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Combinatorics

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Abstract. Combinatorics is a thriving branch of Mathematics with strong links to other areas of Mathematics and the sciences. Many of the modern techniques employed in Combinatorics draw on diverse ideas from Probability, Analysis, Geometry, Topology, and conversely, combinatorial approaches often lead to new insights in these other areas, and in the broader scientific arena (particularly Theoretical Computer Science). The most recent years in Combinatorics have been particularly exciting with spectacular solutions of many longstanding open problems, via an array of new ideas that promise to have many future applications. Leading experts and new talents came together to share and further develop these exciting ideas. Talks emphasized recent breakthroughs in discrete geometry, graph theory, combinatorial applications in Fourier analysis, group theory, combinatorial probability, Ramsey theory, nonabelian additive combinatorics, and design theory.

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Introduction by the Organizers

The most recent edition of the Oberwolfach Combinatorics workshop was held in the first week of January 2026, organised by Jacob Fox (Stanford), Peter Keevash (Oxford) and Wojciech Samotij (Tel Aviv). The participants were drawn from the leading researchers in the field worldwide, including mathematicians from several European countries, Israel, the United Kingdom, Canada and the United States. The program of the workshop comprised twelve 50-minute-long plenary lectures and thirteen shorter, 25-minute talks. To provide a platform for early career researchers to introduce themselves to the international community, we made a

particular effort to encourage them to offer talks and accommodate them in the schedule of the meeting. A lively, one-hour-long problem session, led by Nati Linial (Jerusalem) was held on Tuesday evening. As always, the daily collaborative discussions between various groups of attendees were evident and testament to the success of this series of meetings in advancing the field.

This report contains extended abstracts of all the lectures given at the meeting as well as a summary of the problems presented at the open problem session. The lectures covered a diverse range of topics in combinatorics, including extremal combinatorics, probabilistic combinatorics and structural graph theory, and also its interactions with other areas of mathematics, particularly analysis, geometry and group theory. As ever, the calibre of the results presented and the quality of the exposition in the lectures was first-rate, so this report can only give an overall impression of the significance of the mathematical achievements presented by our colleagues in this workshop. We will now briefly highlight three of the most spectacular achievements that were communicated during the workshop.

The first highlight was the joint work of Zach Hunter, Matthew Kwan, and Lisa Sauermann on anti-concentration properties of the permanent of a random matrix. The motivation for this work comes from the recent spectacular developments on the problem of estimating the singularity probability of the uniformly random $n \times n$ matrix A_n with ± 1 entries, which has been investigated since the 1960s. A series of papers on this problem that spanned several decades culminated in the recent work of Tikhomirov, who proved the optimal estimate on the logarithm of this probability, which one may write succinctly as $\Pr(\det(A_n) = 0) = (1/2 + o(1))^n$. It is reasonable to expect that a similar, exponential in n, upper bound on the probability that the permanent of A_n equals zero (or any other value) holds as well. However, all known proofs of upper bounds on the singularity probability fundamentally relied on linear-algebraic properties of the determinant, and thus cannot be extended to the superficially-similar permanent. As a result, the state-of-the-art on this problem was the comparatively much weaker estimate $\Pr(\mathrm{per}(A_n) = 0) \leq n^{-c}$, valid for sufficiently small $c > 0$, proved by Tao and Vu in 2009. In the first improvement of this fifteen-year old result, Hunter, Kwan, and Sauermann immediately establish the exponential bound

$$\sup_x \Pr(\mathrm{per}(A_n) = x) \leq \exp(-cn)$$

for some $c > 0$, a major breakthrough! Further, their methods are sufficiently robust to yield analogous bounds not only on permanents of random matrices with independent entries but can also to recover the known estimates on the anti-concentration properties of the determinant. The only assumption that is required on the distribution Q of the entires of the random matrix is the very mild anti-concentration bound $\sup_{x \in \mathbb{R}} Q(x) \leq p$ for some fixed $p < 1$.

The second highlight of the workshop was the joint work of Richard Montgomery and Lisa Sauermann establishing an asymptotic form of Rota's Basis Conjecture. This attractive conjecture, posed by Rota in 1989, states that any family of n bases in a matroid of rank n can be rearranged into a family n bases that is transverse

to the original family. Besides being a natural statement, its generality provides a potential unification of two strands of research that have a large literature in their own right, obtained by specialising matroids to (a) real-representable matroids or (b) graphic matroids. The two main results presented in the lecture by Lisa Sauermann were an asymptotic packing theorem, which finds $n - o(n)$ transversal bases in the original family, an an asymptotic covering theorem, which finds a cover of the original family by $n + o(n)$ transversal bases. The proof is highly complex and grapples with the underlying matroidal concepts to a degree not yet see in previous attempts on the conjcture, so that one feels that this work is a very significant step towards the full conjecture.

The third highlight of the workshop was presented by Aleksa Milojević and Istvan Tomon in two consecutive talks, which covered their joint work with Zhihan Jin and Shengtong Zhang on spectral graph theory and its applications to the maximum cut problem in graphs and to Chowla's cosine problem in harmonic analysis. This work was later featured in an article in Quanta. Their main result shows that a graph with average degree d and smallest eigenvalue λ_n satisfies $|\lambda_n| \leq d^{1-\alpha}$ contains a clique of size $d^{1-O(\alpha)}$. One application gives a substantially better upper bound for the minimum of cosine polynomials. Another application shows that the Edwards bound on the maximum cut of a graph can be substantially improved for graphs that don't contain a very large clique.

Workshop: Combinatorics

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Abstracts

Extended VC-dimension and Radon type theorems for unions of convex sets

Noga Alon
(joint work with Shakhar Smorodinsky)

Radon's theorem states that any set of $d+2$ points in R^d can be partitioned into two subsets whose convex hulls intersect. Formally, given a set $P = \{x_1, x_2, \ldots, x_{d+2}\} \subseteq R^d$, there exists a partition of P into two disjoint subsets P_1 and P_2 such that $\mathrm{CH}(P_1) \cap \mathrm{CH}(P_2) \neq \emptyset$, where here and in what follows $CH(X)$ denotes the convex hull of a set of points X. The bound $d+2$ is tight as one can easily see by taking any set of $d+1$ affine-independent points in R^d, that is, the vertices of the d-dimensional simplex.

This fundamental theorem is a cornerstone in discrete geometry, providing insights into the structure of point sets and their convex combinations. It implies various fundamental theorems in geometry including the classical Helly's Theorem, the center point theorem, and more. Its implications extend to machine learning, statistical learning and computational geometry, influencing algorithms for geometric separation, point location, and convex hull computations. The combinatorial notion of VC-dimension is the analog of Radon's bound for abstract set-systems.

Radon's theorem serves as a basis for numerous generalizations and related results, such as the following classical theorem of Tverberg [4], which further enriches our understanding of geometric configurations.

Theorem 1 ([4]). *Let $r \geq 2$ be a fixed integer and $d \geq 1$. Then for any set P of $(r-1)(d+1)+1$ points in R^d there exists a partition of P into r pairwise disjoint sets $P = \bigcup_{i=1}^{r} P_i$ such that $\bigcap_{i=1}^{r} CH(P_i) \neq \emptyset$.*

Note that Radon's theorem is the special case of Tverberg's theorem with $r = 2$.

In our recent paper [1] described here we settle an open problem posed by Kalai in the 70s and reiterated in later surveys (e.g., [2]) that asks for a generalization of Radon's Theorem (and Tverberg's Theorem) to sets which are not necessarily convex nor even connected, but are the union of a bounded number of convex sets.

Definition 2. Let $s \geq 1$ be an integer. A set C in R^d is said to be s-convex if it is the union of s convex sets.

Problem 3 ([2]). *What is the least integer $f = f(d, s, t)$ such that for any set P of f points in R^d there is a partition $P = A \cup B$ such that any s-convex set containing A must intersect any t-convex set containing B? Is it polynomial in the parameters?*

A more general problem is the following:

Problem 4. *What is the least integer $f = f_r(d, s_1, \ldots, s_r)$ such that for any set P of f points in R^d there is a partition into r pairwise disjoint sets $P = \bigcup_{i=1}^r P_i$ such that for any family of sets $C_1, \ldots, C_r$ with $P_i \subset C_i$ where C_i is an s_i-convex set for every $i \in [r]$ we have $\bigcap_{i=1}^r C_i \neq \emptyset$?*

Note that Radon's theorem is equivalent to $f(d, 1, 1) = d+2$ and more generally Tverberg's theorem is equivalent to $f_r(d, 1, \ldots, 1) = (r-1)(d+1)+1$.

Our main results are the following.

Theorem 5. $f(d, s, t) \leq O(dst \log(st+1))$

Theorem 6. $f_r(d, s_1, \ldots, s_r) \leq O\left(dr^2 \cdot \log r) \cdot \Pi_{i=1}^r s_i \cdot \ln(1 + \Pi_{i=1}^r s_i)\right)$

A notable feature of the proofs is that they are mainly combinatorial and therefore work for abstract separable convexity spaces with bounded Radon numbers.

The bounds in both theorems are not too far from being tight, as shown in a very recent follow-up paper [3], where the authors prove that $f(2, s, s) \geq s^2$ and that for $d \geq 2r-2$, $f_r(d, s, s, \ldots, s) \geq (d-2r+4)s^r$.

A key concept in the proof of Theorem 6 which extends the VC-dimension of a hypergraph is defined below.

Definition 7. Let $H = (V, E)$ be a fixed hypergraph. A subset $S \subset V$ is said to be *r-shattered* by E if for any partition of S into r pairwise disjoint sets $S = \bigcup_{i=1}^r S_i$ there exists hyperedges $e_1, \ldots, e_r \in E$ such that $S_i \subset e_i$ for all $i \in [r]$ and $S \cap \bigcap_{i=1}^r e_i = \emptyset$.

The following combinatorial lemma establishes a connection between the usual VC-dimension of a hypergraph and the above extended notion.

Lemma 8. *There exists an absolute constant C such that for every integer d, any hypergraph $H = (V, E)$ with VC-dimension d and every integer $r \geq 2$, every r-shattered set has size at most $Cdr^2 \log r$. This bound is nearly optimal: for every d and r there is a hypergraph with VC-dimension d that admits an r-shattered set of size $\Omega(dr^2)$.*

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MaxCut in H-free graphs: recent results and connections

Aleksa Milojević
(joint work with Zhihan Jin, István Tomon, Shengtong Zhang)

Given a graph G, a *maximum cut* in G, denoted by $\mathrm{mc}(G)$, is the largest number of edges crossing some bipartition $V(G) = A \cup B$ of its vertex set. MaxCut is among the most extensively studied graph parameters, lying at the intersection of theoretical computer science [12, 15, 16], extremal combinatorics [1, 2, 10, 13] and probabilistic graph theory [8, 7, 9]. In theoretical computer science, one is usually interested in efficient approximations of the maximum cut, while in extremal combinatorics the emphasis is on establishing sharp bounds in terms of various graph parameters, such as the number of vertices or edges.

A simple probabilistic argument shows that every graph with m edges has a cut of size at least $m/2$. Indeed, if one chooses a cut uniformly at random, its expected size is exactly $m/2$. The constant $1/2$ is best possible in general, and therefore it is often more natural to measure the *surplus* of a graph G, defined as $\mathrm{surp}(G) = \mathrm{mc}(G) - m/2$. A fundamental result of Edwards [10, 11] asserts that any graph G with m edges satisfies $\mathrm{mc}(G) \geq \frac{m}{2} + \frac{\sqrt{8m+1}-1}{8}$ or, equivalently, that $\mathrm{surp}(G) \geq \frac{\sqrt{8m+1}-1}{8}$, which is sharp when G is a clique on an odd number of vertices.

