Under the hood of a carbon footprint calculator

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We explain the basic linear algebra formulas used in carbon footprints computations.

Have you computed your carbon footprint lately? Wondered where all those numbers came from? Two main methods for calculating carbon footprints are the *input-output* method and the *life-cycle analysis* method, also known as *cradle-to-grave*. Both aim to quantify the emissions of a product or service, but the methods are fundamentally different.

Input-output is an economic approach developed in [2], widely accepted since the 1960s, but only recently adapted to environmental reporting. It uses national or regional data on industries and their interdependencies. It relies on aggregated data about economic sectors (e.g., agriculture, manufacturing) and their emissions, and we shall see in this note (Definition 2), the formula estimating emissions associated with production, supply chains, and services. This is the commonly accepted method for large-scale, economy-wide assessments, is not data-intensive, and relies on publicly available economic data. It will lack detail on specific products and miss the emissions from smaller, niche activities, but offers the warranty that all emissions are taken into account (Proposition 1).

Life-cycle analysis estimates a specific product's environmental impact over its entire life cycle, from raw material extraction to disposal. However, the under or over counting of secondary emissions is only estimated, and the total complexity makes a consistency check impossible to actually carry out.

A concrete question. To estimate the carbon footprint of a T-shirt, for instance, one needs the footprint of the cotton all the way from production of the raw material up to the stitching of the T-shirt, the washing during its lifespan and the disposal of the unusable T-shirt. But to estimate the cotton footprint only, one needs the transport of the water, the production of the pesticides to protect the crop, the fertilizers, all those requiring tractors and fuel, the fuel itself requiring transport and fuel, thus creating loops that are difficult to extricate. And once having estimated all the emissions, how do we know that all the emissions of all the objects that we so compute, actually sum up to all the emissions that we actually produce? The input-output method takes the problem from the other end: it takes *all* the measured emissions, and then attributes them to different sectors, and then computes the carbon footprint of goods from the textile industry according to its direct emissions, combined with all the indirect emissions, as we now explain.

Basic input-output. This method models a closed economy with *n economical sectors*, $S = (S_1, \dots, S_n)$, and comes in the following form:

	S_1	S_2	…	S_n	D	T
S_1	c_{11}	c_{12}	…	c_{1n}	d_1	t_1
S_2	c_{21}	c_{22}	…	c_{2n}	d_2	t_2
…	…	…	…	…	…	…
S_n	c_{n1}	c_{n2}	…	c_{nn}	d_n	t_n
V	v_1	v_2	…	v_n		
T	t_1	t_2	…	t_n		

The $n \times n$ *transaction matrix* is $C = (c_{ij}) \in M_n(\mathbf{R}_+)$ (positive coefficients only), where the entry c_{ij} is the number representing the economical transactions from sector S_i to S_j. The diagonal entries c_{ii} then represent the internal gross revenue of the sector. The *demand vector* $D = (d_1, \dots, d_n)$ expresses the money spent by the population on each sector: d_i is the amount bought from sector S_i, for $i = 1, \dots, n$. The *added value* vector is given by $V = (v_1, \dots, v_n)$ and the *total* vector $T = (t_1, \dots, t_n)$ satisfies the relations

$$t_i = \sum_{j=1}^{n} c_{ij} + d_i = v_i + \sum_{j=1}^{n} c_{ji} \qquad \text{for } i = 1, \dots, n.$$

The value t_i represents the *total output* of the sector S_i, which is also the *total input* because the system is a closed circuit.

Definition 1. The *technical coefficients matrix* $A = (a_{ij})$ is given by

$$a_{ij} = \frac{c_{ij}}{t_j},$$

so that $A = CD_T^{-1}$ where D_T denotes the diagonal matrix with entries $t_1, \ldots, t_n$ on the diagonal.

Recall that the scalar product of two vectors $X = (x_1, \ldots, x_n)$ and $Y^t = (y_1, \ldots, y_n)$ is given by

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i = XY,$$

where X is a *row vector*, i.e., a $1 \times n$ matrix, and Y is a *column vector*, i.e., an $n \times 1$ matrix and XY is the usual matrix multiplication, so it is a real number.

