced by the inverse of (4.1.1).

Let $\mathrm{In}_S(X)$ be the inertia stack of X, as in Definition 3.2.2, and let $\pi_1, \pi_2\colon \mathrm{In}_S(X) \to X$ denote the projection maps.

Definition 4.1.3 ([8, Definition 4.3.6]). Let $A \in D_{\text{ét}}(X, \Lambda)$ be an object that is dualizable over S. The *characteristic class* of A over S, denoted $\mathrm{cc}_{X/S}(A)$, is the element of $H^0(\mathrm{In}_S(X), K_{\mathrm{In}_S(X)/S})$ determined by the composite

$$\Lambda \xrightarrow{\pi_1^* \operatorname{coev}} \pi_1^* \Delta^!(A^\vee \boxtimes A) \to \pi_2^! \Delta^*(A^\vee \boxtimes A) \xrightarrow{\pi_2^! \operatorname{ev}} K_{\mathrm{In}_S(X)/S},$$

where the second arrow is base change.

Characteristic classes behave nicely under pushforward by a proper map and pullback by an open immersion, as the following two propositions demonstrate.

Proposition 4.1.4 ([8, Lemma 4.3.7]). *Let $i\colon U \to X$ be an open immersion of decent v-stacks fine over S. Then $\mathrm{In}_S(i)\colon \mathrm{In}_S(U) \to \mathrm{In}_S(X)$ is also an open immersion. If $A \in D(X, \Lambda)$ is dualizable over S, then so is i^*A, and*

$$\mathrm{cc}_{U/S}(i^*A) = \mathrm{In}_S(i)^* \, \mathrm{cc}_{X/S}(A).$$

Proposition 4.1.5 ([8, Corollary 4.3.9]). *Let $p\colon X \to Y$ be an proper morphism of decent v-stacks fine over S. Then $\mathrm{In}_S(p)\colon \mathrm{In}_S(X) \to \mathrm{In}_S(Y)$ is also proper. If $A \in D(X, \Lambda)$ is dualizable over S, then so is $p_! A$, and*

$$\mathrm{cc}_{X/S}(p_! A) = \mathrm{In}_S(i)_! \, \mathrm{cc}_{Y/S}(A).$$

In Section 4.5, we will show that the behavior of characteristic classes under cohomologically smooth pullback can also sometimes be understood explicitly.

4.2. Geometrization of trace distributions

Now, let Λ denote an arbitrary $\mathbb{Z}_\ell$-algebra (not necessarily killed by a power of ℓ). Let $C_c(G_b(F), \Lambda)$ denote the space of locally constant Λ-valued functions on $G_b(F)$, and let

$$\mathrm{Haar}\big(G_b(F), \Lambda\big) = \mathrm{Hom}_{\Lambda[G_b(F)]}\big(C_c\big(G_b(F), \Lambda\big), \Lambda\big)$$

denote the space of Λ-valued Haar measures on $G_b(F)$. Define

$$\mathrm{HaarDist}\big(G_b(F), \Lambda\big) = \mathrm{Hom}_\Lambda\big(C_c\big(G_b(F), \Lambda\big) \otimes_\Lambda \mathrm{Haar}\big(G_b(F), \Lambda\big), \Lambda\big). \tag{4.2.1}$$

A Λ-representation ρ of $G_b(F)$ is said to be smooth if the stabilizer of any vector is open. A smooth representation ρ is said to be admissible if for every compact open pro-p subgroup $K \subset G_b(F)$, the derived K-invariants of ρ are a perfect complex of Λ-modules. Any admissible smooth representation ρ determines a trace distribution $\mathrm{tr.dist}(\rho) \in \mathrm{HaarDist}(G, \Lambda)$. For any $f \in C_c(G_b(F), \Lambda)$, the operator $\int_{g \in G_b(F)} f(g)\rho(g)$ has image contained in a finitely presented Λ-module, and $\mathrm{tr.dist}(\rho)(f)$ is defined to be its trace.

If in addition Λ is isomorphic to $\mathbb{C}$ and ρ has finite length, then tr.dist(ρ) has a Harish-Chandra character Θ_ρ [10, Theorem 16.3]. Here, Θ_ρ is a locally constant function from the regular semisimple locus $G_b(F)_{\mathrm{rs}}$ of $G_b(F)$ to Λ satisfying

$$(\mathrm{tr.dist}\rho)(f) = \int_{g \in G_b(F)_{\mathrm{rs}}} \Theta_\rho(g) f(g)$$

for all $f \in C_c(G_b(F), \Lambda) \otimes_\Lambda \mathrm{Haar}(G_b(F), \Lambda)$. (Since $G_b(F)_{\mathrm{rs}}$ is not compact, we choose a isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ to make sense of the integral. However, since Θ_ρ is uniquely determined by the values of the integrals for which f has support in a compact subset of $G_b(F)_{\mathrm{rs}}$, Θ_ρ does not depend on the choice of isomorphism.) Since the strongly regular locus is dense in the regular semisimple locus, there is no harm in integrating over $G_b(F)_{\mathrm{sr}}$ instead of $G_b(F)_{\mathrm{rs}}$.

The trace distribution can be reinterpreted as a characteristic class. Let $D(G_b(F), \Lambda)$ denote the derived category of the category of Λ-representations of $G_b(F)$. For any small v-stack X, Fargues–Scholze define a category $D_{\mathrm{lis}}(X, \Lambda)$ [2, VII.6.1]. If Λ is killed by a power of ℓ, then $D_{\mathrm{lis}}(X, \Lambda)$ is equivalent to $D_{\text{ét}}(X, \Lambda)$ by [2, Proposition VII.6.6].

Until the end of Section 4.4, let S be either $\mathrm{Spd}\,\overline{\mathbb{F}}_p$, or $\mathrm{Spd}\,C$ for some algebraically closed nonarchimedean field C of characteristic p.

Proposition 4.2.2 ([2, Proposition VII.7.1]). *There are equivalences*

$$D\big(G_b(F), \Lambda\big) \cong D_{\mathrm{lis}}\big([S/G_b(F)], \Lambda\big) \cong D_{\mathrm{lis}}\big([S/\widetilde{G}_b], \Lambda\big).$$

Proposition 4.2.3 ([8, Example 4.2.5]). *Suppose Λ is killed by a power of ℓ. There is an isomorphism*

$$H^0\big(\mathrm{In}_S\left([S/\underline{G_b(F)}]\right), K_{\mathrm{In}_S([S/\underline{G_b(F)}])/S}\big) \cong \mathrm{HaarDist}\left(G_b(F), \Lambda\right)^{G_b(F)}.$$

Here, $G_b(F)$ acts on HaarDist$(G_b(F), \Lambda)$ *by conjugation.*

Proposition 4.2.4. *Suppose Λ is killed by a power of ℓ.*

(1) *An object of $D(G_b(F), \Lambda)$ is admissible if and only if the corresponding object of $D_{\text{ét}}([S/\underline{G_b(F)}], \Lambda)$ is dualizable over S.*

(2) *Under the isomorphism of Proposition 4.2.3, if ρ is admissible, then*

$$\mathrm{cc}_{[S/\underline{G_b(F)}]/S}(\rho) = \mathrm{tr.dist}(\rho).$$

Proof. The first claim can be proved in the same way as [2, Theorem V.7.1]. The second claim is [8, Proposition 4.4.3]. ∎

4.3. Generalization of distributions to Bun_G

We again assume that Λ is a $\mathbb{Z}_\ell$-algebra killed by a power of ℓ. We are interested in understanding the characteristic classes of objects in $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$. We will focus on the restriction to the strongly regular part of the inertia stack. We begin by determining the space in which these characteristic classes live.

Proposition 4.3.1. *The following statements hold:*

(1) *For each $b \in B(G)$, the map $\underline{G_b(F)} \to \widetilde{G}_b$ induces an isomorphism*

$$\mathrm{In}_S\left([S/\underline{G_b(F)}]\right)_{\mathrm{sr}} \xrightarrow{\sim} \mathrm{In}_S\left([S/\widetilde{G}_b]\right)_{\mathrm{sr}}.$$

(2) *For each $b \in B(G)$, there is an isomorphism*

$$H^0\left(\mathrm{In}_S\left([S/\widetilde{G}_b]\right)_{\mathrm{sr}}, K_{\mathrm{In}_S([S/\widetilde{G}_b])_{\mathrm{sr}}/S}\right) \cong \mathrm{HaarDist}\left(G_b(F)_{\mathrm{sr}}, \Lambda\right)^{G_b(F)}.$$

(3) *There is an isomorphism*

$$H^0\left(\mathrm{In}_S(\mathrm{Bun}_{G,S})_{\mathrm{sr}}, K_{\mathrm{In}_S(\mathrm{Bun}_{G,S})_{\mathrm{sr}}/S}\right) \cong \prod_{b\in B(G)} \mathrm{HaarDist}\left(G_b(F)_{\mathrm{sr}}, \Lambda\right)^{G_b(F)}.$$

Lemma 4.3.2. *The map $c\colon \underline{G_b(F)}_{\mathrm{sr}} \times \widetilde{G}_b^{>0} \to (\widetilde{G}_b)_{\mathrm{sr}}$ that sends $(g,h) \mapsto hgh^{-1}$ is an isomorphism of diamonds.*

Proof. We will show by induction that for each $\lambda \geq 0$, the map

$$c_\lambda\colon \underline{G_b(F)}_{\mathrm{sr}} \times \widetilde{G}_b^{>0}/\widetilde{G}_b^{>\lambda} \to \underline{G_b(F)}_{\mathrm{sr}} \times \widetilde{G}_b^{>0}/\widetilde{G}_b^{>\lambda}$$

defined by $(g,h) \mapsto ghg^{-1}h^{-1}$ is an isomorphism. The case $\lambda = 0$ is trivial.

Let λ be a slope appearing in $\widetilde{G}_b^{>0}$, and suppose that $c_{\lambda'}$ is an isomorphism for $\lambda' < \lambda$. To show that c_λ is also an isomorphism, it suffices to prove that for each affinoid perfectoid $Z = \mathrm{Spa}(R, R^+)$, c_λ induces an isomorphism on Z-points.