In general, if G is a disjoint union of constantly many cliques, then its maximum cut is $m/2 + O(\sqrt{m})$. This naturally raises the question: can one improve this bound if G is far from being a disjoint union of cliques? One way to ensure that a graph is far from a disjoint union of cliques is to forbid some fixed graph H as a subgraph. The study of the MaxCut, and in particular the surplus, in H-free graphs was initiated by Erdős and Lovász in the 1970s [13]. A landmark result in the area is due to Alon [1], who proved that if a graph G has m edges and contains no triangles, then $\mathrm{surp}(G) = \Omega(m^{4/5})$, and this bound is tight.

Extending this line of research, Alon, Bollobás, Krivelevich, and Sudakov [2] conjectured that for every fixed H, there exists a positive constant $\varepsilon_H > 0$ such that every H-free graph G with m edges has surplus at least $\mathrm{surp}(G) \geq \Omega_H(m^{3/4+\varepsilon_H})$.

Two natural classes of graphs H for which this conjecture has been studied are short cycles C_r and complete graphs K_r. The surplus in graphs without short cycles is investigated extensively [2, 3, 5, 14]. On the other hand, determining the minimum surplus in K_r-free graphs is much more difficult. Already Alon, Bollobás, Krivelevich, and Sudakov [2] proved that for every r, there exists $\varepsilon_r > 0$ such that every K_r-free graph has surplus at least $m^{1/2+\varepsilon_r}$. The value of ε_r was improved by Carlson, Kolla, Li, Mani, Sudakov, and Trevisan [6], and further strengthened by Glock, Janzer, and Sudakov [14], who proved $\mathrm{surp}(G) = \Omega_r\big(m^{\frac{1}{2}+\frac{3}{4r-2}}\big)$. Nevertheless, the conjecture of Alon, Bollobás, Krivelevich and Sudakov remains wide open. For many years, it was a tantalizing open problem to find any absolute constant $\varepsilon > 0$, independent of r, such that every K_r-free graph has surplus $\Omega_r(m^{1/2+\varepsilon})$.

Glock, Janzer and Sudakov [14] write *"Arguably, the main open problem is to decide whether there exists a positive absolute constant ε such that any K_r-free graph with m edges has surplus $\Omega_r(m^{1/2+\varepsilon})$."* Here, we present a result that not only resolves this problem, but also shows that one can guarantee such a surplus by forbidding extremely large cliques as well.

Theorem 1. *For every $\delta > 0$ there exists $\varepsilon > 0$ such that the following holds for every sufficiently large m. Let G be a graph with m edges such that G contains no clique of size $m^{1/2-\delta}$. Then G has a cut of size at least $\frac{m}{2} + m^{1/2+\varepsilon}$.*

In the very extreme case, Balla, Hambardzumyan, and Tomon [4] recently showed that graphs with clique number $o(\sqrt{m})$ already have surplus $\omega(m^{1/2})$. Despite the similarity between this result and the previous theorem, there is no implication between the two due to the hidden dependencies. The methods achieving these results are also very different, despite both being algebraic in nature.

Our methods rely on the close relationship between the MaxCut of a graph G and its smallest eigenvalue. Recall that we can associate to every n-vertex graph G the set of eigenvalues of its adjacency matrix, which we denote by $\lambda_1 \geq \cdots \geq \lambda_n$. It is well known that $\mathrm{surp}(G) \leq |\lambda_n| n$ (see e.g. [1]). In many natural cases we have $\mathrm{surp}(G) = \Theta(|\lambda_n| n)$, and so a good way to think about the surplus is as a robust version of the smallest eigenvalue.

More precisely, [17] showed that

$$\mathrm{surp}(G) \geq \sum_{\lambda_i < 0} |\lambda_i| \text{ and } \mathrm{surp}(G) \geq \frac{1}{\Delta(G) \log n} \sum_{\lambda_i < 0} |\lambda_i|^3,$$

where $\Delta(G)$ denotes the maximum degree of G. Using these relations, if G has large negative eigenvalues, one can show that its surplus is large. Thus, it is of interest to study graphs with no very negative eigenvalues. The following theorem guarantees that in this setting, the graph must contain very large cliques.

Theorem 2. *Let $\gamma \in (0,1)$, and let d be a large integer. Let G be a graph of average degree d and assume that the smallest eigenvalue λ_n of G satisfies $|\lambda_n| \leq d^\gamma$. Then G contains a clique of size at least $d^{1-O(\gamma)}$.*

The main technical contribution underlying both of the above results is the following lemma, which shows that if G has no very negative eigenvalues, then almost all the mass in the second moment is carried by the largest eigenvalues. This is encapsulated in the following statement.

Lemma 3. *Let n be an integer, $0 < \varepsilon < 1/4$ be a fixed parameter, and let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of an n-vertex graph G. If $|\lambda_n| \leq n^{1/4-\varepsilon}$, then for every $\delta \in [0,1]$,*

$$\sum_{0 \leq \lambda_i \leq \delta n} \lambda_i^2 \leq O(\delta^\varepsilon n^2).$$

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Chowla's cosine problem

István Tomon
(joint work with Zhihan Jin, Aleksa Milojević, Shengtong Zhang)

In 1948, in the study of certain Dedekind zeta functions, Ankeny and Chowla came across the following question (see [2]): is it true that for every $K > 0$ and sufficiently large $n > 0$, if A is a set of n positive integers, then the minimum of the function $f_A(x) = \sum_{a \in A} \cos(ax)$ is less than $-K$? In 1960, Uchiyama and Uchiyama [9] answered this question affirmatively, but with poor quantitative dependencies.

In 1965, Chowla [3] revisited the problem and made a more precise conjecture, today known as *Chowla's cosine problem*: show that for an n-element set A of positive integers,

$$\min_{x\in[0,2\pi]} f_A(x) = \min_{x\in[0,2\pi]} \sum_{a\in A} \cos(ax) \le -\Omega(\sqrt{n}).$$

In case A can be written as $A = B - B$, where B is a Sidon set, one has $\min_{x\in[0,2\pi]} f(x) = -\Theta(\sqrt{n})$, so if the conjecture is true, the bound $-\Omega(\sqrt{n})$ is the best possible.

The subsequent decades saw a persistent interest in this problem, and the bounds of Uchiyama and Uchiyama were improved by Roth [7] in 1973, who showed that $\min_x f_A(x) \le -\Omega(\sqrt{\log n/\log\log n})$. Soon after, the resolution of the Littlewood L_1-problem in 1981 by Konyagin [5] and McGehee, Pigno and Smith [6] improved this to $\min_x f_A(x) \le -\Omega(\log n)$. It was Bourgain [1] who first broke this logarithmic barrier, and his method was further refined in 2004 by Ruzsa [8] to give the previously best known bound $\min_x f(x) \le -\exp\left(\Omega(\sqrt{\log n})\right)$. Making the next step, we prove the first polynomial bound for Chowla's cosine problem.

Theorem 1. *For any finite set A of positive integers, there exists $x \in [0, 2\pi]$ such that*

$$\sum_{a\in A} \cos(ax) \le -|A|^{1/10-o(1)}.$$

The new key ingredient of the proof of Theorem 1 and the main part of our argument is the following graph-theoretic result.

Theorem 2. *For every $\gamma \in (0, 1/10)$, the following holds for every sufficiently large d. Let G be a graph of average degree d and assume that the smallest eigenvalue λ of G satisfies $|\lambda| \le d^\gamma$. Then G contains a clique of size at least $d^{1-4\gamma}$.*

With this theorem in our hands, Theorem 1 can be proved as follows. We embed A into the cyclic group $\mathbb{Z}/N\mathbb{Z}$ for a sufficiently large prime N and consider the Cayley graph $G = \mathrm{Cay}(\mathbb{Z}/N\mathbb{Z}, A \cup -A)$. It is well-known that the eigenvalues of Cayley graphs correspond to the Fourier coefficients of the generating set, and thus the smallest eigenvalue λ satisfies

$$\lambda = \sum_{a\in A\cup -A} \exp(2\pi iak/N) = 2\sum_{a\in A} \cos(2\pi ak/N)$$

for some $k \in \mathbb{Z}/N\mathbb{Z}$. Hence, $\frac{1}{2}\lambda \ge \min_x f(x)$, so a lower bound on $\min_x f(x)$ yields an upper bound on $|\lambda|$. Then we can apply Theorem 2 to conclude that G contains a very large clique S. Appealing to the transitive symmetry of the Cayley graph, we argue that G contains many large cliques, in particular, $S + t$ is a clique for every $t \in \mathbb{Z}/N\mathbb{Z}$. By studying the interactions of these cliques, we arrive to a contradiction, showing that $|\lambda|$ must be large.

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A large hole in pseudo-random graphs

Michael Krivelevich

(joint work with Sahar Diskin, Itay Markbreit, Maksim Zhukovskii)

We discuss the existence of long induced cycles (also called holes) and paths in pseudo-random graphs. We adopt (n, d, λ)-graphs as a working model for pseudo-random graphs, see [3] for an extensive discussion of (n, d, λ)-graphs and their properties. A graph G is called an (n, d, λ)-graph if it has n vertices, is d-regular, and all non-trivial eigenvalues of G (i.e., of the adjacency matrix of G) are at most λ in their absolute values. The edge distribution in such graphs is governed by the famed Expander Mixing Lemma by Alon and Chung [1].

We show that there exist constants $\delta_1, \delta_2 > 0$ such that if G is an (n, d, λ)–graph with the eigenvalue ratio $\lambda/d \leq \delta_1$, then G contains an induced cycle of length at least $\delta_2 n/d$. This constitutes a substantial improvement over a recent result of Draganić and Keevash [2], who actually argued about long induced paths in pseudo-random graphs. We further demonstrate that, up to a constant factor, this is best possible.

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Long cycles in vertex transitive digraphs

Matija Bucić

(joint work with K. Hendrey, B. Mohar, R. Steiner, and L. Yepremyan)

Finding long paths and cycles in graphs is one of the most classical directions of study in graph theory. Perhaps the most famous instance of this general direction is the question of finding the longest possible cycle, namely one that traverses all the vertices. Such a cycle is called a *Hamilton cycle*, and a graph containing it is said to be *Hamiltonian*. Hamiltonicity is a very classical and extensively studied graph property. In general, it is a hard problem to decide if a given graph is Hamiltonian. In fact, this is one of Karp's famous 21 NP-hard problems [6], and is often used to establish hardness of other computational problems.

This goes a long way towards explaining why there are so many interesting results establishing sufficient conditions for Hamiltonicity. Perhaps the most famous one, featured in essentially every introductory course on graph theory, is Dirac's theorem from 1952, which states that any graph with minimum degree at least $\frac{n}{2}$ is Hamiltonian. A major downside of Dirac's theorem is that it only applies for very dense graphs, leading to a more challenging question of finding natural graph properties that would force Hamiltonicity even for much sparser graphs. Perhaps the most intriguing candidate, first considered in the 1960s, is symmetry. This idea first appeared in a paper by Lovász [8] from 1969, where he conjectured that any connected vertex transitive graph must contain a Hamilton path. Thomassen (see [2]) refined this conjecture in 1978 by asserting that any sufficiently large connected vertex transitive graph is Hamiltonian.

