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Bloch's Principle for Holomorphic Maps into Subvarieties of Semi-Abelian Varieties

by

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Abstract

We generalize a fundamental theorem in higher-dimensional value distribution theory about entire curves in subvarieties X of semi-abelian varieties to the situation of the sequences of holomorphic maps from the unit disc into X. This generalization implies, among other things, that subvarieties of log general type in semi-abelian varieties are pseudo-Kobayashi hyperbolic. As another application, we improve a classical theorem due to Cartan in the 1920s about the system of nowhere vanishing holomorphic functions on the unit disc satisfying Borel's identity.

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Contents

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§1. Introduction

Bloch's principle is a widely recognized guiding principle in the study of complex function theory. The origin of this principle goes back to Bloch's famous dictum "Nihil est in infinito quod non prius fuerit in finito",[1] which appeared in several of his papers published in 1926 (e.g., [4, p. 311]). In many contexts, this statement is interpreted more concretely as the heuristic principle that a family of holomorphic functions in a domain, all of which have a property P, is likely to be normal if P cannot be possessed by non-constant entire functions in the plane (e.g., [38, p. 101]). A typical example is the correspondence between Picard's little theorem and Montel's theorem that a family of holomorphic functions on a domain, all of which omit two values 0 and 1, is normal. There are several very good references for Bloch's principle including [3], [38, Chap. 4], [47].

In the light of his principle, Bloch [4] investigated a theory over the finite disc that corresponds to Borel's generalization of Picard's little theorem (cf. Theorem 1.8). This investigation was succeeded by Cartan [10], who in particular generalized Montel's theorem above (cf. Theorem 1.9). After nearly half a century, Kiernan and Kobayashi [28] interpreted the works of Bloch and Cartan in the context of Kobayashi hyperbolic geometry. These developments are fully explained by Lang [30, Chap. VIII]. We shall discuss an improvement of Cartan's theorem later (cf. Theorem 1.7).

In this paper, we are interested in holomorphic maps into subvarieties of semi-abelian varieties. In the situation over the complex plane $\mathbb{C}$, we have the following fundamental theorem due to Bloch [5], Ochiai [36], Kawamata [22], and Noguchi [34].

Theorem 1.1 (Bloch, Ochiai, Kawamata, Noguchi). *Let A be a semi-abelian variety and let $X \subsetneqq A$ be a proper closed algebraic subvariety of A. Let $f\colon \mathbb{C} \to X$ be a holomorphic map. Then there exists a proper semi-abelian subvariety $B \subsetneqq A$ with the following two properties:*

(1) *$\varpi \circ f\colon \mathbb{C} \to A/B$ is a constant map, where $\varpi\colon A \to A/B$ is the quotient map.*
(2) *$f(\mathbb{C}) \subset \bigcap_{b\in B}(X + b)$.*

[1] According to Schiff [38, p. 101], this may be translated as "Nothing exists in the infinite plane that has not been previously done in the finite disc."

The statement of this theorem is possibly unfamiliar, but convenient to discuss Bloch's principle. An equivalent statement is that the Zariski closure of the image $f(\mathbb{C})$ is a translate of a semi-abelian subvariety $B' \subset A$ (cf. [27, Thm. 3.9.19]). We may take $B \subset A$ in Theorem 1.1 to be this B'. As noted by Noguchi (cf. [35, p. 156]), this theorem includes Borel's generalization of Picard's little theorem (cf. Theorem 1.8). We apply the theorem to the case that A is an algebraic torus $(\mathbb{G}_m)^n$ and $X \subset (\mathbb{G}_m)^n$ is a subvariety defined by the linear equation $x_1 + \cdots + x_n + 1 = 0$ using coordinates $x_1, \dots, x_n$ of $(\mathbb{G}_m)^n \subset \mathbb{C}^n$. For more discussion about Theorem 1.1, we refer the readers to [27, Sec. 3.9] and [35, Sec. 4.8].

Now we are going to discuss a corresponding generalization of Theorem 1.1 for holomorphic mappings from the unit disc $\mathbb{D}$. To state our main theorem, we first introduce one terminology from [30, p. 242]. Let $\gamma > 0$ and let $W \subset \mathbb{C}$ be an open set. An assertion concerning points $w \in W$ will be said to hold for γ-almost all $w \in W$ if it holds for all $w \in W$ possibly except for w contained in at most countably many closed discs such that the sum of the radii is less than γ.

Let $V \subset \bar{A}$ be a Zariski closed set, where $\bar{A}$ is an equivariant compactification of a semi-abelian variety A. See Appendix A for the necessary matters on semi-abelian varieties. Let $B \subset A$ be a semi-abelian variety. We set

$$\mathrm{Sp}_B V = \bigcap_{b \in B} (V + b) \subset V.$$

Then $\mathrm{Sp}_B V \subset \bar{A}$ is a Zariski closed subset.

The following is the main result of this paper. In the following statement, we denote by $\mathrm{Hol}(\mathbb{D}, X)$ the set of all holomorphic mappings from $\mathbb{D}$ to X. For $0 < s < 1$, we set $\mathbb{D}(s) = \{z \in \mathbb{D};\ |z| < s\}$.

Theorem 1.2. *Let A be a semi-abelian variety. Let $X \subsetneqq A$ be a proper closed algebraic subvariety. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic maps in $\mathrm{Hol}(\mathbb{D}, X)$. Then there exist a proper semi-abelian subvariety $B \subsetneqq A$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ with the following two properties:*

(1) *$(\varpi \circ f_{n_k})_{k\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic map $g\colon \mathbb{D} \to A/B$, where $\varpi\colon A \to A/B$ is the quotient map.*

(2) *Let $\bar{A}$ be an equivariant compactification and let $\overline{X} \subset \bar{A}$ be the Zariski closure of X in $\bar{A}$. Then for every $0 < s < 1$, $\gamma > 0$, and open neighborhood $U \subset \bar{A}$ of $\mathrm{Sp}_B\overline{X}$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, we have $f_{n_k}(z) \in U$ for γ-almost all $z \in \mathbb{D}(s)$.*

Before discussing the applications of the theorem, we derive Theorem 1.1 from Theorem 1.2. Given $f\colon \mathbb{C} \to X$, we define a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathrm{Hol}(\mathbb{D}, X)$ by

$\varphi_n(z) = f(nz)$. Then by Theorem 1.2, there exist a subsequence $\{\varphi_{n_k}\}_{k=1}^{\infty}$ and a proper semi-abelian subvariety $B \subsetneqq A$ such that $\{\varpi \circ \varphi_{n_k}\}$ converges uniformly on compact subsets of $\mathbb{D}$ to $g\colon \mathbb{D} \to A/B$. We claim that $\varpi \circ f\colon \mathbb{C} \to A/B$ is constant. So assume on the contrary that $\varpi \circ f$ is non-constant. Then there exists a global holomorphic one-form $\eta \in \Gamma(A/B, \Omega^1_{A/B})$ such that $(\varpi \circ f)^*\eta = \zeta(z)\,dz$ is non-zero on $\mathbb{C}$. Hence there exists $l \geq 0$ such that $\zeta^{(l)}(0) \neq 0$. We set $(\varpi \circ \varphi_n)^*\eta = \xi_n(z)\,dz$. Then $\xi_n(z) = n\zeta(nz)$. Hence $|\xi_n^{(l)}(0)| = |n^{l+1}\zeta^{(l)}(0)| \to \infty$ as $n \to \infty$. On the other hand, ξ_{n_k} converges uniformly on compact subsets of $\mathbb{D}$ to $\xi(z)$ on $\mathbb{D}$, where $g^*\eta = \xi(z)\,dz$. Hence $\xi_{n_k}^{(l)}(0) \to \xi^{(l)}(0)$ as $k \to \infty$. This is a contradiction. Hence $\varpi \circ f$ is constant. To ensure assertion (2) of Theorem 1.1, we take a minimum semi-abelian variety $B \subsetneqq A$ such that $\varpi \circ f$ is constant. By considering the translations by $f(0)$, we may assume without loss of generality that $f(0) = 0_A$, i.e., the identity element of A. Then we have $f(\mathbb{C}) \subset B$. Let $X' \subset B$ be the Zariski closure of $f(\mathbb{C})$. To show $X' = B$, we assume on the contrary that $X' \subsetneqq B$. Then by the argument above applied for $f\colon \mathbb{C} \to X' \subsetneqq B$, we get a proper semi-abelian subvariety $B' \subsetneqq B$ such that $\varpi' \circ f\colon \mathbb{C} \to B/B'$ is constant, where $\varpi'\colon B \to B/B'$ is the quotient map. This contradicts the choice of B. Thus $X' = B$. By $X' \subset X \cap B$, we have $X \cap B = B$. Hence $f(\mathbb{C}) \subset X \cap B \subset \bigcap_{b \in B}(b + X)$ as desired. This completes the implication of Theorem 1.1 from Theorem 1.2.

Note that in the implication above, assertion (2) of Theorem 1.2 plays no role. However, in the applications of Theorem 1.2 below, we need assertion (2).

Next we discuss applications of Theorem 1.2 to Kobayashi hyperbolic geometry. We first introduce some terminologies from [27, p. 245]. Let W be a relatively compact open domain of M, and let Δ be a closed subset of M. We say that W is tautly embedded modulo Δ in M if for each sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Hol}(\mathbb{D}, W)$, one of the following holds:

(1) $(f_n)_{n\in\mathbb{N}}$ has a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges uniformly on compact subsets of $\mathbb{D}$ to some $f \in \mathrm{Hol}(\mathbb{D}, M)$;
(2) for each compact set $K \subset \mathbb{D}$ and each compact set $L \subset M - \Delta$ there exists an integer n_0 such that $f_n(K) \cap L = \emptyset$ for all $n \geq n_0$.

Let $X \subset A$ be a closed algebraic subvariety of a semi-abelian variety A. Let $\overline{X} \subset \bar{A}$ be the compactification, where $\bar{A}$ is an equivariant compactification. We set

$$Z = \left\{x \in \overline{X}; \exists B \subset A, \text{ semi-abelian variety s.t. } \dim(x + B) \geq 1 \text{ and } x + B \subset \overline{X}\right\}.$$

Let $S \subset \overline{X}$ be the Zariski closure of Z. Then $S \subset \overline{X}$ is a Zariski closed set. By [34, Lem. 4.1], S is a proper subset of $\overline{X}$, provided X is of log-general type. As an application of Theorem 1.2, we obtain the following theorem.

Theorem 1.3. *Let $X \subset A$ be a closed algebraic subvariety of a semi-abelian variety A. Let $\bar{A}$ be a smooth equivariant compactification and let $\overline{X} \subset \bar{A}$ be the compactification. Then X is tautly embedded modulo S in $\overline{X}$.*

A theorem of Kiernan and Kobayashi [28] claims that if W is tautly embedded modulo Δ in M, then W is hyperbolically embedded modulo Δ in M (cf. [27, Thm. 5.1.13], [30, p. 37, Thm. 1.4]). Hence by Theorem 1.3, we immediately get the following corollary.

Corollary 1.4. *Let $X \subset A$ be a closed algebraic subvariety of a semi-abelian variety A. Let $\bar{A}$ be a smooth equivariant compactification and let $\overline{X} \subset \bar{A}$ be the compactification. Then X is hyperbolically embedded modulo S in $\overline{X}$. In particular, if X is of log-general type, then X is pseudo-Kobayashi hyperbolic.*

Here we say that a quasi-projective variety X is pseudo-Kobayashi hyperbolic if X is Kobayashi hyperbolic modulo some proper Zariski closed subset. In the last statement of our corollary, X is Kobayashi hyperbolic modulo $S \cap X$, which is a proper Zariski closed subset of X because X is of log-general type.

When A is compact, these results are previously proved in [46]. In the compact case, Theorem 1.2 yields a result on infinitesimal Kobayashi–Royden pseudo-metric F_X defined as follows. Let X be an algebraic variety. For each $x \in X$, we call the set $\breve{T}_x X$ of all 1-jets the tangent cone of X at x, and $\breve{T}X = \bigcup_{x \in X} \breve{T}_x X$ the tangent cone of X (cf. [27, p. 31]). If X is smooth, then $\breve{T}X$ coincides with the usual tangent bundle. For $v \in \breve{T}X$, we set

$$F_X(v) = \inf\{r > 0;\ \exists f\colon \mathbb{D} \to X,\ \text{holomorphic map s.t. } f'(0) = \tfrac{1}{r}v\}.$$

Suppose there exists a holomorphic map $f\colon \mathbb{C} \to X$ such that $f'(0) = v \in \breve{T}X$; then we have $F_X(v) = 0$. However, the converse is not true in general (cf. Example 14.3). Theorem 1.2 yields the following theorem. We do not know whether this statement is true or not for subvarieties of non-compact semi-abelian varieties.

Theorem 1.5. *Let A be an abelian variety. Let $X \subset A$ be a closed subvariety. Suppose $v \in \breve{T}X$ satisfies $F_X(v) = 0$. Then there exists a holomorphic map $f\colon \mathbb{C} \to X$ such that $f'(0) = v$.*

Now we return to the classical topics concerning the Bloch–Cartan theorem. To simplify the complicated indices in the description, we employ the following conventions.

Convention 1.6. For a set E and an infinite set I, an indexed family $(x_i)_{i \in I}$ is a function $I \to E$. When E is an infinite set, we consider E as an indexed family indexed by E itself by the identity map $E \to E$. Let S be a metric space with the distance function d. Let $(f_i)_{i\in I}$ be an indexed family of functions defined in $\mathbb{D}$, and with values in S. We say that $(f_i)_{i\in I}$ converges uniformly on compact subsets on $\mathbb{D}$ to some $f\colon \mathbb{D} \to S$ if for every compact subset $K \subset \mathbb{D}$ and every $\varepsilon > 0$, there exists a finite subset $F \subset I$ such that $d(f_i(z), f(z)) < \varepsilon$ for all $z \in K$ and $i \in I - F$. Of course, when $I = \mathbb{N}$, this definition coincides with the usual definition of uniform convergence on compact subsets on $\mathbb{D}$. If each f_i is continuous, then f is continuous. Indeed, we take a sequence $i_1, i_2, i_3, \ldots$ of distinct elements in I. Then the sequence $(f_{i_k})_{k\in\mathbb{N}}$ converges uniformly on compact subsets on $\mathbb{D}$ to f. Hence f is continuous. Similarly, if S is a complex manifold and each f_i is holomorphic, then f is holomorphic.

Theorem 1.7. *Let $\mathcal{F}$ be an infinite set of p-tuples $f = (f_1, \ldots, f_p)$ of nowhere-vanishing holomorphic functions on $\mathbb{D}$ satisfying the identity*

$$f_1 + f_2 + \cdots + f_p = 0. \tag{1.1}$$

Then there exist disjoint non-empty subsets $I_1, \ldots, I_n \subset \{1, \ldots, p\}$ and an infinite subset $\mathcal{G} \subset \mathcal{F}$ with the following properties:

(1) *$n \geq 1$.*

(2) *Let $k \in \{1, \ldots, n\}$. Then*

   (a) *for all $i, j \in I_k$, the indexed family $(f_i/f_j)_{f\in\mathcal{G}}$ converges uniformly on compact subsets of $\mathbb{D}$ to a nowhere-vanishing holomorphic function;*

   (b) *for all $j \in I_k$, the indexed family $(\sum_{i\in I_k} f_i/f_j)_{f\in\mathcal{G}}$ converges uniformly on compact subsets of $\mathbb{D}$ to 0. In particular, I_k contains at least two elements.*

(3) *Set $I = I_1 \sqcup \cdots \sqcup I_n$. Let $i \in \{1, \ldots, p\} - I$. Let $\varepsilon > 0$, $s \in (0, 1)$, and $\gamma > 0$. Then there exists a finite subset $\mathcal{E} \subset \mathcal{G}$ such that for all $f \in \mathcal{G} - \mathcal{E}$, we have*

$$\frac{|f_i(z)|}{\sqrt{\sum_{j\in I} |f_j(z)|^2}} < \varepsilon$$

for γ-almost all $z \in \mathbb{D}(s)$.

Some historical remarks are required. The identity (1.1) was considered by Borel [7], who generalized Picard's little theorem as follows.

Theorem 1.8 (Borel). *Let $f_1, \ldots, f_p$ be nowhere-vanishing holomorphic functions on $\mathbb{C}$ satisfying the identity (1.1). Then there exists a partition of indices*

$$\{1, \ldots, p\} = I_1 \sqcup \cdots \sqcup I_n$$

such that each I_k satisfies that

(1) *for every $i, j \in I_k$, the quotient f_i/f_j is constant, and*

(2) $\sum_{i\in I_k} f_i = 0$.

By the second conclusion, each I_k has at least two elements. In particular, when $p = 3$, we have $n = 1$ and $I_1 = \{1, 2, 3\}$. This implies Picard's little theorem as follows. Let $g\colon \mathbb{C} \to \mathbb{C} - \{0, 1\}$. We set $f_1(z) = g(z)$, $f_2(z) = 1 - g(z)$, and $f_3(z) = -1$. Then f_1, f_2, and f_3 are nowhere-vanishing holomorphic functions on $\mathbb{C}$ satisfying the identity $f_1 + f_2 + f_3 = 0$. Hence by Borel's theorem, $f_1/f_3 = -g$ is constant, as desired.

As we have already mentioned above, the corresponding theory over the disc $\mathbb{D}$ was investigated by Bloch [4] and Cartan [10, p. 312]. See also [30, Chap. VIII]. Let $\mathcal{F}$ be an infinite set of p-tuples $f = (f_1, \ldots, f_p)$ of nowhere-vanishing holomorphic functions on $\mathbb{D}$ satisfying the identity (1.1). A subset $I \subset \{1, \ldots, p\}$ is called C-class if there exists $i \in I$ such that for all $j \in I$, the sequence $(f_j/f_i)_{f\in\mathcal{F}}$ is uniformly bounded on every compact set of $\mathbb{D}$ and $(\sum_{j\in I} f_j/f_i)_{f\in\mathcal{F}}$ converges uniformly on compact subsets of $\mathbb{D}$ to 0.

Theorem 1.9 (Cartan). *There exists an infinite subset $\mathcal{G} \subset \mathcal{F}$ such that $\{1, \ldots, p\}$ itself is C-class, or there exist two disjoint subsets $I_1, I_2 \subset \{1, \ldots, p\}$ such that both are C-classes.*

Cartan conjectured that there exists an infinite subset $\mathcal{G} \subset \mathcal{F}$ such that the set $\{1, \ldots, p\}$ can be partitioned into C-classes (cf. [10, p. 318], [30, p. 244]). However, this conjecture was disproved by Eremenko [14]. In [15], Eremenko proposed a modified version of Cartan's conjecture and proved it for the case $p = 5$.

Theorem 1.7 implies Theorem 1.9 as follows. We apply Theorem 1.7 to get an infinite subset $\mathcal{G} \subset \mathcal{F}$ and disjoint subsets $I_1, \ldots, I_n \subset \{1, \ldots, p\}$. If $n \geq 2$, there is nothing to do for each I_k is a C-class by the second assertion of Theorem 1.7. Thus we consider the case $n = 1$. Let $j \in I_1$. Then assertion (2a) of Theorem 1.7 implies that $(\sqrt{\sum_{i\in I_1} |f_i|^2}/|f_j|)_{f\in\mathcal{G}}$ converges uniformly on compact subsets of $\mathbb{D}$. Hence the third assertion of Theorem 1.7 reads as follows:

> Let $i \notin I_1$, $\varepsilon > 0$, $s \in (0, 1)$, and $\gamma > 0$. Then there exists a finite subset $\mathcal{E} \subset \mathcal{G}$ such that for all $f \in \mathcal{G} - \mathcal{E}$, we have $|f_i|/|f_j| < \varepsilon$ for γ-almost all $z \in \mathbb{D}(s)$.

Let $i \notin I_1$. Let $K \subset \mathbb{D}$ be a compact set and let $\varepsilon > 0$. We take $s \in (0, 1)$ and $\gamma > 0$ such that $K \subset \mathbb{D}(s - 2\gamma)$. Then by the third assertion of Theorem 1.7, there exists a finite set $\mathcal{E} \subset \mathcal{G}$ such that for all $f \in \mathcal{G} - \mathcal{E}$, we have $|f_i|/|f_j| < \varepsilon$ for γ-almost all

$z \in \mathbb{D}(s)$. Let $f \in \mathcal{G} - \mathcal{E}$. We may take $s' \in (s - 2\gamma, s)$ such that $|f_i|/|f_j| < \varepsilon$ holds over the circle $\partial\mathbb{D}(s')$. Since f_i/f_j is holomorphic, the maximal principle yields that $|f_i|/|f_j| < \varepsilon$ for all $z \in \mathbb{D}(s')$, hence for all $z \in K$. Hence $(f_i/f_j)_{f\in\mathcal{G}}$ converges uniformly to 0 on K. Hence if $n = 1$, then $\{1, \dots, p\}$ is C-class. This completes the derivation of Theorem 1.9 from Theorem 1.7. $\square$

Let us discuss the structure of the proof of Theorem 1.2. Sections 2–12 are devoted to the proof of Theorem 1.2. Although the proof of Theorem 1.2 is lengthy, the structure is rather simple. The proof is based on two key propositions: Propositions 3.20 and 11.3. Proposition 3.20 establishes a new normality criterion for families $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A)$, where limit holomorphic maps may lie in the boundary of A. Then Proposition 11.3 shows that this criterion can be applied to our situation by taking a quotient. More precisely, for a subvariety $X \subsetneqq A$ and a family $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, X)$, it shows the existence of $B \subset A$ such that the family $\{\varpi \circ f\}_{f\in\mathcal{F}}$ in $\mathrm{Hol}(\mathbb{D}, A/B)$ satisfies the criterion from Proposition 3.20, in addition to the concentration of $\mathcal{F}$ to $\mathrm{Sp}_B\overline{X}$. Here, $\varpi\colon A \to A/B$ is the quotient map. The proof of Theorem 1.2 is completed in Section 12 using these two propositions.

Section 12 is again divided into two major steps. The first step is to prove Lemma 12.4, which is a direct consequence of Propositions 3.20 and 11.3. However, Lemma 12.4 is weaker than Theorem 1.2, most notably because the limit morphism $g\colon \mathbb{D} \to \overline{A/B}$ is obtained only in the compactification of A/B rather than in A/B itself. We resolve this issue in the second step of our proof, which constitutes the remainder of Section 12. This step relies heavily on the knowledge of the compactifications of semi-abelian varieties, as discussed in Appendix A. In particular, Lemma A.15 plays a crucial role. Finally, Theorem 1.2 is proved in an equivalent formulation Theorem 12.12.

The proofs of Propositions 3.20 and 11.3 are provided in Sections 4–8 and Sections 9–11, respectively. Before these proofs, Sections 2 and 3 provide the necessary preliminaries. In Section 2, we recall the Demailly jet space, an essential object for the statement of Proposition 3.20. In Section 3, we introduce new concepts, including "horizontally integrable", and state our normality criterion (Proposition 3.20) for a family $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A)$. This criterion states that the horizontal integrability of a certain limit of the Demailly jet lift of $\mathcal{F}$ is sufficient for the family $\mathcal{F}$ to be normal. Therefore, verifying the horizontal integrability is a crucial step for applying Proposition 3.20. As part of the proof for Proposition 11.3, we establish a sufficient condition for horizontal integrability in Section 10. This is done by analyzing the Demailly jet tower with the help of the theory of Hilbert schemes.

The main technical tool for the proof of our normality criterion (Proposition 3.20) is Nevanlinna theory, which we introduce in Section 4. After discussing several estimates in this theory, we complete the proof of Proposition 3.20 in Section 8 (see Section 3.4 for more details). In contrast, the proof of Proposition 11.3 is based on algebro-geometric arguments, and does not require Nevanlinna theory. Consequently, these two proofs are logically independent from each other. Therefore, the corresponding sections (Sections 4–8 and Sections 9–11) can be read independently.

Although we state our normality criterion in the form of Proposition 3.20, we use its direct corollary, Corollary 8.4, instead of Proposition 3.20 itself for the proof of Theorem 1.2. We remark that the statement of Corollary 8.4 does not require Nevanlinna theory, while the proof relies heavily on it. Thus, for a first reading of the proof of Theorem 1.2, one possible approach is to read Sections 1 through 3, then check the statements of Corollary 8.4 and Proposition 11.3, and finally proceed directly to Section 12, where Theorem 1.2 is proved. The additional notation used in the statement of Proposition 11.3 is introduced at the beginning of Section 11.

In each of Sections 13, 14, and 15, we prove Theorems 1.3, 1.5, and 1.7, in this order, as applications of Theorem 1.2. Some needed facts in this paper for semi-abelian varieties are treated in Appendix A. Appendix B is devoted to the proof of an algebro-geometric proposition needed in this paper, namely a flattening result using blow-ups.

Convention 1.10. In this paper, an algebraic variety (or simply a variety) is an integral, separated scheme of finite type over the complex number field $\mathbb{C}$ (cf., e.g., [18, p. 105]). In particular, every variety is reduced, irreducible, and non-empty (cf. [18, Chap. II, Prop. 3.1]). Every variety has a canonically associated complex space structure (cf. [18, p. 439]).

Table 1: List of selected symbols

γ-almost all	Exceptional discs have total radius less than γ	§1
$\mathrm{Sp}_B V$	Zariski closed set defined by $\bigcap_{b\in B}(V+b)$	§1
$\mathbb{D}$	Unit disc	§1
$\mathrm{Hol}(\mathbb{D}, X)$	Set of all holomorphic maps from $\mathbb{D}$ to X	§1
$\mathbb{D}(s)$	Disc of radius s centered at 0	§1
M_k	Demailly jet space of M	§2
$f_{[k]}\colon \mathbb{D}\to M_k$	Lift of $f\colon \mathbb{D}\to M$	§2
M_k^{sing}	Zariski closed subset of M_k defined by (2.2)	§2
$S_{k,A}$	Variety satisfying $(A\times S)_k \simeq A\times S_{k,A}$	§2

$P_{k,A}$	Special case of $S_{k,A}$ when S is a single point	§2
f_A, f_S	Composite of $f\colon \mathbb{D} \to A \times S$ with the projections	§2
$f_{S_{k,A}}\colon \mathbb{D} \to S_{k,A}$	Composite of $f_{[k]}\colon \mathbb{D} \to (A\times S)_k$ & $(A\times S)_k \to S_{k,A}$	§2
$S_{k,A}^{\mathrm{sing}}$	Locus of $S_{k,A}$ satisfying $(A \times S)_k^{\mathrm{sing}} \simeq A \times S_{k,A}^{\mathrm{sing}}$	§2
$f_{P_{k,A}}$, $P_{k,A}^{\mathrm{sing}}$	$f_{S_{k,A}}$ and $S_{k,A}^{\mathrm{sing}}$ when S is a single point	§2
$E_{k,A,A/B}$	Locus of $P_{k,A}$ defined in Def. 2.1	§2
$\mathcal{F} \to Z$	Concentrates to Z outside a small exceptional set	§3.1
$\mathcal{F} \Rightarrow Z$	Still $\mathcal{F} \to Z'$ after Z-admissible modifications	§3.1
$\mathrm{LIM}(\mathcal{F};\ \{E_i\})$	Zariski closed set defined in Def. 3.12	§3.1
$S_{1,A}^*$	Typical horizontally integrable set defined in (3.2)	§3.2
$\Pi(\mathcal{F})$	Set of semi-abelian varieties defined in Def. 3.18	§3.3
$\lvert\cdot\rvert_\omega$	Norm on the tangent space induced by ω	§3.3
$\mathcal{F}_{P_{k,A}}$	Sequence $((f_i)_{P_{k,A}})_{i\in I}$, where $\mathcal{F} = (f_i)_{i\in I}$	§3.3
T_k	$\mathrm{LIM}(\mathcal{F}_{P_{k,A}};\ \{E_{k,A,A/B}\}_{B\in\Pi(\mathcal{F})})$ if it exists	§3.3
$\mathrm{Hol}_m(\mathbb{D}, V)$	Set of all multivalued holomorphic maps	§4
Y_f	Source space of a multivalued holomorphic map f	§4
$\pi_f\colon Y_f \to \mathbb{D}$	Natural proper surjective holomorphic map	§4
λ_Z	Weil function for Z	§4
$N_s(r, f, Z)$	Counting function	§4
$m(r, f, \lambda_Z)$	Proximity function	§4
$T_s(r, f, Z, \lambda_Z)$	Defined by $N_s(r, f, Z) + m(r, f, \lambda_Z) - m(s, f, \lambda_Z)$	§4
$T_s(r, f, \omega)$	Order function	§4
$\mathcal{G} \rightsquigarrow T$	Generalization of $\mathcal{F} \to T$ defined in Def. 5.3	§5.1
$m(r, (g, \Omega, \Omega'), \lambda_Z)$	Generalization of the proximity function	§5.1
$T_s(r, (g, \Omega, \Omega'), \omega)$	Generalization of the order function	§5.1
C_H^1	1-dimensional unlimited Hausdorff content	§8
Λ_k	Non-reduced scheme $\mathrm{Spec}\,\mathbb{C}[\varepsilon]/(\varepsilon^{k+1})$	§9
$\mathcal{D}Z$	Closed subscheme of S_1 induced by $Z \subset S$	§9
$\mathcal{P}_k X$	Closed subscheme of $V \times S_k$ induced by $X \subset V \times S$	§9.1
$J_k S$	Classical jet space	§9.2
$J_k^{\mathrm{reg}} S$	Space of jets with non-zero first derivatives	§9.2
$X_{B,k}$	Closed subscheme of $\bar{A} \times S_{k,A/B}$ defined by (10.1)	§10
$\mathrm{Stab}(V)$	Stabilizer of V	§10
$\mathrm{Stab}^0(V)$	Identity component of $\mathrm{Stab}(V)$	§10
$\Lambda_{X,\bar{A}}(\mathcal{F})$	Set of semi-abelian varieties defined in Def. 11.1	§11
$\mathcal{F}_{A/B}$	Sequence $(\varpi_B \circ f)_{f\in\mathcal{F}}$ where ϖ_B is the quotient	§11
$\mathcal{F}'_{A/B}$	Remove the constant maps from $\mathcal{F}_{A/B}$	§11
$\mathcal{F}_{P_{k,A/B}}$	Sequence $(\mathcal{F}'_{A/B})_{P_{k,A/B}}$	§11

$Z_{k,A/B}$	Zariski closed subset of $P_{k,A/B}$ defined by (11.1)	§11
$X[Z]$	Closed subscheme of X defined by Def. 11.4	§11.1

§2. Demailly jet spaces

We introduce Demailly jet spaces (cf. [12]). Let M be a positive-dimensional smooth algebraic variety. Let $V \subset TM$ be an algebraic vector subbundle, whose bundle rank is positive. Set $\widetilde{M} = P(V)$. Let $\pi\colon \widetilde{M} \to M$ be the projection. We define a vector subbundle $\widetilde{V} \subset T\widetilde{M}$ by the following: for every point $(x,[v]) \in \widetilde{M}$ associated with a vector $v \in V_x \backslash \{0\}$, we set

$$\widetilde{V}_{(x,[v])} = \{\xi \in T_{(x,[v])}\widetilde{M};\ \pi_*(\xi) \in \mathbb{C}v\}, \tag{2.1}$$

where $\pi_*\colon T\widetilde{M} \to TM$ is the induced map. Let $f\colon \mathbb{D} \to M$ be a non-constant holomorphic map. We say that f is tangent to V if $f'(z) \in V_{f(z)}$ for all $z \in \mathbb{D}$. If f is tangent to V, we may define $f_{[1]}\colon \mathbb{D} \to \widetilde{M}$ by $f_{[1]}(z) = (f(z), [f'(z)])$. Then $f_{[1]}$ is tangent to $\widetilde{V}$.

We inductively define the Demailly jet space M_k together with vector subbundle $V_k \subset TM_k$ by

$$(M_0, V_0) = (M, TM), \quad (M_k, V_k) = (\widetilde{M_{k-1}}, \widetilde{V_{k-1}}).$$

For a non-constant holomorphic map $f\colon \mathbb{D} \to M$, we define $f_{[k]}\colon \mathbb{D} \to M_k$ inductively by $f_{[0]} = f$ and $f_{[k]} = (f_{[k-1]})_{[1]}$.

For $k \geq 2$, we define the singular locus $M_k^{\mathrm{sing}} \subset M_k$ as follows. We note $M_k \subset PTM_{k-1}$. We have a natural map $M_{k-1} \to M$, from which we get the relative tangent bundle $T_{M_{k-1}/M} \subset TM_{k-1}$. We set

$$M_k^{\mathrm{sing}} = M_k \cap PT_{M_{k-1}/M}. \tag{2.2}$$

Then $M_k^{\mathrm{sing}} \subset M_k$ is a Zariski closed set. We claim that this is a divisor. To show this, we consider the maps $M_{k-1} \to M_{k-2} \to M$, which induces

$$TM_{k-1} \to TM_{k-2} \to TM.$$

For $v \in TM_{k-1}$, we have $v \in T_{M_{k-1}/M}$ if and only if $(\pi_{k-1})_*(v) \in T_{M_{k-2}/M}$, where $\pi_{k-1}\colon M_{k-1} \to M_{k-2}$ is the natural projection. The rational map $PTM_{k-1} \dashrightarrow PTM_{k-2}$ induces the holomorphic map

$$p\colon PTM_{k-1} - PT_{M_{k-1}/M_{k-2}} \to PTM_{k-2}.$$

Then

$$PT_{M_{k-1}/M} - PT_{M_{k-1}/M_{k-2}} = p^{-1}(PT_{M_{k-2}/M}). \tag{2.3}$$

Note that the subbundle $T_{M_{k-1}/M_{k-2}} \subset TM_{k-1}$ satisfies

$$T_{M_{k-1}/M_{k-2}} \subset V_{k-1}. \tag{2.4}$$

The rank of $T_{M_{k-1}/M_{k-2}}$ is equal to $\dim M - 1$. Set

$$D_k = PT_{M_{k-1}/M_{k-2}} \subset PV_{k-1} = M_k.$$

Then D_k is a divisor on M_k. By (2.3), we have

$$M_k^{\mathrm{sing}} = D_k \cup \pi_k^{-1}(M_{k-1}^{\mathrm{sing}}), \tag{2.5}$$

where $\pi_k\colon M_k \to M_{k-1}$. Hence M_k^{sing} is a divisor on M_k, using the induction on k.

Next we consider the case of semi-abelian varieties. Let A be a semi-abelian variety. Let $m\colon A \times A \to A$ be the natural action such that $(x, a) \mapsto x + a$. This induces

$$m_*\colon T(A \times A) \to TA.$$

We have a subbundle

$$A \times TA \subset T(A \times A).$$

Thus we get

$$(A \times TA)|_{A\times\{0_A\}} \to TA.$$

By $T_{0_A}A = \mathrm{Lie}\, A$, we get

$$\psi\colon A \times \mathrm{Lie}\, A \to TA. \tag{2.6}$$

Then ψ is an isomorphism of vector bundles over A. For each $a \in A$, we denote by $t_a\colon A \to A$ the translation defined by a. This induces an isomorphism

$$(t_a)_*\colon TA \to TA.$$

Then we have

$$(\psi^{-1} \circ (t_a)_* \circ \psi)(x, v) = (x + a, v). \tag{2.7}$$

Let $f \in \mathrm{Hol}(\mathbb{D}, A)$. We define $f_{\mathrm{Lie}\, A}\colon \mathbb{D} \to \mathrm{Lie}\, A$ by the composite of

$$\mathbb{D} \xrightarrow{f'} TA \xrightarrow{\psi^{-1}} A \times \mathrm{Lie}\, A \to \mathrm{Lie}\, A.$$

For $a \in A$, we define $f_a\colon \mathbb{D} \to A$ by $f_a(z) = f(z) + a$. By (2.7), we have

$$(f_a)_{\mathrm{Lie}\, A} = f_{\mathrm{Lie}\, A}. \tag{2.8}$$

We consider the Demailly jet space for the case $M = A \times S$, where S is a smooth algebraic variety. We construct a smooth algebraic variety $S_{k,A}$ and a vector subbundle

$$V_k^{\dagger} \subset TS_{k,A} \times \mathrm{Lie}(A) \tag{2.9}$$

as follows. Set $S_{0,A} = S$ and $V_0^\dagger = TS \times \mathrm{Lie}(A)$. Suppose $S_{k-1,A}$ and $V_{k-1}^\dagger \subset TS_{k-1,A} \times \mathrm{Lie}(A)$ are given. We set

$$S_{k,A} = P(V_{k-1}^\dagger). \tag{2.10}$$

Then $S_{k,A}$ is a smooth algebraic variety. Let $\tau\colon S_{k,A} \to S_{k-1,A}$ be the projection. We have a vector bundle map

$$(\tau_*, \mathrm{id}_{\mathrm{Lie}(A)})\colon TS_{k,A} \times \mathrm{Lie}(A) \to TS_{k-1,A} \times \mathrm{Lie}(A).$$

We define $V_k^\dagger \subset TS_{k,A} \times \mathrm{Lie}(A)$ as follows. For each $(x, [v]) \in S_{k,A}$, where $x \in S_{k-1,A}$ and $v \in V_{k-1}^\dagger \backslash \{0\}$, we set

$$(V_k^\dagger)_{(x,[v])} = \left\{\xi \in T_{(x,[v])}S_{k,A} \times \mathrm{Lie}(A);\ (\tau_*, \mathrm{id}_{\mathrm{Lie}(A)})(\xi) \in \mathbb{C}\cdot v\right\}. \tag{2.11}$$

By the isomorphism (2.6), we have an isomorphism

$$T(A \times S_{k,A}) \simeq A \times TS_{k,A} \times \mathrm{Lie}(A).$$

By this isomorphism, we consider $A \times V_k^\dagger$ as a vector subbundle of $T(A \times S_{k,A})$.

Next we construct an isomorphism

$$\varphi_k\colon A \times S_{k,A} \to (A \times S)_k \tag{2.12}$$

as follows. For $k = 0$, we set $\varphi_0 = \mathrm{id}_{A\times S}$. Note that $(\varphi_0)_*(A \times V_0^\dagger) = V_0$. Suppose we are given an isomorphism $\varphi_{k-1}\colon A \times S_{k-1,A} \to (A \times S)_{k-1}$ such that $(\varphi_{k-1})_*(A \times V_{k-1}^\dagger) = V_{k-1}$. Then the projectivization of $(\varphi_{k-1})_*$ induces an isomorphism $\varphi_k\colon A \times S_{k,A} \to (A \times S)_k$. Under this isomorphism, we have $(\varphi_k)_*(A \times V_k^\dagger) = V_k$. Thus inductively, we have constructed the isomorphism (2.12).

In the following, we identify $(A\times S)_k$ with $A\times S_{k,A}$ by the isomorphism (2.12). When S is a single point, we denote

$$P_{k,A} = \{\mathrm{pt}\}_{k,A}.$$

Then under the isomorphism (2.12), we have $A_k = A \times P_{k,A}$.

Let $f \in \mathrm{Hol}(\mathbb{D}, A \times S)$ be non-constant. We denote by $f_S\colon \mathbb{D} \to S$ the composite of $f\colon \mathbb{D} \to A \times S$ and the second projection $A \times S \to S$. We define

$$f_{S_{k,A}}\colon \mathbb{D} \to S_{k,A} \tag{2.13}$$

as follows. We set $f_{S_{0,A}} = f_S$. Note that $((f_{S_{0,A}})', f_{\mathrm{Lie}\,A})(z) \in V_0^\dagger$. Suppose that $f_{S_{k-1,A}}\colon \mathbb{D} \to S_{k-1,A}$ is given such that $((f_{S_{k-1,A}})', f_{\mathrm{Lie}\,A})(z) \in V_{k-1}^\dagger$. We define $f_{S_{k,A}}\colon \mathbb{D} \to S_{k,A}$ by the projectivization of $((f_{S_{k-1,A}})', f_{\mathrm{Lie}\,A})$. Then we have $((f_{S_{k,A}})', f_{\mathrm{Lie}\,A})(z) \in V_k^\dagger$. Thus we have constructed $f_{S_{k,A}}\colon \mathbb{D} \to S_{k,A}$ inductively for all k. Let $f_A\colon \mathbb{D} \to A$ be the composite of $f\colon \mathbb{D} \to A \times S$ and the first projection $A \times S \to A$. We have $\varphi_k \circ (f_A, f_{S_{k,A}}) = f_{[k]}$, which follows from the construction. For $a \in A$, we set $f_a\colon \mathbb{D} \to A \times S$ by $f_a(z) = (f_A(z) + a, f_S(z))$. Then by (2.8), we have

$$(f_a)_{S_{k,A}} = f_{S_{k,A}} \tag{2.14}$$

for all $a \in A$.

Let $k \geq 2$. We define $S_{k,A}^{\mathrm{sing}} \subset S_{k,A}$ by

$$S_{k,A}^{\mathrm{sing}} = S_{k,A} \cap P(T_{S_{k-1,A}/S} \times \{0\}), \tag{2.15}$$

where $T_{S_{k-1,A}/S} \times \{0\} \subset TS_{k-1,A} \times \mathrm{Lie}\, A$. Then we have

$$A \times S_{k,A}^{\mathrm{sing}} = (A \times S)_k^{\mathrm{sing}} \tag{2.16}$$

under the isomorphism of (2.12). When S is a single point, we denote $f_{S_{k,A}}$ and $S_{k,A}^{\mathrm{sing}}$ by $f_{P_{k,A}}$ and $P_{k,A}^{\mathrm{sing}}$, respectively.

The following definition plays an important role in this paper.

Definition 2.1. Let $B \subset A$ be a semi-abelian subvariety. For $k \geq 1$, we define $E_{k,A,A/B} \subset P_{k,A}$ by

$$E_{k,A,A/B} = P_{k,A} \cap P(TP_{k-1,A} \times \mathrm{Lie}\, B),$$

where $TP_{k-1,A} \times \mathrm{Lie}\, B \subset TP_{k-1,A} \times \mathrm{Lie}\, A$.

Then we have $E_{k,A,A} \subset E_{k,A,A/B}$. Moreover, by $T_{P_{k-1,A}/\{\mathrm{pt}\}} = TP_{k-1,A}$, we have

$$P_{k,A}^{\mathrm{sing}} = E_{k,A,A}. \tag{2.17}$$

Lemma 2.2. *Let $k \geq 1$. Let $\tau\colon P_{k+1,A} \to P_{k,A}$ be the projection. Then we have*

$$\tau^{-1}(E_{k,A,A/B}) \subset E_{k+1,A,A/B}.$$

Proof. By the definition (2.10), we have $P_{k+1,A} = P(V_k^\dagger)$, where $V_k^\dagger \subset TP_{k,A} \times \mathrm{Lie}(A)$. Let $(y, [\xi]) \in P_{k+1,A} \backslash E_{k+1,A,A/B}$, where $y \in P_{k,A}$ and $\xi \in V_k^\dagger \backslash \{0\}$. Then $\xi \notin TP_{k,A} \times \mathrm{Lie}(B)$. Let $y = (x, [v]) \in P(V_{k-1}^\dagger)$, where $x \in P_{k-1,A}$ and $v \in V_{k-1}^\dagger \backslash \{0\}$. Then by the definition of $V_k^\dagger$ (cf. (2.11)), the image of ξ under the map $TP_{k,A} \times \mathrm{Lie}(A) \to TP_{k-1,A} \times \mathrm{Lie}(A)$ is contained in the linear space $\mathbb{C} \cdot v$. Hence $v \notin TP_{k-1,A} \times \mathrm{Lie}(B)$. Hence $y \in P_{k,A} \backslash E_{k,A,A/B}$. Hence we have proved $\tau^{-1}(E_{k,A,A/B}) \subset E_{k+1,A,A/B}$. $\square$

Remark 2.3. Let $k \geq 0$. We have the subbundle $V_k^\dagger \subset TP_{k,A} \times \mathrm{Lie}(A)$ so that $P_{k+1,A} = P(V_k^\dagger)$. Set $S = P_{k,A}$. For each $l \geq 0$, we denote by $V_{l,S}^\dagger \subset TS_{l,A} \times \mathrm{Lie}(A)$ the object in (2.9) so that $S_{l+1,A} = P(V_{l,S}^\dagger)$. Then for each $l \geq 0$, there exists a natural embedding

$$P_{k+l,A} \subset S_{l,A}$$

such that $V_{k+l}^\dagger \subset V_{l,S}^\dagger \cap (TP_{k+l,A} \times \mathrm{Lie}(A))$. This is constructed inductively as follows. For $l = 0$, we note $P_{k,A} = S = S_{0,A}$ and $V_k^\dagger \subset TP_{k,A} \times \mathrm{Lie}(A) = V_{0,S}^\dagger$. We discuss the induction step from l to $l+1$. By $P_{k+l,A} \subset S_{l,A}$ and $V_{k+l}^\dagger \subset V_{l,S}^\dagger \cap (TP_{k+l} \times \mathrm{Lie}(A))$, we have

$$P_{k+l+1,A} = P(V_{k+l}^\dagger) \subset P(V_{l,S}^\dagger) = S_{l+1,A}.$$

The constructions of $V_{k+l+1}^\dagger$ and $V_{l+1,S}^\dagger$ (cf. (2.11)) yield

$$V_{k+l+1}^\dagger \subset V_{l+1,S}^\dagger \cap (TP_{k+l+1,A} \times \mathrm{Lie}(A)).$$

This completes the induction step.

Remark 2.4. As we shall see later in Section 11 (cf. (11.10)), we have a natural morphism $\mu_k \colon P_{k,A} \backslash E_{k,A,A/B} \to P_{k,A/B}$, which plays an important role in the proof of Proposition 11.3.

§3. Sufficient condition for normality: Statement of Proposition 3.20

The goal of this section is to introduce Proposition 3.20. This proposition gives a sufficient condition for a subset of $\mathrm{Hol}(\mathbb{D}, A)$ to be normal, where A is a semi-abelian variety. The proof of this proposition is rather lengthy, so we devote Sections 4–8 to the proof. To state our proposition, we need to prepare some terminology, which we describe below.

§3.1. Family of holomorphic maps and Zariski closed sets

We start from the following two definitions.

Definition 3.1. Let S be a variety and let $Z \subset S$ be a Zariski closed set.

(1) By a *Z-admissible modification* $\varphi \colon S' \to S$, we assume that

(a) φ is projective and birational, and

(b) there exists a Zariski open set $U \subset S$ such that $Z \cap U \neq \emptyset$ and $\varphi^{-1}(U) \to U$ is an isomorphism.

(2) For a Z-admissible modification $\varphi\colon S' \to S$, we define the *minimal transform* $Z' \subset S'$ as follows. Let $\mathcal{U}$ be the set of all Zariski open subsets $U \subset S$ with property (b) above. We set $Z' = \bigcap_{U\in\mathcal{U}} \overline{\varphi^{-1}(Z\cap U)}$, where $\overline{\varphi^{-1}(Z\cap U)} \subset S'$ is the Zariski closure.

To introduce the next definition, we recall the terminology "γ-almost all" already used in Theorem 1.2: Given $\gamma > 0$ and an open set $W \subset \mathbb{C}$, we say that an assertion concerning points $w \in W$ holds for γ-almost all $w \in W$ if the set of points $w \in W$ that do not satisfy the assertion is covered by at most countably many closed discs whose total sum of radii is less than γ.

Definition 3.2. Let $\mathcal{F} = (f_i)_{i\in I}$ be an infinite indexed family in $\mathrm{Hol}(\mathbb{D}, S)$, where S is a variety. Let $Z \subset S$ be a Zariski closed set.

(1) We write $\mathcal{F} \to Z$ if the following holds: For every $0 < s < 1$, $\gamma > 0$, and open neighborhood $U \subset S$ of Z, there exists a finite subset $E \subset I$ such that, for all $i \in I - E$, we have $f_i(z) \in U$ for γ-almost all $z \in \mathbb{D}(s)$.
(2) We write $\mathcal{F} \Rightarrow Z$ if the following hold:
  (a) Let $V \subset S$ be a Zariski closed set such that $Z \not\subset V$. Then $f_i(\mathbb{D}) \not\subset V$ for all $i \in I$ with finite exception.
  (b) Let $S' \to S$ be a Z-admissible modification and let $Z' \subset S'$ be the minimal transform. Then $\mathcal{F} \to Z'$.

Remark 3.3. We supplement the condition $\mathcal{F} \to Z'$ in assertion (2b). There exists a Zariski open set $U \subset S$ such that $\varphi\colon S' \to S$ satisfies assertion (1b) in Definition 3.1. Then by $Z \not\subset (S - U)$, there exists a finite subset $E \subset I$ such that $f_i(\mathbb{D}) \not\subset (S - U)$ for all $i \in I - E$. Then for each $i \in I - E$, there is a natural lift $f_i'\colon \mathbb{D} \to S'$ of f_i. By $\mathcal{F} \to Z'$ in assertion (2b), we mean $(f_i')_{i\in I-E} \to Z'$.

Remark 3.4. If $\mathcal{F} \to Z$, then $\mathcal{G} \to Z$ for all infinite indexed subfamilies $\mathcal{G}$ of $\mathcal{F}$. Here we call an infinite indexed family $\mathcal{G} = (f_j)_{j\in J}$ a subfamily of $\mathcal{F} = (f_i)_{i\in I}$ if $J \subset I$ is an infinite subset. If $\mathcal{F} \Rightarrow Z$, then $\mathcal{G} \Rightarrow Z$ for all infinite indexed subfamilies $\mathcal{G}$ of $\mathcal{F}$.

We prove several basic properties related to Definition 3.2.

Lemma 3.5. *Let $\mathcal{F} = (f_i)_{i\in I}$ be an infinite indexed family in* $\mathrm{Hol}(\mathbb{D}, S)$. *Let Z_1 and Z_2 be Zariski closed subsets of S. If $\mathcal{F} \to Z_1$ and $\mathcal{F} \to Z_2$, then $\mathcal{F} \to Z_1 \cap Z_2$.*

Proof. Set $S_o = S\backslash(Z_1 \cap Z_2)$. Then $(Z_1 \cap S_o) \cap (Z_2 \cap S_o) = \emptyset$. Hence there exist open neighborhoods $U_1 \subset S_o$ of $Z_1 \cap S_o$ and $U_2 \subset S_o$ of $Z_2 \cap S_o$ such that $U_1 \cap U_2 = \emptyset$.

We take $0 < s < 1$, $\gamma > 0$, and an open neighborhood $U \subset S$ of $Z_1 \cap Z_2$. Then $U_1 \cup U \subset S$ is an open neighborhood of Z_1. Hence by $\mathcal{F} \to Z_1$, there exists a finite set $E_1 \subset I$ such that for all $i \in I \backslash E_1$, we have $f_i(z) \in U_1 \cup U$ for $\gamma/2$-almost all $z \in \mathbb{D}(s)$. Similarly, by $\mathcal{F} \to Z_2$, there exists a finite set $E_2 \subset I$ such that for all $i \in I \backslash E_2$, we have $f_i(z) \in U_2 \cup U$ for $\gamma/2$-almost all $z \in \mathbb{D}(s)$. Set $E = E_1 \cup E_2$. Then E is finite. Note that $(U_1 \cup U) \cap (U_2 \cup U) = U$. Hence for all $i \in I \backslash E$, we have $f_i(z) \in U$ for γ-almost all $z \in \mathbb{D}(s)$. Thus $\mathcal{F} \to Z_1 \cap Z_2$. $\square$

Lemma 3.6. *Let $\mathcal{F} = (f_i)_{i \in I}$ be an infinite indexed family in* $\mathrm{Hol}(\mathbb{D}, S)$. *Then there exists a Zariski closed subset Y such that Zariski closed subsets $Z \subset S$ satisfy $\mathcal{F} \to Z$ if and only if $Y \subset Z$. In particular, $\mathcal{F} \to Y$.*

Proof. We denote by $\mathcal{Z}$ the set of all Zariski closed subsets $Z \subset S$ such that $\mathcal{F} \to Z$. We have $\mathcal{F} \to S$, hence $S \in \mathcal{Z}$. Hence $\mathcal{Z} \neq \emptyset$. By the Noetherian property, there exists a minimal element $Y \in \mathcal{Z}$. Then $\mathcal{F} \to Y$. We note that if $Z \in \mathcal{Z}$, then $Y \subset Z$. Indeed, if not, then $Y \cap Z \subsetneqq Y$. By $\mathcal{F} \to Y$ and $\mathcal{F} \to Z$, we have $\mathcal{F} \to Y \cap Z$ (cf. Lemma 3.5), hence $Y \cap Z \in \mathcal{Z}$. This contradicts the choice of Y. Hence $Y \subset Z$. Conversely, if a Zariski closed set $Z \subset S$ satisfies $Y \subset Z$, then by $\mathcal{F} \to Y$, we have $\mathcal{F} \to Z$. Hence $Z \in \mathcal{Z}$. Thus $Z \in \mathcal{Z}$ if and only if $Y \subset Z$. $\square$

Remark 3.7. Suppose $\mathcal{F} = (f_i)_{i \in I}$ converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic map $g \colon \mathbb{D} \to S$. Let $V \subset S$ be the Zariski closure of $g(\mathbb{D})$. Then Y in Lemma 3.6 coincides with V. To check this, we note $\mathcal{F} \to V$, hence $Y \subset V$. To check the converse, we assume on the contrary that $V \not\subset Y$. Then $g(\mathbb{D}) \not\subset Y$. Hence $g^{-1}(Y) \subset \mathbb{D}$ is a discrete subset. Hence we may take a closed disc $K \subset \mathbb{D} \backslash g^{-1}(Y)$ of positive radius. We take an open neighborhood $U \subset S$ of Y such that $g(K) \cap \overline{U} = \emptyset$. Then since $\mathcal{F}$ converges to g uniformly on K, there exists a finite subset $E \subset I$ such that for all $i \in I \backslash E$, we have $f_i(K) \cap U = \emptyset$. This contradicts $\mathcal{F} \to Y$. Hence $V \subset Y$. Thus $V = Y$.

Lemma 3.8. *Let $\varphi \colon S' \to S$ be a morphism and let $Z \subset S$ be a Zariski closed set. Let $\mathcal{F} = (f_i)_{i \in I}$ be an infinite indexed family in* $\mathrm{Hol}(\mathbb{D}, S')$ *such that $(\varphi \circ f_i)_{i \in I} \to Z$. Suppose $\varphi \colon S' \to S$ is proper. Then $\mathcal{F} \to \varphi^{-1}(Z)$.*

Proof. Let $s \in (0, 1)$ and $\gamma > 0$. Let $U \subset S'$ be an open neighborhood of $\varphi^{-1}(Z)$. Since φ is proper, $\varphi(S' \backslash U)$ is a closed subset. Set $W = S \backslash \varphi(S' \backslash U)$. Then $W \subset S$ is an open set such that $Z \subset W$. We have $\varphi^{-1}(W) \subset U$. By $(\varphi \circ f_i)_{i \in I} \to Z$, there exists a finite subset $E \subset I$ such that for all $i \in I \backslash E$ we have $\varphi \circ f_i(z) \in W$ for γ-almost all $z \in \mathbb{D}(s)$. Then for all $i \in I \backslash E$ we have $f_i(z) \in U$ for γ-almost all $z \in \mathbb{D}(s)$. Hence $\mathcal{F} \to \varphi^{-1}(Z)$. $\square$

Lemma 3.9. *Let $\mathcal{F}$ be an infinite indexed family in* $\mathrm{Hol}(\mathbb{D}, S)$*. Let Z_1 and Z_2 be Zariski closed subsets of S. If $\mathcal{F} \Rightarrow Z_1$ and $\mathcal{F} \Rightarrow Z_2$, then either $Z_1 \subset Z_2$ or $Z_2 \subset Z_1$.*

Proof. Assume on the contrary that $Z_1 \not\subset Z_2$ and $Z_2 \not\subset Z_1$. Set $V = Z_1 \cap Z_2$ in the sense of scheme theory, namely $\mathcal{I}_V = \mathcal{I}_{Z_1} + \mathcal{I}_{Z_2}$ for the defining ideal sheaves in $\mathcal{O}_S$. Set $S' = \mathrm{Bl}_V S$ with $\varphi \colon S' \to S$. Let Z_1' and Z_2' be the strict transforms of Z_1 and Z_2, respectively. Then

$$Z_1' \cap Z_2' = \emptyset. \tag{3.1}$$

We prove this. Note that $\varphi^* V \subset S'$ is a Cartier divisor. Hence we may take a closed subscheme $Z_1'' \subset S'$ such that $\varphi^* Z_1 = \varphi^* V + Z_1''$. Indeed, we have $\mathcal{I}_{\varphi^* Z_1} \subset \mathcal{I}_{\varphi^* V} \subset \mathcal{O}_{S'}$, where $\mathcal{I}_{\varphi^* V}$ is an invertible sheaf. So we take $Z_1'' \subset S'$ so that $\mathcal{I}_{Z_1''} = \mathcal{I}_{\varphi^* Z_1} \otimes (\mathcal{I}_{\varphi^* V})^{-1} \subset \mathcal{O}_{S'}$. Then $\mathcal{I}_{Z_1''} \cdot \mathcal{I}_{\varphi^* V} = \mathcal{I}_{\varphi^* Z_1}$. Similarly, there exists $Z_2'' \subset S'$ such that $\mathcal{I}_{Z_2''} \cdot \mathcal{I}_{\varphi^* V} = \mathcal{I}_{\varphi^* Z_2}$. Then by $\mathcal{I}_{\varphi^* Z_1} + \mathcal{I}_{\varphi^* Z_2} = \mathcal{I}_{\varphi^* V}$, we have $(\mathcal{I}_{Z_1''} + \mathcal{I}_{Z_2''}) \cdot \mathcal{I}_{\varphi^* V} = \mathcal{I}_{\varphi^* V}$. Hence $\mathcal{I}_{Z_1''} + \mathcal{I}_{Z_2''} = \mathcal{O}_{S'}$. This shows $Z_1'' \cap Z_2'' = \emptyset$. By $Z_1' \subset Z_1''$ and $Z_2' \subset Z_2''$, we get (3.1).

Now note that $\mathrm{Bl}_V S \to S$ is Z_1-admissible. Hence by $\mathcal{F} \Rightarrow Z_1$, we have $\mathcal{F} \to Z_1'$. Similarly, $\mathcal{F} \to Z_2'$. This is a contradiction. □

Lemma 3.10. *Let $\mathcal{F}$ be an infinite indexed family in* $\mathrm{Hol}(\mathbb{D}, S)$*. Let $Z \subset S$ be a Zariski closed set such that $\mathcal{F} \Rightarrow Z$. Then Z is irreducible.*

Proof. We assume on the contrary that $Z = Z_1 \cup Z_2$. Set $V = Z_1 \cap Z_2$ in the sense of scheme theory. Then $Z_1' \cap Z_2' = \emptyset$ in $\mathrm{Bl}_V S$, where Z_1' and Z_2' are the strict transforms (cf. (3.1)). Let $\varphi \colon \mathrm{Bl}_V S \to S$, which is Z-admissible. Set $U_1 = S - Z_2$. Then $V \cap U_1 = \emptyset$. Hence $\varphi^{-1}(U_1) \to U_1$ is an isomorphism. Moreover, we have $Z \cap U_1 = Z_1 \cap U_1 \neq \emptyset$. Hence the minimal transform $Z' \subset \mathrm{Bl}_Z S$ satisfies $Z' \subset \overline{\varphi^{-1}(Z_1 \cap U_1)} \subset Z_1'$. Similarly $Z' \subset Z_2'$. Hence $Z' = \emptyset$. This contradicts $\mathcal{F} \to Z'$. Hence Z is irreducible. □

Lemma 3.11. *Let $\mathcal{F} = (f_i)_{i \in I}$ be an infinite indexed family in* $\mathrm{Hol}(\mathbb{D}, S)$*. Let $Z \subset S$ be a Zariski closed set such that $\mathcal{F} \Rightarrow Z$. Let $\varphi \colon S' \to S$ be a Z-admissible modification and let $Z' \subset S'$ be the minimal transform. Then $\mathcal{F} \Rightarrow Z'$.*

Proof. If $V \subset S'$ is a Zariski closed set such that $Z' \not\subset V$, then $Z \not\subset \varphi(V)$. For all $i \in I$ with finite exception, we have $f_i(\mathbb{D}) \not\subset \varphi(V)$, hence $f_i'(\mathbb{D}) \not\subset V$, where $f_i' \colon \mathbb{D} \to S'$ is the lift of $f_i \colon \mathbb{D} \to S$. Let $S'' \to S'$ be a Z'-admissible modification and let $Z'' \subset S''$ be the minimal transform of Z'. Then $S'' \to S$ is Z-admissible and the minimal transform of Z coincides with Z''. These easily follow from the irreducibility of Z (cf. Lemma 3.10). Hence $\mathcal{F} \to Z''$. Hence $\mathcal{F} \Rightarrow Z'$. □

Definition 3.12. Let $\mathcal{F}$ be an infinite indexed family in $\mathrm{Hol}(\mathbb{D}, S)$ and let $\{E_i\}_{i\in I}$ be a countable family of Zariski closed sets in S. We define $\mathrm{LIM}(\mathcal{F};\ \{E_i\})$ as the Zariski closed subset of S with the following properties:

(1) $\mathcal{F} \Rightarrow \mathrm{LIM}(\mathcal{F};\ \{E_i\})$ and $\mathrm{LIM}(\mathcal{F};\ \{E_i\}) \not\subset \bigcup_{i\in I} E_i$;

(2) for every proper Zariski closed subset $W \subsetneqq \mathrm{LIM}(\mathcal{F};\ \{E_i\})$ and every infinite indexed subfamily $\mathcal{G}$ of $\mathcal{F}$, either $\mathcal{G} \not\Rightarrow W$ or $W \subset \bigcup_{i\in I} E_i$.

Remark 3.13. We collect some basic properties.

(1) $\mathrm{LIM}(\mathcal{F};\ \{E_i\})$ is unique if it exists. This follows from Lemma 3.9. Indeed, suppose that both $Z_1, Z_2 \subset S$ satisfy the two conditions of Definition 3.12. Then $\mathcal{F} \Rightarrow Z_1$ and $\mathcal{F} \Rightarrow Z_2$. By Lemma 3.9, we may assume that $Z_1 \subset Z_2$. Assume on the contrary that $Z_1 \neq Z_2$. Then by the second condition of Definition 3.12, we have $Z_1 \subset \bigcup_{i\in I} E_i$, for $\mathcal{F} \Rightarrow Z_1$. This contradicts the first condition of Definition 3.12 that $Z_1 \not\subset \bigcup_{i\in I} E_i$. Hence $Z_1 = Z_2$. This shows that $\mathrm{LIM}(\mathcal{F};\ \{E_i\})$ is unique if it exists.

(2) $\mathrm{LIM}(\mathcal{F};\ \{E_i\})$ is irreducible, if it exists. This follows from Lemma 3.10.

(3) If $\mathrm{LIM}(\mathcal{F};\ \{E_i\})$ exists, then for all infinite subfamilies $\mathcal{G}$ of $\mathcal{F}$, $\mathrm{LIM}(\mathcal{G};\ \{E_i\})$ exists and $\mathrm{LIM}(\mathcal{G};\ \{E_i\}) = \mathrm{LIM}(\mathcal{F};\ \{E_i\})$. This follows directly from Definition 3.12 (cf. Remark 3.4).

Lemma 3.14. *Let $\mathcal{F} = (f_i)_{i\in I}$ be an infinite indexed family in $\mathrm{Hol}(\mathbb{D}, S)$. Let $\{E_j\}_{j\in J}$ be a countable family of Zariski closed sets of S. Assume that for each $j \in J$, all but finitely many $i \in I$ satisfy $f_i(\mathbb{D}) \not\subset E_j$. Then there exists an infinite indexed subfamily $\mathcal{G}$ of $\mathcal{F}$ such that $\mathrm{LIM}(\mathcal{G};\ \{E_j\})$ exists.*

Proof. We denote by $\mathcal{Z}$ the set of Zariski closed subsets $Z \subset S$ such that $f_i(\mathbb{D}) \subset Z$ for infinitely many $i \in I$. We have $S \in \mathcal{Z}$, hence $\mathcal{Z} \neq \emptyset$. By the Noetherian property, we may take a minimal $Z \in \mathcal{Z}$. We take an infinite subset $I' \subset I$ such that $f_i(\mathbb{D}) \subset Z$ for all $i \in I'$. Set $\mathcal{F}' = (f_i)_{i\in I'}$. Then $\mathcal{F}' \Rightarrow Z$. By Lemma 3.10, Z is irreducible. We have $Z \not\subset \bigcup E_j$. Indeed, suppose $Z \subset \bigcup E_j$. Then since J is countable, there exists $j \in J$ such that $Z \subset E_j$. Hence $f_i(\mathbb{D}) \subset E_j$ for all $i \in I'$, a contradiction. Thus $Z \not\subset \bigcup E_j$.

For an infinite subfamily $\mathcal{G}$ of $\mathcal{F}'$, we denote by $\mathcal{W}_{\mathcal{G}}$ the set of Zariski closed subsets $W \subset S$ such that $\mathcal{G} \Rightarrow W$ and $W \not\subset \bigcup_{j\in J} E_j$. By the above argument, we have $Z \in \mathcal{W}_{\mathcal{G}}$. Hence $\mathcal{W}_{\mathcal{G}} \neq \emptyset$. By Lemma 3.9 and the Noetherian property, we may take a unique minimal element $W_{\mathcal{G}} \subset \mathcal{W}_{\mathcal{G}}$. If $\mathcal{G}'$ is an infinite subfamily of $\mathcal{G}$, then $W_{\mathcal{G}} \subset \mathcal{W}_{\mathcal{G}'}$. Hence $W_{\mathcal{G}'} \subset W_{\mathcal{G}}$. Hence by the Noetherian property, we may take an infinite subfamily $\mathcal{G}$ of $\mathcal{F}'$ such that $W_{\mathcal{G}} = W_{\mathcal{G}'}$ for all infinite subfamilies $\mathcal{G}'$ of $\mathcal{G}$. Then $\mathrm{LIM}(\mathcal{G};\ \{E_j\}) = W_{\mathcal{G}}$. $\square$

Lemma 3.15. *Let $p\colon S_1 \to S_2$ be a morphism of varieties. Let $\mathcal{F} = (f_i)_{i\in I}$ be an infinite indexed family in* $\mathrm{Hol}(\mathbb{D}, S_1)$*. Let $Z \subset S_1$ be a Zariski closed subset such that $\mathcal{F} \Rightarrow Z$. Then $\{p \circ f_i\}_{i\in I} \Rightarrow \overline{p(Z)}$, where $\overline{p(Z)} \subset S_2$ is the Zariski closure of $p(Z)$.*

Proof. Let $V \subset S_2$ be a Zariski closed set such that $\overline{p(Z)} \not\subset V$. Then we have $Z \not\subset p^{-1}(V)$. We have $f_i(\mathbb{D}) \not\subset p^{-1}(V)$ for all $i \in I$ with finite exception. Hence $p \circ f_i(\mathbb{D}) \not\subset V$ for all $i \in I$ with finite exception.

Now let $S_2' \to S_2$ be a $\overline{p(Z)}$-admissible modification. We may take a Z-admissible modification $S_1' \to S_1$ such that $p'\colon S_1' \to S_2'$ exists. Let $Z' \subset S_1'$ and $\overline{p(Z)}' \subset S_2'$ be the minimal transforms of Z and $\overline{p(Z)}$, respectively. Then $p'(Z') \subset \overline{p(Z)}'$. We have $\mathcal{F} \to Z'$. Hence $\{p' \circ f_i\}_{i\in I} \to \overline{p(Z)}'$. Thus $\{p \circ f_i\}_{i\in I} \Rightarrow \overline{p(Z)}$. $\square$

§3.2. Horizontally integrable

Let S be a smooth algebraic variety. Let $\iota\colon U \hookrightarrow A \times S$ be an immersion, i.e., open of closed immersion. Assume that $q\colon U \to S$ is étale, where q is the composite of the immersion $U \hookrightarrow A \times S$ and the second projection $A \times S \to S$. Then we get an immersion $[\iota_*]\colon PTU \hookrightarrow PT(A\times S)$, which is the projectivization of the differential $\iota_*\colon TU \hookrightarrow T(A \times S)$. By (2.12), we have $PT(A \times S) = A \times S_{1,A}$. Hence by the composite of the immersion $[\iota_*]$ and the second projection $PT(A \times S) \to S_{1,A}$, we get $\iota'\colon PTU \to S_{1,A}$.

Definition 3.16. Let $Z \subset S_{1,A}$ be an irreducible Zariski closed set. We say that Z is *horizontally integrable* if there exists an immersion $\iota\colon U \hookrightarrow A \times S$ such that $q\colon U \to S$ is étale and $Z \cap \iota'(PTU) \subset Z$ is Zariski dense in Z.

We describe a simple example. Let $S \hookrightarrow A \times S$ be an immersion induced from a constant map $S \to A$, and let $S_{1,A}^* \subset S_{1,A}$ be the image of $PTS \to PT(A\times S) \to S_{1,A}$. Namely, we set

$$S_{1,A}^* = P(TS \times \{0\}). \tag{3.2}$$

Then every irreducible Zariski closed set $Z \subset S_{1,A}$ such that $Z \subset S_{1,A}^*$ is horizontally integrable.

In general, let $\iota\colon U \hookrightarrow A \times S$ be an immersion such that $q\colon U \to S$ is étale. Then the induced map $q'\colon U_{1,A} \to S_{1,A}$ is also étale. Let $\varphi\colon U \to A$ be the composite of ι and the first projection $A \times S \to A$. We note that the immersion $\iota\colon U \hookrightarrow A \times S$ implies an isomorphism

$$\Phi\colon A \times U \to A \times U \tag{3.3}$$

over U by

$$\Phi(a,u) = (a - \varphi(u), u). \tag{3.4}$$

This defines an isomorphism $[\Phi_*]\colon PT(A\times U) \to PT(A\times U)$. This induces

$$D\Phi\colon U_{1,A} \to U_{1,A},$$

which is an isomorphism over U: Under the decomposition $PT(A\times U) = A\times U_{1,A}$ (cf. (2.12)), we have $[\Phi_*]((a,u')) = (a - \varphi(\tau(u')), D\Phi(u'))$ for $(a,u') \in A\times U_{1,A}$, where $\tau\colon U_{1,A} \to U$ is the natural map.

Lemma 3.17. *Suppose $Z \subset S_{1,A}$ is horizontally integrable. Let $\iota\colon U \hookrightarrow A\times S$ be an immersion as in Definition 3.16. Then there exists an irreducible component $\Xi \subset U_{1,A}$ of $(q')^{-1}(Z)$ such that $D\Phi(\Xi) \subset U^*_{1,A}$.*

Proof. The immersion ι induces the section $\iota_U\colon U \hookrightarrow A\times U$ of the second projection $A\times U \to U$. Then ι_U induces $\iota'_U\colon PTU \to U_{1,A}$ as above. Note that $\iota'\colon PTU \to S_{1,A}$ is the composite of $\iota'_U\colon PTU \to U_{1,A}$ and $q'\colon U_{1,A} \to S_{1,A}$. Let $\Xi_1,\dots,\Xi_l$ be the irreducible components of $(q')^{-1}(Z)$. Then we have

$$\bigcup_i q'(\iota'_U(PTU)\cap \Xi_i) = \iota'(PTU)\cap Z.$$

There exists Ξ_i such that $q'(\iota'_U(PTU)\cap\Xi_i) \subset Z$ is Zariski dense in Z. We set $\Xi = \Xi_i$. Since q' is étale, we have $\dim\Xi = \dim Z$. Hence $\iota'_U(PTU)\cap\Xi \subset \Xi$ is Zariski dense in Ξ.

Since the composite of $U \xrightarrow{\iota_U} A\times U \xrightarrow{\Phi} A\times U$ is the graph of a constant map, the image of the composite of $PTU \xrightarrow{\iota'_U} U_{1,A} \xrightarrow{D\Phi} U_{1,A}$ is $U^*_{1,A}$. Hence $\iota'_U(PTU) \subset (D\Phi)^{-1}U^*_{1,A}$, so $\Xi\cap\iota'_U(PTU) \subset (D\Phi)^{-1}U^*_{1,A}$. Since $\Xi\cap\iota'_U(PTU)$ is Zariski dense in Ξ and $(D\Phi)^{-1}U^*_{1,A}$ is Zariski closed, we have $\Xi \subset (D\Phi)^{-1}U^*_{1,A}$. This concludes the proof. □

§3.3. Statement of Proposition 3.20

We say that a smooth positive $(1,1)$-form ω_A on a semi-abelian variety A is invariant if ω_A is invariant under the translation of A.

Definition 3.18. Given an infinite indexed family $\mathcal{F} = (f_i)_{i\in I}$ in $\mathrm{Hol}(\mathbb{D}, A)$, we denote by $\Pi(\mathcal{F})$ the set of all semi-abelian subvarieties $B \subset A$ satisfying the following property: there is no infinite subset $J \subset I$ such that $\{|(\varpi_B\circ f_i)'|_{\omega_{A/B}}\}_{i\in J}$ converges uniformly on compact subsets of $\mathbb{D}$ to 0. Here, $\varpi_B\colon A \to A/B$ is the quotient map and $|\cdot|_{\omega_{A/B}}$ is a norm on $T(A/B)$ defined by an invariant $(1,1)$-form $\omega_{A/B}$ on A/B.

Note that every semi-abelian variety contains only countably many semi-abelian varieties (cf. [35, Cor. 5.1.9]). Hence $\Pi(\mathcal{F})$ is a countable set. We note $A \notin \Pi(\mathcal{F})$. Hence if $B \in \Pi(\mathcal{F})$, then $E_{k,A,A/B} \subsetneqq P_{k,A}$ for all $k \geq 1$.

For a non-constant holomorphic map $f: \mathbb{D} \to A \times S$, we recall the notation $f_{S_{k,A}}: \mathbb{D} \to S_{k,A}$ from (2.13). We use the notation $f_{P_{k,A}}: \mathbb{D} \to P_{k,A}$ if $S = \{\mathrm{pt}\}$, where $P_{k,A} = \{\mathrm{pt}\}_{k,A}$. Given an infinite indexed family $\mathcal{F} = (f_i)_{i\in I}$ of non-constant holomorphic maps in $\mathrm{Hol}(\mathbb{D}, A)$, we define an infinite indexed family $\mathcal{F}_{P_{k,A}}$ in $\mathrm{Hol}(\mathbb{D}, P_{k,A})$ by

$$\mathcal{F}_{P_{k,A}} = ((f_i)_{P_{k,A}})_{i\in I}. \tag{3.5}$$

We consider the following assumption for an infinite indexed family $\mathcal{F}$ of non-constant holomorphic maps in $\mathrm{Hol}(\mathbb{D}, A)$.

Assumption 3.19. The Zariski closed subset $\mathrm{LIM}(\mathcal{F}_{P_{k,A}};\ \{E_{k,A,A/B}\}_{B\in\Pi(\mathcal{F})})$ exists for all $k \geq 1$.

If this assumption is satisfied, we write

$$T_k = \mathrm{LIM}(\mathcal{F}_{P_{k,A}};\ \{E_{k,A,A/B}\}_{B\in\Pi(\mathcal{F})}) \subset P_{k,A}.$$

As we shall see later (cf. Lemma 12.2), every infinite indexed family $\mathcal{F}$ in $\mathrm{Hol}(\mathbb{D}, A)$ contains an infinite subfamily which satisfies this assumption.

Now we take an infinite indexed family $\mathcal{F}$ in $\mathrm{Hol}(\mathbb{D}, A)$ which satisfies Assumption 3.19. Let $k \geq 1$. We claim

$$T_k \subset p(T_{k+1}), \tag{3.6}$$

where $p: P_{k+1,A} \to P_{k,A}$ is the natural map. Indeed, by Lemma 3.15, we have $\mathcal{F}_{P_{k,A}} \Rightarrow p(T_{k+1})$. Hence by Lemma 3.9, we have either $T_k \subset p(T_{k+1})$ or $p(T_{k+1}) \subsetneqq T_k$. So assume on the contrary that $p(T_{k+1}) \subsetneqq T_k$. Then by the definition of T_k, we have $p(T_{k+1}) \subset \bigcup_{B\in\Pi(\mathcal{F})} E_{k,A,A/B}$. We have $p^{-1}(E_{k,A,A/B}) \subset E_{k+1,A,A/B}$ (cf. Lemma 2.2). Hence

$$T_{k+1} \subset p^{-1}\left(\bigcup_{B\in\Pi(\mathcal{F})} E_{k,A,A/B}\right) \subset \bigcup_{B\in\Pi(\mathcal{F})} E_{k+1,A,A/B}.$$

This is a contradiction. Hence we have proved (3.6).

We fix $k \geq 0$. For each $l \geq 1$, let $p_l: P_{k+l,A} \to P_{k+1,A}$ be the natural map. Then by (3.6), we have a sequence

$$p_1(T_{k+1}) \subset p_2(T_{k+2}) \subset p_3(T_{k+3}) \subset \cdots.$$

By Remark 3.13 (2), each $p_l(T_{k+l})$ is irreducible. Hence the sequence above stabilizes, i.e., there exists $l_0 \geq 1$ such that $p_l(T_{k+l}) = p_{l_0}(T_{k+l_0})$ for all $l \geq l_0$. We set $Z = p_{l_0}(T_{k+l_0})$, namely

$$Z = \bigcup_{l \geq 1} p_l(T_{k+l}). \tag{3.7}$$

By Remark 2.3, we have $Z \subset P_{k+1,A} \subset (P_{k,A})_{1,A}$.

Proposition 3.20. *Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A)$ be an infinite set of non-constant holomorphic maps which satisfies Assumption 3.19. Suppose that there exists $k \geq 0$ such that $Z \subset P_{k+1,A} \subset (P_{k,A})_{1,A}$ is horizontally integrable, where Z is defined by (3.7). Then $\mathcal{F}$ is normal for every smooth equivariant compactification $\bar{A}$. Namely, for every smooth equivariant compactification $\bar{A}$ and every sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, there exists a subsequence of $(f_n)_{n\in\mathbb{N}}$ which converges uniformly on compact subsets of $\mathbb{D}$ to $g\colon \mathbb{D} \to \bar{A}$.*

This proposition will be proved in Section 8 after the preparation in Sections 4–7.

§3.4. Outline of the proof of Proposition 3.20

We discuss an outline of the proof of Proposition 3.20. The proof heavily relies on Nevanlinna theory. In the next section, we introduce Nevanlinna theory and prove several estimates needed for the proof of Proposition 3.20. In Section 5, we prove a main proposition, Proposition 5.1, for the proof of Proposition 3.20. Here, the assumption that $Z \subset P_{k+1,A}$ is horizontally integrable plays a crucial role. For some technical reasons described below, we apply Proposition 5.1 not to the sequence $((f_n)_{[k]})$ itself, but to a certain modification $((\varphi_n)_{[k]})$ obtained by a translate and by passing to a subsequence, which ensures that Assumption 6.2 is satisfied together with $((\varphi_n)_{P_{k+1,A}})_{n\in\mathbb{N}} \Rightarrow Z$. The conclusion of Proposition 5.1 then yields an estimate in Nevanlinna theory of the following form:

$$\begin{aligned} T_s(r, \varphi_n, \omega_{\bar{A}}) \leq{}& c_1 T_s\big(r, ((\varphi_n)_{[k]})_{P_{k,A}}, \omega_{P_{k,A}}\big) \\ &+ c_2 m\big((s+r)/2, ((\varphi_n)_{[k]})_{P_{k,A}}, \lambda_W\big) + c_3. \end{aligned} \tag{3.8}$$

This modification of the sequence (f_n) to (φ_n), which is performed in Lemma 8.1, is necessary to estimate the right-hand side of (3.8). Thanks to Lemma 8.2, it suffices to work with (φ_n).

In Sections 6 and 7, we estimate the first and second terms on the right-hand side of (3.8), respectively. In Section 6, we introduce a geometric generalization of Nevanlinna's lemma on logarithmic derivatives, namely the logarithmic tautological inequality, and prove Lemma 6.3. This lemma shows that the first term on the right-hand side of (3.8) is very small compared with the left-hand side, thanks to the modification (φ_n). Thus, we can ignore it. In Section 7, we prove Lemma 7.4, again using Lemma 6.3. Here, the fact that $Z \subset P_{k+1,A}$ is constructed in the particular form (3.7) plays a crucial role. This lemma shows that the second term on the right-hand side of (3.8) is also very small compared with the left-hand side, again thanks to the modification (φ_n). From this, we conclude that the left-hand side of (3.8) is bounded, which implies that

$$\int_{\mathbb{D}(s)} \varphi_n^* \omega_{\bar{A}}$$

is also bounded. By the normality condition proved in Lemma 8.3, this allows us to choose a convergent subsequence from (φ_n), thereby proving Proposition 3.20 using Lemma 8.2. This is a rough outline of the proof of Proposition 3.20, omitting several issues that require careful discussion.

The most technically involved part of the proof is Section 5. For more discussion about the proof of Proposition 5.1, see Section 5.1.

§4. Nevanlinna theory

Let V be an algebraic variety. We denote by $\mathrm{Hol}_m(\mathbb{D}, V)$ the set of all multivalued holomorphic maps to V, i.e., each map is represented by

$$\begin{array}{ccc} Y_f & \xrightarrow{f} & V, \\ \big\downarrow{\scriptstyle \pi_f} & & \\ \mathbb{D}, & & \end{array}$$

where f is a holomorphic map from a Riemann surface Y_f with a proper, surjective holomorphic map $\pi_f \colon Y_f \to \mathbb{D}$. By the properness, $\pi_f \colon Y_f \to \mathbb{D}$ is a finite ramified covering with a well-defined covering degree, denoted by $\deg \pi_f$. Namely, if $z \in \mathbb{D}$ is not a branch point, the fiber $\pi_f^{-1}(z)$ consists of exactly $\deg \pi_f$ points.