Let X_R^{alg} denote the schematic Fargues–Fontaine curve over R, and let H denote the inner twisting of $G \times X_R^{\mathrm{alg}}$ by $\mathcal{E}_b$. The slope filtration on $\mathcal{E}_b$ induces a filtration on H.

By the argument of [2, Proposition III.5.1], for $\lambda \leq \lambda'$,

$$(\widetilde{G}_b^{>0}/\widetilde{G}_b^{>\lambda'})(Z) = (\widetilde{G}_b^{>0}/\widetilde{G}_b^{>\lambda})(Z)/(\widetilde{G}_b^{>\lambda'}/\widetilde{G}_b^{>\lambda})(Z).$$

So $c_{\lambda'}^{-1}$ admits a lift to an endomorphism of $(\underline{G_b(F)}_{\mathrm{sr}} \times \widetilde{G}_b^{>0}/\widetilde{G}_b^{>\lambda})(Z)$.

For each $g \in G_b(F)$, the endomorphism of $H^{\geq\lambda}/H^{>\lambda}$ induced by c is injective and linear, so it is an isomorphism. So c induces an automorphism of $(\underline{G_b(F)} \times \widetilde{G}_b^{\geq\lambda}/\widetilde{G}_b^{>\lambda})(Z)$.

Combining these facts, we see that there is a lift of $c_{\lambda'}^{-1}$ that is an inverse of the map of Z-points of c_λ. ∎

Proof of Proposition 4.3.1. The first item follows immediately from Lemma 4.3.2. The second item follows from the first and Proposition 4.2.3. The third item follows from the second and Theorem 3.2.4. ∎

4.4. Action of automorphisms of Banach–Colmez spaces on compactly supported cohomology

As mentioned in Lemma 3.2.5, $\widetilde{G}_b$ is a successive extension of $\underline{G_b(F)}$ by Banach–Colmez spaces. In order to understand the relation between characteristic classes on $[S/\underline{G_b(F)}]$

and characteristic classes on $[S/\widetilde{G}_b]$, it will be useful to understand how endomorphisms of Banach–Colmez spaces act on the cohomology of these spaces.

Lemma 4.4.1. *Let*

$$1 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 1$$

be an exact sequence of group diamonds that are locally spatial and separated and cohomologically smooth over S. Assume the morphisms are locally compactifiable of finite dim.trg.

Let $f\colon S \to [S/\mathcal{G}]$ be the quotient map, and define f', f'' similarly. Let g be an automorphism of $\mathcal{G}$ that respects the exact sequence.

Suppose that $f'_!\Lambda$ and $f''_!\Lambda$ are shifts of the constant sheaf Λ, and that g acts by $\lambda', \lambda'' \in \Lambda^\times$ on $f'_!\Lambda$ and $f''_!\Lambda$, respectively. Then $f_!\Lambda$ is also a shift of the constant sheaf Λ, and g acts by $\lambda'\lambda''$ on $f_!\Lambda$.

Proof. Consider the diagram

$$\begin{tikzcd} S \arrow[r] & {[S/\mathcal{G}']} \arrow[r] \arrow[d] & {[S/\mathcal{G}]} \arrow[d] \\ & S \arrow[r] & {[S/\mathcal{G}'']}. \end{tikzcd} \tag{4.4.2}$$

The square is cartesian, so we may apply base change. ∎

Lemma 4.4.3. *Let $\mathcal{E}$ be an object in $D^b(\mathrm{Coh}(X_C))$ such that $\mathcal{H}^0(\mathcal{E})$ has positive slopes, $\mathcal{H}^{-1}(\mathcal{E})$ has negative slopes, and all other cohomology groups are zero. Let $\mathcal{BC}(\mathcal{E})$ denote the corresponding Banach–Colmez space. Let $f\colon \mathrm{Spd}\, C \to [\mathrm{Spd}\, C/\mathcal{BC}(\mathcal{E})]$ be the quotient map. Then $f_!\Lambda \cong \Lambda[-2\deg \mathcal{E}]$.*

Proof. First, consider the case where $\mathcal{E}$ is a vector bundle with positive slopes. Let

$$\tilde{f}\colon \mathcal{BC}(\mathcal{E}) \to \mathrm{Spd}\, C$$

be the base change of f. By [7, Proposition 4.8], $\tilde{f}_!\Lambda \cong \Lambda[-2\dim \mathcal{E}]$. To show that $f_!\Lambda \cong \Lambda[-2\dim \mathcal{E}]$, we need to show that translation by $\mathcal{BC}(\mathcal{E})$ induces the identity on $\tilde{f}_!\Lambda$. Indeed, $\mathcal{BC}(\mathcal{E})$ is connected, while $\tilde{f}_!\Lambda$ is discrete.

In general, we can find an exact triangle $\mathcal{F} \to \mathcal{G} \to \mathcal{E}$, where $\mathcal{F}$ and $\mathcal{G}$ have positive slopes. There is a corresponding exact sequence $0 \to \mathcal{BC}(\mathcal{F}) \to \mathcal{BC}(\mathcal{G}) \to \mathcal{BC}(\mathcal{E}) \to 0$. By applying this lemma to $\mathcal{F}$ and $\mathcal{G}$ and considering the diagram (4.4.2), we see that the pullback of $f_!\Lambda$ to $[\mathrm{Spd}\, C/\mathcal{BC}(\mathcal{G})]$ is isomorphic to $\Lambda[-2\deg \mathcal{E}]$. So $f_!\Lambda$ must also be isomorphic to $\Lambda[-2\deg \mathcal{E}]$. ∎

Definition 4.4.4. Let $\mathcal{E}$ be as in Lemma 4.4.3, and let g be an automorphism of $\mathcal{E}$. Define $\det g \in F^\times$ as follows. If $\mathcal{E}$ is a vector bundle with positive slopes, then $\det g$ is the image of g in $\mathrm{Aut}(\wedge^{\mathrm{rk}\,\mathcal{E}}\mathcal{E}) \cong F^\times$. If $\mathcal{E}[-1]$ is a vector bundle with negative slopes, then $\det g$ is the image of g^{-1} in $\mathrm{Aut}(\wedge^{\mathrm{rk}\,\mathcal{E}[-1]}\mathcal{E}[-1])$. If $\mathcal{E}$ is torsion, then $\det g = 1$. In general, $\det g$ is the product of the determinants of g acting on the graded pieces of the slope filtration of $\mathcal{E}$.

Lemma 4.4.5. *Let $\mathcal{E}$, $\mathcal{BC}(\mathcal{E})$, f be as in Lemma 4.4.3. Let g be an automorphism of $\mathcal{E}$. Then the action of g on $f_!\Lambda$ is multiplication by $|\det g|^{-1}$, where the norm on F is defined so that $|p| = p^{-[F:\mathbb{Q}_p]}$.*

Proof. We will reduce to a special case using Lemma 4.4.1. Lemma 4.4.3 guarantees that the conditions of Lemma 4.4.1 are met.

Using the slope filtration, we reduce to the case where $\mathcal{E}$ is semistable. First, suppose $\mathcal{E}$ has infinite slope. By dévissage, we can reduce to the case where $\mathcal{BC}(\mathcal{E})$ is a product of copies of $\mathbb{G}_a^\diamond$. Then the lemma is satisfied, since any g has determinant 1 and acts trivially on the compactly supported cohomology of $\mathcal{BC}(\mathcal{E})$.

If $\mathcal{E}$ has finite positive (resp. negative) slope, then by using an exact triangle of the form $\mathcal{E}(-n) \to \mathcal{E} \to \mathcal{E}/\mathcal{E}(-n)$ (resp. $\mathcal{E}[-1](n) \to \mathcal{E}[-1](n)/\mathcal{E}[-1] \to \mathcal{E}$), we reduce to the case where the slope of $\mathcal{E}$ is positive and $\leq [F : \mathbb{Q}_p]$. Using Jordan normal form, we further reduce to the case where g acts by a scalar (and F is replaced by a finite extension). In that case, $\mathcal{BC}(\mathcal{E})$ is the universal cover of the generic fiber of a formal $\mathcal{O}_F$-module E over $\mathcal{O}_C$. Since $\mathcal{O}_F \setminus \{0\}$ generates $F^\times$, we may also assume that the scalar is in $\mathcal{O}_F$, so g induces an endomorphism of E. Using [11, Theorem 7.3.4], we see that g must act on $f_!\Lambda$ by the degree of this endomorphism, which is $|\det g|^{-1}$. ∎

Let $b \in B(G)$. We can also compute the action of endomorphisms of $\widetilde{G}_b^{>0}$ on its compactly supported cohomology.

Let ν_b be the Newton point of b. Recall from [2, Section III.5.1.1] that ν_b and $-\nu_b$ determine a pair of opposite parabolic subgroups P_b^+, P_b^- of the quasisplit inner form of G, and G_b is an inner form of $M_b = P_b^+ \cap P_b^-$. Over some finite extension F' of F, $(G_b)_{F'}$ becomes conjugate to $(M_b)_{F'}$; use this conjugation to identify the two groups. For $g \in G_b(F)$, let $D_b^-(g)$ be defined by

$$D_b^-(g) = \det\left(1 - \operatorname{Ad} g | \operatorname{Lie}\left((P_b^-)_{F'}\right) / \operatorname{Lie}\left((M_b)_{F'}\right)\right). \tag{4.4.6}$$

Then $D_b^-(g)$ lies in F since $\mathrm{Gal}(F'/F)$ acts on g by conjugation. It does not depend on F' or the identification of $(G_b)_{F'}$ and $(M_b)_{F'}$ as conjugates.