These conjectures have attracted an immense amount of work over the years, with multiple surveys on the topic [16, 11, 1, 3] being written, starting as early as the 1980s. Despite this attention, both conjectures remain widely open and most of what is known concerns various additional assumptions under which the conjectures hold. On the other hand, Babai [2] already in 1979 initiated a very general direction of attack, namely of trying to at least find a long cycle in *every* connected vertex transitive graph (without any additional assumptions). In particular, he proved that an n-vertex connected vertex transitive graph always has a cycle of length $\Omega(\sqrt{n})$. In recent years, there have been several improvements over this result, all making interesting connections to a number of other interesting graph theoretic problems [5, 10, 4, 9] leading to the current state of the art of $\Omega(n^{9/14})$ proved in [10].

Here, we are interested in the directed analogue of this problem. Namely, how long a cycle can we find in any n-vertex connected vertex transitive digraph? This question is even more classical, owing to the fact that in the arguably most interesting instance of the problem, namely that of Cayley digraphs, the directed variant is more natural[1] and translates to a natural group rearrangement problem. Indeed, the two oldest papers to consider this problem started from this

[1]The key difference is that in the digraph case, one deals with arbitrary generating sets, not just symmetric ones, and this is more natural from the group theoretic perspective.

group theoretic question and translated it to the Cayley digraph instance of our problem. These were a 1946 paper by Rankin [12], and an independent 1959 paper by Rapaport-Strasser [13]. Both of these works attribute their motivation to Campanology (as well as the "knight tour" problem in the latter case), where various instances of this problem have been considered in practice, more than a century before (see [12, Section 4] or [15, Chapter 15] for more details).

The directed analog of Thomassen's conjecture was first disproved[2] by Trotter and Erdős [14] in 1978, who exhibited an infinite family of connected vertex transitive digraphs without Hamilton cycles. Motivated by this result, Alspach asked already in 1981 whether such graphs need to be at least "nearly" Hamiltonian. Here, to formalize this question, we can use the *perimeter gap*, defined as the difference between the number of vertices and the length of the longest directed cycle in a digraph, as a measure of how far from Hamiltonian a graph is. In particular, Alspach (Question 7 in [1]) asked if for any constant C there exists a connected vertex transitive digraph with perimeter gap larger than C. We answer this question in the affirmative in the following quantitative form.

Theorem 1. *For infinitely many natural numbers n, there exists a connected vertex transitive digraph on n vertices with perimeter gap at least $(1 - o(1)) \ln(n)$.*

Given this result, the question of how long of a cycle we can actually guarantee, raised by Babai in the undirected case already in 1979, becomes even more natural in the directed case. Here, it was not even known whether one can guarantee a cycle of length growing with n. We prove such a result, establishing a directed analog of Babai's result from 1979.

Theorem 2. *In any connected vertex transitive digraph D on $n \geq 2$ vertices, there is a directed cycle of length at least $\Omega(n^{1/3})$.*

We note that neither Babai's result nor the aforementioned recent results improving on it extend to the directed case. This is because all of these proofs at various places crucially rely on the fact that in a 2-connected graph G, any two longest cycles must intersect. However, an analogous statement fails badly for strongly 2-connected digraphs, even if they are assumed to be 2-out- and 2-in-regular, namely, there exist strongly 2-connected 2-regular digraphs with arbitrarily many vertex-disjoint longest directed cycles.

In fact, our arguments provide slightly longer directed paths in place of directed cycles, namely of length at least $\Omega(\sqrt{n})$. Using this, we also get a new proof of Babai's result in the undirected setting where the longest cycle and path are always within a constant factor of each other, see [7].

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Exponential anticoncentration of the permanent

Matthew Kwan
(joint work with Zach Hunter, Lisa Sauermann)

Two important matrix parameters are the *determinant* and *permanent*: for an $n \times n$ matrix $A = (a_{ij})$, they are defined as

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sign}(\pi) \prod_{i=1}^{n} a_{i\pi(i)} \qquad \text{and} \qquad \operatorname{per}(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i\pi(i)}. \tag{1}$$

In combinatorics, computer science and random matrix theory, it is of great interest to understand statistical properties of determinants and permanents, among various classes of discrete matrices. In particular, one of the most intensive directions of research is the study of the singularity probability of random sign matrices. Let A_n be a uniformly random $n \times n$ matrix with ± 1 entries; what is the probability that $\det(A_n) = 0$? Perhaps the foundational theorem in combinatorial random matrix theory, due to Komlós [10], is that $\Pr[\det(A_n) = 0] = o(1)$ as $n \to \infty$. A subsequent breakthrough by Kahn, Komlós and Szemerédi [8] (simplified by

Tao and Vu [14]) showed that the singularity probability is in fact *exponentially small*: that is to say, $\Pr[\det(A_n) = 0] = \exp(-cn)$ for some constant $c > 0$. Quantitative aspects were later improved by Tao and Vu [15] and by Bourgain, Vu and Wood [2], culminating in a celebrated result of Tikhomirov [18] proving the essentially optimal bound $\Pr[\det(A_n) = 0] \le (1/2 + o(1))^n$.

It is reasonable to expect similar results for the permanent. However, the permanent is much more difficult to study than the determinant, and much less is known: until now, the only theorem in this direction is due to Tao and Vu [17], who proved that

$$\Pr[\mathrm{per}(A_n) = 0] \le n^{-c}$$

for some constant $c > 0$ (see also [16] for an earlier, simpler version of their argument that gives a bound of shape $\exp(-c\sqrt{\log n})$). Although Tao and Vu did not attempt to optimise the constant c, they note that their argument cannot hope to obtain a stronger bound than about $1/\sqrt{n}$, owing to their use of the so-called *Erdős–Littlewood–Offord inequality* (they write that "in principle, one can obtain better results by using more advanced Littlewood-Offord inequalities, but it is not clear to the authors how to restructure the rest of the argument so that such inequalities can be exploited").

In this paper, we overcome this barrier, obtaining the first exponential bound.

Theorem 1. *There is an absolute constant $c > 0$ such that the following holds. Let A_n be a uniformly random $n \times n$ matrix with ± 1 entries. Then, for any $x \in \mathbb{R}$,*

$$\Pr[\mathrm{per}(A_n) = x] \le \exp(-cn).$$

It is worth remarking that our proof works in exactly the same way for the determinant instead of the permanent, so as a byproduct this also gives a new proof of the Kahn–Komlós–Szemerédi theorem. All previous work on determinants of random matrices fundamentally used linear-algebraic properties of the determinant; since such properties are unavailable for the permanent, our proof is necessarily very different. In fact, our proof is also very different to the previous approach of Tao and Vu in [17] (which was until now the only known approach to study permanents of random matrices).

General entry distributions. Some of the previous work on determinants and permanents of discrete random matrices readily generalises to other entry distributions. For example, shortly after Komlós' foundational paper showing that $\Pr[\det(A_n) = 0] = o(1)$, in a second paper [11] he proved that the same result holds for a random matrix with independent μ-distributed entries, for any non-degenerate distribution μ (fixing μ and sending $n \to \infty$).

Bourgain, Vu and Wood [2] observed that in fact one does not need to fix the entry distribution; it suffices for the entries to be independent and "uniformly anticoncentrated" in the sense that there is a constant $\varepsilon > 0$ such that each entry a_{ij} satisfies $\sup_z \Pr[a_{ij} = z] \le 1 - \varepsilon$. Specifically, they observed (see [2, Theorem 1.4]) that some of their ideas could be combined with an approach of Tao and Vu [14], to prove that $\Pr[\det(A_n) = 0]$ decays exponentially in this very general setting. Here, we prove the same statement for the permanent.

For a random variable X, we write $Q[X] = \sup_x \Pr[X = x]$ for the maximum point probability of X.

Theorem 2. *For any $0 < p < 1$ there is $c_p > 0$ such that the following holds. Let $A_n \in \mathbb{R}^{n\times n}$ be a random matrix with independent entries, such that every entry ξ of A_n satisfies $Q[\xi] \le p$. Then*

$$Q[\mathrm{per}(A_n)] \le \exp(-c_p n).$$

Note that Theorem 1 is an immediate corollary of Theorem 2 for $p = 1/2$. We remark that while Theorem 1 might be improvable to a super-exponential bound (as conjectured by Vu [20, Conjecture 6.12]), in the more general setting of Theorem 2 an exponential bound is best-possible. Indeed, if the entries in the first row of A_n are each equal to zero with probability p, then with probability p^n the entire first row is zero, in which case $\mathrm{per}(A_n) = 0$.

The range of the permanent. For any set $S \subseteq \mathbb{R}$ and any $n \in \mathbb{N}$, we write $\Phi_S(n)$ for the set of all permanents of $n \times n$ matrices with entries in S.

For any discrete random variable X, it is easy to see that X must be supported on at least $1/Q[X]$ different values. So, by applying Theorem 2 to random matrices whose entries are uniformly sampled from any pair $\{a, b\} \subseteq \mathbb{R}$, we obtain an exponential lower bound on $|\Phi_{\{a,b\}}(n)|$.

Corollary 3. *For any set $S \subseteq \mathbb{R}$ of size at least 2, taking $c_{1/2}$ as in Theorem 2, we have*

$$|\Phi_S(n)| \ge \exp(c_{1/2} n),$$

This result is not new when $S = \{0, 1\}$. Indeed, in this case a classical theorem of Brualdi and Newman [3] says that $\Phi_{\{0,1\}}(n) \supseteq \{0, \dots, 2^{n-1}\}$ (see also the improvements in [5]). However, Corollary 3 significantly improves on previous results when $S = \{-1, 1\}$. Indeed, in 1993 Kräuter [12] showed that $|\Phi_{\{-1,1\}}(n)| \ge n + 1$. This has remained essentially the state of the art until very recently, when Ingram and Razborov [7] showed how to use ideas from additive combinatorics and Diophantine geometry to prove that $|\Phi_{\{-1,1\}}(n)| \ge \exp(c(\log n)^2/\log\log n)$ for some constant $c > 0$. Actually, they explicitly raised the problem (see [7, Problem 1]) of proving that $|\Phi_{\{-1,1\}}(n)|$ grows exponentially in n; this problem is solved by Corollary 3.

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C_4-free subgraphs of high degree with geometric applications

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(joint work with Zach Hunter, Aleksa Milojević, István Tomon)

In 1953, Zarankiewicz [26] asked for the maximum number of edges in an n-vertex graph G that contains no copy of $K_{s,s}$, the complete bipartite graph with two parts of size s. Just a year later, Kővári, Sós, and Turán [18] proved that such graphs have at most $O_s(n^{2-1/s})$ edges—a bound that remains unimproved to this day, although it is known to be tight only for $s \in \{2, 3\}$. On the other hand, a simple application of the probabilistic deletion method yields a lower bound of $\Omega_s(n^{2-2/(s+1)})$ for $s \geq 4$ (see, e.g., [2]), which, up to polylogarithmic factors, is the best known to date.

Over the past two decades, Zarankiewicz-type problems have been extensively studied in various restricted graph families, motivated by questions from both combinatorial geometry and structural graph theory. In many geometric incidence problems, it is natural to study $K_{s,s}$-free graphs, as this condition often

corresponds to an inherent geometric constraint. For instance, the incidence graph of points and lines in $\mathbb{R}^2$ is always $K_{2,2}$-free, while in point-hyperplane incidence graphs, avoiding $K_{s,s}$ reflects a natural non-degeneracy assumption [3, 22].