We introduce the notion of Weil functions (cf., e.g., [44, Def. 2.2.1]). Let V be a projective variety and let $Z \subset V$ be a closed subscheme. A Weil function λ_Z for Z is a continuous function $\lambda_Z \colon V - \operatorname{supp} Z \to \mathbb{R}$ which satisfies the following condition. For each $x \in V$, there are a Zariski open neighborhood $U \subset V$ of x, holomorphic functions $g_1, \ldots, g_l \in \Gamma(U, \mathcal{O}_V)$ which define $Z \cap U$, and a continuous function $\alpha \colon U \to \mathbb{R}$ on U such that

$$\left| \lambda_Z(y) + \log \max_{1 \le i \le l} \{|g_i(y)|\} \right| \le \alpha(y)$$

for all $y \in U - \operatorname{supp}(Z \cap U)$. We summarize the needed properties of Weil functions (cf. [44, Sec. 2.2]):

- A Weil function λ_Z exists for every closed subscheme $Z \subset V$.
- If λ_Z and λ'_Z are Weil functions for Z, then there exists a positive constant γ such that $|\lambda_Z(x) - \lambda'_Z(x)| \leq \gamma$ for all $x \in V - \operatorname{supp} Z$.
- If λ_Z is a Weil function for Z, then λ_Z is bounded from below, namely there exists a constant γ such that $\lambda_Z(x) > \gamma$ for all $x \in V - \operatorname{supp} Z$. In particular, we may choose a Weil function λ_Z such that $\lambda_Z \geq 0$.
- Suppose that $p\colon \widetilde{V} \to V$ is a morphism from another projective variety $\widetilde{V}$. Then $\lambda_Z \circ p$ is a Weil function for the pull-back $p^*Z \subset \widetilde{V}$.
- Let Z, Z' be closed subschemes of V. Let λ_Z and $\lambda_{Z'}$ be Weil functions for Z and Z', respectively. Then $\lambda_Z + \lambda_{Z'}$ is a Weil function for $Z + Z'$, where $Z + Z'$ is defined by $\mathcal{I}_{Z+Z'} = \mathcal{I}_Z \cdot \mathcal{I}_{Z'}$.
- Let Z, Z' be closed subschemes of V. Assume that $\operatorname{supp} Z \subset \operatorname{supp} Z'$. Then there exist positive constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that $\lambda_Z \leq \gamma_1 \lambda_{Z'} + \gamma_2$.
- Assume that V is smooth. Let D be an effective Cartier divisor on V. Let L be a line bundle on V associated to D, and let h be a smooth Hermitian metric on L. Let σ be a section of L associated to D such that $h(\sigma(x), \sigma(x)) \leq 1$ for all $x \in V$. Then $\lambda_D(x) = -\log\sqrt{h(\sigma(x), \sigma(x))}$, where $x \in V - \operatorname{supp} D$, is a Weil function for D with $\lambda_D \geq 0$.

Remark 4.1. In this paper, we always assume $\lambda_Z \geq 0$ for Weil functions unless otherwise specified.

We introduce Nevanlinna theory. Let V be a projective variety. Let $Z \subset V$ be a closed subscheme. Let $f \in \operatorname{Hol}_m(\mathbb{D}, V)$ such that $f(Y_f) \not\subset \operatorname{supp} Z$. For $0 < s < r < 1$, we set

$$N_s(r, f, Z) = \frac{1}{\deg \pi_f} \int_s^r \left(\sum_{y \in Y_f(t)} \operatorname{ord}_y f^* Z \right) \frac{dt}{t},$$

where $Y_f(t) = \pi_f^{-1}(\mathbb{D}(t))$. Let λ_Z be a Weil function for Z. We set

$$m(r, f, \lambda_Z) = \frac{1}{\deg \pi_f} \int_{y \in \partial Y_f(r)} \lambda_Z(f(y)) \frac{d \arg \pi_f(y)}{2\pi}.$$

We set

$$T_s(r, f, Z, \lambda_Z) = N_s(r, f, Z) + m(r, f, \lambda_Z) - m(s, f, \lambda_Z).$$

Assume V is smooth. Let ω be a smooth $(1,1)$-form on V. Let $f \in \operatorname{Hol}_m(\mathbb{D}, V)$. For $0 \leq s < r < 1$, we set

$$T_s(r, f, \omega) = \frac{1}{\deg \pi_f} \int_s^r \frac{dt}{t} \int_{Y_f(t)} f^* \omega.$$

We state the famous first main theorem in our case (e.g., [27, §8.4], [30, p. 219], [35], etc).

First main theorem *Let V be a smooth projective variety. Let $D \subset V$ be an effective Cartier divisor, L be the associated line bundle, and h be a smooth Hermitian metric on L. Let $\omega_{(L,h)}$ be the curvature form for the metrized line bundle (L,h). Let σ be a section of L such that $D = (\sigma = 0)$. Set $\lambda_D(x) = -\log\sqrt{h(\sigma(x),\sigma(x))}$. Then for every $f \in \mathrm{Hol}_m(\mathbb{D}, V)$ with $f(Y_f) \not\subset \operatorname{supp} D$, we have*

$$T_s(r, f, \omega_{(L,h)}) = N_s(r, f, D) + m(r, f, \lambda_D) - m(s, f, \lambda_D) \tag{4.1}$$

for all $0 < s < r < 1$.

Proof. By the Poincaré–Lelong formula, we have

$$2dd^c \log(1/\|\sigma \circ f\|) = -\sum_{y \in Y_f} (\mathrm{ord}_y f^*D)\delta_y + f^*\omega_{(L,h)},$$

where δ_y is Dirac current supported on y. Integrating over $Y_f(t)$, we get

$$2\int_{Y_f(t)} dd^c \log(1/\|\sigma \circ f\|) = -\sum_{Y_f(t)} \mathrm{ord}_y f^*D + \int_{Y_f(t)} f^*\omega_{(L,h)}.$$

Hence, we get

$$\begin{aligned}
-N_s(r, f, D) + T_s(r, f, \omega_{(L,h)}) &= \frac{2}{\deg \pi_f}\int_s^r \frac{dt}{t}\int_{Y_f(t)} dd^c \log\Big(\frac{1}{\|\sigma \circ f\|}\Big)\\
&= \frac{2}{\deg \pi_f}\int_s^r \frac{dt}{t}\int_{\partial Y_f(t)} d^c \log\Big(\frac{1}{\|\sigma \circ f\|}\Big)\\
&= m(r, f, \lambda_D) - m(s, f, \lambda_D).
\end{aligned}$$

This proves (4.1). See also [45, Thm. 3.2]. □

As a corollary, we have the following: Let V be a projective variety, which is not necessarily smooth. Let D and D' be linearly equivalent effective Cartier divisors on V. Then we have

$$|T_s(r, f, D, \lambda_D) - T_s(r, f, D', \lambda_{D'})| \leq c \tag{4.2}$$

for all $0 < s < r < 1$, where c is a positive constant which only depends on the choice of the Weil functions λ_D and $\lambda_{D'}$. Indeed, if D and D' are very ample, we reduce to the case of projective spaces by taking an embedding $\iota\colon V \hookrightarrow \mathbb{P}^k$ so that $D = \iota^*H$ and $D' = \iota^*H'$ for hyperplane sections H and H'. In this case, (4.2) follows from (4.1). In general, we take an effective Cartier divisor $E \subset V$ such that $D + E$ and $D' + E$ are very ample to reduce to the previous case.

Lemma 4.2. *Let V be a projective variety. Let $D \subset V$ be an effective Cartier divisor which is ample. Let $Z \subset V$ be a closed subscheme. Let $\lambda_D \geq 0$ and $\lambda_Z \geq 0$ be Weil functions for D and Z, respectively. Then there exist positive constants $c > 0$, $c' > 0$ such that for all $f \in \mathrm{Hol}_m(\mathbb{D}, V)$ with $f(Y_f) \not\subset \operatorname{supp} D \cup \operatorname{supp} Z$, we have*

$$T_s(r, f, Z, \lambda_Z) \leq cT_s(r, f, D, \lambda_D) + c'$$

for all $0 < s < r < 1$.

Proof. Set $V' = \mathrm{Bl}_Z V$. Let $\varphi \colon V' \to V$ be the projection and let $\varphi^* Z \subset V'$ be the induced closed subscheme. Then $\varphi^* Z$ is a Cartier divisor on V'. We denote by L the associated line bundle. Let M be the ample line bundle on V associated to D. Then there exists a positive integer l such that $\varphi^* M^{\otimes l} \otimes L^{-1}$ is very ample on V' (cf. [18, Chapter II, Prop. 7.10 (b)]). There exists a closed immersion $\iota \colon V' \hookrightarrow \mathbb{P}^k$ such that $\iota^* \mathcal{O}_{\mathbb{P}^k}(1) = \varphi^* M^{\otimes l} \otimes L^{-1}$. Let $H \subset \mathbb{P}^k$ be an effective divisor from $\mathcal{O}_{\mathbb{P}^k}(1)$. Then $\varphi^*(lD)$ and $\varphi^* Z + \iota^* H$ are linearly equivalent Cartier divisors. Let $h_{\mathcal{O}_{\mathbb{P}^k}(1)}$ be the Fubini–Study metric on $\mathcal{O}_{\mathbb{P}^k}(1)$. Let σ be a section of $\mathcal{O}_{\mathbb{P}^k}(1)$ such that $H = (\sigma = 0)$. Set $\lambda_H(x) = -\log \sqrt{h_{\mathcal{O}_{\mathbb{P}^k}(1)}(\sigma(x), \sigma(x))} \geq 0$. By (4.2), we have

$$T_s(r, f, Z, \lambda_Z) + T_s(r, \iota \circ f, H, \lambda_H) \leq lT_s(r, f, D, \lambda_D) + \alpha,$$

where α is a positive constant which only depends on the choices of λ_D, λ_Z, and λ_H. Set $\omega_{\mathbb{P}^k} = c_1(\mathcal{O}_{\mathbb{P}^k}(1), h_{\mathcal{O}_{\mathbb{P}^k}(1)})$. By (4.1), we have

$$T_s(r, \iota \circ f, H, \lambda_H) = T_s(r, \iota \circ f, \omega_{\mathbb{P}^k}) \geq 0.$$

Hence we get

$$T_s(r, f, Z, \lambda_Z) \leq lT_s(r, f, D, \lambda_D) + \alpha.$$

This conclude the proof. □

Lemma 4.3. *Let V be a smooth projective variety and let ω_V be a smooth positive $(1,1)$-form on V. Let $Z \subset V$ be a closed subscheme with a Weil function $\lambda_Z \geq 0$. Then there exists a positive constant $c > 0$ such that for all $f \in \mathrm{Hol}_m(\mathbb{D}, V)$ with $f(Y_f) \not\subset \operatorname{supp} Z$, we have*

$$m(r, f, \lambda_Z) \leq cT_s(r, f, \omega_V) + m(s, f, \lambda_Z) + c$$

for all $s \in (0,1)$ and $r \in (s, 1)$.

Proof. Since $N_s(r, f, Z) \geq 0$, we have

$$m(r, f, \lambda_Z) - m(s, f, \lambda_Z) \leq T_s(r, f, Z, \lambda_Z).$$

Let $D_1, \dots, D_l \subset V$ be effective ample divisors such that $D_1 \cap \cdots \cap D_l = \emptyset$. Let $\lambda_{D_i} \geq 0$ be a Weil function for D_i. We may take D_i such that $f(Y_f) \not\subset \operatorname{supp} D_i$. By Lemma 4.2, we have

$$m(r, f, \lambda_Z) - m(s, f, \lambda_Z) \leq c_i T_s(r, f, D_i, \lambda_{D_i}) + c_i'.$$

By (4.1), we have

$$T(r, f, D_i, \lambda_{D_i}) \leq c_i'' T_s(r, f, \omega_V).$$

We set $c = \max_{1 \leq i \leq l} \max\{c_i c_i'', c_i'\}$ to complete the proof. □

We apply Lemma 4.3 for $f\colon \mathbb{D} \to V$ with $f(0) \notin \operatorname{supp} Z$. Then $m(s, f, \lambda_Z) \to \lambda_Z(f(0))$ when $s \to 0+$. Hence we get

$$m(r, f, \lambda_Z) \leq cT_0(r, f, \omega_V) + \lambda_Z(f(0)) + c. \tag{4.3}$$

Lemma 4.4. *Let $p\colon \Sigma \to V$ be a generically finite surjective morphism of projective varieties, where V is smooth. Let D be an effective Cartier divisor on Σ with a Weil function $\lambda_D \geq 0$. Let ω_V be a smooth positive $(1,1)$-form on V. Let $E \subset \Sigma$ be the exceptional locus of $\varphi\colon \Sigma \to \Sigma^+$, where $\Sigma \to \Sigma^+ \to V$ is the Stein factorization. Let $\lambda_E \geq 0$ be a Weil function for E. Then there exists a positive constant $c > 0$ such that for all $f \in \mathrm{Hol}_m(\mathbb{D}, \Sigma)$ with $f(Y_f) \not\subset \operatorname{supp} D \cup \operatorname{supp} E$, we have*

$$T_s(r, f, D, \lambda_D) \leq cT_s(r, p \circ f, \omega_V) + cm(s, f, \lambda_E) + c$$

for all $0 < s < r < 1$.

Proof. Let $\widetilde{D} \subset \Sigma^+$ be the scheme-theoretic image of the composite $D \hookrightarrow \Sigma \to \Sigma^+$. Then we have $\mathcal{I}_{\varphi^*\widetilde{D}} \subset \mathcal{I}_D \subset \mathcal{O}_\Sigma$. Since $\mathcal{I}_D$ is an invertible sheaf, we have a closed subscheme $Z \subset \Sigma$ such that $\varphi^*\widetilde{D} = D + Z$. Since $\Sigma - E \to \Sigma^+$ is an open immersion, we note that $\operatorname{supp} Z \subset E$. Hence we have a positive constant $c' > 0$ such that $\lambda_Z \leq c'\lambda_E + c'$. Thus

$$\begin{aligned} T_s(r, f, D, \lambda_D) &\leq T_s(r, f, \varphi^*\widetilde{D}, \varphi^*\lambda_{\widetilde{D}}) + m(s, f, \lambda_Z) \\ &\leq T_s(r, \varphi \circ f, \widetilde{D}, \lambda_{\widetilde{D}}) + c'm(s, f, \lambda_E) + c'. \end{aligned}$$

Let $G_1, \dots, G_\nu \subset V$ be effective ample divisors such that $G_1 \cap \cdots \cap G_\nu = \emptyset$. Then since $q\colon \Sigma^+ \to V$ is finite, $q^*G_j \subset \Sigma^+$ are ample. We may take G_j such that $\varphi \circ f(Y_f) \not\subset \operatorname{supp} q^*G_j$. By Lemma 4.2, we get

$$\begin{aligned} T_s(r, \varphi \circ f, \widetilde{D}, \lambda_{\widetilde{D}}) &\leq \gamma_j T_s(r, \varphi \circ f, q^*G_j, \lambda_{G_j} \circ q) + \gamma_j' \\ &= \gamma_j T_s(r, p \circ f, G_j, \lambda_{G_j}) + \gamma_j'. \end{aligned}$$

By (4.1), we have

$$T_s(r, p\circ f, G_j, \lambda_{G_j}) \leq \mu_j T_s(r, p\circ f, \omega_V).$$

We set $c = \max_{1\leq j\leq \nu} \max\{\gamma_j\mu_j, c' + \gamma_j'\}$ to conclude the proof. □

Lemma 4.5. *Let $p\colon \Sigma \to V$ be a generically finite surjective morphism of smooth projective varieties. Let ω_Σ and ω_V be smooth positive $(1,1)$-forms on Σ and V, respectively. Let $E \subset \Sigma$ be the exceptional locus of $\varphi\colon \Sigma \to \Sigma^+$, where $\Sigma \to \Sigma^+ \to V$ is the Stein factorization. Let $\lambda_E \geq 0$ be a Weil function for E. Then there exists a positive constant $c > 0$ such that for all $f \in \mathrm{Hol}_m(\mathbb{D}, \Sigma)$ with $f(Y_f) \not\subset \operatorname{supp} E$, we have*

$$T_s(r, f, \omega_\Sigma) \leq cT_s(r, p\circ f, \omega_V) + cm(s, f, \lambda_E) + c$$

for all $0 < s < r < 1$.

Proof. Let $D_1, \ldots, D_l \subset \Sigma$ be linearly equivalent, effective ample divisors such that $D_1 \cap \cdots \cap D_l = \emptyset$. Let L be the associated line bundle on Σ. Then since L is ample, there exists a smooth Hermitian metric h on L such that the associated curvature form $\omega_{(L,h)}$ is positive. Then there exists a positive constant $\alpha > 0$ such that $\omega_\Sigma < \alpha\omega_{(L,h)}$. Let σ_i be a section of L such that $D_i = (\sigma_i = 0)$. Set $\lambda_{D_i}(x) = -\log \sqrt{h(\sigma_i(x), \sigma_i(x))} \geq 0$. We may take D_i such that $f(Y_f) \not\subset \operatorname{supp} D_i$. Then by (4.1), we have

$$T_s(r, f, \omega_\Sigma) \leq \alpha T_s(r, f, D_i, \lambda_{D_i}).$$

By Lemma 4.4, we have

$$T_s(r, f, D_i, \lambda_{D_i}) \leq c_i T_s(r, p\circ f, \omega_V) + c_i m(s, f, \lambda_E) + c_i$$

for all $0 < s < r < 1$. We set $c = \max_{1\leq i\leq l}\{\alpha c_i\}$ to conclude the proof. □

Remark 4.6. Let V be a smooth projective variety and let $Z \subset V$ be a closed subscheme. Suppose $\mathrm{Bl}_Z V$ is smooth. Then there exists a positive constant $c > 0$ such that for all $f\colon \mathbb{D} \to V$ with $f(\mathbb{D}) \not\subset \operatorname{supp} Z$, we have

$$T_s(r, f, \omega_{\mathrm{Bl}_Z V}) \leq cT_s(r, f, \omega_V) + cm(s, f, \lambda_{\operatorname{supp} Z}) + c \tag{4.4}$$

for all $s \in (0,1)$ and $r \in (s,1)$. This follows from Lemma 4.5 applied to $\mathrm{Bl}_Z V \to V$. Indeed, we have $m(s, f, \lambda_E) \leq cm(s, f, \lambda_{\operatorname{supp} Z})$.

Remark 4.7. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, V)$ be an infinite set, and let $Z \subset V$ be a Zariski closed set. For $s \in (0,1)$, $\gamma > 0$, and an open neighborhood $U \subset V$ of Z, we set

$$\mathcal{F}_{s,\gamma,U} = \big\{f \in \mathcal{F};\ f(z) \in U \text{ for } \gamma\text{-almost all } z \in \mathbb{D}(s)\big\}.$$

Then if $s' \geq s$, $\gamma' \leq \gamma$, and $U' \subset U$, we have $\mathcal{F}_{s',\gamma',U'} \subset \mathcal{F}_{s,\gamma,U}$. We remark the following two facts which directly follow from the definition:

(1) $\mathcal{F} \not\to Z$ if and only if there exist $s \in (0,1)$, $\gamma > 0$, $Z \subset U \subset V$ such that $\mathcal{F} - \mathcal{F}_{s,\gamma,U}$ is infinite.

(2) $\mathcal{G} \not\to Z$ for all infinite subset $\mathcal{G} \subset \mathcal{F}$ if and only if there exist $s \in (0,1)$, $\gamma > 0$, $Z \subset U \subset V$ such that $\mathcal{F}_{s,\gamma,U}$ is finite. Moreover, if $f(\mathbb{D}) \not\subset Z$ for all $f \in \mathcal{F}$, then we may take s, γ, U such that $\mathcal{F}_{s,\gamma,U} = \emptyset$.

Lemma 4.8. *Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, V)$ be an infinite set, and let $Z \subset V$ be a Zariski closed set. Assume that $\mathcal{F} \not\to Z$. Then there exists an infinite subset $\mathcal{G} \subset \mathcal{F}$ such that $f(\mathbb{D}) \not\subset Z$ for all $f \in \mathcal{G}$, and that $\mathcal{G}' \not\to Z$ for all infinite subsets $\mathcal{G}' \subset \mathcal{G}$.*

Proof. We may take $s \in (0,1)$, $\gamma > 0$, $Z \subset U \subset V$ such that $\mathcal{F} - \mathcal{F}_{s,\gamma,U}$ is infinite. Set $\mathcal{G} = \mathcal{F} - \mathcal{F}_{s,\gamma,U}$. Then $\mathcal{G}_{s,\gamma,U} = \mathcal{G} \cap \mathcal{F}_{s,\gamma,U} = \emptyset$. Hence $\mathcal{G}' \not\to Z$ for all infinite subsets $\mathcal{G}' \subset \mathcal{G}$. Note that $f(\mathbb{D}) \not\subset Z$ for all $f \in \mathcal{G}$. □

Lemma 4.9. *Let V be a smooth projective variety. Let ω_V be a smooth positive $(1,1)$-form. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, V)$ be an infinite subset. Let $Z \subset V$ be a closed subscheme with a Weil function λ_Z. Assume that $\mathcal{G} \not\to \operatorname{supp} Z$ for all infinite subsets $\mathcal{G} \subset \mathcal{F}$ and that $f(\mathbb{D}) \not\subset \operatorname{supp} Z$ for all $f \in \mathcal{F}$. Then there exist $\sigma \in (0,1)$ and $\alpha > 0$ such that for all $s \in (\sigma, 1)$, $r \in (s,1)$, and $f \in \mathcal{F}$, we have*

$$m(r, f, \lambda_Z) \leq \frac{\alpha}{(r-s)} T_s(r, f, \omega_V) + \alpha.$$

To prove this lemma, we start from the following consideration. Let $w \in \mathbb{D}(r)$. We set

$$\varphi_{w,r}(z) = r^2 \frac{w - z}{r^2 - \bar{w}z}, \tag{4.5}$$

which is an isomorphism $\varphi_{w,r} \colon \mathbb{D}(r) \to \mathbb{D}(r)$ such that $\varphi_{w,r}(0) = w$. We have $\varphi'_{w,r}(z) = r^2 \frac{|w|^2 - r^2}{(r^2 - \bar{w}z)^2}$. Applying the maximum and minimum principles on $\mathbb{D}(r)$, we get

$$\frac{r - |w|}{r + |w|} \leq |\varphi'_{w,r}(z)| \leq \frac{r + |w|}{r - |w|} \tag{4.6}$$

for all $z \in \overline{\mathbb{D}(r)}$.

Lemma 4.10. *Let $0 < \sigma < r < 1$. Then for all non-negative functions Λ on $\partial\mathbb{D}(r)$ and $w \in \mathbb{D}(\sigma)$, we have*

$$\frac{r-\sigma}{r+\sigma} \int_0^{2\pi} \Lambda(\varphi_{w,r}(re^{i\theta})) \frac{d\theta}{2\pi} \leq \int_0^{2\pi} \Lambda(re^{i\theta}) \frac{d\theta}{2\pi} \leq \frac{r+\sigma}{r-\sigma} \int_0^{2\pi} \Lambda(\varphi_{w,r}(re^{i\theta})) \frac{d\theta}{2\pi}.$$

Proof. We have

$$\int_0^{2\pi} \Lambda(re^{i\theta})\,\frac{d\theta}{2\pi} = \int_0^{2\pi} \Lambda(\varphi_{w,r}(re^{i\theta}))|\varphi'_{w,r}(re^{i\theta})|\,\frac{d\theta}{2\pi}.$$

Hence by (4.6) we have

$$\frac{r-|w|}{r+|w|}\int_0^{2\pi} \Lambda(\varphi_{w,r}(re^{i\theta}))\,\frac{d\theta}{2\pi} \le \int_0^{2\pi} \Lambda(re^{i\theta})\,\frac{d\theta}{2\pi} \le \frac{r+|w|}{r-|w|}\int_0^{2\pi} \Lambda(\varphi_{w,r}(re^{i\theta}))\,\frac{d\theta}{2\pi}.$$

Since $r \in (\sigma, 1)$ and $w \in \overline{\mathbb{D}(\sigma)}$, we have $\frac{r+|w|}{r-|w|} < \frac{r+\sigma}{r-\sigma}$. The proof is completed. $\square$

In the proof of Lemma 4.9, we need the following estimate from [19, Lem. 6.17]: If μ is a mass distribution on $\mathbb{C}$ with finite total mass M and γ is a constant with $0 < \gamma < 1$, then we have

$$\int_{\mathbb{C}} \log\frac{1}{|z-w|}\,d\mu_z \le \tau_\gamma M \tag{4.7}$$

for γ-almost all $w \in \mathbb{C}$, where $\tau_\gamma > 0$ is a positive constant which depends on γ. For instance, we may take $\tau_\gamma = \log(6/\gamma)$. See also [30, Chapter VIII, §3].

Proof of Lemma 4.9. By Remark 4.7, there exist $s_0 \in (0,1)$, $\gamma_0 > 0$, and an open neighborhood $U_0 \subset V$ of $\operatorname{supp} Z$ such that $\mathcal{F}_{s_0,\gamma_0,U_0} = \emptyset$. We fix $\sigma \in (s_0, 1)$. Let $f \in \mathcal{F}$. By Lemma 4.3, we get

$$m(r, f, \lambda_Z) \le cT_\sigma(r, f, \omega_V) + m(\sigma, f, \lambda_Z)$$

for all $r \in (\sigma, 1)$, where $c > 0$ is a positive constant which only depends on ω_V and λ_Z. Given an isomorphism $\varphi_{w,\sigma}\colon \mathbb{D}(\sigma) \to \mathbb{D}(\sigma)$ such that $w \in \mathbb{D}(s_0)$ and $f(w) \notin \operatorname{supp} Z$, we get (cf. (4.3))

$$m(\sigma, f\circ\varphi_{w,\sigma}, \lambda_Z) \le cT_0(\sigma, f\circ\varphi_{w,\sigma}, \omega_V) + \lambda_Z(f(w)).$$

By Lemma 4.10, we have

$$m(\sigma, f, \lambda_Z) \le \frac{\sigma+s_0}{\sigma-s_0} m(\sigma, f\circ\varphi_{w,\sigma}, \lambda_Z) \le \frac{2}{\sigma-s_0} m(\sigma, f\circ\varphi_{w,\sigma}, \lambda_Z).$$

Hence we get

$$m(\sigma, f, \lambda_Z) \le \frac{2c}{\sigma-s_0} T_0(\sigma, f\circ\varphi_{w,\sigma}, \omega_V) + \frac{2}{\sigma-s_0}\lambda_Z(f(w)),$$

so that

$$m(r, f, \lambda_Z) \le cT_\sigma(r, f, \omega_V) + \frac{2c}{\sigma-s_0} T_0(\sigma, f\circ\varphi_{w,\sigma}, \omega_V) + \frac{2}{\sigma-s_0}\lambda_Z(f(w)). \tag{4.8}$$

Now we estimate the right-hand side of this estimate. We apply (4.7). We choose $w \in \mathbb{D}(s_0)$ such that $f(w) \notin U_0$ and

$$\int_{\mathbb{C}} \log \frac{1}{|z-w|} d\mu_z \leq \tau_{\gamma_0} \int_{\mathbb{D}(\sigma)} f^*\omega_V,$$

where $\mu = \mathbb{I}_{\mathbb{D}(\sigma)} f^*\omega_V$. We set $\eta = \sup_{x \in V - U_0} \lambda_Z(x)$. Then we have

$$\lambda_Z(f(w)) \leq \eta. \tag{4.9}$$

We have

$$\begin{aligned}\int_0^\sigma \frac{dt}{t} \int_{\mathbb{D}(t)} (f \circ \varphi_{w,\sigma})^*\omega_V &= \int_{\mathbb{C}} \log \frac{\sigma}{|\xi|} d(\varphi_{w,\sigma}^*\mu)_\xi = \int_{\mathbb{C}} \log \frac{\sigma}{|\varphi_{w,\sigma}^{-1}(z)|} d\mu_z \\ &= \int_{\mathbb{C}} \log \frac{|\sigma - (\overline{w}/\sigma)z|}{|z-w|} d\mu_z \leq (\tau_{\gamma_0} + \log 2) \int_{\mathbb{D}(\sigma)} f^*\omega_V.\end{aligned}$$

Hence

$$T_0(\sigma, f \circ \varphi_{w,\sigma}, \omega_V) \leq \frac{\tau_{\gamma_0} + \log 2}{r-s} T_s(r, f, \omega_V)$$

for $\sigma < s < r < 1$. We have

$$T_\sigma(r, f, \omega_V) \leq \frac{1}{\sigma(r-s)} T_s(r, f, \omega_V) \tag{4.10}$$

for $\sigma < s < r < 1$. We substitute (4.9)–(4.10) into (4.8) to conclude the proof. Here we set

$$\alpha = \max\left\{\frac{c}{\sigma} + \frac{2c(\tau_{\gamma_0} + \log 2)}{\sigma - s_0}, \frac{2\eta}{\sigma - s_0}\right\},$$

which is independent of the choice of $s \in (\sigma, 1)$, $r \in (s, 1)$, and $f \in \mathcal{F}$. □

§5. Main proposition for the proof of Proposition 3.20

We recall the notation $f_{S_{k,A}}$ from (2.13). The following proposition plays an important role in the proof of Proposition 3.20. In particular, it provides an important estimate (5.1) under the assumption that $(f_{S_{1,A}})_{f \in \mathcal{F}} \Rightarrow Z$, where Z is horizontally integrable (see Definition 3.16).

Proposition 5.1. *Let $\bar{A}$ be a smooth projective equivariant compactification of a semi-abelian variety A. Let S be a smooth projective variety. Let $\tau\colon S_{1,A} \to S$ be the natural projection. Let $Z \subset S_{1,A}$ be an irreducible Zariski closed set. Assume that Z is horizontally integrable. Then there exists a proper Zariski closed set $W \subsetneqq \tau(Z)$ with the following property: Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A \times S)$ be an infinite set of non-constant holomorphic maps such that $(f_{S_{1,A}})_{f \in \mathcal{F}} \Rightarrow Z$. Let $\omega_{\bar{A}}$ and ω_S be smooth positive $(1,1)$-forms on $\bar{A}$ and S, respectively. Let $\lambda_W \geq 0$ be a Weil* *function for W. Let $s \in (1/2, 1)$ and $\delta > 0$. Then there exist positive constants $c_1 > 0$, $c_2 > 0$, $c_3 > 0$ such that for all $f \in \mathcal{F}$ with $f_S(\mathbb{D}) \not\subset W$, we have*

$$T_s(r, f_A, \omega_{\bar{A}}) \leq c_1 T_s(r, f_S, \omega_S) + c_2 m((s+r)/2, f_S, \lambda_W) + c_3 \tag{5.1}$$

for all $r \in (s, 1)$ outside some exceptional set $E \subset (s, 1)$ with the linear measure $|E| < \delta$.

In the proof of Proposition 3.20, we apply this proposition to $S = P_{k,A}$, with k and Z as in the statement of Proposition 3.20. See Section 3.4 for the role of Proposition 5.1 in the proof of Proposition 3.20. The rest of this section is devoted to the proof of Proposition 5.1.

§5.1. Outline of the proof of Proposition 5.1

We discuss an outline of the proof of Proposition 5.1. We recall $S_{1,A}^* = P(TS \times \{0\})$ from (3.2). Then $S_{1,A}^* \subset S_{1,A}$ is horizontally integrable. To see the idea of the proof of Proposition 5.1, it is instructive to consider the special case $Z = S_{1,A}^*$. The assumption $(f_{S_{1,A}})_{f\in\mathcal{F}} \Rightarrow S_{1,A}^*$ implies $(f_{S_{1,A}})_{f\in\mathcal{F}} \to S_{1,A}^*$. Since $S_{1,A}^* \subset S_{1,A}$ consists of the locus where the derivative is null in the direction of A, the condition $(f_{S_{1,A}})_{f\in\mathcal{F}} \to S_{1,A}^*$ means that the derivative of f_A is bounded by the derivative of f_S outside small exceptional sets. Thus it is reasonable to expect an estimate of the form (5.1) using an integration argument over $\mathbb{D}$. For a general horizontally integrable Z, we apply Lemma 3.17 to reduce the problem to this setting.

The reduction goes as follows, leaving aside the rigorous details for a moment. Since $Z \subset S_{1,A}$ is horizontally integrable, we may take $U \hookrightarrow A \times S$ as in Definition 3.16. Then we get an isomorphism $\Phi\colon A \times U \to A \times U$ as in (3.3). For each $f \in \mathcal{F}$, we may take a lift $\hat{f} \in \mathrm{Hol}_m(\mathbb{D}, A \times U)$ of f so that $\{(\Phi \circ \hat{f})_{U_{1,A}}\}_{f\in\mathcal{F}}$ becomes very close to $U_{1,A}^*$, thanks to Lemma 3.17. Then the derivative of $(\Phi \circ \hat{f})_A$ is bounded by the derivative of $(\Phi \circ \hat{f})_U$. Thus an estimate of the form (5.1) for $\Phi \circ \hat{f}$, instead of f, is easier to prove. We then use this situation to derive the estimate (5.1) for the original f. See Section 5.2 for the detail of the construction of $\hat{f}$.

However, there are several problems to work out this argument rigorously.

- $\{(\Phi \circ \hat{f})_{U_{1,A}}\}_{f\in\mathcal{F}}$ becomes very close to $U_{1,A}^*$ only on some branch $\Omega_f' \to \Omega_f$ over a domain $\Omega_f \subset \mathbb{D}$. The situation will be explained in Definition 5.3 below.
- We need to compactify U. Moreover, to complete the argument rigorously, we need to take a compactification Σ carefully. This is the theme of Section 5.3. The resulting compactification Σ, together with a semi-positive $(1,1)$-form η on it, are essential for converting the estimate of $\Phi \circ \hat{f}$ into that of f. The Zariski closed subset $W \subsetneqq \tau(Z)$ appears in this process.

- After the compactification of U, the extension of the immersion $U \hookrightarrow A\times S$ in Definition 3.16 may hit the boundary of $\bar{A}\times S$. Consequently, the isomorphism Φ only extends to a rational map $\bar{A}\times\Sigma \dashrightarrow \bar{A}\times\Sigma$, rather than an isomorphism $A\times\Sigma\to A\times\Sigma$. As a result, $(\Phi\circ\hat{f})_A$ may hit the boundary of $\bar{A}$. This issue arises only when A is non-compact. Compare with the isomorphism Φ_0 in the diagram (5.28).

Based on the outline of the proof, one might hope that the proof still works if we replace $\omega_{\bar{A}}$ in the left-hand side of (5.1) with an invariant positive $(1,1)$-form on A. However, this is not the case, primarily because of the third issue above.

Now we introduce Lemma 5.2 below to estimate the left-hand side of (5.1). We recall that a semi-abelian variety A is an algebraic group with an expression

$$0\to T\to A\xrightarrow{\rho} A_0\to 0, \tag{5.2}$$

where A_0 is an abelian variety and $T\simeq\mathbb{G}_m^l$ is an algebraic torus (cf. Appendix A).

Lemma 5.2. *Let $\bar{A}$ be a smooth projective equivariant compactification of a semi-abelian variety A. Let $\rho\colon A\to A_0$ be the canonical quotient as in (5.2). Let $\omega_{\bar{A}}$ be a smooth positive $(1,1)$-form on $\bar{A}$ and ω_{A_0} be an invariant positive $(1,1)$-form on A_0. For an irreducible component D of ∂A, let $\lambda_D\geq 0$ be a Weil function. Then there exists a positive constant $c>0$ such that for all $g\in\mathrm{Hol}(\mathbb{D},A)$, we have*

$$T_s(r,g,\omega_{\bar{A}})\leq cT_s(r,\rho\circ g,\omega_{A_0})+c\sum_{D\subset\partial A}|m(r,g,\lambda_D)-m((s+r)/2,g,\lambda_D)|+c \tag{5.3}$$

for all $s\in(1/2,1)$ and $r\in(s,1)$, where $D\subset\partial A$ runs over all irreducible components of ∂A.

Proof. We first note that

$$T_s(r,g,\omega_{\bar{A}})\leq 4T_\sigma(r,g,\omega_{\bar{A}}) \tag{5.4}$$

for all $1/2<s<r<1$, where $\sigma=(s+r)/2$.

Let $\bar{A}=(\bar{T}\times A)/T$ (cf. Lemma A.6). Then $\bar{T}\subset\bar{A}$ is projective. Since the Picard group of T is trivial, we may take a (not necessarily effective) very ample divisor D_0 on $\bar{T}$ such that $\operatorname{supp} D_0\subset\partial T$. Note that D_0 is T-invariant. Set $D=(D_0\times A)/T$. Then D is a Cartier divisor on $\bar{A}$. We take a Zariski open covering $\{U_i\}$ of A_0 such that $\bar{\rho}^{-1}(U_i)=U_i\times\bar{T}$, where $\bar{\rho}\colon\bar{A}\to A_0$ is the extension of ρ. Then we have $D\cap\bar{\rho}^{-1}(U_i)=D_0\times U_i$. Hence $\mathcal{O}_{\bar{A}}(D)$ is very ample for $\bar{\rho}$ in the sense of [17, Def. 13.52]. Since A_0 is projective, we may take an ample line bundle L on A_0. Then by [17, Prop. 13.65], there exists a positive integer $n\geq 1$ such that

$\mathcal{O}_{\bar{A}}(D) \otimes \bar{\rho}^* L^{\otimes k}$ is ample on $\bar{A}$. We write $D = D^+ - D^-$ to show the positive and negative parts of D. By (4.1), there exist positive constants $\alpha > 0$ and $\alpha' > 0$ such that for all $g \in \mathrm{Hol}(\mathbb{D}, A)$, we have

$$\begin{aligned} T_s(r, g, \omega_{\bar{A}}) &\leq \alpha T_s(r, \rho \circ g, \omega_{A_0}) \\ &\quad + \alpha' T_\sigma(r, g, D^+, \lambda_{D^+}) - \alpha' T_\sigma(r, g, D^-, \lambda_{D^-}) + \alpha \end{aligned} \tag{5.5}$$

for all $1/2 < s < r < 1$ (cf. (5.4)). For $g \in \mathrm{Hol}(\mathbb{D}, A)$, we have $N_\sigma(r, g, D^+) = 0$, hence

$$T_\sigma(r, g, D^+, \lambda_{D^+}) = m(r, g, \lambda_{D^+}) - m(\sigma, g, \lambda_{D^+}).$$

Similarly, we have $T_\sigma(r, g, D^-, \lambda_{D^-}) = m(r, g, \lambda_{D^-}) - m(\sigma, g, \lambda_{D^-})$. Hence

$$T_\sigma(r, g, D^+, \lambda_{D^+}) - T_\sigma(r, g, D^-, \lambda_{D^-}) \leq \sum_{D \subset \partial A} |m(r, g, \lambda_D) - m(\sigma, g, \lambda_D)|,$$

where $D \subset \partial A$ runs over all irreducible components of ∂A. Combining this estimate with (5.5), we conclude the proof of the lemma. Here we set $c = \max\{\alpha, \alpha'\}$. $\square$

In the proof of Proposition 5.1, we estimate the first and second terms on the right-hand side of (5.3) in Lemmas 5.11 and 5.17, respectively. The first term is easier to estimate. The reason is that since $(\rho \circ f_A)^* \omega_{A_0}$ is subharmonic, we could estimate $T_s(r, \rho \circ f_A, \omega_{A_0})$ directly from the information of $\rho \circ f_A$ over Ω_f, based on [46, Lem. 7.1]. The main technique to estimate the second term will be discussed in Sections 5.5–5.7. Namely we first estimate the second term (in the setting of $\Phi \circ \hat{f}$) by arc lengths (cf. Lemma 5.14), then we estimate the arc lengths by the area-length method (cf. Lemma 5.15).

We now return to the first issue from the outline: the concentration of $\{(\Phi \circ \hat{f})_{U_{1,A}}\}_{f \in \mathcal{F}}$ to $U_{1,A}^*$. To examine this concentration, we introduce the following definition.

Definition 5.3. Let V be a variety. Let $\mathcal{G} = ((g_i, \Omega_i, \Omega_i'))_{i \in I}$ be an infinite indexed family of triples $(g_i, \Omega_i, \Omega_i')$, where $g_i \in \mathrm{Hol}_m(\mathbb{D}, V)$, $\Omega_i \subset \mathbb{D}$ is a connected open subset, and $\Omega_i' \subset Y_{g_i}$ is a connected component of $\pi_{g_i}^{-1}(\Omega_i)$. Let $T \subset V$ be a Zariski closed set. We write $\mathcal{G} \rightsquigarrow T$ if the following holds: For every $0 < s < 1$, $\gamma > 0$, and an open subset $U \subset V$ such that $T \subset U$, there exists a finite subset $E \subset I$ such that for every $i \in I - E$, we have

- $z \in \Omega_i$ for γ-almost all $z \in \mathbb{D}(s)$,
- $g_i(\Omega_i') \subset U$.

We prepare some notation in Nevanlinna theory in the context of this definition. Let Σ be a projective variety and let $Z \subset \Sigma$ be a closed subscheme with a Weil function λ_Z. Let (g,Ω,Ω') be a triple as in Definition 5.3, i.e., $g\in \mathrm{Hol}_m(\mathbb{D},\Sigma)$, $\Omega\subset\mathbb{D}$ is a connected open subset, and $\Omega'\subset Y_g$ is a connected component of $\pi_g^{-1}(\Omega)$. Assume $g(Y_g)\not\subset \operatorname{supp} Z$. We set

$$m(r,(g,\Omega,\Omega'),\lambda_Z)=\frac{1}{\deg(\pi_g|_{\Omega'})}\int_{y\in\Omega'\cap\partial Y_g(r)}\lambda_Z(g(y))\frac{d\arg\pi_g(y)}{2\pi}, \tag{5.6}$$

where $\deg(\pi_g|_{\Omega'})$ is the degree of the restriction $\pi_g|_{\Omega'}\colon\Omega'\to\Omega$.

Suppose Σ is smooth. Let ω be a smooth $(1,1)$-form on Σ. We set

$$T_s(r,(g,\Omega,\Omega'),\omega)=\frac{1}{\deg(\pi_g|_{\Omega'})}\int_s^r\frac{dt}{t}\int_{\Omega'\cap Y_g(t)}g^*\omega. \tag{5.7}$$

In the rest of this section, A is a semi-abelian variety, S is a smooth projective variety, and $Z\subset S_{1,A}$ is an irreducible Zariski closed set which is horizontally integrable.

§5.2. Good liftings of $f\in\mathcal{F}$

Since $Z\subset S_{1,A}$ is horizontally integrable, we take $\iota\colon U\hookrightarrow A\times S$ as in Definition 3.16. We get the isomorphism

$$\Phi\colon A\times U\to A\times U$$

from (3.3). We apply Lemma 3.17 to get an irreducible component $\Xi\subset U_{1,A}$ of $(q')^{-1}(Z)$ such that

$$D\Phi(\Xi)\subset U^*_{1,A}, \tag{5.8}$$

where $q'\colon U_{1,A}\to S_{1,A}$ is induced from $q\colon U\to S$. Let $\Theta\subset U$ be the image of Ξ under $U_{1,A}\to U$. Since $U_{1,A}\to U$ is proper, Θ is a Zariski closed subset of U. Let Σ be a smooth compactification of U such that $q\colon U\to S$ extends to a morphism $\bar q\colon\Sigma\to S$. We denote by $\bar\Theta\subset\Sigma$ the Zariski closure of Θ in Σ.

Remark 5.4. We make the following observations.

(1) Let $g\in\mathrm{Hol}_m(\mathbb{D},A\times\Sigma)$ such that $g_\Sigma(Y_g)\not\subset\partial U$. Then we may define

$$\Phi\circ g\colon Y_g\to\bar A\times\Sigma$$

as follows. Set $Q_g=(g_\Sigma)^{-1}(\partial U)$. Then $Q_g\subset Y_g$ is a discrete subset. Thus we get

$$\Phi\circ(g|_{Y_g\backslash Q_g})\colon Y_g\backslash Q_g\to A\times U\subset\bar A\times\Sigma.$$

Since $\bar A\times\Sigma$ is compact and $\Phi\colon\bar A\times\Sigma\dashrightarrow\bar A\times\Sigma$ is rational, $\Phi\circ(g|_{Y_f\backslash Q_f})$ extends to a holomorphic map $\Phi\circ g\colon Y_g\to\bar A\times\Sigma$. By the construction, we have $(\Phi\circ g)_{\bar A}(Y_g)\not\subset\partial A$.

(2) Let $h \in \mathrm{Hol}_m(\mathbb{D}, \bar{A} \times \Sigma)$ be non-constant such that $h_{\bar{A}}(Y_h) \not\subset \partial A$. Then we may define $h_{\Sigma_{1,A}} \colon Y_h \to \Sigma_{1,A}$ as follows. Set $R_h = (h_{\bar{A}})^{-1}(\partial A)$. Then $R_h \subset Y_h$ is a discrete subset. We get $h|_{Y_h \backslash R_h} \colon Y_h \backslash R_h \to A \times \Sigma$. Since h is non-constant, this induces $(h|_{Y_h \backslash Q_h})_{\Sigma_{1,A}} \colon Y_h \backslash Q_h \to \Sigma_{1,A}$. Since $\Sigma_{1,A}$ is compact, we get $h_{\Sigma_{1,A}} \colon Y_h \to \Sigma_{1,A}$.

With this notation, we state the following lemma.

Lemma 5.5. *Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A \times S)$ be an infinite set of non-constant holomorphic maps such that $(f_{S_{1,A}})_{f \in \mathcal{F}} \Rightarrow Z$. Then there exists a finite subset $\mathcal{E} \subset \mathcal{F}$ with the following property: For each $f \in \mathcal{F} \backslash \mathcal{E}$, there exist*

- *a lifting $\hat{f} \in \mathrm{Hol}_m(\mathbb{D}, A \times \Sigma)$ of $f \in \mathrm{Hol}(\mathbb{D}, A \times S)$,*
- *a connected open subset $\Omega_f \subset \mathbb{D}$,*
- *a connected component Ω'_f of $\pi_{\hat{f}}^{-1}(\Omega_f) \subset Y_{\hat{f}}$*

such that

$$\hat{f}_\Sigma(Y_{\hat{f}}) \not\subset \partial U, \tag{5.9}$$

$$((\hat{f}_\Sigma, \Omega_f, \Omega'_f))_{f \in \mathcal{F} \backslash \mathcal{E}} \rightsquigarrow \overline{\Theta}, \tag{5.10}$$

$$\big(((\Phi \circ \hat{f})_{\Sigma_{1,A}}, \Omega_f, \Omega'_f)\big)_{f \in \mathcal{F} \backslash \mathcal{E}} \rightsquigarrow \Sigma^*_{1,A}, \tag{5.11}$$

and

$$\deg \pi_{\hat{f}} \leq [\mathbb{C}(S) : \mathbb{C}(\Sigma)]. \tag{5.12}$$

Before proving this lemma, we need some preparation.

5.2.1. Preliminary lemmas for the proof of Lemma 5.5.

Lemma 5.6. *Let $s \in (0,1)$ and $\gamma > 0$. Let $V \subset \mathbb{D}(s)$ be an open set such that $z \in V$ for γ-almost all $z \in \mathbb{D}(s)$. Let $s' \in (0, s)$. Then there exists an open subset $\Omega \subset V$ such that Ω is connected and $z \in \Omega$ for 2γ-almost all $z \in \mathbb{D}(s')$.*

Proof. Set $K = \overline{\mathbb{D}(s')} \backslash V$. Then K is compact. We first show that there exists a finite correction of closed discs $D_1, \ldots, D_n$ such that

(1) $K \subset \bigcup_{i=1}^n D_i$, and
(2) $\sum_{i=1}^n r_i < 2\gamma$, where r_i is the radius of D_i.

Indeed, let $\{E_i\}$ be a countable set of closed discs such that $\mathbb{D}(s) \backslash V \subset \bigcup E_i$ and the sum of the radii of E_i is less than γ. Let O_i be the open disc whose center is equal to that of E_i and radius is equal to double that of E_i. We have $K \subset \bigcup O_i$. Hence we may take $O_1, O_2, \ldots, O_n$ such that $K \subset O_1 \cup \cdots \cup O_n$ and the sum of the radii of O_i is less than 2γ. Thus the closed discs $\overline{O_1}, \dots, \overline{O_n}$ satisfy our requirements.

Now let n be the minimum such that there exist closed discs $D_1, \dots, D_n$ with properties (1) and (2) above. We claim that $D_i \cap D_j = \emptyset$ for $i \neq j$. To prove this, we suppose on the contrary that there exists $i \neq j$ such that $D_i \cap D_j \neq \emptyset$. We may assume without loss of generality that $i = 1$ and $j = 2$. Let p_1, p_2 be the centers of D_1, D_2, respectively. Let p be the point which divides the line segment $\overline{p_1p_2}$ internally in the ratio $\overline{p_1p} : \overline{pp_2} = r_2 : r_1$. Let D be the closed disc whose center is p and radius is equal to $r_1 + r_2$. Then $D_1 \cup D_2 \subset D$. Hence the closed discs $D, D_3, \dots, D_n$ satisfy properties (1) and (2) above. This contradicts the choice of n. Hence we have proved $D_i \cap D_j = \emptyset$ for $i \neq j$. Now we set $\Omega = \mathbb{D}(s') \backslash \bigcup_{i=1}^n D_i$. Then Ω is open and connected. We have $z \in \Omega$ for 2γ-almost all $z \in \mathbb{D}(s')$. $\square$

Lemma 5.7. *Let $\widetilde{V}$ and V be smooth projective varieties. Let $p\colon \widetilde{V} \to V$ be a proper, surjective, generically finite morphism. Let $\widetilde{V}_0 \subset \widetilde{V}$ be a non-empty Zariski open set such that $p\colon \widetilde{V} \to V$ is quasi-finite on $\widetilde{V}_0$. Let $Z \subset V$ be an irreducible Zariski closed set. Let $\widetilde{Z} \subset \widetilde{V}$ be an irreducible component of $p^{-1}(Z)$ such that $\widetilde{Z} \cap \widetilde{V}_0 \neq \emptyset$. Let $\mathcal{G} = (g_i)_{i \in I}$ be an infinite indexed family in $\mathrm{Hol}(\mathbb{D}, V)$ such that $\mathcal{G} \Rightarrow Z$. Then for all but finitely many $i \in I$, there exist*

- *a lift $\tilde{g}_i \in \mathrm{Hol}_m(\mathbb{D}, \widetilde{V})$ of g_i,*
- *a connected open subset $\Omega_i \subset \mathbb{D}$, and*
- *a connected component Ω'_i of $\pi_{\tilde{g}_i}^{-1}(\Omega_i) \subset Y_{\tilde{g}_i}$*

such that

- $((\tilde{g}_i, \Omega_i, \Omega'_i))_{i \in I} \rightsquigarrow \widetilde{Z}$,
- $\deg \pi_{\tilde{g}_i} \leq [\mathbb{C}(V) : \mathbb{C}(\widetilde{V})]$, *and*
- $\tilde{g}_i(Y_{\tilde{g}_i}) \not\subset \widetilde{V} \backslash \widetilde{V}_0$.

Proof. We construct a Z-admissible modification $V' \to V$ and $\widetilde{Z}$-admissible modification $\widetilde{V}' \to \widetilde{V}$ with the following properties (cf. Definition 3.1):

- p induces a morphism $p'\colon \widetilde{V}' \to V'$ with the following commutative diagram:

$$\begin{array}{ccc} \widetilde{V} & \longleftarrow & \widetilde{V}' \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ V & \longleftarrow & V'. \end{array}$$

- p' is quasi-finite on some neighborhood of $\widetilde{Z}'$, where $\widetilde{Z}' \subset \widetilde{V}'$ is the minimal transform.
- $(p')^{-1}(Z') = \widetilde{Z}' \amalg E$, where $Z' \subset V'$ is the minimal transform.

We construct these objects as follows. We decompose $p^{-1}(Z)$ into irreducible components as $p^{-1}(Z) = \widetilde{Z} \cup F_1 \cup \cdots \cup F_k$ and set $F = F_1 \cup \cdots \cup F_k$. Replacing $\widetilde{V}$ by $\mathrm{Bl}_{\widetilde{Z}\cap F}\widetilde{V}$, we may assume $\widetilde{Z} \cap F_j = \emptyset$ if $p(F_j) = Z$ (cf. the proof of (3.1)). By Lemma B.2, we may take a Z-admissible modification $V' \to V$ such that $\widetilde{V}'|_{Z'} \to Z'$ is flat. We may assume moreover that V' is smooth. Then $\widetilde{V}'|_{Z'} \to Z'$ is an open map. Hence every irreducible component of $(p')^{-1}(Z')$ dominates Z'. Thus $\widetilde{Z}'$ is a connected component of $(p')^{-1}(Z')$. Hence $\widetilde{Z}' \to Z'$ is flat, so finite. Hence p' is quasi-finite on some neighborhood of $\widetilde{Z}'$, and $(p')^{-1}(Z') = \widetilde{Z}' \amalg E$.

Let $T \subset \widetilde{V}'$ be a Zariski open neighborhood of $\widetilde{Z}'$ such that $p'|_T \colon T \to V'$ is quasi-finite and $T \cap E = \emptyset$. Then, by Zariski's main theorem, there exist a compactification $T \subset \bar{T}$ and a finite map

$$p'' \colon \bar{T} \to V'$$

such that $p''|_T = p'|_T$. We have $(p'')^{-1}(Z') = \widetilde{Z}' \amalg E'$. Let $\widetilde{Z}' \subset O_{\widetilde{Z}'}$ and $E' \subset O_{E'}$ be open neighborhoods in $\bar{T}$ such that $O_{\widetilde{Z}'} \cap O_{E'} = \emptyset$. We may assume $O_{\widetilde{Z}'} \subset T$. Note that $p''(\bar{T} - (O_{\widetilde{Z}'} \cup O_{E'})) \subset V'$ is compact and disjoint from Z'. Hence we may take an open neighborhood $Z' \subset O_{Z'}$ which is disjoint from $p''(\bar{T} - (O_{\widetilde{Z}'} \cup O_E))$. Then $(p'')^{-1}(O_{Z'}) \subset O_{\widetilde{Z}'} \cup O_{E'}$. We replace $O_{\widetilde{Z}'}$ by $(p'')^{-1}(O_{Z'}) \cap O_{\widetilde{Z}'}$. Then $p''|_{O_{\widetilde{Z}'}} \colon O_{\widetilde{Z}'} \to O_{Z'}$ is a proper map. Since V' is smooth, by the Remmert open mapping theorem, the finite map $p'' \colon \bar{T} \to V'$ is an open map. Hence $p''|_{O_{\widetilde{Z}'}} \colon O_{\widetilde{Z}'} \to O_{Z'}$ is proper and open.

By $\mathcal{G} \Rightarrow Z$, we have $\mathcal{G} \to Z'$. Here, by removing finite elements from I, we assume that every $g_i \colon \mathbb{D} \to V$ has a unique lift $\mathbb{D} \to V'$ (cf. Remark 3.3), and we continue to use the same notation g_i. Let $O_{Z'} \Supset U_1 \Supset U_2 \Supset \cdots$ be a sequence of open neighborhoods of Z' such that every open neighborhood of Z' in $O_{Z'}$ contains some U_n for sufficiently large n. Set $\gamma_n = \frac{1}{2^{n+1}}$ and

$$I_n = \big\{ i \in I;\ g_i(z) \in U_n \text{ for } \gamma_n\text{-almost all } z \in \mathbb{D}(1-\gamma_n) \big\}.$$

Then we get a descending sequence $I \supset I_1 \supset I_2 \supset \cdots$. By $\mathcal{G} \to Z'$, $I \backslash I_n$ is a finite set for every n. We set $I_\infty = \bigcap_n I_n$.

For $i \in I_1$, we are going to construct a lift $\tilde{g}_i \in \mathrm{Hol}_m(\mathbb{D}, \bar{T})$, a connected open set $\Omega_i \subset \mathbb{D}$, and a connected component $\Omega_i' \subset Y_{\tilde{g}_i}$ of $\pi_{\tilde{g}_i}^{-1}(\Omega_i)$. We first consider the case $i \notin I_\infty$. We fix $i \in I_1 \backslash I_\infty$. We take n_i such that $i \in I_{n_i}$, but $i \notin I_{n_i+1}$. By Lemma 5.6, we may take an open set $\Omega_i \subset g_i^{-1}(U_{n_i}) \cap \mathbb{D}(1-2\gamma_{n_i})$ such that Ω_i is connected and $z \in \Omega_i$ for $2\gamma_{n_i}$-almost all $z \in \mathbb{D}(1-2\gamma_{n_i})$. We take $h \colon \mathcal{Y} \to \bar{T}$ by pull-back, where $\mathcal{Y}$ is a one-dimensional analytic space with finite map $q\colon \mathcal{Y}\to\mathbb{D}$:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "h"] \arrow[d, "q"] & \bar{T} \arrow[d, "p''"] \\ \mathbb{D} \arrow[r, "g_i"] & V'. \end{tikzcd}$$

Let $\mathcal{Y}_1,\dots,\mathcal{Y}_l$ be irreducible components of $\mathcal{Y}$. We may take $\mathcal{Y}_j$, where $1\le j\le l$, such that $h(\mathcal{Y}_j)\cap O_{\widetilde{Z}'}\neq\emptyset$. Then $h|_{\mathcal{Y}_j}\colon \mathcal{Y}_j\to\bar{T}$ defines a lift $\tilde{g}_i\in\mathrm{Hol}_m(\mathbb{D},\bar{T})$ of g_i such that $\tilde{g}_i(Y_{\tilde{g}_i})\cap O_{\widetilde{Z}'}\neq\emptyset$. We take a connected component Ω_i' of $\pi_{\tilde{g}_i}^{-1}(\Omega_i)\subset Y_{\tilde{g}_i}$ such that $\tilde{g}_i(\Omega_i')\cap O_{\widetilde{Z}'}\neq\emptyset$. By $g_i(\Omega_i)\subset U_{n_i}\subset O_{Z'}$, we have $\tilde{g}_i(\Omega_i')\subset O_{\widetilde{Z}'}\cup O_E$. Hence, since Ω_i' is connected, we have

$$\tilde{g}_i(\Omega_i')\subset O_{\widetilde{Z}'}. \tag{5.13}$$

When $i\in I_\infty$, we have $g_i(\mathbb{D})\subset Z'$. Since $\widetilde{Z}'\to Z'$ is a finite map, we have a lift $\tilde{g}_i\colon Y_{\tilde{g}_i}\to\widetilde{Z}'$ of g_i. We set $\Omega_i=\mathbb{D}$ and $\Omega_i'=Y_{\tilde{g}_i}$. Thus we have constructed $\tilde{g}_i\in\mathrm{Hol}_m(\mathbb{D},\bar{T})$, Ω_i, and Ω_i' for all $i\in I_1$.

We claim that, for all $i\in I_n$, we have $\tilde{g}_i(\Omega_i')\subset(p'')^{-1}(U_n)\cap O_{\widetilde{Z}'}$. This is obvious for $i\in I_\infty$ by $\tilde{g}_i(Y_{\tilde{g}_i})\subset\widetilde{Z}'$. When $i\notin I_\infty$, we have $n_i\ge n$, hence $g_i(\Omega_i)\subset U_{n_i}\subset U_n$. Thus $\tilde{g}_i(\Omega_i')\subset(p'')^{-1}(U_n)\cap O_{\widetilde{Z}'}$ (cf. (5.13)). This proves our claim.

Now we prove $((\tilde{g}_i,\Omega_i,\Omega_i'))_{i\in I_1}\rightsquigarrow\widetilde{Z}'$. Let $M\subset\bar{T}$ be an open neighborhood of $\widetilde{Z}'$. Let $s\in(0,1)$ and $\delta>0$. We take a sufficiently large n such that $s<1-2\gamma_n$ and $2\gamma_n<\delta$. Note that $O_{\widetilde{Z}'}\backslash M$ is a closed subset of $O_{\widetilde{Z}'}$. By $(p'')^{-1}(Z')=\Xi'\amalg E'$, we have $(p'')^{-1}(Z')\cap O_{\widetilde{Z}'}=\widetilde{Z}'$. Hence $Z'\cap p''(O_{\widetilde{Z}'}\backslash M)=\emptyset$. Since $p''|_{O_{\widetilde{Z}'}}\colon O_{\widetilde{Z}'}\to O_{Z'}$ is proper, $p''(O_{\widetilde{Z}'}\backslash M)$ is a closed subset of $O_{Z'}$. Hence there exists n' such that $U_{n'}\cap p''(O_{\widetilde{Z}'}\backslash M)=\emptyset$. Hence we have $(p'')^{-1}(U_{n'})\cap O_{\widetilde{Z}'}\subset M$. We set $n''=\max\{n,n'\}$. Then for all $i\in I_{n''}$, we have $\tilde{g}_i(\Omega_i')\subset M$ and $z\in\Omega_i$ for δ-almost all $z\in\mathbb{D}(s)$. Hence $((\tilde{g}_i,\Omega_i,\Omega_i'))_{i\in I_1}\rightsquigarrow\widetilde{Z}'$.

So far, we have considered that $\tilde{g}_i$ are maps into $\bar{T}$. Since the birational map $\bar{T}\dashrightarrow\widetilde{V}'$ is an isomorphism on T and $O_{\widetilde{Z}'}\subset T$, we have $((\tilde{g}_i,\Omega_i,\Omega_i'))_{i\in I_1}\rightsquigarrow\widetilde{Z}'$ under the consideration $\tilde{g}_i\in\mathrm{Hol}_m(\mathbb{D},\widetilde{V}')$. Thus we have $((\tilde{g}_i,\Omega_i,\Omega_i'))_{i\in I_1}\rightsquigarrow\widetilde{Z}$ under the consideration $\tilde{g}_i\in\mathrm{Hol}_m(\mathbb{D},\widetilde{V})$. By the construction, we have $\deg\pi_{\tilde{g}_i}\le[\mathbb{C}(V),\mathbb{C}(\widetilde{V})]$.

Now let $\widetilde{V}_0'\subset\widetilde{V}'$ be the inverse image of $\widetilde{V}_0$ under the map $\widetilde{V}'\to\widetilde{V}$. We define $J\subset I_1$ to be the set of $i\in I_1$ such that $\tilde{g}_i(Y_{\tilde{g}_i})\subset\widetilde{V}'\backslash\widetilde{V}_0'$ under the consideration $\tilde{g}_i\in\mathrm{Hol}_m(\mathbb{D},\widetilde{V}')$. To prove that J is finite, we assume on the contrary that J is infinite. Then there exists an irreducible component D of $\widetilde{V}'\backslash\widetilde{V}_0'$ such that $\tilde{g}_i(Y_{\tilde{g}_i})\subset D$ for infinitely many $i\in J$. By $((\tilde{g}_i,\Omega_i,\Omega_i'))_{i\in I_1}\rightsquigarrow\widetilde{Z}'$, we have $D\cap\widetilde{Z}'\neq\emptyset$. Hence

$T \cap D \neq \emptyset$. Hence $Z' \not\subset p'(D)$ and $g_i(\mathbb{D}) \subset p'(D)$ for infinitely many $i \in J$. This contradicts the assumption $\mathcal{G} \Rightarrow Z$. □

5.2.2. Proof of Lemma 5.5. We prove Lemma 5.5. We first fix a modification of $\Sigma_{1,A}$ as follows. The isomorphism $D\Phi\colon U_{1,A} \to U_{1,A}$ induces a rational map $\Sigma_{1,A} \dashrightarrow \Sigma_{1,A}$. We also have a rational map $\Sigma_{1,A} \dashrightarrow S_{1,A}$. By taking a birational modification $\widetilde{\Sigma_{1,A}} \to \Sigma_{1,A}$, we may assume that

- $p\colon \widetilde{\Sigma_{1,A}} \to S_{1,A}$ is generically finite and surjective, and
- $D\Phi\colon \widetilde{\Sigma_{1,A}} \to \Sigma_{1,A}$ is holomorphic.

Note that the natural map $U_{1,A} \to S_{1,A}$ is étale and $D\Phi$ is holomorphic on $U_{1,A}$. Hence we may assume $U_{1,A} \subset \widetilde{\Sigma_{1,A}}$. Let $\bar{\Xi} \subset \widetilde{\Sigma_{1,A}}$ be the closure of $\Xi \subset U_{1,A}$. Then $\bar{\Xi}$ is an irreducible component of $p^{-1}(Z)$ such that

$$\bar{\Xi} \cap U_{1,A} \neq \emptyset. \tag{5.14}$$

Now we are given an infinite subset $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A \times S)$ of non-constant holomorphic maps so that $(f_{S_{1,A}})_{f\in\mathcal{F}} \Rightarrow Z$. By (5.14) and $(f_{S_{1,A}})_{f\in\mathcal{F}} \Rightarrow Z$, we may apply Lemma 5.7 for $p\colon \widetilde{\Sigma_{1,A}} \to S_{1,A}$ and $(f_{S_{1,A}})_{f\in\mathcal{F}}$. Then there exists a finite subset $\mathcal{E} \subset \mathcal{F}$ such that for each $f \in \mathcal{F}\backslash\mathcal{E}$, there exist a lift $g \in \mathrm{Hol}_m(\mathbb{D}, \widetilde{\Sigma_{1,A}})$ of $f_{S_{1,A}}$, a connected open subset $\Omega_f \subset \mathbb{D}$, and a connected component Ω'_f of $\pi_g^{-1}(\Omega_f) \subset Y_g$ such that

$$((g, \Omega_f, \Omega'_f))_{f\in\mathcal{F}\backslash\mathcal{E}} \rightsquigarrow \bar{\Xi}, \tag{5.15}$$

$$\deg \pi_g \leq [\mathbb{C}(S_{1,A}) : \mathbb{C}(\Sigma_{1,A})], \tag{5.16}$$

and

$$g(Y_g) \not\subset \widetilde{\Sigma_{1,A}}\backslash U_{1,A}. \tag{5.17}$$

Let $\tau'\colon \widetilde{\Sigma_{1,A}} \to \Sigma$ be the natural projection. Then we have the following commutative diagram:

$$\begin{tikzcd} Y_g \arrow[r, "g"] \arrow[d, "\pi_g"'] & \widetilde{\Sigma_{1,A}} \arrow[r, "\tau'"] \arrow[d, "p"] & \Sigma \arrow[d, "\bar{q}"] \\ \mathbb{D} \arrow[r, "f_{S_{1,A}}"'] & S_{1,A} \arrow[r, "\tau"'] & S. \end{tikzcd} \tag{5.18}$$

For $f \in \mathcal{F} \backslash \mathcal{E}$, we set $\hat{f} = (f_A, \tau' \circ g) \in \mathrm{Hol}_m(\mathbb{D}, A \times \Sigma)$. Then we get the following commutative diagram:

$$\begin{tikzcd} Y_{\hat{f}} \arrow[r, "\hat{f}"] \arrow[d, "\pi_{\hat{f}}"'] & A \times \Sigma \arrow[r] \arrow[d] & \Sigma \arrow[d, "\bar{q}"] \\ \mathbb{D} \arrow[r, "f"'] & A \times S \arrow[r] & S. \end{tikzcd} \tag{5.19}$$

Here, $Y_{\hat{f}} = Y_g$. Note that $\tau'(\bar{\Xi}) = \bar{\Theta}$. Hence by (5.15), we get (5.10).

Next we show (5.9) and (5.12). Let $f \in \mathcal{F} \backslash \mathcal{E}$. By (5.17), we get (5.9). We note that $[\mathbb{C}(S) : \mathbb{C}(\Sigma)] = [\mathbb{C}(S_{1,A}) : \mathbb{C}(\Sigma_{1,A})]$. Hence by (5.16), we get (5.12).

Finally, we prove (5.11). We have $p \circ g = f_{S_{1,A}} \circ \pi_g$ (cf. (5.18)) and $p \circ (\hat{f})_{\widetilde{\Sigma_{1,A}}} = f_{S_{1,A}} \circ \pi_{\hat{f}}$ (cf. (5.19)). Hence we get

$$p \circ g = p \circ (\hat{f})_{\widetilde{\Sigma_{1,A}}}. \tag{5.20}$$

By the definition of $\hat{f}$, we get $\tau' \circ g = \hat{f}_\Sigma$. By (5.19), we have $\tau' \circ (\hat{f})_{\widetilde{\Sigma_{1,A}}} = \hat{f}_\Sigma$. Hence we get

$$\tau' \circ g = \tau' \circ (\hat{f})_{\widetilde{\Sigma_{1,A}}}. \tag{5.21}$$

Since $U_{1,A} = S_{1,A} \times_S U$, the two relations (5.20) and (5.21) yield $(\hat{f})_{\widetilde{\Sigma_{1,A}}} = g$. Hence by (5.15), we have

$$\left(((\hat{f})_{\widetilde{\Sigma_{1,A}}}, \Omega_f, \Omega'_f)\right)_{f \in \mathcal{F} \backslash \mathcal{E}} \to \bar{\Xi}. \tag{5.22}$$

By (5.8), we get

$$D\Phi(\bar{\Xi}) \subset \Sigma^*_{1,A}.$$

Thus by (5.22), we get $((D\Phi \circ \hat{f}_{\widetilde{\Sigma_{1,A}}}, \Omega_f, \Omega'_f))_{f \in \mathcal{F} \backslash \mathcal{E}} \to \Sigma^*_{1,A}$. By $D\Phi \circ \hat{f}_{\widetilde{\Sigma_{1,A}}} = (\Phi \circ \hat{f})_{\Sigma_{1,A}}$, we get (5.11). $\square$

§5.3. Good compactification of U

Lemma 5.8. *Let S be a smooth projective variety. Let $q\colon U \to S$ be an étale morphism from a smooth variety U. Let $\Theta \subset U$ be an irreducible Zariski closed set. Then there exist*

- *a smooth compactification Σ of U such that $q\colon U \to S$ extends to a morphism $\bar{q}\colon \Sigma \to S$,*
- *a Zariski open set $\Sigma^o \subset \Sigma$ such that $\bar{\Theta} \subset \Sigma^o$, where $\bar{\Theta} \subset \Sigma$ is the Zariski closure of Θ in Σ,*
- *a smooth semi-positive $(1,1)$-form $\eta \geq 0$ on Σ such that $\eta > 0$ on Σ^o,*

- *a proper Zariski closed subset $W \subset S$ with $W \subsetneqq \bar{q}(\overline{\Theta})$*

such that the following properties hold:

(1) *Let $\Delta \subset \Sigma$ be an irreducible component of ∂U such that $\Delta \cap \overline{\Theta} \neq \emptyset$. Then $\bar{q}(\Delta) \subset W$.*

(2) *Let ω_S be a smooth positive $(1,1)$-form on S and let $\lambda_W \geq 0$ be a Weil function for W. Then there exists a positive constant $c > 0$ such that for all $g \in \mathrm{Hol}_m(\mathbb{D}, \Sigma)$ with $\bar{q} \circ g(Y_g) \not\subset W$, we have*

$$T_s(r, g, \eta) \leq cT_s(r, \bar{q} \circ g, \omega_S) + cm((s+r)/2, \bar{q} \circ g, \lambda_W) + c$$

for all $s \in (1/2, 1)$ and $r \in (s, 1)$.

To prove this lemma, we start from the following lemma.

Lemma 5.9. *Let X be a projective variety. Let V and V' be non-empty Zariski open subsets of X such that $V \cap V'$ is smooth. Then there exists a proper birational morphism $p\colon X' \to X$ from a projective variety X' such that $p^{-1}(V)$ is smooth and $p^{-1}(V') \to V'$ is isomorphic.*

Proof. By a strong desingularization of V, we have a sequence

$$V = V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow \cdots \leftarrow V_k$$

such that

- V_k is smooth, and
- each $V_{i+1} \to V_i$ is a blow-up $V_{i+1} = \mathrm{Bl}_{C_i} V_i$ such that the center $C_i \subset V_i$ satisfies $p_i(C_i) \subset V \backslash (V \cap V')$, where $p_i\colon V_i \to V$ is the natural morphism.

We inductively construct a projective variety X_i with an open immersion $V_i \subset X_i$ as follows. Set $X_0 = X$. Then $V_0 \subset X_0$. Suppose we have constructed X_i with $V_i \subset X_i$. Let $\overline{C_i} \subset X_i$ be the (schematic) closure of $C_i \subset V_i$. We set $X_{i+1} = \mathrm{Bl}_{\overline{C_i}} X_i$. Then the inverse image of $V_i \subset X_i$ under the projection $X_{i+1} \to X_i$ is equal to $V_{i+1} \subset X_{i+1}$. Since X_i is projective, the blow-up X_{i+1} is also projective. Thus we have constructed projective varieties X_i with open immersions $V_i \subset X_i$ for all $i = 0, 1, \ldots, k$.

By the construction, we have extensions $\bar{p}_i\colon X_i \to X$ of p_i. By $p_i(C_i) \subset V \backslash (V \cap V')$, we have $\bar{p}_i(\overline{C_i}) \subset X \backslash V'$. Hence $\bar{p}_i\colon X_i \to X$ are isomorphisms over V' for all i. Moreover, we have $\overline{p_i}^{-1}(V) = V_i$ for all i. We set $X' = X_k$ and $p = \bar{p}_k$ to conclude the proof. □

Proof of Lemma 5.8. We first construct Σ and Σ^o. We apply Zariski's main theorem for the quasi-finite map $q\colon U \to S$. Then we get an open immersion $U \hookrightarrow \Sigma_1$

and a finite map

$$\bar{q}_1 \colon \Sigma_1 \to S.$$

Since S is projective, Σ_1 is projective. Since U is smooth, we may assume that Σ_1 is normal. Set

$$\Sigma_2 = \mathrm{Bl}_{\overline{\Theta}_1 \cap \partial U} \Sigma_1, \tag{5.23}$$

where $\overline{\Theta}_1 \subset \Sigma_1$ is the Zariski closure of $\Theta \subset U$ in Σ_1. Then Σ_2 is projective. Set

$$p_1 \colon \Sigma_2 \to \Sigma_1.$$

Then p_1 is isomorphic over $U \subset \Sigma_1$. Hence $U \subset \Sigma_2$. Let $\overline{\Theta}_2 \subset \Sigma_2$ be the Zariski closure of $\Theta \subset U$ in Σ_2. If $\Delta \subset \Sigma_2$ is an irreducible component of $\Sigma_2 \backslash U$ such that $\Delta \cap \overline{\Theta}_2 \neq \emptyset$, then

$$p_1(\Delta) \subsetneqq \overline{\Theta}_1. \tag{5.24}$$

Let $V \subset \Sigma_2$ be a Zariski open set defined by $V = \Sigma_2 \backslash \bigcup_{\Delta'} \Delta'$, where Δ' runs over all irreducible components of $\Sigma_2 \backslash U$ such that $\Delta' \cap \overline{\Theta}_2 = \emptyset$. Then we have

$$\overline{\Theta}_2 \subset V. \tag{5.25}$$

We define a Zariski closed set $D \subset \Sigma_2$ by $D = \bigcup_{\Delta} \Delta$, where Δ runs over all irreducible components of $\Sigma_2 \backslash U$ such that $\Delta \cap \overline{\Theta}_2 \neq \emptyset$. We have $U = V \cap (\Sigma_2 \backslash D)$, where U is smooth. By Lemma 5.9, there exists a proper birational modification

$$p_2 \colon \Sigma_3 \to \Sigma_2$$

such that $p_2^{-1}(V)$ is smooth and p_2 is isomorphic over $\Sigma_2 \backslash D$. We define Σ by a smooth modification

$$p_3 \colon \Sigma \to \Sigma_3$$

which is an isomorphism over $p_2^{-1}(V) \subset \Sigma_3$. We set $\Sigma^o = (p_3 \circ p_2)^{-1}(V)$. By (5.25), we have $\overline{\Theta} \subset \Sigma^o$.

We construct $W \subset S$ with $W \subsetneqq \bar{q}(\overline{\Theta})$ and prove (1). We denote the exceptional locus of $p_1 \colon \Sigma_2 \to \Sigma_1$ by $E_1 \subset \Sigma_2$. We set

$$W = \bar{q}_1 \circ p_1(E_1 \cup D) \subset S.$$

By (5.23), we have $p_1(E_1) \subsetneqq \overline{\Theta}_1$. By (5.24), we have $p_1(D) \subsetneqq \overline{\Theta}_1$. Hence we have $W \subsetneqq \bar{q}(\overline{\Theta})$. By the definition of $D \subset \Sigma_2$, we obtain assertion (1).

Let $E \subset \Sigma_3$ be the exceptional locus of $\Sigma_3 \to \Sigma_1$. Then we claim

$$\bar{q}_1 \circ p_1 \circ p_2(E) \subset W. \tag{5.26}$$

To prove this, we denote the exceptional locus of $p_2\colon \Sigma_3 \to \Sigma_2$ by $E_2 \subset \Sigma_3$. Then we have $E \subset p_2^{-1}(E_1) \cup E_2$. Hence $p_2(E) \subset E_1 \cup p_2(E_2)$. Since $p_2\colon \Sigma_3 \to \Sigma_2$ is isomorphic over $\Sigma_2 \backslash D$, we have $p_2(E_2) \subset D$. Hence $p_2(E) \subset E_1 \cup D$. Thus we have proved (5.26).

Next we construct η and prove property (2). Since Σ_3 is projective, we have an immersion $\iota\colon \Sigma_3 \hookrightarrow \mathbb{P}^n$. Let $\omega_{\mathbb{P}^n}$ be the Fubini–Study metric on $\mathbb{P}^n$. Let $\varphi\colon \Sigma \to \mathbb{P}^n$ be the composite of $\Sigma \to \Sigma_3 \to \mathbb{P}^n$. We set

$$\eta = \varphi^* \omega_{\mathbb{P}^n}.$$

Then η is semi-positive on Σ and positive on Σ^o, for the composite of $\Sigma^o \hookrightarrow \Sigma \to \Sigma_3$ is an open immersion.

Now we prove (2). Let $H_1, \dots, H_l \subset \mathbb{P}^n$ be hyperplanes such that $H_1 \cap \cdots \cap H_l = \emptyset$. Let h be the Fubini–Study metric on $\mathcal{O}_{\mathbb{P}^n}(1)$. Let τ_i be a section of $\mathcal{O}_{\mathbb{P}^n}(1)$ such that $H_i = (\tau_i = 0)$. Set $\lambda_{H_i}(x) = -\log \sqrt{h(\tau_i(x), \tau_i(x))} \geq 0$. By Lemma 4.4, there exists a positive constant $c_i > 0$ such that for all $h \in \mathrm{Hol}_m(\mathbb{D}, \Sigma_3)$ with $h(Y_h) \not\subset E \cup \iota^* H_i$, we have

$$T_\sigma(r, \iota \circ h, H_i, \lambda_{H_i}) \leq c_i T_\sigma(r, \bar{q}_1 \circ p_1 \circ p_2 \circ h, \omega_S) + c_i m(\sigma, h, \lambda_E) + c_i \tag{5.27}$$

for all $0 < \sigma < r < 1$.

Let $g \in \mathrm{Hol}_m(\mathbb{D}, \Sigma)$ with $\bar{q} \circ g(Y_g) \not\subset W$. We may take H_i such that $\varphi \circ g(Y_g) \not\subset \operatorname{supp} H_i$, where $\varphi\colon \Sigma \to \mathbb{P}^n$ is the morphism above. Then by (4.1), we have

$$T_\sigma(r, g, \eta) = T_\sigma(r, \varphi \circ g, \omega_{\mathbb{P}^n}) = T_\sigma(r, \varphi \circ g, H_i, \lambda_{H_i})$$

for all $0 < \sigma < r < 1$. By (5.26) and (5.27), we have

$$T_\sigma(r, \varphi \circ g, H_i, \lambda_{H_i}) \leq c_i T_\sigma(r, \bar{q} \circ g, \omega_S) + c_i m(\sigma, \bar{q} \circ g, \lambda_W) + c_i.$$

We set $c' = \max_{1 \leq i \leq l}\{c_i\}$. Then we have

$$T_\sigma(r, g, \eta) \leq c' T_\sigma(r, \bar{q} \circ g, \omega_S) + c' m(\sigma, \bar{q} \circ g, \lambda_W) + c'.$$

Now for $1/2 < s < r < 1$ and $\sigma = (s + r)/2$, we have

$$T_s(r, g, \eta) \leq 4 T_\sigma(r, g, \eta) \leq 4c' T_\sigma(r, \bar{q} \circ g, \omega_S) + 4c' m(\sigma, \bar{q} \circ g, \lambda_W) + 4c'.$$

We set $c = 4c'$. By $T_\sigma(r, \bar{q} \circ g, \omega_S) \leq T_s(r, \bar{q} \circ g, \omega_S)$, we conclude the proof. $\square$

Remark 5.10. Let A be a semi-abelian variety and let S be a smooth projective variety. Let $Z \subset S_{1,A}$ be an irreducible Zariski closed set. Assume that Z is horizontally integrable. Then we take and fix the following objects:

- An immersion $U \hookrightarrow A \times S$ as in Definition 3.16.
- An isomorphism $\Phi\colon A \times U \to A \times U$ from (3.3).
- An irreducible component $\Xi \subset U_{1,A}$ of $(q')^{-1}(Z)$ from Lemma 3.17, where $q'\colon U_{1,A} \to S_{1,A}$ is induced from $q\colon U \to S$.
- The image $\Theta \subset U$ of Ξ under $U_{1,A} \to U$. Since $U_{1,A} \to U$ is proper, Θ is an irreducible Zariski closed subset of U.

We may apply Lemma 5.8 for $q\colon U \to S$ and $\Theta \subset U$ to get and fix the following objects:

- A smooth compactification Σ of U such that $q\colon U \to S$ extends to a morphism $\bar{q}\colon \Sigma \to S$.
- A Zariski open set $\Sigma^o \subset \Sigma$ such that $\overline{\Theta} \subset \Sigma^o$, where $\overline{\Theta} \subset \Sigma$ is the Zariski closure of Θ in Σ.
- A smooth semi-positive $(1,1)$-form $\eta \geq 0$ on Σ such that $\eta > 0$ on Σ^o.
- A proper Zariski closed subset $W \subset S$ with $W \subsetneqq \bar{q}(\overline{\Theta})$.