Lemma 4.4.7. *Let $f\colon (\widetilde{G}_b^{>0})_S \to S$ be the structure map. Then $f_!\Lambda$ is a shift of Λ. If $m \in G_b(F)$, then the map $\widetilde{G}_b^{>0} \to \widetilde{G}_b^{>0}$ defined by $n \mapsto m^{-1}n^{-1}mn$ acts on $f_!\Lambda$ by $|D_b^-(m)|^{-1}$.*

Proof. It suffices to prove the lemma in the case $S = \operatorname{Spd} C$; the case $S = \operatorname{Spd} \overline{\mathbb{F}}_p$ then follows by base change.

The group $\widetilde{G}_b^{>0}$ has a filtration whose graded pieces $\widetilde{G}_b^{\geq\lambda}/\widetilde{G}_b^{>\lambda}$ are Banach–Colmez spaces. Let $\hat{f}\colon S \to [S/(\widetilde{G}_b^{>0})_S]$ denote the quotient map; then repeated applications of Lemmas 4.4.1 and 4.4.3 imply that $\hat{f}_!\Lambda$ is a shift of Λ. By base change, $f_!\Lambda$ is also a shift of Λ. To see that the map $n \mapsto m^{-1}n^{-1}mn$ acts by multiplication by $|D_b^-(m)|^{-1}$, apply Lemma 4.4.5 to the quotients $\widetilde{G}_b^{\geq\lambda}/\widetilde{G}_b^{>\lambda}$. ∎

4.5. Characteristic classes under cohomologically smooth pullback

We will prove some results regarding the behavior of characteristic classes under cohomologically smooth pullback.

This subsection is rather technical. The reader is encouraged to read Propositions 4.5.17 and 4.5.18 and Example 4.5.20 for motivation.

We now let S denote an arbitrary decent v-stack. Let $f\colon X \to Y$ be a cohomologically smooth morphism of decent v-stacks fine over S.

Lemma 4.5.1. *Let $A \in D_{\text{ét}}(Y, \Lambda)$ be an object that is dualizable over S. Then f^*A is also dualizable over S.*

Proof. Let p_1, p_2 (resp. $\tilde{p}_1, \tilde{p}_2$) denote the projection maps $Y \times_S Y \to Y$ (resp. $X \times_S X \to X$). We have a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{RHom}}(f^*A, K_{X/S}) \boxtimes_S f^*A \arrow[r] \arrow[d, "\sim"] & \underline{\mathrm{RHom}}(\tilde{p}_1^* f^* A, \tilde{p}_2^! f^* A) \arrow[d, "\sim"] \\
(f \times \mathrm{id})^! (\mathrm{id} \times f)^* \big(\underline{\mathrm{RHom}}(A, K_{Y/S}) \boxtimes_S A\big) \arrow[r] & (f \times \mathrm{id})^! (\mathrm{id} \times f)^* \underline{\mathrm{RHom}}(p_1^* A, p_2^! A).
\end{tikzcd}$$

If the bottom horizontal arrow is an isomorphism, then the top horizontal arrow must be an isomorphism as well. ∎

Proposition 4.5.2. *The following diagram is commutative with cartesian squares.*

$$\begin{tikzcd}
\mathrm{Ins}_S(X) \arrow[r, "\tilde{\pi}_2"] \arrow[d, "\tilde{\Delta}_f"] \arrow[ddd, bend right=60, "\mathrm{Ins}_S(f)"'] & X \arrow[d, "\Delta_f"] & \\
\mathrm{Ins}_S(Y) \times_Y X \arrow[r, "\hat{\pi}_2"] \arrow[dd, "\tilde{f}"] & X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[dd, "f"] \\
& X \arrow[dr, "f"] & \\
\mathrm{Ins}_S(Y) \arrow[rr, "\pi_2"] & & Y.
\end{tikzcd} \tag{4.5.3}$$

Consider the map

$$\Lambda_{\mathrm{Ins}_S(X)} \to \mathrm{Ins}_S(f)^! \Lambda_{\mathrm{Ins}_S(Y)}$$

defined by applying the composite natural transformation

$$\mathrm{Ins}_S(f)^* \pi_2^* \xrightarrow{\sim} \tilde{\pi}_2^* \Delta_f^* p_1^* f^* \xrightarrow{\sim} \tilde{\pi}_2^* \Delta_f^! p_1^! f^* \to \tilde{\Delta}_f^! \hat{\pi}_2^* p_1^! f^* \xrightarrow{\sim} \tilde{\Delta}_f^! \hat{\pi}^* p_2^* f^! \xrightarrow{\sim} \tilde{\Delta}_f^! \tilde{f}^! \pi_2^* \tag{4.5.4}$$

to Λ_Y. Here, the third arrow is base change, and the fourth and fifth arrows are smooth base change. For any $A \in D(X, \Lambda)$ such that (Y, A) is dualizable over S, the morphism

$$\Lambda_{\mathrm{Ins}_S(X)} \xrightarrow{(4.5.4)} \mathrm{Ins}_S(f)^! \Lambda_{\mathrm{Ins}_S(Y)} \xrightarrow{\mathrm{Ins}_S(f)^! \operatorname{cc}_{Y/S}(A)} \mathrm{Ins}_S(f)^! K_{\mathrm{Ins}_S Y/S} \xrightarrow{\sim} K_{\mathrm{Ins}_S X/S} \tag{4.5.5}$$

is $\operatorname{cc}_{X/S}(f^*A)$.

Proof. By considering the diagram

$$\begin{tikzcd}
 & & & \mathrm{In}_S(Y) \arrow[dr] \arrow[dddd, bend left] & \\
\mathrm{In}_S(X) \arrow[r] \arrow[d] & \mathrm{In}_S(Y) \times_Y X \arrow[r] \arrow[d] \arrow[urr] & X \arrow[rr] \arrow[d] & & Y \arrow[dd] \\
X \arrow[r] & X \times_Y X \arrow[r] \arrow[d] & X \times_S X \arrow[r] \arrow[d] & X \arrow[dd] & \\
 & X \arrow[r] \arrow[drr] & X \times_S Y \arrow[rr] & & Y \times_S Y, \\
 & & & Y \arrow[ur] &
\end{tikzcd}$$

we see that the morphism (4.5.5) is computed by the cohomological correspondence

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & & X \arrow[dl] \arrow[dr] & \\
(X, \Lambda_X) & & (X \times_S X, f^! A^\vee \boxtimes f^* A) & & (X, K_{X/S}),
\end{tikzcd}$$

where the left half is obtained by applying f^* to coev_Y and then applying smooth base change, and the right half is obtained by applying $f^!$ to ev_Y and then applying smooth base change. It is then clear that the right half of the correspondence is ev_X. The following lemma verifies that left half is coev_X. ∎

Lemma 4.5.6. *Consider the coevaluation maps*

$$\mathrm{coev}\colon \Lambda \to \Delta^!_{Y/S}(A^\vee \boxtimes A),$$
$$\widetilde{\mathrm{coev}}\colon \Lambda \to \Delta^!_{X/S}(f^! A^\vee \boxtimes f^* A).$$

The following diagram commutes.

$$\begin{tikzcd}
 & \Lambda \arrow[dl, "\widetilde{\mathrm{coev}}"'] \arrow[dr, "f^*\mathrm{coev}"] & \\
\Delta^!_{X/S}(f^! A^\vee \boxtimes f^* A) \arrow[r, "\sim"] & \Delta^!_{X/S}(f \times \mathrm{id})^!(\mathrm{id} \times f)^*(A^\vee \boxtimes A) \arrow[r, "\sim"] & f^* \Delta^!_{Y/S}(A^\vee \boxtimes A).
\end{tikzcd}$$

Here, the lower right arrow is smooth base change.

Proof. This follows from the commutativity of the diagram

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[d, "\sim"] & \underline{\mathrm{RHom}}(f^*A, f^*A) \arrow[r, "\sim"] \arrow[d, "\sim"] & \Delta^!_{X/S}\underline{\mathrm{RHom}}(\tilde{p}_1^* f^* A, \tilde{p}_2^! f^* A) \arrow[r, "\sim"] \arrow[d, "\sim"] & \Delta^!_{X/S}(f^! A^\vee \boxtimes f^* A) \arrow[d, "\sim"] \\
\Lambda \arrow[r] & f^*\underline{\mathrm{RHom}}(A, A) \arrow[r, "\sim"] & f^*\Delta^!_{Y/S}\underline{\mathrm{RHom}}(p_1^* A, p_2^! A) \arrow[r, "\sim"] & f^*\Delta^!_{Y/S}(A^\vee \boxtimes A).
\end{tikzcd}$$

■

In (4.5.4), the only map that is not an isomorphism is base change along Δ_f. The following proposition will allow us to compute the base change explicitly in many cases of interest.

Proposition 4.5.7. *Let $G = M \ltimes N$ be a semi-direct product of group diamonds over S, with N cohomologically smooth. Let M_{reg} be an open subdiamond of M, and let $G_{\mathrm{reg}} = M_{\mathrm{reg}}N$. Suppose that the morphism $c\colon [G_{\mathrm{reg}}//M] \to [G_{\mathrm{reg}}//M]$ sending $mn \mapsto n^{-1}mn$ is an isomorphism.*

Let $f\colon [S/M] \to [S/G]$ denote the map induced by the inclusion $M \to G$. In the notation of Proposition 4.5.2, we can identify $\mathrm{In}_S(Y)$ with $[G//G]$, $\mathrm{In}_S(Y) \times_Y X$ with $[G//M]$, and $\mathrm{In}_S(X)$ with $[M//M]$.