More generally, refined bounds for Zarankiewicz-type problems have been obtained in a variety of settings: for intersection graphs of geometric objects [10, 11, 12, 17, 25], incidence graphs involving points and geometric shapes [7, 13, 17, 22], semialgebraic and semilinear graphs [5, 13, 22], graphs of bounded VC-dimension [13, 16], and for visibility graphs [1, 8]. In all these cases, the classical Kővari-Sós-Turán bound can be substantially improved. For example, it was shown in [13] that a $K_{s,s}$-free graph of VC-dimension d has at most $O_{d,s}(n^{2-1/d})$ edges – a significant improvement over the $O(n^{2-1/s})$ bound when $d \ll s$. In some cases, the bound is even linear: for example, [10] showed that a $K_{s,s}$-free intersection graph of n curves in the plane has at most $O_s(n)$ edges. Beyond merely improving the bounds, these works also bring together a rich toolkit of techniques such as partitioning methods, variants of ε-nets and separator theorems that have deepened our understanding of the underlying geometry of these problems.

These results reveal an interesting general phenomenon: in many restricted graph families, the extremal behavior is not governed by s, as it is in general graphs, but instead reflects intrinsic structural parameters such as the ambient dimension or the VC-dimension, while s may only appear in the constant factor.

In parallel with this line of work, a similar phenomenon has been observed in structural graph theory. One of the first results in this direction is due to Kühn and Osthus [19], who showed that if a $K_{s,s}$-free n-vertex graph G does not contain an induced subdivision of a fixed graph H, then G has at most $O_{H,s}(n)$ edges. A *subdivision* of the graph H is any graph which can be obtained by replacing the edges of H by disjoint paths. Thus, the answer to Zarankiewicz's problem changes substantially in the family of graphs avoiding induced subdivisions of H.

Since then, analogous behavior has been observed in other families of interest, such as graphs avoiding an induced copy of a tree T [15, 24], and families defined by forbidding an induced bipartite graph [6, 15]. In fact, the general phenomenon can be described more abstractly, as follows. If $\mathcal{F}$ is a family of graphs closed under taking induced subgraphs and $\mathcal{F}$ excludes at least one bipartite graph H, then there exists $\varepsilon = \varepsilon(\mathcal{F}) > 0$ such that every $K_{s,s}$-free n-vertex graph in $\mathcal{F}$ has at most $O_s(n^{2-\varepsilon})$ edges. This was implicitly proven by [6, 13, 14], and the value of ε was later improved for various families $\mathcal{F}$ by [4, 15].

In this talk, we focus on Zarankiewicz-type problems in geometrically defined graphs, where the extremal number is expected to be linear or nearly linear. We uncover a new structural result about graphs avoiding high-degree induced C_4-free subgraphs, which we use to improve and unify a wide range of Zarankiewicz-type results for geometric graph families, obtaining tight bounds in each case.

Intersection graphs of curves. The *intersection graph* of a family of sets or geometric objects $\mathcal{F}$ is the graph whose vertices correspond to the elements of $\mathcal{F}$, with an edge between any two sets that intersect. A particularly interesting class of intersection graphs are *string graphs*, which are defined as the intersection

graphs of families of planar curves (which are sometimes also called strings). Here, a *planar curve* stands for the image of a continuous function $\phi : [0, 1] \to \mathbb{R}^2$.

Pach and Sharir asked in [23] what is the largest number of edges in a $K_{s,s}$-free string graph on n vertices. Separator theorems can be used to show that such graphs have at most $2^{O(s)}n$ edges. Further, since string graphs do not contain subdivisions of non-planar graphs as induced subgraphs, and in particular no subdivision of K_5, the theorem of Kühn and Osthus [19] discussed above directly shows that the number of edges in $K_{s,s}$-free string graphs is $O_s(n)$. However, the point of the work of Fox and Pach, together with the subsequent works, is to pinpoint the correct dependence on s.

These bounds were then further improved [12, 21] until Lee [20] established the optimal bound of $O(s \log s)$, using optimal bounds on the size of balanced separators in string graphs. Although the existence of small separators is a powerful tool, with a wide range of theoretical and algorithmic applications, it is only available for a limited selection of graph families. In this work, we provide an alternative proof of the optimal $O(s \log s)$ bound which relies on our key graph-theoretic tool, Theorem 2, and avoids the use of separators altogether.

Theorem 1. *Any n-vertex $K_{s,s}$-free string graph has $O(s(\log s)n)$ edges.*

Key graph-theoretic tool: finding C_4-free subgraphs or dense patches. Our main theoretical contribution, and the unifying theme of this work, is the following statement, which says that every graph G either contains a C_4-free induced subgraph with large average degree or a dense subgraph, whose size is at least the average degree of G.

Theorem 2. *For every positive integer k there exists $c = c(k) > 0$ such that the following holds. Every graph G of average degree at least d either contains an induced C_4-free subgraph with average degree at least k, or it contains a subgraph on d vertices with at least cd^2 edges.*

In a typical application, we attempt to bound the average degree of a $K_{s,s}$-free graph from some geometrically defined class by $d \leq O(s)$, which corresponds to bounding the number of edges by $O(sn)$. To do this, the first step is to bound the number of edges of a C_4-free graph from this family by Cn, for some absolute constant C, which corresponds to addressing the case when $s = 2$. Then, given a $K_{s,s}$-free graph G from this geometric family, we apply our Theorem 2 with any $k > C$, and observe that, since G does not contain C_4-free subgraphs of average degree at least k, it must contain a subgraph on d vertices with cd^2 edges. The second step is to show that in such dense graphs one can find bicliques of size $\Omega(d)$. In particular, this means that $s \geq \Omega(d)$, i.e. that $d \leq O(s)$.

Graphs without dense subgraphs. Finally, we highlight a strengthening of Theorem 2, which can be used to settle a recent conjecture of Fox, Nenadov, and Pham [9]. A natural relaxation of $K_{s,s}$-freeness, introduced in [9], is the notion of (c, t)-sparsity. A graph G is said to be (c, t)-*sparse* if for all subsets $A, B \subseteq V(G)$ with $|A|, |B| \geq t$, there are at most $e(A, B) \leq (1-c)|A||B|$ edges between them. It

is easy to see that a graph is $K_{s,s}$-free if and only if it is $(1/s^2, s)$-sparse. Another motivation for studying (c,t)-sparse graphs is that random graphs $G(n,p)$ are (c,t)-sparse for any $c < 1-p$ and $t = \Omega_{p,c}(\log n)$.

In [9], the authors showed that if $c > 0$ is a constant and T is a tree, then every (c,t)-sparse graph with at least Ctn edges contains an induced copy of T, where $C = C(T,c)$ is sufficiently large with respect to T and c. They conjectured that the same statement holds if instead of a tree T, we seek an induced subdivision of a fixed graph H. We prove this conjecture.

Theorem 3. *For every $c > 0$ and every graph H, there exists a constant C such that if G is a (c,t)-sparse graph with average degree at least Ct, then G contains an induced subdivision of H.*

In particular, this theorem immediately follows from the following strengthening of Theorem 2, after recalling that C_4-free graphs of sufficiently large average degree contain an induced subdivision of H (Kühn and Osthus [19]).

Theorem 4. *Let $k \geq 1, c > 0$, then there exists $C = C(k,c)$ such that the following holds. Let G be a (c,t)-sparse graph with average degree at least Ct. Then G contains a bipartite C_4-free induced subgraph of average degree at least k.*

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On the largest k-product-free subsets of the Symmetric and Alternating Groups

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(joint work with Peter Keevash)

A subset A of a group G is said to be *k-product-free* if for all

$$a_1, a_2, \ldots, a_k \in A, a_1 a_2 \ldots a_k \notin A.$$

A natural question presents itself: How dense is the largest k-product-free subset of G?

Product-free sets in groups have been the subject of extensive study; see the surveys [9] and [12] for an overview. Writing $m_k(G)$ to denote the largest density of a k-product-free subset of G, it is known [6], for example, that $m_2(G) \geqslant 2/7$ for finite abelian groups G. In contrast, in his seminal work on *quasirandom* groups, Gowers [4] showed that there is no constant $c > 0$, so that $m_2(G) \geqslant c$ holds for all finite groups G.

The question for larger k has also received some attention. When $G = (\mathbb{Z}, +)$, Calkin and Erdős [1] conjectured that the extremal upper density is $\rho(k)^{-1}$, where

$$\rho(k) := \min\{l \in \mathbb{N} : l \nmid k - 1\}.$$

This was established by Łuczak and Schoen [11]. Illingworth, Michel, and Scott [7] have recently shown that the result extends to the non-abelian setting, in the case G is the free group over a finite alphabet.

Amongst the class of non-abelian groups, the symmetric groups $\mathrm{Sym}(n)$ are arguably the first to merit consideration. Determining $m_k(G)$ is quite easy for $G = \mathrm{Sym}(n)$ when $k = 2\ell$ is even. The odd permutations form a 2ℓ-product-free

set, and it's not too hard to see that one cannot do better, that is, $m_{2\ell}(\mathrm{Sym}(n)) = 1/2$. So, when k is even, the interesting question to ask is what is the largest set of *even* permutations that form a k-product-free set. In other words, what is $m_k(\mathrm{Alt}(n))$?

When $k = 2$, Crane [5] and Kedlaya [8] provided a construction that showed $m_2(\mathrm{Alt}(n)) \gtrsim n^{-1/2}$. In the other direction, Gowers [4] showed that $m_2(\mathrm{Alt}(n)) \lesssim n^{-1/3}$, which was improved by Eberhard [2] to $n^{-1/2}\log^{7/2}(n)$. In a recent breakthrough, Keevash, Lifshitz, and Minzer [10] showed that the Crane–Kedlaya construction is optimal for sufficiently large n.

Keevash, Lifshitz, and Minzer [10, Problem 7.1] also raised the more general problem of determining $m_k(G)$ for permutation groups G. We provide the following construction of a k-product-free set of permutations, which is analogous to the Calkin–Erdős construction in the integers. Consider the set of permutations

$$\{\sigma \in \mathrm{Sym}(n) : \sigma(i) = i + 1 \pmod{\rho(k)}, \text{for } i \in [\rho(k)]\}.$$

This set is easily seen to be k-product-free and so $m_k(G) \geqslant n^{-\rho(k)}$ for $G = \mathrm{Sym}(n)$ or $\mathrm{Alt}(n)$. We prove the following theorem, which determines the order of growth of $m_k(\mathrm{Sym}(n))$ and $m_k(\mathrm{Alt}(n))$ for $k \geqslant 8$.

Theorem 1. *For $k \geqslant 8$, it holds that $m_k(\mathrm{Alt}(n)) = \Theta_k(n^{-\rho(k)})$. Further, if $k \geqslant 8$ is odd, then $m_k(\mathrm{Sym}(n)) = \Theta_k(n^{-\rho(k)})$.*

In contrast, for $k \leqslant 5$, there are constructions which perform much better than the standard construction above. We already saw this in the case $k = 2$. When $k = 3$, define for $x, y \in [n]$ the set of permutations

$$G_y^x := \{\pi : \pi(x) = y, \pi(y) \neq x\}.$$

Note that G_y^x is 3-product-free and its density is $(1 - o(1))n^{-1}$. In particular, G_y^x is larger than the standard construction, which has density around n^{-3}. We show that the largest 3-product-free sets are of the form G_y^x for sufficiently large n, and further, that stability holds.