Footprints. Economic activities have all sorts of footprints that are correlated to the income and the cost of things. To compute the footprint, we need to have an *emission* vector $E = (e_1, \ldots, e_n)$, where e_i stands for the direct emission of sector S_i. It could be millions of tons of CO_2, measured or estimated directly above the industries of the sector, or square feet measured by the occupancy of the industries, or tons of plastic trash estimated by the waste management authorities/actors, etc. Then $|E| = e_1 + \cdots + e_n$ is the total footprint of the economic system. The *direct intensity* is given by $F = (f_1, \ldots, f_n)$, with

$$f_i = \frac{e_i}{t_i},$$

where e_i is the ith component of the emission vector E, and t_i the one of the total vector T, that is the total output of the sector S_i. In other words, f_i is the print per unit of money. Now, if we compute the inner product

$$\langle D, F \rangle = \sum_{i=1}^{n} d_i f_i,$$

it represents the footprint directly attributed to the consumers, but it does not take into account all the intermediate exchanges that the sectors had in order to create the product. If you buy goods for k dollars to industry S_i, then kf_i indicates the direct footprint of the purchase, but not the total footprint of the goods. The latter has to take into account the exchanges the sector S_i had with the other ones.

Definition 2. The *total intensity vector* $X = (x_1, \ldots, x_n)$ giving the total footprint of the sector S_i by money amount of final consumer demand, is given by

$$X = F + FA + FA^2 + \cdots = F(I - A)^{-1},$$

where I denotes the $n \times n$ identity matrix and A the technical coefficients matrix from Definition 1.

Notice that since most matrices are invertible, we will assume that this one is.

Proposition 1. *The total footprint is completely attributed to the consumers. Namely, with the above notations,* $\langle X, D \rangle = |E|$.

Proof. First notice that the equations $t_i = \sum_{j=1}^{n} c_{ij} + d_i$ for $i = 1, \ldots, n$ can be recast as

$$D = T - C\mathbf{1} = T - AD_T\mathbf{1} = T - AT = (I - A)T,$$

where $\mathbf{1} = (1, \ldots, 1)$, T is the total vector and D is the demand vector, both viewed as column vectors. We compute:

$$\langle X, D \rangle = \langle F(I - A)^{-1}, (I - A)T \rangle = FT = \sum_{i=1}^{n} \frac{e_i}{t_i} t_i = |E|. \qquad \blacksquare$$

Waste. In a closed economy, in order to take into account waste disposal, one needs a sector S_i that runs the waste management. That sector has a large footprint since incinerators produce large quantities of CO_2 (of the order of 1 kg CO_2 per kg of waste incinerated[1]), and is eventually attributed to the consumers through waste management taxes (in France the weight of waste is estimated at half a ton per person and per year). The model also does not discuss demand and waste, assuming that everything is bought and nothing gets wasted.

Corporations. The carbon footprint inventory of a specific firm or corporation is separated into its direct emissions or *scope one*, and its indirect emissions. The latter are separated as *scope two*, consisting of the emissions from acquired electricity, heat, and cooling, and the rest is *scope three*, computed using databases of life cycle analysis.[2]

Inverses. If B is a matrix ε-close to A (say, in terms of the coefficients), then the distance between $(I - A)^{-1}$ and $(I - B)^{-1}$ could a priori be as big as we want, for instance if the matrices both have an eigenvalue close enough to 1. Since all the values are estimated, the model is probably not very stable in an optimization exercise.

Added value. The consumer footprint does not take into account the added value V, which is what the companies make, and the model attributes the footprint solely to the consumers, through the vector D. We could define the *total systemic vector*

$$Y = F(I - A^t)^{-1}$$

and check that $\langle Y, V \rangle = |E|$. The vector Y distributes the total footprint to the added value of the various sectors. That attribution

[1] Municipal waste statistics: https://ec.europa.eu/eurostat/statistics-explained/?title=Municipal_waste_statistics.

[2] Greenhouse Gas Protocol: https://ghgprotocol.org/standards-guidance.

seems more natural than the consumer one, as it links the footprint to the benefits of the companies.[3]

In concrete terms. The tables are quite available, see, e.g., for Switzerland,[4] but a measure of environmental impact for each sector appearing in the table is harder to obtain, especially as it needs to be computed/collected in a homogeneous manner over the different sectors in order to make sense.

Acknowledgements. This note is based on the article by J. Kitzes [1], the talk by J. Steinberger at the Climathics Workshop in 2022,[5] and the third author's talk for the *Commission Romande de Mathématique* in 2024.

References

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[2] W. Leontief, *Input-output economics*. Oxford University Press, New York (1966)

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[3] See also the 2021 opinion piece: https://www.nytimes.com/2021/08/31/opinion/climate-change-carbon-neutral.html.

[4] https://www.bfs.admin.ch/bfs/en/home/statistics/national-economy/input-output.html

[5] https://www.la-matrice.ch/climathics-workshop