The restriction of (4.5.4) to $[M_{\mathrm{reg}}//M]$ is given by

$$\widetilde{\Delta}^*_{f,\mathrm{reg}} \tilde{f}^*_{\mathrm{reg}} \Lambda \xrightarrow{\sim} \widetilde{\Delta}^!_{f,\mathrm{reg}} \tilde{f}^!_{\mathrm{reg}} \Lambda \xrightarrow{\sim} \widetilde{\Delta}^!_{f,\mathrm{reg}} \widehat{\pi}^*_{2,\mathrm{reg}} p_2^* f^! \Lambda \\ \xrightarrow{\sim} \widetilde{\Delta}^!_{f,\mathrm{reg}} c_* c^* \widehat{\pi}^*_{2,\mathrm{reg}} p_2^* f^! \Lambda \xrightarrow{\sim} \widetilde{\Delta}^!_{f,\mathrm{reg}} \tilde{f}^!_{\mathrm{reg}} \Lambda. \tag{4.5.8}$$

Here, the subscript $_{\mathrm{reg}}$ indicates restriction from $[M//M]$ to $[M_{\mathrm{reg}}//M]$, the first arrow uses the fact that $\tilde{f}_{\mathrm{reg}} \widetilde{\Delta}_{f,\mathrm{reg}}$ is an isomorphism, the second and fourth arrows are smooth base change, and the third arrow is induced by the natural transformation $\mathrm{id} \xrightarrow{\sim} c_* c^*$.

Proof. After restricting left column of the diagram (4.5.3) to $[G_{\mathrm{reg}}//G]$, we obtain the following diagram:

$$\begin{tikzcd}
{[M_{\mathrm{reg}}//M]} \arrow[r] \arrow[d] & {[S/M]} \arrow[d] & \\
{[G_{\mathrm{reg}}//M]} \arrow[r, "mn \mapsto n"] \arrow[ddd] & {[N//M]} \arrow[r] \arrow[d] & {[S/M]} \arrow[ddd] \\
& {[S/M]} \arrow[ddr] & \\
& & \\
{[G_{\mathrm{reg}}//G]} \arrow[rr] & & {[S/G].}
\end{tikzcd} \tag{4.5.9}$$

We claim that there is a commutative diagram

$$\begin{tikzcd}
{[G_{\mathrm{reg}}//M]} \arrow[r, "n^{-1}mn \mapsto n"] \arrow[dd] & {[N//M]} \arrow[r] \arrow[d] & {[S/M]} \arrow[dd] \\
& {[S/M]} \arrow[dr] & \\
{[G_{\mathrm{reg}}//G]} \arrow[rr] \arrow[ur] & & {[S/G],}
\end{tikzcd} \tag{4.5.10}$$

where the 2-morphisms filling in the outer and upper right squares match the ones in (4.5.9). Indeed, the 2-morphism in the upper right square is induced by the inclusion map $N \to G$, the 2-morphism in the large square is induced by the constant map $G_{\mathrm{reg}} \to G$ sending everything to the identity, we can choose the 2-morphism in the upper left square to be induced by the constant map $G_{\mathrm{reg}} \to M$ sending everything to the identity, and we can choose the 2-morphism in the bottom triangle to be induced by the map $G_{\mathrm{reg}} \to G$ sending $n^{-1}mn \mapsto n^{-1}$. The composition of the 2-morphisms filling in the smaller squares and triangle matches the 2-morphism filling in the outer square.

The two arrows $[G_{\mathrm{reg}}//M] \to [N//M]$ are related by composition with c. Then a diagram chase shows that (4.5.8) agrees with (4.5.4). ∎

Proposition 4.5.11. *Let G, M, G_{reg}, M_{reg} be as in Proposition 4.5.7. Let*

$$c' : \big[(M_{\mathrm{reg}} \times N)//G\big] \to \big[(M_{\mathrm{reg}} \times N)//G\big]$$

be given by $(m,n) \mapsto (m, m^{-1}n^{-1}mn)$. Let $f : [N//M] \to [S/M]$ be the obvious map.

In the notation of Proposition 4.5.2, we can identify $\mathrm{In}_S(Y)$ with $[M//M]$, $\mathrm{In}_S(Y) \times_Y X$ with $[(M \times N)//M]$, and $\mathrm{In}_S(X) \times_{\mathrm{In}_S(Y)} [M_{\mathrm{reg}}//M]$ with $[M_{\mathrm{reg}}//M]$.

The restriction of (4.5.4) to $[M_{\mathrm{reg}}//M]$ is given by a map analogous to (4.5.8), with c' in place of c.

Proof. After pulling back the left column of (4.5.3) along $[M_{\mathrm{reg}}//M] \to [M//M]$, the diagram becomes

$$\begin{tikzcd}
{[M_{\mathrm{reg}}//M]} \arrow[r] \arrow[d] & {[N//M]} \arrow[d] & \\
{\big[(M_{\mathrm{reg}}\times N)//M\big]} \arrow[r, "(m,n)\mapsto(m^{-1}nm,n)"] \arrow[dd] & {\big[(N\times N)//M\big]} \arrow[r, "(n_1,n_2)\mapsto n_2"] \arrow[d] & {[N//M]} \arrow[dd] \\
& {[N//M]} \arrow[dr] & \\
{[M_{\mathrm{reg}}//M]} \arrow[rr] & & {[S/M].}
\end{tikzcd} \tag{4.5.12}$$

The top left square can be extended to form the following diagram.

$$\begin{tikzcd}
{[M_{\mathrm{reg}}//M]} \arrow[r] \arrow[d] & {[N//M]} \arrow[r] \arrow[d] & {[S/M]} \arrow[d] \\
{\big[(M_{\mathrm{reg}}\times N)//M\big]} \arrow[r, "(m,n)\mapsto(m^{-1}nm,n)"] & {\big[(N\times N)//M\big]} \arrow[r, "(n_1,n_2)\mapsto n_1^{-1}n_2"] & {[N//M].}
\end{tikzcd} \tag{4.5.13}$$

The two arrows $[(M_{\mathrm{reg}} \times N)//M] \to [N//M]$ are related by composition with c'. Then a diagram chase shows that the analogue of (4.5.8) agrees with (4.5.4). ∎

The remainder of this section is devoted to applications of Propositions 4.5.7 and 4.5.11. For these applications, let $S = \operatorname{Spd}\overline{\mathbb{F}}_p$ or $\operatorname{Spd} C$ for an algebraically closed nonarchimedean field C.

Corollary 4.5.14. *Let $f\colon [S/\underline{G_b(F)}] \to [S/\widetilde{G}_b]$ be the map induced by the inclusion $\underline{G_b(F)} \hookrightarrow \widetilde{G}_b$. Let A be a dualizable object in $D_{\text{ét}}([S/\underline{G_b(F)}], \Lambda)$. Then under the isomorphism of Proposition* 4.3.1 (1),

$$\mathrm{cc}_{[S/\underline{G_b(F)}]/S}(A)_{\mathrm{sr}} = |D_b^-|^{-1}\,\mathrm{cc}_{[S/\widetilde{G}_b]/S}(f^*A)_{\mathrm{sr}}.$$

Proof. We will apply Proposition 4.5.7, with $G = \widetilde{G}_b$, $M = \underline{G_b(F)}$, $M_{\mathrm{reg}} = \underline{G_b(F)}_{\mathrm{sr}}$, $N = \widetilde{G}_b^{>0}$. Using Lemma 4.4.7 and smooth base change with respect to the cartesian squares

$$\begin{array}{ccccc} N & \longleftarrow & G_{\mathrm{reg}} & \longrightarrow & [G_{\mathrm{reg}}/\!/M] \\ \downarrow & & \downarrow & & \downarrow \\ S & \longleftarrow & M_{\mathrm{reg}} & \longrightarrow & [G_{\mathrm{reg}}/\!/G], \end{array} \tag{4.5.15}$$

we see that $\tilde{f}_{\mathrm{reg}!}\hat{\pi}^*_{2,\mathrm{reg}}p_2^*f^!\Lambda$ is locally free of rank one, and c acts on it by $|D_b^-|^{-1}$. In other words, if we compose the map

$$\tilde{f}_{\mathrm{reg}!}\hat{\pi}^*_{2,\mathrm{reg}}p_2^*f^!\Lambda \to \tilde{f}_{\mathrm{reg}!}c_*c^*\hat{\pi}^*_{2,\mathrm{reg}}p_2^*f^!\Lambda$$

induced by $\mathrm{id} \to c_*c^*$ with the map

$$\tilde{f}_{\mathrm{reg}!}c_*c^*\hat{\pi}^*_{2,\mathrm{reg}}p_2^*f^!\Lambda \to \tilde{f}_{\mathrm{reg}!}\hat{\pi}^*_{2,\mathrm{reg}}p_2^*f^!\Lambda$$

induced by $\tilde{f}_{\mathrm{reg}!}c_* \xrightarrow{\sim} \tilde{f}_{\mathrm{reg}}$, $c^*\hat{\pi}^*p_2^* \xrightarrow{\sim} \hat{\pi}^*p_2^*$, the result is multiplication by $|D_b^-|^{-1}$.

We claim that the map

$$\hat{\pi}^*_{2,\mathrm{reg}}p_2^*f^!\Lambda \to \tilde{f}^!_{\mathrm{reg}}\tilde{f}_{\mathrm{reg}!}\hat{\pi}^*_{2,\mathrm{reg}}p_2^*f^!\Lambda \tag{4.5.16}$$

induced by $\mathrm{id} \to \tilde{f}^!_{\mathrm{reg}}\tilde{f}_{\mathrm{reg}!}$ is an isomorphism. Indeed, by smooth base change, it is enough to check that $f^!\Lambda \to f^!f_!f^!\Lambda$ is an isomorphism. This map has a left inverse $f^!f_!f^!\Lambda \to f^!\Lambda$ induced by the natural transformation $f_!f^! \to \mathrm{id}$. By repeated applications of [7, Proposition 4.8 (i)] and smooth base change, $f_!f^!\Lambda \to \Lambda$ is an isomorphism, so

$$f^!f_!f^!\Lambda \to f^!\Lambda$$

is an isomorphism as well.