Theorem 2. *Let $G = \mathrm{Sym}(n)$ or $\mathrm{Alt}(n)$. For large n, any 3-product-free set $A \subseteq G$ with $\mu(A) \geqslant (1 - 2^{-18})n^{-1}$ is a subset of some G_y^x.*

We also consider the cross problem, both for its own sake, and as a tool in the other proofs. We call a tuple $(A_0, A_1, \ldots, A_k)$ of subsets of $\mathrm{Alt}(n)$ k-product-free if there are no solutions to the equation $a_1 a_2 \cdots a_k = a_0$, with $a_i \in A_i$. Let $f_k(n)$ denote the maximum of $\min\{\mu(A_0), \ldots, \mu(A_k)\}$ among product-free tuples $(A_0, A_1, \ldots, A_k)$ in $\mathrm{Alt}(n)$. Eberhard [2] proved that $f_2(n) = \frac{\log^{O(1)}(n)}{\sqrt{n}}$, which is tight up to the implicit constant. We show the following result.

Theorem 3. $f_3(n) = \frac{\log^{O(1)}(n)}{n}$.

Note that, as $f_k(n)$ decreases monotonically with k, the theorem implies that $f_k(n) \leqslant \frac{\log^{O(1)}(n)}{n}$ for $k \geqslant 3$. However, letting $\mathcal{U}_{i\to j} := \{\sigma \in \mathrm{Alt}(n) : \sigma(i) = j\}$,

one sees that $(\mathrm{Alt}(n) \setminus \mathcal{U}_{k\to 0}, \mathcal{U}_{1\to 0}, \mathcal{U}_{2\to 1}, \ldots, \mathcal{U}_{k\to k-1})$ is k-product-free and so, in fact, $f_k(n) = \frac{\log^{O(1)}(n)}{n}$ for $k \geqslant 3$.

Our approach builds on the work of Keevash, Lifshitz, and Minzer in the case $k = 2$. The principal technical ingredient is the theory of hypercontractivity in $\mathrm{Sym}(n)$ recently developed by Filmus, Kindler, Lifshitz and Minzer [3]. Our main technical contribution is the introduction of combinatorial gadgets, which we call *diagrams*, to enable a density-increment argument.

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Polynomial Bogolyubov for Special Linear Groups via Tensor Rank

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(joint work with Shai Evra and Guy Kindler)

Bogolyubov's lemma is a foundational tool in additive combinatorics. In the context of vector spaces over finite fields, it states that if $A \subseteq \mathbb{F}_q^n$ is a sufficiently dense subset, the sumset $2A - 2A$ contains a large subspace. The quantitative dependency of the subspace's co-dimension on the density of A is a central question. While Sanders [2] established a quasi-polynomial bound—showing a co-dimension of $O_q(\log^4(1/\alpha))$ for a set of density α—obtaining a polynomial bound remains a major open problem in the abelian setting.

In this work, we investigate the analogue of this problem for the non-abelian group $G = SL_n(\mathbb{F}_q)$ in the regime where the rank n grows. Our main result establishes a polynomial Bogolyubov-type lemma for these groups.

Theorem 1 (Polynomial Bogolyubov for $SL_n(\mathbb{F}_q)$). *There exists an absolute constant $C > 0$ such that for every $n \in \mathbb{N}$, every prime power q, and every subset $A \subseteq SL_n(\mathbb{F}_q)$ of density $\mu(A)$, the set $AA^{-1}AA^{-1}$ contains a subgroup H of density*

$$\mu(H) \geq \mu(A)^C.$$

Moreover, H is a "good groumvirate," meaning it is a conjugate of a subgroup of the form

$$L_k = \left\{ \begin{pmatrix} I_k & 0 \\ 0 & X \end{pmatrix} : X \in SL_{n-k}(\mathbb{F}_q) \right\},$$

where I_k is the $k \times k$ identity matrix.

This result guarantees that not only does the product set contain a large subgroup, but the subgroup itself retains the structure of a special linear group of a slightly smaller rank.

Product Mixing and Global Sets. The proof of Theorem 1 relies on a robust "quasirandomness" framework for $SL_n(\mathbb{F}_q)$. We identify a pseudorandomness condition we call *globalness*. Roughly speaking, a set A is global if it does not correlate significantly with "dictators" (sets of matrices fixing a specific vector mapping) or their low-degree generalizations ("umvirates").

We prove that global sets exhibit strong mixing properties. Specifically, if subsets $A, B \subseteq SL_n(\mathbb{F}_q)$ are global, their convolution $1_A * 1_B$ is close to the constant function in L^2-norm.

Theorem 2 (Global Product Mixing). *Let $A, B \subseteq SL_n(\mathbb{F}_q)$ be global sets of density at least q^{-cn^2}. Then*

$$\|1_A * 1_B - \mu(A)\mu(B)\|_2 \leq q^{-n/4}\mu(A)\mu(B).$$

A corollary of this mixing is that for global sets A, B, C, the product ABC covers the entire group. This allows us to deduce Theorem 1 by first finding a large structured subset (a "good umvirate") inside A where A behaves globally, and then arguing that the product set fills this structure.

Analytic Framework: Tensor Rank and The Bilinear Scheme. To prove the mixing results, we develop a spectral theory for $L^2(SL_n(\mathbb{F}_q))$ that departs from the standard Peter-Weyl decomposition into irreducible representations based on dimension alone. Instead, we use the Gurevich-Howe notion of *tensor rank* [3]. An irreducible representation ρ has tensor rank k if it appears in the decomposition of $\omega^{\otimes k}$ (where ω is the standard permutation representation on $\mathbb{F}_q^n$) but not in $\omega^{\otimes(k-1)}$.

This creates a filtration of $L^2(SL_n(\mathbb{F}_q))$ into levels $V_{=d}$, analogous to the Fourier levels of the Boolean cube. The key technical challenge is to establish "level-d inequalities" which bound the weight of global functions on low tensor rank representations.

To achieve this, we map functions from the non-abelian group $SL_n(\mathbb{F}_q)$ to the abelian "bilinear scheme" $\mathcal{L}(V, V)$ (the space of all linear maps). We leverage

the connection between tensor rank in the group and the concept of *juntas* in the bilinear scheme. We then utilize a strengthened *Bonami-type hypercontractive inequality* for global functions on the bilinear scheme.

Theorem 3 (Bonami-Type Inequality for Bilinear Scheme). *Let $f \in L^2(\mathcal{L}(V,W))$ be a global function of degree d, and let ℓ be a power of 2. Then*

$$\|f\|_\ell^\ell \leq q^{O(d^2\ell^2)}\|f\|_2^2\epsilon^{\ell/2-1}.$$

This hypercontractivity allows us to show that global functions have negligible mass on representations of low tensor rank. Combined with the Sarnak-Xue trace method [5] and dimension lower bounds for tensor rank representations by Guralnick, Larsen, and Tiep [4], this yields the required mixing bounds.

Applications. Beyond the Polynomial Bogolyubov lemma, our methods yield several other results:

- **Product-Free Sets:** We improve Gowers' bounds for product-free sets in $SL_n(\mathbb{F}_q)$. We also provide a stability result: any dense product-free set must have large intersection with a t-umvirate (a stabilizer of a t-tuple of vectors).
- **Approximate Subgroups:** We obtain structural results for K-approximate subgroups in the high-rank regime. We show that any K-approximate subgroup A is contained in the union of few cosets of a polynomial-sized good groumvirate.

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A bipartite Bollobás-Eldridge-Catlin conjecture

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(joint work with Julia Böttcher, Jozef Skokan, Benny Sudakov)

Bollobás, Eldridge and Catlin [2, 3] conjectured that any n-vertex graph H with maximum degree Δ is contained in any n-vertex graph G with minimum degree at least $\left(1 - \frac{1}{\Delta+1}\right)n$. This conjecture remains open, though it is easy for $\Delta = 1$ and was proved for $\Delta = 2$ by Aigner and Brandt [1], and $\Delta = 3$ by Csaba, Shokoufandeh and Szemerédi [5]. For larger Δ, a weaker minimum degree condition than the conjecture was proved by Sauer and Spencer [7] and improved by Kaul, Kostochka and Yu [6].

In 2007, Csaba [4] proved the special case when H is bipartite. In fact, he was able to prove more: the minimum degree required for G can be reduced slightly for any constant Δ and sufficiently large n. Csaba's proof uses the Regularity Lemma, and hence the dependence of n on Δ is poor.

We return to this problem and determine the correct dependence of the minimum degree. There are constants $c, C > 0$ such that, for all sufficiently large n, if $\delta(G) \geq \big(1 - \frac{c \log \Delta}{\Delta}\big)n$, then G contains all bipartite n-vertex graphs H with maximum degree Δ, and there exists a graph G' with $\delta(G') \geq \big(1 - \frac{C \log \Delta}{\Delta}\big)n$ which does not contain all such bipartite graphs. Our proof does not use the Regularity Lemma, and works for any $\Delta \leq n^{1/6}$.

Two open problems are to determine the behaviour if 'bipartite' is replaced with 'tripartite' or generally k-partite for fixed k and sufficiently large Δ, or if H is not required to be partite but F-free for some fixed graph F.

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A polynomial improvement for the odd cycle–complete Ramsey numbers

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(joint work with Marcelo Campos, Marcus Michelen, Florian Pfender and Julian Sahasrabudhe)

The *cycle-complete* Ramsey number $r(C_\ell, K_k)$ is the smallest n so that every graph on n vertices has either a cycle of length ℓ or an independent set of size k. Here our interest lies in the case when ℓ is fixed and odd and k tends to infinity. The study of these numbers goes back to the foundational work of Erdős [4] who proved

$$c_\ell k^{1+\frac{1}{2\ell}} \leq r(C_\ell, K_k) \leq k^{1+\frac{2}{\ell-1}}, \tag{1}$$

where the upper bound holds for all odd $\ell > 1$ (he proved a similar result when ℓ is even) and the lower bound was proved using an early application of the probabilistic method. The lower bound was then improved by Spencer [8] in 1977 who

used the Lovász local lemma to show

$$r(C_\ell, K_k) \geq k^{1+\frac{1}{\ell-2}+o(1)}. \tag{2}$$

While this bound has been improved by logarithmic factors by Bohman and Keevash [1] and by Verstraëte and Mubayi [6], the exponent in (2) has stood as the best known bound on these numbers, apart from the special cases $\ell \in \{5, 7\}$ (see [3, 6]). In this work we give the first polynomial improvement to this bound for all odd $\ell > 7$.

Theorem 1. *For all odd $\ell > 7$, we have that*

$$r(C_\ell, K_k) \geq k^{1+\frac{1}{\ell-2}+\epsilon_\ell+o(1)},$$

where $\epsilon_\ell = \Theta(\ell^{-2})$.

In fact we can take $\epsilon_\ell = ((\ell-2)(2\ell-5))^{-1}$ in the statement of our main theorem.

What is perhaps most interesting about this result is that the bound (2) represents a natural barrier for both this and many other problems in combinatorics. Indeed, a simple way to construct a C_ℓ-free graph with small independence number is to sample $G \sim G(n,p)$ and then delete an edge from each instance of C_ℓ. Here one needs to choose p so that

$$p^\ell n^\ell \ll pn^2,$$

to ensure that only a small fraction of the edges of G are removed. This is known as *the deletion threshold* for the cycle C_ℓ and the bound at (2) reflects the independence number of $G(n,p)$ precisely at this deletion threshold. In many cases in combinatorics we know little about how to obtain good examples that are significantly beyond this threshold.