By $\bar{q}(\overline{\Theta}) = \tau(Z)$, we have $W \subsetneqq \tau(Z)$, where $\tau\colon S_{1,A} \to S$ is the natural map.

§5.4. Estimate for the first term of the right-hand side of (5.3)

Let A be a semi-abelian variety with the quotient $\rho\colon A \to A_0$ as in (5.2). Let S be a smooth projective variety. Let $Z \subset S_{1,A}$ be an irreducible Zariski closed set. Assume that Z is horizontally integrable. We recall $W \subset S$ from Remark 5.10.

Lemma 5.11. *Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A \times S)$ be an infinite set of non-constant holomorphic maps such that $(f_{S_{1,A}})_{f\in\mathcal{F}} \Rightarrow Z$. Let ω_{A_0} be an invariant positive $(1,1)$-form on A_0 and let ω_S be a smooth positive $(1,1)$-form on S. Let $\lambda_W \geq 0$ be a Weil function for W. Let $s \in (1/2, 1)$ and $\delta > 0$. Then there exists a positive constant $\alpha > 0$ such that for all $f \in \mathcal{F}$ with $f_S(\mathbb{D}) \not\subset W$, we have*

$$T_s(r, \rho\circ f_A, \omega_{A_0}) \leq \alpha T_s(r, f_S, \omega_S) + \alpha m((s+r)/2, f_S, \lambda_W) + \alpha$$

for all $r \in (s,1)$ outside some exceptional set $E \subset (s,1)$ with the linear measure $|E| < \delta$.

Proof. We use the objects fixed in Remark 5.10. Note that the isomorphism $\Phi\colon A\times U \to A \times U$ induces an isomorphism $\Phi_0\colon A_0 \times \Sigma \to A_0 \times \Sigma$ with the following commutative diagram:

$$\begin{tikzcd} A\times U \arrow[r, "\Phi"] \arrow[d, "(\rho,i)"'] & A\times U \arrow[d, "(\rho,i)"] \\ A_0\times\Sigma \arrow[r, "\Phi_0"'] & A_0\times\Sigma. \end{tikzcd} \tag{5.28}$$

Here, $i\colon U\to\Sigma$ is the open immersion.

We prove this claim. Let $\varphi\colon U\to A$ be the composite of the immersion $\iota\colon U\hookrightarrow A\times U$ and the first projection $A\times S\to A$. Then $\Phi\colon A\times U\to A\times U$ is defined by $\Phi(a,u)=(a-\varphi(u),u)$ (cf. (3.4)). We have a rational map $\rho\circ\varphi\colon\Sigma\dashrightarrow A_0$. Since A_0 is an abelian variety and Σ is smooth, this rational map extends to a morphism $\rho\circ\varphi\colon\Sigma\to A_0$. Then Φ_0 is defined by $\Phi_0(b,u)=(b-\rho\circ\varphi(u),u)$ for $(b,u)\in A_0\times\Sigma$. Then Φ_0 fits in the commutative diagram (5.28).

We set

$$\delta'=\delta/801. \tag{5.29}$$

We take an open neighborhood $O_{\overline{\Theta}}\subset\Sigma$ of $\overline{\Theta}\subset\Sigma$ such that $\overline{O_{\overline{\Theta}}}\subset\Sigma^o$, where we recall $\Sigma^o\subset\Sigma$ from Remark 5.10. We take a smooth positive $(1,1)$-form ω_Σ on Σ such that

$$\omega_\Sigma\leq\eta \tag{5.30}$$

on $O_{\overline{\Theta}}$, where we recall η from Remark 5.10.

We apply Lemma 5.5 to get a finite subset $\mathcal{E}\subset\mathcal{F}$ such that for each $f\in\mathcal{F}\backslash\mathcal{E}$, we get a lifting $\hat{f}\in\mathrm{Hol}_m(\mathbb{D},A\times\Sigma)$ of $f\in\mathrm{Hol}(\mathbb{D},A\times S)$, a connected open set $\Omega_f\subset\mathbb{D}$, and a connected component $\Omega_f'\subset Y_{\hat{f}}$ of $\pi_{\hat{f}}^{-1}(\Omega_f)$ with the properties (5.9)–(5.12) described in Lemma 5.5. Then we have (cf. (5.10))

$$((\hat{f}_\Sigma,\Omega_f,\Omega_f'))_{f\in\mathcal{F}\backslash\mathcal{E}}\rightsquigarrow\overline{\Theta}.$$

Hence there exists a finite subset $\mathcal{E}_1\subset\mathcal{F}$, where $\mathcal{E}\subset\mathcal{E}_1$, such that for all $f\in\mathcal{F}\backslash\mathcal{E}_1$, we have $z\in\Omega_f$ for δ'-almost all $z\in\mathbb{D}(1-\delta')$ and

$$\hat{f}_\Sigma(\Omega_f')\subset O_{\overline{\Theta}}. \tag{5.31}$$

Let ω_A be an invariant positive $(1,1)$-form on A such that

$$\rho^*\omega_{A_0}\leq\omega_A. \tag{5.32}$$

We take an open neighborhood $O_{\Sigma_{1,A}^*}\subset\Sigma_{1,A}$ of $\Sigma_{1,A}^*$ such that

$$|v_{\mathrm{Lie}\,A}|_{\omega_A}\leq|v_\Sigma|_{\omega_\Sigma} \tag{5.33}$$

for all $(v_\Sigma, v_{\mathrm{Lie}\,A}) \in T\Sigma \times \mathrm{Lie}\,A$ with $[(v_\Sigma, v_{\mathrm{Lie}\,A})] \in O_{\Sigma^*_{1,A}} \subset \Sigma_{1,A}$. We have (cf. (5.11))

$$\big(((\Phi \circ \hat{f})_{\Sigma_{1,A}}, \Omega_f, \Omega'_f)\big)_{f \in \mathcal{F} \backslash \mathcal{E}} \rightsquigarrow \Sigma^*_{1,A}.$$

Hence, there exists a finite subset $\mathcal{E}_2 \subset \mathcal{F}$, where $\mathcal{E}_1 \subset \mathcal{E}_2$, such that for all $f \in \mathcal{F}\backslash\mathcal{E}_2$, we have

$$(\Phi \circ \hat{f})_{\Sigma_{1,A}}(\Omega'_f) \subset O_{\Sigma^*_{1,A}}. \tag{5.34}$$

Now we take $f \in \mathcal{F}\backslash\mathcal{E}_2$ such that $f_S(\mathbb{D}) \not\subset W$. Since $z \in \Omega_f$ for δ'-almost all $z \in \mathbb{D}(1-\delta')$, we may apply [46, Lem. 7.1] to get

$$T_s(r, \rho \circ f_A, \omega_{A_0}) \leq 8 \int_s^r \frac{dt}{t} \int_{\mathbb{D}(t)\cap\Omega_f} (\rho \circ f_A)^* \omega_{A_0} \tag{5.35}$$

for all $r \in (s, 1-\delta')$ outside some exceptional set $E' \subset (s, 1-\delta')$ whose linear measure satisfies

$$|E'| \leq 800\delta'. \tag{5.36}$$

By $\hat{f}_A = f_A \circ \pi_{\hat{f}}$, we have

$$\int_s^r \frac{dt}{t} \int_{\mathbb{D}(t)\cap\Omega_f} (\rho \circ f_A)^* \omega_{A_0} = T_s(r, (\rho \circ \hat{f}_A, \Omega_f, \Omega'_f), \omega_{A_0}). \tag{5.37}$$

Here we recall (5.7) for the definition of the right-hand side.

Let $\Phi_0 \colon A_0 \times \Sigma \to A_0 \times \Sigma$ be the isomorphism above (cf. (5.28)). Let $\nu_1 \colon A_0 \times \Sigma \to A_0$ be the first projection, and $\nu_2 \colon A_0 \times \Sigma \to \Sigma$ be the second projection. Then there exists a positive constant $\alpha_1 > 0$ such that

$$(\nu_1 \circ \Phi_0^{-1})^* \omega_{A_0} \leq \alpha_1 (\nu_1^* \omega_{A_0} + \nu_2^* \omega_\Sigma).$$

We define $\tilde{\rho} \colon A \times \Sigma \to A_0 \times \Sigma$ by $\tilde{\rho}(a, s) = (\rho(a), s)$ for $(a, s) \in A \times \Sigma$. Then, we get

$$\begin{aligned} &T_s(r, (\rho \circ \hat{f}_A, \Omega_f, \Omega'_f), \omega_{A_0}) \\ &\quad \leq \alpha_1 T_s\big(r, ((\Phi_0 \circ \tilde{\rho} \circ \hat{f})_{A_0}, \Omega_f, \Omega'_f), \omega_{A_0}\big) + \alpha_1 T_s(r, (\hat{f}_\Sigma, \Omega_f, \Omega'_f), \omega_\Sigma). \end{aligned} \tag{5.38}$$

By (5.33) and (5.34), we have

$$T_s\big(r, ((\Phi \circ \hat{f})_A, \Omega_f, \Omega'_f), \omega_A\big) \leq T_s(r, (\hat{f}_\Sigma, \Omega_f, \Omega'_f), \omega_\Sigma).$$

By the commutativity of (5.28) and (5.32), we have

$$T_s\big(r, ((\Phi_0 \circ \tilde{\rho} \circ \hat{f})_{A_0}, \Omega_f, \Omega'_f), \omega_{A_0}\big) \leq T_s\big(r, ((\Phi \circ \hat{f})_A, \Omega_f, \Omega'_f), \omega_A\big).$$

Hence we get

$$T_s\big(r,((\Phi_0\circ\tilde{\rho}\circ\hat{f})_{A_0},\Omega_f,\Omega'_f),\omega_{A_0}\big)\le T_s(r,(\hat{f}_\Sigma,\Omega_f,\Omega'_f),\omega_\Sigma). \tag{5.39}$$

Hence by (5.35), (5.37), (5.38), and (5.39), we get

$$T_s(r,\rho\circ f_A,\omega_{A_0})\le 16\alpha_1 T_s(r,(\hat{f}_\Sigma,\Omega_f,\Omega'_f),\omega_\Sigma)$$

for all $r\in(s,1-\delta')\backslash E'$. Using (5.30) and (5.31), we have

$$T_s(r,(\hat{f}_\Sigma,\Omega_f,\Omega'_f),\omega_\Sigma)\le T_s(r,(\hat{f}_\Sigma,\Omega_f,\Omega'_f),\eta)\le(\deg\pi_{\hat{f}})T_s(r,\hat{f}_\Sigma,\eta).$$

Hence we get

$$T_s(r,\rho\circ f_A,\omega_{A_0})\le\alpha_2 T_s(r,\hat{f}_\Sigma,\eta) \tag{5.40}$$

for all $r\in(s,1-\delta')\backslash E'$, where we set $\alpha_2=16\alpha_1[\mathbb{C}(S):\mathbb{C}(\Sigma)]$ (cf. (5.12)). We set $\sigma=(r+s)/2$. By Lemma 5.8, we get

$$T_s(r,\hat{f}_\Sigma,\eta)\le\alpha_3 T_s(r,f_S,\omega_S)+\alpha_3 m(\sigma,f_S,\lambda_W)+\alpha_3, \tag{5.41}$$

where $\alpha_3>0$ is a positive constant which is independent of the choice of $f\in\mathcal{F}\backslash\mathcal{E}_2$. We set $\alpha=\alpha_2\alpha_3$ and $E=E'\cup(1-\delta',1)$. By (5.29) and (5.36), we have $|E|\le\delta$. Then by (5.40) and (5.41), we get the desired estimate for $f\in\mathcal{F}\backslash\mathcal{E}_2$ with $f_S(\mathbb{D})\not\subset W$.

Finally, we enlarge $\alpha>0$ so that

$$\max_{f\in\mathcal{E}_2}T_s(1-\delta,\rho\circ f_A,\omega_{A_0})\le\alpha$$

to complete the proof of the lemma. □

§5.5. Estimate of Weil functions

Let ω_A be an invariant positive $(1,1)$-form on A. Given $x,y\in A$, we denote by $d(x,y)$ the distance with respect to ω_A. Let $\pi\colon\mathbb{C}^n\to A$ be a universal covering, where $n=\dim A$. We denote by $d_{\mathbb{C}^n}$ the Euclidean distance on $\mathbb{C}^n$. Note that $\pi^*\omega_A$ is an invariant positive $(1,1)$-form on $\mathbb{C}^n$ with respect to the additive structure of $\mathbb{C}^n$. Hence there exists a positive constant $\alpha>1$ such that

$$\frac{1}{\alpha}d(x,y)\le d_{\mathbb{C}^n}(\pi^{-1}(x),\pi^{-1}(y))\le\alpha d(x,y) \tag{5.42}$$

for all $x,y\in A$.

Lemma 5.12. *Let $B\subset A$ be a semi-abelian subvariety. Let d_B be the distance function on B with respect to some invariant positive $(1,1)$-form on B. Then* *there exists a positive constant* $\beta > 1$ *such that*

$$\frac{1}{\beta} d_B(x,y) \leq d(x,y) \leq \beta d_B(x,y)$$

for all $x, y \in B$.

Proof. Let $\mathbb{C}^k \to B$ be a universal covering, where $k = \dim B$. Then $B \subset A$ induces an immersion $\mathbb{C}^k \subset \mathbb{C}^n$ into the universal covering of A. This is a linear subspace. Hence for $p, q \in \mathbb{C}^k$, we have $d_{\mathbb{C}^k}(p,q) = d_{\mathbb{C}^n}(p,q)$. Hence by (5.42), we obtain our lemma. □

Lemma 5.13. *Let* $\bar{A}$ *be a smooth equivariant compactification. Let* $D \subset \partial A$ *be an irreducible component with a Weil function* λ_D. *There exists a positive constant* $c > 0$ *such that*

$$\lambda_D(y) \leq \lambda_D(x) + cd(x,y) + c$$

for all $x, y \in A$.

Proof. We first consider the case that A is an algebraic torus, and then prove the general case.

The case of algebraic tori. The proof is by induction on the dimension of A. When $\dim A = 1$, we have $A = \mathbb{G}_m$ and $\bar{A} = \mathbb{P}^1$. We may assume $D = (\infty)$. Then

$$|\lambda_D(x) - \lambda_D(y)| \leq \left|\log^+ |x| - \log^+ |y|\right| \leq \left|\log |x/y|\right|.$$

On the other hand, by (5.42), we have

$$\left|\log |x/y|\right| \leq \alpha d(x,y). \tag{5.43}$$

This shows our estimate in the one-dimensional case.

Now we consider the case $\dim A \geq 2$ and assume that our lemma is true for algebraic tori whose dimension is less than $\dim A$. Let $D \subset \partial A$ be an irreducible component. Let $I \subset A$ be the isotropy group of D. Then $I = \mathbb{G}_m$. Set $B = A/I$. We take a compactification $\overline{B}$ of B such that $\overline{B} = D$. By Lemma A.11, there exist a Zariski open neighborhood $U \subset \bar{A}$ of $D \subset \bar{A}$ and a canonical projection $p\colon U \to D$ which extends $A \to B$. Note that $p\colon U \to D$ is a total space of a line bundle over $\overline{B}$ whose zero section is D. Let $\|\cdot\|$ be a smooth Hermitian metric on the line bundle $p\colon U \to \overline{B}$. We define $\mu\colon U\backslash D \to \mathbb{R}$ by $\mu(x) = \log(1/\|x\|)$ where $x \in U$. We prove the estimate of Lemma 5.13 in several steps.

Step 1. We first show that there exists a positive constant $c_1 > 0$ such that

$$|\mu(x) - \mu(y)| \leq c_1 d(x,y) + c_1 \tag{5.44}$$

for all $x, y \in A$ which satisfy $p(x) = p(y)$. To prove this, we take a finite affine covering $\overline{B} = \bigcup_{i\in I} V_i$ such that $U|_{V_i} = V_i \times \mathbb{C}$. Let $\tau_i \colon U|_{V_i} \to \mathbb{C}$ be the second projection. We may take an open set $W_i \Subset V_i$ such that $\overline{B} = \bigcup_{i\in I} W_i$ and $|\mu(x) - \log|1/\tau_i(x)|| < \gamma_i$ on $x \in p^{-1}(W_i)\backslash D$. We set $\gamma = \max_{i\in I}\gamma_i$. Now given $x, y \in A$ such that $p(x) = p(y)$, we take $i \in I$ such that $p(x) \in W_i$. Then we have

$$|\mu(x) - \mu(y)| \leq \big|\log|1/\tau_i(x)| - \log|1/\tau_i(y)|\big| + 2\gamma.$$

By the one-dimensional case (cf. (5.43)), we have

$$\big|\log|1/\tau_i(x)| - \log|1/\tau_i(y)|\big| \leq \alpha d_{\mathbb{G}_m}(\tau_i(x), \tau_i(y)).$$

Hence we get

$$|\mu(x) - \mu(y)| \leq \alpha d_{\mathbb{G}_m}(\tau_i(x), \tau_i(y)) + 2\gamma.$$

We take $g \in I$ such that $y = g \cdot x$. Since τ_i is $\mathbb{G}_m$ equivariant, we have $\tau_i(y) = g \cdot \tau_i(x)$. Hence we have $d_{\mathbb{G}_m}(\tau_i(x), \tau_i(y)) = d_{\mathbb{G}_m}(e_{\mathbb{G}_m}, g)$. Hence we get

$$|\mu(x) - \mu(y)| \leq \alpha d_{\mathbb{G}_m}(e_{\mathbb{G}_m}, g) + 2\gamma.$$

Similarly, we have $d(x, y) = d(e_A, g)$. By Lemma 5.12, we get $d_I(e_A, g) \leq \beta d(e_A, g)$. Hence we get (5.44) with $c_1 = \max\{\alpha\beta, 2\gamma\}$.

Step 2. By [6, Cor., p. 115], the quotient $A \to B$ has a section $s \colon B \to A$ of group varieties. We next show

$$|\mu(x) - \mu(y)| \leq c_2 d(x, y) + c_2 \tag{5.45}$$

for all $x, y \in s(B)$. By the section $s \colon B \to A$, we get a rational section $\bar{s} \colon \overline{B} \dashrightarrow U$, which is holomorphic and zero-free on B. Set $(s) = E - F$. Then $\mu(s(b)) = \lambda_E(b) - \lambda_F(b)$ for $b \in B$. Then by the induction hypothesis, we have, for $b, b' \in B$ with $x = s(b)$ and $y = s(b')$,

$$\begin{aligned}|\mu(x) - \mu(y)| &= |(\lambda_E(b) - \lambda_F(b)) - (\lambda_E(b') - \lambda_F(b'))| \\ &\leq |\lambda_E(b) - \lambda_E(b')| + |\lambda_F(b) - \lambda_F(b')| \\ &\leq \gamma d_B(b, b') + \gamma.\end{aligned}$$

By Lemma 5.12, we have $d_B(b, b') \leq \alpha d(x, y)$. Hence we get (5.45) with $c_2 = \max\{\alpha\gamma, \gamma\}$.

Step 3. Now we take $x, y \in A$ such that $x \in s(B)$. Then by (5.44) and (5.45), we have

$$\begin{aligned}|\mu(x) - \mu(y)| &\leq \big|\mu(x) - \mu\big(s(p(y))\big)\big| + \big|\mu\big(s(p(y))\big) - \mu(y)\big| \\ &\leq c_2 d\big(x, s(p(y))\big) + c_1 d\big(s(p(y)), y\big) + c_1 + c_2.\end{aligned}$$

By

$$d\big(s(p(y)),y\big) \le d\big(x,s(p(y))\big) + d(x,y),$$

we get

$$|\mu(x)-\mu(y)| \le (c_1+c_2)d\big(x,s(p(y))\big) + c_1 d(x,y) + c_1 + c_2.$$

Since $s(p(x)) = x$, Lemma 5.12 yields $d(x,s(p(y))) \le \alpha d_B(p(x),p(y))$. Hence

$$|\mu(x)-\mu(y)| \le (c_1+c_2)\alpha d_B(p(x),p(y)) + c_1 d(x,y) + c_1 + c_2.$$

Since $p|_A\colon A \to B$ is a group homomorphism, we have

$$d_B(p(x),p(y)) \le \alpha' d(x,y). \tag{5.46}$$

Indeed, $(p|_A)^*\omega_B$ is an invariant $(1,1)$-form on A, where ω_B is the invariant positive $(1,1)$-form on B used to define d_B. Hence there exists a positive constant $\alpha' > 0$ such that $(p|_A)^*\omega_B \le \alpha'\omega_A$. To conclude, we get

$$|\mu(x)-\mu(y)| \le cd(x,y) + c,$$

where $c = \max\{(c_1+c_2)\alpha\alpha' + c_1, c_1 + c_2\}$.

Step 4. Now we take $x, y \in A$ in general. We take $g \in I$ such that $s(p(x)) = g \cdot x$. Then we get

$$|\mu(x)-\mu(y)| = |\mu(g\cdot x)-\mu(g\cdot y)| \le cd(g\cdot x, g\cdot y) + c = cd(x,y) + c.$$

Hence

$$|\lambda_D(x)-\lambda_D(y)| \le |\mu(x)-\mu(y)| \le cd(x,y) + c,$$

which concludes the induction step. Hence we get our lemma in the case of algebraic tori.

The case of general semi-abelian varieties. We treat the general case as in (5.2):

$$0 \to T \to A \xrightarrow{\rho} A_0 \to 0.$$

We pull back this sequence by the universal covering $\pi\colon \mathbb{C}^n \to A_0$ to get

$$0 \to T \to A \times_{A_0} \mathbb{C}^n \xrightarrow{r} \mathbb{C}^n \to 0.$$

We have a section $s\colon \mathbb{C}^n \to A \times_{A_0} \mathbb{C}^n$ of complex Lie groups. Let $p\colon A \times_{A_0} \mathbb{C}^n \to A$ be the natural projection. We set

$$\psi = p \circ s\colon \mathbb{C}^n \to A.$$

Then p and ψ are morphisms of complex Lie groups. We take a closed ball $\mathbb{B} \subset \mathbb{C}^n$ centered at the origin such that $\pi(\mathbb{B}) = A_0$.

We first show that there exists a positive constant $\alpha_1 > 0$ such that for all $g \in T$ and $z, w \in \mathbb{B}$, we have

$$|\lambda_D(g \cdot \psi(z)) - \lambda_D(g \cdot \psi(w))| \leq \alpha_1. \tag{5.47}$$

We prove this. Let $\bar{A}$ correspond to a torus embedding $T \subset \overline{T}$ (cf. Lemma A.6). Note that $\rho\colon A \to A_0$ extends to $\bar{\rho}\colon \bar{A} \to A_0$. Then $r\colon A \times_{A_0} \mathbb{C}^n \to \mathbb{C}^n$ extends to $\bar{r}\colon \bar{A} \times_{A_0} \mathbb{C}^n \to \mathbb{C}^n$. The section $s\colon \mathbb{C}^n \to A \times_{A_0} \mathbb{C}^n$ induces a (non-canonical) splitting $\bar{A} \times_{A_0} \mathbb{C}^n = \overline{T} \times \mathbb{C}^n$ such that the composite $q \circ s\colon \mathbb{C}^n \to T$ with the first projection $q\colon \bar{A} \times_{A_0} \mathbb{C}^n \to \overline{T}$ is the constant map identically equal to e_T:

$$\begin{tikzcd}
\bar{A} \arrow[d, "\bar{\rho}"'] & \bar{A} \times_{A_0} \mathbb{C}^n \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\bar{r}"] & \overline{T}. \\
A_0 & \mathbb{C}^n. \arrow[l, "\pi"]
\end{tikzcd}$$

Note that $p^*\lambda_D(x) - (\lambda_D|_{\rho^{-1}(0)})(q(x))$ is continuous on $\bar{A} \times_{A_0} \mathbb{C}^n$. Since $\bar{r}^{-1}(\mathbb{B})$ is compact, there exists a positive constant $\alpha_1' > 0$ such that

$$|p^*\lambda_D(x) - (\lambda_D|_{\rho^{-1}(0)})(q(x))| < \alpha_1'$$

for all $x \in \bar{r}^{-1}(\mathbb{B})$. Note that $q(g \cdot s(z)) = g$ for all $g \in T$ and $z \in \mathbb{C}^n$. Hence for all $z \in \mathbb{B}$ and $g \in T$, we have

$$|p^*\lambda_D(g \cdot s(z)) - (\lambda_D|_{\rho^{-1}(0)})(g)| \leq \alpha_1'.$$

Hence by

$$\begin{aligned}
&|\lambda_D(g \cdot \psi(z)) - \lambda_D(g \cdot \psi(w))| \\
&\qquad \leq |p^*\lambda_D(g \cdot s(z)) - (\lambda_D|_{\rho^{-1}(0)})(g)| + |p^*\lambda_D(g \cdot s(w)) - (\lambda_D|_{\rho^{-1}(0)})(g)|,
\end{aligned}$$

we get (5.47) with $\alpha_1 = 2\alpha_1'$.

Next we prove that there exists a positive constant $\alpha_2 > 0$ such that for all $g \in T$ and $z, w \in \mathbb{B}$, we have

$$d(g \cdot \psi(z), g \cdot \psi(w)) \leq \alpha_2. \tag{5.48}$$

We prove this. Since $\psi\colon \mathbb{C}^n \to A$ is a group homomorphism, $\psi^*\omega_A$ is an invariant $(1,1)$-form on $\mathbb{C}^n$. Hence we have $d(\psi(z), \psi(w)) \leq \alpha_2' d_{\mathbb{C}^n}(z, w)$ for all $z, w \in \mathbb{C}^n$. See the argument for (5.46). Hence

$$d(g \cdot \psi(z), g \cdot \psi(w)) = d(\psi(z), \psi(w)) \leq \alpha_2' d_{\mathbb{C}^n}(z, w)$$

for all $g \in T$ and $z, w \in \mathbb{C}^n$. Since $\mathbb{B} \subset \mathbb{C}^n$ is compact, there exists a positive constant $\gamma > 0$ such that $d_{\mathbb{C}^n}(z,w) < \gamma$ for all $z, w \in \mathbb{B}$. Hence we get (5.48) with $\alpha_2 = \alpha_2' \gamma$.

Now, for $x, y \in A$, we take $x', y' \in r^{-1}(\mathbb{B})$ such that $p(x') = x$ and $p(y') = y$. We take $g_x, g_y \in T$ such that $g_x \cdot s(r(x')) = x'$ and $g_y \cdot s(r(y')) = y'$. Then by the torus case above, we have

$$|\lambda_D(g_x \cdot \psi(0)) - \lambda_D(g_y \cdot \psi(0))| \leq c_1 d_T(g_x \cdot \psi(0), g_y \cdot \psi(0)) + c_1.$$

By Lemma 5.12, we have $d_T(g_x \cdot \psi(0), g_y \cdot \psi(0)) \leq \beta d(g_x \cdot \psi(0), g_y \cdot \psi(0))$. Hence

$$|\lambda_D(g_x \cdot \psi(0)) - \lambda_D(g_y \cdot \psi(0))| \leq c_1 \beta d(g_x \cdot \psi(0), g_y \cdot \psi(0)) + c_1. \tag{5.49}$$

We have

$$\begin{aligned} |\lambda_D(x) - \lambda_D(y)| \leq & \big|\lambda_D\big(g_x \cdot \psi(r(x'))\big) - \lambda_D(g_x \cdot \psi(0))\big| \\ & + |\lambda_D(g_x \cdot \psi(0)) - \lambda_D(g_y \cdot \psi(0))| \\ & + \big|\lambda_D(g_y \cdot \psi(0)) - \lambda_D\big(g_y \cdot \psi(r(y'))\big)\big|. \end{aligned}$$

Hence by (5.47) and (5.49), we get

$$|\lambda_D(x) - \lambda_D(y)| \leq c_1 \beta d(g_x \cdot \psi(0), g_y \cdot \psi(0)) + c_1 + 2\alpha_1.$$

By (5.48), we have

$$\begin{aligned} d(g_x \cdot \psi(0), g_y \cdot \psi(0)) &\leq d\big(g_x \cdot \psi(0), g_x \cdot \psi(r(x'))\big) + d\big(g_x \cdot \psi(r(x')), g_y \cdot \psi(r(y'))\big) \\ &\quad + d\big(g_y \cdot \psi(r(y')), g_y \cdot \psi(0)\big) \\ &\leq d(x,y) + 2\alpha_2. \end{aligned}$$

Hence we get

$$|\lambda_D(x) - \lambda_D(y)| \leq c d(x,y) + c,$$

where $c = \max\{c_1\beta, 2c_1\alpha_2\beta + c_1 + 2\alpha_1\}$. This concludes the proof. □

§5.6. Application of Lemma 5.13

We recall the notation from (5.6).

Lemma 5.14. *Let $\bar{A}$ be a smooth equivariant compactification of a semi-abelian variety A, and let Σ be a smooth projective variety. Let $D \subset \partial A$ be an irreducible component with a Weil function $\lambda_D \geq 0$. Let ω_A be an invariant positive $(1,1)$-form on A. Then there exists a positive constant $c > 0$ with the following property: Let (g, Ω, Ω') be a triple as in Definition 5.3, where $g \in \mathrm{Hol}_m(\mathbb{D}, \bar{A})$ with $g(Y_g) \not\subset \partial A$. We take $r, r' \in (0,1)$ such that $\partial\mathbb{D}(r) \subset \Omega$ and $\partial\mathbb{D}(r') \subset \Omega$. We take $\theta \in \mathbb{R}$ such* *that the line segment γ connecting $re^{i\theta}$ and $r'e^{i\theta}$ satisfies $\gamma \subset \Omega$. Assume that $\partial\mathbb{D}(r') \cup \partial\mathbb{D}(r) \cup \gamma \subset \Omega$ does not contain the critical values of $\pi_g \colon Y_g \to \mathbb{D}$. Then we have*

$$\begin{aligned} &|m(r,(g,\Omega,\Omega'),\lambda_D) - m(r',(g,\Omega,\Omega'),\lambda_D)| \\ &\quad \le \frac{c}{\deg(\pi_g|_{\Omega'})}\ell_{\omega_A}\big(g\big(\Omega' \cap \pi_g^{-1}(\partial\mathbb{D}(r) \cup \partial\mathbb{D}(r') \cup \gamma)\big)\big) + c, \end{aligned} \tag{5.50}$$

where ℓ_{ω_A} is the length with respect to ω_A.

Proof. By Lemma 5.13, there exists a positive constant $\alpha > 0$ such that

$$|\lambda_D(x) - \lambda_D(y)| \le \alpha d(x,y) + \alpha \tag{5.51}$$

for all $x, y \in A$, where $d(x,y)$ is the distance with respect to ω_A.

Let (g,Ω,Ω') be a triple as in Definition 5.3, where $g \in \mathrm{Hol}_m(\mathbb{D},\bar{A})$ with $g(Y_g) \not\subset \partial A$. For each $z \in \Omega$, we set

$$\mu_g(z) = \frac{1}{\deg(\pi_g|_{\Omega'})}\sum_{y \in \Omega' \cap \pi_g^{-1}(z)} \lambda_D(g(y)).$$

We have

$$m(r,(g,\Omega,\Omega'),\lambda_D) = \int_{z \in \Omega \cap \partial\mathbb{D}(r)} \mu_g(z)\frac{d\arg \pi_g(z)}{2\pi}. \tag{5.52}$$

Let $z, z' \in \Omega$ be two points connected by a smooth arc σ in Ω. Suppose that $\sigma \subset \Omega$ does not contain the critical values of $\pi_g \colon Y_g \to \mathbb{D}$. Then we claim

$$|\mu_g(z) - \mu_g(z')| \le \frac{\alpha}{\deg(\pi_g|_{\Omega'})}\ell_{\omega_A}\big(g(\Omega' \cap \pi_g^{-1}(\sigma))\big) + \alpha. \tag{5.53}$$

To show this, we set $\Omega' \cap \pi_g^{-1}(z) = \{y_1,\dots,y_k\}$ and $\Omega' \cap \pi_g^{-1}(z) = \{y'_1,\dots,y'_k\}$, where $k = \deg(\pi_g|_{\Omega'})$. We may assume that y_i and y'_i are connected by a smooth arc σ_i in Ω', where σ_i is a lift of σ. Then by (5.51), we get

$$|\mu_g(z) - \mu_g(z')| \le \frac{1}{k}\sum_{i=1}^{k}|\lambda_D(g(y_i)) - \lambda_D(g(y'_i))| \le \frac{\alpha}{k}\sum_{i=1}^{k} d(g(y_i), g(y'_i)) + \alpha.$$

By $d(g(y_i), g(y'_i)) \le \ell_{\omega_A}(g(\sigma_i))$, we get (5.53).

Now we take $r, r' \in (0,1)$, $\theta \in \mathbb{R}$, and γ as in Lemma 5.14. We have $\partial\mathbb{D}(r) \subset \Omega$, $\partial\mathbb{D}(r') \subset \Omega$, and $\gamma \subset \Omega$. Moreover, $\partial\mathbb{D}(r') \cup \partial\mathbb{D}(r) \cup \gamma \subset \Omega$ does not contain the critical values of $\pi_g \colon Y_g \to \mathbb{D}$. By (5.53), we have

$$|\mu_g(re^{i\theta'}) - \mu_g(re^{i\theta})| \le \frac{\alpha}{\deg(\pi_g|_{\Omega'})}\ell_{\omega_A}\big(g\big(\Omega' \cap \pi_g^{-1}(\partial\mathbb{D}(r))\big)\big) + \alpha$$

for all $\theta' \in \mathbb{R}$. Hence by (5.52), we have

$$|m(r,(g,\Omega,\Omega'),\lambda_D) - \mu_g(re^{i\theta})| \le \frac{\alpha}{\deg(\pi_g|_{\Omega'})}\ell_{\omega_A}\big(g\big(\Omega' \cap \pi_g^{-1}(\partial\mathbb{D}(r))\big)\big) + \alpha.$$

Similarly we have

$$|m(r',(g,\Omega,\Omega'),\lambda_D) - \mu_g(r'e^{i\theta})| \le \frac{\alpha}{\deg(\pi_g|_{\Omega'})}\ell_{\omega_A}\big(g\big(\Omega' \cap \pi_g^{-1}(\partial\mathbb{D}(r'))\big)\big) + \alpha.$$

By (5.53), we have

$$|\mu_g(re^{i\theta}) - \mu_g(r'e^{i\theta})| \le \frac{\alpha}{\deg(\pi_g|_{\Omega'})}\ell_{\omega_A}\big(g(\Omega' \cap \pi_g^{-1}(\gamma))\big) + \alpha.$$

The last three estimates yield (5.50), where $c = 3\alpha$. □

§5.7. Application of the area-length method

For $r \in (\frac{1}{2}, 1)$ and $\theta \in [0, 2\pi]$, we denote by $\gamma_{r,\theta} \subset \mathbb{D}$ the line segment connecting $\frac{1}{2}e^{i\theta}$ and $re^{i\theta}$. Let η be a smooth semi-positive $(1,1)$-form on a smooth projective variety Σ. We denote by ℓ_η the length of curves in Σ with respect to η. The next lemma is an application of the area-length method.

Lemma 5.15. *Let $f \in \mathrm{Hol}_m(\mathbb{D}, \Sigma)$ and $s \in (\frac{1}{2}, 1)$. Let $\delta > 0$.*

(1) *There exists a subset $E_1 \subset (s,1)$ with $|E_1| \le \delta$ such that, for all $r \in (s,1)\backslash E_1$, we have*

$$\ell_\eta\big(f\big(\pi_f^{-1}(\partial\mathbb{D}(r))\big)\big) \le c_1(\deg \pi_f)^2 T_s(r,f,\eta) + c_1(\deg \pi_f)^2,$$

where $c_1 > 0$ is a positive constant which only depends on δ.

(2) *There exists a subset $E_2 \subset (s,1)$ with $|E_2| \le \delta$ such that, for all $r \in (s,1)\backslash E_2$, we have*

$$\ell_\eta\big(f(\pi_f^{-1}(\gamma_{r,\theta}))\big) \le c_2(\deg \pi_f) T_s(r,f,\eta) + c_2(\deg \pi_f)$$

for all $\theta \in (0, 2\pi)$ outside some exceptional set $E_3 \subset (0, 2\pi)$ with linear measure $|E_3| < \delta$. Here, $c_2 > 0$ is a positive constant which only depends on δ.

Proof. Let $f^*\eta = \varphi^2 \pi_f^*(dx \wedge dy)$, where $z = x + iy$. Set

$$A(r) = \int_{Y_f(r)} f^*\eta$$

and

$$T(r) = T_s(r,f,\eta).$$

Then we have

$$A(r) = r(\deg \pi_f)T'(r) \leq (\deg \pi_f)T'(r).$$

By

$$A(r) = \int_0^r dt \int_{\pi_f^{-1}(\partial\mathbb{D}(t))} \varphi^2 t \, d\arg \pi_f,$$

we have

$$A'(r) = \int_{\pi_f^{-1}(\partial\mathbb{D}(r))} \varphi^2 r \, d\arg \pi_f.$$

Set $L(r) = \ell_\eta(f(\pi_f^{-1}(\partial\mathbb{D}(r))))$. Then

$$L(r) = \int_{\pi_f^{-1}(\partial\mathbb{D}(r))} \varphi r \, d\arg \pi_f.$$

Hence by the Cauchy–Schwarz inequality, we get

$$L(r)^2 \leq 2\pi r(\deg \pi_f) \int_{\pi_f^{-1}(\partial\mathbb{D}(r))} \varphi^2 r \, d\arg \pi_f \leq 2\pi(\deg \pi_f)A'(r).$$

We estimate the right-hand side. Note that since η is semi-positive, we have $A'(r) \geq 0$ for all $r \in (0,1)$. Hence we may apply the Borel growth lemma (Lemma 5.16 below) for $A(r)$. Letting $\varepsilon = \sqrt{2} - 1$ and $\delta' = \delta/2$, we have

$$A'(r) \leq \frac{2}{\varepsilon\delta'} \max\{1, A(r)^{1+\varepsilon}\} \leq \frac{2}{\varepsilon\delta'}(\deg \pi_f)^{1+\varepsilon} \max\{1, (T'(r))^{1+\varepsilon}\}$$

for all $r \in (s,1)$ outside some exceptional set $E_1' \subset (s,1)$ of linear measure less than δ'. Again by Lemma 5.16, we have

$$(T'(r))^{1+\varepsilon} \leq \Big(\frac{2}{\varepsilon\delta'}\Big)^{1+\varepsilon} \max\{1, T(r)^{(1+\varepsilon)^2}\}$$

for all $r \in (s,1)$ outside some exceptional set $E_1'' \subset (s,1)$ of linear measure less than δ'. Hence

$$A'(r) \leq \Big(\frac{2}{\varepsilon\delta'}\Big)^{2+\varepsilon} (\deg \pi_f)^{1+\varepsilon} \max\{1, T(r)^{(1+\varepsilon)^2}\}$$

for all $r \in (s,1)$ outside $E_1 = E_1' \cup E_1''$, where $|E_1| < \delta$. Then we get

$$L(r)^2 \leq 2\pi\Big(\frac{2}{\varepsilon\delta'}\Big)^{2+\varepsilon} (\deg \pi_f)^{2+\varepsilon} \max\{1, T(r)^2\},$$

thus

$$\begin{aligned} L(r) &\leq \sqrt{2\pi}\Big(\frac{2}{\varepsilon\delta'}\Big)^{1+\varepsilon/2} (\deg \pi_f)^{1+\varepsilon/2} \max\{1, T(r)\} \\ &\leq c_1(\deg \pi_f)^2 T(r) + c_1(\deg \pi_f)^2 \end{aligned}$$

for all $r \in (s,1)$ outside E_1. Here we set $c_1 = \sqrt{2\pi}(\frac{2}{\varepsilon\delta'})^{1+\varepsilon/2}$, which only depends on δ. This is the first estimate.

Next we prove the second estimate. Set $L_\gamma(r,\theta) = \ell_\eta(f(\pi_f^{-1}(\gamma_{r,\theta})))$. Then

$$L_\gamma(r,\theta) = \int_{\pi_f^{-1}(\gamma_{r,\theta})} \varphi\, d|\pi_f|.$$

Since $|\pi_f| > \frac{1}{2}$ on $\pi_f^{-1}(\gamma_{r,\theta})$, we have

$$L_\gamma(r,\theta) \leq 2\int_{\pi_f^{-1}(\gamma_{r,\theta})} \varphi|\pi_f|\, d|\pi_f|.$$

By the Cauchy–Schwarz inequality, we get

$$L_\gamma(r,\theta)^2 \leq 4(\deg \pi_f)\int_{\pi_f^{-1}(\gamma_{r,\theta})} \varphi^2|\pi_f|\, d|\pi_f|,$$

hence

$$\int_0^{2\pi} L_\gamma(r,\theta)^2\, d\theta \leq 4(\deg \pi_f)A(r) \leq 4(\deg \pi_f)^2 T'(r).$$

By Lemma 5.16, letting $\varepsilon = 1$, we have

$$\int_0^{2\pi} L_\gamma(r,\theta)^2 d\theta \leq \frac{8(\deg \pi_f)^2}{\delta} \max\{1, T(r)^2\}$$

for all $r \in (s,1)\backslash E_2$ with $|E_2| < \delta$. Then, for each $r \in (s,1)\backslash E_2$, we get

$$L_\gamma(r,\theta)^2 \leq \frac{8(\deg \pi_f)^2}{\delta^2} \max\{1, T(r)^2\}$$

for all $\theta \in (0,2\pi)$ outside some exceptional set E_3 with $|E_3| < \delta$. Hence

$$L_\gamma(r,\theta) \leq \frac{2\sqrt{2}\deg \pi_f}{\delta} \max\{1, T(r)\}$$

for all $\theta \in (0,2\pi)\backslash E_3$. Thus we obtain the second estimate by letting $c_2 = 2\sqrt{2}/\delta$, which only depends on δ. □

Lemma 5.16. *Let g be a continuously differentiable, increasing function on $[s,1)$ with $g(s) \geq 0$. Let $\delta > 0$ and $0 < \varepsilon \leq 1$. Then we have*

$$g'(r) \leq \frac{2}{\varepsilon\delta} \max\{1, g(r)^{1+\varepsilon}\}$$

for all $r \in (s,1)$ outside a set E with $|E| < \delta$.

Proof. Set

$$E = \left\{r \in (s,1);\ g'(r) > \tfrac{2}{\varepsilon\delta} \max\{1, g(r)^{1+\varepsilon}\}\right\}.$$

If $E=\emptyset$, then our assertion is trivial. Suppose $E\neq\emptyset$. We have

$$|E|<\frac{\varepsilon\delta}{2}\int_E \frac{g'(r)}{\max\{1,g(r)^{1+\varepsilon}\}}\,dr\leq \frac{\varepsilon\delta}{2}\int_s^1 \frac{g'(r)}{\max\{1,g(r)^{1+\varepsilon}\}}\,dr.$$

We have the following three cases.

Case 1*: $g(r)\geq 1$ for all $r\in[s,1)$.* Then we have

$$\int_s^1 \frac{g'(r)}{\max\{1,g(r)^{1+\varepsilon}\}}\,dr=\int_s^1 \frac{g'(r)}{g(r)^{1+\varepsilon}}\,dr=\lim_{t\to 1-0}\Big[\frac{-1}{\varepsilon g(r)^{\varepsilon}}\Big]_s^t\leq\frac{1}{\varepsilon}.$$

Case 2*: $g(r)\leq 1$ for all $r\in[s,1)$.* Then we have

$$\int_s^1 \frac{g'(r)}{\max\{1,g(r)^{1+\varepsilon}\}}\,dr=\int_s^1 g'(r)\,dr\leq 1.$$

Case 3*: Otherwise, we have $g(s)<1$ and $\lim_{r\to 1-0}g(r)>1$.* We set $\kappa=\sup\{r\in[s,1);\ g(r)\leq 1\}$. Then we have $s<\kappa<1$ and $g(\kappa)=1$. Hence we have

$$\begin{aligned}\int_s^1 \frac{g'(r)}{\max\{1,g(r)^{1+\varepsilon}\}}\,dr&=\int_s^\kappa g'(r)\,dr+\int_\kappa^1 \frac{g'(r)}{g(r)^{1+\varepsilon}}\,dr\\ &\leq 1+\lim_{t\to 1-0}\Big[\frac{-1}{\varepsilon g(r)^{\varepsilon}}\Big]_\kappa^t\leq\frac{2}{\varepsilon}.\end{aligned}$$

Thus in all cases, we have proved $|E|<\delta$. □

§5.8. Estimate for the second term of the right-hand side of (5.3)

Let A be a semi-abelian variety with a smooth projective equivariant compactification $\bar{A}$. Let S be a smooth projective variety. Let $Z\subset S_{1,A}$ be an irreducible Zariski closed set. Assume that Z is horizontally integrable. We recall $W\subset S$ from Remark 5.10.

Lemma 5.17. *Let $D\subset\bar{A}$ be an irreducible component of ∂A with a Weil function $\lambda_D\geq 0$. Let $\mathcal{F}\subset\mathrm{Hol}(\mathbb{D},A\times S)$ be an infinite set of non-constant holomorphic maps such that $(f_{S_{1,A}})_{f\in\mathcal{F}}\Rightarrow Z$. Let ω_S be a smooth positive $(1,1)$-form on S and let $\lambda_W\geq 0$ be a Weil function for W. Let $s\in(1/2,1)$ and $\delta>0$. Then there exists a positive constant $\alpha>0$ such that for all $f\in\mathcal{F}$ with $f_S(\mathbb{D})\not\subset W$, we have*

$$\begin{aligned}&|m(r,f_A,\lambda_D)-m((s+r)/2,f_A,\lambda_D)|\\ &\qquad\leq\alpha T_s(r,f_S,\omega_S)+\alpha m((s+r)/2,f_S,\lambda_W)+\alpha\end{aligned}$$

for all $r\in(s,1)\backslash E$, where $E\subset(s,1)$ is an exceptional set with $|E|<\delta$.

Proof. We recall the objects fixed in Remark 5.10. Set $\partial U = \Sigma - U$ with a Weil function $\lambda_{\partial U} \geq 0$. Then by Lemma 5.8 (1), we get

$$\bigcup_{\Delta} \bar{q}(\Delta) \subset W, \tag{5.54}$$

where Δ runs over all irreducible components $\Delta \subset \partial U$ such that $\Delta \cap \overline{\Theta} \neq \emptyset$. Here, $\bar{q}\colon \Sigma \to S$ is the extension of $q\colon U \to S$. Let $K \subset \Sigma$ be a compact neighborhood of $\overline{\Theta} \subset \Sigma$. We assume that $K \subset \Sigma^o$ and

$$\Delta' \cap K = \emptyset \tag{5.55}$$

for all irreducible components $\Delta' \subset \partial U$ such that $\Delta' \cap \overline{\Theta} = \emptyset$. By (5.54) and (5.55), there exists a positive constant $\alpha_1 > 0$ such that

$$\lambda_{\partial U}(x) \leq \alpha_1 \lambda_W(q(x)) + \alpha_1 \tag{5.56}$$

for all $x \in K$.

Since $\bar{A}$ is an equivariant compactification, the isomorphism $\Phi\colon A \times U \to A \times U$ extends to a morphism

$$\overline{\Phi}\colon \bar{A} \times U \to \bar{A} \times U$$

by the definition (3.4). Then the inverse $\overline{\Phi}^{-1}\colon \bar{A} \times U \to \bar{A} \times U$ induces a rational map

$$\overline{\Phi}^{-1}\colon \bar{A} \times \Sigma \dashrightarrow \bar{A} \times \Sigma,$$

which is holomorphic over $\bar{A} \times U \subset \bar{A} \times \Sigma$. Let $\nu_1\colon \bar{A} \times \Sigma \to \bar{A}$ be the first projection and let $\nu_2\colon \bar{A} \times \Sigma \to \Sigma$ be the second projection. We claim that there exists a positive constant $\alpha_2 > 0$ such that

$$|\lambda_D(\nu_1(x)) - \lambda_D(\nu_1 \circ \Phi^{-1}(x))| \leq \alpha_2 \lambda_{\partial U}(\nu_2(x)) + \alpha_2 \tag{5.57}$$

for all $x \in A \times U \subset \bar{A} \times \Sigma$. Indeed, since $D \subset \bar{A}$ is A-invariant, we have $\overline{\Phi}(D \times U) = D \times U$. Hence we have

$$(\nu_1 \circ \overline{\Phi}^{-1})^{-1}(D)|_{\bar{A} \times U} = \nu_1^{-1}(D)|_{\bar{A} \times U}$$

over $\bar{A} \times U$. Thus by [44, Prop. 2.2.9 (7)], we get (5.57).

Now we are given an infinite subset $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A \times S)$ of non-constant holomorphic maps so that $(f_{S_{1,A}})_{f \in \mathcal{F}} \Rightarrow Z$. We apply Lemma 5.5 to get a finite subset $\mathcal{E} \subset \mathcal{F}$ such that for each $f \in \mathcal{F} \backslash \mathcal{E}$, we get a lifting $\hat{f} \in \mathrm{Hol}_m(\mathbb{D}, A \times \Sigma)$ of $f \in \mathrm{Hol}(\mathbb{D}, A \times S)$, a connected open set $\Omega_f \subset \mathbb{D}$, and a connected component $\Omega'_f \subset Y_{\hat{f}}$ of $\pi_{\hat{f}}^{-1}(\Omega_f)$ with the properties (5.9)–(5.12) described in Lemma 5.5.

Let $s \in (1/2, 1)$, $\delta > 0$ be given. Set

$$\delta' = \delta/11.$$

We have (cf. (5.10))

$$((\hat{f}_\Sigma, \Omega_f, \Omega'_f))_{f\in\mathcal{F}\backslash\mathcal{E}} \rightsquigarrow \overline{\Theta}.$$

Hence, there exists a finite subset $\mathcal{E}_1 \subset \mathcal{F}$ with $\mathcal{E} \subset \mathcal{E}_1$ such that for all $f \in \mathcal{F}\backslash\mathcal{E}_1$, we have $z \in \Omega_f$ for δ'-almost all $z \in \mathbb{D}(1-\delta')$ and

$$\hat{f}_\Sigma(\Omega'_f) \subset K. \tag{5.58}$$

Let ω_Σ be a smooth positive $(1,1)$-form on Σ such that

$$\omega_\Sigma \leq \eta \tag{5.59}$$

on K. Here we recall η from Remark 5.10. We take an open neighborhood $O \subset \Sigma_{1,A}$ of $\Sigma^*_{1,A}$ such that

$$|v_{\mathrm{Lie}\,A}|_{\omega_A} \leq |v_\Sigma|_{\omega_\Sigma} \tag{5.60}$$

for all $(v_\Sigma, v_{\mathrm{Lie}\,A}) \in T\Sigma \times \mathrm{Lie}\,A$ with $[(v_\Sigma, v_{\mathrm{Lie}\,A})] \in O \subset \Sigma_{1,A}$. We have (cf. (5.11))

$$\big(((\Phi\circ\hat{f})_{\Sigma_{1,A}}, \Omega_f, \Omega'_f)\big)_{f\in\mathcal{F}\backslash\mathcal{E}} \rightsquigarrow \Sigma^*_{1,A}.$$

Hence, we may take a finite subset $\mathcal{E}_2 \subset \mathcal{F}$ with $\mathcal{E}_1 \subset \mathcal{E}_2$ such that for all $f \in \mathcal{F}\backslash\mathcal{E}_2$, we have

$$(\Phi\circ\hat{f})_{\Sigma_{1,A}}(\Omega'_f) \subset O. \tag{5.61}$$

Let $f \in \mathcal{G}\backslash\mathcal{E}_2$ with $f_S(\mathbb{D}) \not\subset W$. Then by (5.56) and (5.58), we have

$$m(r, (\hat{f}_\Sigma, \Omega_f, \Omega'_f), \lambda_{\partial U}) \leq \alpha_1 m(r, f_S, \lambda_W) + \alpha_1 \tag{5.62}$$

for all $r \in (0,1)$. We set

$$E_1 = \big\{r \in (s, 1-\delta');\ (\mathbb{D}\backslash\Omega_f)\cap\partial\mathbb{D}(r) \neq \emptyset\big\} \cup \big\{r \in (s, 1-\delta');\ (\mathbb{D}\backslash\Omega_f)\cap\partial\mathbb{D}(\sigma) \neq \emptyset\big\},$$

where $\sigma = (s+r)/2$. By adding finite points to E_1, we may assume that if $r \in (s, 1-\delta')\backslash E_1$, then $\partial\mathbb{D}(r) \cup \partial\mathbb{D}(\sigma)$ is contained in Ω_f and does not contain the critical values of $\pi_{\hat{f}}\colon Y_{\hat{f}} \to \mathbb{D}$. Then we have

$$|E_1| \leq 6\delta'.$$

For all $r \in (s, 1-\delta')\backslash E_1$, we have

$$m(r, f_A, \lambda_D) = m(r, (\hat{f}_A, \Omega_f, \Omega'_f), \lambda_D)$$

and

$$m(\sigma, f_A, \lambda_D) = m(\sigma, (\hat{f}_A, \Omega_f, \Omega'_f), \lambda_D).$$

By (5.57) and (5.62), we get

$$\big|m\big(r,((\Phi\circ\hat{f})_A,\Omega_f,\Omega'_f),\lambda_D\big)-m(r,(\hat{f}_A,\Omega_f,\Omega'_f),\lambda_D)\big|\le\alpha_3 m(r,f_S,\lambda_W)+\alpha_3,$$

where $\alpha_3=\alpha_2\alpha_1+\alpha_2$. Similarly, we have

$$\big|m\big(\sigma,((\Phi\circ\hat{f})_A,\Omega_f,\Omega'_f),\lambda_D\big)-m(\sigma,(\hat{f}_A,\Omega_f,\Omega'_f),\lambda_D)\big|\le\alpha_3 m(\sigma,f_S,\lambda_W)+\alpha_3.$$

Thus for $r\in(s,1-\delta')\backslash E_1$, we have

$$\begin{aligned}&|m(r,f_A,\lambda_D)-m(\sigma,f_A,\lambda_D)|\\&\quad=|m(r,(\hat{f}_A,\Omega_f,\Omega'_f),\lambda_D)-m(\sigma,(\hat{f}_A,\Omega_f,\Omega'_f),\lambda_D)|\\&\quad\le\big|m\big(r,((\Phi\circ\hat{f})_A,\Omega_f,\Omega'_f),\lambda_D\big)-m\big(\sigma,((\Phi\circ\hat{f})_A,\Omega_f,\Omega'_f),\lambda_D\big)\big|\\&\qquad+\alpha_3 m(r,f_S,\lambda_W)+\alpha_3 m(\sigma,f_S,\lambda_W)+2\alpha_3.\end{aligned}$$

Hence

$$\begin{aligned}&|m(r,f_A,\lambda_D)-m(\sigma,f_A,\lambda_D)|\\&\quad\le\big|m\big(r,((\Phi\circ\hat{f})_A,\Omega_f,\Omega'_f),\lambda_D\big)-m\big(\sigma,((\Phi\circ\hat{f})_A,\Omega_f,\Omega'_f),\lambda_D\big)\big|\\&\qquad+\alpha_3 m(r,f_S,\lambda_W)+\alpha_3 m(\sigma,f_S,\lambda_W)+2\alpha_3.\end{aligned}\tag{5.63}$$

Next we claim

$$\begin{aligned}&\big|m\big(r,((\Phi\circ\hat{f})_A,\Omega_f,\Omega'_f),\lambda_D\big)-m\big(\sigma,((\Phi\circ\hat{f})_A,\Omega_f,\Omega'_f),\lambda_D\big)\big|\\&\quad\le\alpha_4 T_s(r,\hat{f}_\Sigma,\eta)+\alpha_4\end{aligned}\tag{5.64}$$

for all $r\in(s,1)\backslash E_2$, where $E_2\subset(0,1)$ is an exceptional set with $|E_2|<11\delta'$. Here, $\alpha_4>0$ is a positive constant which does not depend on the choice of $f\in\mathcal{F}\backslash\mathcal{E}_2$.

We prove this. Set $\gamma_{r,\theta}\colon te^{i\theta}$, $1/2\le t\le r$. If $\partial\mathbb{D}(\sigma)+\gamma_{r,\theta}+\partial\mathbb{D}(r)\subset\Omega_f$, then by (5.60) and (5.61), we have

$$\begin{aligned}&\ell_{\omega_A}\big((\Phi\circ\hat{f})_{\bar{A}}\big(\Omega'_f\cap\pi_{\hat{f}}^{-1}(\partial\mathbb{D}(r)\cup\partial\mathbb{D}(\sigma)\cup\gamma_{r,\theta})\big)\big)\\&\quad\le\ell_{\omega_\Sigma}\big(\hat{f}_\Sigma\big(\Omega'_f\cap\pi_{\hat{f}}^{-1}(\partial\mathbb{D}(r)\cup\partial\mathbb{D}(\sigma)\cup\gamma_{r,\theta})\big)\big).\end{aligned}$$

By (5.58) and (5.59), we have

$$\begin{aligned}&\ell_{\omega_\Sigma}\big(\hat{f}_\Sigma\big(\Omega'_f\cap\pi_{\hat{f}}^{-1}(\partial\mathbb{D}(r)\cup\partial\mathbb{D}(\sigma)\cup\gamma_{r,\theta})\big)\big)\\&\quad\le\ell_\eta\big(\hat{f}_\Sigma\big(\Omega'_f\cap\pi_{\hat{f}}^{-1}(\partial\mathbb{D}(r)\cup\partial\mathbb{D}(\sigma)\cup\gamma_{r,\theta})\big)\big).\end{aligned}$$

Thus by Lemma 5.14, we get

$$\begin{aligned}&\big|m\big(r,((\Phi\circ\hat{f})_{\bar{A}},\Omega_f,\Omega'_f),\lambda_D\big)-m\big(\sigma,((\Phi\circ\hat{f})_{\bar{A}},\Omega_f,\Omega'_f),\lambda_D\big)\big|\\&\quad\le\frac{c}{\deg(\pi_{\hat{f}}|_{\Omega'_{\hat{f}}})}\ell_\eta\big(\hat{f}_\Sigma\big(\pi_{\hat{f}}^{-1}(\partial\mathbb{D}(r)\cup\partial\mathbb{D}(\sigma)\cup\gamma_{r,\theta})\big)\big)+c,\end{aligned}\tag{5.65}$$

provided that $\partial\mathbb{D}(r)\cup\partial\mathbb{D}(\sigma)\cup\gamma_{r,\theta}\subset\Omega_f$ does not contain the critical values of $\pi_{\hat f}\colon Y_{\hat f}\to\mathbb{D}$. Here, $c>0$ is a positive constant which appears in Lemma 5.14, hence is independent of the choice of $f\in\mathcal{F}\backslash\mathcal{E}_2$, r, and θ.

Now we apply Lemma 5.15 to get

$$\ell_\eta\big(\hat f_\Sigma\big(\pi_{\hat f}^{-1}(\partial\mathbb{D}(r))\big)\big)\le\alpha_5(\deg\pi_{\hat f})^2T_s(r,\hat f_\Sigma,\eta)+\alpha_5(\deg\pi_{\hat f})^2,$$

for all $r\in(s,1)$ outside some exceptional set $E_3\subset(s,1)$ with

$$|E_3|<\delta'.$$

Here, $\alpha_5>0$ only depends on δ'. We define $E_4\subset(s,1)$ by $r\in E_4$ if and only if $(s+r)/2\in E_3$. We have

$$|E_4|\le 2|E_3|\le 2\delta'.$$

Then for $r\in(s,1-\delta')\backslash(E_1\cup E_4)$, we have

$$\ell_\eta\big(\hat f_\Sigma\big(\pi_{\hat f}^{-1}(\partial\mathbb{D}(\sigma))\big)\big)\le\alpha_5(\deg\pi_{\hat f})^2T_s(\sigma,\hat f_\Sigma,\eta)+\alpha_5(\deg\pi_{\hat f})^2,$$

and $\partial\mathbb{D}(\sigma)\subset\Omega_f$. Thus for $r\in(s,1-\delta')\backslash(E_1\cup E_3\cup E_4)$, the estimate (5.65) yields

$$\begin{aligned}&\big|m\big(r,((\Phi\circ\hat f)_{\bar A},\Omega_f,\Omega_f'),\lambda_D\big)-m\big(\sigma,((\Phi\circ\hat f)_{\bar A},\Omega_f,\Omega_f'),\lambda_D\big)\big|\\&\qquad\le 2c\alpha_5(\deg\pi_{\hat f})T_s(r,\hat f_\Sigma,\eta)+2c\alpha_5(\deg\pi_{\hat f})+c\\&\qquad\quad+\frac{c}{\deg(\pi_{\hat f}|_{\Omega_{\hat f}'})}\ell_\eta\big(\hat f_\Sigma(\pi_{\hat f}^{-1}(\gamma_{r,\theta}))\big)\end{aligned}$$

provided $\gamma_{r,\theta}$ is contained in Ω_f and does not contain the critical values of $\pi_{\hat f}\colon Y_{\hat f}\to\mathbb{D}$. By Lemma 5.15 (2), there exists $E_5\subset(s,1)$ with

$$|E_5|<\delta'$$

such that for each $r\in(s,1-\delta')\backslash E_5$, we may choose $\theta\in(0,2\pi)$ such that $\gamma_{r,\theta}\subset\Omega_f$, $\gamma_{r,\theta}$ does not contain the critical values of $\pi_{\hat f}\colon Y_{\hat f}\to\mathbb{D}$, and

$$\ell_\eta\big(\hat f_\Sigma(\pi_{\hat f}^{-1}(\gamma_{r,\theta}))\big)\le\alpha_6(\deg\pi_{\hat f})T_s(r,\hat f_\Sigma,\eta)+\alpha_6(\deg\pi_{\hat f}).$$

Here, $\alpha_6>0$ only depends on δ'. Hence, we get

$$\begin{aligned}&\big|m\big(r,((\Phi\circ\hat f)_{\bar A},\Omega_f,\Omega_f'),\lambda_D\big)-m\big(\sigma,((\Phi\circ\hat f)_{\bar A},\Omega_f,\Omega_f'),\lambda_D\big)\big|\\&\qquad\le c(2\alpha_5+\alpha_6)(\deg\pi_{\hat f})T_s(r,\hat f_\Sigma,\eta)+c(2\alpha_5+\alpha_6)(\deg\pi_{\hat f})+c\end{aligned}$$

for all $r\in(s,1)\backslash E_2$, where

$$E_2=E_1\cup E_3\cup E_4\cup E_5\cup(1-\delta',1). \tag{5.66}$$

We have

$$|E_2| < 11\delta' = \delta.$$

This concludes the proof for (5.64), where we set $\alpha_4 = c(2\alpha_5+\alpha_6)[\mathbb{C}(S):\mathbb{C}(\Sigma)]+c$.

Now, by (5.63) and (5.64), we get

$$\begin{aligned}&|m(r,f_A,\lambda_D)-m(\sigma,f_A,\lambda_D)|\\ &\qquad\le \alpha_7 T_s(r,\hat{f}_\Sigma,\eta)+\alpha_7 m(r,f_S,\lambda_W)+\alpha_7 m(\sigma,f_S,\lambda_W)+\alpha_7\end{aligned}$$

for all $r\in(s,1)\backslash E_2$, where $\alpha_7 = 2\alpha_3+\alpha_4$. Here we note that $E_1\cup(1-\delta',1)\subset E_2$ (cf. (5.66)). By Lemma 5.8, we have

$$T_s(r,\hat{f}_\Sigma,\eta)\le \alpha_8 T_s(r,\bar{q}\circ\hat{f}_\Sigma,\omega_S)+\alpha_8 m(\sigma,f_S,\lambda_W)+\alpha_8$$

for all $r\in(1/2,1)$. Here, $\alpha_8>0$ is a positive constant which does not depend on the choice of $f\in\mathcal{F}\backslash\mathcal{E}_2$. Hence we get

$$\begin{aligned}&|m(r,f_A,\lambda_D)-m(\sigma,f_A,\lambda_D)|\\ &\qquad\le \alpha_9 T_s(r,f_S,\omega_S)+\alpha_9 m(r,f_S,\lambda_W)+\alpha_9 m(\sigma,f_S,\lambda_W)+\alpha_9\end{aligned}$$

for all $r\in(s,1)\backslash E_2$, where $\alpha_9=\alpha_7\alpha_8+\alpha_7$. By Lemma 4.3, we get

$$m(r,f_S,\lambda_W)\le \alpha_{10}T_\sigma(r,f_S,\omega_S)+m(\sigma,f_S,\lambda_W)+\alpha_{10}$$

for all $r\in(1/2,1)$. Here, $\alpha_{10}>0$ is a positive constant which does not depend on the choice of $f\in\mathcal{F}\backslash\mathcal{E}_2$. Hence we get

$$|m(r,f_A,\lambda_D)-m(\sigma,f_A,\lambda_D)|\le \alpha T_s(r,f_S,\omega_S)+\alpha m(\sigma,f_S,\lambda_W)+\alpha \tag{5.67}$$

for all $r\in(s,1)\backslash E_2$, where $\alpha=\max\{2\alpha_9,\alpha_9\alpha_{10}+\alpha_9\}$. This proves the desired estimate for $f\in\mathcal{F}\backslash\mathcal{E}_2$ with $f_S(\mathbb{D})\not\subset W$.

Finally, we enlarge $\alpha>0$ so that

$$2\sup_{r\in(s,1-\delta)} m(r,f,\lambda_D)\le\alpha$$

for all $f\in\mathcal{E}_2$. Then (5.67) is valid for all $f\in\mathcal{F}$ with $f_S(\mathbb{D})\not\subset W$. □

§5.9. Proof of Proposition 5.1

We complete the proof of Proposition 5.1. We take $W\subset S$ as in Remark 5.10. Then we have $W\subsetneqq\tau(Z)$ (cf. Remark 5.10). Let $\mathcal{F}\subset\mathrm{Hol}(\mathbb{D},A\times S)$, $\omega_{\bar{A}}$, ω_S, λ_W, $s\in(1/2,1)$, $\delta>0$ be given as in Proposition 5.1. Let $\rho\colon A\to A_0$ be the canonical quotient (cf. (5.2)) and let ω_{A_0} be an invariant positive $(1,1)$-form on A_0. Note that we are given a smooth projective equivariant compactification $\bar{A}$. For an irreducible component $D\subset\bar{A}$ of ∂A, let $\lambda_D\ge0$ be a Weil function.

Now let $f \in \mathcal{F}$ such that $f_S(\mathbb{D}) \not\subset W$. By Lemma 5.2, we get

$$T_s(r, f_A, \omega_{\bar{A}}) \leq c' T_s(r, \rho \circ f_A, \omega_{A_0}) + c' \sum_{D \subset \partial A} |m(r, f_A, \lambda_D) - m(\sigma, f_A, \lambda_D)| + c'$$

for all $r \in (s, 1)$, where $c' > 0$ is independent of the choice of $f \in \mathcal{F}$. By Lemma 5.11, we get

$$T_s(r, \rho \circ f_A, \omega_{A_0}) \leq \alpha T_s(r, f_S, \omega_S) + \alpha m((s+r)/2, f_S, \lambda_W) + \alpha$$

for all $r \in (s, 1) \backslash E'$ with $|E'| < \delta/2$. Here, $\alpha > 0$ is independent of the choice of $f \in \mathcal{F}$. Let $D_1, \ldots, D_l$ be the irreducible components of ∂A. By Lemma 5.17, we get

$$\begin{aligned} &|m(r, f_A, \lambda_{D_i}) - m((s+r)/2, f_A, \lambda_{D_i})| \\ &\qquad \leq \beta_i T_s(r, f_S, \omega_S) + \beta_i m((s+r)/2, f_S, \lambda_W) + \beta_i \end{aligned}$$

for all $r \in (s, 1) \backslash E_i$, where $E_i \subset (s, 1)$ is an exceptional set with $|E_i| < \delta/2l$. Here, $\beta_i > 0$ is independent of the choice of $f \in \mathcal{F}$. We set $c_1 = c_2 = c'(\alpha + \sum_{i=1}^{l} \beta_i)$, $c_3 = c' + c'(\alpha + \sum_{i=1}^{l} \beta_i)$, and $E = E' \cup \bigcup_{i=1}^{l} E_i$. Then $|E| < \delta$. This shows (5.1), where $c_1 > 0$, $c_2 > 0$, $c_3 > 0$ are independent of the choice of $f \in \mathcal{F}$. This completes the proof of Proposition 5.1. $\square$

Remark 5.18. Let us observe from the proof of Proposition 5.1 why it is natural to work with Demailly jet spaces rather than classical jet spaces in our paper. This is because the projectivization of the tangent spaces behaves better than the tangent spaces themselves. Our proposition follows from Lemmas 5.11 and 5.17. For the proof of Lemma 5.11, we take an open neighborhood $O_{\Sigma^*_{1,A}} \subset \Sigma_{1,A}$ of $\Sigma^*_{1,A}$ over which (5.33) holds. We note that $\Sigma^*_{1,A} \subset \Sigma_{1,A}$ is the projectivization of $T\Sigma \times \{0\} \subset T\Sigma \times \operatorname{Lie} A$ (cf. (3.2)). However, unlike its projectivization, we cannot find an open neighborhood of $T\Sigma \times \{0\}$ over which (5.33) is satisfied. This is because the condition fails to hold over an arbitrary small neighborhood of the zero section of $T\Sigma \times \operatorname{Lie} A$. A similar issue arises in the proof of Lemma 5.17 concerning the estimate (5.60). Therefore, working with the projectivization is a more suitable approach. This is why it is natural to use Demailly jet spaces in our paper.

§6. Application of logarithmic tautological inequality

The purpose of this section is to prove Lemma 6.3. This is an application of the logarithmic tautological inequality, which is a generalization of Nevanlinna's lemma on logarithmic derivatives.

§6.1. Logarithmic tautological inequality

Let X be a smooth projective variety with a smooth, positive $(1,1)$-form ω_X. Let $D \subset X$ be a simple normal crossing divisor. We set

$$\overline{T}X(-\log D) = P(TX(-\log D) \oplus \mathcal{O}_X),$$

which is a smooth compactification of $TX(-\log D)$. Let $\partial TX(-\log D)$ be the Cartier divisor on the boundary which corresponds to a section of $\mathcal{O}_{\overline{T}X(-\log D)}(1)$. If $f \in \mathrm{Hol}(\mathbb{D}, X)$ is holomorphic with $f(\mathbb{D}) \not\subset D$, then the derivative of f induces a holomorphic map $f' \colon \mathbb{D} \to \overline{T}X(-\log D)$.

Lemma 6.1. *Let $\varepsilon > 0$, $\delta > 0$, and $s \in (0,1)$. Let $\lambda_{\partial TX(-\log D)}$ and λ_D be Weil functions for $\partial TX(-\log D) \subset \overline{T}X(-\log D)$ and $D \subset X$, respectively. Then there exists a positive constant $\mu > 0$ such that for all $f \in \mathrm{Hol}(\mathbb{D}, X)$ with $f(\mathbb{D}) \not\subset D$, we have*

$$m(r, f', \lambda_{\partial TX(-\log D)}) \leq \varepsilon T_s(r, f, \omega_X) + \varepsilon m(r, f, \lambda_D) + \mu$$

for all $r \in (s,1)$ outside some exceptional set of linear measure less than δ.

This lemma is a variant of the estimate for entire curves $f \colon \mathbb{C} \to X$ due to R. Kobayashi [26] and McQuillan [32]. We refer to Vojta [43, Thm. A.2] for the precise statement and simplified proof. We remark that [43, Thm. A.2] implies the classical Nevanlinna lemma on logarithmic derivatives when applied to entire curves $f \colon \mathbb{C} \to \mathbb{P}^1$ and $D = (0) + (\infty)$, while Vojta's proof of [43, Thm. A.2] is based on Nevanlinna's lemma. In the following, we follow another method described in [45, Thm. 4.8], which does not use Nevanlinna's lemma. See also [9, Sec. 2].

Proof of Lemma 6.1. The proof divides into three steps. The following proof is similar to the proof of [45, Thm. 4.8].

Step 1. Let $\overline{T}X = P(TX \oplus \mathcal{O}_X)$ be a smooth compactification of TX and let ∂TX be the Cartier divisor on the boundary which corresponds to a section of $\mathcal{O}_{\overline{T}X}(1)$. Let $\lambda_{\partial TX}$ be the Weil function of ∂TX defined by $\lambda_{\partial TX}(v) = \log \sqrt{1 + |v|^2_{\omega_X}}$ for $v \in TX$, where $|\cdot|_{\omega_X}$ is a norm on TX defined by ω_X. We prove the following estimate for all $g \in \mathrm{Hol}_m(\mathbb{D}, X)$ with $g(Y_g) \not\subset D$:

$$m(r, g', \lambda_{\partial TX(-\log D)}) \leq m(r, g, \lambda_D) + m(r, g', \lambda_{\partial TX}) + \mu_1 \tag{6.1}$$

for all $r \in (0,1)$. Here, $\mu_1 > 0$ is a positive constant which only depends on the choices of Weil functions.

We prove this. The natural morphism $\iota_1 \colon TX(-\log D) \to TX$ induces a birational map

$$\psi \colon \overline{T}X \dashrightarrow \overline{T}X(-\log D).$$

Let $Z \subset \overline{T}X$ be the indeterminacy locus of ψ. Let $p\colon \overline{T}X \to X$ be the projection. Then we have

$$(\psi|_{\overline{T}X\backslash Z})^*\partial TX(-\log D) = (p^*D + \partial TX)|_{\overline{T}X\backslash Z}. \tag{6.2}$$

Let $\alpha\colon \widetilde{\overline{T}X} \to \overline{T}X$ be a modification such that ψ induces a morphism $\widetilde{\psi}\colon \widetilde{\overline{T}X} \to \overline{T}X(-\log D)$. Then there exists an effective Cartier divisor $E \subset \widetilde{\overline{T}X}$ such that

$$\widetilde{\psi}^*\partial TX(-\log D) = \alpha^*(p^*D + \partial TX) - E. \tag{6.3}$$

Indeed, by (6.2), we have

$$(\psi|_{\overline{T}X\backslash Z})^*\mathcal{O}_{\overline{T}X(-\log D)}(1) = (p^*\mathcal{O}_X(D) \otimes \mathcal{O}_{\overline{T}X}(1))|_{\overline{T}X\backslash Z}. \tag{6.4}$$

Let $p^{\log}\colon \overline{T}X(-\log D) \to X$ be the projection. There exists a natural surjection

$$(p^{\log})^*(p^{\log})_*\mathcal{O}_{\overline{T}X(-\log D)}(1) \to \mathcal{O}_{\overline{T}X(-\log D)}(1). \tag{6.5}$$

We pull this back by $\psi|_{\overline{T}X\backslash Z}\colon \overline{T}X\backslash Z \to \overline{T}X(-\log D)$ and combine with (6.4). Then we have the following surjection on $\overline{T}X\backslash Z$:

$$p^*(p^{\log})_*\mathcal{O}_{\overline{T}X(-\log D)}(1)|_{\overline{T}X\backslash Z} \to p^*\mathcal{O}_X(D) \otimes \mathcal{O}_{\overline{T}X}(1)|_{\overline{T}X\backslash Z}.$$

Since Z has codimension greater than 1, this morphism extends over the whole of $\overline{T}X$. We denote by K the kernel of the following surjection obtained from (6.5):

$$\widetilde{\psi}^*(p^{\log})^*(p^{\log})_*\mathcal{O}_{\overline{T}X(-\log D)}(1) \to \widetilde{\psi}^*\mathcal{O}_{\overline{T}X(-\log D)}(1).$$

Then the composition of

$$K \to \alpha^*p^*(p^{\log})_*\mathcal{O}_{\overline{T}X(-\log D)}(1) \to \alpha^*(p^*\mathcal{O}_X(D) \otimes \mathcal{O}_{\overline{T}X}(1))$$

is a zero map on $\alpha^{-1}(\overline{T}X\backslash Z)$, hence on $\widetilde{\overline{T}X}$. Hence we get a morphism

$$\widetilde{\psi}^*\mathcal{O}_{\overline{T}X(-\log D)}(1) \to \alpha^*(p^*\mathcal{O}_X(D) \otimes \mathcal{O}_{\overline{T}X}(1)).$$

Hence there exists an effective Cartier divisor E on $\widetilde{\overline{T}X}$ such that (6.3) is valid. Hence, we get (6.1).

Step 2. We estimate $m(r, g', \lambda_{\partial TX})$, where

$$m(r, g', \lambda_{\partial TX}) = \frac{1}{\deg \pi_g} \int_{y \in \partial Y_g(r)} \log \sqrt{1 + |g'(y)|^2_{\omega_X}}\, \frac{d \arg \pi_g(y)}{2\pi}.$$

Using the concavity of log, we have

$$m(r, g', \lambda_{\partial TX}) \leq \frac{1}{2} \log\left(1 + \frac{1}{\deg \pi_g} \int_{y \in \partial Y_g} |g'(y)|^2_{\omega_X}\, \frac{d \arg \pi_g(y)}{2\pi}\right).$$

We set

$$\tau(r) = \frac{1}{\deg \pi_g} \int_s^r dt \int_{Y_g(t)} g^*\omega_X.$$

Then we have

$$\frac{1}{2\pi r}\frac{d^2}{dr^2}\tau(r) = \frac{1}{\deg \pi_g} \int_{y\in\partial Y_g(r)} |g'(y)|^2_{\omega_X} \frac{d\arg \pi_g(y)}{2\pi}.$$

Hence

$$m(r, g', \lambda_{\partial TX}) \leq \frac{1}{2}\log\Big(1 + \frac{1}{2\pi r}\frac{d^2}{dr^2}\tau(r)\Big).$$

Hence for $r > s$, we have

$$m(r, g', \lambda_{\partial TX}) \leq \frac{1}{2}\log\Big(1 + \frac{1}{2\pi s}\frac{d^2}{dr^2}\tau(r)\Big).$$

Now we apply Lemma 5.16 twice. We have

$$\begin{aligned}\frac{d^2}{dr^2}\tau(r) &\leq \frac{4}{\delta}\max\{1, (\tau'(r))^2\} \\ &\leq \frac{4}{\delta}\max\Big\{1, \Big(\frac{4}{\delta}\max\{1, \tau(r)^2\}\Big)^2\Big\} \\ &= \frac{4^3}{\delta^3}\max\{1, \tau(r)^4\} \\ &\leq \frac{4^3}{\delta^3} + \frac{4^3}{\delta^3}\tau(r)^4\end{aligned}$$

for $r \in (s, 1)$ outside some exceptional set E with $|E| < \delta$. Hence by $\tau(r) \leq T_s(r, g, \omega_X)$, we get

$$m(r, g', \lambda_{\partial TX}) \leq \frac{1}{2}\log(1 + c + cT_s(r, g, \omega_X)^4) \tag{6.6}$$

for $r \in (s, 1)$ outside E, where $c = \frac{4^3}{2\pi s\delta^3}$.

Step 3. We take a positive integer l such that $\frac{1}{l} < \varepsilon$. There exists a ramification covering $\varphi\colon X' \to X$ such that (1) X' is smooth, (2) $D' := (\varphi^*D)_{\mathrm{red}}$ is a normal crossing, (3) $lD' \subset \varphi^*D$ (cf. [23, Thm. 17]). We take a holomorphic map $g\colon Y \to X'$, where Y is a Riemann surface with a proper surjective holomorphic map $\pi_g\colon Y \to \mathbb{D}$, with the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{g} & X' \\ {\scriptstyle \pi_g}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathbb{D} & \xrightarrow[f]{} & X. \end{array}$$

Then we have

$$lm(r,g,\lambda_{D'}) \le m(r,g,\lambda_{\varphi^*D}) = m(r,f,\lambda_D).$$

The morphism $TX'(-\log D') \to TX(-\log D)$ induces a rational map

$$\Phi\colon \overline{T}X'(-\log D') \dashrightarrow \overline{T}X(-\log D).$$

Let $Z' \subset \overline{T}X'(-\log D')$ be the indeterminacy locus of Φ. Then we have

$$(\Phi|_{\overline{T}X'(-\log D')\backslash Z'})^*(\partial TX(-\log D)) = \partial TX'(-\log D')|_{\overline{T}X'(-\log D')\backslash Z'}. \tag{6.7}$$

Let $\beta\colon \widetilde{\overline{T}X'(-\log D')} \to \overline{T}X'(-\log D')$ be a modification such that Φ induces a morphism $\widetilde{\Phi}\colon \widetilde{\overline{T}X'(-\log D')} \to \overline{T}X(-\log D)$. Then by an argument similar to that used to justify (6.3), we have

$$\widetilde{\Phi}^*(\partial TX(-\log D)) = \beta^*(\partial TX'(-\log D')) - E', \tag{6.8}$$

where $E' \subset \widetilde{\overline{T}X'(-\log D')}$ is an effective Cartier divisor.