Using the isomorphism (4.5.16) along with the fact that $\tilde{f}_{\mathrm{reg}}\widetilde{\Delta}_{f,\mathrm{reg}}$ is an isomorphism, we see that c also acts by multiplication by $|D_b^-|^{-1}$ on $\widetilde{\Delta}^!_{f,\mathrm{reg}}\hat{\pi}^*_{2,\mathrm{reg}}p_2^*f^!\Lambda$. Now applying Proposition 4.5.7 yields the desired result. ∎

Proposition 4.5.17. *Let ρ be a representation of $G_b(F)$, and let $i_b\colon [S/\underline{G_b(F)}] \to \mathrm{Bun}_{G,S}$ be the usual map.*

(1) *Under the isomorphism of Proposition* 4.3.1 (1),

$$|D_b^-|^{-1}\,\mathrm{cc}_{\mathrm{Bun}_{G,S}/S}(i_{b*}\rho)|_{\mathrm{In}_S(\mathrm{Bun}^b_{G,S})_{\mathrm{sr}}} = \mathrm{cc}_{[S/\underline{G_b(F)}]/S}(\rho)_{\mathrm{sr}}.$$

(2) *If $b' \in B(G)$ is not a specialization of b, then*

$$\mathrm{cc}_{\mathrm{Bun}_{G,S}/S}(i_{b*}\rho)|_{\mathrm{In}_S(\mathrm{Bun}^{b'}_{G,S})_{\mathrm{sr}}} = 0.$$

Proof. The first claim follows from Corollary 4.5.14 and Proposition 4.1.5.

The to prove the second claim, observe that i_b is a composite of the open immersion $\mathrm{Bun}^b_{G,S} \to \overline{\mathrm{Bun}^b_{G,S}}$ and the closed immersion $\overline{\mathrm{Bun}^b_{G,S}} \to \mathrm{Bun}_{G,S}$. So this claim follows from Propositions 4.1.4 and 4.1.5. ■

If b' is not a specialization of b, then the situation is more complicated, as we will see in Example 4.5.20.

Proposition 4.5.18. *Let $f\colon \mathcal{M}_{b,S} \to \mathrm{Bun}_{G,S}$ be the chart defined in* [2, Definition V.3.2], *and let A be a dualizable object in $D_{\text{ét}}(\mathrm{Bun}_{G,S}, \Lambda)$. The base change of $\mathrm{In}_S(\mathcal{M}_{b,S}) \to \mathrm{In}_S(\mathrm{Bun}_{G,S})$ to $\mathrm{In}_S(\mathrm{Bun}^b_{G,S})_{\mathrm{sr}}$ is an isomorphism. Under this isomorphism,*

$$|D_b^-|^{-1}\,\mathrm{cc}_{\mathrm{Bun}_{G,S}/S}(A)_{b,\mathrm{sr}} = \mathrm{cc}_{\mathcal{M}_{b,S}/S}(f^*A)_{b,\mathrm{sr}}.$$

Proof. We can identify $(\widetilde{G}_b)_S$ with a substack of $\mathrm{Bun}_{G,S}$. By Theorem 3.2.4, $[(\widetilde{G}_b)_{\mathrm{sr}}//\widetilde{G}_b]_S$ is identified with an open substack of $\mathrm{In}_S(\mathrm{Bun}_G)$. Then same arguments as in Proposition 4.5.7 and Corollary 4.5.14 apply. We use smooth base change with respect to the cartesian square

$$\begin{tikzcd}
[S/\underline{G_b(F)}] \arrow[r] \arrow[d] & \mathcal{M}_{b,S} \arrow[d] \\
[S/\widetilde{G}_b] \arrow[r] & \mathrm{Bun}_{G,S}
\end{tikzcd}$$

to show that the calculations of Corollary 4.5.14 still apply in this setting. ■

Corollary 4.5.19. *In the situation of Proposition 4.5.11, suppose that $N = \mathcal{BC}(\mathcal{E})$, where $\mathcal{E}$ is as in Lemma 4.4.3. Then the map (4.5.4) is given by multiplication by the function $M \to F^\times$ sending $m \mapsto |\det(m-1)|^{-1}$. Here, $\det$ denotes the determinant of the action on $\mathcal{E}$, as defined in Definition 4.4.4.*

Proof. This follows from Lemma 4.4.5 and Proposition 4.5.11 in the same way that Corollary 4.5.14 follows from Lemma 4.4.7 and Proposition 4.5.7. ■

Example 4.5.20. Let $G = \mathrm{GL}_2$. Let $b, b' \in B(G)$ correspond to the vector bundles $\mathcal{O}(1/2)$ and $\mathcal{O} \oplus \mathcal{O}(1)$, respectively, on the Fargues–Fontaine curve. Then $G_b(F)$ is isomorphic to the group of units in a nontrivial quaternion algebra over F, and $G_{b'}(F)$ is isomorphic to $(F^\times)^2$.

Let $i_{b*}\colon [S/\underline{G_b(F)}] \to \mathrm{Bun}_{G,S}$ denote the inclusion map. We will use Proposition 4.5.18 and Corollary 4.5.19 to compute the strongly regular b'-part of the characteristic class of $i_{b*}\Lambda$.

Let $\pi_{b'}\colon \mathcal{M}_{b',S} \to \mathrm{Bun}_{G,S}$ be the chart defined in [2, Definition V.3.2]. By Proposition 4.5.18,

$$\mathrm{cc}_{\mathrm{Bun}_{G,S}/S}(i_{b*}\Lambda)_{b',\mathrm{sr}} = |1 - t_1/t_2|\,\mathrm{cc}_{\mathcal{M}_{b',S}}(\pi_{b'}^* i_{b*}\Lambda)_{b',\mathrm{sr}}.$$

Here, t_1, t_2 are coordinates on $G_{b'}(F) \cong (F^\times)^2$. There is an exact triangle

$$\Lambda \to f^* i_{b*}\Lambda \to j_!\Lambda[-1],$$

where $j\colon [S/\underline{G_b(F)}] \hookrightarrow \mathcal{M}_{b',S}$ is the inclusion of the closed stratum of $\mathcal{M}_{b',S}$ in $\mathcal{M}_{b',S}$. So

$$\mathrm{cc}_{\mathcal{M}_{b',S}}(\pi_{b'}^* i_{b*}\Lambda) = \mathrm{cc}_{\mathcal{M}_{b',S}}(\Lambda) - \mathrm{cc}_{\mathcal{M}_{b',S}}(j_!\Lambda).$$

By Proposition 4.1.5,

$$\mathrm{cc}_{\mathcal{M}_{b'}}(j_!\Lambda) = \mathrm{In}(j)_!\,\mathrm{cc}_{[S/G_{b'}(F)]}(\Lambda).$$

The characteristic class $\mathrm{cc}_{[S/\underline{G_{b'}(F)}]/S}(\Lambda)$ is the distribution that has Harish-Chandra character 1. We will abuse notation slightly and write

$$\mathrm{cc}_{[S/\underline{G_{b'}(F)}]/S}(\Lambda) = 1.$$

Recall that $\mathcal{M}_{b',S}$ is isomorphic to $(\mathcal{BC}(\mathcal{O}(-1))/\underline{G_{b'}(F)})_S$. By Proposition 4.5.11 applied to $\mathcal{M}_{b',S} \to [S/\underline{G_b(F)}]$,

$$\mathrm{cc}_{\mathcal{M}_{b',S}/S}(\Lambda)_{b',\mathrm{sr}} = |1 - t_2/t_1|\,\mathrm{cc}_{[S/\underline{G_b(F)}]/S}(\Lambda)_{b',\mathrm{sr}} = |1 - t_2/t_1|.$$

Putting everything together, we obtain

$$\mathrm{cc}_{\mathrm{Bun}_{G,S}/S}(i_{b*\Lambda})_{b',\mathrm{sr}} = |1 - t_1/t_2||1 - t_2/t_1| - |1 - t_1/t_2|.$$

In Example 5.3.5, we will use this calculation to compute the image of the compactly supported cohomology of the Drinfeld upper half plane in $\mathrm{Groth}(\mathrm{GL}_2(F))$.

Example 4.5.21. Let G be a connected reductive group over F, let $b \in B(G)$ be basic, and let $b' \in B(G)$ be an element whose only generalization is b. By a similar calculation to Example 4.5.20, $\mathrm{cc}_{\mathrm{Bun}_G/S}(i_{b*}\Lambda)_{b',\mathrm{sr}}$ has Harish-Chandra character

$$\prod_{\alpha\in\Phi^+\cup\Phi^-} \big|1 - \alpha(t)\big| - \prod_{\alpha\in\Phi^-} \big|1 - \alpha(t)\big|.$$

Here, Φ^+, Φ^- denote the set of roots of G that are positive (resp. negative) with respect to the parabolic P_b^+.

These examples rely on the fact that $\pi_{b'}^* i_{b*}\Lambda$ is an extension of a constant sheaf by a shift of a sheaf supported at a single point. To go further, it will be necessary to understand the characteristic classes of more complicated constructible sheaves on $\mathcal{M}_{b'}$.

5. Hecke correspondences and trace distributions

Now, we will use Theorem 3.2.4, Proposition 2.1.4, and Proposition 4.5.18 to generalize [8, Section 6]. We will recall the setup of [8] and explain how to generalize it.

5.1. Local shtuka spaces and Hecke correspondences

Let F be a finite extension of $\mathbb{Q}_p$. Let G be a connected reductive group over F, let $b \in B(G)$, and let μ be a conjugacy class of cocharacters $\mathbb{G}_{m,\overline{F}} \to G_{\overline{F}}$. Let $D(G_b(F), \overline{\mathbb{Q}}_\ell)$ denote the derived category of smooth $\overline{\mathbb{Q}}_\ell$-representations of $G_b(F)$. We will now recall the functor

$$\rho \mapsto R\Gamma(G, b, \mu)[\rho]$$

from $D(G_b(F), \overline{\mathbb{Q}}_\ell)$ to $D(G(F), \overline{\mathbb{Q}}_\ell)$ that was mentioned in the introduction. We also recall its relation to the Hecke operators defined in [2, Section IX].