In this work we show that we can push well beyond this deletion threshold for the cycle complete Ramsey numbers, using a new idea of randomly superimposing blow-ups of random graphs. This idea is due to Hefty, Horn, King and Pfender [5], which was inspired by a previous idea of Campos, Jenssen, Michelen and Sahasrabudhe [2]. In [5], this yields the lower bound on the extreme off diagonal Ramsey numbers $R(3,k)$

$$R(3,k) \geq \left(\frac{1}{2} + o(1)\right)\frac{k^2}{\log k},$$

which is within a factor of $2 + o(1)$ of the best upper bound due to Shearer [7], and is conjectured to be sharp [2].

It seems reasonable to guess, for odd ℓ, that Erdős's bound (1) gives the correct exponent,

$$r(C_\ell, K_k) = k^{1+\frac{2}{\ell-1}+o(1)},$$

which has been tentatively conjectured and supported by some recent conditional results [3]. It appears that our construction cannot be directly used to obtain an improvement to the lower bound beyond a factor of n^{ϵ_ℓ} where $\epsilon_\ell = \Theta(\ell^{-2})$. It would be particularly interesting to obtain an improvement of the form n^{ϵ_ℓ} where $\epsilon_\ell = \Theta(\ell^{-1})$.

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An improved upper bound for the multicolour Ramsey number of odd cycles

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(joint work with Maria Axenovich, Wouter Cames van Batenburg, Lukas Michel, Mathieu Rundström)

The *k-colour Ramsey number* $R_k(H)$ of a graph H is the smallest integer n such that every k-edge-colouring of the complete graph K_n contains a monochromatic copy of H. For the triangle, the notoriously difficult Schur-Erdős problem asks to determine the growth rate of $R_k(C_3)$. In 1917, Schur [12] showed that

$$\Omega(3^k) \leq R_k(C_3) \leq O(k!).$$

Since then, the upper bound has remained unchanged up to small improvements to the constant factor, while the lower bound has been improved to $\Omega(3.28^k)$ by Ageron, Casteras, Pellerin, Portella, Rimmel, and Tomasik [1]. Erdős conjectured that $R_k(C_3) = 2^{\Theta(k)}$ [3] and offered monetary awards for proving this conjecture and solving some related problems.

For longer odd cycles, Bondy and Erdős [2] and Erdős and Graham [5] obtained the bounds

$$\ell \cdot 2^k + 1 \leq R_k(C_{2\ell+1}) \leq 2\ell \cdot (k+2)!.$$

If k is fixed, it turns out that the lower bound is sharp as Jenssen and Skokan [9] proved that $R_k(C_{2\ell+1}) = \ell \cdot 2^k + 1$ for all sufficiently large ℓ. However, this is not true if ℓ is fixed. Indeed, Day and Johnson [4] showed that for all ℓ there exists some $\delta = \delta(\ell) > 0$ such that $R_k(C_{2\ell+1}) \geq 2\ell \cdot (2+\delta)^{k-1}$ for all sufficiently large k.

Usually, if the number of colours is large, longer odd cycles should be easier to find than shorter odd cycles. For instance, Fox [6] conjectured that for every $\varepsilon > 0$ there exists some ℓ such that $R_k(C_{2\ell+1}) \leq k^{\varepsilon k}$ for all sufficiently large k. Li [10] even made the stronger conjecture that $R_k(C_{2\ell+1}) \leq o(k!^{1/\ell})$ as $k \to \infty$. Nevertheless, the gap between these conjectures and the best upper bounds known

remained large. Li [10] proved that $R_k(C_5) \le c^k \cdot \sqrt{k!}$ for some constant c, which was later extended by Lin and Chen [11] to longer odd cycles by showing that for $\ell \ge 2$ we have $R_k(C_{2\ell+1}) \le c^k \cdot \sqrt{k!}$ for some constant c that only depends on ℓ.[1] Only under the wide open additional assumption that each Ramsey graph for $R_k(C_{2\ell+1})$ is nearly regular[2], Li [10] showed that this bound could be improved to $R_k(C_{2\ell+1}) \le c^k \cdot k!^{1/\ell}$.

In this paper, we prove the conjecture of Fox.

Theorem 1. *For $k, \ell \in \mathbb{N}$,*

$$R_k(C_{2\ell+1}) \le (4\ell-2)^k \cdot k^{k/\ell} + 1.$$

Using the well-known inequality $(k/e)^k < k!$, this bound implies that $R_k(C_{2\ell+1}) \le c^k \cdot k!^{1/\ell} + 1$ for $c := (4\ell-2)e^{1/\ell}$, and so this establishes the conditional upper bound of Li unconditionally.

In addition to finding a monochromatic odd cycle of a specific length, there has also been considerable interest in finding any short monochromatic odd cycle. It is easy to see that there is a k-edge-colouring of K_{2^k} without any monochromatic odd cycles, but that every k-edge-colouring of K_{2^k+1} contains such a cycle. Motivated by this, in 1973, Erdős and Graham [5, Question (iii) in Section 6] asked for the smallest integer $L(k)$ such that every k-edge-colouring of K_{2^k+1} contains a monochromatic odd cycle of length at most $L(k)$.

Day and Johnson [4] proved that $L(k) \ge 2^{\Omega(\sqrt{\log k})}$. Recently, Girão and Hunter [7] obtained the first non-trivial upper bound, showing that $L(k) \le (2^k + 1)/k^{1-o(1)}$. Using an algebraic approach, Janzer and Yip [8] improved this to $L(k) \le O(k^{3/2} \cdot 2^{k/2})$.

Both Girão and Hunter [7] and Janzer and Yip [8] also discussed the more general problem of finding short monochromatic odd cycles in colourings of complete graphs with more vertices. In the regime where the number of vertices is $(2+\delta)^k$ for some small $\delta > 0$, the methods of both papers can guarantee a monochromatic odd cycle of length at most $O_\delta(k)$. The dependence on δ is better in [8], where the length of the cycle is $O(\delta^{-1/2} \cdot k)$ if $\delta < 1$. With our method, we can find significantly shorter monochromatic odd cycles, unless δ tends to 0 very quickly.

Theorem 2. *For $k \in \mathbb{N}$ and $b > 2$, every k-edge-colouring of K_n with $n > b^k$ contains a monochromatic odd cycle of length at most $2\lceil \log_{b/2} k \rceil + 1$.*

In particular, if $b = 2 + \delta$ for $0 < \delta < 1$, this result guarantees a cycle of length $O(\delta^{-1} \cdot \log k)$. In addition, the result also applies to larger values of b. If $b = k^\varepsilon$, this yields a result similar to Theorem 1, but without controlling the exact cycle length. We remark that Theorem 2 does not give any non-trivial result for the Erdős–Graham problem, where $\delta \approx 1/(k \cdot 2^{k-1})$.

[1] These results were proved independently by Fox [6], but were never published.

[2] That is, there is an absolute constant ε such that for all sufficiently large k, the minimum degree of every Ramsey graph for $R_k(C_{2\ell+1})$ is at least an ε-fraction of its average degree.

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More on balancing extensions

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(joint work with Max Aires)

All (new) results mentioned here are from [1] and [2].

Definitions. We use P for both a poset and its underlying set. Elements x, y of P are *comparable*, denoted $x \sim y$, if $x < y$ or $x > y$. The sets of maximal and minimal elements of $X \subseteq P$ are $\max(X)$ and $\min(X)$. An *ideal* of P is $I \subseteq P$ satisfying $x < y \in I \Rightarrow x \in I$. A *chain* is a linearly ordered set, and an *antichain* is a set with no relations $x < y$. The *height*, $\mathrm{ht}(P)$, is the maximum size of a chain of P. We assume throughout that P has size n.

A *(linear) extension* of P may be viewed as either an order-preserving bijection from P to $[n]$, or a linear ordering of P respecting the poset relations. The set of extensions is $E(P)$ and $e(P) = |E(P)|$. A uniformly random extension is either f or $\prec$ depending on whether we are thinking of bijections or orderings. We use $p(x \prec y)$ for $\mathbb{P}(f(x) < f(y))$.

Central quantities are: $\delta_{xy} = \min\{p(x \prec y), p(y \prec x)\}$; $\delta(P) = \max \delta_{xy}$ (the max over distinct $x, y \in P$); and $h(x) = h_P(x) = \mathbb{E}f(x)$, the *average height* of x.

Balance. Perhaps the best known of all poset problems is:

Conjecture 1 (The "1/3-2/3 Conjecture")**.** *If P is finite and not a chain then*

$$\delta(P) \geq 1/3. \tag{1}$$

This was proposed by Kislitsyn [11], Fredman and Stanley (both circa 1975), and Linial [13], in large part motivated by questions about sorting. From the algorithmic standpoint what's wanted is existence of *some* universal $\delta > 0$ such that $\delta(P) \geq \delta$ for every P not a chain.

Such existence was shown in [9], with $\delta = 3/11$ and a proof based mainly on the Aleksandrov-Fenchel Inequalities. Easier proofs with smaller δ's, based on the Brunn-Minkowski Theorem, were given (independently) in [10] and [8]. The current best bound, $(5-\sqrt{5})/10$ due to Brightwell, Felsner and Trotter [3], is *best possible* for the infinite version of the problem. At this writing there is no known proof of *any* fixed (positive) lower bound that does not depend on real geometry.

While Conjecture 1 is tight, one expects to do better except in quite special situations, e.g. as in the following natural guess from [9].

Conjecture 2 ("Kahn-Saks Conjecture"). *If $w(P) \to \infty$, then*

$$\delta(P) \to 1/2. \tag{2}$$

The best previous progress here is a celebrated result of Komlós [12]:

Theorem 3. *If $|\min(P)| = \Omega(n)$ then* (2) *holds (as $n \to \infty$).*

Another interesting example: Chan, Pak and Panova [4] show (2) for large Young diagrams of *bounded* width.

The next theorem summarizes some of what we can say regarding (2).

Theorem 4. *If P satisfies any one of the following then* (2) *holds:*

(a) $|\min(P)| \gg \log n$;

(b) $w(P) = \Omega(n)$;

(c) $\min_{x \in P} h(x) \gg 1$.

Each of (a),(b) greatly improves on Theorem 3, and (b) gives (2) for any fixed $\mathrm{ht}(P)$, which is new even for $\mathrm{ht}(P) = 3$. A nice point of (c) is that, like Conjecture 2 (and unlike other results here), it assumes only that something is large in *absolute* terms, rather than in terms of n.

The following result of Friedman [5] gives the most general conditions previously known to imply (1), which [5] shows in the stronger form

$$\delta(P) > 1/e - o(1). \tag{3}$$

The $1/e$ is natural, corresponding to Grunbaum's result [6] that at least $1/e$ of the volume of a compact convex $K \subseteq \mathbb{R}^d$ lies on each side of any hyperplane through its centroid. (An approximate version of [6] for log-concave densities is important for the work discussed here.)

Theorem 5 ([5]). *If P satisfies either of the following then* (3) *holds:*

(a) $|\min(P)| \gg \sqrt{n}$;

(b) $\mathrm{ht}(P) < 2\log_2 \log n - \omega(1)$.

Theorem 5(a), like Theorem 3, is (much) weaker than Theorem 4(a), while (b) is subsumed by (b) of the next result (which follows easily from (c)).

Theorem 6. *If P satisfies any one of the following then* (3) *holds:*

(a) $w(P) \gg \sqrt{n}$;

(b) $\mathrm{ht}(P) \ll n$;

(c) $\log e(P) \gg n$.