We prove this. By (6.7), we have the following on $\overline{T}X'(-\log D')\backslash Z'$:

$$(\Phi|_{\overline{T}X'(-\log D')\backslash Z'})^*\mathcal{O}_{\overline{T}X(-\log D)}(1) = \mathcal{O}_{\overline{T}X'(-\log D')}(1)|_{\overline{T}X'(-\log D')\backslash Z'}. \tag{6.9}$$

We pull back (6.5) by

$$\Phi|_{\overline{T}X'(-\log D')\backslash Z'}\colon \overline{T}X'(-\log D')\backslash Z' \to \overline{T}X(-\log D)$$

and combine with (6.9). Then, on $\overline{T}X'(-\log D')\backslash Z'$, we have the following surjection:

$$\begin{aligned}((p')^{\log})^*(p^{\log})_*\mathcal{O}_{\overline{T}X(-\log D)}(1)|_{\overline{T}X'(-\log D')\backslash Z'}\\ \to \mathcal{O}_{\overline{T}X'(-\log D')}(1)|_{\overline{T}X'(-\log D')\backslash Z'},\end{aligned}$$

where $(p')^{\log}\colon \overline{T}X'(-\log D') \to X'$ is the projection. Since Z' has codimension greater than 1, this morphism extends over whole $\overline{T}X'(-\log D')$. We denote by K the kernel of the following surjection obtained from (6.5):

$$\widetilde{\Phi}^*(p^{\log})^*(p^{\log})_*\mathcal{O}_{\overline{T}X(-\log D)}(1) \to \widetilde{\Phi}^*\mathcal{O}_{\overline{T}X(-\log D)}(1).$$

Then the composition of

$$K \to \beta^*((p')^{\log})^*(p^{\log})_*\mathcal{O}_{\overline{T}X(-\log D)}(1) \to \beta^*\mathcal{O}_{\overline{T}X'(-\log D')}(1)$$

is a zero map on $\beta^{-1}(\overline{T}X'(-\log D')\backslash Z')$, hence on $\widetilde{\overline{T}X'(-\log D')}$. Hence we get a morphism

$$\widetilde{\Phi}^*\mathcal{O}_{\overline{T}X(-\log D)}(1) \to \beta^*\mathcal{O}_{\overline{T}X'(-\log D')}(1).$$

Hence there exists an effective Cartier divisor E' on $\widetilde{\bar{T}X'(-\log D')}$ such that (6.8) is valid.

Now by (6.8), we have

$$m(r, f', \lambda_{\partial TX(-\log D)}) \leq m(r, g', \lambda_{\partial TX'(-\log D')}) + \mu_2.$$

Here, $\mu_2 > 0$ is a positive constant which only depends on the choices of Weil functions. Using (6.1) for the pair (X', D') instead of (X, D), we get

$$\begin{aligned} m(r, f', \lambda_{\partial TX(-\log D)}) &\leq m(r, g', \lambda_{\partial TX'(-\log D')}) + \mu_2 \\ &\leq m(r, g, \lambda_{D'}) + m(r, g', \lambda_{\partial TX'}) + \mu_1 + \mu_2 \\ &\leq \frac{1}{l} m(r, f, \lambda_D) + m(r, g', \lambda_{\partial TX'}) + \mu_1 + \mu_2 \\ &\leq \varepsilon m(r, f, \lambda_D) + m(r, g', \lambda_{\partial TX'}) + \mu_1 + \mu_2. \end{aligned}$$

By (6.6), for the pair $(X', \omega_{X'})$ instead of (X, ω_X), we get

$$m(r, f', \lambda_{\partial TX(-\log D)}) \leq \varepsilon m(r, f, \lambda_D) + \frac{1}{2}\log(1 + c + cT_s(r, g, \omega_{X'})^4) + \mu_1 + \mu_2$$

for all $r \in (s, 1)$ outside some exceptional set of linear measure less than δ. Since $\varphi\colon X' \to X$ is finite, there exist positive constants $c' > 0$ and $c'' > 0$ such that

$$T_s(r, g, \omega_{X'}) \leq c' T_s(r, g, \varphi^*\omega_X) + c'' = c' T_s(r, f, \omega_X) + c''$$

for all $r \in (s, 1)$, where c' and c'' are independent of the choice of g (cf. Lemma 4.5). Hence we get

$$\begin{aligned} m(r, f', \lambda_{\partial TX(-\log D)}) \leq{}& \varepsilon m(r, f, \lambda_D) \\ &+ \frac{1}{2}\log\bigl(1 + c + c(c' T_s(r, f, \omega_X) + c'')^4\bigr) + \mu_1 + \mu_2 \end{aligned}$$

for all $r \in (s, 1)$ outside some exceptional set of linear measure less than δ. We take a positive constant $\mu > 0$ such that

$$\frac{1}{2}\log(1 + c + c(c'x + c'')^4) + \mu_1 + \mu_2 \leq \varepsilon x + \mu$$

for $x \geq 0$. Then we obtain our estimate. □

§6.2. The case of semi-abelian varieties

Let $\bar{A}$ be an equivariant compactification of a semi-abelian variety A. Let $\mathcal{G}$ be an infinite indexed family in $\mathrm{Hol}(\mathbb{D}, A)$. We consider the following assumption.

Assumption 6.2. Let $D \subset \partial A$ be an irreducible component. Then $\mathcal{G}' \not\to D$ for every infinite subfamily $\mathcal{G}'$ of $\mathcal{G}$.

We recall $\Pi(\mathcal{G})$ from Definition 3.18. In this subsection, we prove the following lemma.

Lemma 6.3. *Let A be a semi-abelian variety and let S be a smooth projective variety. Let $\bar{A}$ be a smooth projective equivariant compactification. Let $\omega_{\bar{A}}$ and ω_S be smooth, positive $(1,1)$-forms on $\bar{A}$ and S, respectively. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A \times S)$ be an infinite set such that $\mathcal{F}_A = (f_A)_{f \in \mathcal{F}}$ satisfies Assumption 6.2. Assume that $\{0\} \in \Pi(\mathcal{F}_A)$ and that f_A is non-constant for all $f \in \mathcal{F}$.*

Let $k \in \mathbb{Z}_{\geq 1}$. Let $\omega_{S_{k,A}}$ be a smooth, positive $(1,1)$-form on $S_{k,A}$. Then there exists $\sigma \in (0,1)$ with the following property: Let $s \in (\sigma, 1)$, $\varepsilon > 0$, and $\delta > 0$. Then there exist positive constants $\mu_1 > 0$, $\mu_2 > 0$ such that for all $f \in \mathcal{F}$, the estimate

$$T_s(r, f_{S_{k,A}}, \omega_{S_{k,A}}) \leq \varepsilon T_s(r, f_A, \omega_{\bar{A}}) + \mu_1 T_s(r, f_S, \omega_S) + \mu_2$$

holds for all $r \in (s, 1)$ outside some exceptional set of linear measure less than δ.

To prove this, we prepare several lemmas.

Lemma 6.4. *Let $\bar{A}$ be a smooth projective equivariant compactification. Let $\mathcal{F} = (f_i)_{i \in I}$ be an infinite indexed family in $\mathrm{Hol}(\mathbb{D}, A)$ such that Assumption 6.2 is satisfied. Then there exists $\sigma \in (0,1)$ with the following property: Let $s \in (\sigma, 1)$ and $\delta > 0$. Then there exists a positive constant $c > 0$ such that for all $i \in I$, we have*

$$m(r, f_i, \lambda_{\partial A}) \leq c T_s(r, f_i, \omega_{\bar{A}}) + c$$

for all $r \in (s + \delta, 1)$.

Proof. We first consider the case that the subset

$$\mathcal{F}_o = \{f_i;\ i \in I\} \subset \mathrm{Hol}(\mathbb{D}, A)$$

is finite. In this case, our estimate follows from Lemma 4.3. Thus we assume that $\mathcal{F}_o$ is infinite. Then $\mathcal{F}_o$ satisfies Assumption 6.2. By replacing $\mathcal{F}$ by $\mathcal{F}_o$, we may assume that $\mathcal{F}$ is a subset of $\mathrm{Hol}(\mathbb{D}, A)$.

Let $D_1, \ldots, D_n$ be the irreducible components of ∂A. Then for each $j = 1, \ldots, n$, by Assumption 6.2, we apply Lemma 4.9 to get $\sigma_j \in (0,1)$ and $\alpha_j > 0$ such that, for $s \in (\sigma_j, 1)$ and $\delta > 0$, we have

$$m(r, f, \lambda_{D_j}) \leq \frac{\alpha_j}{\delta} T_s(r, f, \omega_{\bar{A}}) + \alpha_j$$

for all $r \in (s+\delta, 1)$ and all $f \in \mathcal{F}$. We set $\sigma = \max\{\sigma_1, \dots, \sigma_n\}$ and $\alpha = \alpha_1 + \cdots + \alpha_n$. Let $s \in (\sigma, 1)$ and $\delta > 0$. Then for all $f \in \mathcal{F}$, we have

$$m(r, f, \lambda_{\partial A}) = \sum_{j=1}^{n} m(r, f, \lambda_{D_j}) \leq \frac{\alpha}{\delta} T_s(r, f, \omega_{\bar{A}}) + \alpha$$

for all $r \in (s+\delta, 1)$. We set $c = \max\{\alpha, \alpha/\delta\}$ to conclude the proof. □

Let V_1 and V_2 be smooth algebraic varieties. Let $p_1 \colon V_1 \times V_2 \to V_1$ be the first projection and let $p_2 \colon V_1 \times V_2 \to V_2$ be the second projection. Let ω_{V_1} and ω_{V_2} be smooth $(1,1)$-forms on V_1 and V_2, respectively. We set

$$\omega_{V_1 \times V_2} = p_1^* \omega_{V_1} + p_2^* \omega_{V_2}. \tag{6.10}$$

Let $w \in \mathbb{D}(r)$. We recall $\varphi_{w,r} \colon \mathbb{D}(r) \to \mathbb{D}(r)$ from (4.5).

Lemma 6.5. *Let S be a smooth projective variety. Let $\bar{A}$ be a smooth projective equivariant compactification. Let $\omega_{\bar{A}}$ and ω_S be smooth, positive $(1,1)$-forms on $\bar{A}$ and S, respectively. Let ω_A be an invariant positive $(1,1)$-form on A. Let $\mathcal{F} \subset$ $\mathrm{Hol}(\mathbb{D}, A \times S)$ be an infinite set of holomorphic maps such that $\mathcal{F}_A = (f_A)_{f \in \mathcal{F}}$ satisfies Assumption 6.2. Then there exists $\sigma \in (0,1)$ with the following property: Let $\varepsilon > 0$, $s \in (\sigma, 1)$, and $\delta > 0$. Then there exists a positive constant $\mu > 0$ such that for all $f \in \mathcal{F}$ and $w \in \mathbb{D}(\sigma)$, we have*

$$\int_0^{2\pi} \log |(f \circ \varphi_{w,r})'(re^{i\theta})|_{\omega_{A\times S}} \frac{d\theta}{2\pi} \leq \varepsilon T_s(r, f, \omega_{\bar{A}\times S}) + \mu$$

for all $r \in (s, 1)$ outside some exceptional set of linear measure less than δ.

Proof. We first apply Lemma 6.4 for $\mathcal{F}_A = (f_A)_{f \in \mathcal{F}}$ to get $\sigma \in (0,1)$. Let $\varepsilon > 0$, $s \in (\sigma, 1)$, $\delta > 0$ be given.

Set $D = \partial A \times S \subset \bar{A} \times S$. We first note that by Lemma A.16, we have

$$T(\bar{A} \times S)(-\log D) = T\bar{A}(-\log \partial A) \times TS = \bar{A} \times \mathrm{Lie}\, A \times TS.$$

Hence $\omega_{A\times S}$ defines a Hermitian metric on $T(\bar{A} \times S)(-\log D)$. Hence we may define the Weil function by

$$\lambda_{\partial T(\bar{A}\times S)(-\log D)}(v) = \log \sqrt{1 + |v|^2_{\omega_{A\times S}}}, \tag{6.11}$$

where $v \in T(\bar{A} \times S)(-\log D)$.

By Lemma 6.4, there exists $c > 0$ such that

$$m(r, f_A, \lambda_{\partial A}) \leq cT_s(r, f_A, \omega_{\bar{A}}) + c$$

for all $f \in \mathcal{F}$ and $r \in (s+\delta', 1)$, where $\delta' = \delta/2$. Set $\varepsilon' = \frac{(s-\sigma)\varepsilon}{(s+\sigma)(1+c)}$. By Lemma 6.1, there exists $\mu_0 > 0$ such that

$$m(r, f', \lambda_{\partial T(\bar{A}\times S)(-\log D)}) \leq \varepsilon' T_s(r, f, \omega_{\bar{A}\times S}) + \varepsilon' m(r, f, \lambda_D) + \mu_0$$

for all $f \in \mathcal{F}$ and $r \in (s, 1)$ outside some exceptional set of linear measure less than δ'. Hence we have

$$m(r, f', \lambda_{\partial T(\bar{A}\times S)(-\log D)}) \leq \varepsilon'(1+c) T_s(r, f, \omega_{\bar{A}\times S}) + \varepsilon' c + \mu_0 \tag{6.12}$$

for all $f \in \mathcal{F}$ and $r \in (s, 1)$ outside some exceptional set of linear measure less than $2\delta'$.

Now we have

$$|(f\circ\varphi_{r,w})'(z)|_{\omega_{A\times S}} = |f'\circ\varphi_{r,w}(z)|_{\omega_{A\times S}} \times |\varphi'_{r,w}(z)|.$$

Hence by (4.6), we have

$$\int_0^{2\pi} \log|(f\circ\varphi_{w,r})'(re^{i\theta})|_{\omega_{A\times S}} \frac{d\theta}{2\pi} \leq \int_0^{2\pi} \log|f'\circ\varphi_{w,r}(re^{i\theta})|_{\omega_{A\times S}} \frac{d\theta}{2\pi} + \frac{s+\sigma}{s-\sigma}$$

for all $r \in (s, 1)$. Using (6.11), we have

$$\begin{aligned}\int_0^{2\pi} \log|f'\circ\varphi_{r,w}(re^{i\theta})|_{\omega_{A\times S}} \frac{d\theta}{2\pi} &\leq \int_0^{2\pi} \log\sqrt{1+|f'\circ\varphi_{r,w}(re^{i\theta})|^2_{\omega_{A\times S}}} \frac{d\theta}{2\pi} \\ &= m(r, f'\circ\varphi_{r,w}, \lambda_{\partial T(\bar{A}\times S)(-\log D)}).\end{aligned}$$

By Lemma 4.10, we have

$$m(r, f'\circ\varphi_{r,w}, \lambda_{\partial T(\bar{A}\times S)(-\log D)}) \leq \frac{s+\sigma}{s-\sigma} m(r, f', \lambda_{\partial T(\bar{A}\times S)(-\log D)})$$

for all $r \in (s, 1)$. Hence

$$\int_0^{2\pi} \log|(f\circ\varphi_{w,r})'(re^{i\theta})|_{\omega_{A\times S}} \frac{d\theta}{2\pi} \leq \frac{s+\sigma}{s-\sigma} m(r, f', \lambda_{\partial T(\bar{A}\times S)(-\log D)}) + \frac{s+\sigma}{s-\sigma}$$

for all $r \in (s, 1)$. Combining this with (6.12), we get our lemma. Here we set $\mu = \frac{s+\sigma}{s-\sigma}(\varepsilon' c + \mu_0 + 1)$. □

Let μ be a non-negative mass on $\mathbb{D}(r)$. For $|w| < s < r$, we define

$$T_s^w(r, \mu) = \int_s^r \frac{dt}{t} \int_{\mathbb{D}(t)} \varphi^*_{w,r}\mu.$$

Lemma 6.6. *Let $0 < \sigma < s < r < 1$. Then for all non-negative mass μ on $\mathbb{D}(r)$ and $w \in \mathbb{D}(\sigma)$, we have*

$$\frac{s(s-\sigma)}{s+\sigma} T_s^w(r, \mu) \leq T_s(r, \mu) \leq \frac{s+\sigma}{s(s-\sigma)} T_s^w(r, \mu).$$

To prove this lemma, we prepare the following two estimates: Let $|w| < \sigma < s < r < 1$; then we have

$$|\varphi_{w,r}(z)| \geq \frac{s+\sigma}{s-\sigma}(|z|-r)+r, \tag{6.13}$$

$$|\varphi_{w,r}(z)| \leq \frac{s-\sigma}{s+\sigma}(|z|-r)+r, \tag{6.14}$$

for all $z \in \mathbb{D}(r)$. Indeed, we have

$$r^2\frac{|z|-|w|}{r^2-|w|\,|z|} \leq |\varphi_{w,r}(z)| \leq r^2\frac{|w|+|z|}{r^2+|w|\,|z|} \tag{6.15}$$

for all $z \in \mathbb{D}(r)$. We have

$$r^2\frac{|z|-|w|}{r^2-|w|\,|z|} = r\frac{r+|w|}{r^2-|w|\,|z|}(|z|-r)+r.$$

For $z \in \mathbb{D}(r)$, we have

$$\frac{r+|w|}{r^2-|w|\,|z|} \leq \frac{r+|w|}{r(r-|w|)} \leq \frac{s+\sigma}{r(s-\sigma)}.$$

Hence combining these with (6.15), we get (6.13). To prove (6.14), we note

$$r^2\frac{|w|+|z|}{r^2+|w|\,|z|} = r\frac{r-|w|}{r^2+|w|\,|z|}(|z|-r)+r.$$

For $z \in \mathbb{D}(r)$, we have

$$\frac{r-|w|}{r^2+|w|\,|z|} \geq \frac{r-|w|}{r(r+|w|)} \geq \frac{s-\sigma}{r(s+\sigma)}.$$

Hence combining these with (6.15), we get (6.14).

Proof of Lemma 6.6. We set

$$U_s^w(r,\mu) = \int_s^r dt\int_{\mathbb{D}(t)}\varphi_{w,r}^*\mu.$$

Then we have

$$U_s^w(r,\mu) \leq T_s^w(r,\mu) \leq \frac{1}{s}U_s^w(r,\mu)$$

for all $r \in (s,1)$.

We fix $r \in (s,1)$ and $w \in \mathbb{D}(\sigma)$. For $x \in (0,r)$, we set

$$\Lambda(x) = \min\{r-x, r-s\}.$$

Then we have

$$U_s^w(r,\mu) = \int_{\mathbb{D}(r)}\Lambda(|z|)\varphi_{w,r}^*\mu_z.$$

In particular,

$$U_s^0(r,\mu)=\int_{\mathbb{D}(r)}\Lambda(|z|)\mu_z.$$

Firstly, by (6.13), we have

$$r-|\varphi_{w,r}(z)|\leq \frac{s+\sigma}{s-\sigma}(r-|z|)$$

for $|z|\leq r$. Hence we have

$$\Lambda(|\varphi_{w,r}(z)|)\leq r-|\varphi_{w,r}(z)|\leq \frac{s+\sigma}{s-\sigma}\Lambda(|z|)$$

for $s\leq |z|\leq r$. By $\Lambda(|\varphi_{w,r}(z)|)\leq r-s$, this holds for all $|z|\leq r$. Hence we get

$$U_s^0(r,\mu)=\int_{\mathbb{D}(r)}\Lambda(|z|)\mu_z=\int_{\mathbb{D}(r)}\Lambda(|\varphi_{w,r}(z)|)\varphi_{w,r}^*\mu_z\leq \frac{s+\sigma}{s-\sigma}U_s^w(r,\mu).$$

Thus

$$T_s(r,\mu)\leq \frac{1}{s}U_s^0(r,\mu)\leq \frac{s+\sigma}{s(s-\sigma)}T_s^w(r,\mu).$$

Next by (6.14), we have

$$\frac{s-\sigma}{s+\sigma}(r-|z|)\leq r-|\varphi_{w,r}(z)|$$

for $|z|\leq r$. Hence we have

$$\Lambda(|\varphi_{w,r}(z)|)\geq \frac{s-\sigma}{s+\sigma}\Lambda(|z|)$$

for $|z|\leq r$, provided $s\leq|\varphi_{w,r}(z)|\leq r$. By $\Lambda(|z|)\leq r-s$, this holds for all $|z|\leq r$. Hence we get

$$U_s^0(r,\mu)=\int_{\mathbb{D}(r)}\Lambda(|z|)\mu_z=\int_{\mathbb{D}(r)}\Lambda(|\varphi_{w,r}(z)|)\varphi_{w,r}^*\mu_z\geq \frac{s-\sigma}{s+\sigma}U_s^w(r,\mu).$$

Thus

$$T_s(r,\mu)\geq U_s^0(r,\mu)\geq \frac{s(s-\sigma)}{s+\sigma}T_s^w(r,\mu).$$

This concludes the proof of our lemma. □

Proof of Lemma 6.3. We prove our lemma by the induction on k. Thus we first consider the case $k=1$. We set $D=\partial(A\times S)$. Then $D\subset \bar{A}\times S$ is a simple normal crossing divisor. By Lemma A.16, we note that $\omega_{A\times S}$ induces a Hermitian metric on the tautological line bundle $\mathcal{O}_{PT(\bar{A}\times S)(-\log D)}(1)$ on $PT(\bar{A}\times S)(-\log D)$. Let $\omega_{\mathcal{O}_{PT(\bar{A}\times S)(-\log D)}(1)}$ be the associated curvature form for this tautological line bundle. Similarly, we note that $\omega_{A\times S}$ induces a Hermitian metric on the tautological line bundle $\mathcal{O}_{P(TS\times \mathrm{Lie}\,A)}(1)$ on $S_{1,A}$. Let $\omega_{\mathcal{O}_{P(TS\times \mathrm{Lie}\,A)}(1)}$ be the associated curvature form for this tautological line bundle. By $PT(\bar{A}\times S)(-\log D)=\bar{A}\times S_{1,A}$, we have

$$q^*\omega_{\mathcal{O}_{P(TS\times \mathrm{Lie}\,A)}(1)}=\omega_{\mathcal{O}_{PT(\bar{A}\times S)(-\log D)}(1)},$$

where $q\colon \bar{A}\times S_{1,A}\to S_{1,A}$ is the projection. There exist positive constants α_1 and α_2 such that

$$\omega_{S_{1,A}}\le \alpha_1\tau^*\omega_S+\alpha_2\omega_{\mathcal{O}_{P(TS\times \mathrm{Lie}\,A)}(1)}, \tag{6.16}$$

where $\tau\colon S_{1,A}\to S$.

We first choose $\sigma\in(0,1)$ which appears in Lemma 6.5. We modify σ as follows. Let ω_A be an invariant positive $(1,1)$-form on A. For each $n\in\mathbb{Z}_{\ge 2}$, let $\mathcal{E}_n\subset\mathcal{F}$ be the set of $f\in\mathcal{F}$ such that

$$\sup_{z\in\mathbb{D}(1-\frac{1}{n})}|f_A'(z)|_{\omega_A}\le\frac{1}{n}.$$

Then we have $\mathcal{E}_2\supset\mathcal{E}_3\supset\cdots$. By the assumption $\{0\}\in\Pi(\mathcal{F}_A)$, we may take $n_0\in\mathbb{Z}_{\ge 2}$ such that $\mathcal{E}_{n_0}$ is finite. Since f_A is non-constant for all $f\in\mathcal{F}$, we may take $n_1\ge n_0$ such that $\mathcal{E}_{n_1}=\emptyset$. By enlarging σ if necessary, we may assume that $1-\frac{1}{n_1}\le\sigma<1$. Then for all $f\in\mathcal{F}$, we have

$$\sup_{z\in\mathbb{D}(\sigma)}|f_A'(z)|_{\omega_A}\ge 1-\sigma.$$

For each $f\in\mathcal{F}$, we take $w_f\in\overline{\mathbb{D}(\sigma)}$ such that

$$|f_A'(w_f)|_{\omega_A}=\sup_{z\in\mathbb{D}(\sigma)}|f_A'(z)|_{\omega_A}.$$

Let $s\in(\sigma,1)$ and let $f\in\mathcal{F}$. For $r\in(s,1)$, we take $\varphi_{w_f,r}\colon\mathbb{D}(r)\to\mathbb{D}(r)$ as in (4.5) and set $g=f\circ\varphi_{w_f,r}$. Then by (6.16), we have

$$T_s(r,g_{S_{1,A}},\omega_{S_{1,A}})\le\alpha_1T_s(r,g_S,\omega_S)+\alpha_2T_s(r,g_{[1]},\omega_{\mathcal{O}_{PT(\bar{A}\times S)(-\log D)}(1)}).$$

By Lemma 6.6, we have

$$T_s(r,f_{S_{1,A}},\omega_{S_{1,A}})\le\frac{s+\sigma}{s(s-\sigma)}T_s(r,g_{S_{1,A}},\omega_{S_{1,A}})$$

and

$$T_s(r,f_S,\omega_S)\ge\frac{s(s-\sigma)}{s+\sigma}T_s(r,g_S,\omega_S).$$

Thus we get

$$T_s(r, f_{S_{1,A}}, \omega_{S_{1,A}}) \leq \frac{\alpha_1(s+\sigma)^2}{s^2(s-\sigma)^2} T_s(r, f_S, \omega_S) + \frac{\alpha_2(s+\sigma)}{s(s-\sigma)} T_s(r, g_{[1]}, \omega_{\mathcal{O}_{PT(\bar{A}\times S)(-\log D)}(1)}). \tag{6.17}$$

Next we estimate the second term of the right-hand side. We claim

$$T_s(r, g_{[1]}, \omega_{\mathcal{O}_{PT(\bar{A}\times S)(-\log D)}(1)}) \leq \int_0^{2\pi} \log |g'(re^{i\theta})|_{\omega_{A\times S}} \frac{d\theta}{2\pi} - \int_0^{2\pi} \log |g'(se^{i\theta})|_{\omega_{A\times S}} \frac{d\theta}{2\pi}, \tag{6.18}$$

where $|\cdot|_{\omega_{A\times S}}$ is the length with respect to $\omega_{A\times S}$. This is obtained as follows. The metric $\omega_{A\times S}$ defines a Hermitian metric $|\cdot|_{\omega_{A\times S}}$ on $\mathcal{O}_{PT(\bar{A}\times S)(-\log D)}(-1)$, whose curvature form is $-\omega_{\mathcal{O}_{PT(\bar{A}\times S)(-\log D)}(1)}$. By the Poincaré–Lelong formula, we have

$$-(g_{[1]})^*\omega_{\mathcal{O}_{PT(\bar{A}\times S)(-\log D)}(1)} = [(g')^*Z] - 2dd^c \log |g'|_{\omega_{A\times S}}$$

as currents on $\mathbb{D}$, where Z is the zero section of $\mathcal{O}_{PT(\bar{A}\times S)(-\log D)}(-1)$. By the Jensen formula, we get (6.18). Compare with the proof of (4.1).

Now let $\varepsilon > 0$ and $\delta > 0$. We set $\varepsilon' = \frac{s(s-\sigma)\varepsilon}{\alpha_2(s+\sigma)}$. By Lemma 6.5, we get

$$\int_0^{2\pi} \log |g'(re^{i\theta})|_{\omega_{A\times S}} \frac{d\theta}{2\pi} \leq \varepsilon' T_s(r, f, \omega_{\bar{A}\times S}) + \mu$$

for $r \in (s, 1)$ outside some exceptional set of linear measure less than δ. Here, $\mu > 0$ is a positive constant which appears in Lemma 6.5. In particular, μ is independent of the choices of $f \in \mathcal{F}$ and w_f. Since $\log |g'_A|_{\omega_A}$ is subharmonic, we have

$$\int_0^{2\pi} \log |g'(se^{i\theta})|_{\omega_{A\times S}} \frac{d\theta}{2\pi} \geq \int_0^{2\pi} \log |g'_A(se^{i\theta})|_{\omega_A} \frac{d\theta}{2\pi} \geq \log |g'_A(0)|_{\omega_A}.$$

Note that by (4.6), we have

$$|g'_A(0)|_{\omega_A} = |f'_A(w_f)|_{\omega_A} |\varphi'_{w_f,r}(0)| \geq (1-\sigma)\frac{s-\sigma}{s+\sigma}.$$

Hence, taking into account these estimate in (6.18), we get

$$T_s(r, g_{[1]}, \omega_{\mathcal{O}_{PT(\bar{A}\times S)(-\log D)}(1)}) \leq \varepsilon' T_s(r, f, \omega_{\bar{A}\times S}) + \mu' \tag{6.19}$$

for $r \in (s, 1)$ outside some exceptional set of linear measure less than δ. Here we set $\mu' = \mu + \log \frac{s+\sigma}{(1-\sigma)(s-\sigma)}$.

Combining (6.17) with (6.19), we get

$$T_s(r, f_{S_{1,A}}, \omega_{S_{1,A}}) \leq \varepsilon T_s(r, f_A, \omega_{\bar{A}}) + \mu_1 T_s(r, f_S, \omega_S) + \mu_2$$

for $r \in (s, 1)$ outside some exceptional set of linear measure less than δ. Here we set $\mu_1 = \frac{\alpha_1(s+\sigma)^2}{s^2(s-\sigma)^2} + \frac{\alpha_2(s+\sigma)\varepsilon'}{s(s-\sigma)}$, $\mu_2 = \frac{\alpha_2(s+\sigma)\mu'}{s(s-\sigma)}$ to conclude the proof of the case $k = 1$.

We assume that $k \geq 2$ and the lemma is true for $k - 1$. By the construction of $S_{k,A}$, we have $S_{k,A} \subset (S_{k-1,A})_{1,A}$ (cf. (2.10)). Hence there exists a positive constant $\alpha > 0$ such that $\omega_{S_{k,A}} \leq \alpha\omega_{(S_{k-1,A})_{1,A}}$ on $S_{k,A}$. Hence we have

$$T_s(r, f_{S_{k,A}}, \omega_{S_{k,A}}) \leq \alpha T_s(r, (f_{[k-1]})_{(S_{k-1,A})_{1,A}}, \omega_{(S_{k-1,A})_{1,A}}). \tag{6.20}$$

By the first step applied to $A \times S_{k-1,A}$ and $\{f_{[k-1]};\ f \in \mathcal{F}\} \subset \mathrm{Hol}(\mathbb{D}, A \times S_{k-1,A})$, we get $\sigma' \in (0, 1)$ with the following property: For $s \in (\sigma', 1)$, $\varepsilon > 0$, and $\delta > 0$, there exist $\mu_1' = \mu_1'(s, \varepsilon, \delta) > 0$ and $\mu_2' = \mu_2'(s, \varepsilon, \delta) > 0$ such that for all $f \in \mathcal{F}$, we have

$$\begin{aligned} &T_s(r, (f_{[k-1]})_{(S_{k-1,A})_{1,A}}, \omega_{(S_{k-1,A})_{1,A}}) \\ &\quad \leq \frac{\varepsilon}{2\alpha} T_s(r, f_A, \omega_{\bar{A}}) + \mu_1' T_s(r, (f_{[k-1]})_{S_{k-1,A}}, \omega_{S_{k-1,A}}) + \mu_2' \end{aligned} \tag{6.21}$$

for all $r \in (s, 1)$ outside some exceptional set of linear measure less than $\delta/2$. Now, by the induction hypothesis, we get $\sigma \in (\sigma', 1)$ with the following property: For $s \in (\sigma, 1)$, $\varepsilon > 0$, and $\delta > 0$, there exist $\mu_1'' > 0$ and $\mu_2'' > 0$ such that for all $f \in \mathcal{F}$, we have

$$\begin{aligned} &T_s(r, (f_{[k-1]})_{S_{k-1,A}}, \omega_{S_{k-1,A}}) \\ &\quad \leq \frac{\varepsilon}{2\alpha\mu_1'(s, \varepsilon, \delta)} T_s(r, f_A, \omega_{\bar{A}}) + \mu_1'' T_s(r, f_S, \omega_S) + \mu_2'' \end{aligned} \tag{6.22}$$

holds for $r \in (s, 1)$ outside some exceptional set of linear measure less than $\delta/2$. Hence by (6.20)–(6.22), our σ satisfies the required property to conclude the induction step. Here we set $\mu_1 = \alpha\mu_1'\mu_1''$ and $\mu_2 = \alpha(\mu_1'\mu_2'' + \mu_2')$. □

§7. Nevanlinna theory and blow-ups

In this section we shall establish Lemma 7.4. For this purpose, we start from a general estimate (cf. Lemma 7.2).

§7.1. A general estimate

We recall Definitions 3.1 and 3.2 for the terminologies used in the following lemma.

Lemma 7.1. *Let S be a smooth projective variety. Let $W \subset S$ be an irreducible Zariski closed set. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, S)$ be an infinite subset such that $\mathcal{F} \not\Rightarrow W$ and $f(\mathbb{D}) \not\subset W$ for all $f \in \mathcal{F}$ with finite exception. Then there exist an infinite subset*

$\mathcal{F}' \subset \mathcal{F}$ and a W-admissible modification $\varphi\colon S' \to S$ such that the following two properties hold:

1. *$S' = \mathrm{Bl}_Y S$ for some closed subscheme $Y \subset S$ such that $\operatorname{supp} Y \subsetneqq W$ and $f(\mathbb{D}) \not\subset \operatorname{supp} Y$ for all $f \in \mathcal{F}'$.*
2. *$\mathcal{F}' \not\to W'$, where $W' \subset S'$ is the strict transform.*

Moreover, we may take S' to be smooth.

Proof. We consider the following two cases.

Case 1. There exists a Zariski closed set $Z \subset S$ such that $W \not\subset Z$ and $f(\mathbb{D}) \subset Z$ for infinitely many $f \in \mathcal{F}$. In this case, we take an infinite subset $\mathcal{F}' \subset \mathcal{F}$ such that $f(\mathbb{D}) \subset Z$ for all $f \in \mathcal{F}'$. Set $Y = W \cap Z$ as a closed subscheme. Then $\operatorname{supp} Y \subsetneqq W$. Set $S' = \mathrm{Bl}_Y S$. Then $Z' \cap W' = \emptyset$ in S', where Z' and W' are the strict transforms of Z and W, respectively (cf. (3.1)). By $\operatorname{supp} Y \subset W$, we have $f(\mathbb{D}) \not\subset \operatorname{supp} Y$ for all $f \in \mathcal{F}'$ with finite exception. Hence by removing these finite elements from $\mathcal{F}'$, we may assume $f(\mathbb{D}) \not\subset \operatorname{supp} Y$ for all $f \in \mathcal{F}'$. Hence we may consider as $f(\mathbb{D}) \subset Z'$ for all $f \in \mathcal{F}'$. Hence $\mathcal{F}' \not\to W'$.

Case 2. Otherwise, there exists a W-admissible modification $\widehat{S} \to S$ such that $\mathcal{F} \not\to \widehat{W}$, where $\widehat{W} \subset \widehat{S}$ is the minimal transform. By [18, p. 171, Ex. 7.11 (c)], there exists a closed subscheme $Z \subset S$ such that $\widehat{S} = \mathrm{Bl}_Z S$ and $W \not\subset \operatorname{supp} Z$. Set $Y = W \cap Z$ as closed subschemes of S. Then $\operatorname{supp} Y \subsetneqq W$. Since we are in Case 2, $W \not\subset \operatorname{supp} Z$ implies that only finitely many $f \in \mathcal{F}$ satisfy $f(\mathbb{D}) \subset \operatorname{supp} Z$. By removing these finite elements from $\mathcal{F}$, we get $\mathcal{F}' \subset \mathcal{F}$ so that $f(\mathbb{D}) \not\subset \operatorname{supp} Z$ for all $f \in \mathcal{F}'$.

Now we set $S' = \mathrm{Bl}_Y S$ with $\varphi\colon S' \to S$. There exists a closed subscheme $Z^\dagger \subset S'$ such that $\mathcal{I}_{\varphi^* Z} = \mathcal{I}_{\varphi^* Y} \cdot \mathcal{I}_{Z^\dagger}$. Indeed, since $\varphi^* Y \subset S'$ is a Cartier divisor, $\mathcal{I}_{\varphi^* Y}$ is an invertible sheaf. By $\mathcal{I}_{\varphi^* Z} \subset \mathcal{I}_{\varphi^* Y} \subset \mathcal{O}_{S'}$, we take $Z^\dagger \subset S'$ such that $\mathcal{I}_{Z^\dagger} = \mathcal{I}_{\varphi^* Z} \otimes (\mathcal{I}_{\varphi^* Y})^{-1} \subset \mathcal{O}_{S'}$. Similarly, there exists $W^\dagger \subset S'$ such that $\mathcal{I}_{\varphi^* W} = \mathcal{I}_{\varphi^* Y} \cdot \mathcal{I}_{W^\dagger}$. Then by $\mathcal{I}_{\varphi^* Z} + \mathcal{I}_{\varphi^* W} = \mathcal{I}_{\varphi^* Y}$, we have $\mathcal{I}_{\varphi^* Y} \cdot (\mathcal{I}_{Z^\dagger} + \mathcal{I}_{W^\dagger}) = \mathcal{I}_{\varphi^* Y}$ so that $\mathcal{I}_{Z^\dagger} + \mathcal{I}_{W^\dagger} = \mathcal{O}_{S'}$. Hence $Z^\dagger \cap W^\dagger = \emptyset$. Note that $W' \subset W^\dagger$, where $W' \subset S'$ is the strict transform. Hence we get

$$Z^\dagger \cap W' = \emptyset.$$

We set $\widetilde{S} = \mathrm{Bl}_{Z^\dagger} S'$. Then since $(\varphi \circ p)^* Z$ is a Cartier divisor, $\widetilde{S} \to \widehat{S}$ exists:

$$\begin{tikzcd}
S' \arrow[r, equal] & \mathrm{Bl}_Y S \arrow[d, "\varphi"'] & \mathrm{Bl}_{Z^\dagger} S' \arrow[l, "p"'] \arrow[d] \arrow[r, equal] & \widetilde{S} \\
& S & \mathrm{Bl}_Z S \arrow[l] \arrow[r, equal] & \widehat{S}.
\end{tikzcd}$$

Let $\widetilde{W} \subset \widetilde{S}$ be the strict transform. Then by $\mathcal{F} \not\to \widehat{W}$, we have $\mathcal{F}' \not\to \widetilde{W}$. On the other hand, by $Z^\dagger \cap W' = \emptyset$, the morphism $p\colon \widetilde{S} \to S'$ is an isomorphism on a neighborhood of $\widetilde{W}$. Hence $\mathcal{F}' \not\to W'$.

Now suppose S' is not smooth. Then we take a birational modification $S'' \to S'$ such that S'' is smooth. Since S is smooth, we may assume that $S'' \to S$ is an isomorphism over $S \backslash \operatorname{supp} Y$. By [18, p. 171, Ex. 7.11 (c)], we may take a closed subscheme $Y' \subset S$ such that $S'' = \mathrm{Bl}_{Y'} S$ and $\operatorname{supp} Y' \subset \operatorname{supp} Y$. We replace S' by S''. The proof of the lemma is completed. □

We recall Definition 3.12 for the terminology used in the following lemma.

Lemma 7.2. *Let S be a smooth projective variety with a smooth positive $(1,1)$-form ω_S. Let $W \subset S$ be a Zariski closed set with a Weil function $\lambda_W \geq 0$. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, S)$ be an infinite set of holomorphic maps and let $\{E_i\}_{i \in I}$ be a countable set of Zariski closed subsets of S. Assume that $\mathrm{LIM}(\mathcal{F}, \{E_i\}_{i\in I})$ exists and $\mathrm{LIM}(\mathcal{F}, \{E_i\}_{i\in I}) \not\subset W$. Then there exist $\sigma \in (0,1)$, $\beta > 0$, $k \in \mathbb{Z}_{\geq 0}$, a Zariski closed set $E \subset \bigcup E_i$ with a Weil function $\lambda_E \geq 0$, and an infinite subset $\mathcal{G} \subset \mathcal{F}$ with the following two properties:*

(1) *For all $f \in \mathcal{G}$, we have $f(\mathbb{D}) \not\subset E \cup W$.*

(2) *Let $s \in (\sigma, 1)$ and $f \in \mathcal{G}$. Then we have*

$$m(r, f, \lambda_W) \leq \frac{\beta}{(r-s)^k} T_s(r, f, \omega_S) + \frac{\beta}{(r-s)^k} m(s, f, \lambda_E) + \frac{\beta}{(r-s)^k},$$

for all $r \in (s, 1)$.

Proof. The proof is by Noetherian induction on W. Let $\mathcal{P}$ be the set of all Zariski closed set $W \subset S$ such that our lemma is false for W. To show $\mathcal{P} = \emptyset$, we assume on the contrary that $\mathcal{P} \neq \emptyset$. We take a minimal element $W \in \mathcal{P}$. In the following, we shall show that our lemma is true for W.

Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, S)$ and $\{E_i\}_{i\in I}$ be the objects which appear in our lemma. Thus $\mathrm{LIM}(\mathcal{F}, \{E_i\}_{i\in I})$ exists and $\mathrm{LIM}(\mathcal{F}, \{E_i\}_{i\in I}) \not\subset W$.

We first observe that W is irreducible. Otherwise, we have $W = W_1 \cup W_2$. Let $\lambda_{W_1} \geq 0$ and $\lambda_{W_2} \geq 0$ be Weil functions such that $\lambda_W \leq \lambda_{W_1} + \lambda_{W_2}$. By $W_1 \subsetneqq W$, we have $W_1 \notin \mathcal{P}$. Note that $\mathrm{LIM}(\mathcal{F}, \{E_i\}_{i\in I}) \not\subset W_1$. Hence we may take $\sigma_1 \in (0,1)$, $\beta_1 > 0$, $k_1 \in \mathbb{N}$, a Zariski closed set $E_1 \subset \bigcup E_i$ with a Weil function $\lambda_{E_1} \geq 0$, and an infinite subset $\mathcal{G}_1 \subset \mathcal{F}$ such that for all $f \in \mathcal{G}_1$, we have $f(\mathbb{D}) \not\subset E_1 \cup W_1$ and

$$m(r, f, \lambda_{W_1}) \leq \frac{\beta_1}{(r-s)^{k_1}} T_s(r, f, \omega_S) + \frac{\beta_1}{(r-s)^{k_1}} m(s, f, \lambda_{E_1}) + \frac{\beta_1}{(r-s)^{k_1}},$$

where $\sigma_1 < s < r < 1$. Now $\mathrm{LIM}(\mathcal{G}_1, \{E_i\}_{i\in I})$ exists and

$$\mathrm{LIM}(\mathcal{G}_1, \{E_i\}_{i\in I}) = \mathrm{LIM}(\mathcal{F}, \{E_i\}_{i\in I}) \not\subset W_2.$$

Hence by $W_2 \notin \mathcal{P}$, we may take $\sigma_2 \in (0,1)$, $\beta_2 > 0$, $k_2 \in \mathbb{N}$, a Zariski closed set $E_2 \subset \bigcup E_i$ with a Weil function $\lambda_{E_2} \geq 0$, and an infinite subset $\mathcal{G}_2 \subset \mathcal{G}_1$ such that for all $f \in \mathcal{G}_2$, we have $f(\mathbb{D}) \not\subset E_2 \cup W_2$ and

$$m(r, f, \lambda_{W_2}) \leq \frac{\beta_2}{(r-s)^{k_2}} T_s(r, f, \omega_S) + \frac{\beta_2}{(r-s)^{k_2}} m(s, f, \lambda_{E_2}) + \frac{\beta_1}{(r-s)^{k_2}},$$

where $\sigma_2 < s < r < 1$. We set $\sigma = \max\{\sigma_1, \sigma_2\}$, $\beta = \beta_1 + \beta_2$, $k = \max\{k_1, k_2\}$, $E = E_1 \cup E_2$, $\mathcal{G} = \mathcal{G}_2$. We take a Weil function λ_E such that $\lambda_E \geq \max\{\lambda_{E_1}, \lambda_{E_2}\}$. Then by $m(r, f, \lambda_W) \leq m(r, f, \lambda_{W_1}) + m(r, f, \lambda_{W_2})$, the two properties of our lemma are satisfied. Hence $W \notin \mathcal{P}$, a contradiction. Hence we may assume that W is irreducible.

We consider two cases.

Case 1*:* $W \subset \bigcup E_i$. In this case, we set $\sigma = 1/2$, $E = W$, and $\lambda_E = \lambda_W$. By $\mathrm{LIM}(\mathcal{F};\ \{E_i\}) \not\subset W$, only finitely many $f \in \mathcal{F}$ satisfy $f(\mathbb{D}) \subset W$. We remove these f from $\mathcal{F}$ to get an infinite subset $\mathcal{G} \subset \mathcal{F}$. Then $f(\mathbb{D}) \not\subset E \cup W$ for all $f \in \mathcal{G}$. By Lemma 4.3, there exists a positive constant $c > 0$ such that

$$m(r, f, \lambda_W) \leq cT_s(r, f, \omega_S) + cm(s, f, \lambda_W)$$

for all $s \in (\sigma, 1)$, $r \in (s, 1)$, and $f \in \mathcal{G}$. By $\lambda_W = \lambda_E$, our lemma is true for W. Here we set $\beta = c$ and $k = 0$.

Case 2*:* $W \not\subset \bigcup E_i$. By $\mathrm{LIM}(\mathcal{F};\ \{E_i\}) \not\subset W$ and $W \not\subset \bigcup E_i$, we have $\mathcal{F} \not\Rightarrow W$ (cf. Lemma 3.9). By Lemma 7.1, there exist an infinite subset $\mathcal{F}' \subset \mathcal{F}$ and a closed subscheme $\mathcal{Z} \subset S$ such that

(1) $\operatorname{supp} \mathcal{Z} \subsetneqq W$,
(2) $f(\mathbb{D}) \not\subset \operatorname{supp} \mathcal{Z}$ for all $f \in \mathcal{F}'$,
(3) $\mathcal{F}' \not\to W'$, where $W' \subset \mathrm{Bl}_{\mathcal{Z}} S$ is the strict transform.

Set $S' = \mathrm{Bl}_{\mathcal{Z}} S$. We may assume that S' is smooth. By Lemma 4.8, replacing $\mathcal{F}'$ by its infinite subset, we may assume that $\mathcal{F}'' \not\to W'$ for all infinite subsets $\mathcal{F}'' \subset \mathcal{F}'$ and

$$f(\mathbb{D}) \not\subset W' \tag{7.1}$$

for all $f \in \mathcal{F}'$. Let $\lambda_{W'} \geq 0$ be a Weil function for W' and let $\omega_{S'}$ be a smooth positive $(1,1)$-form on S'. By Lemma 4.9, there exist $\sigma_0 \in (0,1)$ and $\alpha > 0$ such that for all $\rho \in (\sigma_0, 1)$, $r \in (\rho, 1)$, and $f \in \mathcal{F}'$, we have

$$m(r, f, \lambda_{W'}) \leq \frac{\alpha}{r-\rho} T_\rho(r, f, \omega_{S'}) + \alpha. \tag{7.2}$$

Set $Z = \operatorname{supp} \mathcal{Z}$ with a Weil function $\lambda_Z \geq 0$. By $Z \subsetneqq W$, we have $Z \notin \mathcal{P}$. Note that $\mathrm{LIM}(\mathcal{F}';\ \{E_i\})$ exists and $\mathrm{LIM}(\mathcal{F}';\ \{E_i\}) = \mathrm{LIM}(\mathcal{F};\ \{E_i\})$. Hence $\mathrm{LIM}(\mathcal{F}';\ \{E_i\}) \not\subset Z$. Thus by the induction hypothesis, we get $\sigma_1 \in (0,1)$, $\beta_1 > 0$, $k_1 \in \mathbb{Z}_{\geq 0}$, $E \subset \bigcup E_i$ with $\lambda_E \geq 0$, and $\mathcal{G} \subset \mathcal{F}'$ such that for all $s \in (\sigma_1, 1)$, $r \in (s, 1)$, and $f \in \mathcal{G}$, we have

$$m(r, f, \lambda_Z) \leq \frac{\beta_1}{(r-s)^{k_1}} T_s(r, f, \omega_S) + \frac{\beta_1}{(r-s)^{k_1}} m(s, f, \lambda_E) + \frac{\beta_1}{(r-s)^{k_1}} \tag{7.3}$$

and $f(\mathbb{D}) \not\subset E \cup Z$. By (7.1), we have

$$f(\mathbb{D}) \not\subset E \cup W \tag{7.4}$$

for all $f \in \mathcal{G}$.

Now we set $\sigma = \max\{\sigma_0, \sigma_1\}$. Let $s \in (\sigma, 1)$ and $r \in (s, 1)$. Set $\rho = (s+r)/2$. Then by (4.4), there exists a positive constant $c > 0$ such that

$$T_\rho(r, f, \omega_{S'}) \leq c T_\rho(r, f, \omega_S) + c m(\rho, f, \lambda_Z) + c$$

for all $f \in \mathcal{G}$. Then by (7.3) applied to $r = \rho$, we get

$$T_\rho(r, f, \omega_{S'}) \leq \frac{\beta_2}{(r-s)^{k_1}} T_s(r, f, \omega_S) + \frac{\beta_2}{(r-s)^{k_1}} m(s, f, \lambda_E) + \frac{\beta_2}{(r-s)^{k_1}},$$

where $\beta_2 = \max\{c + 2^{k_1} c\beta_1, 2^{k_1} c\beta_1\}$. Combining this with (7.2), we get

$$\begin{aligned} m(r, f, \lambda_{W'}) \leq{} & \frac{2\alpha\beta_2}{(r-s)^{k_1+1}} T_s(r, f, \omega_S) \\ & + \frac{2\alpha\beta_2}{(r-s)^{k_1+1}} m(s, f, \lambda_E) + \frac{\alpha + 2\alpha\beta_2}{(r-s)^{k_1+1}}. \end{aligned} \tag{7.5}$$

There exists a positive constant $c' > 0$ such that $m(r, f, \lambda_W) \leq c' m(r, f, \lambda_Z) + m(r, f, \lambda_{W'})$ for all $f \in \mathcal{G}$. Combining this with (7.3) and (7.5), we get for all $f \in \mathcal{G}$,

$$m(r, f, \lambda_W) \leq \frac{\beta}{(r-s)^k} T_s(r, f, \omega_S) + \frac{\beta}{(r-s)^k} m(s, f, \lambda_E) + \frac{\beta}{(r-s)^k}.$$

Here we set $\beta = \alpha + c'\beta_1 + 2\alpha\beta_2$ and $k = k_1 + 1$. This and (7.4) imply $W \notin \mathcal{P}$.

Now in both cases above, we have $W \notin \mathcal{P}$. This is a contradiction. Thus $\mathcal{P} = \emptyset$. We have proved our lemma. □

§7.2. The case of semi-abelian varieties

We recall $E_{k,A,A/B} \subset P_{k,A}$ from Definition 2.1 and $\Pi(\mathcal{G})$ from Definition 3.18. We use the convention from (6.10) in the proof.

Lemma 7.3. *Let $\bar{A}$ be a smooth projective equivariant compactification. Let $\omega_{\bar{A}}$ be a smooth positive $(1,1)$-form on $\bar{A}$. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A)$ be an infinite set such that $\mathcal{F}$ satisfies Assumption 6.2. Let $B \subset A$ be a semi-abelian subvariety such that $B \in \Pi(\mathcal{F})$ and that $\varpi_B \circ f$ is non-constant for all $f \in \mathcal{F}$, where $\varpi_B \colon A \to A/B$ is the quotient. Let $k \in \mathbb{Z}_{\geq 1}$. Let $\lambda_{E_{k,A,A/B}} \geq 0$ be a Weil function for $E_{k,A,A/B} \subset P_{k,A}$. Then there exists $\sigma \in (0,1)$ with the following property: Let $s \in (\sigma, 1)$, $\varepsilon > 0$, and $\delta > 0$. Then there exists a positive constant $\beta > 0$ such that for all $f \in \mathcal{F}$, the estimate*

$$m(r, f_{P_{k,A}}, \lambda_{E_{k,A,A/B}}) \leq \varepsilon T_s(r, f, \omega_{\bar{A}}) + \beta$$

holds for all $r \in (s,1)$ outside some exceptional set of linear measure less than δ.

Proof. We first take $\sigma_0 \in (0,1)$ as follows. Let $\omega_{A/B}$ be an invariant positive $(1,1)$-form on A/B. For each $n \in \mathbb{Z}_{\geq 2}$, let $\mathcal{E}_n \subset \mathcal{F}$ be the set of $f \in \mathcal{F}$ such that

$$\sup_{z \in \mathbb{D}(1-\frac{1}{n})} |(\varpi_B \circ f)'(z)|_{\omega_{A/B}} \leq \frac{1}{n}.$$

Then we have $\mathcal{E}_2 \supset \mathcal{E}_3 \supset \cdots$. By the assumption $B \in \Pi(\mathcal{F})$, we may take $n_0 \in \mathbb{Z}_{\geq 2}$ such that $\mathcal{E}_{n_0}$ is finite. Since $\varpi_B \circ f$ is non-constant for all $f \in \mathcal{F}$, we may take $n_1 \geq n_0$ such that $\mathcal{E}_{n_1} = \emptyset$. We take $\sigma_0 \in (0,1)$ such that $1 - \frac{1}{n_1} \leq \sigma_0 < 1$. Then we have

$$\sup_{z \in \mathbb{D}(\sigma_0)} |(\varpi_B \circ f)'(z)|_{\omega_{A/B}} \geq 1 - \sigma_0$$

for all $f \in \mathcal{F}$. For each $f \in \mathcal{F}$, we take $w_f \in \overline{\mathbb{D}(\sigma_0)}$ such that

$$|(\varpi_B \circ f)'(w_f)|_{\omega_{A/B}} \geq 1 - \sigma_0. \tag{7.6}$$

Next we take $\sigma \in (\sigma_0, 1)$ as follows. Let $\omega_{P_{k-1,A}}$ be a smooth positive $(1,1)$-form on $P_{k-1,A}$. We note that by $B \in \Pi(\mathcal{F})$, we have $\{0\} \in \Pi(\mathcal{F})$. We first apply Lemma 6.3 to get $\sigma_1 \in (\sigma_0, 1)$ with the following property: For $s \in (\sigma_1, 1)$, $\varepsilon > 0$, and $\delta > 0$, there exists a positive constant $\mu_1 = \mu_1(s, \varepsilon, \delta) > 0$ such that for all $f \in \mathcal{F}$, we have

$$T_s(r, f_{P_{k-1,A}}, \omega_{P_{k-1,A}}) \leq \frac{\varepsilon}{2} T_s(r, f, \omega_{\bar{A}}) + \mu_1 \tag{7.7}$$

for all $r \in (s,1)$ outside some exceptional set of linear measure less than $\delta/2$. Let ω_A be an invariant positive $(1,1)$-form on A. By Lemma 6.5, we get $\sigma \in (\sigma_1, 1)$

with the following property: For $s \in (\sigma, 1)$, $\varepsilon > 0$, and $\delta > 0$, there exists a positive constant $\mu_2 = \mu_2(s, \varepsilon, \delta) > 0$ such that for all $f \in \mathcal{F}$ and $w \in \mathbb{D}(\sigma)$, we have

$$(7.8)\quad \int_0^{2\pi} \log |(f_{[k-1]} \circ \varphi_{w,r})'(re^{i\theta})|_{\omega_{A\times P_{k-1,A}}} \frac{d\theta}{2\pi} \leq \frac{\varepsilon}{2} T_s(r, f_{[k-1]}, \omega_{\bar{A}\times P_{k-1,A}}) + \mu_2$$

for all $r \in (s, 1)$ outside some exceptional set of linear measure less than $\delta/2$. Here we recall $\varphi_{w,r} \colon \mathbb{D}(r) \to \mathbb{D}(r)$ from (4.5).

Let $s \in (\sigma, 1)$, $\varepsilon > 0$, and $\delta > 0$. We set $\varepsilon' = \varepsilon \frac{s-\sigma}{s+\sigma}$. Let $f \in \mathcal{F}$. Given $r \in (s, 1)$, we set $g = f \circ \varphi_{w_f,r}$. Then we have $g_{[k-1]} = f_{[k-1]} \circ \varphi_{w_f,r}$. Hence by (7.7) and (7.8) both applied to ε', we get

$$(7.9)\quad \int_0^{2\pi} \log |(g_{[k-1]})'(re^{i\theta})|_{\omega_{A\times P_{k-1,A}}} \frac{d\theta}{2\pi} \leq \varepsilon' T_s(r, f, \omega_{\bar{A}}) + \mu_1' + \mu_2'$$

for all $r \in (s, 1)$ outside some exceptional set of linear measure less than δ. Here we set $\mu_1' = \mu_1(s, \varepsilon', \delta)$ and $\mu_2' = \mu_2(s, \varepsilon', \delta)$.

Let $p \colon A \times P_{k-1,A} \to A/B$ be the composite of the first projection and ϖ_B. Then there exists a positive constant $c > 0$ such that for all $v \in T(A \times P_{k-1,A})$ with $[v] \in A \times P_{k,A} \subset PT(A \times P_{k-1,A})$, we have

$$\lambda_{E_{k,A,A/B}}([v]_{P_{k,A}}) \leq \log\Big(\frac{|v|_{\omega_{A\times P_{k-1,A}}}}{|p'(v)|_{\omega_{A/B}}}\Big) + c,$$

where $[v]_{P_{k,A}} \in P_{k,A}$ is the image of $[v] \in A \times P_{k,A}$ under the second projection $A \times P_{k,A} \to P_{k,A}$. Hence for all $f \in \mathcal{F}$ and $r \in (s, 1)$, we have

$$m(r, g_{P_{k,A}}, \lambda_{E_{k,A,A/B}}) \leq \int_0^{2\pi} \log \frac{|(g_{[k-1]})'(re^{i\theta})|_{\omega_{A\times P_{k-1,A}}}}{|(p \circ g_{[k-1]})'(re^{i\theta})|_{\omega_{A/B}}} \frac{d\theta}{2\pi} + c.$$

Since $\lambda_{E_{k,A,A/B}} \geq 0$ and $g_{[k]} = f_{[k]} \circ \varphi_{w_f,r}$, Lemma 4.10 yields that

$$\begin{aligned} m(r, f_{P_{k,A}}, \lambda_{E_{k,A,A/B}}) &\leq \frac{r+\sigma}{r-\sigma} m(r, g_{P_{k,A}}, \lambda_{E_{k,A,A/B}}) \\ &\leq \frac{s+\sigma}{s-\sigma} m(r, g_{P_{k,A}}, \lambda_{E_{k,A,A/B}}) \end{aligned}$$

for all $r \in (s, 1)$. Hence we have

$$m(r, f_{P_{k,A}}, \lambda_{E_{k,A,A/B}}) \leq \frac{s+\sigma}{s-\sigma} \int_0^{2\pi} \log \frac{|(g_{[k-1]})'(re^{i\theta})|_{\omega_{A\times P_{k-1,A}}}}{|(p \circ g_{[k-1]})'(re^{i\theta})|_{\omega_{A/B}}} \frac{d\theta}{2\pi} + c\frac{s+\sigma}{s-\sigma}$$

for all $r \in (s,1)$. Combining this estimate with (7.9), we get

$$
\begin{aligned}
m(r, f_{P_{k,A}}, \lambda_{E_{k,A,A/B}}) \leq \varepsilon T_s(r, f, \omega_{\bar{A}}) \\
&+ \frac{s+\sigma}{s-\sigma} \int_0^{2\pi} \log \frac{1}{|(p \circ g_{[k-1]})'(re^{i\theta})|_{\omega_{A/B}}} \frac{d\theta}{2\pi} \\
&+ \frac{s+\sigma}{s-\sigma} (\mu_1' + \mu_2' + c)
\end{aligned} \tag{7.10}
$$

for all $r \in (s,1)$ outside some exceptional set of linear measure less than δ.

Next, by the subharmonicity of $\log |(\varpi_B \circ g)'|_{\omega_{A/B}}$, we get

$$
\begin{aligned}
\int_0^{2\pi} \log \frac{1}{|(p \circ g_{[k-1]})'(re^{i\theta})|_{\omega_{A/B}}} \frac{d\theta}{2\pi} &= \int_0^{2\pi} \log \frac{1}{|(\varpi_B \circ g)'(re^{i\theta})|_{\omega_{A/B}}} \frac{d\theta}{2\pi} \\
&\leq \log \frac{1}{|(\varpi_B \circ g)'(0)|_{\omega_{A/B}}}
\end{aligned}
$$

for all $r \in (s,1)$. By (4.6) and (7.6), we have

$$
|(\varpi_B \circ g)'(0)|_{\omega_{A/B}} = |(\varpi_B \circ f)'(w_f)|_{\omega_{A/B}} |\varphi_{w_f,r}'(0)| \geq \frac{(1-\sigma)(s-\sigma)}{s+\sigma}.
$$

Hence

$$
\int_0^{2\pi} \log \frac{1}{|(p \circ g_{[k-1]})'(re^{i\theta})|_{\omega_{A/B}}} \frac{d\theta}{2\pi} \leq \log\Big(\frac{s+\sigma}{(1-\sigma)(s-\sigma)} \Big) \tag{7.11}
$$

for all $r \in (s,1)$. Using (7.10) and (7.11) and letting

$$
\beta = \frac{s+\sigma}{s-\sigma} \Big(\mu_1' + \mu_2' + \log\Big(\frac{s+\sigma}{(1-\sigma)(s-\sigma)} \Big) + c \Big),
$$

we complete the proof. □

Lemma 7.4. *Let $k \in \mathbb{Z}_{\geq 1}$. Let $\bar{A}$ be a smooth projective equivariant compactification. Let $\omega_{\bar{A}}$ be a smooth positive $(1,1)$-form on $\bar{A}$. Let $W \subset P_{k,A}$ be a closed subscheme with a Weil function $\lambda_W \geq 0$. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A)$ be an infinite set such that $\mathcal{F}$ satisfies Assumption 6.2 and $\{0\} \in \Pi(\mathcal{F})$. Let $\Pi' \subset \Pi(\mathcal{F})$ be a subset such that $\mathrm{LIM}(\mathcal{F}_{P_{k,A}};\ \{E_{k,A,A/B}\}_{B \in \Pi'})$ exists, where $\mathcal{F}_{P_{k,A}} = (f_{P_{k,A}})_{f \in \mathcal{F}}$. Assume that $\mathrm{LIM}(\mathcal{F}_{P_{k,A}};\ \{E_{k,A,A/B}\}_{B \in \Pi'}) \not\subset \mathrm{supp}\, W$. Then there exist $\sigma \in (0,1)$ and an infinite subset $\mathcal{G} \subset \mathcal{F}$ with the following property: Let $s \in (\sigma, 1)$, $\varepsilon > 0$, $\delta > 0$. Then there exists a positive constant $\beta > 0$ such that, for all $f \in \mathcal{G}$, we have*

$$
m(r, f_{P_{k,A}}, \lambda_W) \leq \varepsilon T_s(r, f_A, \omega_{\bar{A}}) + \beta
$$

for all $r \in (s,1)$ outside some exceptional set of linear measure less than δ.

Proof. We first consider the case that the subset

$$\{f_{P_{k,A}};\ f \in \mathcal{F}\} \subset \mathrm{Hol}(\mathbb{D}, P_{k,A}) \tag{7.12}$$

is finite. We may choose an infinite subset $\mathcal{G} \subset \mathcal{F}$ by removing finite elements from $\mathcal{F}$ such that $f_{P_{k,A}}(\mathbb{D}) \not\subset \operatorname{supp} W$ for all $f \in \mathcal{G}$. Then our estimate is valid for this $\mathcal{G}$ and any $\sigma \in (0,1)$. Indeed, we just need to set

$$\beta = \max_{f\in\mathcal{G}} \sup_{r\in(s,1-\delta)} m(r, f_{P_{k,A}}, \lambda_W).$$

Hence in the following, we assume that the set (7.12) is infinite. Replacing $\mathcal{F}$ by its infinite subset, we may assume that the set (7.12) is infinite and the map $\mathcal{F} \ni f \mapsto f_{P_{k,A}} \in \mathrm{Hol}(\mathbb{D}, P_{k,A})$ is injective. Hence we may assume that the infinite indexed family $\mathcal{F}_{P_{k,A}}$ is an infinite subset of $\mathrm{Hol}(\mathbb{D}, P_{k,A})$.

There exists a positive constant $c > 0$ such that $\lambda_W \leq c\lambda_{\operatorname{supp} W} + c$. Hence we may assume that W is reduced. Let $\omega_{P_{k,A}}$ be a smooth positive $(1,1)$-form on $P_{k,A}$. We apply Lemma 7.2 to obtain $\sigma \in (0,1)$, $\beta_0 > 0$, $l \in \mathbb{Z}_{\geq 0}$, $\mathcal{G} \subset \mathcal{F}$, and a Zariski closed set $E \subset \bigcup_{B\in\Pi'} E_{k,A,A/B}$ with a Weil function $\lambda_E \geq 0$ such that

$$\begin{aligned} m(r, f_{P_{k,A}}, \lambda_W) \leq{}& \frac{\beta_0}{(r-\rho)^l} T_\rho(r, f_{P_{k,A}}, \omega_{P_{k,A}}) \\ &+ \frac{\beta_0}{(r-\rho)^l} m(\rho, f_{P_{k,A}}, \lambda_E) + \frac{\beta_0}{(r-\rho)^l} \end{aligned} \tag{7.13}$$

for all $\sigma < \rho < r < 1$ and $f \in \mathcal{G}$. We may take a finite subset $\Lambda \subset \Pi'$ such that letting $E_\Lambda = \bigcup_{B\in\Lambda} E_{k,A,A/B}$, we have $E \subset E_\Lambda$. Note that $\mathcal{G}$ satisfies Assumption 6.2. By removing finite elements from $\mathcal{G}$, we may assume that $\varpi_B \circ f$ is non-constant for all $B \in \Lambda$ and $f \in \mathcal{G}$. By enlarging $\sigma \in (0,1)$, if necessary, we may assume that σ fits in Lemma 7.3 for each $B \in \Lambda$. Note that $\{0\} \in \Pi(\mathcal{G})$. Hence, again by enlarging $\sigma \in (0,1)$, we may assume that σ fits in Lemma 6.3.

Now we fix $s \in (\sigma, 1)$, $\varepsilon > 0$, and $\delta > 0$. We set $\varepsilon' = \frac{\varepsilon\delta^l}{2^{2l+1}\beta_0}$. We take $f \in \mathcal{G}$. We apply Lemma 7.3 to get

$$m(\rho, f_{P_{k,A}}, \lambda_{E_\Lambda}) \leq \varepsilon' T_s(\rho, f, \omega_{\bar{A}}) + \beta_1$$

for all $\rho \in (s,1)$ outside some exceptional set of linear measure less than $\delta/4$. Hence we may take $\rho \in (s, s+\delta/4)$ which satisfies this property. Hence by (7.13), we get

$$m(r, f_{P_{k,A}}, \lambda_W) \leq \frac{4^l\beta_0}{\delta^l} T_s(r, f_{P_{k,A}}, \omega_{P_{k,A}}) + \varepsilon' \frac{4^l\beta_0}{\delta^l} T_s(r, f, \omega_{\bar{A}}) + \frac{4^l\beta_0(1+\beta_1)}{\delta^l}$$

for all $r \in (s+\delta/2, 1)$. We apply Lemma 6.3 to get

$$T_s(r, f_{P_{k,A}}, \omega_{P_{k,A}}) \leq \varepsilon' T_s(r, f_A, \omega_{\bar{A}}) + \mu$$

for all $r \in (s,1)$ outside some exceptional set of linear measure less than $\delta/2$. Combining these two estimates, we get the estimate of our lemma. Here we set $\beta = \frac{4^l \beta_0(1+\beta_1+\mu)}{\delta^l}$. □

§8. Proof of Proposition 3.20

In this section, we prove Proposition 3.20 after preparing three lemmas. Let $\bar{A}$ be a smooth equivariant compactification of a semi-abelian variety A. Given an infinite indexed family $\mathcal{F}$ in $\mathrm{Hol}(\mathbb{D}, A)$, we set

$$\begin{aligned}\mathcal{I}(\mathcal{F}) = \big\{&D;\text{ irreducible component of } \partial A \text{ s.t.}\\ &\exists \mathcal{F}' \text{ a subfamily of } \mathcal{F} \text{ such that } \mathcal{F}' \to D\big\}.\end{aligned}$$

For an irreducible component $D \subset \partial A$, let $I_D \subset A$ be the isotropy group for D. Then $I_D = \mathbb{G}_m$. Set $I_{\mathcal{F}} = I_{D_1} \cdots I_{D_l} \subset A$ where $\{D_1, \ldots, D_l\} = \mathcal{I}(\mathcal{F})$. In the next lemma, for $\tau \in A$, we use the same notation for the map $\tau\colon \bar{A} \to \bar{A}$ defined by $a \mapsto a + \tau$.

Lemma 8.1. *Let $\mathcal{F} = (f_i)_{i\in I}$ be an infinite indexed family in* $\mathrm{Hol}(\mathbb{D}, A)$*. For $i \in I$, there exists $\tau_i \in I_{\mathcal{F}}$ such that $(\tau_i \circ f_i)_{i\in I}$ satisfies Assumption 6.2.*

In the following proof, we use the notation of the 1-dimensional unlimited Hausdorff content defined as follows. Let $K \subset \mathbb{C}$. We set

$$\begin{aligned}C^1_H(K) = \inf\big\{\textstyle\sum_j r_j;\ \exists \text{ a countable cover of } K\\ \text{by closed discs with radii } r_j > 0\big\}.\end{aligned} \tag{8.1}$$

Proof of Lemma 8.1. If $\mathcal{I}(\mathcal{F}) = \emptyset$, then our lemma is trivial. Hence we assume $\mathcal{I}(\mathcal{F}) \neq \emptyset$. We take $D \in \mathcal{I}(\mathcal{F})$. For each $i \in I$, we shall show that there exists $\tau_{i,D} \in I_D$ such that the indexed family $\mathcal{F}' = (\tau_{i,D} \circ f_i)_{i\in I}$ satisfies the following two properties:

- $D \notin \mathcal{I}(\mathcal{F}')$.
- If $E \notin \mathcal{I}(\mathcal{F})$, then $E \notin \mathcal{I}(\mathcal{F}')$.

We are going to construct $\tau_{i,D}$. For each $E \notin \mathcal{I}(\mathcal{F})$, there exist $s_E \in (0,1)$, $\gamma_E > 0$, an open neighborhood $U_E \subset \bar{A}$ of E, and a finite subset $J_E \subset I$ such that

$$C^1_H(\mathbb{D}(s_E)\backslash f_i^{-1}(U_E)) \geq \gamma_E \tag{8.2}$$

for all $i \in I\backslash J_E$ (cf. Remark 4.7). We may assume moreover that if $E \cap D = \emptyset$, then $U_E \Subset \bar{A} - D$. We set $s = \max_{E\notin\mathcal{I}(\mathcal{F})}\{s_E\}$, $\gamma = \min_{E\notin\mathcal{I}(\mathcal{F})}\{\gamma_E\}$, and $\gamma' = \gamma/2$.

We apply Lemma A.11 to get $p\colon W \to D$, where $W \subset \bar{A}$ is a Zariski open neighborhood of D and W is a total space of a line bundle over D. Let $\|\cdot\|$ be a smooth Hermitian metric on this line bundle $p\colon W \to D$. We set $U_D = \{x \in W;\ \|x\| < 1\}$. Then $U_D \subset \bar{A}$ is an open neighborhood of D. By replacing $\|\cdot\|$ if necessary, we may assume that $U_D \cap U_E = \emptyset$ if $D \cap E = \emptyset$. Note that I_D acts on W by the scalar product (cf. Lemma A.11). Hence for $a \in I_D$, we may define its norm $|a|$ by $|a| = \|a\cdot x\|/\|x\|$ for $x \in W - D$. We set $U'_D = \{x \in W;\ \|x\| < 1/2\}$. Then $U'_D \Subset U_D$.

Let $i \in I$. We choose $\tau_{i,D} \in I_D$ in the two cases below.

Case 1: $f_i(z) \in U'_D$ for γ'-almost all $z \in \mathbb{D}(s)$. We set

$$\sigma_i = \sup\bigl\{|\tau|;\ \tau \in I_D \text{ and } \tau \circ f_i(z) \in U'_D \text{ for } \gamma'\text{-almost all } z \in \mathbb{D}(s)\bigr\}.$$

Note that if $\tau \in I_D$ satisfies $|\tau| \min_{z\in\overline{\mathbb{D}(s)}} \|f_i(z)\| \geq 1/2$, then $\tau \circ f_i(z) \notin U'_D$ for all $z \in \mathbb{D}(s)$. Hence $\sigma_i < \infty$. We take $\tau_{i,D} \in I_D$ such that $|\tau_{i,D}| = \sigma_i$. Then we have

$$|\tau_{i,D}| \geq 1. \tag{8.3}$$

This follows from $\sigma_i \geq |e_{I_D}|$, where e_{I_D} is the identity element of I_D.

Case 2: Otherwise, we set $\tau_{i,D} = e_{I_D}$. We set $\Omega_i = \mathbb{D}(s) \cap (\tau_{i,D} \circ f_i)^{-1}(U_D)$. We define $G \subset I$ by the set of $i \in I$ such that f_i belongs to Case 1. By the definition of $\tau_{i,D}$, we have

$$C^1_H(\mathbb{D}(s)\backslash\Omega_i) < \gamma' \tag{8.4}$$

for all $i \in G$. We show that $\mathcal{F}' = (\tau_{i,D} \circ f_i)_{i\in I}$ satisfies our requirement. We first observe that $D \notin \mathcal{I}(\mathcal{F}')$. To show this, let $U''_D = \{x \in W;\ \|x\| < 1/4\}$ so that $U''_D \Subset U'_D$. Then we have $C^1_H(\mathbb{D}(s)\backslash(\tau_{i,D} \circ f_i)^{-1}(U''_D)) \geq \gamma'$ for all $i \in I$. Hence $D \notin \mathcal{I}(\mathcal{F}')$.

Next we take $E \notin \mathcal{I}(\mathcal{F})$. We first consider the case $E \cap D = \emptyset$. By $U_E \cap U_D = \emptyset$ and (8.4), we have $C^1_H(\mathbb{D}(s) \cap (\tau_{i,D} \circ f_i)^{-1}(U_E)) < \gamma'$, provided $i \in G$. By (8.2), we have $C^1_H(\mathbb{D}(s_E)\backslash(\tau_{i,D} \circ f_i)^{-1}(U_E)) \geq \gamma_E$ for all $i \in I\backslash(G \cup J_E)$. Hence $E \notin \mathcal{I}(\mathcal{F}')$ as desired.

In the following, we consider the case $E \cap D \neq \emptyset$. We take an open neighborhood $U'_E \subset U_E$ of E so that $U'_E \cap U_D \subset U_D - p^{-1}(p(\overline{U_D} - U_E))$. Then for $\tau \in I_D$ with $|\tau| \geq 1$, we have

$$U'_E \cap U_D \subset \tau(U_E). \tag{8.5}$$

By $\tau_{i,D} \circ f_i(\Omega_i) \subset U_D$, we have

$$(\tau_{i,D} \circ f_i)^{-1}(U'_E) \cap \Omega_i \subset (\tau_{i,D} \circ f_i)^{-1}(U'_E \cap U_D).$$

By (8.3) and (8.5), we have

$$(\tau_{i,D}\circ f_i)^{-1}(U'_E\cap U_D)\subset(\tau_{i,D}\circ f_i)^{-1}(\tau_{i,D}(U_E))=f_i^{-1}(U_E).$$

Hence we get

$$(\tau_{i,D}\circ f_i)^{-1}(U'_E)\cap\Omega_i\subset f_i^{-1}(U_E). \tag{8.6}$$

By (8.2) and the definitions of s and γ, we have $C^1_H(\mathbb{D}(s)\backslash f_i^{-1}(U_E))\geq\gamma$ for all $i\in I\backslash J_E$. Then (8.4) yields that $C^1_H(\Omega_i\backslash f_i^{-1}(U_E))\geq\gamma'$ for all $i\in G\backslash J_E$. Hence by (8.6), we have $C^1_H(\Omega_i\backslash(\tau_{i,D}\circ f_i)^{-1}(U'_E))\geq\gamma'$, thus $C^1_H(\mathbb{D}(s)\backslash(\tau_{i,D}\circ f_i)^{-1}(U'_E))\geq\gamma'$ for all $i\in G\backslash J_E$. By (8.2), the same holds for all $i\in I\backslash J_E$. Thus $E\notin\mathcal{I}(\mathcal{F}')$ as desired.

Now starting from $\mathcal{F}$, we set $\mathcal{F}_1=\mathcal{F}'$. Inductively, we get $\mathcal{F}_2,\mathcal{F}_3,\dots$ by $\mathcal{F}_{k+1}=\mathcal{F}'_k$. Then by the above consideration, we have $\mathcal{I}(\mathcal{F}_k)\supsetneqq\mathcal{I}(\mathcal{F}_{k+1})$. Hence we get $\mathcal{I}_{\mathcal{F}_n}=\emptyset$ for some n. Then $\mathcal{F}_n$ satisfies Assumption 6.2. $\square$

Lemma 8.2. *Let $p\colon W\to V$ be a vector bundle, where V is a smooth projective variety. We consider $V\subset W$ by the zero-section. Let $\mathcal{F}=(f_i)_{i\in I}$ be an infinite indexed family in* $\mathrm{Hol}(\mathbb{D},W)$. *Assume that $\mathcal{F}\to V$ and that $\{p\circ f_i\}_{i\in I}$ converges uniformly on compact subsets of $\mathbb{D}$ to $g\colon\mathbb{D}\to V$. Then $\mathcal{F}$ converges uniformly on compact subsets of $\mathbb{D}$ to g.*

Proof. We equip W with a smooth Hermitian metric. We denote by $\|\cdot\|_W$ the associated Euclidean norm on each fiber of $p\colon W\to V$. Let $O_1,\dots,O_n\subset V$ be an open set in the standard topology of V such that $V\subset O_1\cup\cdots\cup O_n$. Assume that $W|_{O_j}=\mathbb{C}^k\times O_j$ and that letting $\varphi_j\colon W|_{O_j}\to\mathbb{C}^k$ be the first projection, there exists a positive constant $c_j>1$ such that

$$\frac{1}{c_j}\|\varphi_j(x)\|\leq\|x\|_W\leq c_j\|\varphi_j(x)\| \tag{8.7}$$

for all $x\in W|_{O_j}$. Let δ be a distance function on V which determines the standard topology on V. Let $O'_j\Subset O_j$ be an open set such that $V\subset O'_1\cup\cdots\cup O'_l$. There exists a positive constant $\alpha>0$ such that for all $j=1,\dots,n$, we have

$$\delta(\overline{O'_j},V-O_j)>\alpha. \tag{8.8}$$

Let $\tau\colon V\times W\to\mathbb{R}_{\geq0}$ be defined by

$$\tau(x,y)=\delta(x,p(y))+\|y\|_W.$$

Let $r\in(0,\alpha/2)$ and $s\in(0,1)$. We claim that there exists a finite set $E\subset I$ such that, for all $i\in I\backslash E$ and $z\in\mathbb{D}(s)$, we have

$$\tau(g(z),f_i(z))<r. \tag{8.9}$$

We prove this. Let $s' \in (s,1)$. Since $\{p\circ f_i\}_{i\in I}$ converges uniformly on compact subsets of $\mathbb{D}$ to g, there exists a finite subset $E_1 \subset I$ such that for all $i \in I\backslash E_1$ and $z \in \mathbb{D}(s')$, we have

$$\delta(g(z), p\circ f_i(z)) < r/2. \tag{8.10}$$

Set $c = \max_{1\leq j\leq n}\{c_j\}$ and

$$U = \{x \in W;\ \|x\|_W < r/2c^2\}.$$

Then since $\|\cdot\|_W\colon W \to \mathbb{R}_{\geq 0}$ is continuous, $U \subset W$ is an open neighborhood of $V \subset W$. We take $\gamma \in (0, \frac{s'-s}{2})$ such that if $\Delta \subset \mathbb{D}(s')$ is a disc of radius γ, then for all $z, z' \in \Delta$, we have

$$\delta(g(z), g(z')) < \alpha/2. \tag{8.11}$$

By $\mathcal{F} \to V$, there exists a finite subset $E_2 \subset I$ such that, for all $i \in I\backslash E_2$, we have

$$C_H^1(\mathbb{D}(s')\backslash f_i^{-1}(U)) < \gamma', \tag{8.12}$$

where $\gamma' = \gamma/4$. Set $E = E_1 \cup E_2$. For all $i \in I\backslash E$ and $z \in \mathbb{D}(s)$, we claim

$$\|f_i(z)\|_W < r/2. \tag{8.13}$$

To show this, we take $z \in \mathbb{D}(s)$ and $i \in I\backslash E$. Given $\rho \in (0,\gamma)$, we denote by Δ_ρ the disc of radius ρ centered at z. By $\gamma < \frac{s'-s}{2}$, we have $\overline{\Delta_\rho} \subset \mathbb{D}(s')$. We take O_j' such that $g(z) \in O_j'$. To show $p\circ f_i(\Delta_\rho) \subset O_j$, we take $w \in \Delta_\rho$. Then by (8.11), we have $\delta(g(z), g(w)) < \alpha/2$. By $w \in \mathbb{D}(s')$, the estimate (8.10) yields $\delta(g(w), p\circ f_i(w)) < \alpha/4$. Hence we have $\delta(g(z), p\circ f_i(w)) < \alpha$. Hence by (8.8), we have $p\circ f_i(w) \in O_j$. This shows $p\circ f_i(\Delta_\rho) \subset O_j$, thus $f_i(\Delta_\rho) \subset W|_{O_j}$. By (8.12), there exists $\rho \in (0,\gamma)$ such that $f_i(\partial\Delta_\rho) \subset U$. By (8.7), we have $\|\varphi_j\circ f_i(w)\| < r/2c_j$ for all $w \in \partial\Delta_\rho$. Since $\|\varphi_j\circ f_i\|$ is subharmonic on Δ_ρ, the maximum principle yields $\|\varphi_j\circ f_i(z)\| < r/2c_j$. By (8.7), we get (8.13). Thus by (8.10) and (8.13), we get (8.9).