Hecke operators are defined using the diagram of v-stacks

$$\begin{tikzcd} & \mathrm{Hecke} \arrow[r] \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \mathrm{Hecke}^{\mathrm{loc}} \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Div}^1 . \end{tikzcd}$$

Here, $\mathrm{Div}^1 = (\mathrm{Spd}\, \breve{F})/\,\mathrm{Frob}^{\mathbb{Z}}$ is the diamond classifying degree 1 Cartier divisors on the Fargues–Fontaine curve, Hecke is the stack classifying modifications of vector bundles on the Fargues–Fontaine curve that are isomorphisms away from a single point of Div^1, and $\mathrm{Hecke}^{\mathrm{loc}}$ is the local Hecke stack $[L^+G \setminus LG/L^+G]$.

The stack $\mathrm{Hecke}^{\mathrm{loc}}$ has a stratification by Schubert cells. For any conjugacy class of cocharacters μ of G, there is a diagram

$$\begin{tikzcd} & \mathrm{Hecke}_{\leq\mu} \arrow[r] \arrow[dl, "h_{1,\leq\mu}"'] \arrow[dr, "h_{2,\leq\mu}"] & \mathrm{Hecke}^{\mathrm{loc}}_{\leq\mu} \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times \mathrm{Div}^1 . \end{tikzcd}$$

The maps $h_{1,\leq\mu}$ and $h_{2,\leq\mu}$ are proper [2, Section I.2].

Let C be the completion of the algebraic closure of $\overline{F}$. The local shtuka space $\mathrm{Sht}_{G,b,\mu,C}$ is the fiber product

$$\mathrm{Hecke}_{\leq\mu} \times_{\mathrm{Bun}_G \times \mathrm{Bun}_G \times \mathrm{Div}^1} \mathrm{Spd}\, C,$$

where the map from $\mathrm{Spd}\, C$ to the first copy of Bun_G corresponds to the G-bundle $\mathcal{E}_{b,C}$, the map from $\mathrm{Spd}\, C$ to the second copy of Bun_G corresponds to the trivial G-bundle $\mathcal{E}_{1,C}$, and the map $\mathrm{Spd}\, C \to \mathrm{Div}^1$ is induced by the inclusion $\breve{F} \hookrightarrow C$.

The space $\mathrm{Sht}_{G,b,\mu,C}$ is a locally spatial diamond. It admits an action of $G(F) \times \widetilde{G}_b(C)$, and in particular of the subgroup $G(F) \times G_b(F)$. For any compact open subgroup $K \subset G(F)$, the quotient $\mathrm{Sht}_{G,b,\mu,K,C} = \mathrm{Sht}_{G,b,\mu,C}/K$ is also a locally spatial diamond [15, Section 23].

Choose a prime $\ell \neq p$. For each nonnegative integer n, the geometric Satake equivalence [2, Section VI] associates with μ an object $\mathcal{S}_{\mu,n}$ in $D_{\text{ét}}(\text{Hecke}^{\text{loc}}, \mathbb{Z}/\ell^n\mathbb{Z})$. (One sometimes uses a different normalization that is defined over $(\mathbb{Z}/\ell^n\mathbb{Z})[q^{1/2}]$, where q is the size of the residue field of F, in order to trivialize a cyclotomic twist in the Weil group action. But since we are ignoring the Weil group action, we use a normalization that is always defined over $\mathbb{Z}/\ell^n\mathbb{Z}$.) We will also write $\mathcal{S}_{\mu,n}$ for the pullback of this object to $\text{Sht}_{G,b,\mu,K,C}$. Define

$$R\Gamma_c(\text{Sht}_{G,b,\mu,K,C}, \mathcal{S}_\mu) = \varinjlim_U \varprojlim_n R\Gamma_c(U, \mathcal{S}_{\mu,n}),$$

where U runs over quasicompact open subdiamonds of $\text{Sht}_{G,b,\mu,K,C}$. For any object ρ of $D(G_b(F), \overline{\mathbb{Q}}_\ell)$, define

$$R\Gamma(G,b,\mu)[\rho] = \varinjlim_{K\subset G(F)} \text{RHom}_{G_b(F)}\big(R\Gamma_c(\text{Sht}_{G,b,\mu,K,C}, \mathcal{S}_\mu) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell, \rho\big).$$

It is an object of $D(G(F), \overline{\mathbb{Q}}_\ell)$. By [8, Proposition 6.4.5], if ρ is admissible (resp. of finite length), then $R\Gamma(G,b,\mu)[\rho]$ is as well.

For any $\mathbb{Z}_\ell$-algebra Λ, there is a Hecke operator

$$T_\mu \colon D_{\text{lis}}(\text{Bun}_G, \Lambda) \to D_{\text{lis}}(\text{Bun}_G, \Lambda)^{BW_E}.$$

We will also write T_μ for the induced functor $D_{\text{lis}}(\text{Bun}_{G,C}, \Lambda) \to D_{\text{lis}}(\text{Bun}_{G,C}, \Lambda)$.

Proposition 5.1.1 ([8, Proposition 6.4.5]). *For any triple G, b, μ, and any object ρ of $D(G_b(F), \overline{\mathbb{Q}}_\ell)$,*

$$R\Gamma(G,b,\mu)[\rho] \cong i_1^* T_{-\mu} i_{b*}\rho.$$

Here, i_b denotes the composite

$$\big[\text{Spd}\, C/\underline{G_b(F)}\big] \to [\text{Spd}\, C/\widetilde{G}_b] \xrightarrow{\sim} \text{Bun}_{G,C}^b \to \text{Bun}_{G,C},$$

and i_1 is defined similarly.

As in [8], we will deduce information about the Hecke operators for $\Lambda = \overline{\mathbb{Q}}_\ell$ from the Hecke operators for $\Lambda = \overline{\mathbb{Z}}_\ell/\ell^n$. If Λ is killed by a power of ℓ, then we can identify $D_{\text{lis}}(\text{Bun}_G, \Lambda)$ with $D_{\text{ét}}(\text{Bun}_G, \Lambda)$ and $D_{\text{lis}}(\text{Bun}_G, \Lambda)^{BW_E}$ with $D_{\text{ét}}(\text{Bun}_G \times [*/\underline{W_E}], \Lambda)$. The latter can further be identified with a full subcategory of $D_{\text{ét}}(\text{Bun}_G \times \text{Div}^1, \Lambda)$. The induced functor $D_{\text{ét}}(\text{Bun}_G, \Lambda) \to D_{\text{ét}}(\text{Bun}_G \times \text{Div}^1, \Lambda)$ is given by

$$A \mapsto h_{2*}(h_1^* A \otimes \mathcal{S}_{\mu,n}) = h_{2,\leq\mu*}(h_{1,\leq\mu}^* A \otimes \mathcal{S}_{\mu,n}),$$

where n is chosen so that ℓ^n kills Λ.

The problem of computing the trace distribution is therefore reduced to the problem of determining how the characteristic classes interact with the operations appearing in Proposition 5.1.1. The analysis can be summarized as follows.

- Proposition 4.5.17 computes the restriction of $\mathrm{cc}_{\mathrm{In}_C(\mathrm{Bun}_{G,C})/C}(i_{b*}\rho)$ to some of the strata of $\mathrm{In}_C(\mathrm{Bun}_{G,C})_{\mathrm{sr}}$. Our lack of knowledge about what happens on the remaining strata is the reason that we are unable to extend Theorem 1.2 to the entire strongly regular locus.
- The pullback h_1^* and tensor product with $\mathcal{S}_\mu$ are treated in [8, Proposition 6.4.8]. We make no changes to this part of their argument.
- Proposition 4.1.5 reduces the problem of analyzing the pushforward h_{2*} to the problem of characterizing the fibers of h_2. This analysis is performed in Section 5.2.
- Proposition 4.1.4 handles the pullback i_1^*.

Section 5.3 combines all of the ingredients.

5.2. Fixed points in Hecke correspondences

Let $\mathrm{Hecke}^{b,*}$ denote preimage of Bun_G^b in Hecke under h_1, and let $\mathrm{Hecke}^{*,1}$ denote the preimage of $\mathrm{Bun}_G^b \times \mathrm{Div}^1$ in Hecke under h_2. Let $\mathrm{Hecke}^{b,1}$ denote the intersection of $\mathrm{Hecke}^{b,*}$ and $\mathrm{Hecke}^{*,1}$

The stack $\mathrm{In}(\mathrm{Hecke}^{b,1})$ parametrizes triples (g, g', ϕ), where $g \in G(F)$, g' is a point of $\widetilde{G}_b$, and $\phi\colon \mathcal{E}_1 \dashrightarrow \mathcal{E}_b$ is a modification, up to equivalence. Two triples (g, g', ϕ) and (h, h', ψ) are considered equivalent if there exist $\gamma \in G(F)$ and a point γ' of $\widetilde{G}_b$ such that $h = \gamma g \gamma^{-1}$, $g' = \gamma' h' \gamma'^{-1}$, $\psi = \gamma' \phi \gamma^{-1}$.

Suppose we have a triple (g, g', ϕ) such that g is strongly regular, with centralizer T. Then by Proposition 2.1.4, the modification ϕ must come from a modification of T-bundles. Let $\mathrm{inv}[b](g, g') \in B(T)$ denote the isomorphism class of the T-bundle associated with $\mathcal{E}_b$.

For any torus T over F, let $X_*(T)$ denote its cocharacter lattice. Both $B(T)$ and $X_*(T)$ have the structure of an abelian group. There is also an action of $\Gamma = \mathrm{Gal}(\overline{F}/F)$ on $X_*(T)$.

Lemma 5.2.1. *The following statements hold:*

(1) *Any homomorphism $\beta_{\mathbb{G}_m}\colon X_*(\mathbb{G}_m) \to B(\mathbb{G}_m)$ extends uniquely to a natural transformation $\beta\colon X_*(-)_\Gamma \to B(-)$ of functors from the category of tori over F to the category of abelian groups. If $\beta_{\mathbb{G}_m}$ is an isomorphism, then β is also an isomorphism.*

(2) *Let $\beta\colon X_*(-)_\Gamma \to B(-)$ be the natural transformation such that $\beta_{\mathbb{G}_m}$ sends the identity cocharacter of $X_*(\mathbb{G}_m)$ to the class of a uniformizer of $\breve{F}$ in $B(\mathbb{G}_m)$. Let T be a torus over F. Then there is a modification of T-bundles $\mathcal{E}_1 \dashrightarrow \mathcal{E}_b$ with parameter μ if and only if $\beta_T(\mu) = b$.*

Proof. The first item follows from [13, Lemma 2.2 and Proposition 2.3].