Theorems 4 and 6(a) are *local* in that the well-balanced pairs can be chosen from $\min(P)$ in Theorem 4(a,c), and any antichain of size $\Omega(w(P))$ in the other cases. In this sense Theorems 4(a,c) and 6(a) are essentially best possible, but we expect Theorem 4(b) is not.

Parameters. Much of what we do involves understanding and exploiting relations between several old and new parameters of a poset, including: $h(x)$; $\sigma^2(x) = \mathrm{Var}(f(x))$;

$$\pi(x) = |\{y \in P - x : y \not\sim x\}|;$$

$$\mathrm{win}(x) = \mathbb{E}[r(x) - q(x)],$$

where $r(x) = \min\{f(y) : y > x)\}$ and $q(x) = \max\{f(y) : y < x)\}$; and, with $P = \{x_1, \ldots, x_n\}$ and $h(x_1) \leq \cdots \leq h(x_n)$,

$$\mathrm{gap}(P) = \max\{h(x_1), h(x_2) - h(x_1), \ldots, h(x_n) - h(x_{n-1}), n + 1 - h(x_n)\}.$$

For $\alpha \in \{\sigma^2, \pi, \mathrm{win}\}$, we set $\alpha(P) = \max\{\alpha(x) : x \in P\}$.

For parameters α and β, we use $\alpha \rightsquigarrow \beta$ to mean that β tends to infinity if α does. Some of these implications are trivial or nearly so, namely $\alpha \rightsquigarrow \pi$ for $\alpha \in \{\mathrm{gap}, w, \mathrm{win}, \sigma\}$, and $\mathrm{win} \rightsquigarrow \sigma$; but the next three are more substantial.

Theorem 7. $w \rightsquigarrow \mathrm{win}$.

Theorem 8. $\mathrm{gap} \rightsquigarrow w$.

Theorem 9. $\pi \rightsquigarrow \sigma$.

(So we also have $w \rightsquigarrow \sigma$ and $\mathrm{gap} \rightsquigarrow \mathrm{win}$.) On the other hand it's easy to see that there are no $\rightsquigarrow$'s for the sequence $(\sigma, \mathrm{win}, w, \mathrm{gap})$, and that $\pi \not\rightsquigarrow \mathrm{win}$ (so also $\pi \not\rightsquigarrow w, \mathrm{gap}$). The following possible salvage of the trivially false $\sigma \rightsquigarrow \mathrm{win}$ has been particularly frustrating.

Conjecture 10. *For every P and x,* $\mathrm{win}(x) = \Omega(\sigma^2(x)/n)$.

We don't even know that one can't have $\sigma^2(x) = \Omega(n^2)$ while $\mathrm{win}(x) = O(1)$ (respectively the largest and smallest possible values).

The implications $w \rightsquigarrow \mathrm{win} \rightsquigarrow \sigma \rightsquigarrow \pi$ are again "local," the first in the sense that a large antichain must contain an element with large win, and the others in the sense that (e.g.) large $\mathrm{win}(x)$ implies large $\sigma(x)$. But $\pi \rightsquigarrow \sigma$ is *not* local (e.g. let P consist of two unrelated chains of size $n/2$ with x minimal in one of them),

and Theorem 9, the most difficult of the results reported here, was a surprise. We close with a few comments related to Theorems 7-9.

The following quantified version of Theorem 7, which is sharp when $P = A$, is essential for Theorem 4(b) and all parts of Theorem 6.

Theorem 11. *For any ideal A in a poset P,* $\sum_{x\in A} \mathrm{win}(x) \geq (n+1)|A|^2/n$.

Regarding Theorem 8, we mention two possible strengthenings, the first of which, which would be definitive, is an old conjecture [7].

Conjecture 12. *For any ideal A,* $\max\{h(x) : x \in A\} \geq |A| - |\max(A)| + 1$.

This is true (mainly because of [14]) and tight when $A = P$, but in general we don't know that $|A| - \max\{h(x) : x \in A\}$ is bounded by *any* function of $|\max(A)|$.

Conjecture 13. *For any P,* $\mathrm{gap}(P) = O(w(P))$.

This, with Theorem 4, would have a nice corollary which, curiously, we don't know to be true:

if P contains some x comparable to no other element then it satisfies (2).

(It is *not* true that such an x must itself belong to a pair witnessing (2).)

To conclude, we mention two easy applications of the combination of Theorem 9 and a relative of Theorem 8, viz.

Theorem 14. *For fixed w, P of width at most w, and $x \in P$, $\delta_x \to 1/2$ as $\sigma(x) \to \infty$.*

The first application concerns another old conjecture [7]:

Conjecture 15. *If $\pi(P) \to \infty$ then* (2) *holds.*

This seems a considerable—and perhaps less plausible, since its hypothesis is so weak—strengthening of Conjecture 2; but, quite unexpectedly, the two turn out to be equivalent:

Theorem 16. *Conjecture 2 implies Conjecture 15.*

Finally, the next statement greatly extends the (difficult) result of [4] mentioned following Therorem 3 (since a Young diagram is just an ideal of $\mathbb{N}^2$).

Theorem 17. *For fixed d and w, and P an ideal of $\mathbb{N}^d$ with $w(P) \in [2, w]$,*

$$\delta(P) \to 1/2 \text{ as } |P| \to \infty.$$

To get this from Theorems 9 and 14, we mainly need to add the easy fact that $\pi(P)$ is large whenever P is a large ideal of $\mathbb{N}^d$ and not a chain.

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Complexity measures of tree decompositions

Maria Chudnovsky

For a graph G, a *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi\colon V(T) \to 2^{V(G)}$ with the following properties:

(1) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
(2) For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
(3) For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

This is a way to represent a general graph using a tree (which is structurally a much simpler object). Such a representation is a useful tool for understanding the structure of a graph; it also often has algorithmic applications.

There are several ways to measure the complexity of a given tree decomposition. Classically, the *width* of (T, χ) is the maximum cardinality of $\chi(t)$, where the maximum is taken over all $t \in V(T)$. The *treewidth* of a graph G, denoted by $tw(G)$, is the smallest width of a tree decomposition of G.

We denote by $W_{t\times t}$ the t-by-t hexagonal grid, and by $K_{t,t}$ the complete bipartite graph with both sides of the bipartition of size t. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. A graph class $\mathcal{C}$ is said to be *of bounded treewidth* if there exists an integer c such that $tw(G) < c$ for every $G \in \mathcal{C}$.

Traditionally tree decompositions were studied in the context of graph minors; the dichotomy between graph classes of bounded and unbounded treewidth is completely understood from this perspective. This is captured in the celebrated *Grid Minor Theorem* [6].

Theorem 1. *For every positive integer t there exists an integer $c(t)$ such that every graph that does not contain a minor isomorphic to $W_{t\times t}$ has treewidth at most $c(t)$.*

An equivalent statement can be formulated in terms of subgraphs [6]:

Theorem 2. *For every positive integer t there exists an integer $c(t)$ such that every graph that does not contain a subgraph isomorphic to a subdivision of $W_{t\times t}$ has treewidth at most $c(t)$.*

A graph H is an *induced minor* of a graph G if H can be obtained from an induced subgraph of G by contracting edges, and deleting all parallel edges created in the process. In the last decade much attention has been directed to understanding the behavior of tree decompositions in the context of induced subgraphs, and more recently induced minors. For a positive integer t, we denote by $\mathcal{C}_t$ the class of $\{K_{t,t}, W_{t\times t}\}$-induced-minor-free graphs, and by $\mathcal{C}_t^*$ be the subclass of $\mathcal{C}_t$ consisting of all graphs with no complete subgraph of size t.

A necessary condition for getting a class of bounded treewidth is that a large complete graph is forbidden as a subgraph, and a large complete bipartite graph is forbidden as an induced minor. Unfortunately, a direct analogue of Theorems 1 and 2 is false even if these additional conditions are taken into account. However, several variants of these theorems remain of interest.

First, instead of studying graph classes of bounded treewidth, one can ask whether, given a graph class $\mathcal{C}$ there exists a slow-growing function $f_{\mathcal{C}}$ such that for every n-vertex graph $G \in \mathcal{C}$, $tw(G) < f(n)$. The following conjecture has become known in the area:

Conjecture 3. *For every $t \in \mathbb{N}$ there is an integer $d = d(t)$ such that every n-vertex graph $G \in \mathcal{C}_t^*$ with $n > 2$ satisfies $tw(G) \leq \log^d n$.*

This is still open, but a weaker result where the poly-logarithmic bound on treewidth is replaced by a subpolynomial one was proved in [1].

Theorem 4. *For every $t \in \mathbb{N}$, there exist $\epsilon = \epsilon(t) \in (0, 1]$ and $c = c(t) \in \mathbb{N}$ such that every n-vertex graph G in $\mathcal{C}_t^*$ satisfies $tw(G) \leq 2^{c \log^{1-\epsilon} n}$.*

Next, one can use parameters other then cardinality to measure the complexity of a tree decomposition. Let G be a graph, $S \subseteq V(G)$ be a vertex set in G and r be a positive integer. The *distance-r-independence number* of S, denoted by $\alpha_r(S)$, is the size of the largest subset $I \subseteq S$ such that for every pair $u, v \in I$, the distance between u and v in G is greater than r. We define α_r of a tree decomposition, and tree-α_r of a graph analogously to width and treewidth. Tree-α_1 was introduced in [5] as a tool for studying tree decompositions of graph classes containing complete graphs of unbounded size, and the following conjecture was proposed (where $\omega(G)$ is the largest size of a complete subgraph of G):

Conjecture 5. *Let $\mathcal{C}$ be a class of graphs closed under induced subgraphs. Then, if there exists a function f such that for every $G \in \mathcal{C}$ it holds that $tw(G) \leq f(\omega(G))$, then $\mathcal{C}$ has bounded tree-independence number.*

Conjecture 5 received a significant amount of attention in the graph theoretic community, and was disproved in [4] (even if f is a polynomial function). However, the following variant is true [3]:

Theorem 6. *Let $\mathcal{C}$ be a class of graphs closed under induced subgraphs. Then, for every positive integer c there exists a positive integer d with the following property. If for every n-vertex graph $G \in \mathcal{C}$ with $n > 2$ it holds that $tw(G) \leq (\omega(G) \log n)^c$, then for for every n-vertex graph $G \in \mathcal{C}$ with $n > 2$ it holds that tree-$\alpha_1(G) \leq \log^d n$.*

The following is the best possible analogue of Conjecture 3 when large complete graphs are permitted:

Conjecture 7. *For every positive integer t there is an integer $d = d(t)$ such that every n-vertex graph $G \in \mathcal{C}_t$ with $n > 2$ satisfies tree-$\alpha_1(G) \leq \log^d n$.*

The following is shown in [2]:

Theorem 8. *For every positive integer t there is an integer $d = d(t)$ such that every n-vertex graph $G \in \mathcal{C}_t$ with $n > 2$ satisfies tree-$\alpha_{16(\log n+1)}(G) \leq \log^d n$.*

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Covering high-dimensional Euclidean spaces with random translates of a given convex body

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(joint work with Boris Bukh, Jun Gao, Xizhi Liu, Shumin Sun)