For $r > 0$ and $x \in V$, we set $B_x(r) = \{y \in W;\ \tau(x,y) < r\}$. Note that $\tau(x,y)$ is continuous with respect to $y \in W$. Hence $B_x(r) \subset W$ is an open subset. Let d be a distance function on W which induces the topology on W. Let $\varepsilon > 0$. Then we may take a positive constant $r_\varepsilon > 0$ such that

$$\sup_{y\in B_x(r_\varepsilon)} d(x,y) < \varepsilon \tag{8.14}$$

for all $x \in V$. To prove this, we note that for each $x \in V$, we may take $r_{\varepsilon,x} > 0$ such that $\sup_{y\in B_x(r_{\varepsilon,x})} d(x,y) < \varepsilon/4$. We consider the open covering $\{B_x(r_{\varepsilon,x}/2)\}_{x\in V}$

of $V \subset W$. Since V is compact, there exist $x_1, \ldots, x_l \in V$ such that $V \subset B_{x_1}(r_{\varepsilon,x_1}/2) \cup \cdots \cup B_{x_l}(r_{\varepsilon,x_l}/2)$. We set $r_\varepsilon = \min_{1\leq j\leq l}\{r_{\varepsilon,x_j}/2\}$. Let $x \in V$. Then there exists x_j such that $x \in B_{x_j}(r_{\varepsilon,x_j}/2)$. Let $y \in B_x(r_\varepsilon)$. Then we have

$$\tau(x_j, y) \leq \delta(x_j, x) + \tau(x, y) < r_{\varepsilon,x_j}.$$

Hence $y \in B_{x_j}(r_{\varepsilon,x_j})$. Thus $d(x, y) \leq d(x_j, x) + d(x_j, y) < \varepsilon/2$. This proves (8.14).

Now we prove that $\mathcal{F}$ converges uniformly on compact subsets of $\mathbb{D}$ to g. Let $s \in (0, 1)$ and let $\varepsilon > 0$. Then by (8.9) applied to $r = \min\{\alpha/3, r_\varepsilon\}$, there exists a finite subset $E \subset I$ such that for all $i \in I\backslash E$, we have

$$\tau(g(z), f_i(z)) < r_\varepsilon$$

for all $z \in \mathbb{D}(s)$. By (8.14), we have $d(g(z), f_i(z)) < \varepsilon$ for all $z \in \mathbb{D}(s)$ and $i \in I\backslash E$. Thus $\mathcal{F}$ converges uniformly on compact subsets of $\mathbb{D}$ to g. $\square$

Lemma 8.3. *Let $\bar{A}$ be a smooth equivariant compactification of A and let $\omega_{\bar{A}}$ be a smooth positive $(1, 1)$-form on $\bar{A}$. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A)$ be an infinite set of holomorphic maps which satisfies Assumption 6.2. Assume that for all $0 < r < 1$, we have*

$$\sup_{f\in\mathcal{F}}\left\{\int_{\mathbb{D}(r)} f^*\omega_{\bar{A}}\right\} < \infty.$$

Then there exists an infinite subset $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{G}$ converges uniformly on compact subsets of $\mathbb{D}$ to some $g\colon \mathbb{D} \to A$.

Proof. The proof divides into three steps.

Step 1. We first consider the case that A is an abelian variety. In this case, we have $\bar{A} = A$. We may assume that ω_A is a positive invariant $(1, 1)$-form. Let $\sigma \in (0, 1)$. We shall show that $\mathcal{F}$ is equicontinuous on $\mathbb{D}(\sigma)$. Namely we prove

$$\sup_{w\in\mathbb{D}(\sigma)} \sup_{f\in\mathcal{F}}\{|f'(w)|_{\omega_A}\} < \infty.$$

Let $s \in (\sigma, 1)$. We set

$$c = \sup_{f\in\mathcal{F}}\left\{\int_{\mathbb{D}(s)} f^*\omega_A\right\}.$$

Let $w \in \mathbb{D}(\sigma)$. Recall $\varphi_{w,s}\colon \mathbb{D}(s) \to \mathbb{D}(s)$ from (4.5). Then

$$\int_{\mathbb{D}(s)} (f \circ \varphi_{w,s})^*\omega_A = \int_{\mathbb{D}(s)} f^*\omega_A \leq c.$$

For $r \in (0, s)$, we set

$$u(r) = \int_0^{2\pi} |(f \circ \varphi_{w,s})'(re^{i\theta})|^2_{\omega_A}\, d\theta.$$

Since $|(f\circ\varphi_{w,s})'(z)|^2_{\omega_A}$ is subharmonic on $\mathbb{D}(s)$, we have $2\pi|(f\circ\varphi_{w,s})'(0)|^2_{\omega_A} \leq u(r)$. Hence

$$\int_{\mathbb{D}(s)} (f\circ\varphi_{w,s})^*\omega_A = \int_0^s u(r)r\,dr \geq \pi s^2|(f\circ\varphi_{w,s})'(0)|^2_{\omega_A}.$$

This shows

$$|(f\circ\varphi_{w,s})'(0)|_{\omega_A} \leq \sqrt{\frac{c}{\pi s^2}}.$$

Hence by (4.6), we have

$$|f'(w)|_{\omega_A} \leq \frac{s+\sigma}{s-\sigma}\sqrt{\frac{c}{\pi s^2}}.$$

Hence $\mathcal{F}$ is equicontinuous on $\mathbb{D}(\sigma)$. By the Arzelá–Ascoli theorem, there exists a subsequence $\mathcal{G}\subset\mathcal{F}$ such that $\mathcal{G}$ converges uniformly on compact subsets of $\mathbb{D}$ to some $g\colon\mathbb{D}\to A$.

Step 2. We consider the case $A=(\mathbb{G}_m)^k$. We are given an equivariant compactification $\bar{A}$. Then by Lemma 6.4, there exists $s_0\in(0,1)$ such that for all $r\in(s_0,1)$, we have

$$\sup_{f\in\mathcal{F}}\{m(r,f,\lambda_{\partial A})\}<\infty. \tag{8.15}$$

Let $p_i\colon A\to\mathbb{G}_m$ be the i-th projection. Let $\tau_i\colon W_i\to\bar{A}$ be a smooth modification such that $p_i\colon A\to\mathbb{G}_m$ extends to $\overline{p_i}\colon W_i\to\mathbb{P}^1$. Then we have

$$\operatorname{supp}\overline{p_i}^*((0)+(\infty))\subset\operatorname{supp}\tau_i^*\partial A.$$

Hence by (8.15), we get

$$\sup_{f\in\mathcal{F}}\{m(r,p_i\circ f,\lambda_{(0)+(\infty)})\}<\infty \tag{8.16}$$

for all $r\in(s_0,1)$. Note that $m(r,p_i\circ f,\lambda_{(0)+(\infty)})$ is an increasing function on r. Hence (8.16) holds for all $r\in(0,1)$. Thus by Montel's theorem (cf. [30, Thm. 1.6, p. 230] or [13, p. 233]), there exists an infinite subset $\mathcal{G}_1\subset\mathcal{F}$ such that $(p_1\circ f)_{f\in\mathcal{G}_1}$ converges uniformly on compact subsets of $\mathbb{D}$ to $g_1\colon\mathbb{D}\to\mathbb{C}$. By (8.16), we have $g_1\not\equiv 0$. Hence we have $g_1\colon\mathbb{D}\to\mathbb{G}_m$. By the same argument, there exists an infinite subset $\mathcal{G}_2\subset\mathcal{G}_1$ such that $(p_2\circ f)_{f\in\mathcal{G}_2}$ converges uniformly on compact subsets of $\mathbb{D}$ to $g_2\colon\mathbb{D}\to\mathbb{C}$. Again we have $g_2\colon\mathbb{D}\to\mathbb{G}_m$. In this way, we get $\mathcal{G}_1\supset\mathcal{G}_2\supset\cdots\supset\mathcal{G}_k$. We set $\mathcal{G}=\mathcal{G}_k$. Then $\mathcal{G}$ is an infinite set and converges uniformly on compact subsets of $\mathbb{D}$ to $(g_1,\dots,g_k)\colon\mathbb{D}\to(\mathbb{G}_m)^k$.

Step 3. We consider the general case with $0 \to T \to A \xrightarrow{p} A_0 \to 0$. By Lemma A.6, there exists an equivariant compactification $\overline{T}$ such that $\bar{A} = (\overline{T} \times A)/T$. We take the universal covering $\widetilde{A_0} \to A_0$. We consider $0 \to T \to A \times_{A_0} \widetilde{A_0} \xrightarrow{\tilde{p}} \widetilde{A_0} \to 0$, which has a splitting. Then we have $\bar{A} \times_{A_0} \widetilde{A_0} = \overline{T} \times \widetilde{A_0}$. Let $q\colon \bar{A} \times_{A_0} \widetilde{A_0} \to \overline{T}$ be the first projection and $\pi\colon \bar{A} \times_{A_0} \widetilde{A_0} \to \bar{A}$ be the natural map. We continue to write $p\colon \bar{A} \to A_0$ and $\tilde{p}\colon \bar{A} \times_{A_0} \widetilde{A_0} \to \widetilde{A_0}$:

$$\begin{tikzcd}
\bar{A} \arrow[d, "p"'] & \bar{A} \times_{A_0} \widetilde{A_0} \arrow[l, "\pi"'] \arrow[r, "q"] \arrow[d, "\tilde{p}"] & \overline{T} \\
A_0 & \widetilde{A_0}. \arrow[l]
\end{tikzcd}$$

Now we claim that there exists an infinite subset $\mathcal{F}' \subset \mathcal{F}$ such that $(p \circ f)_{f \in \mathcal{F}'}$ converges uniformly on compact subsets of $\mathbb{D}$. Indeed, if $\{p \circ f;\ f \in \mathcal{F}\} \subset \mathrm{Hol}(\mathbb{D}, A_0)$ is a finite subset, we take an infinite subset $\mathcal{F}' \subset \mathcal{F}$ such that $p \circ f$ is all the same for all $f \in \mathcal{F}'$; otherwise we apply the first step for the infinite subset $\{p \circ f;\ f \in \mathcal{F}\} \subset \mathrm{Hol}(\mathbb{D}, A_0)$ to get an infinite subset $\mathcal{F}' \subset \mathcal{F}$ such that $(p \circ f)_{f \in \mathcal{F}'}$ converges uniformly on compact subsets of $\mathbb{D}$. For each $f \in \mathcal{F}'$, we take a lifting $\tilde{f}\colon \mathbb{D} \to A \times_{A_0} \widetilde{A_0}$. We may assume that $(\tilde{p} \circ \tilde{f})_{f \in \mathcal{F}'}$ converges uniformly on compact subsets of $\mathbb{D}$ to $h\colon \mathbb{D} \to \widetilde{A_0}$. For each fixed $r \in (0,1)$, there exists a compact set $K_r \subset \widetilde{A_0}$ such that $h(\overline{\mathbb{D}(r)}) \subset K_r^o$, where K_r^o is the interior of K_r. Then for all $f \in \mathcal{F}'$ with finite exception, we have $\tilde{p} \circ \tilde{f}(\overline{\mathbb{D}(r)}) \subset K_r^o$. Let $\omega_{\overline{T}}$ be a smooth positive $(1,1)$-form on $\overline{T}$. Note that there exists a positive constant $c_r > 0$ such that $q^* \omega_{\overline{T}} \le c_r \pi^* \omega_{\bar{A}}$ on $\tilde{p}^{-1}(K_r)$. Hence we have

$$\sup_{f \in \mathcal{F}'} \left\{ \int_{\mathbb{D}(r)} (q \circ \tilde{f})^* \omega_{\overline{T}} \right\} < \infty.$$

Next we claim that $(q \circ \tilde{f})_{f \in \mathcal{F}'}$ satisfies Assumption 6.2. Let $D \subset \partial T$ be an irreducible component. Then $(D \times A)/T$ is an irreducible component of ∂A with $\pi(q^{-1}(D)) = (D \times A)/T$. Since $\mathcal{F}'$ satisfies Assumption 6.2, there exist an open neighborhood $(D \times A)/T \subset U_0 \subset \bar{A}$, $s_0 \in (0,1)$, $\gamma_0 > 0$, and a finite subset $\mathcal{E} \subset \mathcal{F}'$ such that

$$C_H^1(\mathbb{D}(s_0) \backslash f^{-1}(U_0)) \ge \gamma_0$$

for all $f \in \mathcal{F}' \backslash \mathcal{E}$ (cf. Remark 4.7). Note that $q(\tilde{p}^{-1}(K_{s_0}) \backslash \pi^{-1}(U_0)) \subset \overline{T}$ is compact. Set $W = \overline{T} \backslash q(\tilde{p}^{-1}(K_{s_0}) \backslash \pi^{-1}(U_0))$. Then W is an open neighborhood of D. We have $q^{-1}(W) \cap \tilde{p}^{-1}(K_{s_0}) \subset \pi^{-1}(U_0)$. Hence for all $f \in \mathcal{F}' \backslash \mathcal{E}$ with $\tilde{p} \circ \tilde{f}(\overline{\mathbb{D}(s_0)}) \subset K_{s_0}^o$, we have

$$C_H^1(\mathbb{D}(s_0) \backslash (q \circ \tilde{f})^{-1}(W)) \ge \gamma_0.$$

Thus $(q\circ\tilde{f})_{f\in\mathcal{F}'}$ satisfies Assumption 6.2. Hence there exists an infinite subset $\mathcal{G}\subset\mathcal{F}'$ such that $(q\circ\tilde{f})_{f\in\mathcal{G}}$ converges uniformly on compact subsets of $\mathbb{D}$. Indeed, if $\{q\circ\tilde{f};\ f\in\mathcal{F}'\}\subset \mathrm{Hol}(\mathbb{D},T)$ is a finite subset, we take an infinite subset $\mathcal{G}\subset\mathcal{F}'$ such that $q\circ\tilde{f}$ is all the same for all $f\in\mathcal{G}$; otherwise we apply the second step for the infinite subset $\{q\circ\tilde{f};\ f\in\mathcal{F}'\}\subset\mathrm{Hol}(\mathbb{D},T)$ to get an infinite subset $\mathcal{G}\subset\mathcal{F}'$ such that $(q\circ\tilde{f})_{f\in\mathcal{G}}$ converges uniformly on compact subsets of $\mathbb{D}$. Then $\mathcal{G}$ converges uniformly on compact subsets of $\mathbb{D}$. □

Proof of Proposition 3.20. We note that Z is an irreducible Zariski closed set of $P_{k+1,A}$. In the following, we fix $l\geq 1$ such that the natural map $T_{k+l}\to Z$ is surjective under the map $P_{k+l,A}\to P_{k+1,A}$. We divide the proof into the following several steps.

Step 1. We are given a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ and a smooth equivariant compactification $\bar{A}$. For a while, we assume furthermore that $\bar{A}$ is projective. Replacing $(f_n)_{n\in\mathbb{N}}$ by its subsequence, we may assume that the sequence consists of distinct elements of $\mathcal{F}$, for otherwise the existence of a convergent subsequence is obvious. Replacing $(f_n)_{n\in\mathbb{N}}$ by its subsequence, we may assume that for every irreducible component $D\subset\partial A$, we have either (1) $(f_n)_{n\in\mathbb{N}}\to D$, or (2) $(f_{n_k})_{k\in\mathbb{N}}\not\to D$ for every subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$. This is achieved as follows. Let $D_1,\dots,D_k$ be all irreducible components of ∂A. We define a subsequence $\mathcal{G}_1$ of (f_n) as follows. If (f_n) contains a subsequence $(f_{n'})$ such that $(f_{n'})\to D_1$, then we set $\mathcal{G}_1=(f_{n'})$. Otherwise, we set $\mathcal{G}_1=(f_n)$. If $\mathcal{G}_1$ contains a subsequence $(f_{n''})$ such that $(f_{n''})\to D_2$, then we set $\mathcal{G}_2=(f_{n''})$. Otherwise we set $\mathcal{G}_2=\mathcal{G}_1$. Continue this process to get $\mathcal{G}_k$. Then $\mathcal{G}_k$ satisfies our requirement. We replace (f_n) by $\mathcal{G}_k$.

We denote by $\mathcal{I}$ the set of all irreducible components D_i of ∂A such that $(f_n)\to D_i$. We set $V=\bigcap_{D_i\in\mathcal{I}}D_i$. When $\mathcal{I}=\emptyset$, we read $V=\bar{A}$. By Lemma 3.5, we have

$$(f_n)\to V. \tag{8.17}$$

Step 2. We may assume that $\{0\}\in\Pi((f_n)_{n\in\mathbb{N}})$, for otherwise the existence of a convergent subsequence is obvious. We apply Lemma 8.1 to get an element τ_n of the isotropy group for V such that $(\tau_n\circ f_n)_{n\in\mathbb{N}}$ satisfies Assumption 6.2. We take $p\colon W\to V$ from Lemma A.11, where $W\subset\bar{A}$ is the total space of the vector bundle over V. Then we have

$$p\circ\tau_n\circ f_n=p\circ f_n \tag{8.18}$$

for all $n\in\mathbb{N}$ in $\mathrm{Hol}(\mathbb{D},V)$.

We set $S=P_{k,A}$. Then we have $P_{k+1,A}\subset S_{1,A}$. Since all elements of $\mathcal{F}$ are non-constant, $\tau_n\circ f_n\colon\mathbb{D}\to A$ is non-constant for all $n\in\mathbb{N}$. Hence we get

$(\tau_n \circ f_n)_{[k]} \colon \mathbb{D} \to A \times S$, which yields

$$((\tau_n \circ f_n)_{[k]})_{S_{1,A}} \colon \mathbb{D} \to S_{1,A}.$$

Then under the inclusion $\iota \colon P_{k+1,A} \hookrightarrow S_{1,A}$, we have $((\tau_n \circ f_n)_{[k]})_{S_{1,A}} = \iota \circ (\tau_n \circ f_n)_{P_{k+1,A}}$. We note $Z \subset P_{k+1,A} \subset S_{1,A}$.

Now we claim that $(((\tau_n \circ f_n)_{[k]})_{S_{1,A}})_{n\in\mathbb{N}} \Rightarrow Z$. Indeed, we have

$$(\tau_n \circ f_n)_{P_{k+l,A}} = (f_n)_{P_{k+l,A}} \tag{8.19}$$

for all $n \in \mathbb{N}$ (cf. (2.14)). By $\mathcal{F}_{P_{k+l,A}} \Rightarrow T_{k+l}$ and Remark 3.4, we have

$$((f_n)_{P_{k+l,A}})_{n\in\mathbb{N}} \Rightarrow T_{k+l}.$$

Hence $((\tau_n \circ f_n)_{P_{k+l,A}})_{n\in\mathbb{N}} \Rightarrow T_{k+l}$. Thus by Lemma 3.15, we get

$$((\tau_n \circ f_n)_{P_{k+1,A}})_{n\in\mathbb{N}} \Rightarrow Z.$$

Hence again by Lemma 3.15, we get $(((\tau_n \circ f_n)_{[k]})_{S_{1,A}})_{n\in\mathbb{N}} \Rightarrow Z$.

In the following, we are going to prove that there exists a subsequence $(\tau_{n_k} \circ f_{n_k})_{k\in\mathbb{N}}$ which converges uniformly on compact subsets on $\mathbb{D}$ to a holomorphic map $\mathbb{D} \to A$. Set

$$\mathcal{G} = \{\tau_n \circ f_n;\ n \in \mathbb{N}\} \subset \mathrm{Hol}(\mathbb{D}, A).$$

If $\mathcal{G}$ is finite, the existence of such $(\tau_{n_k} \circ f_{n_k})_{k\in\mathbb{N}}$ is obvious. Hence in the following, we assume that $\mathcal{G}$ is infinite. We note the following:

- $\{0\} \in \Pi(\mathcal{G})$.
- $\mathcal{G}$ satisfies Assumption 6.2.
- $((\varphi_{[k]})_{S_{1,A}})_{\varphi\in\mathcal{G}} \Rightarrow Z$, where $\varphi_{[k]} \colon \mathbb{D} \to A \times S$.

These properties follow from the discussion of this step (Step 2).

Step 3. Let $\omega_{\bar{A}}$ and ω_S be smooth positive $(1,1)$-forms on $\bar{A}$ and S, respectively. We apply Lemma 6.3 to get $\sigma_1 \in (0,1)$ with the following property: Let $s \in (\sigma_1, 1)$, $\varepsilon > 0$, $\delta > 0$. Then there exists $\mu > 0$ such that, for all $\varphi \in \mathcal{G}$, we have

$$T_s(r, \varphi_S, \omega_S) \leq \varepsilon T_s(r, \varphi, \omega_{\bar{A}}) + \mu \tag{8.20}$$

for all $r \in (s, 1)$ outside some exceptional set with the linear measure less than δ.

Next by (8.19) and Remark 3.13 (3), we have

$$\mathrm{LIM}(\mathcal{G}_{P_{k+l,A}}, \{E_{k+l,A,A/B}\}_{B\in\Pi(\mathcal{F})}) = T_{k+l}.$$

Since $Z \subset S_{1,A}$ is horizontally integrable, we may take a Zariski closed subset $W \subsetneqq t(Z) \subset S$ which appears in Proposition 5.1, where $t \colon S_{1,A} \to S$ is the natural projection. Let $\pi\colon P_{k+l,A} \to S$ be the natural projection. By $\pi(T_{k+l}) = t(Z)$, we have $T_{k+l} \not\subset \pi^{-1}(W)$. Let $\lambda_W \geq 0$ be a Weil function for W. Then $\lambda_W \circ \pi$ is a Weil function for $\pi^* W \subset P_{k+l,A}$. By $\Pi(\mathcal{G}) \supset \Pi(\mathcal{F})$, we may apply Lemma 7.4 to get an infinite subset $\mathcal{G}' \subset \mathcal{G}$ and $\sigma_2 \in (0,1)$ with the following property: Let $s \in (\sigma_2, 1)$, $\varepsilon > 0$, $\delta > 0$. Then there exists $\beta > 0$ such that, for all $\varphi \in \mathcal{G}'$, we have

$$m(r, \varphi_{P_{k+l,A}}, \lambda_W \circ \pi) \leq \varepsilon T_s(r, \varphi, \omega_{\bar{A}}) + \beta \tag{8.21}$$

for all $r \in (s,1)$ outside some exceptional set with the linear measure less than δ.

We set $\sigma = \{1/2, \sigma_1, \sigma_2\}$. Note that we have $\varphi_S(\mathbb{D}) \not\subset W$ for all $\varphi \in \mathcal{G}'$, which follows from (8.21).

Step 4. Now we fix $s \in (\sigma, 1)$ arbitrary. We set

$$\delta = (1-s)/5.$$

By Proposition 5.1 applied to $\{\varphi_{[k]};\ \varphi \in \mathcal{G}\} \subset \mathrm{Hol}(\mathbb{D}, A \times S)$, there exist $c_1 > 0$, $c_2 > 0$, $c_3 > 0$ such that for all $\varphi \in \mathcal{G}$ with $\varphi_S(\mathbb{D}) \not\subset W$, we have

$$T_s(r, \varphi, \omega_{\bar{A}}) \leq c_1 T_s(r, \varphi_S, \omega_S) + c_2 m((s+r)/2, \varphi_S, \lambda_W) + c_3$$

for all $r \in (s,1)$ outside some exceptional set with the linear measure less than δ. Letting $\varepsilon = 1/3c_2$ in (8.21), we obtain that, for all $\varphi \in \mathcal{G}'$, we have

$$m((s+r)/2, \varphi_S, \lambda_W) \leq \frac{1}{3c_2} T_s(r, \varphi, \omega_{\bar{A}}) + \beta$$

for all $r \in (s,1)$ outside some exceptional set of the linear measure less than 2δ. Hence for all $\varphi \in \mathcal{G}'$, we get

$$T_s(r, \varphi, \omega_{\bar{A}}) \leq \frac{1}{3} T_s(r, \varphi, \omega_{\bar{A}}) + c_1 T_s(r, \varphi_S, \omega_S) + c_2 \beta_2 + c_3$$

for all $r \in (s,1)$ outside an exceptional set of the linear measure less than 3δ. By (8.20), letting $\varepsilon = 1/3c_1$, we obtain that for all $\varphi \in \mathcal{G}'$, we have

$$T_s(r, \varphi, \omega_{\bar{A}}) \leq \frac{2}{3} T_s(r, \varphi, \omega_{\bar{A}}) + c_1 \mu + c_2 \beta_2 + c_3$$

for all $r \in (s,1)$ outside an exceptional set of the linear measure less than 4δ. Thus for all $\varphi \in \mathcal{G}'$, we get

$$T_s(r, \varphi, \omega_{\bar{A}}) \leq c$$

for all $r \in (s,1)$ outside some exceptional set of linear measure less than 4δ, where $c = 3(c_1\mu + c_2\beta_2 + c_3)$. We may apply this estimate for some $r \in (s+\delta, 1)$. Hence we get

$$T_s(s+\delta, \varphi, \omega_{\bar{A}}) \leq c,$$

thus

$$\int_{\mathbb{D}(s)} \varphi^* \omega_{\bar{A}} \leq \frac{(s+\delta)c}{\delta}$$

for all $\varphi \in \mathcal{G}'$. Hence by Lemma 8.3, there exists a subsequence $(\tau_{n_k} \circ f_{n_k})_{k\in\mathbb{N}}$ which converges to a holomorphic map $g\colon \mathbb{D} \to A$.

Step 5. Now we show that $(f_{n_k})_{k\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{D}$ to $p \circ g$, where $p\colon W \to V$. Note that $(p \circ \tau_{n_k} \circ f_{n_k})_{k\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{D}$ to $p \circ g$. By (8.18), $(p \circ f_{n_k})_{k\in\mathbb{N}}$ also converges uniformly on compact subsets of $\mathbb{D}$ to $p \circ g$. We apply Lemma 8.2. Then by (8.17), $(f_{n_k})_{k\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{D}$ to $p \circ g$.

Step 6. So far, we have assumed that $\bar{A}$ is projective. Now let $\bar{A}$ be an arbitrary smooth equivariant compactification. We may take an equivariant modification $q\colon \hat{A} \to \bar{A}$ such that $\hat{A}$ is smooth and projective (cf. Lemma A.8). Then we may take a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges uniformly on compact subsets of $\mathbb{D}$ to $g\colon \mathbb{D} \to \hat{A}$. Then $(f_{n_k})_{k\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{D}$ to $q \circ g\colon \mathbb{D} \to \bar{A}$. This concludes the proof. □

Corollary 8.4. *Let $\mathcal{F} = (f_i)_{i\in I}$ be an infinite indexed family of non-constant holomorphic maps in* $\mathrm{Hol}(\mathbb{D}, A)$ *which satisfies Assumption* 3.19. *Suppose that there exists $k \geq 0$ such that $Z \subset P_{k+1,A} \subset (P_{k,A})_{1,A}$ is horizontally integrable, where Z is defined by* (3.7). *Let $\bar{A}$ be a smooth equivariant compactification. Then there exists an infinite subfamily $\mathcal{G}$ of $\mathcal{F}$ such that $\mathcal{G}$ converges uniformly on compact subsets of $\mathbb{D}$ to $g\colon \mathbb{D} \to \bar{A}$.*

Proof. Set $\mathcal{F}_o = \{f_i;\ i \in I\} \subset \mathrm{Hol}(\mathbb{D}, A)$. We may assume that the map $I \to \mathcal{F}_o$ is a finite-to-one mapping. Indeed, otherwise we may take an infinite subset $J \subset I$ such that f_i are all the same for all $i \in J$, hence the assertion is obvious. Thus we assume the map $I \to \mathcal{F}_o$ is a finite-to-one mapping. In particular, $\mathcal{F}_o$ is an infinite set. We note that $\Pi(\mathcal{F}_o) = \Pi(\mathcal{F})$. By taking a section of $I \to \mathcal{F}_o$, we take an infinite subset $I_o \subset I$ such that $I_o \to \mathcal{F}_o$ is bijective. By Remark 3.13 (3), we have

$$\mathrm{LIM}((\mathcal{F}_o)_{P_{k,A}};\ \{E_{k,A,A/B}\}_{B\in\Pi(\mathcal{F}_o)}) = \mathrm{LIM}(\mathcal{F}_{P_{k,A}};\ \{E_{k,A,A/B}\}_{B\in\Pi(\mathcal{F})}).$$

Hence the assumption of Proposition 3.20 is satisfied for $\mathcal{F}_o$. Hence by Proposition 3.20 for $\mathcal{F}_o$, we may take an infinite subset $\mathcal{G}_o \subset \mathcal{F}_o$ such that $\mathcal{G}_o$ converges uniformly on compact subsets of $\mathbb{D}$. Then we may take an infinite subset $J \subset I_o$ such that $J \to \mathcal{G}_o$ is bijective. We set $\mathcal{G} = (f_i)_{i\in J}$ to conclude the proof. □

§9. Families of closed subschemes and regular jets

We set $\Lambda_k = \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^{k+1})$. Then we have a sequence of closed immersions

$$\Lambda_0 \hookrightarrow \Lambda_1 \hookrightarrow \Lambda_2 \hookrightarrow \cdots . \tag{9.1}$$

Let S be a smooth variety. Given a morphism $\eta\colon \Lambda_k \to S$ of schemes, we obtain the derivative $\eta'\colon \Lambda_{k-1} \to TS$ of η. This map satisfies the following: Let φ be a local holomorphic function on S. Let $d\colon \mathcal{O}_S \to \Omega_S^1$ be the derivation. Then we have

$$(\eta')^* d\varphi = \frac{d}{d\varepsilon}\eta^*\varphi, \tag{9.2}$$

where $\frac{d}{d\varepsilon}\colon \mathbb{C}[\varepsilon]/(\varepsilon^{k+1}) \to \mathbb{C}[\varepsilon]/(\varepsilon^k)$ is the derivation. A regular k-jet is a morphism $\eta\colon \Lambda_k \to S$ such that $\eta'(0) \notin 0_{TS}$, where $0_{TS} \subset TS$ is the zero section. Hence by the composite of η' and $TS - 0_{TS} \to S_1$, we get $\eta_{[1]}\colon \Lambda_{k-1} \to S_1$ into the projectivization $S_1 = P(TS)$. Inductively, we obtain $\eta_{[l]}\colon \Lambda_{k-l} \to S_l$ into Demailly jet space S_l for $1 \leq l \leq k$. In particular, we get a point $\eta_{[k]}(0) \in S_k$.

Let $Z \subset S$ be a closed subscheme. We define a closed subscheme $\mathcal{D}Z \subset S_1$ as follows. Let $W \subset S$ be an affine open set where $Z \cap W$ is defined by $\varphi_1, \ldots, \varphi_n \subset \Gamma(W, \mathcal{O}_W)$. Then we define the closed subscheme $\widetilde{Z \cap W} \subset TW$ by $\varphi_1, \ldots, \varphi_n, d\varphi_1, \ldots, d\varphi_n$. Then this definition of $\widetilde{Z \cap W}$ does not depend on the choice of generators $\varphi_1, \ldots, \varphi_n$, so well defined over W. In general, we cover S by open affines $\{W_i\}$ and make closed subschemes $\widetilde{Z \cap W_i} \subset TW_i$. Then we glue these subschemes and define the subscheme $\widetilde{Z} \subset TS$. By the construction, $\widetilde{Z}$ is invariant under the $\mathbb{C}^*$-action on the fibers of $TS \to S$. Thus we get a closed subscheme $\mathcal{D}Z = \widetilde{Z}/\mathbb{C}^* \subset S_1$.

§9.1. Family of closed subschemes

Let V and S be smooth varieties and let $X \subset V \times S$ be a closed subscheme. For $k \geq 0$, we define a closed subscheme $\mathcal{P}_k X \subset V \times S_k$ inductively as follows. We have $V \times TS = T_{(V\times S)/V} \subset T(V \times S)$. Hence we get $V \times S_1 \subset (V \times S)_1$. We define $\mathcal{P}_1 X \subset V \times S_1$ by the restriction of $\mathcal{D}X \subset (V \times S)_1$ onto $V \times S_1 \subset (V \times S)_1$. Now suppose we get a closed subscheme $\mathcal{P}_k X \subset V \times S_k$. We have $S_{k+1} \subset (S_k)_1$. Thus we define $\mathcal{P}_{k+1} X \subset V \times S_{k+1}$ by the restriction of

$$\mathcal{P}_1(\mathcal{P}_k X) \subset V \times (S_k)_1 \tag{9.3}$$

onto $V \times S_{k+1} \subset V \times (S_k)_1$.

Lemma 9.1. *Let V and S be smooth varieties and let $X \subset V \times S$ be a closed subscheme. Let $\xi\colon \Lambda_k \to S$ be a regular k-jet. Let $s_0 = \xi(0) \in S$ and $s_k = \xi_{[k]}(0) \in S_k$. Assume that the natural map $(\mathcal{P}_k X)_{s_k} \to X_{s_0}$ is an isomorphism as schemes.*

Then $X_{\Lambda_k} = X_{s_0} \times \Lambda_k$ as closed subschemes of $V \times \Lambda_k$. Here, X_{Λ_k} is the pull-back of $X \to S$ by ξ.

Before proving this lemma, we start from a preliminary observation. Let $W \subset V$ be an affine open subset and set $Z = X \cap (W \times S)$. For $\nu = 0, \dots, k$, let $I_\nu \subset \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_{k-\nu}]$ be the defining ideal of $(\mathcal{P}_\nu Z)_{\Lambda_{k-\nu}}$, where $(\mathcal{P}_\nu Z)_{\Lambda_{k-\nu}}$ is the pull-back of $\mathcal{P}_\nu Z \subset W \times S_\nu$ by $\xi_{[\nu]} \colon \Lambda_{k-\nu} \to S_\nu$. We denote $\frac{\partial}{\partial \varepsilon} \colon \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_{k-\nu}] \to \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_{k-\nu-1}]$ by $\mathrm{id}_{\mathbb{C}[W]} \otimes \frac{d}{d\varepsilon}$.

Claim. If $h \in I_\nu$, then $\frac{\partial}{\partial \varepsilon} h \in I_{\nu+1}$.

We prove this claim. Let $U \subset S_\nu$ be an affine open subset such that $\xi_{[\nu]}(0) \in U$. Suppose that $\mathcal{P}_\nu Z$ is defined by $f_1, \dots, f_n \in \Gamma(W \times U, \mathcal{O}_{W \times U})$ on $W \times U$. Then $(\mathcal{P}_\nu Z)_{\Lambda_{k-\nu}} \subset W \times \Lambda_{k-\nu}$ is defined by $g_1, \dots, g_n \in \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_{k-\nu}]$, where $g_1, \dots, g_n$ are the images of $f_1, \dots, f_n$ under the natural map $\mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[U] \to \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_{k-\nu}]$. We note that (9.2) yields the following commutative diagram of $\mathbb{C}[W]$-modules:

$$\begin{tikzcd}
\mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[U] \arrow[r, "\mathrm{id}_{\mathbb{C}[W]} \otimes \xi_{[\nu]}^*"] \arrow[d, "\mathrm{id}_{\mathbb{C}[W]} \otimes d_U"'] & \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_{k-\nu}] \arrow[d, "\frac{\partial}{\partial \varepsilon}"] \\
\mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[TU] \arrow[r, "\mathrm{id}_{\mathbb{C}[W]} \otimes (\xi'_{[\nu]})^*"'] & \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_{k-\nu-1}].
\end{tikzcd}$$

Hence $(\mathcal{P}_{\nu+1} Z)_{\Lambda_{k-\nu-1}}$ is defined by

$$\bar{g}_1, \dots, \bar{g}_n, \frac{\partial}{\partial \varepsilon} g_1, \dots, \frac{\partial}{\partial \varepsilon} g_n.$$

Here, $\bar{g}_1, \dots, \bar{g}_k$ are the images of $g_1, \dots, g_k$ under $\mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_{k-\nu}] \to \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_{k-\nu-1}]$, where $\mathbb{C}[\Lambda_{k-\nu-1}] = \mathbb{C}[\Lambda_{k-\nu}]/(\varepsilon^{k-\nu})$ (cf. (9.1)).

Now let $h \in I_\nu$. Then there exist $b_1, \dots, b_n \in \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_{k-\nu}]$ such that $h = b_1 g_1 + \cdots + b_n g_n$. Then we have $\frac{\partial}{\partial \varepsilon} h = \bar{g}_1 \frac{\partial}{\partial \varepsilon} b_1 + \bar{b}_1 \frac{\partial}{\partial \varepsilon} g_1 + \cdots + \bar{g}_n \frac{\partial}{\partial \varepsilon} b_n + \bar{b}_n \frac{\partial}{\partial \varepsilon} g_n$. Hence $\frac{\partial}{\partial \varepsilon} h \in I_{\nu+1}$. This proves our claim.

Proof of Lemma 9.1. We may assume that S is affine. Let $X \subset V \times S$ be defined locally on an affine open $W \subset V$ by $f_1, \dots, f_n$. Then $X_{\Lambda_k} \subset V \times \Lambda_k$ is defined locally by $g_1, \dots, g_n \in \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_k]$, where $g_1, \dots, g_n$ are the images of $f_1, \dots, f_n$ under the natural map $\mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[S] \to \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_k]$. Let $\mathfrak{X} \subset V \times \Lambda_k$ be the constant family $\mathfrak{X} = (X_{s_0}) \times \Lambda_k$ over Λ_k. Then $\mathfrak{X}$ is defined locally by $g_1|_{\varepsilon=0}, \dots, g_n|_{\varepsilon=0}$. Hence it is enough to show

$$(g_1, \dots, g_n) = (g_1|_{\varepsilon=0}, \dots, g_n|_{\varepsilon=0})$$

as ideals of $\mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_k]$.

We first claim

$$\frac{\partial^{\nu} g_i}{\partial \varepsilon^{\nu}}|_{\varepsilon=0} \in (g_1|_{\varepsilon=0}, \dots, g_n|_{\varepsilon=0}) \tag{9.4}$$

for $\nu = 0, \dots, k$, where $(g_1|_{\varepsilon=0}, \dots, g_n|_{\varepsilon=0}) \subset \mathbb{C}[W]$. To prove this, we set $s_\nu = \xi_{[\nu]}(0) \in S_\nu$. Then we have

$$(\mathcal{P}_k X)_{s_k} \subset (\mathcal{P}_{k-1} X)_{s_{k-1}} \subset \cdots \subset (\mathcal{P}_1 X)_{s_1} \subset X_{s_0}.$$

Thus by the assumption $(\mathcal{P}_k X)_{s_k} = X_{s_0}$, we have $(\mathcal{P}_\nu X)_{s_\nu} = X_{s_0}$. By the claim above, we have $\frac{\partial^\nu g_i}{\partial \varepsilon^\nu} \in I_\nu$. Hence $\frac{\partial^\nu g_i}{\partial \varepsilon^\nu}|_{\varepsilon=0} \in \mathbb{C}[W]$ is contained in the defining ideal of $(\mathcal{P}_k X)_{s_k} \cap W$, hence that of $X_{s_0} \cap W$. Note that $X_{s_0} \cap W$ is defined on W by $(g_i|_{\varepsilon=0})_{i=1,\dots,n}$. This proves (9.4).

Now we have

$$g_i = g_i|_{\varepsilon=0} + \varepsilon \frac{\partial g_i}{\partial \varepsilon}|_{\varepsilon=0} + \cdots + \frac{1}{k!}\varepsilon^k \frac{\partial^k g_i}{\partial \varepsilon^k}|_{\varepsilon=0}. \tag{9.5}$$

Hence by (9.4), we get $(g_1, \dots, g_n) \subset (g_1|_{\varepsilon=0}, \dots, g_n|_{\varepsilon=0})$.

Next we show $(g_1|_{\varepsilon=0}, \dots, g_n|_{\varepsilon=0}) \subset (g_1, \dots, g_n)$. For this we only show $g_1|_{\varepsilon=0} \in (g_1, \dots, g_n)$, for the other indices are treated in the same manner. Let $J \subset (g_1, \dots, g_n)$ be the subset of the elements of the form

$$A = g_1|_{\varepsilon=0} + \varepsilon A_1 + \cdots + \varepsilon^k A_k,$$

where $A_1, \dots, A_k \in (g_1|_{\varepsilon=0}, \dots, g_n|_{\varepsilon=0}) \subset \mathbb{C}[W]$. By (9.4) and (9.5), we have $g_1 \in J$. Hence $J \neq \emptyset$. For each element $A \in J$, we set

$$\mu_A = \min\{j;\ A_j \neq 0\},$$

where we set $\mu_A = k+1$ if $A_j = 0$ for all $j = 1, \dots, k$. We put $\mu = \max_{A \in J} \mu_A$. What we want to show is that $\mu = k+1$. So suppose $\mu \le k$. Take $A \in J$ such that $\mu_A = \mu$. Then

$$A = g_1|_{\varepsilon=0} + \varepsilon^\mu A_\mu + \cdots + \varepsilon^k A_k.$$

By $A_\mu \in (g_1|_{\varepsilon=0}, \dots, g_n|_{\varepsilon=0})$, there exist $b_1, \dots, b_n \in \mathbb{C}[W]$ such that

$$A_\mu = b_1 g_1|_{\varepsilon=0} + \cdots + b_n g_n|_{\varepsilon=0}.$$

Then by (9.4) and (9.5), we have

$$A' = A - \varepsilon^\mu b_1 g_1 - \cdots - \varepsilon^\mu b_n g_n \in J.$$

Moreover, we have $\mu_{A'} > \mu$. This is a contradiction. Hence we have $\mu = k+1$, thus $g_1|_{\varepsilon=0} \in (g_1, \dots, g_n)$. This completes the proof of our lemma. □

§9.2. Regular jets and Demailly jet spaces

Let S be a smooth variety. We are going to introduce the jet space J_kS as in [45, Sec. 4.2]. This definition is equivalent to the usual definition described in [35, Sec. 4.6.1]. See Remark 9.2 below.

Set $J_0S = S$ and $J_1S = TS$. For $k \geq 1$, the space J_kS is a smooth variety with an embedding

$$J_kS \overset{\iota_k}{\hookrightarrow} TJ_{k-1}S,$$

where we set $\iota_1\colon J_1S \to TS$ to be the identity map. We define J_kS inductively as follows. So we suppose that the smooth variety J_kS and the embedding $\iota_k\colon J_kS \hookrightarrow TJ_{k-1}S$ are given. Let $\varpi_k\colon J_kS \to J_{k-1}S$ be the composite of $\iota_k\colon J_kS \hookrightarrow TJ_{k-1}S$ and the projection $TJ_{k-1}S \to J_{k-1}S$. Then $\varpi_k\colon J_kS \to J_{k-1}S$ induces the following commutative diagram:

$$\begin{tikzcd} TJ_kS \arrow[r, "p_k"] \arrow[d, "(\varpi_k)_*"'] & J_kS \arrow[d, "\varpi_k"] \\ TJ_{k-1}S \arrow[r, "p_{k-1}"'] & J_{k-1}S. \end{tikzcd}$$

This induces the morphism

$$\mu_k\colon TJ_kS \to J_kS \times_{J_{k-1}S} TJ_{k-1}S.$$

The graph of $\iota_k\colon J_kS \to TJ_{k-1}S$ defines the closed immersion

$$\hat{\iota}_k\colon J_kS \hookrightarrow J_kS \times_{J_{k-1}S} TJ_{k-1}S.$$

Then $\iota_{k+1}\colon J_{k+1}S \hookrightarrow TJ_kS$ is defined by the base change of $\hat{\iota}_k$ by μ_k:

$$\begin{tikzcd} J_{k+1}S \arrow[r, "\iota_{k+1}"] \arrow[d, "(\mu_k)'"'] & TJ_kS \arrow[d, "\mu_k"] \\ J_kS \arrow[r, "\hat{\iota}_k"'] & J_kS \times_{J_{k-1}S} TJ_{k-1}S. \end{tikzcd}$$

We get a map $(\mu_k)'\colon J_{k+1}S \to J_kS$ from this base change. Let $\varpi_{k+1}\colon J_{k+1}S \to J_kS$ be the composite of $\iota_{k+1}\colon J_{k+1}S \hookrightarrow TJ_kS$ and the projection $p_k\colon TJ_kS \to J_kS$. Then we note

$$(\mu_k)' = \varpi_{k+1}. \tag{9.6}$$

Indeed, we have $(\mu_k)' = \mathrm{id}_{J_kS} \circ (\mu_k)' = r_1 \circ \hat{\iota}_k \circ (\mu_k)'$, where $r_1\colon J_kS \times_{J_{k-1}S} TJ_{k-1}S \to J_kS$ is the first projection. On the other hand, we have $r_1 \circ \hat{\iota}_k \circ$

$(\mu_k)' = r_1 \circ \mu_k \circ \iota_{k+1} = p_k \circ \iota_{k+1}$. Hence we get (9.6). By [45, Cor. 4.4], the map $\varpi_{k+1}\colon J_{k+1}S \to J_kS$ is smooth. Hence $J_{k+1}S$ is smooth.

Remark 9.2. Assume that an affine open $W \subset S$ admits coordinate functions $\chi_1, \dots, \chi_n$ such that TW splits as $TW = W \times \mathbb{C}^n$ and $(d\chi_1, \dots, d\chi_n)$ defines the second projection, where $n = \dim S$. Then by [45, Cor. 4.6], we have $J_kW = W \times \mathbb{C}^{kn}$ and $(d\chi_1, \dots, d\chi_n, \dots, d^k\chi_1, \dots, d^k\chi_n)$ defines the second projection. This shows that our definition of J_kS coincides with the usual definition described in [35, Sec. 4.6.1].

The space $\iota_{k+1}\colon J_{k+1}S \hookrightarrow TJ_kS$ is characterized as follows.

Lemma 9.3. *A map $\eta\colon W \to TJ_kS$ from a scheme W factors $\iota_{k+1}\colon J_{k+1}S \hookrightarrow TJ_kS$ if and only if $\iota_k \circ p_k \circ \eta = (\varpi_k)_* \circ \eta$. In particular, if we consider the case $W = \operatorname{Spec}\mathbb{C}$, we have*

$$J_{k+1}S = \big\{x \in TJ_kS;\ \iota_k \circ p_k(x) = (\varpi_k)_*(x)\big\}. \tag{9.7}$$

Proof. To prove this, we suppose that $\eta\colon W \to TJ_kS$ satisfies $\iota_k \circ p_k \circ \eta = (\varpi_k)_* \circ \eta$. We claim that

$$\mu_k \circ \eta = \hat{\iota}_k \circ p_k \circ \eta. \tag{9.8}$$

To show this we note that

$$r_1 \circ \mu_k \circ \eta = p_k \circ \eta = r_1 \circ \hat{\iota}_k \circ p_k \circ \eta. \tag{9.9}$$

We also have

$$r_2 \circ \mu_k \circ \eta = (\varpi_k)_* \circ \eta, \tag{9.10}$$

where $r_2\colon J_kS \times_{J_{k-1}S} TJ_{k-1}S \to TJ_{k-1}S$ is the second projection of the fiber product:

$$\begin{tikzcd} J_kS \times_{J_{k-1}S} TJ_{k-1}S \arrow[r, "r_2"] \arrow[d, "r_1"'] & TJ_{k-1}S \arrow[d, "p_{k-1}"] \\ J_kS \arrow[r, "\varpi_k"'] & J_{k-1}S. \end{tikzcd}$$

By our assumption $\iota_k \circ p_k \circ \eta = (\varpi_k)_* \circ \eta$, we have

$$(\varpi_k)_* \circ \eta = \iota_k \circ p_k \circ \eta = r_2 \circ \hat{\iota}_k \circ p_k \circ \eta. \tag{9.11}$$

Hence by (9.10) and (9.11), we have $r_2 \circ \mu_k \circ \eta = r_2 \circ \hat{\iota}_k \circ p_k \circ \eta$. Combining this with (9.9), we get (9.8) by the universal property of the fiber product. Hence by (9.8), the universal property of the fiber product yields the map $(\eta, p_k \circ \eta)\colon W \to J_{k+1}S$

such that the composition with $\iota_{k+1}\colon J_{k+1}S \to TJ_kS$ is equal to η. Hence η factors $\iota_{k+1}\colon J_{k+1}S \hookrightarrow TJ_kS$.

To prove the converse, we claim

$$\iota_k \circ p_k \circ \iota_{k+1} = (\varpi_k)_* \circ \iota_{k+1}. \tag{9.12}$$

Indeed, by (9.6), we have $(\mu_k)' = p_k \circ \iota_{k+1}$. Hence we have the following equalities:

$$\iota_k \circ p_k \circ \iota_{k+1} = r_2 \circ \hat{\iota}_k \circ p_k \circ \iota_{k+1} = r_2 \circ \hat{\iota}_k \circ (\mu_k)' = r_2 \circ \mu_k \circ \iota_{k+1} = (\varpi_k)_* \circ \iota_{k+1}.$$

Hence if $\eta\colon W \to TJ_kS$ factors $\iota_{k+1}\colon J_{k+1}S \hookrightarrow TJ_kS$, we have $\iota_k \circ p_k \circ \eta = (\varpi_k)_* \circ \eta$. □

Let $l \geq k$. Given a morphism $\eta\colon \Lambda_l \to S$, we obtain $j_k\eta\colon \Lambda_{l-k} \to J_kS$ such that

$$\iota_k \circ j_k\eta = (j_{k-1}\eta)'. \tag{9.13}$$

Indeed, to show the existence of $j_k\eta$ by induction on k, we assume the existence for k. Then we have

$$\iota_k \circ p_k \circ (j_k\eta)' = \iota_k \circ j_k\eta = (j_{k-1}\eta)' = (\varpi_k)_* \circ (j_k\eta)'.$$

Hence, by Lemma 9.3, the map $(j_k\eta)'\colon \Lambda_{l-k-1} \to TJ_kS$ factors $\iota_{k+1}\colon J_{k+1}S \hookrightarrow TJ_kS$. Thus we get $j_{k+1}\eta\colon \Lambda_{l-k-1} \to J_{k+1}S$ such that $\iota_{k+1} \circ j_{k+1}\eta = (j_k\eta)'$.

Let $J_k^{\mathrm{reg}}S \subset J_kS$ be the Zariski open which is the inverse image of $TS - \{0_{TS}\}$ under the map $J_kS \to TS$, provided $k \geq 1$. We set $J_0^{\mathrm{reg}}S = S$. Note that the map $J_{k+1}S \to J_kS$ induces $J_{k+1}^{\mathrm{reg}}S \to J_k^{\mathrm{reg}}S$.

Lemma 9.4. *Let S and M be smooth algebraic varieties. Let $V \subset TM$ be an algebraic vector subbundle and let $\widetilde{M} = P(V)$. Let $k \geq 0$. Let $\varphi\colon J_k^{\mathrm{reg}}S \to M$ be a morphism such that the induced map $\varphi_*\colon TJ_k^{\mathrm{reg}}S \to TM$ satisfies*

$$\varphi_*(\iota_{k+1}(J_{k+1}^{\mathrm{reg}}S)) \subset V\backslash\{0_V\}.$$

Let $\Phi\colon J_{k+1}^{\mathrm{reg}}S \to \widetilde{M}$ be the composite of the following morphisms:

$$J_{k+1}^{\mathrm{reg}}S \xrightarrow{\varphi_* \circ \iota_{k+1}} V\backslash\{0_V\} \xrightarrow{\tau} \widetilde{M}.$$

Then the induced map $\Phi_*\colon TJ_{k+1}^{\mathrm{reg}}S \to T\widetilde{M}$ satisfies $\Phi_*(\iota_{k+2}(J_{k+2}^{\mathrm{reg}}S)) \subset \widetilde{V}\backslash\{0_V\}$, where $\widetilde{V} \subset T\widetilde{M}$ is defined by (2.1).

Proof. We first show that the following diagram is commutative:

$$
\begin{tikzcd}
J^{\mathrm{reg}}_{k+1}S \arrow[r, "\Phi"] \arrow[d, "\varpi_{k+1}"'] & \widetilde{M} \arrow[d, "\pi"] \\
J^{\mathrm{reg}}_{k}S \arrow[r, "\varphi"'] & M.
\end{tikzcd}
\tag{9.14}
$$

Indeed, $\pi \circ \Phi$ is the composite of

$$J^{\mathrm{reg}}_{k+1}S \xrightarrow{\iota_{k+1}} TJ^{\mathrm{reg}}_{k}S \xrightarrow{\varphi_*} TM \xrightarrow{q} M.$$

By the definition of the derivation, we have $q \circ \varphi_* = \varphi \circ p_k$, i.e., the following diagram is commutative:

$$
\begin{tikzcd}
TJ^{\mathrm{reg}}_{k}S \arrow[r, "\varphi_*"] \arrow[d, "p_k"'] & TM \arrow[d, "q"] \\
J^{\mathrm{reg}}_{k}S \arrow[r, "\varphi"'] & M.
\end{tikzcd}
$$

Hence we get

$$\pi \circ \Phi = q \circ \varphi_* \circ \iota_{k+1} = \varphi \circ p_k \circ \iota_{k+1} = \varphi \circ \varpi_{k+1}.$$

This shows that (9.14) is commutative.

Now by (9.14), we get the following commutative diagram:

$$
\begin{tikzcd}
J^{\mathrm{reg}}_{k+2}S \arrow[r, "\iota_{k+2}"] \arrow[d, "\varpi_{k+2}"'] & TJ^{\mathrm{reg}}_{k+1}S \arrow[r, "\Phi_*"] \arrow[d, "(\varpi_{k+1})_*"'] & T\widetilde{M} \arrow[r, "\tilde{q}"] \arrow[d, "\pi_*"] & \widetilde{M} \arrow[d, "\pi"] \\
J^{\mathrm{reg}}_{k+1}S \arrow[r, "\iota_{k+1}"'] & TJ^{\mathrm{reg}}_{k}S \arrow[r, "\varphi_*"'] & TM \arrow[r, "q"'] & M.
\end{tikzcd}
\tag{9.15}
$$

Indeed, the only non-trivial part is the relation $(\varpi_{k+1})_* \circ \iota_{k+2} = \iota_{k+1} \circ \varpi_{k+2}$. To show this, we note $(\varpi_{k+1})_* \circ \iota_{k+2} = \iota_{k+1} \circ p_{k+1} \circ \iota_{k+2}$, which follows from Lemma 9.3 (cf. (9.12)). Thus by $p_{k+1} \circ \iota_{k+2} = \varpi_{k+2}$, we get $(\varpi_{k+1})_* \circ \iota_{k+2} = \iota_{k+1} \circ \varpi_{k+2}$. Thus the above diagram (9.15) is commutative.

We take $y \in J^{\mathrm{reg}}_{k+2}S$. We want to show

$$\Phi_* \circ \iota_{k+2}(y) \in \widetilde{V}_{\tilde{q} \circ \Phi_* \circ \iota_{k+2}(y)}. \tag{9.16}$$

Let $\tau \colon V \backslash \{0_V\} \to \widetilde{M}$ be the projection. By the definition of Φ, we have

$$\Phi \circ p_{k+1} \circ \iota_{k+2} = \Phi \circ \varpi_{k+2} = \tau \circ \varphi_* \circ \iota_{k+1} \circ \varpi_{k+2} = \tau \circ \pi_* \circ \Phi_* \circ \iota_{k+2}, \tag{9.17}$$

where the last equality follows from the commutativity of (9.15). Since Φ_* is the derivation of $\Phi\colon J^{\mathrm{reg}}_{k+1}S \to \widetilde{M}$, we have $\tilde{q}\circ\Phi_* = \Phi\circ p_{k+1}$. Combining this with (9.17), we have

$$\tilde{q}\circ\Phi_*\circ\iota_{k+2} = \Phi\circ p_{k+1}\circ\iota_{k+2} = \tau\circ\pi_*\circ\Phi_*\circ\iota_{k+2}.$$

Thus we observe that $\tilde{q}(\Phi_*\circ\iota_{k+2}(y)) \in \widetilde{M}$ is the image of $\pi_*(\Phi_*\circ\iota_{k+2}(y))$ under the map $\tau\colon V\backslash\{0\}\to\widetilde{M}$. Hence by the definition of $\widetilde{V}\subset T\widetilde{M}$ (cf. (2.1)), we get (9.16). By the assumption of φ, we have $\pi_*(\Phi_*\circ\iota_{k+2}(y))\notin 0_{TM}$. Hence $\Phi_*\circ\iota_{k+2}(y)\notin 0_{T\widetilde{M}}$. Hence $\Phi_*\circ\iota_{k+2}(y)\in\widetilde{V}\backslash\{0_V\}$. The proof is completed. □

For each $k\geq 0$, we define a morphism $\varphi_k\colon J^{\mathrm{reg}}_kS\to S_k$ inductively as follows. We recall from Section 2 the vector subbundle $V_k\subset TS_k$ so that $S_{k+1}=P(V_k)$. We set $\varphi_0\colon S\to S$ to be the identity map, where $S_0=S$ and $J^{\mathrm{reg}}_0S=S$. Then we have $(\varphi_0)_*(\iota_1(J^{\mathrm{reg}}_1S))\subset V_0\backslash\{0\}$, where $V_0=TS$. Suppose we have constructed $\varphi_k\colon J^{\mathrm{reg}}_kS\to S_k$ which satisfies $(\varphi_k)_*(\iota_{k+1}(J^{\mathrm{reg}}_{k+1}S))\subset V_k\backslash\{0\}$. Then we define $\varphi_{k+1}\colon J^{\mathrm{reg}}_{k+1}S\to S_{k+1}$ by the composite of the following morphisms:

$$J^{\mathrm{reg}}_{k+1}S \xrightarrow{(\varphi_k)_*\circ\iota_{k+1}} V_k\backslash\{0\}\to S_{k+1}.$$

Then by Lemma 9.4, we have $(\varphi_{k+1})_*(\iota_{k+2}(J^{\mathrm{reg}}_{k+2}S))\subset V_{k+1}\backslash\{0\}$. Hence we have constructed $\varphi_k\colon J^{\mathrm{reg}}_kS\to S_k$ inductively. By the construction, we have

$$(\varphi_k)_*(\iota_{k+1}(J^{\mathrm{reg}}_{k+1}S))\subset V_k\backslash\{0\} \tag{9.18}$$

for all $k\geq 0$. By (9.14), the following diagram commutes for all $k\geq 1$:

$$\begin{tikzcd} J^{\mathrm{reg}}_kS \arrow[r, "\varphi_k"] \arrow[d, "\varpi_k"'] & S_k \arrow[d, "\pi_k"] \\ J^{\mathrm{reg}}_{k-1}S \arrow[r, "\varphi_{k-1}"'] & S_{k-1}. \end{tikzcd}$$

Hence for each $s\in J^{\mathrm{reg}}_{k-1}S$, we get the restriction map

$$\varphi_k|_{\varpi_k^{-1}(s)}\colon \varpi_k^{-1}(s)\to\pi_k^{-1}(\varphi_{k-1}(s)). \tag{9.19}$$

Here, $\varpi_k^{-1}(s)\simeq\mathbb{C}^{\dim S}$ and $\pi_k^{-1}(\varphi_{k-1}(s))\simeq\mathbb{P}^{\dim S-1}$. We set $S^{\mathrm{reg}}_k=S_k\backslash S^{\mathrm{sing}}_k$ (cf. (2.2)).

Lemma 9.5. *Let $k\geq 1$. Then for each $s\in J^{\mathrm{reg}}_{k-1}S$, the map (9.19) is smooth and $\varphi_k(\varpi_k^{-1}(s))=\pi_k^{-1}(\varphi_{k-1}(s))\cap S^{\mathrm{reg}}_k$.*

Proof. We prove this by induction on k. Thus we assume the lemma for k and prove the lemma for $k+1$. We take $s \in J_k^{\mathrm{reg}}S$. We first note that

$$(\varphi_k)_{*,s}(T_{J_k^{\mathrm{reg}}S/J_{k-1}^{\mathrm{reg}}S,s}) = T_{S_k/S_{k-1},\varphi_k(s)}, \tag{9.20}$$

where $(\varphi_k)_{*,s}\colon T_sJ_k^{\mathrm{reg}}S \to T_{\varphi_k(s)}S_k$ is the induced map. We prove this from the induction hypothesis as follows. We have

$$\begin{aligned} T_{J_k^{\mathrm{reg}}S/J_{k-1}^{\mathrm{reg}}S,s} &= T_s\big(\varpi_k^{-1}(\varpi_k(s))\big) \\ &\xrightarrow{(\varphi_k)_{*,s}} T_{\varphi_k(s)}\big(\pi_k^{-1}\big(\varphi_{k-1}(\varpi_k(s))\big)\big) = T_{S_k/S_{k-1},\varphi_k(s)}, \end{aligned}$$

where we note the smoothness of ϖ_k and π_k on the first and last equalities. By the induction hypothesis, (9.19) is smooth for k. Hence we get (9.20).

Now we consider the following commutative diagram, where $(\varpi_k)_* \circ \iota_{k+1} = \iota_k \circ \varpi_{k+1}$ follows from (9.12) (cf. (9.15)):

$$\begin{tikzcd} J_{k+1}^{\mathrm{reg}}S \arrow[r, "\iota_{k+1}"] \arrow[d, "\varpi_{k+1}"] & TJ_k^{\mathrm{reg}}S \arrow[r, "(\varphi_k)_*"] \arrow[d, "(\varpi_k)_*"] & TS_k \arrow[d, "(\pi_k)_*"] & V_k\backslash\{0\} \arrow[l] \arrow[r] & S_{k+1} \arrow[d, "\pi_{k+1}"'] \\ J_k^{\mathrm{reg}}S \arrow[r, "\iota_k"'] & TJ_{k-1}^{\mathrm{reg}}S \arrow[r, "(\varphi_{k-1})_*"'] & TS_{k-1} & V_{k-1}\backslash\{0\} \arrow[l] \arrow[r] & S_k. \end{tikzcd} \tag{9.21}$$

By (9.7), we have

$$\iota_{k+1}(\varpi_{k+1}^{-1}(s)) = \big\{x \in T_sJ_k^{\mathrm{reg}}S;\ (\varpi_k)_*(x) = \iota_k(s)\big\}, \tag{9.22}$$

where $\iota_k(s) \in T_{\varpi_k(s)}J_{k-1}^{\mathrm{reg}}S$. We focus on the following two linear maps from (9.21):

$$\begin{tikzcd} T_sJ_k^{\mathrm{reg}}S \arrow[r, "(\varphi_k)_{*,s}"] \arrow[d, "(\varpi_k)_{*,s}"'] & T_{\varphi_k(s)}S_k. \\ T_{\varpi_k(s)}J_{k-1}^{\mathrm{reg}}S & \end{tikzcd}$$

By (9.22), we have

$$(\varpi_k)_{*,s}^{-1}(\iota_k(s)) = \iota_{k+1}(\varpi_{k+1}^{-1}(s)).$$

Hence by $(\varpi_k)_{*,s}^{-1}(0) = T_{J_k^{\mathrm{reg}}S/J_{k-1}^{\mathrm{reg}}S,s}$, we note that $\iota_{k+1}(\varpi_{k+1}^{-1}(s))$ is a translate of the linear subspace $T_{J_k^{\mathrm{reg}}S/J_{k-1}^{\mathrm{reg}}S,s}$ in the linear space $T_sJ_k^{\mathrm{reg}}S$. Hence by (9.20), we observe that $(\varphi_k)_{*,s}(\iota_{k+1}(\varpi_{k+1}^{-1}(s)))$ is a translate of $T_{S_k/S_{k-1},\varphi_k(s)}$. By (9.18), we have

$$(\varphi_k)_{*,s}\big(\iota_{k+1}(\varpi_{k+1}^{-1}(s))\big) \subset V_{k,\varphi_k(s)}\backslash\{0\}. \tag{9.23}$$

On the other hand, we have $T_{S_k/S_{k-1},\varphi_k(s)} \subset V_{k,\varphi_k(s)}$ (cf. (2.4)) and $PV_{k,\varphi_k(s)} = \pi_{k+1}^{-1}(\varphi_k(s))$. By $\varpi_k(s) \in J_{k-1}^{\mathrm{reg}}S$, the induction hypothesis yields that $\varphi_k(s) \in S_k^{\mathrm{reg}}$. Hence the hyperplane $PT_{S_k/S_{k-1},\varphi_k(s)} \subset PV_{k,\varphi_k(s)}$ is equal to $\pi_{k+1}^{-1}(\varphi_k(s)) \cap S_{k+1}^{\mathrm{sing}}$ (cf. (2.5)). Since $(\varphi_k)_{*,s}(\iota_{k+1}(\varpi_{k+1}^{-1}(s)))$ is the translate of $T_{S_k/S_{k-1},\varphi_k(s)}$ in the linear space $V_{k,\varphi_k(s)}$, (9.23) yields that

$$(\varphi_k)_{*,s}\big(\iota_{k+1}(\varpi_{k+1}^{-1}(s))\big) \to \pi_{k+1}^{-1}(\varphi_k(s)) \cap S_{k+1}^{\mathrm{reg}}$$

is an isomorphism under the restriction of the projectivization $V_{k,\varphi_k(s)} \backslash \{0\} \to PV_{k,\varphi_k(s)}$. Now we look at the two morphisms

$$\iota_{k+1}(\varpi_{k+1}^{-1}(s)) \xrightarrow{(\varphi_k)_{*,s}} (\varphi_k)_{*,s}\big(\iota_{k+1}(\varpi_{k+1}^{-1}(s))\big) \to \pi_{k+1}^{-1}(\varphi_k(s)) \cap S_{k+1}^{\mathrm{reg}},$$

where the first map is a translate of the linear map $T_{J_kS/J_{k-1}S,s} \to T_{S_k/S_{k-1},\varphi_k(s)}$ which is surjective by (9.20). Hence $\varpi_{k+1}^{-1}(s) \to \pi_{k+1}^{-1}(\varphi_k(s)) \cap S_{k+1}^{\mathrm{reg}}$ is smooth and $\varphi_{k+1}(\varpi_{k+1}^{-1}(s)) = \pi_{k+1}^{-1}(\varphi_k(s)) \cap S_{k+1}^{\mathrm{reg}}$. This completes the induction step. □

Lemma 9.6. *For each $k \geq 1$ we have $\varphi_k(J_k^{\mathrm{reg}}S) = S_k^{\mathrm{reg}}$.*

Proof. The proof is by induction on k. The case $k = 1$ is trivial. So we assume the case $k-1$ and prove the case k. Let $x \in S_k^{\mathrm{reg}}$. Then $\pi_k(x) \in S_{k-1}^{\mathrm{reg}}$. Hence by the induction hypothesis, there exists $s \in J_{k-1}^{\mathrm{reg}}S$ such that $\varphi_{k-1}(s) = \pi_k(x)$. Then by Lemma 9.5, there exists $s' \in \varpi_k^{-1}(s)$ such that $\varphi_k(s') = x$. □

Lemma 9.7. *Let $\eta\colon \Lambda_l \to S$ be a regular l-jet. Then we have $\varphi_k \circ j_k\eta = \eta_{[k]}$ as elements in $\mathrm{Hom}(\Lambda_{l-k}, S_k)$.*

Proof. The proof is by induction on k. The case $k = 0$ is trivial. So we assume the case $k-1$ and prove the case k. By the induction hypothesis, we have $(\varphi_{k-1})_* \circ (j_{k-1}\eta)' = (\eta_{[k-1]})'$. By (9.13), we have $(\varphi_{k-1})_* \circ (j_{k-1}\eta)' = (\varphi_{k-1})_* \circ \iota_k \circ j_k\eta$. Thus we get $(\varphi_{k-1})_* \circ \iota_k \circ j_k\eta = (\eta_{[k-1]})'$. We composite these maps with $V_{k-1}\backslash\{0\} \to S_k$. Then by the definitions of φ_k and $\eta_{[k]}$, we have $\varphi_k \circ j_k\eta = \eta_{[k]}$. This completes the induction step. □

Lemma 9.8. *Let $w \in J_k^{\mathrm{reg}}S$. Then there exists a regular k-jet $\eta\colon \Lambda_k \to S$ such that $j_k\eta(0) = w$.*

Proof. We first consider the case $S = \mathbb{A}^1$. Let x be the coordinate of $\mathbb{A}^1$. Then $J_k\mathbb{A}^1 = \mathbb{A}^{k+1}$, where $x, dx, \dots, d^kx$ are the coordinate functions of $J_k\mathbb{A}^1$. Let $w = (w_0, w_1, \dots, w_k)$. We define $\eta\colon \Lambda_k \to \mathbb{A}^1$ by

$$x = w_0 + w_1\varepsilon + \frac{w_2}{2!}\varepsilon^2 + \cdots + \frac{w_k}{k!}\varepsilon^k.$$

Then we have $j_k\eta(0) = w$. By $w \in J_k^{\mathrm{reg}}S$, we have $w_1 \neq 0$. Hence η is regular. This proves our lemma when $S = \mathbb{A}^1$.

Next we consider the case $S = \mathbb{A}^n$. In this case, we have the natural splitting $J_k\mathbb{A}^n = (J_k\mathbb{A}^1)^n$. Let $p_i\colon J_k\mathbb{A}^n \to J_k\mathbb{A}^1$ be the i-th projection. We take $\eta_i\colon \Lambda_k \to \mathbb{A}^1$ such that $j_k\eta_i(0) = p_i(w)$. We set $\eta = (\eta_1, \dots, \eta_k)$. Then we have $j_k\eta(0) = w$. By $w \in J_k^{\mathrm{reg}}\mathbb{A}^n$, there exists i such that $p_i(w) \in J_k^{\mathrm{reg}}\mathbb{A}^1$. Then η_i is regular. Hence η is regular. This proves our lemma when $S = \mathbb{A}^n$.

In general, we may assume that S is affine and has local coordinate functions $\chi_1, \dots, \chi_n$ described in Remark 9.2. Then $\chi = (\chi_1, \dots, \chi_n)\colon S \to \mathbb{A}^n$ is étale. This induces $\chi_*\colon J_kS \to J_k\mathbb{A}^n$ so that $d^jx_i \circ \chi_* = d^j\chi_i$. We have $\chi_*(w) \in J_k^{\mathrm{reg}}\mathbb{A}^n$. By the previous step, there exists a regular k-jet $\xi\colon \Lambda_k \to \mathbb{A}^n$ such that $j_k\xi(0) = \chi_*(w)$. We take $\eta\colon \Lambda_k \to S$ such that $\chi \circ \eta = \xi$ and $\eta(0)$ is the image of w under $J_kS \to S$. Then by $\chi_* \circ j_k\eta = j_k\xi$, we have $j_k\eta(0) = w$. □

Corollary 9.9. *Let $w \in S_k^{\mathrm{reg}}$. Then there exists a regular k-jet $\eta\colon \Lambda_k \to S$ such that $\eta_{[k]}(0) = w$.*

Proof. By Lemma 9.6, there exists $w' \in J_k^{\mathrm{reg}}S$ such that $\varphi_k(w') = w$. By Lemma 9.8, there exists a regular k-jet $\eta\colon \Lambda_k \to S$ such that $j_k\eta(0) = w'$. By Lemma 9.7, we have $\eta_{[k]}(0) = w$. This concludes the proof of the corollary. See also [12, Thm. 6.8]. □

§9.3. One lemma for regular jets

Lemma 9.10. *Let S be a smooth variety. Let $Z \subset S$ be a closed subscheme. Then there exists a positive integer k with the following property: Let $\eta\colon \Lambda_k \to S$ be a regular k-jet with non-zero first derivative $\eta'(0) = v \in T_{\eta(0)}S$, where $v \neq 0$. Assume that η factors $Z \subset S$. Then there exists an irreducible component $Z' \subset Z_{\mathrm{red}}$ such that $[v] \in \mathcal{D}Z' \subset S_1$.*

Proof. We may assume that S is affine. We first prove a weaker statement $[v] \in \mathcal{D}(Z_{\mathrm{red}})$. Let $Z_{\mathrm{red}} \subset S$ be defined by $\varphi_1, \dots, \varphi_l$. We consider the map $\Phi\colon S \to \mathbb{A}^l$ defined by $\varphi_1, \dots, \varphi_l$. We have $\Phi^{-1}(0) = Z_{\mathrm{red}}$. There exists k such that $0_k = \mathrm{Spec}(\mathcal{O}_{0,\mathbb{A}^l}/\mathfrak{m}^k) \subset \mathbb{A}^l$ satisfies $Z \subset \Phi^*0_k$. Note that $\varphi_1, \dots, \varphi_l, d\varphi_1, \dots, d\varphi_l$ defines $\mathcal{D}(Z_{\mathrm{red}}) \subset S_1$. Let $\eta\colon \Lambda_k \to S$ be a regular k-jet with non-zero first derivative $\eta'(0) = v \in T_{\eta(0)}S$, where $v \neq 0$, such that η factors $Z \subset S$. To show $[v] \in \mathcal{D}(Z_{\mathrm{red}})$, we assume on the contrary that $[v] \notin \mathcal{D}(Z_{\mathrm{red}})$. Then $\Phi \circ \eta\colon \Lambda_k \to \mathbb{A}^l$ is regular. Since $\Phi \circ \eta$ factors $0_k \subset \mathbb{A}^l$, we have $((\Phi \circ \eta)^*x_1)^k = 0, \dots, ((\Phi \circ \eta)^*x_l)^k = 0$, where $(\Phi \circ \eta)^*x_1, \dots, (\Phi \circ \eta)^*x_l \in \mathbb{C}[\varepsilon]/(\varepsilon^{k+1})$. Since $\Phi \circ \eta$ is regular, we may take $i = 1, \dots, l$ such that $(\Phi \circ \eta)^*x_i = a_1\varepsilon + a_2\varepsilon^2 + \cdots + a_k\varepsilon^k$ with $a_1 \neq 0$. Then $((\Phi \circ \eta)^*x_i)^k = a_1^k\varepsilon^k \neq 0$. This is a contradiction. Hence $[v] \in \mathcal{D}(Z_{\mathrm{red}})$.

Now let $Z_1, \dots, Z_l$ be irreducible closed subschemes of S such that $\operatorname{supp} Z_1, \dots, \operatorname{supp} Z_l$ are the irreducible components of $\operatorname{supp} Z$ and $Z \subset Z_1 + \cdots + Z_l$. Let $k_1, \dots, k_l$ be positive integers which are obtained in the previous step for $Z_1, \dots, Z_l$. Set $k = k_1 + \cdots + k_l$. Let $\eta\colon \Lambda \to S$ be a regular k-jet with non-zero first derivative $\eta'(0) = v \in T_{\eta(0)}S$, where $v \neq 0$, such that η factors $Z \subset S$, hence $Z_1 + \cdots + Z_l$. Let $\eta_i\colon \Lambda_{k_i} \to S$, where $\Lambda_{k_i} = \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^{k_i+1})$, be induced from η. Then there exists i such that η_i factors Z_i. Indeed, to see this, we consider closed subschemes $\eta^* Z_i \subset \Lambda_k$. We may write the defining ideal of $\eta^* Z_i$ as $(\varepsilon^{m_i}) \subset \mathbb{C}[\varepsilon]/(\varepsilon^{k+1})$. By $Z \subset Z_1 + \cdots + Z_l$, we have $\eta^* Z_1 + \cdots + \eta^* Z_l = \Lambda_k$. The defining ideal of $\eta^* Z_1 + \cdots + \eta^* Z_l$ is $(\varepsilon^{m_1 + \cdots + m_l}) \subset \mathbb{C}[\varepsilon]/(\varepsilon^{k+1})$. Hence $m_1 + \cdots + m_l \geq k + 1$. We may take i such that $m_i \geq k_i + 1$. Hence $\eta_i\colon \Lambda_{k_i} \to S$ factors Z_i. Then by the previous step, we have $[v] \in \mathcal{D}((Z_i)_{\mathrm{red}})$. Note that $(Z_i)_{\mathrm{red}}$ is an irreducible component of Z_{red}. □

§10. Sufficient condition for horizontal integrability

The main result of this section is Lemma 10.1. This gives a sufficient condition for $Z \subset S_{1,A/B}$ to be horizontally integrable (cf. Section 3.2). This lemma is used in the proof of Proposition 11.3.

Let $\bar{A}$ be an equivariant compactification of a semi-abelian variety A. Let S be a smooth variety. Let $X \subset \bar{A} \times S$ be a closed subscheme. Let $\mathcal{X} \subset \bar{A} \times A \times S$ be the pull-back of X by the action $m\colon \bar{A} \times A \times S \to \bar{A} \times S$ defined by $(x, a, s) \mapsto (x + a, s)$ so that $\mathcal{X}_{(a,s)} = X_s - a$. Suppose that $X \subset \bar{A} \times S$ is B-invariant. Then by Lemma A.19 we have a closed subscheme $\mathcal{X}_B \subset \bar{A} \times (A/B) \times S$ such that $\mathcal{X}$ is the pull-back of $\mathcal{X}_B$ by the quotient $\bar{A} \times A \times S \to \bar{A} \times (A/B) \times S$ on the second factor. We get the closed subscheme $\mathcal{P}_k \mathcal{X}_B \subset \bar{A} \times ((A/B) \times S)_k$. By the isomorphism (2.12), we have

$$S_{k,A/B} = \{0_{A/B}\} \times S_{k,A/B} \subset (A/B) \times S_{k,A/B} = ((A/B) \times S)_k.$$

Using this immersion $S_{k,A/B} \subset ((A/B) \times S)_k$, we define a closed subscheme $X_{B,k} \subset \bar{A} \times S_{k,A/B}$ by

$$X_{B,k} = (\mathcal{P}_k \mathcal{X}_B)|_{\bar{A} \times S_{k,A/B}}. \tag{10.1}$$

Let $p_{k,B}\colon \bar{A} \times S_{k,A/B} \to S_{k,A/B}$ be the second projection. Let $p_{k,B}|_{X_{B,k}}\colon X_{B,k} \to S_{k,A/B}$ be the composite of the closed immersion $X_{B,k} \hookrightarrow \bar{A} \times S_{k,A/B}$ and $p_{k,B}$. Given $y \in S_{k,A/B}$, we denote by $(X_{B,k})_y \subset \bar{A}$ the scheme-theoretic fiber of $p_{k,B}|_{X_{B,k}}\colon X_{B,k} \to S_{k,A/B}$ over $y \in S_{k,B}$. For $k \geq l$, we have a natural morphism

$\mathcal{P}_k\mathcal{X}_B \to \mathcal{P}_l\mathcal{X}_B$. This induces the following commutative diagram:

$$\begin{array}{ccc} X_{B,k} & \longrightarrow & X_{B,l} \\ {\scriptstyle p_{k,B}|_{X_{B,k}}}\downarrow & & \downarrow{\scriptstyle p_{l,B}|_{X_{B,l}}} \\ S_{k,A/B} & \longrightarrow & S_{l,A/B}. \end{array}$$

For $y \in S_{k,A/B}$, let $y' \in S_{l,A/B}$ be the image of y under the map $S_{k,A/B} \to S_{l,A/B}$. Then we have the map $(X_{B,k})_y \to (X_{B,l})_{y'}$ of the closed subschemes of $\bar{A}$.

Let $V \subset \bar{A}$ be a closed subscheme. We define $\mathrm{Stab}(V) \subset A$ by $a \in \mathrm{Stab}(V)$ if and only if $a + V = V$ as closed subschemes of $\bar{A}$. Let $\mathrm{Stab}^0(V)$ be the connected component of $\mathrm{Stab}(V)$ containing the identity element of A. Then $\mathrm{Stab}^0(V) \subset A$ is a semi-abelian subvariety.

Let $k \geq 2$. We recall $S_{k,A}^{\mathrm{sing}} \subset S_{k,A}$ from (2.15). Set $S_{k,A}^{\mathrm{reg}} = S_{k,A} \backslash S_{k,A}^{\mathrm{sing}}$. Then we have $(A \times S)_k^{\mathrm{reg}} = A \times S_{k,A}^{\mathrm{reg}}$ (cf. (2.16)). The purpose of this section is to prove the following lemma.

Lemma 10.1. *Let $\bar{A}$ be an equivariant compactification of A, where $\bar{A}$ is projective. Let $B \subsetneqq A$ be a proper semi-abelian subvariety. Let $Z \subset S_{1,A/B}$ be an irreducible Zariski closed set. Let $X \subset \bar{A} \times S$ be a B-invariant closed subscheme such that $Z \subset p_{1,B}(X_{B,1})$, where $p_{1,B} \colon \bar{A} \times S_{1,A/B} \to S_{1,A/B}$ is the second projection. Assume that for every integer $k \geq 2$, there exists a non-empty Zariski open subset $O_k \subset Z$ such that for every $y \in O_k$ the following hold:*

(1) $\mathrm{Stab}^0((X_{B,1})_y) = B$.
(2) *The natural map $(X_{B,1})_y \to X_{\tau(y)}$ is an isomorphism as schemes, where $\tau \colon S_{1,A/B} \to S$ is the induced map.*
(3) *There exists $y_k \in S_{k,A/B}^{\mathrm{reg}}$ such that the image of y_k under $S_{k,A/B} \to S_{1,A/B}$ is y, and that the map $(X_{B,k})_{y_k} \to (X_{B,1})_y$ is an isomorphism as schemes.*

Then $Z \subset S_{1,A/B}$ is horizontally integrable.

According to Definition 3.16, the conclusion of Lemma 10.1 reads the existence of an immersion $U \hookrightarrow (A/B) \times S$ with the following properties:

- $q \colon U \to S$ is étale.
- $p'(PTU) \cap Z \subset Z$ is Zariski dense in Z, where $p' \colon (A/B) \times S_{1,A/B} \to S_{1,A/B}$ is the second projection.

Before proving Lemma 10.1, we start from algebro-geometric lemmas.

Lemma 10.2. *Let Σ and S be algebraic varieties such that $\dim \Sigma = \dim S$. Let $p\colon \Sigma \to S$ be unramified. Assume that S is smooth. Then $p\colon \Sigma \to S$ is smooth, hence étale.*

Proof. Set $d = \dim \Sigma = \dim S$. We first show that Σ is smooth. We have the exact sequence (cf. [18, Chapter II, Prop. 8.11])

$$p^*\Omega_S \to \Omega_\Sigma \to \Omega_{\Sigma/S} \to 0.$$

Since $p\colon \Sigma \to S$ is unramified, we have $\Omega_{\Sigma/S} = 0$ (cf. [31, p. 221]). Hence the morphism $p^*\Omega_S \to \Omega_\Sigma$ is surjective. Since $p^*\Omega_S$ is locally free of rank d, we have $\dim \Omega_\Sigma \otimes \mathbb{C}(x) \leq d$ for all $x \in \Sigma$. For general $x \in \Sigma$, we have $\dim \Omega_\Sigma \otimes \mathbb{C}(x) = d$, so by the upper semi-continuity, this holds for all $x \in \Sigma$. Since Σ is reduced, this shows Ω_Σ is locally free of rank d (cf. [18, Chapter II, Ex. 5.8]). Hence Σ is smooth.