The rule that sends a cocharacter μ of T to the isomorphism class of the T-bundle obtained by modifying the trivial T-bundle by μ is a natural transformation $X_*(-) \to B(-)$. Since the action of Γ on the Fargues–Fontaine curve $X_{F,C}$ does not change isomorphism classes of vector bundles, this natural transformation factors through $X_*(-)_\Gamma \to B(-)$. The characterization of the map $X_*(\mathbb{G}_m) \to B(\mathbb{G}_m)$ follows from [1, Section 8.2.1.1]. ∎

Definition 5.2.2. Let Rel_b be the set of equivalence classes of triples (g, g', λ), where $g \in G(F)_{\mathrm{sr}}$, $g' \in G_b(F)_{\mathrm{sr}}$ are stably conjugate, and $\lambda \in X_*(T)$ has the property that its image in $X_*(T)_\Gamma \cong B(T)$ is the same as $\mathrm{inv}[b](g, g')$. Here, T is the centralizer of g in G. Two triples (g_1, g'_1, λ_1) and (g_2, g'_2, λ_2) are considered equivalent if there exist $h \in G(F)$, $h' \in G_b(F)$ satisfying $(g_2, g'_2, \lambda_2) = (hg_1h^{-1}, h'g'_1h'^{-1}, h\lambda_1h^{-1})$.

For any conjugacy class of cocharacters μ of G, let $\mathrm{Rel}_{b,\mu}$ be the subset of Rel_b consisting of triples (g, g', λ) for which $\lambda \le \mu$.

The above definition is slightly different from [8, Definition 3.2.4], which only required that λ and $\mathrm{inv}[b](g, g')$ have the same image in $\pi_1(G)_\Gamma$. The two definitions agree on the elliptic locus, by [8, Theorem 6.2.3].

The following extends [8, Corollary 6.2.4].

Lemma 5.2.3. *The inclusions*

$$\mathrm{In}_C(\mathrm{Hecke}^{b,*}_{G,\le\mu,C})_{\mathrm{sr}} \hookleftarrow \mathrm{In}_C(\mathrm{Hecke}^{b,1}_{G,\le\mu,C})_{\mathrm{sr}} \hookrightarrow \mathrm{In}_C(\mathrm{Hecke}^{*,1}_{G,\le\mu,C})_{\mathrm{sr}}$$

are open and closed immersions.

There is a homeomorphism

$$\left|\mathrm{In}_C(\mathrm{Hecke}^{b,1}_{G,\le\mu,C})_{\mathrm{sr}}\right| \cong \mathrm{Rel}_{b,\mu}\,.$$

Proof. The first claim follows from Theorem 3.2.4, and the second follows from Proposition 2.1.4 and Lemma 5.2.1. ∎

5.3. Computing trace distributions

Lemma 5.3.1. *Let T be a maximal torus of G. Then the map*

$$\left[T(F)_{\mathrm{sr}}//T(F)\right] \to \left[G(F)_{\mathrm{sr}}//G(F)\right]$$

induced by the inclusion $T \hookrightarrow G$ identifies $[T(F)_{\mathrm{sr}}//T(F)]$ with an open and closed substack of $[G(F)_{\mathrm{sr}}//G(F)]$.

Hence for any $\mathbb{Z}_\ell$-algebra Λ, $\mathrm{HaarDist}(T(F)_{\mathrm{sr}}, \Lambda)$ is naturally identified with a direct summand of $\mathrm{HaarDist}(G(F)_{\mathrm{sr}}, \Lambda)^{G(F)}$.

Proof. The map $T(F)_{\mathrm{sr}} \times G(F) \to G(F)$ sending $(t, g) \mapsto gtg^{-1}$ is a smooth map of locally F-analytic manifolds. By [16, Section II.III.10], the image of the map is open. Since the centralizer in $G(F)$ of any element of $T(F)_{\mathrm{sr}}$ is $T(F)$, it follows that $[T(F)_{\mathrm{sr}}//T(F)]$ can be identified with an open substack of $[G(F)_{\mathrm{sr}}//G(F)]$. The complement of the image can be covered with inertia stacks of other tori, which are open. So the image is also closed. ∎

Define a function

$$\mathcal{T}^{G_b\to G}_{b,\mu} : \mathrm{HaarDist}\left(G_b(F)_{\mathrm{sr}}, \Lambda\right)^{G_b(F)} \to \mathrm{HaarDist}\left(G(F)_{\mathrm{sr}}, \Lambda\right)^{G(F)}$$

as follows. Given $\phi \in \mathrm{HaarDist}(G(F)_{\mathrm{sr}}, \Lambda)^{G(F)}$ and a maximal torus T of G, denote the restriction of ϕ to $\mathrm{HaarDist}(T(F)_{\mathrm{sr}}, \Lambda)$ by $\phi|_T$. For any $\phi \in \mathrm{HaarDist}(G_b(F)_{\mathrm{sr}}, \Lambda)^{G_b(F)}$, define $\mathcal{T}_{b,\mu}^{G_b \to G}(\phi)$ such that for any $g \in G(F)_{\mathrm{sr}}$,

$$\mathcal{T}_{b,\mu}^{G_b \to G}(\phi)\big|_{Z_G(g)} = (-1)^{\langle \mu, 2\rho_G \rangle} \sum_{(g,g',\lambda) \in \mathrm{Rel}_b} \dim r_\mu[\lambda] \phi|_{Z_{G_b}(g')}. \tag{5.3.2}$$

Here, $Z_G(g)$ and $Z_{G_b}(g')$ denote the centralizers of g and g', respectively, ρ_G is half the sum of the positive roots of G, r_μ is the representation of the Langlands dual group $\widehat{G}$ with highest weight μ, and we use the fact that g and g' are stably conjugate to identify $Z_G(g)$ with $Z_{G_b}(g')$.

The following extends [8, Proposition 6.4.7].

Proposition 5.3.3. *Let Λ be a $\mathbb{Z}_\ell$-algebra that is killed by a power of ℓ, and let ρ be an admissible representation of $G_b(F)$ with coefficients in λ. Then we have an equality*

$$\mathrm{tr.dist}\big(R\Gamma(G,b,\mu)[\rho]_{\mathrm{sr}}\big) = \sum_{b' \in \overline{\{b\}}} \mathcal{T}_{b',\mu}^{G_{b'} \to G} \, \mathrm{cc}_{\mathrm{Bun}_{G,C}/C}(i_{b*}\rho)_{b',\mathrm{sr}}$$

in $\mathrm{HaarDist}(G(F)_{\mathrm{sr}}, \Lambda)^{G(F)}$.

Proof. We will abbreviate Hecke_C as H, $\mathrm{Bun}_{G,C}$ as B, and In_C as In.

For $b' \in \overline{\{b\}}$, we consider the following diagram, which extends [8, (6.4.2)].

$$\begin{tikzcd}
\mathrm{In}(H^{b',1})_{\mathrm{sr}} \arrow[rr, "\tilde{i}_{b'}"] \arrow[dd, "\tilde{i}'_1"] & & \mathrm{In}(H^{*,1})_{\mathrm{sr}} \arrow[r, "\mathrm{In}(h_2^{*,1})_{\mathrm{sr}}"] \arrow[d, "j'_1"] & \mathrm{In}(B^1)_{\mathrm{sr}} \arrow[d, "j_1"] \\
& & \mathrm{In}(H^{*,1}) \arrow[r, "\mathrm{In}(h_2^{*,1})"] \arrow[d, "\mathrm{In}(i'_1)"] & \mathrm{In}(B^1) \arrow[d, "\mathrm{In}(i_1)"] \\
\mathrm{In}(H^{b',*})_{\mathrm{sr}} \arrow[r, "j'_{b'}"] \arrow[d, "\mathrm{In}(h_1^{b',*})_{\mathrm{sr}}"] & \mathrm{In}(H^{b',*}) \arrow[r, "\mathrm{In}(i'_{b'})"] \arrow[d, "\mathrm{In}(h_1^{b',*})"] & \mathrm{In}(H) \arrow[r, "\mathrm{In}(h_2)"] \arrow[d, "\mathrm{In}(h_1)"] & \mathrm{In}(B) \\
\mathrm{In}(B^{b'})_{\mathrm{sr}} \arrow[r, "j_{b'}"] & \mathrm{In}(B^{b'}) \arrow[r, "\mathrm{In}(i_{b'})"] & \mathrm{In}(B). &
\end{tikzcd}$$

As in the proof of [8, Proposition 6.4.7], we have

$$j_1^* \,\mathrm{cc}_{B^1/C}\left(i_* T_V^\vee (i_b)_* \rho\right) \cong \left(\mathrm{In}(h_2^{*,1})_{\mathrm{sr}}\right)_* (j'_1)^* \,\mathrm{In}(i'_1)^* \,\mathrm{cc}_{H/C}\left(h_1^*(i_b)_*\rho \otimes \mathcal{S}_{V^\vee}\right).$$

By Theorem 3.2.4,

$$\mathrm{In}(H)_{\mathrm{sr}} = \bigsqcup_{b' \in B(G)} \mathrm{In}(H^{b',*})_{\mathrm{sr}}.$$

The characteristic class of $(i_b)_*\rho$ is only supported on $\overline{\{b\}}$. Therefore,

$$\begin{aligned}(j_1')^* \operatorname{In}(i_1')^* \operatorname{cc}_{H/C}\big(h_1^*(i_b)_*\rho \otimes \mathcal{S}_{V^\vee}\big) \\ &= \sum_{b' \in \overline{\{b\}}} (j_1')^* \operatorname{In}(i_1')^* \operatorname{In}(i_{b'}')_*(j_{b'}')_*(j_{b'}')^* \operatorname{In}(i_{b'})^* \operatorname{cc}_{H/C}\big(h_1^*(i_b)_*\rho \otimes \mathcal{S}_{V^\vee}\big) \\ &= \sum_{b' \in \overline{\{b\}}} (\tilde{i}_{b'}')_*(\tilde{i}_1')^*(j_{b'}')^* \operatorname{In}(i_{b'}')^* \operatorname{cc}_{H/C}\big(h_1^*(i_b)_*\rho \otimes \mathcal{S}_{V^\vee}\big),\end{aligned}$$

where we used proper base change in the last line.