Since $p^*\Omega_S \to \Omega_\Sigma$ is surjective, the induced map $\Omega_S \otimes \mathbb{C}(p(x)) \to \Omega_\Sigma \otimes \mathbb{C}(x)$ is surjective for all $x \in \Sigma$. Note that $\Omega_S \otimes \mathbb{C}(p(x)) \to \Omega_\Sigma \otimes \mathbb{C}(x)$ is the dual of the induced map on the tangent spaces $p_*\colon T_{\Sigma,x} \to T_{S,p(x)}$. Hence $p_*\colon T_{\Sigma,x} \to T_{S,p(x)}$ is injective for all $x \in \Sigma$. Hence by $\dim \Sigma = \dim S$, we obtain that $p_*\colon T_{\Sigma,x} \to T_{S,p(x)}$ is surjective for all $x \in \Sigma$. By [18, Chapter III, Prop. 10.4], $p\colon \Sigma \to S$ is smooth. See also [31, p. 141]. □

Next we apply the previous lemma to prove the following lemma.

Lemma 10.3. *Let S and W be smooth algebraic varieties. Let $p\colon W \to S$ be a smooth morphism. Let $\Sigma_0 \subset W$ be an irreducible Zariski closed set and $x \in \Sigma_0$. Assume that $p|_{\Sigma_0}\colon \Sigma_0 \to S$ is unramified at $x \in \Sigma_0$. Then there exists an irreducible Zariski closed set $\Sigma \subset W$ such that $\Sigma_0 \subset \Sigma$ and $p|_\Sigma\colon \Sigma \to S$ is étale at x.*

Proof. Let $h_1, \ldots, h_k$ be local defining functions of $\Sigma_0 \subset W$ around x. Let $L = p^{-1}(p(x))$ and set $s = \dim L$. Since p is smooth, $L \subset W$ is a smooth subvariety. Since $p|_{\Sigma_0}$ is unramified at $x \in \Sigma_0$, we have

$$\big\{v \in T_xL;\ (dh_1|_{T_xL})(v) = \cdots = (dh_k|_{T_xL})(v) = 0\big\} = \{0\}.$$

Since T_xL is an s-dimensional vector space, we may assume that

$$\big\{v \in T_xL;\ (dh_1|_{T_xL})(v) = \cdots = (dh_s|_{T_xL})(v) = 0\big\} = \{0\}.$$

We take $\Sigma \subset W$ such that Σ is defined by $h_1 = \cdots = h_s = 0$ around x. Since the images of $h_1, \ldots, h_s$ in $\mathfrak{m}_x/\mathfrak{m}_x^2$ are linearly independent, Σ is regular at x (cf. [17, Prop. B.74 (3)]). Hence we may assume that Σ is irreducible and reduced. Moreover, $p|_\Sigma\colon \Sigma \to S$ is unramified at x, and $\dim \Sigma \geq \dim W - s = \dim S$. Hence $\dim \Sigma = \dim S$. Since unramified is an open condition, there exists a non-empty

Zariski open set $\Sigma' \subset \Sigma$ such that $x \in \Sigma'$ and $\Sigma' \to S$ is unramified. Since S is smooth, $\Sigma' \to S$ is étale (cf. Lemma 10.2). Hence $\Sigma \to S$ is étale at x. Since Σ_0 is irreducible, we have $\Sigma_0 \subset \Sigma$. $\square$

Lemma 10.4. *Let M, Σ, and H be algebraic varieties. Let $c\colon M \times \Sigma \to H$ be a dominant morphism. Let $V \subset \Sigma$ be a closed subvariety. Assume that there exists a non-empty Zariski open set $W \subset M \times V$ such that the restriction $c|_W \colon W \to \overline{c(M \times V)}$ is étale. Let $a_0 \in M$ satisfy $(\{a_0\} \times V) \cap W \neq \emptyset$. Let $F \subset M \times \Sigma$ be an irreducible component of* $\operatorname{supp} c^{-1}(\overline{c(\{a_0\} \times V)})$ *such that $\{a_0\} \times V \subset F$. Then we have the following:*

(1) *For $t \in V$ with $(a_0, t) \in W$, the induced map $F \to \Sigma$ is unramified at $(a_0, t) \in F$.*
(2) *Suppose moreover that the restriction $c|_{M \times V} \colon M \times V \to H$ is dominant and generically finite. Then $F \to \Sigma$ is dominant and generically finite.*

Proof. We take $t \in V$ such that $(a_0, t) \in W$. Let F_t be the scheme-theoretic fiber of the map $F \to \Sigma$ over $t \in V$. Then $F_t = F \cap (M \times \{t\})$, where the intersection is taken scheme theoretically. We are going to prove $\mathcal{O}_{F_t,(a_0,t)} = \mathbb{C}$. Since $c|_W \colon W \to \overline{c(M \times V)}$ is étale, the scheme-theoretic intersection $F \cap (M \times V)$ coincides with $\{a_0\} \times V$ on some Zariski open neighborhood $U \subset W$ of $(a_0, t) \in W$. Namely $F \cap U = \{a_0\} \times V$. Hence

$$F_t \cap U = U \cap (\{a_0\} \times V) \cap (M \times \{t\}) = \{(a_0, t)\},$$

where $\{(a_0, t)\}$ is a reduced scheme. Hence $F_t \cap U = \operatorname{Spec} \mathbb{C}$. Since $F_t \hookrightarrow M \times \Sigma$ factors $F_t \hookrightarrow M \times V \hookrightarrow M \times \Sigma$, we have $\mathcal{O}_{F_t,(a_0,t)} = \mathbb{C}$. Hence $F \to \Sigma$ is unramified at (a_0, t).

Next suppose $c|_{M \times V} \colon M \times V \to H$ is dominant and generically finite. Note that all the irreducible components of fibers of $M \times \Sigma \to H$ have dimension greater than or equal to $\dim(M \times \Sigma) - \dim H$. Hence we have

$$\dim F \geq \dim(M \times \Sigma) - \dim H + \dim \overline{c(\{a_0\} \times V)}.$$

Since $c|_{M \times V} \colon M \times V \to H$ is dominant and generically finite, we have $\dim H = \dim(M \times V)$. By the choice of a_0, we have $\dim \overline{c(\{a_0\} \times V)} = \dim V$. Hence

$$\dim F \geq \dim(M \times \Sigma) - \dim(M \times V) + \dim V = \dim \Sigma.$$

Thus $\dim F \geq \dim \Sigma$. Since $F \to \Sigma$ is unramified at $(a_0, t) \in F$, where $(a_0, t) \in W$, the map $F \to \Sigma$ is dominant and generically finite. $\square$

Lemma 10.5. *Let M, Σ, and H be algebraic varieties. Let $c\colon M \times \Sigma \to H$ be a dominant morphism. Assume that for generic $s \in \Sigma$, the restriction $c|_{M \times \{s\}}$:*

$M \times \{s\} \to H$ is quasi-finite. Then there exists a closed subvariety $V \subset \Sigma$ such that $c|_{M\times V}\colon M \times V \to H$ is dominant and generically finite.

Proof. Set $d = \dim(M \times \Sigma) - \dim H$. Let $s \in \Sigma$ and $m \in M$ satisfy the following:

- $c|_{M\times\{s\}}\colon M \times \{s\} \to H$ is quasi-finite.
- Set $h = c((m,s)) \in H$ and $c^{-1}(h) = X$. Then all irreducible components of X have dimension equal to d (cf. [18, Chapter II, Ex. 3.22]).

Let $p\colon M \times \Sigma \to \Sigma$ be the second projection. By $(m,s) \in X$, we have $s \in p(X)$. Set $\overline{p(X)} = Y$. Then all irreducible components of Y have dimension equal to or less than d. We take a closed subvariety $V \subset \Sigma$ of codimension equal to d such that $\{s\} = \mathrm{supp}(V \cap Y)$ on some Zariski open neighborhood $U \subset \Sigma$ of $s \in \Sigma$. By $p^{-1}(s) \cap X = c|_{M\times\{s\}}^{-1}(h)$, the set $p^{-1}(s) \cap X$ is finite. Hence

$$c^{-1}(h) \cap (M \times V) \cap (M \times U) = p^{-1}(s) \cap X$$

consists of finite points. We note that this set is non-empty for it contains the point (m,s).

Now we consider the map $c|_{M\times V}\colon M \times V \to H$. Then $(c|_{M\times V})^{-1}(h)$ contains zero-dimensional irreducible components. Moreover, $\dim(M \times V) = \dim H$. Hence $c|_{M\times V}\colon M \times V \to H$ is dominant and generically finite. □

Proof of Lemma 10.1. The proof divides into several steps. In the following argument, we fix a projective embedding $\bar{A} \subset \mathbb{P}^N$.

Step 1. Let $Z_o = \tau(Z) \subset S$. Let $X_{Z_o} \subset \bar{A} \times Z_o$ be the base change. Let P be the Hilbert polynomial of generic fibers of $X_{Z_o} \to Z_o$. Let $Z_o^P \subset Z_o$ be a non-empty Zariski open set such that all the fibers over the points of Z_o^P have Hilbert polynomial P. Then since Z_o^P is integral, $X_{Z_o^P} \to Z_o^P$ is flat (cf. [18, Chapter III, Thm. 9.9]). We denote by $\{\mathcal{S}_1, \dots, \mathcal{S}_l\}$ the flattening stratification of the coherent sheaf $\mathcal{O}_X$ on $\mathbb{P}^N \times S$. (See [39, Thm. 4.2.11] for the existence of such a stratification.) Namely, each $\mathcal{S}_i$ is a locally closed subscheme of S such that

- $S = \coprod \mathcal{S}_i$, and
- a map $T \to S$ from a scheme T factors $\coprod \mathcal{S}_i \to S$ if and only if $X_T \to T$ is flat.

We may choose $S^P \in \{\mathcal{S}_i\}$ such that $Z_o^P \subset S$ factors as

$$Z_o^P \subset S^P.$$

Then $S^P \subset S$ is a locally closed subscheme.

There exists a non-empty Zariski open set $S^o \subset S$ such that $S^P \hookrightarrow S$ factors $S^P \hookrightarrow S^o \subset S$, where $S^P \hookrightarrow S^o$ is a closed immersion. Replacing S by S^o, we may assume that $S^P \subset S$ is a closed subscheme. Under this reduction, the relation $Z_o^P \subset S^P$ implies $Z_o \subset S^P$. Hence $Z_o^P = Z_o$.

Step 2. Let $\mathcal{X} \subset \bar{A} \times A \times S$ be the pull-back of X by the action $m\colon \bar{A} \times A \times S \to \bar{A} \times S$, where $(x, a, s) \mapsto (x + a, s)$ so that

$$\mathcal{X}_{(a,s)} = X_s - a.$$

Since X is B-invariant, $\mathcal{X}$ is B-invariant under the B-action $\bar{A}\times A\times S \to \bar{A}\times A\times S$ defined by $(x, a, s) \mapsto (x, a+b, s)$ for $b \in B$. Hence there exists a closed subscheme $\mathcal{X}_B \subset \bar{A}\times(A/B)\times S$ such that $\mathcal{X}$ is the pull-back of $\mathcal{X}_B$ by the quotient $\bar{A}\times A\times S \to \bar{A} \times (A/B) \times S$ on the second factor (cf. Lemma A.19).

Let $T \to (A/B) \times S$ be a map from a scheme T. Let $(\mathcal{X}_B)_T \to T$ be the pull-back of $\mathcal{X}_B \to (A/B) \times S$ by this map $T \to (A/B) \times S$. Let $X_T \to T$ be the pull-back of $X \to S$ by the composition of $T \to (A/B) \times S$ and the second projection $(A/B) \times S \to S$.

Claim 1. The morphism $(\mathcal{X}_B)_T \to T$ is flat if and only if $X_T \to T$ is flat.

We prove this. Let $\mathcal{X}' \subset \bar{A} \times A \times S$ be the pull-back of $X \subset \bar{A} \times S$ by the map $\bar{A} \times A \times S \to \bar{A} \times S$ defined by $(x, a, s) \mapsto (x, s)$. Let $\mathcal{X}'_B$ be the pull-back of X by the map $\bar{A} \times (A/B) \times S \to \bar{A} \times S$ defined by $(x, a', s) \mapsto (x, s)$. Then $\mathcal{X}'$ is the pull-back of $\mathcal{X}'_B$ by the quotient $\bar{A} \times A \times S \to \bar{A} \times (A/B) \times S$ on the second factor. Note that the isomorphism $\kappa\colon \bar{A} \times A \times S \to \bar{A} \times A \times S$ defined by $(x, a, s) \mapsto (x + a, a, s)$ induces the isomorphism $\kappa|_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}'$ over $A \times S$.

Let $T' \to A \times S$ be the pull-back of $T \to (A/B) \times S$ by the quotient map $A\times S \to (A/B)\times S$. Then the induced map $T' \to T$ is faithfully flat, for $A\times S \to (A/B) \times S$ is faithfully flat (cf. Remark A.1). Let $\mathcal{X}_{T'} \to T'$ be the pull-back of $\mathcal{X} \to A \times S$ by the map $T' \to A \times S$. Then we have the following Cartesian diagram:

$$\begin{CD}
\mathcal{X}_{T'} @>>> (\mathcal{X}_B)_T \\
@VVV @VVV \\
T' @>>> T.
\end{CD}$$

Hence

$$(\mathcal{X}_B)_T \to T \text{ is flat if and only if } \mathcal{X}_{T'} \to T' \text{ is flat.} \tag{10.2}$$

Indeed, if $\mathcal{X}_{T'} \to T'$ is flat, the composite map $\mathcal{X}_{T'} \to T$ is flat. Note that the faithful flatness of $T' \to T$ yields that of $\mathcal{X}_{T'} \to (\mathcal{X}_B)_T$, for the faithful flatness is stable under the fiber product. Hence by [17, Cor. 14.12], we obtain that $(\mathcal{X}_B)_T \to T$ is flat. Conversely, the flatness of $(\mathcal{X}_B)_T \to T$ yields that of $\mathcal{X}_{T'} \to T'$, for the flatness is stable under the fiber product. Thus we obtain (10.2).

The isomorphism κ induces an isomorphism $\mathcal{X}_{T'} \to (\mathcal{X}')_{T'}$ over T'. Hence

$$\mathcal{X}_{T'} \to T' \text{ is flat if and only if } (\mathcal{X}')_{T'} \to T' \text{ is flat.}$$

For the same reason as (10.2), we note that

$$(\mathcal{X}')_{T'} \to T' \text{ is flat if and only if } (\mathcal{X}'_B)_T \to T \text{ is flat.}$$

Note that $(\mathcal{X}'_B)_T = X_T$. Hence $(\mathcal{X}_B)_T \to T$ is flat if and only if $X_T \to T$ is flat. This concludes the proof of the claim.

Now, we note that $X_{S^P} \to S^P$ is flat. Hence by the claim above, the morphism

$$(\mathcal{X}_B)_{(A/B)\times S^P} \to (A/B) \times S^P$$

is flat. Let Hilb be the Hilbert scheme of closed subschemes of $\bar{A}$ with Hilbert polynomials P (cf. [39, Thm. 4.3.4]). By $(\mathcal{X}_B)_{(A/B)\times S^P} \to (A/B) \times S^P$, we have the classification map

$$c\colon (A/B) \times S^P \to \mathrm{Hilb}.$$

Step 3. For each $(a, s) \in (A/B) \times S^P$, we set $E_{(a,s)} = c^*(c(a, s))$, which is the scheme-theoretic fiber of c over the point $c(a, s) \in \mathrm{Hilb}$. Then $E_{(a,s)} \subset (A/B)\times S^P$ is a closed subscheme. We claim that

$$E_{(a,s)} = a + E_{(0,s)}. \tag{10.3}$$

We prove this. For $t\colon T \to (A/B) \times S$, we get $t - a\colon T \to (A/B) \times S$ by the translation. We denote by $(\mathcal{X}_B)_t \subset \bar{A}\times T$ the pull-back of $\mathcal{X}_B$ by $t\colon T \to (A/B)\times S$. Similarly for $(\mathcal{X}_B)_{t-a}$. Let $a' \in A$ be a point whose image under $A \to A/B$ is equal to a. Then we note that

$$(\mathcal{X}_B)_t = (\mathcal{X}_B)_{t-a} - a'. \tag{10.4}$$

To show this, it is enough to consider the case that $t = \mathrm{id}\colon (A/B)\times S \to (A/B)\times S$. Let $t'\colon A \times S \to A \times S$ be the identity map and $t' - a'\colon A \times S \to A \times S$ be the map defined by $(x, s) \mapsto (x - a', s)$. Then

$$\mathcal{X}_{t'} = \mathcal{X}_{t'-a'} - a'.$$

Note that $\mathcal{X}_{t'}$ (resp. $\mathcal{X}_{t'-a'} - a'$) is the pull-back of $(\mathcal{X}_B)_t$ (resp. $(\mathcal{X}_B)_{t-a} - a'$) under the quotient $\bar{A} \times A \times S \to \bar{A} \times (A/B) \times S$ on the second factor. Moreover, the induced maps $\mathcal{X}_{t'} \to (\mathcal{X}_B)_t$ and $\mathcal{X}_{t'-a'} - a' \to (\mathcal{X}_B)_{t-a} - a'$ are the categorical quotients (cf. Lemma A.19). Hence we get (10.4).

Now we note that $t\colon T \to (A/B) \times S^P$ factors $E_{(a,s)}$ if and only if

$$(\mathcal{X}_B)_t = (\mathcal{X}_B)_{(a,s)} \times T$$

as closed subschemes of $\bar{A} \times T$. We note that

$$(\mathcal{X}_B)_{(a,s)} \times T = \mathcal{X}_{(a',s)} \times T = (X_s - a') \times T = (X_s \times T) - a'.$$

Hence $t\colon T \to (A/B) \times S^P$ factors $E_{(a,s)}$ if and only if

$$(\mathcal{X}_B)_t = (X_s \times T) - a'$$

as closed subschemes of $\bar{A} \times T$. Using this, we note that $t - a\colon T \to (A/B) \times S^P$ factors $E_{(0,s)}$ if and only if

$$(\mathcal{X}_B)_{t-a} = X_s \times T$$

as closed subschemes of $\bar{A} \times T$. Hence (10.4) yields that $t\colon T \to (A/B) \times S^P$ factors $E_{(a,s)}$ if and only if $t - a\colon T \to (A/B) \times S$ factors $E_{(0,s)}$. Thus $E_{(a,s)} = a + E_{(0,s)}$. This completes the proof of (10.3).

Step 4. For each $[v] \in Z$, we have $\tau([v]) \in Z_o \subset S^P$. Hence for each $a \in A/B$, we may consider $E_{(a,\tau([v]))}$. We prove the following.

Claim 2. For all $[v] \in \left(\bigcap_{k\geq 2} O_k\right) \subset Z$ and all $a \in A/B$, there exists an irreducible component E of $\operatorname{supp} E_{(a,\tau([v]))}$ such that

$$(a, [v]) \in \mathcal{D}E \subset (A/B) \times S_{1,A/B} \tag{10.5}$$

under the identification $(A/B) \times S_{1,A/B} = ((A/B) \times S)_1$. Here, $E \subset (A/B) \times S$ is a Zariski closed subset.

We prove this. Denoting $S' = (A/B) \times S$, we have $(S')_k = (A/B) \times S_{k,A/B}$. For each $k \geq 2$, by our assumptions (2), (3), and Corollary 9.9, we may take a regular k-jet $\eta\colon \Lambda_k \to S'$ with $\eta'(0) = (0, v)$ such that $(\mathcal{X}_B)_{(0,\tau([v]))} = (\mathcal{P}_k\mathcal{X}_B)_{\eta_{[k]}(0)}$, where $\Lambda_k = \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^{k+1})$. Hence by Lemma 9.1, we have

$$(\mathcal{X}_B)_\eta = (\mathcal{X}_B)_{(0,\tau([v]))} \times \Lambda_k. \tag{10.6}$$

Hence $(\mathcal{X}_B)_\eta \to \Lambda_k$ is flat. Let $\eta_2\colon \Lambda_k \to S$ be the composition with η and the second projection $(A/B) \times S \to S$. Then by Claim 1 in Step 2, $X_{\eta_2} \to \Lambda_k$ is flat. By $\eta_2(0) = \tau([v]) \in Z_o$, $\eta_2\colon \Lambda_k \to S$ factors S^P. Thus $\eta\colon \Lambda_k \to S'$ factors $(A/B) \times S^P$, hence $E_{(0,\tau([v]))}$ (cf. (10.6)). This holds for all k. Hence by Lemma 9.10, we have

$$(0, [v]) \in \mathcal{D}E'$$

for some irreducible component E' of $\operatorname{supp} E_{(0,\tau([v]))}$. By (10.3), we have

$$E_{(a,\tau([v]))} = a + E_{(0,\tau([v]))}.$$

Hence $E = a + E'$ is an irreducible component of $\mathrm{supp}(E_{(a,\tau([v]))})$. By $\mathcal{D}E = a + \mathcal{D}E'$, we complete the proof of the claim.

Step 5. We are going to take a closed subvariety $V \subset Z_o$ such that the restriction

$$c|_{(A/B)\times V} : (A/B) \times V \to \overline{c((A/B) \times Z_o)}$$

is dominant, and generically finite. Here, $\overline{c((A/B) \times Z_o)} \subset \mathrm{Hilb}$ is the Zariski closure in Hilb. Note that the set $\tau(O_2) \subset Z_o$ is dense and constructible (cf. [18, Chapter II, Ex. 3.19]), hence contains non-empty Zariski open set. Hence for generic $t \in Z_o$, by assumptions (1), (2), we have $\mathrm{Stab}^0(X_t) = B$. Hence the restriction

$$c|_{(A/B)\times\{t\}} \colon (A/B) \times \{t\} \to \mathrm{Hilb}$$

is quasi-finite. We apply Lemma 10.5 to $(A/B) \times Z_o \to \overline{c((A/B) \times Z_o)}$ to get our $V \subset Z_o$.

Let $W \subset (A/B) \times V$ be a non-empty Zariski open subset such that $W \to \overline{c((A/B) \times Z_o)}$ is étale. We fix $a_0 \in A/B$ such that

$$(\{a_0\} \times V) \cap W \neq \emptyset.$$

Step 6. We construct Zariski closed subsets F_o and Σ_0 of $(A/B) \times S$. (After $U \hookrightarrow (A/B) \times S$ is constructed, these would be irreducible components of $\overline{U} \cap p^{-1}(Z_o)$ and $\overline{U} \cap p^{-1}(\mathrm{supp}\, S^P)$, where $p\colon (A/B) \times S \to S$ is the second projection.) We consider the restriction

$$c|_{(A/B)\times Z_o} \colon (A/B) \times Z_o \to \overline{c((A/B) \times Z_o)} \subset \mathrm{Hilb}.$$

We take an irreducible component F_o of a Zariski closed set

$$(c|_{(A/B)\times Z_o})^{-1}(\overline{c(\{a_0\} \times V)})$$

such that $\{a_0\} \times V \subset F_o$. Here, $\overline{c(\{a_0\} \times V)} \subset \mathrm{Hilb}$ is the Zariski closure in Hilb. We have a map

$$p|_{F_o} \colon F_o \to Z_o$$

by the composition of the closed immersion $F_o \hookrightarrow (A/B) \times Z_o$ and the second projection $p|_{(A/B)\times Z_o} \colon (A/B) \times Z_o \to Z_o$. Then by Lemma 10.4 (2), the map $p|_{F_o} \colon F_o \to Z_o$ is dominant and generically finite.

Next we construct $\Sigma_0 \subset (A/B) \times S$ so that $F_o \subset \Sigma_0$. Let F be an irreducible component of $F_o \times_{Z_o} Z$ such that the natural maps $F \to F_o$ and $p'|_F \colon F \to Z$ are dominant, where $p' \colon (A/B) \times S_{1,A/B} \to S_{1,A/B}$ is the second projection. We remark that $F_o \times_{Z_o} Z$ is a Zariski closed set of $(A/B) \times Z$. Hence $F \subset (A/B) \times Z$ is a Zariski closed subset. Let $\Theta \subset (A/B) \times S^P$ be a Zariski closed subset defined by

$\Theta = c^{-1}(\overline{c(\{a_0\} \times V)})$. Then $\Theta \subset (A/B) \times S$ is a Zariski closed subset. Let $(a, [v]) \in \bigcap_{k \geq 2}(p'|_F)^{-1}(O_k) \subset F$. We take an irreducible component E of $\mathrm{supp}\, E_{(a,\tau([v]))}$ as in Claim 2 of Step 4. By $(a, \tau([v])) \in F_o$, we have $c((a, \tau([v]))) \in \overline{c(\{a_0\} \times V)}$. Hence $E \subset \Theta$. Hence there exists an irreducible component Θ' of Θ such that $E \subset \Theta'$. Then $\Theta' \subset (A/B) \times S$ is a Zariski closed subset. By (10.5), we have $(a, [v]) \in \mathcal{D}\Theta'$. Hence, denoting by $\Theta_1, \ldots, \Theta_l$ all irreducible components of Θ, we have

$$\bigcap_k (p'|_F)^{-1}(O_k) \subset \mathcal{D}\Theta_1 \cup \cdots \cup \mathcal{D}\Theta_l.$$

Since $\mathcal{D}\Theta_i \subset (A/B) \times S$ are closed subschemes, $\mathrm{supp}(F \cap \mathcal{D}\Theta_i)$ are Zariski closed subsets of F. Note that $(p'|_F)^{-1}(O_k) \subset F$ is a non-empty Zariski open set for each k. Hence we may choose Σ_0 from $\Theta_1, \ldots, \Theta_l$ such that

$$F \subset \mathcal{D}\Sigma_0. \tag{10.7}$$

Since the natural map $F \to F_o$ is surjective, we have

$$F_o \subset \Sigma_0.$$

In particular, $\{a_0\} \times V \subset \Sigma_0$. Hence we may apply Lemma 10.4 (1) for

$$(A/B) \times S^P \to \overline{c((A/B) \times S^P)}$$

to get that $p|_{\Sigma_0} \colon \Sigma_0 \to S$ is unramified at generic $(a_0, t) \in \{a_0\} \times V$, where $p \colon (A/B) \times S \to S$ is the second projection.

Step 7. Now we apply Lemma 10.3 to take $\Sigma \subset (A/B) \times S$ from $\Sigma_0 \subset (A/B) \times S$ such that $\Sigma_0 \subset \Sigma$ and $\Sigma \to S$ is generically finite and étale at generic $(a_0, t) \in \{a_0\} \times V \subset \Sigma_0$. We take a Zariski open $U \subset \Sigma$ such that $q \colon U \to S$ is étale, where q is the restriction of p onto U. We may take U so that $(\{a_0\} \times V) \cap U \neq \emptyset$, hence $F_o \cap U \neq \emptyset$. Hence $F_o \cap U \subset F_o$ is Zariski dense. By (10.7), this shows that $F \cap PTU \subset F$ is Zariski dense, where $PTU = (\mathcal{D}\Sigma)|_U$ in $(A/B) \times S_{1,A/B}$. Hence $p'(F \cap PTU) \subset Z$ is Zariski dense in Z. Hence $p'(PTU) \cap Z \subset Z$ is Zariski dense in Z. This shows that the immersion $U \hookrightarrow (A/B) \times S$ satisfies the property of Definition 3.16. Hence Z is horizontally integrable. This completes the proof of the lemma. □

§11. Verifying the normality condition: Existence of horizontally integrable Z

The purpose of this section is to prove Proposition 11.3 below. To state this proposition, we introduce some terminology. We recall $\Pi(\mathcal{F})$ from Definition 3.18.

Definition 11.1. Let $X \subset A$ be a closed subvariety. Let $\bar{A}$ be an equivariant compactification. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, X)$ be an infinite set of holomorphic maps. We define $\Lambda_{X,\bar{A}}(\mathcal{F})$ to be the set of all semi-abelian subvarieties $B \subset A$ such that

(1) $\mathcal{F} \to \mathrm{Sp}_B \overline{X}$, where $\overline{X} \subset \bar{A}$ is the compactification, and
(2) $B \notin \Pi(\mathcal{F})$.

Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A)$ be an infinite set of holomorphic maps. For $B \in \Pi(\mathcal{F})$, denoting the quotient map by $\varpi_B \colon A \to A/B$, we set

$$\mathcal{F}_{A/B} = (\varpi_B \circ f)_{f \in \mathcal{F}}.$$

This is an infinite indexed family in $\mathrm{Hol}(\mathbb{D}, A/B)$. Then $\mathcal{F}_{A/B}$ contains at most finitely many constant maps. We remove these constant maps from $\mathcal{F}_{A/B}$ to get an infinite subfamily $\mathcal{F}'_{A/B}$ of $\mathcal{F}_{A/B}$. We consider the following assumption for $\mathcal{F}$.

Assumption 11.2. For every $B \in \Pi(\mathcal{F})$, the infinite indexed family $\mathcal{F}'_{A/B}$ in $\mathrm{Hol}(\mathbb{D}, A/B)$ satisfies Assumption 3.19.

As we shall see later (cf. Lemma 12.3), every infinite subset $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A)$ of non-constant holomorphic maps contains an infinite subset which satisfies this assumption. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A)$ be an infinite set of holomorphic maps which satisfies Assumption 11.2. Let $B \in \Pi(\mathcal{F})$. Then there exists a unique $T_{k,A/B} \subset P_{k,A/B}$ as in Assumption 3.19. We define $Z_{k,A/B} \subset P_{k,A/B}$ by

$$Z_{k,A/B} = \bigcup_{l \geq 0} \mathrm{Im}(T_{k+l,A/B} \hookrightarrow P_{k+l,A/B} \to P_{k,A/B}). \tag{11.1}$$

Then by Lemma 3.15, we have $\mathcal{F}_{P_{k,A/B}} \Rightarrow Z_{k,A/B}$, where we set $\mathcal{F}_{P_{k,A/B}} = (\mathcal{F}'_{A/B})_{P_{k,A/B}}$ (cf. (3.5)).

Proposition 11.3. *Let $X \subset A$ be a closed subvariety and let $\bar{A}$ be a smooth equivariant compactification, where $\bar{A}$ is projective. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, X)$ be an infinite set of holomorphic maps which satisfies Assumption 11.2. Assume that $\Lambda_{X,\bar{A}}(\mathcal{F}) = \emptyset$. Then there exists a semi-abelian subvariety $B \subset A$ such that $\mathrm{Sp}_B \overline{X} \neq \emptyset$ with the following properties:*

(1) *$B \in \Pi(\mathcal{F})$.*
(2) *There exists k such that $Z_{k+1,A/B} \subset P_{k+1,A/B}$ is horizontally integrable, where*

$$Z_{k+1,A/B} = \bigcup_{l \geq 1} \mathrm{Im}(T_{k+l,A/B} \hookrightarrow P_{k+l,A/B} \to P_{k+1,A/B}).$$

(3) *$\mathcal{F} \to \mathrm{Sp}_B \overline{X}$.*

We note that the assumption $\Lambda_{X,\bar{A}}(\mathcal{F})=\emptyset$ in this proposition reads $X \subsetneqq A$. Indeed, if $X=A$, then we have $A\in\Lambda_{X,\bar{A}}(\mathcal{F})$, hence $\Lambda_{X,\bar{A}}(\mathcal{F})\neq\emptyset$.

§11.1. First series of auxiliary lemmas

In this subsection, we prove several lemmas related to the following definition.

Definition 11.4. Let S be a variety and let X be a scheme of finite type over $\mathbb{C}$. Let $\psi\colon X\to S$ be a morphism and let $Z\subset S$ be a Zariski closed set. Let $\mathcal{U}$ be the set of all Zariski open sets $U\subset S$ such that $Z\cap U\neq\emptyset$. For $U\in\mathcal{U}$, we consider the scheme-theoretic closure $\overline{\psi^{-1}(U)}\subset X$. We set $X[Z]=\bigcap_{U\in\mathcal{U}}\overline{\psi^{-1}(U)}$, which is a closed subscheme of X.

Remark 11.5. Assume that Z is irreducible. Then there exists $U\in\mathcal{U}$ such that $X[Z]=\overline{\psi^{-1}(U)}$. Indeed, by the Noetherian property, there exist $U_1,\dots,U_k\in\mathcal{U}$ such that $X[Z]=\overline{\psi^{-1}(U_1)}\cap\cdots\cap\overline{\psi^{-1}(U_k)}$. Set $U=U_1\cap\cdots\cap U_k$. Then since Z is irreducible, we have $U\in\mathcal{U}$. Then we have $X[Z]\subset\overline{\psi^{-1}(U)}\subset\overline{\psi^{-1}(U_1)}\cap\cdots\cap\overline{\psi^{-1}(U_k)}$. Hence $X[Z]=\overline{\psi^{-1}(U)}$.

Lemma 11.6. *Let $X\to S$ and $\varphi\colon S\to T$ be morphisms, where S and T are varieties and X is a scheme of finite type over $\mathbb{C}$. Let $\psi\colon X\to T$ be the composite of $X\to S\to T$. Let $V\subset S$ and $W\subset T$ be irreducible Zariski closed subsets such that $W\subset\overline{\varphi(V)}$. Then $X[V]\subset X[W]$. Moreover, assume that there exists a Zariski open set $U\subset T$ such that $U\cap W\neq\emptyset$ and that $\psi^{-1}(U)$ is integral. Then $X[V]=X[W]$.*

Proof. By Remark 11.5, we may take a Zariski open $U_1\subset T$ such that $U_1\cap W\neq\emptyset$ and

$$X[W]=\overline{\psi^{-1}(U_1)}. \tag{11.2}$$

We have $\varphi^{-1}(U_1)\cap V\neq\emptyset$. Hence by the definition of $X[V]$, we have

$$X[V]\subset\overline{\psi^{-1}(U_1)}=X[W].$$

Now we assume moreover that $\psi^{-1}(U)$ is integral for some Zariski open $U\subset T$ such that $U\cap W\neq\emptyset$. We take $U_1\subset T$ as above. We may assume $U_1\subset U$. By $\varphi^{-1}(U_1)\cap V\neq\emptyset$, we may take $U_2\subset\varphi^{-1}(U_1)$ such that $\overline{p^{-1}(U_2)}=X[V]$, where $p\colon X\to S$ (cf. Remark 11.5). Since $\psi^{-1}(U_1)$ is integral, the inclusion $p^{-1}(U_2)\subset\psi^{-1}(U_1)$ is schematic dense. Hence $\psi^{-1}(U_1)\subset X[V]$. Thus by (11.2), we get $X[W]\subset X[V]$. This concludes the proof of the lemma. $\square$

Lemma 11.7. *Let T be a variety and let $W\subset T$ be an irreducible Zariski closed subset. Let $\psi\colon X\to T$ and $\psi'\colon X'\to T$ be morphisms from schemes of finite* *type over $\mathbb{C}$. Let $p\colon X \hookrightarrow X'$ be a closed immersion over T. Assume that there exists a Zariski open set $U \subset T$ such that $U \cap W \neq \emptyset$ and the induced map $\psi^{-1}(U) \to (\psi')^{-1}(U)$ is an isomorphism. Then $X[W] = X'[W]$.*

Proof. The immersion p induces a closed immersion $X[W] \hookrightarrow X'[W]$. Hence we prove the converse. By Remark 11.5, we may take a Zariski open $U \subset T$ such that $U \cap W \neq \emptyset$ and $\overline{\psi^{-1}(U)} = X[W]$. By replacing U by a smaller Zariski open set, we may assume moreover that the induced map $\psi^{-1}(U) \to (\psi')^{-1}(U)$ is an isomorphism. Hence the scheme-theoretic closure $\overline{(\psi')^{-1}(U)} \subset X'$ factors $X[W]$, where $X[W]$ is a closed subscheme of X' by $p\colon X \hookrightarrow X'$. This shows $X'[W] \subset X[W]$. $\square$

We recall Definition 3.1. Let $\varphi\colon S \to T$ be a morphism of varieties. Let $V \subset S$ and $W \subset T$ be irreducible Zariski closed subsets such that $W \subset \overline{\varphi(V)}$. Let $\psi\colon T' \to T$ be a W-admissible modification and set $S' = (S \times_T T')[W']$, where $W' \subset T'$ is the minimal transform. Then the induced map $S' \to S$ is a V-admissible modification. We denote by $V' \subset S'$ the minimal transform of $V \subset S$. In this situation, we have the following lemma.

Lemma 11.8. *Let $X \to S$ and $\varphi\colon S \to T$ be morphisms, where S and T are varieties and X is a scheme of finite type over $\mathbb{C}$. Let $V \subset S$ and $W \subset T$ be irreducible Zariski closed subsets such that $W \subset \overline{\varphi(V)}$. Let $T' \to T$ be a W-admissible modification and set $S' = (S \times_T T')[W']$. Then*

$$(X \times_S S')[V'] \subset (X \times_T T')[W'].$$

Assume moreover that X is integral. Then $(X \times_S S')[V'] = (X \times_T T')[W']$.

Proof. We first prove

$$(X \times_S S')[W'] = (X \times_T T')[W']. \tag{11.3}$$

By the closed immersion $S' \hookrightarrow S \times_T T'$, we get a closed immersion $X \times_S S' \hookrightarrow X \times_T T'$. Note that the following is the fiber product:

$$\begin{tikzcd} X \times_S S' \arrow[r] \arrow[d] & X \times_T T' \arrow[r, equal] \arrow[d] & X \times_S (S \times_T T') \\ S' \arrow[r] & S \times_T T'. & \end{tikzcd} \tag{11.4}$$

By Remark 11.5, we may take a Zariski open $U_1 \subset T'$ such that $U_1 \cap W' \neq \emptyset$, and a Zariski open set $S \times_T U_1 \subset S \times_T T'$ is schematic dense in $S' \subset S \times_T T'$. In particular, we have an open immersion

$$S \times_T U_1 \subset S'. \tag{11.5}$$

Hence the map $S' \to S \times_T T'$ over T' is an isomorphism over $U_1 \subset T'$. Hence the closed immersion $X \times_S S' \hookrightarrow X \times_T T'$ is an isomorphism over $U_1 \subset T'$. Hence by Lemma 11.7, we get (11.3).

Set $X' = X \times_S S'$. Let $\psi \colon X' \to T'$ be the natural map. Since (11.4) is a fiber product, the open immersion (11.5) yields

$$\psi^{-1}(U_1) = X \times_T U_1. \tag{11.6}$$

Now let $\varphi' \colon S' \to T'$ be the induced map. We have $W' \subset \overline{\varphi'(V')}$. We apply Lemma 11.6 to get $X'[V'] \subset X'[W']$. Hence combining with (11.3), we get

$$(X \times_S S')[V'] \subset (X \times_T T')[W'].$$

We assume moreover that X is integral. We take a Zariski open set $U_2 \subset T'$ such that $T' \to T$ is an isomorphism over U_2 and $U_2 \cap W' \neq \emptyset$. We may assume $U_2 \subset U_1$. Then by (11.6), we may consider $\psi^{-1}(U_2)$ as a Zariski open set of X. Hence $\psi^{-1}(U_2)$ is integral. Hence by Lemma 11.6, we get $X'[V'] = X'[W']$. Hence by (11.3), we conclude the proof of our lemma. □

Lemma 11.9. *Let $Z \subset S$ be an irreducible Zariski closed set. Let $S' \to S$ be a Z-admissible modification and let $S'' \to S'$ be a Z'-admissible modification, where $Z' \subset S'$ is the minimal transform. Let $X \to S$ be a morphism. Then we have*

$$(X \times_S S'')[Z''] = ((X \times_S S')[Z'] \times_{S'} S'')[Z''],$$

where $Z'' \subset S''$ is the minimal transform.

Proof. The closed immersion $(X \times_S S')[Z'] \hookrightarrow X \times_S S'$ induces a closed immersion

$$((X \times_S S')[Z'] \times_{S'} S'')[Z''] \hookrightarrow (X \times_S S'')[Z''].$$

We shall show that this is an isomorphism. By Remark 11.5, we may take a Zariski open set $U_1 \subset S'$ such that $Z' \cap U_1 \neq \emptyset$ and

$$(X \times_S S')[Z'] = \overline{X \times_S U_1}.$$

We denote the natural map by $\varphi \colon S'' \to S'$. By Remark 11.5, we may take a Zariski open set $U_2 \subset \varphi^{-1}(U_1)$ such that $U_2 \cap Z'' \neq \emptyset$ and

$$((X \times_S S')[Z'] \times_{S'} S'')[Z''] = \overline{(X \times_S S')[Z'] \times_{S'} U_2}.$$

Hence, we have

$$\begin{aligned} X \times_S U_2 = (X \times_S U_1) \times_{S'} U_2 &\subset (X \times_S S')[Z'] \times_{S'} U_2 \\ &\subset ((X \times_S S')[Z'] \times_{S'} S'')[Z'']. \end{aligned}$$

Then by the definition of $(X \times_S S'')[Z'']$, we have $(X \times_S S'')[Z''] \subset \overline{X \times_S U_2}$. Hence we get

$$(X \times_S S'')[Z''] \subset ((X \times_S S')[Z'] \times_{S'} S'')[Z''].$$

This completes the proof of our lemma. □

In the following two lemmas, we consider a closed subscheme $X \subset \mathbb{P}^N \times S$.

Lemma 11.10. *Let $S' \to S$ be a Z-admissible modification of varieties, where $Z \subset S$ is an irreducible Zariski closed set. Let $X \to S$ be a projective morphism such that $X|_Z \to Z$ is flat, where $X|_Z = X \times_S Z$. Let $Z' \subset S'$ be the minimal transform. Set $X' = (X \times_S S')[Z']$ and $X'|_{Z'} = X' \times_{S'} Z'$. Then $X'|_{Z'} = X|_Z \times_Z Z'$. In particular, $X'|_{Z'} \to Z'$ is flat.*

Proof. The closed immersion $X' \hookrightarrow X \times_S S'$ induces a closed immersion

$$X'|_{Z'} \hookrightarrow (X \times_S S')|_{Z'}. \tag{11.7}$$

By Remark 11.5, we may take a Zariski open $U \subset S'$ such that $U \cap Z' \neq \emptyset$ and $\overline{X \times_S U} = X'$. In particular, we have an open immersion

$$(X \times_S S')|_{Z' \cap U} \subset X'|_{Z'}. \tag{11.8}$$

Since the composite of $Z' \hookrightarrow S' \to S$ factors through $Z \hookrightarrow S$, we have

$$(X \times_S S')|_{Z'} = X|_Z \times_Z Z'. \tag{11.9}$$

In particular, the morphism $(X \times_S S')|_{Z'} \to Z'$ is flat. Hence by Lemma B.1, the inclusion $(X \times_S S')|_{Z' \cap U} \subset (X \times_S S')|_{Z'}$ is scheme-theoretic dense. Hence by (11.8), we get $(X \times_S S')|_{Z'} \subset X'|_{Z'}$. Thus by (11.7), we get $X'|_{Z'} = (X \times_S S')|_{Z'}$. By (11.9), we get $X'|_{Z'} = X|_Z \times_Z Z'$. □

Lemma 11.11. *Let $X \subset \mathbb{P}^N \times S$ be a closed subscheme where S is a variety. Let $Z \subset S$ be an irreducible Zariski closed set. Then there exists a Z-admissible modification $S' \to S$ such that $(X \times_S S')[Z']|_{Z'} \to Z'$ is flat, where $Z' \subset S'$ is the minimal transform and $(X \times_S S')[Z']|_{Z'} = (X \times_S S')[Z'] \times_{S'} Z'$.*

Proof. We apply Lemma B.2 to get a Z-admissible modification $S' \to S$ and a Zariski open set $U \subset S'$ such that $U \cap Z' \neq \emptyset$ and that $X'|_{Z'} \to Z'$ is flat, where $X' \subset X \times_S S'$ is the scheme-theoretic closure of $X \times_S U$. Then we have $(X \times_S S')[Z'] \subset X'$. Hence $(X \times_S S')[Z'][Z'] \subset X'[Z']$. By Lemma 11.9, applied to $S'' = S'$, we get

$$(X \times_S S')[Z'][Z'] = (X \times_S S')[Z'].$$

Hence $(X \times_S S')[Z'] \subset X'[Z']$. On the other hand, by $X' \subset X \times_S S'$, we get $X'[Z'] \subset (X \times_S S')[Z']$. Thus $X'[Z'] = (X \times_S S')[Z']$. We apply Lemma 11.10 to $X' \to S'$ and $\mathrm{id}_{S'} \colon S' \to S'$. The conclusion is $X'[Z']|_{Z'} = X'|_{Z'}$. Hence $(X \times_S S')[Z']|_{Z'} = X'|_{Z'}$. Thus $(X \times_S S')[Z']|_{Z'} \to Z'$ is flat. $\square$

§11.2. Second series of auxiliary lemmas

Lemma 11.12. *Let $X \subset \bar{A} \times S$ be a closed subscheme, where S is integral and $\bar{A}$ is a projective, equivariant compactification. Assume that the induced map $X \to S$ is surjective. Then there exist a non-empty Zariski open set $U \subset S$ and a semi-abelian subvariety $C \subset A$ such that $\mathrm{Stab}^0(X_y) = C$ for all $y \in U$.*

Proof. Since S is integral, by replacing S by its non-empty Zariski open set, we may assume that $X \to S$ is flat. For each $B \subset A$, we set $V_B = \{y \in S;\ B \subset \mathrm{Stab}^0(X_y)\}$. Then $V_B \subset S$ is a Zariski closed set. Indeed, let P be the Hilbert polynomial of X_y for some (hence for all) $y \in S$. Set $Y = \bigcap_{b \in B}(X + b) \subset X$, where the intersection is taken scheme theoretically. Then $y \in V_B$ if and only if the Hilbert polynomial of Y_y is equal to P. Hence V_B is a Zariski closed set (cf. [46, Lem. 3.1]).

Now let $\mathcal{X} \subset \bar{A} \times A \times S$ be the pull-back of X by the action $m \colon \bar{A} \times A \times S \to \bar{A} \times S$, where $(x, a, s) \mapsto (x + a, s)$. Then $\mathcal{X} \to A \times S$ is flat. Let $c \colon A \times S \to \mathrm{Hilb}$ be the classification map. Let $\varphi \colon A \times S \to \mathrm{Hilb} \times S$ be the induced map such that $\varphi(a, s) = (c(a, s), s)$. Then for each $y \in S$, we have $\mathrm{supp}(\varphi^{-1}(\varphi(0, y))) \subset A \times \{y\} = A$. We have

$$\mathrm{supp}\bigl(\varphi^{-1}(\varphi(0, y))\bigr) = \mathrm{Stab}(X_y).$$

For each integer $d \geq 0$, let $E_d \subset S$ be the set of $y \in S$ such that

$$\dim_{(0,y)} \varphi^{-1}(\varphi(0, y)) \geq d.$$

Then $E_d \subset S$ is a Zariski closed set. We take d such that $E_d = S$ and $E_{d+1} \subsetneqq S$. Then for each $y \in S - E_{d+1}$, we have $\dim \mathrm{Stab}^0(X_y) = d$. Let $B_1, B_2, \ldots$ be the set of d-dimensional semi-abelian subvarieties of A. Then $S - E_{d+1} \subset \bigcup V_{B_i}$. Hence there exists B_i such that $S = V_{B_i}$. For $y \in S - E_{d+1}$, we have $B_i \subset \mathrm{Stab}^0(X_y)$. Hence $\mathrm{Stab}^0(X_y) = B_i$. We set $C = B_i$ and $U = S - E_{d+1}$ to conclude the proof. $\square$

In the following lemma, we recall the definition $X_{\{0\},k} \subset \bar{A} \times S_{k,A}$ from (10.1). We have the isomorphism $(A \times S)_k \simeq A \times S_{k,A}$ as in (2.12).

Lemma 11.13. *Let $X \subset \bar{A} \times S$ be a closed subscheme, where S is a smooth algebraic variety and $\bar{A}$ is a smooth equivariant compactification. Let $f \in \mathrm{Hol}(\mathbb{D}, A \times S)$ satisfy $f(\mathbb{D}) \subset \mathrm{supp}\, X$, where f is non-constant. Then $f_{[k]}(\mathbb{D}) \subset \mathrm{supp}\, X_{\{0\},k}$.*

Proof. We define $g\colon \mathbb{D}\times\mathbb{D}\to A\times A\times S$ by

$$g(z,w)=(f_A(z),f_A(w)-f_A(z),f_S(w)).$$

Let $m\colon \bar{A}\times A\times S\to \bar{A}\times S$ be defined by $(x,a,s)\mapsto(x+a,s)$. Then $m\circ g(z,w)=(f_A(w),f_S(w))\in \operatorname{supp} X$. Hence $g(\mathbb{D}\times\mathbb{D})\subset \operatorname{supp}\mathcal{X}$, where $\mathcal{X}\subset A\times A\times S$ is a closed subscheme obtained by the pull-back of $X\subset \bar{A}\times S$ by $m\colon \bar{A}\times A\times S\to\bar{A}\times S$. By taking the k-th derivative for w, we get $\partial_w^k g\colon \mathbb{D}\times\mathbb{D}\to A\times(A\times S)_k$.

Claim. $\partial_w^k g(\mathbb{D}\times\mathbb{D})\subset\operatorname{supp}\mathcal{P}_k\mathcal{X}$.

Proof. We prove this by the induction on k. The case $k=0$ is obvious. So we assume the case k and prove the case for $k+1$. By

$$\partial_w^k g\colon \mathbb{D}\times\mathbb{D}\to\mathcal{P}_k\mathcal{X}\subset A\times(A\times S)_k,$$

we get

$$\partial_w(\partial_w^k g)\colon \mathbb{D}\times\mathbb{D}\to\mathcal{D}\mathcal{P}_k\mathcal{X}\subset(A\times(A\times S)_k)_1.$$

Since $\partial_w(\partial_w^k g)\colon \mathbb{D}\times\mathbb{D}\to A\times(A\times S)_{k+1}\subset(A\times(A\times S)_k)_1$, we get

$$\partial_w(\partial_w^k g)(\mathbb{D}\times\mathbb{D})\subset\mathcal{D}\mathcal{P}_k\mathcal{X}\cap(A\times(A\times S)_{k+1})=\mathcal{P}_{k+1}\mathcal{X}.$$

This completes the induction step. □

By (2.14), we have $\partial_w^k g=(f_A(z),f_A(w)-f_A(z),f_{S_{k,A}}(w))$. Restricting this to the diagonal $\mathbb{D}\subset\mathbb{D}\times\mathbb{D}$, we get

$$\partial_w^k g\circ\Delta(z)=(f_A(z),0_A,f_{S_{k,A}}(z))\in A\times(A\times S)_k|_{A\times\{0\}\times S_{k,A}}.$$

Hence by $X_{\{0\},k}=\mathcal{P}_k\mathcal{X}\cap(\bar{A}\times\{0\}\times S_{k,A})$, the claim above implies $\partial_w^k g\circ\Delta(\mathbb{D})\subset\operatorname{supp}X_{\{0\},k}$. Now under the isomorphism

$$\psi_k\colon A\times(A\times S)_k|_{A\times\{0\}\times S_{k,A}}\to(A\times S)_k,$$

we have $\psi_k\circ\partial_w^k g\circ\Delta=f_{[k]}$. Hence, we have $f_{[k]}(\mathbb{D})\subset\operatorname{supp}X_{\{0\},k}$. □

Lemma 11.14. *Let $X\subset\bar{A}$ be a closed subvariety, where $\bar{A}$ is a smooth equivariant compactification. Let $\mathcal{F}\subset\operatorname{Hol}(\mathbb{D},A\cap X)$ be an infinite set of non-constant holomorphic maps. Let $Z\subset P_{k,A}$ be a Zariski closed subset such that $\mathcal{F}_{P_{k,A}}\Rightarrow Z$. Let $P'_{k,A}\to P_{k,A}$ be a Z-admissible modification and let $Z'\subset P'_{k,A}$ be the minimal transform. Set $X'_{\{0\},k}=(X_{\{0\},k}\times_{P_{k,A}}P'_{k,A})[Z']$. Then we have $f'_{[k]}(\mathbb{D})\subset X'_{\{0\},k}$ for all $f\in\mathcal{F}$ with finite exception, where $f'_{[k]}\colon\mathbb{D}\to A\times P'_{k,A}$ is the lift of $f_{[k]}\colon\mathbb{D}\to A\times P_{k,A}$.*

Proof. By Lemma 3.10, Z is irreducible. By Remark 11.5, we may take a Zariski open set $U \subset P'_{k,A}$ such that $U \cap Z' \neq \emptyset$ and $X'_k = \overline{\psi^{-1}(U)}$, where $\psi\colon X_k \times_{P_{k,A}} P'_{k,A} \to P'_{k,A}$ is the projection. We may assume moreover that the map $P'_{k,A} \to P_{k,A}$ is isomorphic on U. By Lemma 3.11, we have $\mathcal{F}_{P'_{k,A}} \Rightarrow Z'$. Hence we have $f_{P'_{k,A}}(\mathbb{D}) \cap U \neq \emptyset$ for all $f \in \mathcal{F}$ with finite exception. We set $\Omega_f = f^{-1}_{P'_{k,A}}(U)$. Then for all $f \in \mathcal{F}$ with finite exception, $\mathbb{D} - \Omega_f$ is discrete and $f_{P'_{k,A}}(\Omega_f) \subset U$. By Lemma 11.13, we have $f_{[k]}(\mathbb{D}) \subset X_{\{0\},k}$. This shows $f'_{[k]}(\Omega_f) \subset \psi^{-1}(U)$, hence $f'_{[k]}(\mathbb{D}) \subset X'_{\{0\},k}$ for all $f \in \mathcal{F}$ with finite exception. □

§11.3. Demailly jet spaces and quotient maps

We recall Definition 2.1. Let $B \subset A$ be a semi-abelian subvariety. For each $k \geq 1$, the quotient $A \to A/B$ canonically induces the map

$$\mu_k\colon P_{k,A} \backslash E_{k,A,A/B} \to P_{k,A/B} \tag{11.10}$$

inductively as follows. When $k = 1$, we have $P_{1,A} = P(\mathrm{Lie}(A))$, $E_{1,A,A/B} = P(\mathrm{Lie}(B))$, and $P_{1,A/B} = P(\mathrm{Lie}(A/B))$. The map μ_1 is defined by the projectivization of the quotient map $\mathrm{Lie}(A) \to \mathrm{Lie}(A/B)$. Suppose we have μ_{k-1}. Then by $(\mu_{k-1})_*\colon T(P_{k-1,A} \backslash E_{k-1,A,A/B}) \to TP_{k-1,A/B}$ and the quotient map $\mathrm{Lie}(A) \to \mathrm{Lie}(A/B)$, we get

$$T(P_{k-1,A} \backslash E_{k-1,A,A/B}) \times \mathrm{Lie}(A) \to TP_{k-1,A/B} \times \mathrm{Lie}(A/B). \tag{11.11}$$

The kernel of this map is contained in

$$T(P_{k-1,A} \backslash E_{k-1,A,A/B}) \times \mathrm{Lie}(B) \subset T(P_{k-1,A} \backslash E_{k-1,A,A/B}) \times \mathrm{Lie}(A).$$

Let $\tau\colon P_{k,A} \to P_{k-1,A}$ be the projection. Then the restriction of the projectivization of (11.11) onto $P_{k,A}$ yields the map

$$(P_{k,A} \backslash \tau^{-1}(E_{k-1,A,A/B})) \backslash E_{k,A,A/B} \to P_{k,A/B}.$$

Hence by Lemma 2.2, we get $\mu_k\colon P_{k,A} \backslash E_{k,A,A/B} \to P_{k,A/B}$, which is the map (11.10) for k.

Now let $f\colon \mathbb{D} \to A$ be a holomorphic map. Let $f_{A/B}\colon \mathbb{D} \to A/B$ be the composition of f and the quotient map $A \to A/B$. Assume that $f_{A/B}$ is non-constant. Then we get $f_{P_{k,A}}\colon \mathbb{D} \to P_{k,A}$ and $(f_{A/B})_{P_{k,A/B}}\colon \mathbb{D} \to P_{k,A/B}$, where $f_{P_{k,A}}(\mathbb{D}) \not\subset E_{k,A,A/B}$. Then by the construction above, we have

$$(f_{A/B})_{P_{k,A/B}} = \mu_k \circ f_{P_{k,A}}.$$

Lemma 11.15. *Let $C \subset A$ be a semi-abelian subvariety such that $B \subset C \subset A$. Then $\mu_k^{-1}(E_{k,A/B,A/C}) \subset E_{k,A,A/C} \cap (P_{k,A} \backslash E_{k,A,A/B})$.*

Proof. Let $(x,[v]) \in P_{k,A}\backslash E_{k,A,A/C}$, where $x \in P_{k-1,A}$ and $v \in V_{k-1}^{\dagger}\backslash\{0\} \subset TP_{k-1,A} \times \mathrm{Lie}(A)$ (cf. (2.10)). Then $v \notin TP_{k-1,A} \times \mathrm{Lie}(C)$. Then the image of v under the map (11.11) is not contained in $TP_{k-1,A/B} \times \mathrm{Lie}(C/B)$. Hence $\mu_k((x,[v])) \notin E_{k,A/B,A/C}$. This shows

$$\mu_k^{-1}(E_{k,A/B,A/C}) \subset E_{k,A,A/C}$$

in $P_{k,A}\backslash E_{k,A,A/B}$. □

We recall $Z_{k,A/B} \subset P_{k,A/B}$ from (11.1).

Lemma 11.16. *Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, A)$ be an infinite set of holomorphic maps which satisfies Assumption 11.2. Let $B, C \in \Pi(\mathcal{F})$ such that $B \subset C$. Let $P'_{k,A/B} \to P_{k,A/B}$ be a $Z_{k,A/B}$-admissible modification such that the rational map $P_{k,A/B} \dashrightarrow P_{k,A/C}$ induces a morphism $\mu_k \colon P'_{k,A/B} \to P_{k,A/C}$. Then $Z_{k,A/C} \subset \mu_k(Z'_{k,A/B})$, where $Z'_{k,A/B} \subset P'_{k,A/B}$ is the minimal transform.*

Proof. We take $l \in \mathbb{Z}_{\geq 0}$ such that $T_{k+l,A/C} \to Z_{k,A/C}$ and $T_{k+l,A/B} \to Z_{k,A/B}$ are surjective maps. The rational map $P_{k+l,A/B} \dashrightarrow P_{k+l,A/C}$ is holomorphic outside $E_{k+l,A/B,A/C} \subset P_{k+l,A/B}$. By $C/B \in \Pi(\mathcal{F}_{A/B})$, we have $T_{k+l,A/B} \not\subset E_{k+l,A/B,A/C}$. Hence there exists a $T_{k+l,A/B}$-admissible modification $P'_{k+l,A/B} \to P_{k+l,A/B}$ such that the rational map $P_{k+l,A/B} \dashrightarrow P_{k+l,A/C}$ induces a regular map

$$\mu_{k+l} \colon P'_{k+l,A/B} \to P_{k+l,A/C}.$$

By $\mathcal{F}_{P_{k+l,A/B}} \Rightarrow T_{k+l,A/B}$, we have $\mathcal{F}_{P'_{k+l,A/B}} \Rightarrow T'_{k+l,A/B}$ (cf. Lemma 3.11). Hence by Lemma 3.15, we get

$$\mathcal{F}_{P_{k+l,A/C}} \Rightarrow \mu_{k+l}(T'_{k+l,A/B}).$$

By Lemma 3.9, we have either $T_{k+l,A/C} \subset \mu_{k+l}(T'_{k+l,A/B})$ or $\mu_{k+l}(T'_{k+l,A/B}) \subsetneqq T_{k+l,A/C}$. To show

$$T_{k+l,A/C} \subset \mu_{k+l}(T'_{k+l,A/B}),$$

assume on the contrary that $\mu_{k+l}(T'_{k+l,A/B}) \subsetneqq T_{k+l,A/C}$. By Lemma 3.10, $\mu_{k+l}(T'_{k+l,A/B})$ is irreducible. Hence by the definition of $T_{k+l,A/C}$, there exists $C'/C \in \Pi(\mathcal{F}_{A/C})$ such that $\mu_{k+l}(T'_{k+l,A/B}) \subset E_{k+l,A/C,A/C'}$. Thus $T'_{k+l,A/B} \subset \mu_{k+l}^{-1}(E_{k+l,A/C,A/C'})$. Since $T_{k+l,A/B}$ is irreducible, this implies $T_{k+l,A/B} \subset E_{k+l,A/B,A/C'}$ (cf. Lemma 11.15). By $C'/B \in \Pi(\mathcal{F}_{A/B})$, this contradicts the definition of $T_{k+l,A/B}$. Hence $T_{k+l,A/C} \subset \mu_{k+l}(T'_{k+l,A/B})$.

Now $Z_{k,A/C} \subset P_{k,A/C}$ is contained in the image of $T'_{k+l,A/B} \subset P'_{k+l,A/B}$ under the composition of $\mu_{k+l} \colon P'_{k+l,A/B} \to P_{k+l,A/C}$ and $P_{k+l,A/C} \to P_{k,A/C}$. Thus we get $Z_{k,A/C} \subset \mu_k(Z'_{k,A/B})$. □

Let B and C be semi-abelian subvarieties of A such that $B \subset C$. Let $S \subset \bar{A}$ be a Zariski closed set which is C-invariant. Then we get $S_{B,k} \subset \bar{A} \times P_{k,A/B}$ and $S_{C,k} \subset \bar{A} \times P_{k,A/C}$ (cf. (10.1)).

Lemma 11.17. *Let $\varphi\colon \bar{A} \times (P_{k,A/B} \backslash E_{k,A/B,A/C}) \to \bar{A} \times P_{k,A/C}$ be the map induced from the regular map $P_{k,A/B} \backslash E_{k,A/B,A/C} \to P_{k,A/C}$. Then we have*

$$\varphi^*(S_{C,k}) = S_{B,k}|_{P_{k,A/B} \backslash E_{k,A/B,A/C}}.$$

Proof. Let $\mathcal{S} \subset \bar{A} \times A$ be the pull-back of $S \subset \bar{A}$ by the action $m\colon \bar{A} \times A \to \bar{A}$ so that $\mathcal{S}_a = S - a$, where $\mathcal{S}_a \subset \bar{A}$ is the fiber of $\mathcal{S} \to A$ over $a \in A$. Let $\mathcal{S}_B \subset \bar{A} \times (A/B)$ be defined so that the pull-back of $\mathcal{S}_B$ by $\bar{A} \times A \to \bar{A} \times (A/B)$ is equal to $\mathcal{S}$. We define $\mathcal{S}_C \subset \bar{A} \times (A/C)$ similarly. Then the pull-back of $\mathcal{S}_C$ by $\bar{A} \times (A/B) \to \bar{A} \times (A/C)$ is equal to $\mathcal{S}_B$. Let $\phi\colon \bar{A} \times ((A/B) \times P_{k,A/B}) \dashrightarrow \bar{A} \times ((A/C) \times P_{k,A/C})$ be the induced rational map, which is regular on $\bar{A} \times ((A/B) \times (P_{k,A/B} \backslash E_{k,A/B,A/C}))$. Then we have

$$\phi^*(\mathcal{P}_k \mathcal{S}_C) = (\mathcal{P}_k \mathcal{S}_B)|_{(A/B) \times (P_{k,A/B} \backslash E_{k,A/B,A/C})}.$$

Hence by (10.1), we get $\varphi^*(S_{C,k}) = S_{B,k}|_{P_{k,A/B} \backslash E_{k,A/B,A/C}}$. □

§11.4. Main lemma for the proof of Proposition 11.3

Let A be a semi-abelian variety and let $B \subset A$ be a semi-abelian subvariety. Given a Zariski closed set $V \subset \bar{A}$, we set $Y = \mathrm{Sp}_B V$. Then the Zariski closed set $Y \subset \bar{A}$ is B-invariant. Hence we get the closed subscheme $Y_{B,k} \subset \bar{A} \times P_{k,A/B}$. For each $y \in P_{k,A/B}$, the fiber of $Y_{B,k} \to P_{k,A/B}$ over y is denoted by $(Y_{B,k})_y$, which is a closed subscheme of $\bar{A}$.

Let $X \subset A$ be a closed subvariety and let $\bar{A}$ be a smooth equivariant compactification. Let $\overline{X} \subset \bar{A}$ be the Zariski closure. In this subsection, we write $\overline{X}_k = (\overline{X})_{\{0\},k}$ for short. Then $\overline{X}_k \subset \bar{A} \times P_{k,A}$. Although this $\overline{X}_k$ is not the same as the Demailly jet space of $\overline{X}$ discussed in Section 2, no confusion will occur.

We recall $Z_{k,A/B} \subset P_{k,A/B}$ from (11.1). We set $Z_k = Z_{k,A} \subset P_{k,A}$ for short.

Lemma 11.18. *Let A be a non-trivial semi-abelian variety and let $X \subset A$ be a closed subvariety. Let $\bar{A}$ be a smooth equivariant compactification, which is projective. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, X)$ be an infinite set of holomorphic maps which satisfies Assumption 11.2. Assume that $\Lambda_{X,\bar{A}}(\mathcal{F}) = \emptyset$. Then there exists $B \subset A$ such that $\mathrm{Sp}_B \overline{X} \neq \emptyset$ with the following properties:*

(1) *$B \in \Pi(\mathcal{F})$, in particular $B \subsetneqq A$.*

(2) $Z_{k,A/B} \subset p_{k,B}(Y_{B,k})$ *for* $k \gg 1$, *where* $Y = \mathrm{Sp}_B \overline{X}$ *and* $p_{k,B}\colon \bar{A} \times P_{k,A/B} \to P_{k,A/B}$ *is the second projection.*

(3) $\mathrm{Stab}^0((Y_{B,k})_y) = B$ *for generic* $y \in Z_{k,A/B}$ *and* $k \gg 1$.

(4) *For* $k \gg 1$, *there exist a* Z_k*-admissible modification* $\widehat{P}_{k,A} \to P_{k,A}$ *with a regular map* $\sigma\colon \widehat{P}_{k,A} \to P_{k,A/B}$ *and a Zariski closed set* $V_k \subset \widehat{P}_{k,A}$ *such that*

  (a) $Z_{k,A/B} \subset \sigma(V_k)$ *and* $V_k \subset \widehat{Z}_k$, *where* $\widehat{Z}_k \subset \widehat{P}_{k,A}$ *is the minimal transform;*

  (b) *set* $\widehat{X}_k = (\overline{X}_k \times_{P_{k,A}} \widehat{P}_{k,A})[\widehat{Z}_k] \subset \bar{A} \times \widehat{P}_{k,A}$; *then* $\widehat{X}_k|_{\widehat{Z}_k} \to \widehat{Z}_k$ *is flat;*

  (c) *let* $P'_{k,A/B} \to P_{k,A/B}$ *be a* $Z_{k,A/B}$*-admissible modification with the minimal transform* $Z'_{k,A/B} \subset P'_{k,A/B}$, *let* $\widehat{P}'_{k,A} = (\widehat{P}_{k,A} \times_{P_{k,A/B}} P'_{k,A/B})[Z'_{k,A/B}]$ *and* $V'_k = (V_k \times_{P_{k,A/B}} P'_{k,A/B})[Z'_{k,A/B}] \subset \widehat{P}'_{k,A}$; *then* $\mathcal{F}_{\widehat{P}'_{k,A}} \to V'_k$;

  (d) *the image of* $\mathrm{supp}\, \widehat{X}_k|_{V_k} \subset \bar{A} \times \widehat{P}_{k,A}$ *under the map* $\bar{A} \times \widehat{P}_{k,A} \to \bar{A} \times P_{k,A/B}$ *is contained in* $\mathrm{supp}\, Y_{B,k} \subset \bar{A} \times P_{k,A/B}$.

(5) *If* $k \gg 1$, *then for generic* $y \in Z_{k,A/B}$, *the natural map* $(Y_{B,k})_y \to (Y_{B,k-1})_{y_0}$ *is an isomorphism, where* $y_0 \in P_{k-1,A/B}$ *is the image of* y *under the map* $P_{k,A/B} \to P_{k-1,A/B}$.

In the proof of this lemma, we use the following notation: Let $V \subset \bar{A} \times S$ be a Zariski closed set. Let $B \subset A$ be a semi-abelian variety. We set

$$\mathrm{Sp}_B V = \bigcap_{b \in B} (V + b) \subset V.$$

Then $\mathrm{Sp}_B V \subset \bar{A} \times S$ is a Zariski closed subset.

Proof of Lemma 11.18. Let $\mathcal{B}$ be the set of all semi-abelian subvarieties $B \subset A$ such that B satisfies the three properties (1), (4), and $\mathrm{Sp}_B \overline{X} \neq \emptyset$. To show that $\mathcal{B}$ is non-empty, we shall prove $\{0\} \in \mathcal{B}$. We note that $\{0\} \in \Pi(\mathcal{F})$. Indeed, by $\mathrm{Sp}_{\{0\}} \overline{X} = \overline{X}$, we have $\mathcal{F} \to \mathrm{Sp}_{\{0\}} \overline{X}$. Hence if $\{0\} \notin \Pi(\mathcal{F})$, then $\{0\} \in \Lambda_{X,\bar{A}}(\mathcal{F})$. This contradicts $\Lambda_{X,\bar{A}}(\mathcal{F}) = \emptyset$. Hence $\{0\} \in \Pi(\mathcal{F})$. To prove that $\{0\}$ satisfies (4), we take a Z_k-admissible modification $\sigma\colon \widehat{P}_{k,A} \to P_{k,A}$ such that $\widehat{X}_k|_{\widehat{Z}_k} \to \widehat{Z}_k$ is flat, where $\widehat{X}_k$ and $\widehat{Z}_k$ are defined as in the statement of the lemma. The existence of such a modification follows from Lemma 11.11. We set $V_k = \widehat{Z}_k$. Then (4) is satisfied. Thus $\{0\} \in \mathcal{B}$. In particular, $\mathcal{B}$ is non-empty.

We remark that if $B \in \mathcal{B}$, then (2) is satisfied for sufficiently large k satisfying (4). Indeed, by $\mathcal{F}_{P_{k,A}} \Rightarrow Z_k$, we may apply Lemma 11.14 to get $\widehat{Z}_k \subset p_k(\widehat{X}_k)$, where we continue to write the induced map $p_k\colon \bar{A} \times \widehat{P}_{k,A} \to \widehat{P}_{k,A}$. Hence by $V_k \subset \widehat{Z}_k$ (cf. (4a)), we have $V_k = p_k(\widehat{X}_k|_{V_k})$. By (4d), we have $\sigma(V_k) \subset p_{k,B}(Y_{B,k})$, where $p_{k,B}\colon \bar{A} \times P_{k,A/B} \to P_{k,A/B}$ is the second projection. By $Z_{k,A/B} \subset \sigma(V_k)$ (cf. (4a)), we get $Z_{k,A/B} \subset p_{k,B}(Y_{B,k})$. Hence (2) is true for $B \in \mathcal{B}$.

Claim 1. If $B \in \mathcal{B}$, then assertion (5) is satisfied.

Proof. Since $Z_{k,A/B}$ is integral, the generic flatness yields that there exists a non-empty Zariski open set $U_k \subset Z_{k,A/B}$ such that $Y_{B,k}|_{Z_{k,A/B}} \to Z_{k,A/B}$ is flat over U_k. We may assume that the image of U_k under $Z_{k,A/B} \to Z_{k-1,A/B}$ is contained in U_{k-1}. For each k, note that the Hilbert polynomials of the fibers $(Y_{B,k})_y$ are all the same for $y \in U_k$. We denote this polynomial by H_k. For $y \in U_k$, we have $(Y_{B,k})_y \subset (Y_{B,k-1})_{y_0} \subset \bar{A}$, where $y_0 \in P_{k-1,A/B}$ is the image of y under the map $P_{k,A/B} \to P_{k-1,A/B}$. Hence $H_k \leq H_{k-1}$. Thus we get

$$H_k \geq H_{k+1} \geq H_{k+2} \geq \cdots .$$

By [46, Lem. 8.2], there exists k_0 such that $H_{k_0} = H_{k_0+1} = \cdots$. Hence if $k \geq k_0+1$, we have $(Y_{B,k})_y = (Y_{B,k-1})_{y_0}$. This completes the proof. □

In the following, we shall prove that a maximal element in $\mathcal{B}$ satisfies (3). We take $B \in \mathcal{B}$. We consider $Y_{B,k}|_{Z_{k,A/B}} \to Z_{k,A/B}$. Then by property (2), this map is surjective. Set $C_k = \mathrm{Stab}^0(Y_{B,k})_y \subset A$ for generic $y \in Z_{k,A/B}$ (cf. Lemma 11.12). Note that $(Y_{B,k})_y \subset \bar{A}$ is B-invariant. Hence

$$B \subset C_k.$$

By Claim 1 above, there exists C such that $C_k = C$ for all sufficiently large k.

We shall show $C \in \mathcal{B}$. By the construction, we have $\mathrm{Sp}_C Y \neq \emptyset$. Hence by $\mathrm{Sp}_C Y \subset \mathrm{Sp}_C \overline{X}$, we have $\mathrm{Sp}_C \overline{X} \neq \emptyset$. By $B \in \mathcal{B}$, we may take $\widehat{P}_{k,A} \to P_{k,A}$, $\sigma\colon \widehat{P}_{k,A} \to P_{k,A/B}$, and $V_k \subset \widehat{P}_{k,A}$, which are described in (4). Here and in what follows, we assume that k is sufficiently large, satisfying (4) for B. Using Lemma 11.11, we take a $Z_{k,A/B}$-admissible modification $P'_{k,A/B} \to P_{k,A/B}$ such that

- $V'_k|_{Z'_{k,A/B}} \to Z'_{k,A/B}$ is flat, where $Z'_{k,A/B} \subset P'_{k,A/B}$ is the minimal transform and $V'_k \subset \widehat{P}'_{k,A}$ is defined by $V'_k = (V_k \times_{P_{k,A/B}} P'_{k,A/B})[Z'_{k,A/B}]$ as in the statement of (4c).

We present the following commutative diagram:

$$\begin{array}{ccc} \widehat{P}'_{k,A} & \xrightarrow{\sigma'} & P'_{k,A/B} \\ \downarrow & & \downarrow \\ \widehat{P}_{k,A} & \xrightarrow[\sigma]{} & P_{k,A/B}. \end{array}$$

Note that by $\{0\} \in \Pi(\mathcal{F})$, Lemma 11.16 yields that $Z_{k,A/B} \subset \sigma(\widehat{Z}_k)$. Hence $\widehat{P}'_{k,A} \to \widehat{P}_{k,A}$ is a $\widehat{Z}_k$-admissible modification. Hence we may define the minimal transform $\widehat{Z}'_k \subset \widehat{P}'_{k,A}$. Set $\widehat{X}'_k = (\widehat{X}_k \times_{\widehat{P}_{k,A}} \widehat{P}'_{k,A})[\widehat{Z}'_k] \subset \bar{A} \times \widehat{P}'_{k,A}$. We set

$$W_k = \operatorname{supp} V'_k|_{Z'_{k,A/B}} \subset \widehat{P}'_{k,A}.$$

We claim the following.

Claim 2. The image of $\operatorname{supp} \widehat{X}'_k|_{W_k} \subset \bar{A} \times \widehat{P}'_{k,A}$ under the map $\bar{A} \times \widehat{P}'_{k,A} \to \bar{A} \times P_{k,A/B}$ is contained in $\mathrm{Sp}_{C_k}(\operatorname{supp} Y_{B,k})$.

Proof. By the definition of C_k, there exists a dense Zariski open $U \subset Z_{k,A/B}$ such that $\operatorname{supp} Y_{B,k}|_U \subset \mathrm{Sp}_{C_k}(\operatorname{supp} Y_{B,k})$. By shrinking U, we may assume that $U \subset Z'_{k,A/B}$. Since $V'_k|_{Z'_{k,A/B}} \to Z'_{k,A/B}$ is flat, Lemma B.1 yields that $W_k|_U \subset W_k$ is dense. Since $\widehat{X}_k|_{\widehat{Z}_k} \to \widehat{Z}_k$ is flat, Lemma 11.10 yields that $\widehat{X}'_k|_{\widehat{Z}'_k} \to \widehat{Z}'_k$ is flat. By $W_k \subset V'_k \subset \widehat{Z}'_k$, we get that $\widehat{X}'_k|_{W_k} \to W_k$ is flat. Hence Lemma B.1 yields that $\operatorname{supp} \widehat{X}'_k|_{(W_k|_U)} \subset \operatorname{supp} \widehat{X}'_k|_{W_k}$ is dense.

Now by (4d), the image of $\operatorname{supp} \widehat{X}'_k|_{(W_k|_U)}$ under $\bar{A} \times \widehat{P}'_{k,A} \to \bar{A} \times P_{k,A/B}$ is contained in $\operatorname{supp} Y_{B,k}|_U$, hence $\mathrm{Sp}_{C_k}(\operatorname{supp} Y_{B,k})$. Thus the image of $\operatorname{supp} \widehat{X}'_k|_{W_k}$ is contained in $\mathrm{Sp}_{C_k}(\operatorname{supp} Y_{B,k})$. $\square$

Claim 3. Let $P''_{k,A/B} \to P'_{k,A/B}$ be a $Z'_{k,A/B}$-admissible modification. Let $Z''_{k,A/B} \subset P''_{k,A/B}$ be the minimal transform. Set

$$\widehat{P}''_{k,A} = (\widehat{P}'_{k,A} \times_{P'_{k,A/B}} P''_{k,A/B})[Z''_{k,A/B}]$$

and $W_k^+ = (W_k \times_{P'_{k,A/B}} P''_{k,A/B})[Z''_{k,A/B}] \subset \widehat{P}''_{k,A}$. Then $\mathcal{F}_{\widehat{P}''_{k,A}} \to W_k^+$.

We present the following commutative diagram:

$$\begin{tikzcd}
\widehat{P}''_{k,A} \arrow[r, "\sigma''"] \arrow[d] & P''_{k,A/B} \arrow[d] \\
\widehat{P}'_{k,A} \arrow[r, "\sigma'"] & P'_{k,A/B}.
\end{tikzcd}$$

Proof of Claim 3. Set $V''_k = (V'_k \times_{P'_{k,A/B}} P''_{k,A/B})[Z''_{k,A/B}] \subset \widehat{P}''_{k,A}$. Since $V'_k|_{Z'_{k,A/B}} \to Z'_{k,A/B}$ is flat, Lemma 11.10 yields that $V''_k|_{Z''_{k,A/B}} = V'_k|_{Z'_{k,A/B}} \times_{Z'_{k,A/B}} Z''_{k,A/B}$. Similarly, applying Lemma 11.10 to $V'_k|_{Z'_{k,A/B}} \to P'_{k,A/B}$, we have

$$((V'_k|_{Z'_{k,A/B}}) \times_{P'_{k,A/B}} P''_{k,A/B})[Z''_{k,A/B}] = V'_k|_{Z'_{k,A/B}} \times_{Z'_{k,A/B}} Z''_{k,A/B}.$$

Note that $W_k^+ = \operatorname{supp}(((V'_k|_{Z'_{k,A/B}}) \times_{P'_{k,A/B}} P''_{k,A/B})[Z''_{k,A/B}])$. Hence we get

$$\operatorname{supp} V''_k|_{Z''_{k,A/B}} = W_k^+.$$

Now by Lemma 11.9, we have $V''_k = (V_k \times_{P_{k,A/B}} P''_{k,A/B})[Z''_{k,A/B}]$ and $\widehat{P}''_{k,A} = (\widehat{P}_{k,A} \times_{P_{k,A/B}} P''_{k,A/B})[Z''_{k,A/B}]$. Hence by (4c), we have $\mathcal{F}_{\widehat{P}''_{k,A}} \to V''_k$. By Assumption 11.2, we have $\mathcal{F}_{P''_{k,A/B}} \to Z''_{k,A/B}$. Thus by Lemmas 3.5 and 3.8, we get $\mathcal{F}_{\widehat{P}''_{k,A}} \to W_k^+$. This concludes the proof of the claim. $\square$

We are going to prove $C \in \mathcal{B}$. We first prove $C \in \Pi(\mathcal{F})$. By Lemma 11.9, we have $\widehat{X}'_k = (\overline{X}_k \times_{P_{k,A}} \widehat{P}'_{k,A})[\widehat{Z}'_k]$. Hence by Lemma 11.14, we have

$$f_{[k]}(\mathbb{D}) \subset \widehat{X}'_k$$

for all $f \in \mathcal{F}$ with finite exception. We have $\mathcal{F}_{\widehat{P}'_{k,A}} \to W_k$. Hence by Lemma 3.8, $\{f_{[k]}\}_{f\in\mathcal{F}} \to \widehat{X}'_k|_{W_k}$. By Claim 2 above, we have $\mathcal{F} \to \mathrm{Sp}_{C_k} Y$, hence $\mathcal{F} \to \mathrm{Sp}_{C_k} \overline{X}$. Hence by $\Lambda_{X,\bar{A}}(\mathcal{F}) = \emptyset$, we get $C_k \in \Pi(\mathcal{F})$. Thus $C \in \Pi(\mathcal{F})$. In particular, we have $C \neq A$.

By $C \in \Pi(\mathcal{F})$, the definition of $T_{k,A/B}$ yields that $T_{k,A/B} \not\subset E_{k,A/B,A/C}$. Hence

$$Z_{k,A/B} \not\subset E_{k,A/B,A/C}. \tag{11.12}$$

Note that the rational map $P_{k,A/B} \dashrightarrow P_{k,A/C}$ is regular outside $E_{k,A/B,A/C}$. Thus we may moreover assume for the $Z_{k,A/B}$-admissible map $P'_{k,A/B} \to P_{k,A/B}$ that

- $\mu\colon P'_{k,A/B} \to P_{k,A/C}$ exists.

We get the following:

$$\begin{tikzcd} \widehat{P}'_{k,A} \arrow[r, "\sigma'"] \arrow[d] & P'_{k,A/B} \arrow[r, "\mu"] \arrow[d] & P_{k,A/C} \\ \widehat{P}_{k,A} \arrow[r, "\sigma"'] & P_{k,A/B}. & \end{tikzcd}$$

By Lemma 11.16, we have

$$Z_{k,A/C} \subset \mu(Z'_{k,A/B}). \tag{11.13}$$

Now we shall show that $\tau\colon \widehat{P}'_{k,A} \to P_{k,A/C}$ and $W_k \subset \widehat{P}'_{k,A}$ satisfy condition (4). By Lemma 11.9, we have $\widehat{X}'_k = (X_k \times_{P_{k,A}} \widehat{P}'_{k,A})[\widehat{Z}'_k]$. Note that $Z_{k,A/C} \subset \tau(W_k)$ follows from $\sigma'(W_k) = Z'_{k,A/B}$ (cf. (11.13)). Also by $V_k \subset \widehat{Z}_k$, we have $W_k \subset V'_k \subset \widehat{Z}'_k$. These show (4a). Since $\widehat{X}_k|_{\widehat{Z}_k} \to \widehat{Z}_k$ is flat, Lemma 11.10 yields that $\widehat{X}'_k|_{\widehat{Z}'_k} \to \widehat{Z}'_k$ is flat. This shows (4b).

Next we prove (4c). Let $P'_{k,A/C} \to P_{k,A/C}$ be a $Z_{k,A/C}$-admissible modification. We consider the following:

$$\begin{array}{ccccc} \widehat{P}''_{k,A} & \xrightarrow{\sigma''} & P''_{k,A/B} & \longrightarrow & P'_{k,A/C} \\ {\scriptstyle\psi}\downarrow & & \downarrow & & \downarrow \\ \widehat{P}'_{k,A} & \xrightarrow{\sigma'} & P'_{k,A/B} & \xrightarrow{\mu} & P_{k,A/C}. \end{array}$$

Here, $P''_{k,A/B} = (P'_{k,A/B} \times_{P_{k,A/C}} P'_{k,A/C})[Z'_{k,A/C}]$. Then $P''_{k,A/B} \to P'_{k,A/B}$ is a $Z'_{k,A/B}$-admissible modification (cf. (11.13)). Since $\widehat{P}'_{k,A}$ is integral, we may apply Lemma 11.8 to get $\widehat{P}''_{k,A} = (\widehat{P}'_{k,A} \times_{P_{k,A/C}} P'_{k,A/C})[Z'_{k,A/C}]$ (cf. (11.13)). Set $W'_k = (W_k \times_{P_{k,A/C}} P'_{k,A/C})[Z'_{k,A/C}]$. Then by Lemma 11.8, we get $W_k^+ \subset W'_k$ (cf. (11.13)). Thus by Claim 3, we get $\mathcal{F}_{\widehat{P}''_{k,A}} \to W'_k$. This shows (4c).

Finally, to check (4d), we prove that the image $\widehat{X}'_k|_{W_k} \subset \bar{A} \times \widehat{P}'_{k,A} \to \bar{A} \times P_{k,A/C}$ is contained in $S_{C,k} \subset \bar{A} \times P_{k,A/C}$, where $S = \mathrm{Sp}_C \overline{X}$. We have $\mathrm{Sp}_C Y = S$. Hence we get $\mathrm{Sp}_C(Y_{B,k}) = S_{B,k}$. Hence Lemma 11.17 yields

$$\mathrm{Sp}_C(Y_{B,k})|_{(P_{k,A/B} \backslash E_{k,A/B,A/C})} \subset \varphi^{-1}(S_{C,k}), \tag{11.14}$$

where $\varphi\colon \bar{A} \times (P_{k,A/B} \backslash E_{A/B,A/C,k}) \to \bar{A} \times P_{k,A/C}$. We take $U \subset P_{k,A/B}$ such that $U \cap Z_{k,A/B} \neq \emptyset$ and $P'_{k,A/B} \to P_{k,A/B}$ is an isomorphism over U. We consider as $U \subset P'_{k,A/B}$. By (11.12), we may assume

$$U \subset P_{k,A/B} - E_{k,A/B,A/C}.$$

Since $V'_k|_{Z'_{k,A/B}} \to Z'_{k,A/B}$ is flat, Lemma B.1 yields that $W_k|_{(Z'_{k,A/B} \cap U)} \subset W_k$ is dense. By Claim 2 and (11.14), the image of $\widehat{X}'_k|_{W_k|_{(Z'_{k,A/B} \cap U)}}$ under $\bar{A} \times \widehat{P}'_{k,A} \to \bar{A} \times P_{k,A/C}$ is contained in $S_{C,k}$, provided k is sufficiently large so that $C_k = C$. Since $\widehat{X}'_k|_{W_k} \to W_k$ is flat, Lemma B.1 yields that the inclusion $\widehat{X}'_k|_{W_k|_{(Z'_{k,A/B} \cap U)}} \subset \widehat{X}'_k|_{W_k}$ is dense. Hence the image $\mathrm{supp}\, \widehat{X}'_k|_{W_k} \subset \bar{A} \times \widehat{P}'_{k,A} \to \bar{A} \times P_{k,A/C}$ is contained in $S_{C,k} \subset \bar{A} \times P_{k,A/C}$. Thus we have proved $C \in \mathcal{B}$.

Now we finish the proof of the lemma. We take maximal $B \subset A$ such that $B \in \mathcal{B}$. Set $C = \mathrm{Stab}^0(Y_{B,k})_y \subset A$ for generic $y \in Z_{k,A/B}$ and sufficiently large k. Then $B \subset C$. By $C \in \mathcal{B}$ and the maximal property of $B \in \mathcal{B}$, we have $C = B$. This concludes the proof of Lemma 11.18. □

§11.5. Proof of Proposition 11.3

Now we prove Proposition 11.3. Since $\mathcal{F}$ satisfies Assumption 11.2 and $\Lambda_{X,\bar{A}}(\mathcal{F}) = \emptyset$, we may take B as in Lemma 11.18. Then $B \in \Pi(\mathcal{F})$. In particular, $B \neq A$.

Next we prove property (2) of Proposition 11.3. We take sufficiently large k_0 such that properties (2), (3), (4), and (5) of Lemma 11.18 are true for $k \geq k_0$. For each $k \in \mathbb{Z}_{k\geq 1}$, let $Z_{k,A/B} \subset P_{k,A/B}$ be defined by (11.1). Then $Z_{k,A/B}$ is an irreducible Zariski closed set such that $T_{k,A/B} \subset Z_{k,A/B}$. Set $Y = \mathrm{Sp}_B\overline{X}$. If $k \geq k_0$, then for generic $y \in Z_{k,A/B}$, the natural map $(Y_{B,k})_y \to (Y_{B,k-1})_{y_0}$ is an isomorphism, where $y_0 \in P_{k-1,A/B}$ is the image of y under the map $P_{k,A/B} \to P_{k-1,A/B}$ (cf. Lemma 11.18 (5)). We set $S = P_{k_0,A/B}$. Then $P_{k_0+l,A/B} \subset S_{l,A/B}$ (cf. Remark 2.3). We consider $Z_{k_0+l,A/B} \subset S_{l,A/B}$. We set $Z = Z_{k_0+1,A/B} \subset S_{1,A/B}$. By Lemma 10.1, Z is horizontally integrable. Indeed, assumption (1) of Lemma 10.1 follows from Lemma 11.18 (3). Assumption (2) of Lemma 10.1 directly follows from Lemma 11.18 (5). By $B \in \Pi(\mathcal{F})$, the definition of $T_{k_0+l,A/B}$ yields that $T_{k_0+l,A/B} \not\subset E_{k_0+l,A/B,A/B}$ as subsets of $P_{k_0+l,A/B}$. By $P^{\mathrm{sing}}_{k_0+l,A/B} = E_{k_0+l,A/B,A/B}$ (cf. (2.17)), we get $T_{k_0+l,A/B} \not\subset P^{\mathrm{sing}}_{k_0+l,A/B}$, so $Z_{k_0+l,A/B} \not\subset P^{\mathrm{sing}}_{k_0+l,A/B}$. Note that

$$P_{k_0+l,A/B} \subset S_{l,A/B} \subset P(TS_{l-1,A/B} \times \mathrm{Lie}(A/B)).$$

Then we have

$$\begin{aligned} S^{\mathrm{sing}}_{l,A/B} \cap P_{k_0+l,A/B} &= P(T_{S_{l-1,A/B}/S} \times \{0\}) \cap P_{k_0+l,A/B} \\ &\subset P(TS_{l-1,A/B} \times \{0\}) \cap P_{k_0+l,A/B} = P^{\mathrm{sing}}_{k_0+l,A/B}. \end{aligned}$$

Thus $Z_{k_0+l,A/B} \not\subset S^{\mathrm{sing}}_{l,A/B}$. Hence for generic $y \in Z$, there exists $y_l \in Z_{k_0+l,A/B} - S^{\mathrm{sing}}_{l,A/B}$ such that $(Y_{B,k_0+1})_y = (Y_{B,k_0+l})_{y_l}$. We note that

$$Y_{B,k_0+l} = (Y_{B,k_0})_{B,l} \cap (\bar{A} \times P_{k_0+l,A/B})$$

for $l \geq 1$, which follows from the definitions (9.3) and (10.1). Hence assumption (3) of Lemma 10.1 is satisfied. Thus Z is horizontally integrable.

Now we prove $\mathcal{F} \to \mathrm{Sp}_B\overline{X}$. We take $\widehat{P}_{k,A} \to P_{k,A}$ and $\widehat{X}_k \subset \bar{A} \times \widehat{P}_{k,A}$ as in Lemma 11.18 (4), where we fix $k \geq k_0$. By Lemma 11.14, we have $f_{[k]}(\mathbb{D}) \subset \widehat{X}_k$ for all $f \in \mathcal{F}$ with finite exception. Hence by $\mathcal{F}_{\widehat{P}_{k,A}} \to V_k$ (cf. Lemma 11.18 (4c)), we have $\{f_{[k]}\}_{f\in\mathcal{F}} \to \widehat{X}_k|_{V_k}$ (cf. Lemma 3.8). Hence by Lemma 11.18 (4d), we have $\mathcal{F} \to \mathrm{Sp}_B\overline{X}$. This completes the proof of Proposition 11.3. □

§12. Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2.

Lemma 12.1. *Let $\mathcal{F} = (f_i)_{i\in I}$ be an infinite indexed family of holomorphic maps in $\mathrm{Hol}(\mathbb{D}, A)$. Then replacing $\mathcal{F}$ by its infinite subfamily, we have $\Pi(\mathcal{G}) = \Pi(\mathcal{F})$ for all infinite subfamilies $\mathcal{G}$ of $\mathcal{F}$.*

Proof. There are only countably many semi-abelian subvarieties $B \subset A$ (cf. [35, Cor. 5.1.9]). So we enumerate them as $B_1, B_2, \dots$. If this sequence is finite, say it terminates at B_L, we extend it to an infinite sequence by defining $B_k = B_L$ for all $k \geq L$. If I contains an infinite subset $I' \subset I$ such that $(|(\varpi_{B_1} \circ f_i)'|_{\omega_{A/B_1}})_{i\in I'}$ converges uniformly on compact subsets of $\mathbb{D}$ to 0, then we set $I_1 = I'$. If there is no such I', then we set $I_1 = I$. If I_1 contains an infinite subset $I' \subset I_1$ such that $\{|(\varpi_{B_2} \circ f_i)'|_{\omega_{A/B_2}}\}_{i\in I'}$ converges uniformly on compact subsets of $\mathbb{D}$ to 0, then we set $I_2 = I'$. If there is no such I', then we set $I_2 = I_1$. We continue this process to get an infinite decreasing sequence of infinite sets $I \supset I_1 \supset I_2 \supset \cdots$. We define an infinite sequence $i_1, i_2, i_3, \dots$ of distinct elements in I so that for each $l \geq 1$, we have

$$i_k \in I_l \tag{12.1}$$

for all $k \geq l$. This sequence is constructed inductively as follows. We take $i_1 \in I_1$. Suppose distinct elements $i_1, i_2, \dots, i_n$ are chosen so that (12.1) holds for all $k, l \in \mathbb{N}$ with $1 \leq l \leq k \leq n$. Then we choose $i_{n+1} \in I_{n+1} - \{i_1, \dots, i_n\}$. Thus we have constructed the sequence $i_1, i_2, \dots$ with the desired property (12.1).

Now we set $J = \{i_1, i_2, i_3, \dots\}$. Then for all semi-abelian subvarieties B, we have either

- $(|(\varpi_B \circ f_i)'|_{\omega_{A/B}})_{i\in J}$ converges uniformly on compact subsets of $\mathbb{D}$ to 0, or
- no infinite subfamily of $(|(\varpi_B \circ f_i)'|_{\omega_{A/B}})_{i\in J}$ converges uniformly on compact subsets of $\mathbb{D}$ to 0.

Hence replacing $\mathcal{F}$ by $(f_j)_{j\in J}$, we get $\Pi(\mathcal{G}) = \Pi(\mathcal{F})$ for all infinite subfamilies $\mathcal{G}$ of $\mathcal{F}$. □

Lemma 12.2. *Let $\mathcal{F} = (f_i)_{i\in I}$ be an infinite indexed family of non-constant holomorphic maps in* $\mathrm{Hol}(\mathbb{D}, A)$*. Then there exists an infinite subfamily $\mathcal{G}$ of $\mathcal{F}$ such that all infinite subfamilies of $\mathcal{G}$ satisfy Assumption* 3.19.

Proof. By replacing $\mathcal{F}$ by its infinite subfamily, we have

$$\Pi(\mathcal{F}') = \Pi(\mathcal{F}) \tag{12.2}$$

for all infinite subfamilies $\mathcal{F}'$ of $\mathcal{F}$ (cf. Lemma 12.1). Let $k \geq 1$ and $B \in \Pi(\mathcal{F})$. Then only finitely many $i \in I$ satisfy $(f_i)_{P_{k,A}}(\mathbb{D}) \subset E_{k,A,A/B}$, for if $(f_i)_{P_{k,A}}(\mathbb{D}) \subset E_{k,A,A/B}$, then $(\varpi_B \circ f_i)' = 0$. Hence, given an infinite subfamily $\mathcal{F}'$ of $\mathcal{F}$, we may apply Lemma 3.14 to get an infinite subfamily $\mathcal{H}$ of $\mathcal{F}'$ such that $\mathrm{LIM}(\mathcal{H}_{P_{k,A}}, \{E_{k,A,A/B}\}_{B\in\Pi(\mathcal{F})})$ exists.

We apply the argument above for $k = 1$ to get an infinite subfamily $\mathcal{F}_1 = (f_i)_{i\in I_1}$, where $I_1 \subset I$, such that $\mathrm{LIM}((\mathcal{F}_1)_{P_{1,A}}, \{E_{1,A,A/B}\}_{B\in\Pi(\mathcal{F})})$ exists. Again we apply the argument above for $k=2$ to get an infinite subfamily $\mathcal{F}_2=(f_i)_{i\in I_2}$, where $I_2\subset I_1$, such that $\mathrm{LIM}((\mathcal{F}_2)_{P_{2,A}},\{E_{2,A,A/B}\}_{B\in\Pi(\mathcal{F})})$ exists. Continue this process to get an infinite decreasing sequence of infinite sets $I\supset I_1\supset I_2\supset\dots$. We define a countable infinite subset $J=\{i_1,i_2,i_3,\dots\}$ of I such that (12.1) holds for all $l\geq 1$ and $k\geq l$ (cf. the proof of Lemma 12.1). Set $\mathcal{G}=(f_i)_{i\in J}$. Then $\mathrm{LIM}(\mathcal{G}_{P_{k,A}},\{E_{k,A,A/B}\}_{B\in\Pi(\mathcal{F})})$ exists for all $k\geq 1$ (cf. Remark 3.13 (3)). Let $\mathcal{H}$ be an infinite subfamily of $\mathcal{G}$. Then $\mathrm{LIM}(\mathcal{H}_{P_{k,A}},\{E_{k,A,A/B}\}_{B\in\Pi(\mathcal{F})})$ exists for all $k\geq 1$ (cf. Remark 3.13 (3)). Hence, by (12.2), $\mathrm{LIM}(\mathcal{H}_{P_{k,A}},\{E_{k,A,A/B}\}_{B\in\Pi(\mathcal{H})})$ exists for all $k\geq 1$. Thus $\mathcal{H}$ satisfies Assumption 3.19. The proof is completed. □

Lemma 12.3. *Let $\mathcal{F}\subset\mathrm{Hol}(\mathbb{D},A)$ be an infinite set of non-constant holomorphic maps. By replacing $\mathcal{F}$ by its infinite subset, we may assume Assumption 11.2.*

Proof. By replacing $\mathcal{F}$ by its infinite subset, we may assume (12.2) for all infinite subsets $\mathcal{F}'\subset\mathcal{F}$ (cf. Lemma 12.1). We enumerate the elements of $\Pi(\mathcal{F})$ as $B_1,B_2,\dots$. If this sequence is finite, say it terminates at B_L, we extend it to an infinite sequence by defining $B_k=B_L$ for all $k\geq L$. We apply Lemma 12.2 to get an infinite subset $\mathcal{F}_1\subset\mathcal{F}$ such that all infinite subfamilies of $(\mathcal{F}_1)_{A/B_1}$ satisfy Assumption 3.19. Again we apply Lemma 12.2 to get an infinite subset $\mathcal{F}_2\subset\mathcal{F}_1$ such that all infinite subfamilies of $(\mathcal{F}_2)_{A/B_2}$ satisfy Assumption 3.19. We continue this process to get an infinite descending sequence of infinite sets $\mathcal{F}\supset\mathcal{F}_1\supset\mathcal{F}_2\supset\cdots$. We define a countably infinite subset $\mathcal{G}=\{f_1,f_2,f_3,\dots\}$ such that $f_k\in\mathcal{F}_l$ for all $l\geq 1$ and $k\geq l$ (cf. the proof of Lemma 12.1). Then, for all $B\in\Pi(\mathcal{F})$, $\mathcal{G}_{A/B}$ satisfies Assumption 3.19. By (12.2), we have $\Pi(\mathcal{G})=\Pi(\mathcal{F})$. Hence for all $B\in\Pi(\mathcal{G})$, $\mathcal{G}_{A/B}$ satisfies Assumption 3.19. Hence $\mathcal{G}$ satisfies Assumption 11.2. □

Lemma 12.4. *Let $X\subsetneqq A$. Let $\bar{A}$ be a smooth equivariant compactification and let $\overline{X}\subset\bar{A}$ be the compactification. Let $\mathcal{F}\subset\mathrm{Hol}(\mathbb{D},X)$ be an infinite set of holomorphic maps. Then there exist a semi-abelian subvariety $B\subset A$ and an infinite subset $\mathcal{G}\subset\mathcal{F}$ with the following two properties:*

1. *Let $\overline{A/B}$ be a smooth equivariant compactification. Then there exists an infinite subset $\mathcal{H}\subset\mathcal{G}$ such that $\{\varpi\circ f\}_{f\in\mathcal{H}}$ converges uniformly on compact subsets of $\mathbb{D}$ to $g\colon\mathbb{D}\to\overline{A/B}$, where $\varpi\colon A\to A/B$ is the quotient map.*
2. $\mathcal{G}\to\mathrm{Sp}_B\overline{X}$.

Proof. We may assume that $\mathcal{F}$ contains only finitely many constant maps, for otherwise our lemma is trivial by setting $B=\{0\}$ and $\mathcal{G}$ is the set of constant mappings in $\mathcal{F}$. We remove those constant maps from $\mathcal{F}$ and assume that all elements of $\mathcal{F}$ are non-constant.

We first reduce to the case $\bar{A}$ is projective. When this is not satisfied, we may take an equivariant modification $\hat{A} \to \bar{A}$ such that $\hat{A}$ is smooth and projective (cf. Lemma A.8). Let $\widehat{X} \subset \hat{A}$ be the compactification. Then the image of $\mathrm{Sp}_B\widehat{X} \subset \widehat{X}$ under the natural map $\widehat{X} \to \overline{X}$ is contained in $\mathrm{Sp}_B\overline{X}$. Hence $\mathcal{G} \to \mathrm{Sp}_B\widehat{X}$ implies $\mathcal{G} \to \mathrm{Sp}_B\overline{X}$. Hence we have reduced to the case that $\bar{A}$ is projective.

By Lemma 12.3, we may take an infinite subset $\mathcal{G} \subset \mathcal{F}$ such that $\mathcal{G}$ satisfies Assumption 11.2. We prove the lemma in two cases.

If $\Lambda_{X,\bar{A}}(\mathcal{G}) \neq \emptyset$, then we take $B \in \Lambda_{X,\bar{A}}(\mathcal{G})$. Then $\mathcal{G} \to \mathrm{Sp}_B\overline{X}$ and there exists an infinite subset $\mathcal{H} \subset \mathcal{G}$ such that $\{|(\varpi \circ f)'|_{\omega_{A/B}}\}_{f\in\mathcal{H}}$ converges uniformly on compact subsets of $\mathbb{D}$ to 0. Hence replacing $\mathcal{H}$ by its infinite subset, we conclude that $\{\varpi \circ f\}_{f\in\mathcal{H}}$ converges to some constant map. Thus our lemma is valid in this case.

If $\Lambda_{X,\bar{A}}(\mathcal{G}) = \emptyset$, then we take B as in Proposition 11.3. We have $\mathcal{G} \to \mathrm{Sp}_B\overline{X}$. By Proposition 11.3 (1), we have $B \in \Pi(\mathcal{G})$. Hence an infinite indexed family $\mathcal{G}_{A/B}$ in $\mathrm{Hol}(\mathbb{D}, A/B)$ contains only finitely many constant maps and satisfies Assumption 3.19. By Proposition 11.3 (2), we get k such that $Z_{k+1,A/B} \subset P_{k+1,A/B}$ is horizontally integrable, where $Z_{k+1,A/B}$ is defined in (11.1). Let $\overline{A/B}$ be a smooth equivariant compactification. By Corollary 8.4, we may choose an infinite subset $\mathcal{H} \subset \mathcal{G}$ such that $\{\varpi \circ f\}_{f\in\mathcal{H}}$ converges uniformly on compact subsets of $\mathbb{D}$ to $g\colon \mathbb{D} \to \overline{A/B}$. The proof is completed. $\square$

Now to prove Theorem 1.2, Lemma 12.4 is weak for two reasons. Firstly, the limit morphism $g\colon \mathbb{D} \to \overline{A/B}$ is obtained only in the compactification $\overline{A/B}$, not in A/B itself. Secondly, the choices of B and $\mathcal{G}$ depend on the compactification $\bar{A}$. In the remainder of this section, we shall resolve these issues.

Let $B \subset A$ be a semi-abelian subvariety. Let $\hat{A}$ and $\overline{A/B}$ be equivariant compactifications with an equivariant map $p\colon \hat{A} \to \overline{A/B}$. We consider the following assumption.

Assumption 12.5. For every $x \in \hat{A}$, if a semi-abelian subvariety $C \subset A$ satisfies $C \cdot p(x) = p(x)$, then $C \cdot x \subset B \cdot x$ as subsets of $\hat{A}$.

By choosing an equivariant blow-up $\varphi\colon \hat{A} \to \bar{A}$ and a particular equivariant compactification $\overline{A/B}$, we may assume this assumption. We may assume that $\overline{A/B}$ is smooth. This is a consequence of Lemma A.15.

Lemma 12.6. *Let $X \subsetneqq A$ be a proper closed subvariety. Let $B \subset A$ be a semi-abelian subvariety. Let $\overline{A/B}$ be a smooth equivariant compactification and let $\hat{A}$ be an equivariant compactification such that an equivariant map $p\colon \hat{A} \to \overline{A/B}$ with Assumption 12.5 exists. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, X)$ be an infinite set of holomorphic maps such that $\mathcal{F} \to \mathrm{Sp}_B\widehat{X}$, where $\widehat{X} \subset \hat{A}$ is the Zariski closure of $X \subset A$. Assume that*

$(p\circ f)_{f\in\mathcal{F}}$ converges uniformly on compact subsets of $\mathbb{D}$ to $g\colon \mathbb{D}\to\overline{A/B}$ such that $g(\mathbb{D})\subset\partial(A/B)$. Then there exists a semi-abelian subvariety $B'\subset A$ with $B\subset B'$ such that the following two properties hold:

(1) *$(p'\circ f)_{f\in\mathcal{F}}$ converges uniformly on compact subsets of $\mathbb{D}$ to $h\colon\mathbb{D}\to A/B'$, where $p'\colon A\to A/B'$ is the quotient map.*
(2) *$\mathcal{F}\to \mathrm{Sp}_{B'}\widehat{X}$.*

Proof. Let $Y\subset\widehat{X}$ be the smallest Zariski closed set such that $\mathcal{F}\to Y$ (cf. Lemma 3.6). We have

$$Y\subset \mathrm{Sp}_B(\widehat{X}). \tag{12.3}$$

Note that $p(Y)$ is the smallest Zariski closed set such that $(p\circ f)_{f\in\mathcal{F}}\to p(Y)$. Hence $p(Y)$ is the Zariski closure of $g(\mathbb{D})$ (cf. Remark 3.7). By $g(\mathbb{D})\subset\partial(A/B)$, we have $p(Y)\subset\partial(A/B)$. Let $V\subset\overline{A/B}$ be the Zariski closure of $p(Y)+(A/B)\subset\overline{A/B}$. Then since $p(Y)$ is irreducible, V is irreducible. Note that V is A/B-invariant. Hence by Lemma A.11, there exists an A/B-invariant Zariski open set $W\subset\overline{A/B}$ such that $V\subset W$ and that an equivariant morphism $\tau\colon W\to V$ exists. Let $I\subset A/B$ be the isotropy group for V. Let $O\subset V$ be the unique dense A/B-orbit. Then O is isomorphic to $(A/B)/I$ and $O\cap p(Y)\neq\emptyset$. Hence

$$g(\mathbb{D})\not\subset V\backslash O. \tag{12.4}$$

Since $(p\circ f)_{f\in\mathcal{F}}$ converges uniformly on compact subsets on $\mathbb{D}$ to g, the indexed family $(\tau\circ p\circ f)_{f\in\mathcal{F}}$ converges uniformly on compact subsets on $\mathbb{D}$ to $\tau\circ g=g$. We have $\tau\circ p\circ f(\mathbb{D})\subset O$ for all $f\in\mathcal{F}$. Hence by (12.4), we have $g(\mathbb{D})\subset O$. Let B' be $B\subset B'\subset A$ such that $B'/B=I$. Then under the isomorphism $O\simeq A/B'$, we have $p'\circ f=\tau\circ p\circ f$. Hence we have proved (1).

In the following, we shall prove $\mathcal{F}\to\mathrm{Sp}_{B'}(\widehat{X})$. Note that the action of B' on $p(Y)$ is trivial. Hence by Assumption 12.5, we have $B'+Y\subset B+Y$. Since $\mathrm{Sp}_B(\widehat{X})$ is B-invariant, we have $B+Y\subset\mathrm{Sp}_B\widehat{X}$ (cf. (12.3)). Hence we get

$$B'+Y\subset\mathrm{Sp}_B\widehat{X}\subset\widehat{X}.$$

Hence $Y\subset\mathrm{Sp}_{B'}\widehat{X}$, so $\mathcal{F}\to\mathrm{Sp}_{B'}\widehat{X}$. □

Lemma 12.7. *Let $X\subsetneqq A$ be a proper closed subvariety. Then there exists a smooth equivariant compactification $\bar{A}$ such that $\mathrm{Sp}_A(\overline{X})=\emptyset$.*

Proof. Let $\tilde{A}$ be an equivariant compactification and let $\widetilde{X}\subset\tilde{A}$ be the closure. Set $Z=\bigcap_{a\in A}(a+\widetilde{X})$ as a scheme. By $X\neq A$, we have $\operatorname{supp}Z\subset\partial A$. Let

$m\colon A\times\tilde{A}\to\tilde{A}$ be the A-action on the equivariant compactification $\tilde{A}$. We first prove

$$m^*Z=A\times Z. \tag{12.5}$$

Indeed, we have $a+Z=Z$ for all $a\in A$. Hence by Lemma A.17, we have $A\times Z\subset m^*Z$. To prove the converse $m^*Z\subset A\times Z$, we consider an isomorphism $\mu\colon A\times\tilde{A}\to A\times\tilde{A}$ defined by $(a,x)\mapsto(a,m(-a,x))$. By Lemma A.17, we also have $A\times Z\subset\mu^*(A\times Z)$. Note that $m\circ\mu\colon A\times\tilde{A}\to\tilde{A}$ is the second projection. Hence $\mu^*m^*Z=A\times Z$. Thus $\mu^*m^*Z\subset\mu^*(A\times Z)$, hence $m^*Z\subset A\times Z$. Thus we get (12.5).

We claim that the blow-up $\hat{A}=\mathrm{Bl}_Z\tilde{A}$ is an equivariant compactification. Indeed, the map $m\colon A\times\tilde{A}\to\tilde{A}$ induces $m'\colon A\times\mathrm{Bl}_Z\tilde{A}\to\tilde{A}$. Note that $A\times\mathrm{Bl}_Z\tilde{A}=\mathrm{Bl}_{A\times Z}(A\times\tilde{A})$. Hence by (12.5), $(m')^*Z$ is a Cartier divisor. Hence m' factors as $A\times\mathrm{Bl}_Z\tilde{A}\to\mathrm{Bl}_Z\tilde{A}$. Hence we have proved that $\mathrm{Bl}_Z\tilde{A}$ is an equivariant compactification.

Now we may take finite points $a_1,\dots,a_l\in A$ such that $Z=\bigcap_{i=1}^l(a_i+\widetilde{X})$ as a scheme. Then we have

$$\bigcap_{i=1}^{l}(a_i+\widehat{X})=\emptyset, \tag{12.6}$$

where $\widehat{X}\subset\hat{A}$ is the Zariski closure of X. We prove this. Let $\varphi\colon\hat{A}\to\tilde{A}$ be the morphism. Since $\varphi^*Z\subset\hat{A}$ is a Cartier divisor, $\mathcal{I}_{\varphi^*Z}\subset\mathcal{O}_{\hat{A}}$ is an invertible sheaf. By $\mathcal{I}_{\varphi^*(a_i+\widetilde{X})}\subset\mathcal{I}_{\varphi^*Z}\subset\mathcal{O}_{\hat{A}}$, we take $Y_i\subset\hat{A}$ such that $\mathcal{I}_{Y_i}=\mathcal{I}_{\varphi^*(a_i+\widetilde{X})}\otimes(\mathcal{I}_{\varphi^*Z})^{-1}\subset\mathcal{O}_{\hat{A}}$. Then by $\mathcal{I}_{\varphi^*(a_1+\widetilde{X})}+\cdots+\mathcal{I}_{\varphi^*(a_l+\widetilde{X})}=\mathcal{I}_{\varphi^*Z}$, we have $\mathcal{I}_{\varphi^*Z}\cdot(\mathcal{I}_{Y_1}+\cdots+\mathcal{I}_{Y_l})=\mathcal{I}_{\varphi^*Z}$ so that $\mathcal{I}_{Y_1}+\cdots+\mathcal{I}_{Y_l}=\mathcal{O}_{\hat{A}}$. Hence $Y_1\cap\cdots\cap Y_l=\emptyset$. Note that $a_i+\widehat{X}\subset Y_i$. Hence we get (12.6). By $\mathrm{Sp}_A(\widehat{X})\subset\bigcap_{i=1}^l(a_i+\widehat{X})$, we have $\mathrm{Sp}_A(\widehat{X})=\emptyset$. By Lemma A.7, we may take a smooth equivariant modification $\bar{A}\to\hat{A}$. Then $\bar{A}$ satisfies our assertion. (See also [35, Prop. 5.6.7].) $\square$

Let $X\subset A$ be a closed subvariety and $\mathcal{F}\subset\mathrm{Hol}(\mathbb{D},X)$ be an infinite set. Given an equivariant compactification $\bar{A}$, we set

$$Q(\bar{A},\mathcal{F})=\{B\subset A;\ \mathcal{F}\to\mathrm{Sp}_B\overline{X}\},$$

where $\overline{X}\subset\bar{A}$ is the Zariski closure. Then $\{0\}\in Q(\bar{A},\mathcal{F})$, hence $Q(\bar{A},\mathcal{F})\neq\emptyset$. If $\mathcal{G}\subset\mathcal{F}$ is an infinite subset, then we have $Q(\bar{A},\mathcal{F})\subset Q(\bar{A},\mathcal{G})$. Indeed, if $B\in Q(\bar{A},\mathcal{F})$, then $\mathcal{F}\to\mathrm{Sp}_B\overline{X}$. Hence $\mathcal{G}\to\mathrm{Sp}_B\overline{X}$, which shows $B\in Q(\bar{A},\mathcal{G})$.

Lemma 12.8. *Replacing $\mathcal{F}$ by its infinite subset, we have $Q(\bar{A},\mathcal{F})=Q(\bar{A},\mathcal{G})$ for all infinite subsets $\mathcal{G}\subset\mathcal{F}$ and all smooth equivariant compactifications $\bar{A}$.*

Proof. The pairs $(\hat{A}, B)$ of smooth equivariant compactifications $\hat{A}$ and semi-abelian subvarieties $B \subset A$ are countable (cf. Lemma A.10 and [35, Cor. 5.1.9]). We enumerate them as $(\hat{A}_1, B_1), (\hat{A}_2, B_2), \ldots$. If this sequence is finite, say it terminates at $(\hat{A}_L, B_L)$, we extend it to an infinite sequence by defining $(\hat{A}_k, B_k) = (\hat{A}_L, B_L)$ for all $k \geq L$. In the following, we construct an infinite descending sequence $\mathcal{F} \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$ of infinite subsets such that for all $k \geq 1$, one of the following occurs:

(1) $\mathcal{F}_k \to \mathrm{Sp}_{B_k} \widehat{X}_k$, where $\widehat{X}_k \subset \hat{A}_k$ is the Zariski closure of X in $\hat{A}_k$, or
(2) $\mathcal{F}' \not\to \mathrm{Sp}_{B_k} \widehat{X}_k$ for all infinite subset $\mathcal{F}' \subset \mathcal{F}_k$.

The construction is the following. If there exists an infinite subset $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{F}' \to \mathrm{Sp}_{B_1} \widehat{X}_1$, then we set $\mathcal{F}_1 = \mathcal{F}'$, otherwise we set $\mathcal{F}_1 = \mathcal{F}$. If there exists $\mathcal{F}' \subset \mathcal{F}_1$ such that $\mathcal{F}' \to \mathrm{Sp}_{B_2} \widehat{X}_2$, then we set $\mathcal{F}_2 = \mathcal{F}'$, otherwise $\mathcal{F}_2 = \mathcal{F}_1$. In this way, we get $\mathcal{F} \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$.

We define an infinite sequence $f_1, f_2, f_3, \ldots$ of distinct elements in $\mathcal{F}$ so that for each $l \geq 1$, we have

$$f_k \in \mathcal{F}_l \tag{12.7}$$

for all $k \geq l$. This sequence is constructed inductively as follows. We take $f_1 \in \mathcal{F}_1$. Suppose distinct elements $f_1, f_2, \ldots, f_n$ are chosen so that (12.7) holds for all $k, l \in \mathbb{N}$ with $1 \leq l \leq k \leq n$. Then we choose $f_{n+1} \in \mathcal{F}_{n+1} - \{f_1, \ldots, f_n\}$. Thus we have constructed the sequence $f_1, f_2, \ldots$ with the desired property (12.7).

Now we set $\mathcal{F}_o = \{f_1, f_2, \ldots\}$. Then $\mathcal{F}_o \subset \mathcal{F}$ is an infinite subset. Let $\mathcal{G} \subset \mathcal{F}_o$ be an infinite subset and $\bar{A}$ be a smooth equivariant compactification. We claim $Q(\bar{A}, \mathcal{F}_o) = Q(\bar{A}, \mathcal{G})$. To show this, it is enough to show $Q(\bar{A}, \mathcal{G}) \subset Q(\bar{A}, \mathcal{F}_o)$. Let $B \in Q(\bar{A}, \mathcal{G})$. We may take k such that $(\bar{A}, B) = (\hat{A}_k, B_k)$. Since $\mathcal{G} \cap \mathcal{F}_k$ is infinite, $\mathcal{G} \to \mathrm{Sp}_{B_k} \widehat{X}_k$ yields that $\mathcal{F}_k \to \mathrm{Sp}_{B_k} \widehat{X}_k$. Hence $\mathcal{F}_o \to \mathrm{Sp}_{B_k} \widehat{X}_k$, so that $B \in Q(\bar{A}, \mathcal{F}_o)$. Hence $Q(\bar{A}, \mathcal{G}) \subset Q(\bar{A}, \mathcal{F}_o)$. We replace $\mathcal{F}$ by $\mathcal{F}_o$ to conclude the proof. □

In the following, we write $Q(\bar{A}) = Q(\bar{A}, \mathcal{F})$ if no confusion may occur.

Lemma 12.9. *The set $Q(\bar{A})$ contains a finite number of maximal elements, i.e., there exist $B_1, \ldots, B_l \in Q(\bar{A})$ such that for all $B \in Q(\bar{A})$, there exists B_i such that $B \subset B_i$.*

Proof. By Lemma 3.6, there exists a Zariski closed set $Y \subset \overline{X}$ such that for Zariski closed subsets $V \subset \overline{X}$, we have $\mathcal{F} \to V$ if and only if $Y \subset V$. If $B \in Q(\bar{A})$, then $Y \subset \mathrm{Sp}_B \overline{X}$. Since $\mathrm{Sp}_B \overline{X}$ is B-invariant, we have $B+Y \subset \mathrm{Sp}_B \overline{X}$, hence $B+Y \subset \overline{X}$. Conversely, if $B \subset A$ is a semi-abelian subvariety such that $B + Y \subset \overline{X}$, then we have $Y \subset \mathrm{Sp}_B \overline{X}$. Hence we have $\mathcal{F} \to \mathrm{Sp}_B \overline{X}$, so $B \in Q(\bar{A})$. Thus $B \in Q(\bar{A})$ if and only if $B + Y \subset \overline{X}$.

Let $\varphi\colon A \times Y \to \bar{A}$ be the restriction of the A-action $A \times \bar{A} \to \bar{A}$. Then $\varphi^{-1}(\overline{X}) \subset A \times Y$ is Zariski closed. We define $Z \subset A$ to be the set of $a \in A$ such that $\{a\} \times Y \subset \varphi^{-1}(\overline{X})$. Then $Z \subset A$ is Zariski closed. We have

$$B \in Q(\bar{A}) \Leftrightarrow B \subset Z. \tag{12.8}$$

We set $\mathcal{U} = \mathrm{Lie}(A) - \{0\}$. Let $\phi\colon \mathbb{C} \times \mathcal{U} \to A$ be an analytic map defined by $\phi(z,u) = \exp(zu)$. For $t \in \mathbb{C}$, we define $\iota_t\colon \mathcal{U} \to \mathbb{C} \times \mathcal{U}$ by $\iota_t(u) = (t,u)$. We set

$$\mathcal{Z} = \bigcap_{t \in \mathbb{C}} (\phi \circ \iota_t)^{-1}(Z).$$

Then $\mathcal{Z} \subset \mathcal{U}$ is an analytic subset such that

$$u \in \mathcal{Z} \Leftrightarrow \phi_u(\mathbb{C}) \subset Z, \tag{12.9}$$

where $\phi_u\colon \mathbb{C} \to A$ is a one-parameter group defined by $\phi_u(z) = \exp(zu)$. For each $b \in \mathbb{C}^*$, we have $\phi_{bu}(z) = \phi_u(bz)$. Hence $\mathcal{Z} \subset \mathcal{U}$ is invariant under the natural $\mathbb{C}^*$-action on $\mathcal{U}$. Thus $\mathcal{Z}/\mathbb{C}^*$ is an analytic subset of $\mathcal{U}/\mathbb{C}^* = \mathbb{P}(\mathrm{Lie}(A))$. Indeed, let $\Omega \subset \mathbb{P}(\mathrm{Lie}(A))$ be an open set with a local analytic section $\sigma\colon \Omega \to \mathcal{U}$ of the natural projection $\mathcal{U} \to \mathbb{P}(\mathrm{Lie}(A))$. Then we have $\sigma^{-1}(\mathcal{Z}) = (\mathcal{Z}/\mathbb{C}^*) \cap \Omega$. Since $\sigma^{-1}(\mathcal{Z}) \subset \Omega$ is an analytic subset, $\mathcal{Z}/\mathbb{C}^*$ is an analytic subset of $\mathbb{P}(\mathrm{Lie}(A))$. Hence by Chow's lemma, $\mathcal{Z}/\mathbb{C}^*$ is an algebraic subset of $\mathbb{P}(\mathrm{Lie}(A))$.

For $[u] \in \mathbb{P}(\mathrm{Lie}(A))$, where $u \in \mathrm{Lie}(A) - \{0\}$, let $B_{[u]}$ be the Zariski closure of $\phi_u(\mathbb{C}) \subset A$. Then $B_{[u]} \subset A$ is a semi-abelian subvariety. Then by (12.9), we have

$$[u] \in \mathcal{Z}/\mathbb{C}^* \Leftrightarrow B_{[u]} \subset Z. \tag{12.10}$$

Hence $[u] \in \mathcal{Z}/\mathbb{C}^*$ if and only if $\mathbb{P}(\mathrm{Lie}(B_{[u]})) \subset \mathcal{Z}/\mathbb{C}^*$. Thus

$$\mathcal{Z}/\mathbb{C}^* = \bigcup_{[u] \in \mathcal{Z}/\mathbb{C}^*} \mathbb{P}(\mathrm{Lie}(B_{[u]})).$$

Note that there are only countably many semi-abelian subvarieties of A (cf. [35, Cor. 5.1.9]). Hence there exist $[u_1], \dots, [u_l] \in \mathcal{Z}/\mathbb{C}^*$ such that

$$\mathcal{Z}/\mathbb{C}^* = \mathbb{P}(\mathrm{Lie}(B_{[u_1]})) \cup \cdots \cup \mathbb{P}(\mathrm{Lie}(B_{[u_l]})).$$

Now if $B \in Q(\bar{A})$, then $B \subset Z$ (cf. (12.8)). Hence we have $\mathbb{P}(\mathrm{Lie}(B)) \subset \mathcal{Z}/\mathbb{C}^*$ (cf. (12.10)). Hence there exists $B_{[u_i]}$ such that $B \subset B_{[u_i]}$. This concludes the proof. $\square$

Let Σ be the set of all semi-abelian subvarieties of A. Let $\mathcal{Q}$ be the set of all subset $Q \subset \Sigma$ such that

(1) Q contains a finite number of maximal elements, and
(2) if $B, B' \in \Sigma$ such that $B \in Q$ and $B' \subset B$, then $B' \in Q$.

Then $\mathcal{Q}$ satisfies the following.

Lemma 12.10. *Let $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$ be a descending sequence in $\mathcal{Q}$. Then there exists $k \geq 1$ such that $Q_k = Q_{k+1} = Q_{k+2} = \cdots$.*

Proof. For each $Q \in \mathcal{Q}$, we denote by $Q_{\max} \subset Q$ the set of the maximal elements in Q. Then $Q_{\max} \subset Q$ is a finite subset. We set

$$P_Q = \bigcup_{B \in Q_{\max}} B.$$

Then $P_Q \subset A$ is a Zariski closed set. We claim that $Q \subset Q'$ if and only if $P_Q \subset P_{Q'}$. Indeed, assume $Q \subset Q'$. Then for $B \in Q_{\max}$, we have $B \in Q'$, hence there exists $B' \in Q'_{\max}$ such that $B \subset B'$. This shows $B \subset P_{Q'}$, hence $P_Q \subset P_{Q'}$. Conversely, suppose $P_Q \subset P_{Q'}$. Let $B \in Q$. Then $B \subset P_Q \subset P_{Q'}$. Hence there exists $B' \in Q'_{\max}$ such that $B \subset B'$. Thus by property (2) for $\mathcal{Q}$, we have $B \in Q'$. Hence $Q \subset Q'$. Thus we have proved that $Q \subset Q'$ if and only if $P_Q \subset P_{Q'}$.

Now let $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$ be a descending sequence in $\mathcal{Q}$. Then we have $P_{Q_1} \supset P_{Q_2} \supset P_{Q_3} \supset \cdots$. Thus by the Noetherian property, there exists k such that $P_{Q_k} = P_{Q_{k+1}} = P_{Q_{k+2}} = \cdots$. Then we have $Q_k = Q_{k+1} = Q_{k+2} = \cdots$. The proof is completed. □

For each equivariant compactification $\bar{A}$, we have $Q(\bar{A}) \in \mathcal{Q}$. Indeed, by Lemma 12.9, $Q(\bar{A})$ contains a finite number of maximal elements. If $B \in Q(\bar{A})$ and $B' \subset B$, then by $\mathrm{Sp}_B\overline{X} \subset \mathrm{Sp}_{B'}\overline{X}$, we have $\mathcal{F} \to \mathrm{Sp}_{B'}\overline{X}$, hence $B' \in Q(\bar{A})$. Thus $Q(\bar{A}) \in \mathcal{Q}$. For an equivariant compactification $\hat{A} \to \bar{A}$, we have $Q(\hat{A}) \subset Q(\bar{A})$. Indeed, the image of $\mathrm{Sp}_B(\widehat{X}) \subset \hat{A}$ under the map $\hat{A} \to \bar{A}$ is contained in $\mathrm{Sp}_B(\overline{X}) \subset \bar{A}$, which implies that if $B \in Q(\hat{A})$, then $B \in Q(\bar{A})$.

Lemma 12.11. *There exists an equivariant compactification $\bar{A}$ such that for every equivariant compactification $\hat{A} \to \bar{A}$, we have $Q(\bar{A}) = Q(\hat{A})$. Moreover, we may take $\bar{A}$ to be smooth.*

Proof. Assume on the contrary that we get a sequence $\bar{A}_1 \leftarrow \bar{A}_2 \leftarrow \cdots$ such that $Q(\bar{A}_1) \supsetneqq Q(\bar{A}_2) \supsetneqq \cdots$. But this does not occur from Lemma 12.10. Hence there exists $\bar{A}$ of the desired property. If our $\bar{A}$ is not smooth, we may replace $\bar{A}$ by its smooth equivariant blow-up. This exists by Lemma A.7. □

We reduce Theorem 1.2 to the following equivalent statement.

Theorem 12.12. *Let A be a semi-abelian variety. Let $X \subsetneqq A$ be a proper closed algebraic subvariety. Let $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, X)$ be an infinite set of holomorphic maps. Then there exist a proper semi-abelian subvariety $B \subsetneqq A$ and an infinite subset $\mathcal{G} \subset \mathcal{F}$ with the following two properties:*

(1) *$(\varpi \circ f)_{f\in\mathcal{G}}$ converges uniformly on compact subsets of $\mathbb{D}$ to $g\colon \mathbb{D} \to A/B$, where $\varpi\colon A \to A/B$ is the quotient map.*

(2) *Let $\bar{A}$ be an equivariant compactification and let $\overline{X} \subset \bar{A}$ be the compactification. Then $\mathcal{G} \to \mathrm{Sp}_B\overline{X}$.*

Proof. By Lemma 12.8, replacing $\mathcal{F}$ by its infinite subset, we may assume that

$$Q(\tilde{A}, \mathcal{F}) = Q(\tilde{A}, \mathcal{G}) \tag{12.11}$$

for every smooth equivariant compactification $A \subset \tilde{A}$ and every infinite subset $\mathcal{G} \subset \mathcal{F}$. By Lemma 12.11, we may take a smooth equivariant compactification $\bar{A}$ such that

$$Q(\bar{A}, \mathcal{F}) = Q(\hat{A}, \mathcal{F}) \tag{12.12}$$

for all $\hat{A} \to \bar{A}$.

We claim that for every equivariant compactification $\tilde{A}$ and infinite subset $\mathcal{G} \subset \mathcal{F}$, we have

$$Q(\bar{A}, \mathcal{G}) \subset Q(\tilde{A}, \mathcal{G}). \tag{12.13}$$

To prove this, we apply Lemmas A.7 and A.9 to get a smooth equivariant blow-up $\hat{A} \to \bar{A}$ such that $\hat{A} \to \tilde{A}$ exists. Then by (12.11) and (12.12), we get

$$Q(\bar{A}, \mathcal{G}) = Q(\bar{A}, \mathcal{F}) = Q(\hat{A}, \mathcal{F}) = Q(\hat{A}, \mathcal{G}).$$

On the other hand, the existence of the map $\hat{A} \to \tilde{A}$ yields $Q(\hat{A}, \mathcal{G}) \subset Q(\tilde{A}, \mathcal{G})$. Hence we get (12.13).

Now we apply Lemma 12.4 for $\mathcal{F}$ and $\bar{A}$ to get $B_0 \subset A$ and $\mathcal{G} \subset \mathcal{F}$. Then we have $\mathcal{G} \to \mathrm{Sp}_{B_0}\overline{X}$, hence $B_0 \in Q(\bar{A}, \mathcal{G})$. We choose an equivariant blow-up $\varphi\colon \hat{A} \to \bar{A}$ and an equivariant compactification $\overline{A/B_0}$ such that $p\colon \hat{A} \to \overline{A/B_0}$ exists and satisfies Assumption 12.5 (cf. Lemma A.15). We may assume that $\overline{A/B_0}$ is smooth. By (12.13), we have $B_0 \in Q(\hat{A}, \mathcal{G})$. Thus $\mathcal{G} \to \mathrm{Sp}_{B_0}(\widehat{X})$. By Lemma 12.4, we get an infinite subset $\mathcal{H} \subset \mathcal{G}$ with a limit $g\colon \mathbb{D} \to \overline{A/B_0}$ of $(p \circ f)_{f\in\mathcal{H}}$.

We first consider the case $g(\mathbb{D}) \subset A/B_0$. We take arbitrary $\tilde{A}$. Then by (12.13), we have $B_0 \in Q(\tilde{A}, \mathcal{G})$. Hence $\mathcal{H} \to \mathrm{Sp}_{B_0}\widetilde{X}$. Note that B_0 satisfies $\mathrm{Sp}_{B_0}(\widetilde{X}) \neq \emptyset$ for all $\tilde{A}$. Hence by Lemma 12.7, we have $B_0 \subsetneqq A$. We replace $\mathcal{G}$ by $\mathcal{H}$ and set $B = B_0$ to conclude the proof in the case $g(\mathbb{D}) \subset A/B_0$.

Next we assume $g(\mathbb{D}) \not\subset A/B_0$. Then we have $g(\mathbb{D}) \subset \partial(A/B_0)$. Then by Lemma 12.6, we get B with $B_0 \subset B \subset A$ such that $\{\varpi \circ f\}_{f\in\mathcal{H}}$ converges uniformly on compact subsets of $\mathbb{D}$ to $g\colon \mathbb{D} \to A/B$ and that $\mathcal{H} \to \mathrm{Sp}_B(\widehat{X})$. Hence the existence of $\hat{A} \to \bar{A}$ shows that $\mathcal{H} \to \mathrm{Sp}_B\overline{X}$. We take arbitrary $\tilde{A}$. Then by (12.13), we have $B \in Q(\tilde{A}, \mathcal{H})$. Hence $\mathcal{H} \to \mathrm{Sp}_B\widetilde{X}$. Our B satisfies $\mathrm{Sp}_B(\widetilde{X}) \neq \emptyset$ for all $\tilde{A}$. Hence by Lemma 12.7, we have $B \subsetneqq A$. We replace $\mathcal{G}$ by $\mathcal{H}$ to conclude the proof. $\square$

Proof of Theorem 1.2. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Hol}(\mathbb{D}, X)$. We may assume that the set $\mathcal{F} = \{f_n;\ n \in \mathbb{N}\} \subset \mathrm{Hol}(\mathbb{D}, X)$ is infinite, for otherwise, we may take a subsequence consisting of the same maps, hence the theorem is trivially valid for $B = \{0\}$. Then we apply Theorem 12.12 to get B and $\mathcal{G} \subset \mathcal{F}$. To conclude the proof, we take a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of distinct elements in $\mathcal{G}$. $\square$

§13. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. We recall the Zariski closed subset $S \subset \overline{X}$ from the statement of Theorem 1.3.

Lemma 13.1. *Let $X \subsetneqq A$ be a closed algebraic subvariety of a semi-abelian variety A. Let $\bar{A}$ be a smooth equivariant compactification and let $\overline{X} \subset \bar{A}$ be the compactification. Let $(f_i)_{i\in I}$ be an infinite indexed family of holomorphic maps in* $\mathrm{Hol}(\mathbb{D}, X)$ *such that $(f_i)_{i\in I} \not\to S$. Then there exists an infinite subset $J \subset I$ such that $(f_i)_{i\in J}$ converges uniformly on compact subsets of $\mathbb{D}$ to some $h\colon \mathbb{D} \to \overline{X}$.*

In the following proof, we recall C^1_H from (8.1).

Proof. Set $\mathcal{F} = \{f_i;\ i \in I\} \subset \mathrm{Hol}(\mathbb{D}, X)$. Then we may assume that the map $I \to \mathcal{F}$ defined by $i \mapsto f_i$ is a finite-to-one mapping, for otherwise we may choose an infinite indexed subfamily which consists of the same elements in $\mathrm{Hol}(\mathbb{D}, X)$. In particular, $\mathcal{F}$ is infinite. By $(f_i)_{i\in I} \not\to S$, we have $\mathcal{F} \not\to S$. We shall show the existence of an infinite subset $\mathcal{G} \subset \mathcal{F}$ which converges uniformly on compact subsets of $\mathbb{D}$ to some $h\colon \mathbb{D} \to \overline{X}$.

By replacing $\mathcal{F}$ by its infinite subset, we may assume that $\mathcal{F}' \not\to S$ for all infinite subsets $\mathcal{F}' \subset \mathcal{F}$ (cf. Lemma 4.8). We take $B \subset A$ and $\mathcal{G} \subset \mathcal{F}$ as in Theorem 12.12. If $B = \{0\}$, then our lemma is valid. Hence we assume $\dim B \geq 1$. Let $Z \subset \partial X$ be the locus where B acts trivially. Then $\mathrm{Sp}_B\overline{X} \subset S \cup Z$, hence $\mathcal{G} \to S \cup Z$. Let $\mathcal{Z}$ be the set of all Zariski closed sets $Z' \subset Z$ such that $\mathcal{G} \to S \cup Z'$. By $\mathcal{G} \to S \cup Z$, we have $Z \in \mathcal{Z}$, hence $\mathcal{Z} \neq \emptyset$. By the Noetherian property, we may take a minimum element $Z' \in \mathcal{Z}$. By $\mathcal{G} \not\to S$, we have $Z' \neq \emptyset$. Let T be an irreducible component of Z'. Set $S_T = S \cup \overline{(Z' \backslash T)}$. By $\overline{(Z' \backslash T)} \subsetneqq Z'$, we have

$\overline{(Z'\backslash T)} \notin \mathcal{Z}$. Hence

$$\mathcal{G} \not\to S_T. \tag{13.1}$$

Let $C \subset A$ be the isotropy group of the generic points of T. Then $B \subset C$. Let $V \subset \partial A$ be the Zariski closure of $A + T \subset \partial A$. Then C is the isotropy group of V. Since T is irreducible, V is irreducible. Let

$$p\colon W \to V$$

be the vector bundle described in Lemma A.11 so that $V = \overline{A/C}$, where $V \subset W$ is considered a zero-section. Note that

$$S \cup Z' \subset S_T \cup T \subset S_T \cup V.$$

By $\mathcal{G} \to S \cup Z'$, we have

$$\mathcal{G} \to S_T \cup V. \tag{13.2}$$

From the limit map $g_0\colon \mathbb{D} \to A/B$, we get $g\colon \mathbb{D} \to A/C \subset V$, which is the limit of $(p \circ f)_{f\in\mathcal{G}}$.

We are going to prove $\mathcal{G} \to V$. We first show $g(\mathbb{D}) \not\subset S_T \cap V$. To prove this, we assume on the contrary that $g(\mathbb{D}) \subset S_T \cap V$. Let $W \subset \overline{W}$ be a compactification such that $\varphi\colon \overline{W} \to \bar{A}$ and $\bar{p}\colon \overline{W} \to V$ exists. By (13.2), we have $\mathcal{G} \to \varphi^{-1}(S_T) \cup V$. Since we are assuming $g(\mathbb{D}) \subset S_T \cap V$, we have $(p \circ f)_{f\in\mathcal{G}} \to S_T \cap V$ (cf. Remark 3.7). Hence by Lemma 3.8, we have $\mathcal{G} \to \bar{p}^{-1}(S_T \cap V)$. Hence $\mathcal{G} \to (\varphi^{-1}(S_T) \cup V) \cap \bar{p}^{-1}(S_T \cap V)$ (cf. Lemma 3.5). By

$$V \cap \bar{p}^{-1}(S_T \cap V) = S_T \cap V,$$

we have

$$\begin{aligned}(\varphi^{-1}(S_T) \cup V) \cap \bar{p}^{-1}(S_T \cap V) &= (\varphi^{-1}(S_T) \cap \bar{p}^{-1}(S_T \cap V)) \cup (V \cap \bar{p}^{-1}(S_T \cap V)) \\ &\subset \varphi^{-1}(S_T).\end{aligned}$$

Hence $\mathcal{G} \to \varphi^{-1}(S_T)$, so $\mathcal{G} \to S_T$. This contradicts (13.1). Hence $g(\mathbb{D}) \not\subset S_T \cap V$.

Now we replace $\mathcal{G}$ by its infinite subset so that $\mathcal{G}' \not\to S_T$ for all infinite subsets $\mathcal{G}' \subset \mathcal{G}$ (cf. Lemma 4.8). We take an open neighborhood $S_T \subset U_0$ and positive constants $s_0 \in (0,1)$ and $\gamma_0 > 0$ such that for all $f \in \mathcal{G}\backslash\mathcal{E}_1$, where $\mathcal{E}_1 \subset \mathcal{G}$ is a finite subset, we have

$$C_H^1(\mathbb{D}(s_0)\backslash f^{-1}(U_0)) \geq \gamma_0. \tag{13.3}$$

To prove $\mathcal{G} \to V$, we take an open neighborhood $V \subset U_1$ and positive constants $s \in (s_0, 1)$ and $\gamma \in (0, \gamma_0)$ arbitrary. We denote by $K \subset \mathbb{D}$ the finite union of small open discs centered at the points of the finite set $g^{-1}(S_T \cap V) \cap \overline{\mathbb{D}(\frac{s+1}{2})}$ so that $g^{-1}(S_T \cap V) \cap \overline{\mathbb{D}(\frac{s+1}{2})} \subset K$ and

$$C^1_H(K) < \gamma/4. \tag{13.4}$$

We take an open neighborhood $U_3 \subset W$ of $S_T \cap V$ such that $U_3 \subset p^{-1}(U_3 \cap V)$ and $g(\overline{\mathbb{D}(\frac{s+1}{2})} \backslash K) \subset V \backslash \overline{U_3 \cap V}$. Since $(p \circ f)_{f \in \mathcal{G}}$ converges uniformly on $\overline{\mathbb{D}(\frac{s+1}{2})}$ to g, we have $p \circ f(\overline{\mathbb{D}(\frac{s+1}{2})} \backslash K) \subset V \backslash \overline{U_3 \cap V}$ for all $f \in \mathcal{G} \backslash \mathcal{E}_2$, where $\mathcal{E}_2 \subset \mathcal{G}$ is a finite subset. Hence by $U_3 \subset p^{-1}(U_3 \cap V)$, we have $f(\overline{\mathbb{D}(\frac{s+1}{2})} \backslash K) \subset \bar{A} \backslash \overline{U_3}$ for all $f \in \mathcal{G} \backslash \mathcal{E}_2$. By replacing U_0 and U_1 by smaller open neighborhoods of S_T and V, respectively, we may assume that $U_0 \cap U_1 \subset U_3$. Hence we have $f(\mathbb{D}((s+1)/2) \backslash K) \subset \bar{A} \backslash (\overline{U_0 \cap U_1})$, so $\mathbb{D}((s+1)/2) \cap f^{-1}(\overline{U_0 \cap U_1}) \subset K$. Hence by (13.4), we have

$$C^1_H(\mathbb{D}((s+1)/2) \cap f^{-1}(\overline{U_0 \cap U_1})) < \gamma/4$$

for all $f \in \mathcal{G} \backslash \mathcal{E}_2$. Now by (13.2), for all $f \in \mathcal{G} \backslash \mathcal{E}_3$, where $\mathcal{E}_3 \subset \mathcal{G}$ is a finite subset, we have

$$C^1_H(\mathbb{D}((s+1)/2) \backslash f^{-1}(U_0 \cup U_1)) < \gamma/4.$$

Then for all $f \in \mathcal{G} \backslash (\mathcal{E}_2 \cup \mathcal{E}_3)$, we have

$$C^1_H(\mathbb{D}((s+1)/2) \backslash f^{-1}(U'_0 \cup U'_1)) < \gamma/2,$$

where $U'_0 = U_0 \backslash \overline{U_0 \cap U_1}$ and $U'_1 = U_1 \backslash \overline{U_0 \cap U_1}$. By Lemma 5.6, there exists an open set $\Omega_f \subset \mathbb{D}(s) \cap f^{-1}(U'_0 \cup U'_1)$ such that Ω_f is connected and

$$C^1_H(\mathbb{D}(s) \backslash \Omega_f) < \gamma.$$

Note that $U'_0 \cap U'_1 = \emptyset$. Since $f(\Omega_f)$ is connected, we have either $f(\Omega_f) \subset U'_0$ or $f(\Omega_f) \subset U'_1$. By $\gamma_0 > \gamma$ and (13.3), we have $f(\Omega_f) \not\subset U_0$ for all $f \in \mathcal{G} \backslash (\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3)$. Hence $f(\Omega_f) \subset U'_1 \subset U_1$. Thus $\mathcal{G} \to V$.

Now by Lemma 8.2, $\mathcal{G}$ converges uniformly on compact subsets of $\mathbb{D}$ to $g \colon \mathbb{D} \to V \cap \overline{X}$. We take an infinite subset $J \subset I$ such that $f_i \in \mathcal{G}$ for all $i \in J$. Then $(f_i)_{i \in J}$ converges uniformly on compact subsets of $\mathbb{D}$ to g. This completes the proof of the lemma. □

Proof of Theorem 1.3. If $X = A$, then $S = A$. Hence our theorem is trivial. In the following, we assume $X \subsetneqq A$.

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of holomorphic maps in $\mathrm{Hol}(\mathbb{D}, X)$. We assume that condition (2) of the definition of “tautly embedded modulo” is not satisfied. Then there exist compact sets $K \subset \mathbb{D}$ and $L \subset \overline{X} - S$ such that, by replacing $(f_n)_{n \in \mathbb{N}}$ by its subsequence, we have $f_n(K) \cap L \neq \emptyset$ for all $n \in \mathbb{N}$.

We shall show

$$(f_n)_{n\in\mathbb{N}} \not\to S.$$

To prove this, we assume, on the contrary, that $(f_n)_{n\in\mathbb{N}} \to S$. We take an open neighborhood $U \subset \overline{X}$ of S such that $\bar{U} \cap L = \emptyset$. For each $n \in \mathbb{N}$, there exists $a_n \in K$ such that $f_n(a_n) \in L$, which follows from $f_n(K) \cap L \neq \emptyset$. We choose an automorphism $\varphi_n : \mathbb{D} \to \mathbb{D}$ so that $\varphi_n(0) = a_n$. Then $f_n \circ \varphi_n(0) \in L$. In particular, $f_n \circ \varphi_n(0) \notin \bar{U}$. We define $\delta_n > 0$ by

$$\delta_n = \sup\{t \in (0,1);\ f_n \circ \varphi_n(\overline{\mathbb{D}(t)}) \subset \overline{X} - \bar{U}\}.$$

Then we have

$$f_n \circ \varphi_n(\mathbb{D}(\delta_n)) \cap U = \emptyset \tag{13.5}$$

and

$$f_n \circ \varphi_n(\partial\mathbb{D}(\delta_n)) \cap \bar{U} \neq \emptyset. \tag{13.6}$$

Since we are assuming $(f_n)_{n\in\mathbb{N}} \to S$, we have

$$\lim_{n\to\infty} \delta_n = 0. \tag{13.7}$$

Set $r_n = \frac{1}{2\delta_n}$. We define $g_n : \mathbb{D}(r_n) \to X$ by $g_n(z) = f_n \circ \varphi_n(2\delta_n z)$. Here we continue to write $\mathbb{D}(r) = \{z \in \mathbb{C};\ |z| < r\}$ for $r \in \mathbb{R}_{>0}$ without assuming $r < 1$.

We define a descending sequence of infinite subsets $\mathbb{N} \supset I_2 \supset I_2 \supset \cdots$ inductively as follows. Set $I_0 = \mathbb{N}$. Suppose I_{l-1} is defined. By (13.7), we may take an infinite subset $I'_l \subset I_{l-1}$ such that $\delta_i < \frac{1}{2l}$ for all $i \in I'_l$. Hence $r_i > l$ for all $i \in I'_l$. Hence for each $i \in I'_l$, we have the restriction $g_i|_{\mathbb{D}(l)} : \mathbb{D}(l) \to X$. We get an infinite indexed family $(g_i|_{\mathbb{D}(l)})_{i\in I'_l}$ in $\mathrm{Hol}(\mathbb{D}(l), X)$. By (13.5), we have $g_i(\mathbb{D}(1/2)) \cap U = \emptyset$ for all $i \in I'_l$. Hence $(g_i|_{\mathbb{D}(l)})_{i\in I'_l} \not\to S$, where we consider $\mathbb{D}(l)$ as "$\mathbb{D}$". We apply Lemma 13.1. Then we get an infinite subset $I_l \subset I'_l$ such that $(g_i|_{\mathbb{D}(l)})_{i\in I_l}$ converges uniformly on compact subsets of $\mathbb{D}(l)$ to $h_l : \mathbb{D}(l) \to \overline{X}$. Since h_{l-1} is the limit of $(g_i|_{\mathbb{D}(l-1)})_{i\in I_l}$, which is a subfamily of $(g_i|_{\mathbb{D}(l-1)})_{i\in I_{l-1}}$, we have $h_l|_{\mathbb{D}(l-1)} = h_{l-1}$.

Now we get a holomorphic map $h : \mathbb{C} \to \overline{X}$ such that $h|_{\mathbb{D}(l)} = h_l$ for all $l \in \mathbb{N}$. We shall show that h is non-constant. Note that $h_1 : \mathbb{D} \to \overline{X}$ is the limit of $(g_i|_{\mathbb{D}})_{i\in I_1}$. By (13.6), we have $g_i(\partial\mathbb{D}(1/2)) \cap \bar{U} \neq \emptyset$ for all $i \in I_1$. Thus $h_1(\partial\mathbb{D}(1/2)) \cap \bar{U} \neq \emptyset$. On the other hand, by $g_i(0) \in L$ for all $i \in I_1$, we have $h_1(0) \in L$. By $L \cap \bar{U} = \emptyset$, h_1 is non-constant. Hence h is non-constant.

For each irreducible component $D \subset \partial A$, we have either $h(\mathbb{C}) \subset D$ or $h(\mathbb{C}) \cap D = \emptyset$. Thus there exists an A-orbit $O \subset \bar{A}$ such that $h(\mathbb{C}) \subset O$. Note that O is isomorphic to a semi-abelian variety. Hence the Zariski closure of $h(\mathbb{C})$ is a translate of a semi-abelian subvariety of O (cf. Theorem 1.1, or [27, Thm. 3.9.19]). Hence $h(\mathbb{C}) \subset S$. This contradicts $h(0) = h_1(0) \in L$. Hence $(f_n)_{n\in\mathbb{N}} \not\to S$.

Now by Lemma 13.1, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges uniformly on compact subsets of $\mathbb{D}$ to $f\colon \mathbb{D} \to \overline{X}$. Thus we have proved that X is tautly embedded modulo S in $\overline{X}$. □

Remark 13.2. While the compact case of Theorem 1.3 was proved in [46, Thm. 1.4], the strategy of that proof does not seem to extend to the non-compact case. This limitation arises because a key geometric lemma from that proof, [46, Lem. 4.2], fails to generalize to the non-compact setting. We remark that the compact case, [46, Thm. 1.4], can be proved more directly from Theorem 1.2 (cf. Remark 14.2).

§14. Proof of Theorem 1.5

When A is compact, we may eliminate the term "γ-almost" from statement (2) in Theorem 1.2 to get the following.

Corollary 14.1. *Let A be an abelian variety. Let $X \subsetneqq A$ be a proper closed algebraic subvariety. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic maps in* $\mathrm{Hol}(\mathbb{D}, X)$. *Then there exist a proper abelian subvariety $B \subsetneqq A$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ with the following two properties:*

(1) *$(\varpi \circ f_{n_k})_{k\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic map $g\colon \mathbb{D} \to A/B$, where $\varpi\colon A \to A/B$ is the quotient map.*
(2) *For every $0 < s < 1$ and open neighborhood $U \subset A$ of $\mathrm{Sp}_B X$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, we have $f_{n_k}(z) \in U$ for all $z \in \mathbb{D}(s)$.*

Proof. We apply Theorem 1.2 to get $B \subsetneqq A$ and $(f_{n_k})_{k\in\mathbb{N}}$. Assertion (1) follows from that of Theorem 1.2. To prove assertion (2), we set $Z = \varpi(\mathrm{Sp}_B X)$. Then since each fiber of $\varpi\colon A \to A/B$ consists of one B-orbit, we have

$$\varpi^{-1}(Z) = \mathrm{Sp}_B X.$$

Let $U \subset A$ be an open neighborhood of $\mathrm{Sp}_B X$. Then since $A\backslash U$ is compact, $\varpi(A\backslash U) \subset A/B$ is a closed set. We have $Z \cap \varpi(A\backslash U) = \emptyset$. Note that $Z \subset A/B$ is a Zariski closed set. Hence there exists an open neighborhood $W \subset A/B$ of Z such that $W \cap \varpi(A\backslash U) = \emptyset$. Then $\varpi^{-1}(W) \subset U$.

Now let $s \in (0, 1)$. By $(f_{n_k})_{k\in\mathbb{N}} \to \mathrm{Sp}_B X$, we have $(\varpi \circ f_{n_k})_{k\in\mathbb{N}} \to Z$. Hence $g(\mathbb{D}) \subset Z$ (cf. Remark 3.7). Hence there exists $k_0 \in \mathbb{N}$ such that $\varpi \circ f_{n_k}(\mathbb{D}(s)) \subset W$ for all $k \geq k_0$. Hence we have $f_{n_k}(\mathbb{D}(s)) \subset U$ for all $k \geq k_0$. □

Remark 14.2. When A is compact, Theorem 1.3 is easily derived from Corollary 14.1 as follows. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of holomorphic maps in $\mathrm{Hol}(\mathbb{D},X)$. We assume that condition (2) of the definition of "tautly embedded modulo" is not satisfied. Then there exist compact sets $K\subset\mathbb{D}$ and $L\subset X-S$ such that, by replacing $(f_n)_{n\in\mathbb{N}}$ by its subsequence, we have $f_n(K)\cap L\neq\emptyset$ for all $n\in\mathbb{N}$. We apply Corollary 14.1 to get $B\subset A$ and $(f_{n_k})_{k\in\mathbb{N}}$. To prove $\dim B=0$, we assume on the contrary that $\dim B>0$. Then $\mathrm{Sp}_B X\subset S$. Hence assertion (2) of Corollary 14.1 contradicts the condition $f_n(K)\cap L\neq\emptyset$ for all $n\in\mathbb{N}$. Hence $\dim B=0$. Then by assertion (1) of Corollary 14.1, the sequence $(f_{n_k})_{k\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{D}$. Thus Theorem 1.3 for the compact case is re-proved.

Proof of Theorem 1.5. The case $X=A$ is trivial, so we assume $X\subsetneqq A$. Let $v\in\breve{T}_xX$ satisfy $F_X(v)=0$, where $x\in X$. We consider $v\in TA$ by the natural inclusion $\breve{T}X\subset TA$. There exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Hol}(\mathbb{D},X)$ such that $f_n'(0)=nv$. Then $f_n(0)=x$. We apply Corollary 14.1 to get $B\subsetneqq A$ and $(f_{n_k})_{k\in\mathbb{N}}$. We first show $\varpi_*(v)=0$, where $\varpi\colon A\to A/B$ is the quotient map. Note that $\varpi\circ f_{n_k}\colon\mathbb{D}\to A/B$ converges uniformly on compact subsets of $\mathbb{D}$ to $g\colon\mathbb{D}\to A/B$. Hence $(\varpi\circ f_{n_k})'(0)=n_k\varpi_*(v)$ converges to $g'(0)$. Thus we have $\varpi_*(v)=0$.

Now by assertion (2) of Corollary 14.1, we have

$$x\in\mathrm{Sp}_B X$$

(thanks to the absence of the term "γ-almost" in Corollary 14.1). Hence $x+B\subset X$. Since $\varpi_*(v)=0$, we have $v\in T(x+B)$. Hence there exists $f\colon\mathbb{C}\to(x+B)\subset X$ such that $f'(0)=v$. The proof is completed. □

The following example shows that the condition $F_X(v)=0$ does not necessarily imply the existence of $f\colon\mathbb{C}\to X$ with $f'(0)=v$.

Example 14.3. Let X be a smooth surface which is Kobayashi hyperbolic. Let $\widetilde{X}\to X$ be a blow-up along one point $p\in X$. Let $E\subset\widetilde{X}$ be the exceptional divisor. Let $a_1,a_2,a_3\in E$ be three distinct points. Then $\widetilde{X}-Z$ is Brody hyperbolic, where $Z=\{a_1,a_2,a_3\}$. Namely there is no non-constant holomorphic map $f\colon\mathbb{C}\to\widetilde{X}-Z$. On the other hand, for $q\in E-Z\subset\widetilde{X}-Z$ and $v\in T_qE$, we have $F_{\widetilde{X}-Z}(v)=0$. To show this, we take an open neighborhood W of E which is biholomorphic to

$$\{(x,y,[s,t])\in\mathbb{C}^2\times\mathbb{P}^1;\ |x|<1,\ |y|<1,\ xt=ys\}.$$

We may assume that $q\in E$ corresponds to $(0,0,[0,1])$. For $n\in\mathbb{N}$, we define $f_n\in\mathrm{Hol}(\mathbb{D},W)$ by $f_n(z)=(z^3,\frac{1}{n}z^2,[nz,1])$. Set $v=f_1'(0)$. Then $v\in T_qE$ and $v\neq 0$. We have $f_n'(0)=nv$. We have $f_n(\mathbb{D}-\{0\})\subset W-E$ and $f_n(0)=q\in\widetilde{X}-Z$. Hence $f_n(\mathbb{D})\subset\widetilde{X}-Z$. Thus $F_{\widetilde{X}-Z}(v)=0$.

Remark 14.4. We do not know whether a similar statement for Theorem 1.5 holds for non-compact semi-abelian varieties.

§15. Proof of Theorem 1.7

We have an equivariant compactification $(\mathbb{G}_m)^{p-1} \subset \mathbb{P}^{p-1}$ such that the inclusion is defined by $(g_1, \dots, g_{p-1}) \mapsto [g_1 : \cdots : g_{p-1} : 1]$. The action $(\mathbb{G}_m)^{p-1} \times \mathbb{P}^{p-1} \to \mathbb{P}^{p-1}$ is defined by

$$(g_1, \dots, g_{p-1}) \cdot [x_1 : \cdots : x_{p-1} : x_p] = [g_1 x_1 : \cdots : g_{p-1} x_{p-1} : x_p]. \tag{15.1}$$

Here, $x_1, \dots, x_p$ are the homogeneous coordinates of $\mathbb{P}^{p-1}$. Let $V \subset (\mathbb{G}_m)^{p-1}$ be a closed subvariety such that $\overline{V} \subset \mathbb{P}^{p-1}$ is defined by $x_1 + x_2 + \cdots + x_p = 0$.

Let $\mathcal{I} = \{I_1, \dots, I_l\}$ be a disjoint partition $I_1 \sqcup \cdots \sqcup I_l = \{1, \dots, p\}$. We define a subtorus $G_{\mathcal{I}} \subset (\mathbb{G}_m)^{p-1}$ as follows. For $i \in \{1, \dots, p\}$, we define $\tau(i) \in \{1, \dots, l\}$ by $i \in I_{\tau(i)}$. We define a linear subspace $L(\mathcal{I}) \subset \mathbb{C}^p$ by the equations $x_i = x_j$ for all $i, j \in \{1, \dots, p\}$ such that $\tau(i) = \tau(j)$. We consider the immersion $(\mathbb{G}_m)^{p-1} \subset \mathbb{C}^p$ given by $(g_1, \dots, g_{p-1}) \mapsto (g_1, \dots, g_{p-1}, 1)$. We set

$$G_{\mathcal{I}} = (\mathbb{G}_m)^{p-1} \cap L(\mathcal{I}).$$

Then $G_{\mathcal{I}} \subset (\mathbb{G}_m)^{p-1}$ is a subtorus.

Lemma 15.1. *We have*

$$\mathrm{Sp}_{G_{\mathcal{I}}} \overline{V} = \left\{ [x_1 : \cdots : x_p] \in \mathbb{P}^{p-1};\ \textstyle\sum_{i \in I_k} x_i = 0 \text{ for all } 1 \le k \le l \right\}.$$

Proof. Note that $\mathrm{Sp}_{G_{\mathcal{I}}} \overline{V}$ is defined by the simultaneous equations

$$g_1 x_1 + \cdots + g_{p-1} x_{p-1} + x_p = 0 \quad \text{for all } (g_1, \dots, g_{p-1}) \in G_{\mathcal{I}}. \tag{15.2}$$

By changing the indices of $I_1, \dots, I_l$, we may assume $p \in I_l$. We have an isomorphism $(\mathbb{G}_m)^{l-1} \to G_{\mathcal{I}}$ by $(a_1, \dots, a_{l-1}) \mapsto (a_{\tau(1)}, \dots, a_{\tau(p-1)})$, where we set $a_l = 1$. For each $k \in \{1, \dots, l\}$, we choose $\iota(k) \in \{1, \dots, p\}$ such that $\iota(k) \in I_k$. Then ι is a section of $\tau \colon \{1, 2, \dots, p\} \to \{1, \dots, l\}$. For $(g_1, \dots, g_{p-1}) \in G_{\mathcal{I}}$, we have

$$g_1 x_1 + \cdots + g_{p-1} x_{p-1} + x_p = \sum_{k=1}^{l-1} \left(g_{\iota(k)} \sum_{i \in I_k} x_i \right) + \sum_{i \in I_l} x_i.$$

Hence (15.2) is equivalent to $\sum_{i \in I_k} x_i = 0$ for all $k \in \{1, \dots, l\}$. The proof is completed. □

Lemma 15.2. *Let $G \subset (\mathbb{G}_m)^{p-1}$ be a subtorus. Then there exists a disjoint partition $\mathcal{I} = \{I_1, \dots, I_l\}$ of $\{1, 2, \dots, p\}$ such that*

(1) $G \subset G_{\mathcal{I}}$, *and*

(2) $\mathrm{Sp}_{G_{\mathcal{I}}}\overline{V} = \mathrm{Sp}_G\overline{V}$.

Proof. By the inclusion $(\mathbb{G}_m)^{p-1} \subset \mathbb{C}^p$, we have $G \subset \mathbb{C}^p$. Let $\chi_1, \ldots, \chi_p \in \mathrm{Hom}(G, \mathbb{G}_m)$ be the composite of $G \subset \mathbb{C}^p$ and the i-th projections $\mathbb{C}^p \to \mathbb{C}$, where $\chi_p \equiv 1$. Then $\chi_1, \ldots, \chi_p$ are group homomorphisms. We define an equivalence relation $\sim$ on $\{1, \ldots, p\}$ such that $i \sim j$ if and only if $\chi_i = \chi_j$. Thus we get a disjoint partition $\mathcal{I} = \{I_1, \ldots, I_l\}$ of $\{1, \ldots, p\}$ such that each I_k is an equivalence class of the equivalence relation.

Let $L \subset \mathbb{C}^p$ be the linear subspace spanned by $G \subset \mathbb{C}^p$. We claim

$$L = L(\mathcal{I}). \tag{15.3}$$

We prove this. By the definition of $\mathcal{I}$, we have $\chi_i = \chi_j$ if and only if $\tau(i) = \tau(j)$. Hence we have $G \subset L(\mathcal{I})$. Hence $L \subset L(\mathcal{I})$. Let $\iota\colon \{1, \ldots, l\} \to \{1, \ldots, p\}$ be a section of $\tau\colon \{1, \ldots, p\} \to \{1, \ldots, l\}$. Note that $x_{\iota(1)}|_{L(\mathcal{I})}, \ldots, x_{\iota(l)}|_{L(\mathcal{I})}$ form a basis of the dual space of $L(\mathcal{I})$, where $x_1, \ldots, x_p$ are the coordinate functions of $\mathbb{C}^p$. On the other hand, $\{\chi_{\iota(k)}\}_{k \in \{1,\ldots,l\}} \subset \mathrm{Hom}(G, \mathbb{G}_m)$ is linearly independent (cf. [6, Lem. 8.1]). Hence $x_{\iota(1)}|_L, \ldots, x_{\iota(l)}|_L$ are linearly independent on the dual space of L. Hence $\dim L \geq l = \dim L(\mathcal{I})$. Thus we get (15.3).