Again following the argument of loc. cit., we obtain

$$\begin{aligned}(\tilde{i}_{b'}')_*(\tilde{i}_1')^*(j_{b'}')^* \operatorname{In}(i_{b'}')^* \operatorname{cc}_{H/C}\big(h_1^*(i_b)_*\rho \otimes \mathcal{S}_{V^\vee}\big) \\ &= (\tilde{i}_{b'}')_*(\tilde{i}_1')^*\big(j_{b'}^* \operatorname{In}(i_{b'})^* \operatorname{cc}_{B/C}\big((i_b)_*\rho\big) \boxtimes_{\operatorname{In}(B_m^{\mathrm{loc}})_{\mathrm{sr}}} \operatorname{cc}_{H_m^{\mathrm{loc}}/C}(\mathcal{S}_{V^\vee})_{\mathrm{sr}}\big).\end{aligned}$$

Then we have a commutative diagram

$$\begin{tikzcd}
H^0\big(\operatorname{In}(B^{b'})_{\mathrm{sr}}, K_{\operatorname{In}(B^{b'})/C}\big) \arrow[r, "\sim"] \arrow[d, "-\boxtimes_{\operatorname{In}(B_m^{\mathrm{loc}})_{\mathrm{sr}}} \operatorname{cc}_{H_m^{\mathrm{loc}}/C}(\mathcal{S}_{V^\vee})_{\mathrm{sr}}"] & \operatorname{HaarDist}\big(G_{b'}(F)_{\mathrm{sr}}, \Lambda\big)^{G_{b'}(F)} \arrow[d, "q_1^*(-)\otimes(-1)^{\langle 2\rho_G,\mu\rangle \operatorname{rank} V^\vee}[-]"] \\
H^0\big(\operatorname{In}(H^{b',*})_{\mathrm{sr}}, K_{\operatorname{In}(H^{b',*})/C}\big) \arrow[r, "\sim"] \arrow[d, "(\tilde{i}_1')^*"] & \operatorname{HaarDist}\big(\pi_0\big(\operatorname{Fix}(\alpha_{b'})_{\mathrm{sr}}\big), \Lambda\big)^{G_{b'}(F)} \arrow[d, "(p_1)^*"] \\
H^0\big(\operatorname{In}(H^{b',1})_{\mathrm{sr}}, K_{\operatorname{In}(H^{b',1})/C}\big) \arrow[r, "\sim"] \arrow[d, "(\tilde{i}_{b'}')_*"] & \operatorname{HaarDist}\big(\pi_0\big(\operatorname{Fix}(\alpha_{\mathrm{Sht}})_{\mathrm{sr}}\big), \Lambda\big)^{G(F)\times G_{b'}(F)} \arrow[d, "(p_2)_*"] \\
H^0\big(\operatorname{In}(H^{*,1})_{\mathrm{sr}}, K_{\operatorname{In}(H^{*,1})/C}\big) \arrow[r, "\sim"] \arrow[d, "\operatorname{In}(h_2^{*,1})"] & \operatorname{HaarDist}\big(\operatorname{Fix}(\alpha_1)_{\mathrm{sr}}, \Lambda\big)^{G(F)} \arrow[d, "(q_2)_*"] \\
H^0\big(\operatorname{In}(B^1)_{\mathrm{sr}}, K_{\operatorname{In}(B^1)/C}\big) \arrow[r, "\sim"] & \operatorname{HaarDist}\big(G(F)_{\mathrm{sr}}, \Lambda\big)^{G(F)}.
\end{tikzcd}$$

With these modifications, the argument of [8] also works in our more general situation. ■

Finally, we arrive at our main result.

Theorem 5.3.4. *Let ρ be an admissible finite length representation of $G_b(F)$, and let $\pi = R\Gamma(G, b, \mu)[\rho]$. Let $g \in G(F)$. Suppose that for all specializations b' of b in $B(G, \mu)$, g is not stably conjugate to any element of $G_{b'}(F)$. Then*

$$\Theta_\pi(g) = (-1)^{\langle \mu, 2\rho_G\rangle} \sum_{(g,g',\lambda)\in \operatorname{Rel}_b} \dim r_\mu[\lambda]\Theta_\rho(g') \prod_{\alpha\in\Phi^+} \big|1-\alpha(\lambda)\big|^{-1}.$$

Here, ρ_G is half the sum of the positive roots of G, Φ^+ is the set of roots of G that are positive with respect to the parabolic P_b^+, and the norm $|\cdot|$ is defined so that $|p| = p^{-[F:\mathbb{Q}_p]}$.

Proof. This follows from Proposition 5.3.3 in the same way that [8, Theorem 6.5.2] follows from [8, Proposition 6.4.7]. There is a factor of

$$\prod_{\alpha\in\Phi^-} \left|1-\alpha(\lambda)\right|$$

coming from Proposition 4.5.17 and a factor of

$$\prod_{\alpha\in\Phi^+\cup\Phi^-} \left|1-\alpha(\lambda)\right|^{-1}$$

coming from the Weyl integration formula. Here, Φ^- is the set of roots of G that are negative with respect to P_b^+. ∎

In some cases (such as in Examples 4.5.20–4.5.21), we have more information about the characteristic class of $i_{b*}\rho$, and we can compute Θ_π for a larger set of g.

Example 5.3.5. Let $G = \mathrm{GL}_2$, let b correspond to the vector bundle $\mathcal{O}(1/2)$ on the Fargues–Fontaine curve, and let μ be minuscule. Let $\pi = R\Gamma(\mathrm{Sht}_{G,b,\mu})[\rho]$, where ρ is the trivial representation of $G_b(F)$. Then it is known that the image of π in $\mathrm{Groth}(\mathrm{GL}_2(F),\overline{\mathbb{Q}}_\ell)$ is $\mathrm{St}-1$, where St is the Steinberg representation and 1 is the trivial representation.

Hansen [6] suggested that it would be interesting to see how the above formula for π arises from the characteristic class framework. We will give a derivation, using Example 4.5.20 and Proposition 5.3.3. It suffices to show that π and $\mathrm{St}-1$ have the same Harish-Chandra character.

The Harish-Chandra character of the trivial representation is given by

$$\Theta_1(g) = 1.$$

The representation $\mathrm{St}\oplus 1$ is induced from the trivial representation of a Borel subgroup of $\mathrm{GL}_2(F)$. By van Dijk's formula [18, Theorem 3], if λ_1, λ_2 are the eigenvalues of g, then

$$\Theta_{\mathrm{St}}(g)+\Theta_1(g) = \begin{cases} |1-\lambda_1/\lambda_2|^{-1}+|1-\lambda_2/\lambda_1|^{-1}, & \lambda_1,\lambda_2\in F^\times,\\ 0, & \text{otherwise.}\end{cases}$$

(Here, we have defined the norm on F by $|p| = p^{-[F:\mathbb{Q}_p]}$.) So

$$\Theta_{\mathrm{St}}(g)-\Theta_1(g) = \begin{cases} -2+|1-\lambda_1/\lambda_2|^{-1}+|1-\lambda_2/\lambda_1|^{-1}, & \lambda_1,\lambda_2\in F^\times,\\ -2, & \text{otherwise.}\end{cases} \tag{5.3.6}$$

We need to show that $\Theta_\pi(g)$ equals the right-hand side of (5.3.6).

The set $B(G,\mu)$ contains two elements. One of these is b. Denote the other by b'; it corresponds to the vector bundle $\mathcal{O}\oplus\mathcal{O}(1)$.

Let g be a strongly regular element of $G(F)$. If the eigenvalues of g are not in $F^\times$, then g is stably conjugate to an element of $G_b(F)$, but not to an element of $G_{b'}(F)$. If g'

is stably conjugate to g, then $\Theta_\rho(g') = 1$. Then Theorem 5.3.4 implies that $\Theta_\pi(g) = -2$, in agreement with (5.3.6).

Now suppose g has eigenvalues $\lambda_1, \lambda_2 \in F^\times$. Then g is stably conjugate to the elements (λ_1, λ_2) and (λ_2, λ_1) of $G_{b'}(F)$, but not to any element of $G_b(F)$. By Example 4.5.20, the Harish-Chandra character of the b'-part of the characteristic class of ρ takes the values

$$|1-\lambda_1/\lambda_2||1-\lambda_2/\lambda_1| - |1-\lambda_1/\lambda_2|,$$
$$|1-\lambda_1/\lambda_2||1-\lambda_2/\lambda_1| - |1-\lambda_2/\lambda_1|$$

at (λ_1, λ_2) and (λ_2, λ_1), respectively. Now apply Proposition 5.3.3. After multiplying by the factor of $(-1)^{\langle \mu, 2\rho_G\rangle} = -1$ appearing in (5.3.2), and dividing by a factor of

$$|1-\lambda_1/\lambda_2||1-\lambda_2/\lambda_1|$$

coming from the Weyl integration formula, we obtain

$$\Theta_\pi(g) = -2 + |1-\lambda_1/\lambda_2|^{-1} + |1-\lambda_2/\lambda_1|^{-1},$$

in agreement with (5.3.6).

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Daniel R. Gulotta
Department of Mathematics, University of Utah, 155 S 1400 E, Salt Lake City, UT 84112, USA; dgulotta@alum.mit.edu
Author IDs: zbMATH gulotta.daniel-r MR 823263 ORCID 0000-0002-6825-0458