Now we have $G \subset G_{\mathcal{I}}$, which follows from $G \subset L(\mathcal{I})$. Hence we have $\mathrm{Sp}_{G_{\mathcal{I}}}\overline{V} \subset \mathrm{Sp}_G\overline{V}$. It remains to prove $\mathrm{Sp}_G\overline{V} \subset \mathrm{Sp}_{G_{\mathcal{I}}}\overline{V}$. Note that $\mathrm{Sp}_G\overline{V}$ is defined by simultaneous equations

$$b_1x_1 + \cdots + b_{p-1}x_{p-1} + x_p = 0 \quad \text{for all } (b_1, \ldots, b_{p-1}) \in G \subset (\mathbb{G}_m)^{p-1}. \tag{15.4}$$

For $g = (g_1, \ldots, g_{p-1}, 1) \in G_{\mathcal{I}} \subset \mathbb{C}^p$, we have $g \in L(\mathcal{I})$. Thus by (15.3), we have $g \in L$. Hence there exist $(b_{1,1}, \ldots, b_{1,p-1}, 1), \ldots, (b_{s,1}, \ldots, b_{s,p-1}, 1) \in G \subset \mathbb{C}^p$ and $\alpha_1, \ldots, \alpha_s \in \mathbb{C}$ such that

$$(g_1, \ldots, g_{p-1}, 1) = \alpha_1(b_{1,1}, \ldots, b_{1,p-1}, 1) + \cdots + \alpha_s(b_{s,1}, \ldots, b_{s,p-1}, 1).$$

Hence the solutions of the simultaneous equations (15.4) satisfy the equation

$$g_1x_1 + \cdots + g_{p-1}x_{p-1} + x_p = 0.$$

Thus $\mathrm{Sp}_G\overline{V} \subset \mathrm{Sp}_{G_{\mathcal{I}}}\overline{V}$. □

Lemma 15.3. *Suppose $G = (\mathbb{G}_m)^{p-1}$. Then we have $\mathrm{Sp}_G\overline{V} = \emptyset$.*

Proof. Note that $\mathrm{Sp}_G\overline{V}$ is defined by simultaneous equations

$$g_1x_1 + \cdots + g_{p-1}x_{p-1} + x_p = 0 \quad \text{for all } (g_1, \ldots, g_{p-1}) \in (\mathbb{G}_m)^{p-1},$$

which has only a trivial solution. Hence $\mathrm{Sp}_G\overline{V} = \emptyset$. □

Proof of Theorem 1.7. We consider $\mathbb{P}^{p-1}$ as an equivariant compactification of $(\mathbb{G}_m)^{p-1}$, where the action is defined by (15.1). Let $V \subset (\mathbb{G}_m)^{p-1}$ be a closed subvariety such that $\overline{V} \subset \mathbb{P}^{p-1}$ is defined by

$$x_1 + x_2 + \cdots + x_p = 0,$$

where $x_1, \ldots, x_p$ are homogeneous coordinates of $\mathbb{P}^{p-1}$. For each $f = (f_1, \ldots, f_p) \in \mathcal{F}$, we consider a holomorphic map $\hat{f}\colon \mathbb{D} \to \mathbb{P}^{p-1}$ defined by $\hat{f}(z) = [f_1(z) : \cdots : f_p(z)]$. Then we have $\hat{f}(\mathbb{D}) \subset V$, hence $\hat{f} \in \mathrm{Hol}(\mathbb{D}, V)$. Hence $(\hat{f})_{f\in\mathcal{F}}$ is an infinite indexed family in $\mathrm{Hol}(\mathbb{D}, V)$. We replace $\mathcal{F}$ by its countably infinite subset. Then we may consider $(\hat{f})_{f\in\mathcal{F}}$ as a sequence in $\mathrm{Hol}(\mathbb{D}, V)$. We continue to write this sequence $\mathcal{F}$.

We apply Theorem 1.2 to get $B \subset (\mathbb{G}_m)^{p-1}$ and an infinite subset $\mathcal{G} \subset \mathcal{F}$. Then by $(\hat{f})_{f\in\mathcal{G}} \to \mathrm{Sp}_B\overline{V}$, we have $\mathrm{Sp}_B\overline{V} \neq \emptyset$. We apply Lemma 15.2 to get a disjoint partition $\mathcal{I} = \{I_1, \ldots, I_l\}$ of $\{1, \ldots, p\}$ such that $B \subset G_{\mathcal{I}}$ and $\mathrm{Sp}_{G_{\mathcal{I}}}\overline{V} = \mathrm{Sp}_B\overline{V}$. Then we have $\mathrm{Sp}_{G_{\mathcal{I}}}\overline{V} \neq \emptyset$. Hence by Lemma 15.3, we have $G_{\mathcal{I}} \neq (\mathbb{G}_m)^{p-1}$. Since the limit function satisfies $\mathbb{D} \to (\mathbb{G}_m)^{p-1}/B$, the sequence $(\hat{f})_{f\in\mathcal{G}}$ converges under the quotient $(\mathbb{G}_m)^{p-1}/G_{\mathcal{I}}$. Hence by replacing B by $G_{\mathcal{I}}$, we may assume that $B = G_{\mathcal{I}}$.

We take I_k, where $1 \leq k \leq l$. By Lemma 15.1, we have $\mathrm{Sp}_B\overline{V} \subset \{\sum_{j\in I_k} x_j = 0\}$ as Zariski closed subsets of $\mathbb{P}^{p-1}$. Hence we have

$$(\hat{f})_{f\in\mathcal{G}} \to \Big\{ \sum_{j\in I_k} x_j = 0 \Big\}. \tag{15.5}$$

Let $\psi_k\colon \mathbb{P}^{p-1} \dashrightarrow \mathbb{P}^{|I_k|-1}$ be defined by $[x_1 : \cdots : x_p] \mapsto [x_j]_{j\in I_k}$. Here we note that $\mathbb{P}^0 = \mathrm{pt}$, when $|I_k| = 1$. Then ψ_k is regular on $(\mathbb{G}_m)^{p-1} \subset \mathbb{P}^{p-1}$. We have $(\mathbb{G}_m)^{|I_k|-1} \subset \mathbb{P}^{|I_k|-1}$. Note that ψ_k induces a group homomorphism

$$\psi_k|_{(\mathbb{G}_m)^{p-1}}\colon (\mathbb{G}_m)^{p-1} \to (\mathbb{G}_m)^{|I_k|-1}.$$

This $\psi_k|_{(\mathbb{G}_m)^{p-1}}$ is invariant under the action of B on $(\mathbb{G}_m)^{p-1}$, hence factors the quotient map $(\mathbb{G}_m)^{p-1} \to (\mathbb{G}_m)^{p-1}/B$. Hence the sequence $(\psi_k \circ \hat{f})_{f\in\mathcal{G}}$ converges uniformly on compact subsets of $\mathbb{D}$ to $g_k\colon \mathbb{D} \to (\mathbb{G}_m)^{|I_k|-1}$. Set $H_k = \{\sum_{j\in I_k} x_j = 0\} \subset \mathbb{P}^{|I_k|-1}$. We note that $H_k = \emptyset$ if $|I_k| = 1$.

Claim. Suppose $g_k(\mathbb{D}) \not\subset H_k$. Let $\varepsilon \in (0,1)$, $s \in (0,1)$, and $\gamma > 0$. Then there exists a finite subset $\mathcal{E} \subset \mathcal{G}$ such that for all $f \in \mathcal{G}\backslash\mathcal{E}$, we have

$$\frac{\sqrt{\sum_{j\in I_k} |f_j(z)|^2}}{\sqrt{\sum_{1\leq i\leq p} |f_i(z)|^2}} < \frac{\varepsilon}{2p} \tag{15.6}$$

for $\frac{\gamma}{2p}$-almost all $z \in \mathbb{D}(s)$.

We prove this. We define $E_k \subset \mathbb{P}^{p-1}$ by $E_k = \bigcap_{i \in I_k} \{x_i = 0\}$. Then ψ_k induces a regular map $\varphi_k \colon \mathrm{Bl}_{E_k}\mathbb{P}^{p-1} \to \mathbb{P}^{|I_k|-1}$. Let $E'_k \subset \mathrm{Bl}_{E_k}\mathbb{P}^{p-1}$ be the exceptional divisor. Let $\mu \colon \mathbb{P}^{p-1} \to \mathbb{R}_{\geq 0}$ be defined by

$$\mu([x_1 : \cdots : x_p]) = \frac{\sqrt{\sum_{j \in I_k} |x_j|^2}}{\sqrt{\sum_{1 \leq i \leq p} |x_i|^2}}.$$

Set $U = \{\mu < \varepsilon/2p\} \subset \mathbb{P}^{p-1}$. Then U is an open neighborhood of E_k. Let $U' \subset \mathrm{Bl}_{E_k}\mathbb{P}^{p-1}$ be the inverse image of U under $\mathrm{Bl}_{E_k}\mathbb{P}^{p-1} \to \mathbb{P}^{p-1}$. Then U' is an open neighborhood of E'_k. By $g_k(\mathbb{D}) \not\subset H_k$, we may take an open neighborhood $W_0 \subset \mathbb{P}^{|I_k|-1}$ of H_k such that

$$C^1_H(\mathbb{D}(s) \cap g_k^{-1}(W_0)) < \gamma/4p.$$

Here we recall the notation C^1_H from (8.1). We take an open neighborhood $W \Subset W_0$ of H_k. Then since $(\varphi_k \circ \hat{f})_{f \in \mathcal{G}}$ converges uniformly on compact subsets of $\mathbb{D}$ to g_k, we have

$$\mathbb{D}(s) \cap \hat{f}^{-1}(\varphi_k^{-1}(W)) \subset \mathbb{D}(s) \cap g_k^{-1}(W_0)$$

for all but finitely many $f \in \mathcal{G}$. Hence we have

$$C^1_H\big(\mathbb{D}(s) \cap \hat{f}^{-1}(\varphi_k^{-1}(W))\big) < \gamma/4p$$

for all but finitely many $f \in \mathcal{G}$. By (15.5), we have $(\hat{f})_{f \in \mathcal{G}} \to E'_k \cup \varphi_k^* H_k$. Hence, we have

$$C^1_H\big(\mathbb{D}(s) \backslash \hat{f}^{-1}(U' \cup \varphi_k^{-1}(W))\big) < \gamma/4p$$

for all but finitely many $f \in \mathcal{G}$. Hence we have

$$C^1_H(\mathbb{D}(s) \backslash \hat{f}^{-1}(U)) < \gamma/2p$$

for all but finitely many $f \in \mathcal{G}$. This proves our claim.

We define $\Lambda \subset \{1, \dots, l\}$ to be the set of $k \in \{1, \dots, l\}$ such that $g_k(\mathbb{D}) \not\subset H_k$. If $|I_k| = 1$, then $k \in \Lambda$. We note that (15.6) shows that $\Lambda \neq \{1, \dots, l\}$. By changing the indexes, we may assume that $\{1, \dots, n\} = \{1, \dots, l\} \backslash \Lambda$. Then we have $n \geq 1$.

We show that $I_1, \dots, I_n$ satisfy assertions (2) and (3) of Theorem 1.7. We take $k \in \{1, \dots, n\}$. Then $|I_k| \geq 2$. For $i, j \in I_k$, we define $\tau_{ij} \colon (\mathbb{G}_m)^{p-1} \to \mathbb{G}_m$ by $(b_1, \dots, b_{p-1}) \mapsto b_i/b_j$, where we set $b_p = 1$. Then τ_{ij} factors $\psi_k|_{(\mathbb{G}_m)^{p-1}} \colon (\mathbb{G}_m)^{p-1} \to (\mathbb{G}_m)^{|I_k|-1}$. Hence $(\tau_{ij} \circ \hat{f})_{f \in \mathcal{G}}$ converges uniformly on compact subsets of $\mathbb{D}$ to the composite of $g_k \colon \mathbb{D} \to (\mathbb{G}_m)^{|I_k|-1}$ and $\kappa_{ij} \colon (\mathbb{G}_m)^{|I_k|-1} \to \mathbb{G}_m$. Note that $\tau_{ij} \circ \hat{f} = f_i/f_j$. Hence $(f_i/f_j)_{f \in \mathcal{G}}$ converges uniformly on compact subsets of $\mathbb{D}$ to $\kappa_{ij} \circ g_k \colon \mathbb{D} \to \mathbb{G}_m$, which is a nowhere-vanishing holomorphic function on $\mathbb{D}$. By

$g_k(\mathbb{D}) \subset H_k$, we have

$$\sum_{i \in I_k} \kappa_{ij} \circ g_k = 0.$$

Hence $(\sum_{i\in I_k} f_i/f_j)_{f\in\mathcal{G}}$ converges uniformly on compact subsets of $\mathbb{D}$ to 0. Thus assertion (2) of Theorem 1.7 is true.

We prove Theorem 1.7 (3). Set $I = I_1 \sqcup \cdots \sqcup I_n$. We have

$$\frac{\sqrt{\sum_{i\in I}|f_i|^2}}{\sqrt{\sum_{1\leq i\leq p}|f_i|^2}} \geq \frac{\sqrt{\sum_{1\leq i\leq p}|f_i|^2} - \sum_{j\notin I}|f_j|}{\sqrt{\sum_{1\leq i\leq p}|f_i|^2}} = 1 - \sum_{j\notin I}\frac{|f_j|}{\sqrt{\sum_{1\leq i\leq p}|f_i|^2}}.$$

Hence by (15.6), we have

$$\frac{\sqrt{\sum_{1\leq i\leq p}|f_i(z)|^2}}{\sqrt{\sum_{i\in I}|f_i(z)|^2}} < 1+\varepsilon < 2$$

for $\frac{\gamma}{2}$-almost all $z \in \mathbb{D}(s)$. Combining this with (15.6), we get assertion (3) of Theorem 1.7. The proof of Theorem 1.7 is completed. $\square$

Appendix A. Semi-abelian varieties

In this appendix, we describe semi-abelian varieties. We only treat the definitions and properties which are needed in this paper. There are several good references on semi-abelian varieties including [8, Sec. 5.4], [21], [35, Chap. 5], [40, Chap. VI]. All algebraic groups are defined over $\mathbb{C}$.

A semi-abelian variety A is an algebraic group with a (unique) expression

$$0 \to T \to A \to A_0 \to 0,$$

where A_0 is an abelian variety and $T \simeq \mathbb{G}_m^l$ is an algebraic torus. Then A is smooth, connected, and commutative (cf. [8, Rem. 5.4.2 (ii)]). By [8, Thm. 2.7.2], the map $A \to A_0$ is a T-torsor, i.e., we have the following Cartesian diagram:

$$\begin{tikzcd} A \arrow[d] & T \times A \arrow[l, "\varphi"'] \arrow[d, "p"] \\ A_0 & A. \arrow[l] \end{tikzcd}$$

Here, $\varphi(a,t) = a+t$ and p is the second projection.

Remark A.1. Let A be a semi-abelian variety and let $B \subset A$ be a connected algebraic subgroup. By [8, Cor. 5.4.6], B is a semi-abelian variety. We have a quotient $q\colon A \to A/B$, which is a B-torsor (cf. [8, Thm. 2.7.2]). In particular, q is faithfully flat and quasi-compact. By [8, Cor. 5.4.6], A/B is a semi-abelian variety.

By an equivariant compactification $\bar{A}$ of A, we mean that (1) $\bar{A}$ is compact, (2) an open immersion $A \subset \bar{A}$ exists, and (3) the group morphism $A \times A \to A$ extends to $A \times \bar{A} \to \bar{A}$.

Appendix A.1. Construction of an equivariant compactification

The main purpose of this subsection is to introduce an equivariant compactification $\bar{A}$ of A constructed from an equivariant compactification $\bar{T}$ of T. This compactification is described in [29, p. 1414], [42, Lem. 2.2].

Let V be an algebraic variety which admits a T-action. Then we have a T-action on $V \times A$ defined by $(x, a) \mapsto (x - t, a + t)$ for $t \in T$.

Lemma A.2. *The categorical quotient $V \times A \to (V \times A)/T$ exists. Namely, every T-invariant morphism $V \times A \to W$ factors uniquely through $V \times A \to (V \times A)/T$.*

Proof. Let $\{U_i\}$ be a Zariski open covering of A_0 with T-equivariant trivialization $\phi_i\colon A|_{U_i} \to T \times U_i$ (cf. [40, p. 169], [41, Lem. 2.2], [33, Prop. 16.55]). Then for each i, j, we get an isomorphism

$$\phi_j \circ \phi_i^{-1}|_{T\times(U_i\cap U_j)}\colon T \times (U_i \cap U_j) \to T \times (U_i \cap U_j).$$

Let $s_{ij}\colon U_i \cap U_j \to T$ be defined by $\phi_j \circ \phi_i^{-1}|_{T\times(U_i\cap U_j)}(0_T, u) = (s_{ij}(u), u)$. Note that A is reconstructed from a gluing of trivial T-torsors $T \times U_i$ by the Čech cocycle $\{s_{ij}\} \in \check{\mathrm{H}}^1(\{U_i\}, T)$. Now we glue $V \times U_i$ by the same Čech cocycle $\{s_{ij}\}$ to get $(V \times A)/T$. The T-invariant morphism $V \times A \to (V \times A)/T$ is described as follows. For each i, we define a map

$$\mu_i\colon V \times T \times U_i \to V \times U_i$$

by $\mu_i(x, t, u) = (x + t, u)$. Then μ_i is T-invariant under the T-action on $V \times T \times U_i$ defined by $(x, t, u) \mapsto (x - \tau, t + \tau, u)$, where $\tau \in T$. We note that μ_i is a T-torsor with respect to this T-action. The space $V \times A$ is described by a gluing of spaces $V \times T \times U_i$ by the isomorphisms $V \times T \times (U_i \cap U_j) \to V \times T \times (U_i \cap U_j)$ defined by $(x, t, u) \mapsto (x, t + s_{ij}(u), u)$. Then we may glue μ_i to get a T-torsor

$$\mu\colon V \times A \to (V \times A)/T.$$

In particular, μ is a categorical quotient (cf. [8, Prop. 2.6.4]). We remark that the space $(V \times A)/T$ does not depend on the choices of $\{U_i\}$ and ϕ_i. □

Remark A.3. Suppose a T-equivariant map $f\colon V' \to V$ exists. Then by the above construction, f induces $f'\colon (V' \times A)/T \to (V \times A)/T$. If f is an open (resp. closed) immersion, then f' is an open (resp. closed) immersion. If V is smooth, then $(V \times A)/T$ is smooth. Indeed, $(V \times A)/T$ is constructed by a gluing of the spaces $V \times U_i$, which are smooth.

Lemma A.4. *Let $\bar{T}$ be an equivariant compactification of T. Then $(\bar{T} \times A)/T$ is an equivariant compactification of A.*

Proof. We have $A = (T \times A)/T$. Hence the open immersion $T \subset \bar{T}$ induces an open immersion $A \subset (\bar{T} \times A)/T$. The A-action on $(\bar{T} \times A)/T$ is described as follows. Let

$$h\colon A \times (\bar{T} \times A) \to A \times ((\bar{T} \times A)/T)$$

be defined by $h(a,t,a') = (a, \mu(t,a'))$, where $\mu\colon \bar{T} \times A \to (\bar{T} \times A)/T$ is the quotient map. Then h is a T-torsor, hence a categorical quotient under the T-action on $A \times (\bar{T} \times A)$ defined by

$$(a,t,a') \mapsto (a, t - \tau, a' + \tau), \tag{A.1}$$

where $\tau \in T$. Let

$$\varphi\colon A \times (\bar{T} \times A) \to (\bar{T} \times A)/T \tag{A.2}$$

be defined by $\varphi(a,t,a') = \mu(t, a + a')$. Then φ is invariant under the T-action (A.1). Hence φ is the composite of h and a unique map

$$\psi\colon A \times ((\bar{T} \times A)/T) \to (\bar{T} \times A)/T. \tag{A.3}$$

This is our A-action. Note that the map $A \to A_0$ extends to $(\bar{T} \times A)/T \to A_0$, which is proper. Hence $(\bar{T} \times A)/T$ is compact. □

Lemma A.5. *Let $\bar{T}$ be an equivariant compactification of T. Let $Z \subset \bar{T}$ be an irreducible locally closed set which is T-invariant. Then $(Z \times A)/T \subset (\bar{T} \times A)/T$ is an A-invariant, irreducible locally closed set. Moreover, every A-invariant, irreducible locally closed set $V \subset (\bar{T} \times A)/T$ is obtained in this way.*

Proof. By the construction, $(Z \times A)/T \subset (\bar{T} \times A)/T$ is a locally closed set (cf. Remark A.3). We have $\varphi(A \times Z \times A) = (Z \times A)/T$, where φ is given as in (A.2). Hence $\psi(A \times ((Z \times A)/T)) = (Z \times A)/T$, where ψ is given as in (A.3). Hence $(Z \times A)/T \subset (\bar{T} \times A)/T$ is A-invariant.

Let $V \subset (\bar{T} \times A)/T$ be an A-invariant, irreducible locally closed set. Set $Z = V \cap \bar{T}$, where $\bar{T}$ is identified with the fibers of $(\bar{T} \times A)/T \to A_0$. Then we have $V = (Z \times A)/T$. □

Lemma A.6. *Let $\bar{A}$ be a smooth equivariant compactification of A. Then there exists a smooth equivariant compactification $T \subset \bar{T}$ so that $\bar{A} = (\bar{T} \times A)/T$.*

Proof. There exists a rational map $\bar{A} \dashrightarrow A_0$, but every rational map from smooth variety to an abelian variety is regular. So we have a regular map $p\colon \bar{A} \to A_0$. Note that this map is A-equivariant. Indeed, denoting the actions by $m\colon A \times \bar{A} \to \bar{A}$ and $m'\colon A \times A_0 \to A_0$, we have two maps $p \circ m\colon A \times \bar{A} \to A_0$ and $m' \circ (\mathrm{id}_A, p)\colon A \times \bar{A} \to A_0$. These coincide on $A \times A \subset A \times \bar{A}$. Hence $p \circ m = m' \circ (\mathrm{id}_A, p)$, which shows that $p\colon \bar{A} \to A_0$ is A-equivariant. Since $\bar{A}$ is smooth, there exists a non-empty Zariski open set $U \subset A_0$ such that $p^{-1}(U) \to U$ is smooth ([18, Chapter III, Cor. 10.7]). By translation, $p\colon \bar{A} \to A_0$ is smooth.

Now set $\overline{T} = p^{-1}(0_{A_0})$. Then $\overline{T}$ is a smooth equivariant compactification of T. By restricting $m\colon A \times \bar{A} \to \bar{A}$ to $A \times \overline{T} \subset A \times \bar{A}$, we get $A \times \overline{T} \to \bar{A}$. This map is T-invariant. Hence we get $\psi\colon (\overline{T} \times A)/T \to \bar{A}$. Note that the restriction $\psi|_A\colon A \to \bar{A}$ is the open immersion. Hence ψ is a map over A_0. Note that ψ is injective on the fiber $\overline{T} \subset (\overline{T} \times A)/T$ over $0_{A_0} \in A_0$. Hence ψ is injective. Since $\bar{A}$ is smooth, Zariski's main theorem yields that ψ is an open immersion. Since $(\overline{T} \times A)/T$ is complete, ψ is an isomorphism. Thus $\bar{A}$ is obtained by $(\overline{T} \times A)/T$. $\square$

Appendix A.2. Fans and torus embeddings

We describe torus embeddings associated to fans. We basically follow the notation described in [37]. See also [11, 16, 24].

Let $N = \mathbb{Z}^r$. Set $N_{\mathbb{R}} = N \otimes \mathbb{R}$, $M = \mathrm{Hom}(N, \mathbb{Z})$, and $M_{\mathbb{R}} = M \otimes \mathbb{R}$. A subset $C \subset N_{\mathbb{R}}$ is a *convex rational polyhedral cone* if there exist $n_1, \dots, n_s \in N$ such that $C = \mathbb{R}_{\geq 0} n_1 + \cdots + \mathbb{R}_{\geq 0} n_s$. We set

$$C^{\vee} = \{u \in M_{\mathbb{R}};\ u(x) \geq 0 \text{ for all } x \in C\}.$$

Then $C^{\vee} \subset M_{\mathbb{R}}$ is a convex rational polyhedral cone (cf. [37, p. 2]). A subset $F \subset C$ is called a *face* and is denoted $F \prec C$ if there exists $u \in C^{\vee}$ such that $F = C \cap \{u = 0\}$. Then F is a convex rational polyhedral cone (cf. [37, p. 2]). We call C *strongly convex* if $C \cap (-C) = \{0\}$. A *fan* in N is a non-empty collection Δ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying the following conditions:

(1) Every face of any $\sigma \in \Delta$ is contained in Δ.

(2) For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

We call Δ *complete* if Δ is finite and $\bigcup_{\sigma \in \Delta} \sigma = N_{\mathbb{R}}$. We call Δ *non-singular* if for all $\sigma \in \Delta$, there exists a $\mathbb{Z}$-basis $n_1, \dots, n_r$ of N such that $\sigma = \mathbb{R}_{\geq 0} n_1 + \cdots + \mathbb{R}_{\geq 0} n_s$, where $s \leq r$.

Let $N' = \mathbb{Z}^{r'}$ and let Δ' be a fan in $N'_{\mathbb{R}}$. A *map of fans* $\varphi\colon (N', \Delta') \to (N, \Delta)$ is a $\mathbb{Z}$-linear map $\varphi\colon N' \to N$ such that for each $\sigma' \in \Delta'$, there exists $\sigma \in \Delta$ such that $\varphi(\sigma') \subset \sigma$. Here we continue to write the scalar extension as $\varphi\colon N'_{\mathbb{R}} \to N_{\mathbb{R}}$.

Let T be an algebraic torus. Let $M = \mathrm{Hom}(T, \mathbb{G}_m)$ and $N = \mathrm{Hom}(\mathbb{G}_m, T)$. A *torus embedding* $T \subset T\,\mathrm{emb}(\Delta)$ is defined from a fan (N, Δ) as follows (cf. [37, Thm. 1.4]): For each $\sigma \in \Delta$, we construct the affine scheme $\mathrm{Spec}(\mathbb{C}[M \cap \sigma^\vee])$. Then for $\sigma' \in \Delta$ with $\sigma' \prec \sigma$, we get the open immersion $U_{\sigma'} \subset U_\sigma$. By this, we may glue $\{U_\sigma\}_{\sigma \in \Delta}$ to get $T\,\mathrm{emb}(\Delta)$. Then $T\,\mathrm{emb}(\Delta)$ is normal and contains $T = U_{\{0\}}$ as a dense Zariski open set. Moreover, T acts on $T\,\mathrm{emb}(\Delta)$. A torus embedding $T\,\mathrm{emb}(\Delta)$ is complete if and only if Δ is complete, and smooth if and only if Δ is non-singular. Conversely, by Sumihiro's theorem, every normal, equivariant compactification $\bar{T}$ of T is a torus embedding associated to some complete fan ([37, Thm. 1.5]). Let $\psi\colon T' \to T$ be a homomorphism of algebraic tori. Then we have $\psi_*\colon N' \to N$, where $N' = \mathrm{Hom}(\mathbb{G}_m, T')$. Let Δ' be a fan in N'. Suppose $\psi_*\colon (N', \Delta') \to (N, \Delta)$ is a map of fans. Then ψ extends to an equivariant map $T'\,\mathrm{emb}(\Delta') \to T\,\mathrm{emb}(\Delta)$ (cf. [37, Thm. 1.13]).

Let Δ be a complete fan in N. Set $\bar{T} = T\,\mathrm{emb}\,\Delta$. For each $\sigma \in \Delta$, there exists a unique T-orbit $\mathrm{orb}(\sigma) \subset U_\sigma$ which is Zariski closed in U_σ. Then by [37, Prop. 1.6 (iv)], we have

$$\coprod_{\tau \prec \sigma} \mathrm{orb}(\tau) = U_\sigma. \tag{A.4}$$

Set $\sigma^\perp = \{u \in M_{\mathbb{R}};\ u(x) = 0 \text{ for all } x \in \sigma\}$. Then $\mathrm{orb}(\sigma)$ is identified with $\mathrm{Spec}\,\mathbb{C}[M \cap \sigma^\perp]$, where the closed immersion $\mathrm{orb}(\sigma) \subset U_\sigma$ is described by the surjection $\mathbb{C}[M \cap \sigma^\vee] \to \mathbb{C}[M \cap \sigma^\perp]$ defined by $\chi^m \mapsto 0$ for all $m \in (M \cap \sigma^\vee) \backslash (M \cap \sigma^\perp)$ (cf. [37, Prop. 1.6 (iv)]). Here, for the notation χ^m we refer the readers to [16, p. 15]. We note that $\mathrm{Spec}\,\mathbb{C}[M \cap \sigma^\perp]$ is a quotient torus defined by the inclusion $\mathbb{C}[M \cap \sigma^\perp] \subset \mathbb{C}[M]$. The inclusion $\mathbb{C}[M \cap \sigma^\perp] \subset \mathbb{C}[M \cap \sigma^\vee]$ yields the map

$$U_\sigma \to \mathrm{orb}(\sigma) \tag{A.5}$$

whose restriction to $T \subset U_\sigma$ is the quotient map $T \to \mathrm{orb}(\sigma)$. The closed immersion $\mathrm{orb}(\sigma) \hookrightarrow U_\sigma$ is a section of the map (A.5).

Let $I_\sigma \subset T$ be the kernel of the quotient map $T \to \mathrm{orb}(\sigma)$. Then I_σ is the isotropy group for the points of $\mathrm{orb}(\sigma) \subset U_\sigma$. Set $M_\sigma = M/(M \cap \sigma^\perp)$. We claim that $I_\sigma \subset T$ is a subtorus and

$$\mathrm{Hom}(I_\sigma, \mathbb{G}_m) = M_\sigma. \tag{A.6}$$

Indeed, since $M \cap \sigma^\perp \subset M$ is saturated, M_σ is a free $\mathbb{Z}$-module. Hence we get an exact sequence of free $\mathbb{Z}$-modules:

$$0 \to M \cap \sigma^\perp \to M \to M_\sigma \to 0.$$

This yields the exact sequence of tori (cf. [33, Thm. 12.9])

$$1 \to \operatorname{Spec}\mathbb{C}[M_\sigma] \to \operatorname{Spec}\mathbb{C}[M] \to \operatorname{Spec}\mathbb{C}[M \cap \sigma^\perp] \to 1.$$

Hence we have $I_\sigma = \operatorname{Spec}\mathbb{C}[M_\sigma]$. This shows that I_σ is a torus such that (A.6).

Appendix A.3. Smooth modifications

Lemma A.7. *Given an equivariant compactification $\bar{A}$, we may take a smooth equivariant compactification $\hat{A}$ and an equivariant morphism $\hat{A} \to \bar{A}$.*

Proof. Let $\bar{T}$ be the Zariski closure of $T \cdot 0_A \subset A$ in $\bar{A}$. Then $T \subset \bar{T}$ is an equivariant compactification. We have a canonical map $\psi\colon (\bar{T} \times A)/T \to \bar{A}$. We have a map $T \times \bar{T} \to \bar{T}$ which extends the group morphism $T \times T \to T$. Let $\widetilde{T} \to \bar{T}$ be the normalization. This is an isomorphism over T. Hence $T \subset \widetilde{T}$. Since $T \times \widetilde{T}$ is normal, the composite $T \times \widetilde{T} \to T \times \bar{T} \to \bar{T}$ factors $\widetilde{T} \to \bar{T}$. Hence we get $T \times \widetilde{T} \to \widetilde{T}$, which extends the group morphism $T \times T \to T$. Hence $\widetilde{T}$ is an equivariant compactification of T. Then by Sumihiro's theorem, $T \subset \widetilde{T}$ is a torus embedding. Hence by [37, p. 23], there exist a smooth equivariant compactification $\widehat{T}$ and an equivariant morphism $\widehat{T} \to \widetilde{T}$. Set $\hat{A} = (\widehat{T} \times A)/T$. Then we have A-equivariant morphisms $\hat{A} \to (\bar{T} \times A)/T \to \bar{A}$. Note that $\hat{A}$ is a smooth equivariant compactification. Compare with [25, Thm. 1.1]. □

Lemma A.8. *Given a smooth equivariant compactification $\bar{A}$, we may take a smooth projective equivariant compactification $\hat{A}$ and an equivariant morphism $\hat{A} \to \bar{A}$.*

Proof. By Lemma A.6, we have $\bar{A} = (\bar{T} \times A)/T$. By [37, Prop. 2.17], there exists an equivariant modification $\widehat{T} \to \bar{T}$ such that $\widehat{T}$ is projective. By [37, p. 23], we may assume that $\widehat{T}$ is smooth. Then $\hat{A} = (A \times \widehat{T})/T$ is smooth. We claim that this is also projective. Indeed, let $D = \sum_j n_j D_j$ be a T-invariant, very ample divisor on $\widehat{T}$. We set $E = \sum_j n_j (D_j \times A)/T$. Then E is a divisor on $\hat{A}$. Let $p\colon \hat{A} \to A_0$ be the canonical projection. Let $\{U_i\}$ be a Zariski open covering of A_0 such that $p^{-1}(U_i) = \widehat{T} \times U_i$. Then $p^{-1}(U_i) \cap E = D \times U_i$, which shows that $\mathcal{O}_{\hat{A}}(E)$ is p-very ample relative to A_0 (cf. [17, Def. 13.52]). Since A_0 is projective, there exists an ample line bundle L on A_0 such that $p^*L(E)$ is ample on $\hat{A}$ (cf. [17, Prop. 13.65]). Hence $\hat{A}$ is projective. □

Lemma A.9. *Let $\bar{A}$ and $\tilde{A}$ be equivariant compactifications. Then there exists a smooth equivariant compactification $\hat{A}$ such that equivariant morphisms $\hat{A} \to \bar{A}$ and $\hat{A} \to \tilde{A}$ exist.*

Proof. Let $A \subset A \times A$ be the diagonal, which induces $A \subset \bar{A} \times \tilde{A}$. Let $V \subset \bar{A} \times \tilde{A}$ be the Zariski closure of this. Then V is an A-equivariant compactification with equivariant maps $V \to \bar{A}$ and $V \to \tilde{A}$. By Lemma A.7, there exists a smooth equivariant compactification $\hat{A}$ with an equivariant morphism $\hat{A} \to V$. The composites of this with $V \to \bar{A}$ and $V \to \tilde{A}$ yield $\hat{A} \to \bar{A}$ and $\hat{A} \to \tilde{A}$. □

Lemma A.10. *Every semi-abelian variety admits only countably many smooth equivariant compactifications.*

Proof. A smooth equivariant compactification $\bar{A}$ is written as $(\overline{T} \times A)/T$, where $\overline{T}$ is a smooth equivariant compactification (cf. Lemma A.6). By Sumihiro's theorem, every smooth, equivariant compactification $\overline{T}$ of T is a torus embedding associated to some complete fan ([37, Thm. 1.5]). There are only countably many complete fans. Hence there are only countably many smooth equivariant compactifications of T, hence of A. □

Appendix A.4. Orbit closure and isotropy group

Let $\bar{A} = (\overline{T} \times A)/T$, where $\overline{T} = T\,\mathrm{emb}\,\Delta$. For each $\sigma \in \Delta$, we set $O_\sigma = (\mathrm{orb}(\sigma) \times A)/T$. Then $O_\sigma \subset \bar{A}$ is an A-orbit. By (A.4), we have $\bar{A} = \coprod_{\sigma\in\Delta} O_\sigma$. Let $I \subset A$ be the isotropy group for the points of O_σ. Since I acts trivially on the abelian variety A_0, we get $I \subset T$. Then I is the isotropy group for the points of $\mathrm{orb}(\sigma) \subset \overline{T}$. Hence $I \subset T$ is a subtorus and (A.6) yields

$$\mathrm{Hom}(I, \mathbb{G}_m) = M/(M \cap \sigma^\perp). \tag{A.7}$$

Let $V \subset \bar{A}$ be an irreducible Zariski closed set which is A-invariant. Then there exists a unique A-orbit $O_\sigma \subset \bar{A}$ such that V is the Zariski closure of O_σ. Suppose $\bar{A}$ is smooth. Then by [37, Cor. 1.7], the closure of $\mathrm{orb}(\sigma) \subset \overline{T}$ is smooth. Hence V is smooth.

Lemma A.11. *Let $\bar{A}$ be a smooth equivariant compactification. Let $V \subset \bar{A}$ be an irreducible Zariski closed set which is A-invariant. Then there exist an A-invariant Zariski open set $W \subset \bar{A}$ with $V \subset W$ and an equivariant morphism $p\colon W \to V$ such that $p\colon W \to V$ is a total space of a vector bundle over V whose zero-section is the inclusion map $V \hookrightarrow W$. Moreover, V is covered by Zariski open subsets V_0 such that the following properties hold:*

(1) *$p\colon W \to V$ is a trivial vector bundle $\mathbb{C}^l \times V_0 \to V_0$ on V_0.*
(2) *Let $O \subset \bar{A}$ be the A-orbit such that V is the closure of O. Let I be the isotropy group for the points of $O \subset \bar{A}$. Then there exists a decomposition $I = \mathbb{G}_m^l$ such that $(t_1, \dots, t_l) \in \mathbb{G}_m^l$ acts $(z_1, \dots, z_l, y) \in \mathbb{C}^l \times V_0$ as $(z_1, \dots, z_l, y) \mapsto (t_1 z_1, \dots, t_l z_l, y)$.*

Proof. We first consider the case $A = T$. Set $M = \mathrm{Hom}(T, \mathbb{G}_m)$ and $N = \mathrm{Hom}(\mathbb{G}_m, T)$. Let Δ be the fan defining $\overline{T}$, i.e., $\overline{T} = T\,\mathrm{emb}(\Delta)$. By [37, Prop. 1.6], we may take $\tau \in \Delta$ such that V is the closure of $\mathrm{orb}(\tau)$. We may take $v_1, , \ldots, v_l \in N$ such that $\tau = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_l$ and $\tau \cap N = \mathbb{Z}_{\geq 0}v_1 + \cdots + \mathbb{Z}_{\geq 0}v_l$. Let $\overline{N} = N/N_\tau$, where $N_\tau = \mathbb{Z}(\tau \cap N)$. Then $\overline{N}$ is the dual of $M \cap \tau^\perp$. We have

$$\mathrm{orb}(\tau) = \mathrm{Spec}\,\mathbb{C}[M \cap \tau^\perp].$$

For each $\sigma \in \Delta$ with $\tau \prec \sigma$, we set $\bar{\sigma} = (\sigma + \mathbb{R}\tau)/\mathbb{R}\tau \subset \overline{N}_{\mathbb{R}}$. Then $\bar{\sigma} \subset \overline{N}_{\mathbb{R}}$ is a strongly convex rational polyhedral cone (cf. [37, Prop. A.8]). Set $\overline{U}_{\bar{\sigma}} = \mathrm{Spec}\,\mathbb{C}[(M \cap \tau^\perp) \cap \bar{\sigma}^\vee]$. The inclusion $\mathbb{C}[(M \cap \tau^\perp) \cap \bar{\sigma}^\vee] \subset \mathbb{C}[M \cap \sigma^\vee]$ yields $U_\sigma \to \overline{U}_{\bar{\sigma}}$. We have a closed immersion $\overline{U}_{\bar{\sigma}} \hookrightarrow U_\sigma$, which is obtained by the surjection $\mathbb{C}[M \cap \sigma^\vee] \to \mathbb{C}[(M \cap \tau^\perp) \cap \bar{\sigma}^\vee]$ defined by $\chi^m \mapsto 0$ for all $m \in (M \cap \sigma^\vee)\backslash(M \cap \tau^\perp \cap \bar{\sigma}^\vee)$. This inclusion $\overline{U}_{\bar{\sigma}} \hookrightarrow U_\sigma$ is the section of the map $U_\sigma \to \overline{U}_{\bar{\sigma}}$.

We remark that the map $U_\sigma \to \overline{U}_{\bar{\sigma}}$ is a trivial vector bundle whose zero-section is the inclusion map $\overline{U}_{\bar{\sigma}} \hookrightarrow U_\sigma$. Indeed, the sequence

$$0 \to N_\tau \to N \to \overline{N} \to 0$$

has a (non-canonical) splitting $N = N_\tau \oplus \overline{N}$ such that $\sigma = \tau \oplus \bar{\sigma}$, where we view $\tau \subset N_{\mathbb{R}}$ as $\tau \subset (N_\tau)_{\mathbb{R}}$. Considering the dual

$$0 \to M \cap \tau^\perp \to M \to M_\tau \to 0,$$

we get the associated splitting $M = M_\tau \oplus (M \cap \tau^\perp)$. Hence we get

$$\mathbb{C}[M \cap \sigma^\vee] = \mathbb{C}[M_\tau \cap \tau^\vee] \otimes_{\mathbb{C}} \mathbb{C}[(M \cap \tau^\perp) \cap \bar{\sigma}^\vee].$$

We have $\mathrm{Spec}\,\mathbb{C}[M_\tau \cap \tau^\vee] = \mathbb{C}^l$, hence $U_\sigma \simeq \mathbb{C}^l \times \overline{U}_{\bar{\sigma}}$. The inclusion $\overline{U}_{\bar{\sigma}} \subset U_\sigma$ coincides with $\{0\} \times \overline{U}_{\bar{\sigma}} \subset \mathbb{C}^l \times \overline{U}_{\bar{\sigma}}$. Hence the map $U_\sigma \to \overline{U}_{\bar{\sigma}}$ is a trivial vector bundle whose zero-section is the inclusion map $\overline{U}_{\bar{\sigma}} \hookrightarrow U_\sigma$. Moreover, the stabilizer I of $\mathrm{orb}(\tau)$ is identified with $(\mathbb{G}_m)^l \subset \mathbb{C}^l$. Note that the ambiguity of the splitting $N = N_\tau \oplus \overline{N}$ yields an isomorphism $\mathbb{C}^l \times \overline{U}_{\bar{\sigma}} \simeq \mathbb{C}^l \times \overline{U}_{\bar{\sigma}}$ which is defined by the multiplication given by $\overline{U}_{\bar{\sigma}} \to I$.

Now set $\overline{\Delta} = \{\bar{\sigma};\ \sigma \in \Delta,\ \tau \prec \sigma\}$. Then, by [37, Cor. 1.7], $\overline{\Delta}$ is a fan in $\overline{N}_{\mathbb{R}}$ and $V = \mathrm{orb}(\tau)\,\mathrm{emb}(\overline{\Delta})$. Hence $V = \bigcup_{\bar{\sigma} \in \overline{\Delta}} \overline{U}_{\bar{\sigma}}$. We set

$$W = \bigcup_{\sigma \in \Delta, \tau \prec \sigma} U_\sigma.$$

Then the maps $U_\sigma \to \overline{U}_{\bar{\sigma}}$ glue together to get a vector bundle $W \to V$. Indeed, the coordinate transformations of this vector bundle are given by the multiplications by I. Since U_σ are T-invariant, W is T-invariant. This completes the proof when $A = T$.

Next we consider the general case. We write $V = (V' \times A)/T$, where $V' \subset \overline{T}$ is a T-invariant Zariski closed set. Then we may take a Zariski open set $W' \subset \overline{T}$ and $p\colon W' \to V'$ by the previous consideration. We set $W = (W' \times A)/T$, where W' is T-invariant. Then $W \subset \bar{A}$ is an A-invariant Zariski open set and $V \subset W$. This yields $W \to V$ as desired. □

Remark A.12. Let $\bar{A}$ be a smooth equivariant compactification. Then the boundary ∂A is a simple normal crossing divisor. Indeed, for each $x \in \partial A$, we take a unique A-orbit $O \subset \bar{A}$ such that $x \in O$ and set $V = \bar{O}$. Then V is smooth. We take $W \subset \bar{A}$ as in Lemma A.11. Let $V_0 \subset V$ be described in Lemma A.11 so that $x \in V_0$. We may assume $V_0 \subset O$. Then $(\partial A) \cap (\mathbb{C}^l \times V_0) = \bigcup_{i=1}^{l} H_i \times V_0$, where $H_i \subset \mathbb{C}^l$ is defined by $H_i = \{z_i = 0\}$. Hence $(\partial A) \cap (\mathbb{C}^l \times V_0)$ is a simple normal crossing divisor on $\mathbb{C}^l \times V_0$. Since $x \in \partial A$ is arbitrary, ∂A is simple normal crossing.

Appendix A.5. Modification of equivariant maps

The purpose of this subsection is to prove Lemma A.15.

Lemma A.13. *Let $\psi\colon N \to N'$ be surjective. Let Δ be a fan in N and let Δ' be a fan in N'. Set $\Delta'' = \{\sigma \cap \psi^{-1}(\sigma');\ \sigma \in \Delta,\ \sigma' \in \Delta'\}$, where we continue to write $\psi\colon N_{\mathbb{R}} \to N'_{\mathbb{R}}$. Then Δ'' is a fan in N.*

Before proving this lemma, we remark the following. Given a convex rational polyhedral cone $C \subset N_{\mathbb{R}}$, we have $(C^\vee)^\vee = C$ (cf. [37, Thm. A.1]). For another convex rational polyhedral cone $C' \subset N_{\mathbb{R}}$, we have $(C \cap C')^\vee = C^\vee + (C')^\vee$ (cf. [37, Thm. A.1]). In particular, $C \cap C'$ is a convex rational polyhedral cone. Moreover, if C is strongly convex, then $C \cap C'$ is strongly convex. By [37, Prop. A.9], the following two statements are equivalent:

(1) The subset $C \cap C' \subset C$ is a face of C.
(2) For every $x, x' \in C$, if $x + x' \in C \cap C'$, then both $x \in C \cap C'$ and $x' \in C \cap C'$.

The non-trivial part is the implication (2) ⇒ (1). The converse is trivial. Indeed, suppose $C \cap C'$ is a face of C. Then there exists $u \in C^\vee$ such that $C \cap C' = C \cap \{u = 0\}$. If $x, x' \in C$ satisfy $x + x' \in C \cap C'$, then $u(x) \geq 0$, $u(x') \geq 0$, and $u(x + x') = 0$. Hence $u(x) = u(x') = 0$, thus $x, x' \in C \cap C'$.

Proof of Lemma A.13 (cf. [2, Lem. 3.2]). Note that $\psi^{-1}(\sigma')$ is a convex rational polyhedral cone. Since σ is strongly convex, $\sigma \cap \psi^{-1}(\sigma')$ is a strongly convex rational polyhedral cone. Hence Δ'' is a non-empty collection of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$.

We first show that a face of $\sigma\cap\psi^{-1}(\sigma')$ is contained in Δ''. We claim

$$(\sigma\cap\psi^{-1}(\sigma'))^\vee=\sigma^\vee+\psi^*((\sigma')^\vee), \tag{A.8}$$

where $\psi^*\colon M'_{\mathbb{R}}\to M_{\mathbb{R}}$ is the dual map. Indeed, by $(\sigma\cap\psi^{-1}(\sigma'))^\vee=\sigma^\vee+(\psi^{-1}(\sigma'))^\vee$, it is enough to show

$$(\psi^{-1}(\sigma'))^\vee=\psi^*((\sigma')^\vee).$$

Note that

$$(\psi^{-1}(\sigma'))^\vee\supset\psi^*((\sigma')^\vee)$$

is trivial. To show the converse, let $u\in(\psi^{-1}(\sigma'))^\vee$. Let $K\subset N_{\mathbb{R}}$ be the kernel of ψ. Then $K\subset\psi^{-1}(\sigma')$. Hence $K\subset\{u\geq 0\}$. By $-K=K$, we get $K\subset\{u=0\}$. Hence there exists $v\in M'_{\mathbb{R}}$ such that $u=v\circ\psi$. By $u\in(\psi^{-1}(\sigma'))^\vee$, we have $\sigma'\subset\{v\geq 0\}$. Hence $v\in(\sigma')^\vee$. This shows $u\in\psi^*((\sigma')^\vee)$, hence $(\psi^{-1}(\sigma'))^\vee\subset\psi^*((\sigma')^\vee)$. Thus we get (A.8).

Now a face of $\sigma\cap\psi^{-1}(\sigma')$ is defined by $\sigma\cap\psi^{-1}(\sigma')\cap\{u=0\}$ for some $u\in(\sigma\cap\psi^{-1}(\sigma'))^\vee$. By (A.8), we have $u=v+v'\circ\psi$ for some $v\in\sigma^\vee$ and $v'\in(\sigma')^\vee$. We claim that

$$\sigma\cap\psi^{-1}(\sigma')\cap\{u=0\}=(\sigma\cap\{v=0\})\cap\psi^{-1}(\sigma'\cap\{v'=0\}). \tag{A.9}$$

Indeed, the relation "$\supset$" is trivial. To show the converse, let $x\in\sigma\cap\psi^{-1}(\sigma')\cap\{u=0\}$. Then we have $v(x)\geq 0$ and $v'\circ\psi(x)\geq 0$. By $u(x)=0$, we have $v(x)=0$ and $v'\circ\psi(x)=0$. Hence $x\in(\sigma\cap\{v=0\})\cap\psi^{-1}(\sigma'\cap\{v'=0\})$. This shows (A.9). Set $\tau=\sigma\cap\{v=0\}$ and $\tau'=\sigma'\cap\{v'=0\}$. Then τ and τ' are faces of σ and σ', respectively. Hence $\tau\in\Delta$ and $\tau'\in\Delta'$. By (A.9), we have

$$\sigma\cap\psi^{-1}(\sigma')\cap\{u=0\}=\tau\cap\psi^{-1}(\tau')\in\Delta''.$$

Next we show that $(\sigma_1\cap\psi^{-1}(\sigma_1'))\cap(\sigma_2\cap\psi^{-1}(\sigma_2'))$ is a face of $\sigma_1\cap\psi^{-1}(\sigma_1')$. We take $x,y\in\sigma_1\cap\psi^{-1}(\sigma_1')$ such that $x+y\in(\sigma_1\cap\psi^{-1}(\sigma_1'))\cap(\sigma_2\cap\psi^{-1}(\sigma_2'))$. Then we have $x+y\in\sigma_1\cap\sigma_2$. Since $\sigma_1\cap\sigma_2$ is a face of σ_1, and $x,y\in\sigma_1$, we get $x,y\in\sigma_1\cap\sigma_2$. Also we have $\psi(x+y)=\psi(x)+\psi(y)\in\sigma_1'\cap\sigma_2'$. Since $\sigma_1'\cap\sigma_2'$ is a face of σ_1', and $\psi(x),\psi(y)\in\sigma_1'$, we get $\psi(x),\psi(y)\in\sigma_1'\cap\sigma_2'$. Hence $x,y\in(\sigma_1\cap\psi^{-1}(\sigma_1'))\cap(\sigma_2\cap\psi^{-1}(\sigma_2'))$. By [37, Prop. A.9], we have

$$(\sigma_1\cap\psi^{-1}(\sigma_1'))\cap(\sigma_2\cap\psi^{-1}(\sigma_2'))\prec(\sigma_1\cap\psi^{-1}(\sigma_1')).$$

Hence Δ'' is a fan. □

Lemma A.14. *Let $\psi\colon N\to N'$. Let Δ be a complete fan in N. Then there exist a non-singular, complete fan Δ' in N' and a finite subdivision $\widetilde{\Delta}$ of Δ such that for all $\sigma\in\widetilde{\Delta}$, there exists $\sigma'\in\Delta'$ such that $\psi(\sigma)=\sigma'$.*

This is contained in the proof of [1, Prop. 4.4]. We give a proof for the sake of completeness. Before proving this lemma, we remark the following. If $\Delta_1,\dots,\Delta_l$ are complete fans in N, then $\Delta=\{\sigma_1\cap\cdots\cap\sigma_l;\ \sigma_1\in\Delta_1,\dots,\sigma_l\in\Delta_l\}$ is a complete fan in N. Indeed, for the case $l=2$, Lemma A.13 applied to the identity map $N\to N$ yields that Δ is a fan. The same holds for all l by induction. The completeness is obvious. Each $\sigma_1\in\Delta_1$ is a union of cones in Δ. Indeed, we have

$$\sigma_1=\bigcup_{\sigma_2\in\Delta_2,\dots,\sigma_l\in\Delta_2}\sigma_1\cap\sigma_2\cap\cdots\cap\sigma_l,$$

for $\Delta_2,\dots,\Delta_l$ are complete.

Proof of Lemma A.14. For each $\sigma\in\Delta$, we may take a complete fan Δ'_σ in N' such that $\psi(\sigma)$ is a union of cones in Δ'_σ. Indeed, by [37, Thm. A.3], there exist strongly convex rational polyhedral cones $\tau_1,\dots,\tau_l$ such that $\psi(\sigma)=\tau_1\cup\cdots\cup\tau_l$. By [37, Prop. 1.12], there exist complete fans $\Delta'_1,\dots,\Delta'_l$ such that $\tau_i\in\Delta'_i$. We set $\Delta'_\sigma=\{\sigma'_1\cap\cdots\cap\sigma'_l;\ \sigma'_i\in\Delta'_i\}$. Then by Lemma A.13, Δ'_σ is a complete fan. Each $\tau_i\in\Delta'_i$ is a union of cones in Δ'_σ. Hence $\psi(\sigma)$ is a union of cones in Δ'_σ, as desired.

We set $\Delta'=\{\bigcap_{\sigma\in\Delta}\kappa_\sigma;\ \kappa_\sigma\in\Delta'_\sigma\}$. Then by Lemma A.13, Δ' is a complete fan in N'. For all $\sigma\in\Delta$, $\psi(\sigma)$ is a union of cones in Δ'. By replacing Δ' by its finite subdivision, we may assume that Δ' is non-singular.

We set $\widetilde{\Delta}=\{\psi^{-1}(\tau)\cap\sigma;\ \tau\in\Delta',\ \sigma\in\Delta\}$. Then by Lemma A.13, $\widetilde{\Delta}$ is a complete fan in N. We shall show that these $\widetilde{\Delta}$ and Δ' satisfy our requirement. We have $\psi(\psi^{-1}(\tau)\cap\sigma)=\tau\cap\psi(\sigma)$. Hence it is enough to show

$$\tau\cap\psi(\sigma)\in\Delta'.$$

We may take $\tau_1,\dots,\tau_k\in\Delta'$ such that $\psi(\sigma)=\tau_1\cup\cdots\cup\tau_k$. Hence we have

$$\tau\cap\psi(\sigma)=(\tau\cap\tau_1)\cup\cdots\cup(\tau\cap\tau_k).$$

Each $\tau\cap\tau_i$ is a face of τ. We take $x,y\in\tau$ such that $x+y\in\tau\cap\psi(\sigma)$. Then we have $x+y\in\tau\cap\tau_i$ for some τ_i. Since $\tau\cap\tau_i$ is a face of τ, we have $x,y\in\tau\cap\tau_i$. Hence $x,y\in\tau\cap\psi(\sigma)$. By [37, Prop. A.9], $\tau\cap\psi(\sigma)$ is a face of τ. Hence $\tau\cap\psi(\sigma)\in\Delta'$. This completes the proof of our lemma. □

Lemma A.15. *Let $\bar{A}$ be a smooth equivariant compactification and let $B\subset A$ be a semi-abelian subvariety. Then there exists an equivariant compactification $\hat{A}=(\widehat{T}\times A)/T$ with an equivariant morphism $\hat{A}\to\bar{A}$ such that the following properties hold:*

(1) *There exists a smooth equivariant compactification $\widehat{A/B}$ such that an equivariant morphism $p\colon\hat{A}\to\widehat{A/B}$ exists.*

(2) *For every $x \in \hat{A}$, if a semi-abelian subvariety $C \subset A$ satisfies $C \cdot p(x) = p(x)$, then $C \cdot x \subset B \cdot x$ as subsets of $\hat{A}$.*

Proof. We have canonical expressions $0 \to T_A \to A \to A_0 \to 0$ and $0 \to T_B \to B \to B_0 \to 0$. Then we have the canonical expression

$$0 \to T_A/T_B \to A/B \to A_0/B_0 \to 0.$$

Set $N = \mathrm{Hom}(\mathbb{G}_m, T_A)$. Let Δ be the fan in N such that $\overline{T}_A = T_A\,\mathrm{emb}(\Delta)$, where $\bar{A} = (\overline{T}_A \times A)/T_A$. Set $N' = \mathrm{Hom}(\mathbb{G}_m, T_A/T_B)$. Then we have $\psi\colon N \to N'$. By Lemma A.14, we may find a non-singular, complete fan Δ' in N' and a finite subdivision $\widehat{\Delta}$ of Δ such that for all $\sigma \in \widehat{\Delta}$, there exists $\sigma' \in \Delta'$ such that $\psi(\sigma) = \sigma'$. Set $\hat{A} = (\widehat{T}_A \times A)/T_A$ and $\widehat{A/B} = (\widehat{T_A/T_B} \times A/B)/(T_A/T_B)$, where $\widehat{T}_A = T_A\,\mathrm{emb}(\widehat{\Delta})$ and $\widehat{T_A/T_B} = T_A/T_B\,\mathrm{emb}(\Delta')$. Then $\widehat{T_A/T_B}$ is smooth. Hence $\widehat{A/B}$ is smooth. By [37, p. 19], we get equivariant morphisms $p\colon \hat{A} \to \widehat{A/B}$ and $\hat{A} \to \bar{A}$.

We shall show that $p\colon \hat{A} \to \widehat{A/B}$ satisfies assertion (2). Let $x \in \hat{A}$. Set $I_x = \{a \in A;\ a \cdot x = x\}$. We take $\sigma \in \widehat{\Delta}$ such that $x \in (\mathrm{orb}(\sigma) \times A)/T \subset \hat{A}$. Set $M_\sigma = M \cap \sigma^\perp$, where $M = \mathrm{Hom}(T_A, \mathbb{G}_m)$. By (A.7), we have

$$\mathrm{Hom}(I_x, \mathbb{G}_m) = M/M_\sigma,$$

where $I_x \subset T_A$ is a subtorus. Similarly, set $J_{p(x)} = \{a \in A/B;\ a \cdot p(x) = p(x)\}$. We take $\sigma' \in \Delta'$ such that

$$\psi(\sigma) = \sigma'. \tag{A.10}$$

Let $\mathrm{orb}(\sigma') \subset \widehat{T_A/T_B}$ be the corresponding orbit. Denoting the canonical map by $q\colon \widehat{T}_A \to \widehat{T_A/T_B}$, we have $q(\mathrm{orb}(\sigma)) = \mathrm{orb}(\sigma')$ (cf. [16, p. 56], [11, Lem. 3.3.21]). In particular, we have $p(x) \in (\mathrm{orb}(\sigma') \times (A/B))/(T_A/T_B) \subset \widehat{A/B}$. Hence by (A.7), we have

$$\mathrm{Hom}(J_{p(x)}, \mathbb{G}_m) = M'/M'_{\sigma'},$$

where $M' = \mathrm{Hom}(T_A/T_B, \mathbb{G}_m)$ and $M'_{\sigma'} = M' \cap (\sigma')^\perp$. The quotient map $T_A \to T_A/T_B$ induces a morphism $I_x \to J_{p(x)}$ of tori. This induces $\mathrm{Hom}(J_{p(x)}, \mathbb{G}_m) \to \mathrm{Hom}(I_x, \mathbb{G}_m)$, namely $M'/M'_{\sigma'} \to M/M_\sigma$. This fits into the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M'_{\sigma'} \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & M'/M'_{\sigma'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_\sigma \arrow[r] & M \arrow[r] & M/M_\sigma \arrow[r] & 0.
\end{tikzcd}$$

We claim that the map $I_x \to J_{p(x)}$ is surjective. Indeed, by (A.10), we have $M'_{\sigma'} = M' \cap M_\sigma$. Hence the map $M'/M'_{\sigma'} \to M/M_\sigma$ is injective. Thus we get the injection $\mathrm{Hom}(J_{p(x)}, \mathbb{G}_m) \hookrightarrow \mathrm{Hom}(I_x, \mathbb{G}_m)$. Since the image of the map $I_x \to J_{p(x)}$ is a subtorus of $J_{p(x)}$ (cf. [20, Prop. B, p. 54]), this image should coincide with $J_{p(x)}$. Thus the map $I_x \to J_{p(x)}$ is surjective.

Now let $C \subset A$ satisfy $C \cdot p(x)$. Then $(C+B)/B \subset J_{p(x)}$. Hence $(C+B)/B \subset (I_x + B)/B$, so $C \subset I_x + B$. Thus $C \cdot x \subset (I_x + B) \cdot x = B \cdot x$. This completes the proof. □

Appendix A.6. Logarithmic tangent space

We recall the isomorphism $A \times \mathrm{Lie}\, A \simeq TA$ from (2.6). Let $\bar{A}$ be a smooth equivariant compactification of A. Then ∂A is a simple normal crossing divisor on $\bar{A}$ (cf. Remark A.12).

Lemma A.16. *The isomorphism* (2.6) *extends to an isomorphism*

$$\bar{\psi}\colon \bar{A} \times \mathrm{Lie}\, A \to T\bar{A}(-\log(\partial A)).$$

Proof (cf. [35, Prop. 5.4.3]). We denote by

$$\overline{m}\colon \bar{A} \times A \to \bar{A}$$

the natural action defined by $(x, a) \mapsto x + a$. By $\overline{m}^*(\partial A) = (\partial A) \times A$, we have

$$T(\bar{A} \times A)\big(-\log((\partial A) \times A)\big) \to T\bar{A}(-\log(\partial A)).$$

We have the subbundle

$$\bar{A} \times TA \subset T(\bar{A} \times A)\big(-\log((\partial A) \times A)\big).$$

Thus we get

$$\bar{A} \times TA \to T\bar{A}(-\log(\partial A)). \tag{A.11}$$

We restrict this to $\bar{A} \times \{0_A\} \subset \bar{A} \times A$ to get

$$\bar{\psi}\colon \bar{A} \times \mathrm{Lie}\, A \to T\bar{A}(-\log(\partial A)),$$

which extends (2.6). We shall show that $\bar{\psi}$ is an isomorphism.

Let $x \in \bar{A}$ be an arbitrary point. It is enough to show that the induced map

$$\bar{\psi}_x\colon \mathrm{Lie}\, A \to T_x\bar{A}(-\log(\partial A))$$

is injective. Let $O \subset \bar{A}$ be a unique A-orbit such that $x \in O$. Let $I \subset A$ be the isotropy group for $x \in \bar{A}$. Then we have an isomorphism $O \simeq A/I$. Set $V = \overline{O}$.

By Lemma A.11, there exist an A-invariant Zariski open $W \subset \bar{A}$ with $V \subset W$ and an A-equivariant morphism $p\colon W \to V$. We take a Zariski open neighborhood $V_0 \subset V$ of x such that $V_0 \subset O$ and $p^{-1}(V_0) = \mathbb{A}^l \times V_0$. Set $W_0 = p^{-1}(V_0)$. Then W_0 is I-invariant and $(t_1, \dots, t_l) \in I$ acts $(y_1, \dots, y_l, v) \in \mathbb{A}^l \times V_0$ as $(t_1 y_1, \dots, t_l y_l, v)$. We denote this action by

$$\mu\colon W_0 \times I \to W_0.$$

Note that μ is the restriction of $\overline{m}$. The restriction of (A.11) on $W_0 \times TI \subset A \times TA$ induces

$$W_0 \times TI \to TW_0(-\log(\partial A)).$$

By the dual, we get

$$\mu^* \Omega^1_{W_0}(\log(\partial A)) \to \Omega^1_{(W_0 \times I)/W_0}. \tag{A.12}$$

The image of the sections $dy_1/y_1, \dots, dy_l/y_l \in \Omega^1_{W_0}(\log(\partial A))$ on W_0 under the map (A.12) are $dt_1/t_1, \dots, dt_l/t_l$. This shows that the map (A.12) is surjective over $W_0 \times I$. Considering the dual on $(x, e_I) \in W_0 \times I$, we observe that the restriction

$$\bar{\psi}_x|_{\mathrm{Lie}\, I}\colon \mathrm{Lie}\, I \to T_x \bar{A}(-\log(\partial A))$$

is injective.

Next we have the following commutative diagram:

$$\begin{tikzcd}
W \times A \arrow[r, "\overline{m}"] \arrow[d, "q"'] & W \arrow[d, "p"] \\
V \times (A/I) \arrow[r, "\overline{m}_V"'] & V.
\end{tikzcd}$$

This induces the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Lie}\, A \arrow[r, "\overline{m}_*"] \arrow[d, "q_*"'] & T_x W \arrow[d, "p_*"] \\
\mathrm{Lie}(A/I) \arrow[r, "(\overline{m}_V)_*"'] & T_x V.
\end{tikzcd}$$

By $x \in A/I$, we note that $(\overline{m}_V)_*$ is an isomorphism. Hence $\ker \overline{m}_* \subset \mathrm{Lie}\, I$. Since $\overline{m}_*$ factors as

$$\mathrm{Lie}\, A \xrightarrow{\bar{\psi}_x} T_x \bar{A}(-\log(\partial A)) \longrightarrow T_x W,$$

we get $\ker \bar{\psi}_x \subset \mathrm{Lie}\, I$. This shows that $\bar{\psi}_x$ is injective. $\square$

Appendix A.7. An application of faithfully flat descent

Lemma A.17. *Let V and Σ be varieties. Let $Z \subset V$ be a closed subscheme. Let $Y \subset V \times \Sigma$ be a closed subscheme such that $Y_s = Z$ for all $s \in \Sigma$. Then $Z \times \Sigma \subset Y$ as closed subschemes of $V \times \Sigma$.*

Proof. It is enough to prove the case that V and Σ are affine. Thus we assume $V = \operatorname{Spec} R$ and $\Sigma = \operatorname{Spec} S$. Let $I_Z \subset R$ and $I_Y \subset R \otimes_{\mathbb{C}} S$ be the ideals associated to $Z \subset V$ and $Y \subset V \times \Sigma$, respectively. To show $I_Y \subset I_Z \otimes_{\mathbb{C}} S$, we take $\varphi \in I_Y$. Then we may write

$$\varphi = f_1 \otimes g_1 + \cdots + f_l \otimes g_l,$$

where $f_1, \ldots, f_l \in R$ and $g_1, \ldots, g_l \in S$. We take this expression so that l is a minimum. We define a subset $L \subset \mathbb{C}^l$ such that $(a_1, \ldots, a_l) \in L$ if and only if $a_1 f_1 + \cdots + a_l f_l \in I_Z$. Then $L \subset \mathbb{C}^l$ is a linear subspace. To prove $L = \mathbb{C}^l$, we assume on the contrary that $L \subsetneqq \mathbb{C}^l$. Then we may take a non-zero $(\lambda_1, \ldots, \lambda_l) \in \mathbb{C}^l$ such that $\lambda_1 a_1 + \cdots + \lambda_l a_l = 0$ for all $(a_1, \ldots, a_l) \in L$. We may assume without loss of generality that $\lambda_1 = 1$. Now for all $s \in \Sigma$, we have $Y_s = Z$. Hence $g_1(s) f_1 + \cdots + g_l(s) f_l \in I_Z$. Hence $(g_1(s), \ldots, g_l(s)) \in L$, so $\lambda_1 g_1(s) + \cdots + \lambda_l g_l(s) = 0$ for all $s \in \Sigma$. Since S is an integral domain, we have $\lambda_1 g_1 + \cdots + \lambda_l g_l = 0$ in S. By $\lambda_1 = 1$, we have $g_1 = -(\lambda_2 g_2 + \cdots + \lambda_l g_l)$. Hence $\varphi = (f_2 - \lambda_2 f_1) \otimes g_2 + \cdots + (f_l - \lambda_l f_1) \otimes g_l$. This contradicts the minimality of l. Hence $L = \mathbb{C}^l$. Now we have $(1, 0, \ldots, 0) \in L$, hence $f_1 \in I_Z$. Similarly, we have $f_i \in I_Z$ for all i. Hence $\varphi \in I_Z \otimes_{\mathbb{C}} S$. Thus $I_Y \subset I_Z \otimes_{\mathbb{C}} S$. This shows $Z \times \Sigma \subset Y$. □

Let A be a semi-abelian variety and let $B \subset A$ be a semi-abelian subvariety. Let Σ be a variety. Then we get a projection $p \colon A \times \Sigma \to (A/B) \times \Sigma$, which yields the following fiber product:

$$\begin{array}{ccc} B \times A \times \Sigma & \xrightarrow{q_2} & A \times \Sigma \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p} \\ A \times \Sigma & \xrightarrow[p]{} & (A/B) \times \Sigma. \end{array}$$

Here, for $(b, a, s) \in B \times A \times \Sigma$, we have $q_1(b, a, s) = (a, s)$ and $q_2(b, a, s) = (b + a, s)$.

Lemma A.18. *Let $Z \subset A \times \Sigma$ be a closed subscheme which is B-invariant in the sense that $b + Z = Z$ for all $b \in B$. Then $q_1^*(Z) = q_2^*(Z)$ as closed subschemes of $B \times A \times \Sigma$.*

Proof. Note that $q_1^*(Z) = B \times Z$. Let $\pi \colon B \times A \times \Sigma \to B$ be the first projection. Then for $b \in B$, we have $q_2^*(Z) \cap \pi^{-1}(b) = (-b) + Z = Z$. Hence by Lemma A.17, we have $q_1^*(Z) \subset q_2^*(Z)$. We have an isomorphism $\tau\colon B\times A\times\Sigma \to B\times A\times\Sigma$ defined by $\tau(b,a,s) = (b, a-b, s)$. By $q_1 = q_2\circ\tau$, we have $\tau^*q_2^*(Z) = B\times Z$. We have $q_1\circ\tau(b,a,s) = (a-b,s)$. Hence for $b\in B$, we have $\tau^*q_1^*(Z)\cap\pi^{-1}(b) = b+Z = Z$. Hence by Lemma A.17, we have $\tau^*q_2^*(Z)\subset\tau^*q_1^*(Z)$. Hence $q_2^*(Z)\subset q_1^*(Z)$. Thus we get $q_1^*(Z) = q_2^*(Z)$. $\square$

Lemma A.19. *Let A be a semi-abelian variety and let $B\subset A$ be a semi-abelian subvariety. Let Σ be a variety and let $Z\subset A\times\Sigma$ be a closed subscheme which is B-invariant in the sense that $b+Z=Z$ for all $b\in B$. Then there exists a closed subscheme $Z_0\subset (A/B)\times\Sigma$ such that $Z = p^*(Z_0)$, where $p\colon A\times\Sigma\to(A/B)\times\Sigma$ is the projection. Moreover, the natural map $Z\to Z_0$ is a categorical quotient, i.e., all B-invariant morphisms $Z\to Y$ factor uniquely through $Z\to Z_0$.*

Proof. To show the existence of Z_0, we apply the theory of faithfully flat descent. By Lemma A.18, we have $q_1^*(Z) = q_2^*(Z)$ as closed subschemes of $B\times A\times\Sigma$. The cocycle condition on $B\times B\times A\times\Sigma$ (cf. [17, equation (14.21.1)]) reduces to the obvious identity $\mathrm{id}_{B\times B\times Z}\circ\mathrm{id}_{B\times B\times Z} = \mathrm{id}_{B\times B\times Z}$. Since the closed immersion $Z\hookrightarrow A\times\Sigma$ is affine, this descent data yields a closed subscheme $Z_0\subset(A/B)\times\Sigma$ such that $Z = p^*(Z_0)$ (cf. [17, Cor. 14.86]). See also [8, Rem. 2.6.2 (iv)]. The restriction $p|_Z\colon Z\to Z_0$ is the base change of p, hence a B-torsor. Hence $Z\to Z_0$ is a categorical quotient (cf. [8, Prop. 2.6.4]). $\square$

Appendix B. Flattening via blow-ups

Lemma B.1. *Let S be a scheme of finite type over $\mathbb{C}$. Let $X\subset\mathbb{P}^N\times S$ be a closed subscheme such that $X\to S$ is flat. Assume that S is reduced. Let $U\subset S$ be a dense Zariski open set. Then the scheme-theoretic closure of $X|_U$ is X. In particular,* $\operatorname{supp} X|_U\subset\operatorname{supp} X$ *is dense.*

Proof. Let $Y\subset\mathbb{P}^N\times S$ be the scheme-theoretic closure of $X|_U$. Then $Y\subset X$ and $Y|_U = X|_U$. Let P be the Hilbert polynomial for each X_s where $s\in S$. Take $s\in S-U$. Let P' be the Hilbert polynomial of Y_s.

Claim. $P' = P$.

We prove this. By $Y_s\subset X_s$, we have $P'\leq P$, i.e., $P'(m)\leq P(m)$ for $m\gg 1$ (cf. [46, Lem. 3.3]). On the other hand, we have $P\leq P'$ by upper semi-continuity of the Hilbert polynomial. Indeed, let $p\colon Y\to S$ be the composite of the closed immersion $Y\hookrightarrow\mathbb{P}^N\times S$ and the second projection $\mathbb{P}^N\times S\to S$. Let $\mathcal{L}$ be the invertible sheaf on Y obtained by the pull-back of $\mathcal{O}_{\mathbb{P}^N}(1)$ by the composite of $Y\hookrightarrow\mathbb{P}^N\times S$ and the first projection $\mathbb{P}^N\times S\to\mathbb{P}^N$. For $t\in S$, we denote by P_{Y_t}

the Hilbert polynomial of Y_t. Hence for $m \gg 1$, we have

$$P_{Y_t}(m) = \dim H^0(Y_t, \mathcal{L}_t^{\otimes m}).$$

By [39, p. 201, Step 3], there exists $l > 0$ such that for every $t \in S$ and every $m \geq l$, the natural morphism

$$(p_*\mathcal{L}^{\otimes m}) \otimes \mathbb{C}(t) \to \Gamma(Y_t, \mathcal{L}_t^{\otimes m})$$

is an isomorphism, and

$$H^j(Y_t, \mathcal{L}_t^{\otimes m}) = \{0\}$$

for all $j > 0$. Hence $P_{Y_t}(m) = \dim\{(p_*\mathcal{L}^{\otimes m}) \otimes \mathbb{C}(t)\}$ for every $t \in S$ and every $m \geq l$. Since the set $\{P_{Y_t}\}_{t\in S}$ is finite (cf. [39, p. 201, Step 2]), we may assume, by enlarging l, that the condition $P \leq P_{Y_t}$ is equivalent to $P(l) \leq P_{Y_t}(l)$ for every $t \in S$. The function $t \mapsto \dim\{(p_*\mathcal{L}^{\otimes l}) \otimes \mathbb{C}(t)\}$ is upper semi-continuous. Therefore the set $\{t \in S;\ P \leq P_{Y_t}\}$ is a Zariski closed subset of S that contains U, and thus coincides with S. This shows $P \leq P'$, which concludes the proof of the claim.

Now, since S is reduced, we observe that $Y \to S$ is flat. Moreover, we have $Y_s = X_s$ for all $s \in S$. Hence $Y = X$. See [17, Prop. 14.26]. □

Lemma B.2. *Let $X \subset \mathbb{P}^N \times S$ be a closed subscheme where S is a variety. Let $Z \subset S$ be an irreducible Zariski closed set. Then there exist a projective birational modification $\varphi\colon S' \to S$ and a non-empty Zariski open set $U \subset S$ with the following properties:*

(1) *$U \cap Z \neq \emptyset$.*
(2) *$\varphi^{-1}(U) \to U$ is an isomorphism.*
(3) *Let $p\colon X \to S$ be the projection and let $X' \subset \mathbb{P}^N \times S'$ be the scheme-theoretic closure of $p^{-1}(U) \subset X \times_S S'$. Then $X'|_{Z'} \to Z'$ is flat, where $Z' \subset S'$ is the Zariski closure of $\varphi^{-1}(Z \cap U) \subset S'$ and $X'|_{Z'} = X' \times_{S'} Z'$.*

Proof. This follows from [46, Lem. 3.1] as follows. For each $s \in S$, let P_{X_s} be the Hilbert polynomial of $X_s \subset \mathbb{P}^N$. Let $\mathcal{P}$ be the set of all Hilbert polynomials of the closed subschemes of $\mathbb{P}^N$. Then $\mathcal{P}$ is a totally ordered set by $P_1 \leq P_2$ if and only if $P_1(m) \leq P_2(m)$ for all large m. Moreover, $\mathcal{P}$ is well ordered (cf. [46, Lem. 8.2]). Note that $\{P_{X_s}\}_{s\in S}$ is a finite set (cf. [39, p. 201, Step 2]). Let $P_{\max}$ be the maximal element in $\{P_{X_s}\}_{s\in S}$. We set

$$T = \begin{cases} \{s \in S;\ P_{X_s} = P_{\max}\} & \text{if } X \neq \emptyset, \\ \emptyset & \text{if } X = \emptyset. \end{cases}$$

Then $T \subset S$ is a Zariski closed subset (cf. [46, Lem. 3.1]).

The proof of our lemma is by transfinite induction on $P_{\max} \in \mathcal{P}$. So we assume that our lemma is true when $P_{\max} < P$ and consider the case $P_{\max} = P$. If $X = \emptyset$, then our lemma is obvious. Hence in the following, we assume $X \neq \emptyset$. Suppose $Z \subset T$. Then for all $s \in Z$, we have $P_{X_s} = P$. Since Z is integral, $X|_Z \to Z$ is flat (cf. [18, Chapter III, Thm. 9.9]). Hence our lemma is true for $S' = S$ and $U = S$. So we consider the case $Z \not\subset T$. Then $T \neq S$. By [46, Lem. 3.1], there exists a closed subscheme $\mathcal{T}$ with $\operatorname{supp} \mathcal{T} = T$ such that if $\widehat{P} = \max\{P_{\widehat{X}_s}\}_{s \in \widehat{S}}$, then $\widehat{P} < P$, where $\widehat{S} = \mathrm{Bl}_{\mathcal{T}} S$ and $\widehat{X} \subset \mathbb{P}^N \times \widehat{S}$ is the strict transform. This strict transform is defined under the isomorphism $\mathbb{P}^N \times \widehat{S} = \mathrm{Bl}_{\mathbb{P}^N \times \mathcal{T}}(\mathbb{P}^N \times S)$. Let $\widehat{Z} \subset \widehat{S}$ be the strict transform.

Now by the induction hypothesis, there exist a projective birational modification $\psi \colon S' \to \widehat{S}$ and a non-empty Zariski open set $U_0 \subset \widehat{S}$ with the following properties:

- $U_0 \cap \widehat{Z} \neq \emptyset$.
- $\psi^{-1}(U_0) \to U_0$ is an isomorphism.
- Let $\hat{p} \colon \widehat{X} \to \widehat{S}$ be the projection and let $\widetilde{X} \subset \mathbb{P}^N \times S'$ be the scheme-theoretic closure of $\hat{p}^{-1}(U_0) \subset \widehat{X} \times_{\widehat{S}} S'$. Then $\widetilde{X}|_{Z'} \to Z'$ is flat, where $Z' \subset S'$ is the Zariski closure of $\psi^{-1}(\widehat{Z} \cap U_0) \subset S'$.

We denote by $\varphi \colon S' \to S$ the composite of $S' \to \widehat{S} \to S$. Note that $\widehat{S} \to S$ is an isomorphism over $S - T$. Hence we may consider $S - T \subset S$ as an open subset of $\widehat{S}$. Set $U = (S - T) \cap U_0$. Then we may consider U as an open subset of S. Then we have $U \cap Z \neq \emptyset$ and $\varphi^{-1}(U) \to U$ is an isomorphism. Since Z is irreducible, the Zariski closure of $\varphi^{-1}(U \cap Z)$ in S' is equal to Z'. Let $X' \subset \mathbb{P}^N \times S'$ be the scheme-theoretic closure of $p^{-1}(U) \subset X \times_S S'$. Then we have $X' \subset \widetilde{X}$, hence $X'|_{Z'} \subset \widetilde{X}|_{Z'}$. On the other hand, we have $X'|_{\varphi^{-1}(U)} = \widetilde{X}|_{\varphi^{-1}(U)}$. Hence by Lemma B.1, the scheme-theoretic closure of $X'|_{\varphi^{-1}(U) \cap Z'} \subset \widetilde{X}|_{Z'}$ is $\widetilde{X}|_{Z'}$. Hence $\widetilde{X}|_{Z'} \subset X'|_{Z'}$, hence $\widetilde{X}|_{Z'} = X'|_{Z'}$. Hence $X'|_{Z'} \to Z'$ is flat. Hence by transfinite induction, the proof is completed. □

Lemma B.3. *In Lemma B.2, if S is smooth, then S' can be taken as smooth.*

Proof. We apply Lemma B.2 to get S' and U. Suppose S' is not smooth. Then we take a smooth modification $S'' \to S'$. Since S is smooth, we may assume that $S'' \to S'$ is an isomorphism over $\varphi^{-1}(U)$. Let $\psi \colon S'' \to S$ be the induced map. Then $\psi \colon S'' \to S$ and $U \subset S$ satisfy properties (1) and (2) in the statement of Lemma B.2. We prove (3). We consider $U \subset S''$. Let $X'' \subset \mathbb{P}^N \times S''$ be the scheme-theoretic closure of $p^{-1}(U) \subset X \times_S S''$. Then we have $X'' \subset X' \times_{S'} S'' \subset X \times_S S''$. This implies the closed immersion $X''|_{Z''} \subset (X' \times_{S'} S'')|_{Z''}$, where $Z'' \subset S''$ is the Zariski closure of $\psi^{-1}(Z \cap U) \subset S''$. Since $Z'' \to S'$ factors $Z' \to S'$, we have $(X' \times_{S'} S'')|_{Z''} = (X'|_{Z'}) \times Z''$. Hence we get $X''|_{Z''} \subset (X'|_{Z'}) \times Z''$. Note that $X''|_{Z'' \cap U} = (X'|_{Z'}) \times (Z'' \cap U) = p^{-1}(U)|_{Z'' \cap U}$. Since $(X'|_{Z'}) \times Z'' \to Z''$ is flat, Lemma B.1 yields $X''|_{Z''} = (X'|_{Z'}) \times Z''$. Hence $X''|_{Z''} \to Z''$ is flat. We replace S' by S'' to complete the proof. $\square$

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