Let $K \subseteq \mathbb{R}^n$ be a *convex body* (i.e. a compact convex set with non-empty interior). A set $P \subseteq \mathbb{R}^n$ is called a *K-covering* of a set $A \subseteq \mathbb{R}^n$ if A is a subset of

$$P + K := \{\mathbf{p} + \mathbf{x} \colon \mathbf{p} \in P \text{ and } \mathbf{x} \in K\},$$

that is, the union of the translates of K by the elements of P contains every point of A. Let $\mathrm{vol}(A)$ denote the Lebesgue measure of A. For a measurable $A \subseteq \mathbb{R}^n$ with $\mathrm{vol}(A) < \infty$, the *K-covering density* of a set $P \subseteq \mathbb{R}^n$ over A is defined as

$$\vartheta_K(P, A) := \sum_{\mathbf{p} \in P} \frac{\mathrm{vol}\big(A \cap (K + \mathbf{p})\big)}{\mathrm{vol}(A)}.$$

The *K-covering density* of P over $\mathbb{R}^n$ is defined by

$$\vartheta_K(P) := \limsup_{r \to \infty} \vartheta_K\big(P, [-r, r]^n\big).$$

The fundamental result of Rogers [Rog57] states that, for every convex body $K \subseteq \mathbb{R}^n$, there exists a covering of $\mathbb{R}^n$ by translates of K with density at most $(1 + o(1))n \ln n$ as $n \to \infty$. If K is a ball then the covering density was improved to $(1/2 + o(1))n \ln n$ by Dumer [Dum07]. Both of these constructions can be obtained by taking a set X of N random centres in a fundamental domain of a suitable lattice Λ in $\mathbb{R}^n$ and extending this periodically. If Λ is a *packing* lattice for K (that is, the translates of K by the elements of Λ are pairwise disjoint) and its determinant is at most $\mathrm{e}^{O(n)}\,\mathrm{vol}(K)$ then the proofs in [Rog57, Dum07] show that, for an appropriate choice of N, a typical N-set X gives a covering of $\mathbb{R}^n$ *whp* (i.e. with probability $1 - o(1)$ as $n \to \infty$).

On the other hand, the best known lower bound on covering density for balls is $\big(\mathrm{e}^{-3/2} + o(1)\big)\, n$ by Coxeter, Few and Rogers [CFR59]. Closing this gap for ball covering is a major open problem.

Recently, we proved the following results in [BGL$^+$25, GLPS26].

Theorem 1.3 in [BGL$^+$25] shows that, for any given convex body $K \subseteq \mathbb{R}^n$, the above random construction, under the extra assumption that Λ is a packing lattice for $K - K$, fails whp to be a K-covering if the appropriately defined density is strictly less than $(1/2 + o(1))n \ln n$. Thus the above results give that, for every convex body $K \subseteq \mathbb{R}^n$, the density threshold for a random set (defined as above) to produce a K-covering is between $(1/2 + o(1))n \log n$ and $(1 + o(1))n \log n$, with the lower bound being sharp when K is a ball. Interestingly, [BGL$^+$25, Theorem 1.5] shows that a "worst" convex body is a cube for which the threshold is $(1 + o(1))n \log n$. Theorems 1.3 and 1.5 in [BGL$^+$25] are proved by applying the second moment to a suitable discrete net.

Also, [BGL$^+$25, Theorem 1.1] shows that there are ball coverings of $\mathbb{R}^n$ attaining the asymptotically best known density $(1/2 + o(1))\, n \ln n$ such that, additionally, every point of $\mathbb{R}^n$ is covered at most $(1.79556... + o(1))\, n \ln n$ times. This strengthens a result of Erdős and Rogers [ER62] who had the maximum multiplicity at most $(\mathrm{e} + o(1))\, n \ln n$.

The paper [GLPS26] considers *lattice coverings* (where the set of the centers has to be a lattice) and shows that there exists a lattice covering of $\mathbb{R}^n$ by Euclidean balls of equal radius with density $O\big(n \ln^\beta n\big)$ as $n \to \infty$, where

$$\beta := \frac{1}{2} \log_2 \left(\frac{8\pi \mathrm{e}}{3\sqrt{3}} \right) = 1.85837... .$$

This improves upon the previously best known upper bound by Rogers [Rog59] of $O\big(n \ln^{\alpha} n\big)$, where $\alpha := \frac{1}{2}\log_2(2\pi e) = 2.0471...$. The proof in [GLPS26] uses the same idea as in [Rog59]: we start with a random lattice Λ' in $\mathbb{R}^{n-k}$ (which is known to behave as a Poisson point process with respect to the fraction of the space which is covered by $\Lambda' + K'$, provided $\mathrm{vol}(K') \le e^{O(n)} \det(\Lambda')$) and then analyse the effect of adding d extra dimensions (and d random vectors) on the covering properties of $K' \times B^d$, where B^d is a d-dimensional ball. While Rogers [Rog59] uses $d = 1$, we show that taking $d = 2$ can lead to the above improvement. It is unclear what the best possible upper bound given by this method is.

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The entropy method and the mixture bound

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(joint work with Ting-Wei Chao, Maya Sankar)

The mixture bound was first rigorously introduced in [4] to derive hypergraph Turán results, although its trace can already be found in [2] where it was applied to give an entropic proof of Lovász's version of the Kruskal–Katona theorem. In this talk, I will focus on a new application of the mixture bound to the rainbow triangle problem.

The rainbow triangle problem asks the following: in a simple graph with R red edges, G green edges and B blue edges, what is the maximum number of rainbow triangles? If we take a properly-colored K_4 and blow each vertex up into an independent set of size N, then there are $2N^2$ edges of each color, and there are $4N^3$ rainbow triangles. A way to write this is that there are $\sqrt{2RGB}$ rainbow triangles. In [2], we show that there are always at most $\sqrt{2RGB}$ rainbow triangles, showing that this construction is optimal. The original proof used entropy method to some extent, but there was still a large portion of the proof that was not phrased in terms of entropies. We later found a proof that was purely entropic in [3], which was based on a carefully crafted injection and an encoding argument. There is also a recent work [1] by Balogh, Bradshaw, Garcia and Lidický that gave another injection-based proof inspired by a flag-algebraic proof which is significantly

simpler than the earlier proofs. However, the proof that I will present today is a new proof using the mixture bound that is due to Maya Sankar. Using the idea in the proof, we can generalize the rainbow triangle theorem to more colors and show that a simple r-edge-colored graph with m edges has at most $\sqrt{\frac{2(r-2)}{9r}}m^{3/2}$ rainbow triangles. This is tight when there is an affine plane of order $r-1$, where we take the $(r-1)^2$ points as the vertices and connect any two points with an edge colored with the slope of the line connecting the two points.

In this talk, we will still focus on the 3-colored case. Let us begin by examining what a naïve application of Shearer's inequality would give. Let T be the number of rainbow triangles. Let (X_1, X_2, X_3) be a uniformly chosen rainbow triangle ordered uniformly. Then by Shearer's inequality,

$$2\log_2 6T = 2H(X_1, X_2, X_3) \leq 3H(X_1, X_2)$$

To compute $H(X_1, X_2)$, we use the chain rule and the uniform bound to get

$$H(X_1, X_2) = H(\chi(X_1X_2)) + H(X_1, X_2 \mid \chi(X_1X_2)) \leq \log_2 3 + \frac{1}{3}\log_2 8RGB.$$

and rearranging gives $T \leq \sqrt{6RGB}$. To improve this, we will use the mixture bound.

A mixture Z of random variables $X_1, \ldots, X_n$ is sampled by first sampling $i \in [n]$ and then sampling X_i. A special case of the mixture bound says that if $X_1, \ldots, X_n$ are random variables with disjoint supports, then there is a mixture Z of $X_1, \ldots, X_n$ so that

$$\sum_{i=1}^{n} 2^{H(X_i)} \leq 2^{H(Z)}.$$

Now let Y_i be a conditional resampling of X_i given X_{i-1}, X_{i+1} and $\chi(X_iX_{i-1})$. Let Z_{i+1} be a conditional resampling of X_{i+1} given X_{i-1}, Y_i. Then it can be checked that $(X_1, X_2, X_3, Y_i, Z_{i+1})_{i=1}^3$ and $(X_1, X_2, X_3, Z_{i+1}, Z_i)_{i=1}^3$ have disjoint supports. By the mixture bound and symmetry, there is a mixture (X_1, X_2, X_3, A, B) of the six 5-tuples such that

$$6 \cdot 2^{H(X_1,X_2,X_3,Y_i,Z_{i+1})} \leq 2^{H(X_1,X_2,X_3,A,B)} \leq 2^{H(X_1,X_2,X_3)+H(X_1,X_2)},$$

showing that

$$H(Y_i, Z_{i+1} \mid X_1, X_2, X_3) \leq H(X_1, X_2) - \log_2 6.$$

This factor of 6 contains the factor of $\sqrt{3}$ that we aim to improve from the naïve Shearer bound, and in fact, with some simple entropic calculation, this gives the desired upper bound.

Let us end with an open problem. When there is an affine plane of order $r-1$ (say when $r-1$ is a prime power), we know that the upper bound $\sqrt{\frac{2(r-2)}{9r}}m^{3/2}$ can be attained with some simple r-edge-colored graph. In fact, when such an affine plane does not exist, we can improve this upper bound to $(\sqrt{\frac{2(r-2)}{9r}} - \varepsilon)m^{3/2}$ for some ε only depending on r. It would be interesting to determine this constant

exactly for r when there is no affine planes of order $r-1$. For example, when $r = 7$ or $r = 11$.

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Asymptotically-tight packing and covering with transversal bases in Rota's basis conjecture

Lisa Sauermann

(joint work with Richard Montgomery)

Rota's basis conjecture is a famous conjecture concerning the rearrangeability of bases in a vector space or matroid. Rota posed it in 1989 (see [9, Conjecture 4]), both in the special case of vector spaces, and for general matroids. Given any n bases $B_1, \dots, B_n$ of an n-dimensional vector space (or, more generally, in a matroid of rank n), the conjecture states that the elements in each of the bases $B_1, \dots, B_n$ can be ordered in such a way that the first elements of $B_1, \dots, B_n$ together form a basis, the second elements of $B_1, \dots, B_n$ together form a basis, and so on. In other words, the (multi-)set, $B_1 \cup \dots \cup B_n$ can be decomposed into n disjoint bases each consisting of exactly one element from each of the original bases $B_1, \dots, B_n$. Saying that such a basis containing exactly one element from each of $B_1, \dots, B_n$ is a *transversal basis*, Rota's basis conjecture can be restated as follows.

Conjecture 1 (Rota's Basis Conjecture)**.** *Any collection of n disjoint bases in a rank-n matroid can be decomposed into n transversal bases.*

Despite a great deal of attention, including as the subject of the 12th Polymath project [12] in 2017, Conjecture 1 remains widely open, even for the case of vector spaces (i.e., representable matroids). The conjecture has only been confirmed in some very special cases, including for 'paving matroids' by Geelen and Humphries [6], for 'strongly base orderable matroids' by Wild [14], and for matroids of rank at most 4 by Cheung [3]. As discovered by Huang and Rota [9] (and later simplified by Onn [10]), Rota's basis conjecture for real-representable matroids of rank n, for even n, can be reduced to the Alon–Tarsi conjecture for Latin squares of order n. The Alon–Tarsi conjecture was proved when $n-1$ is prime by Drisko [5] and when $n+1$ is prime by Glynn [8], thus together giving a proof of Conjecture 1 for real-representable matroids of rank n when $n-1$ or $n+1$ is an odd prime.

In addition to these special cases, there is some further evidence towards Conjecture 1. In 2006, Aharoni and Berger [1] proved that the natural fractional relaxation of Conjecture 1 holds. More recently, the speaker [13] showed that the conjecture holds with high probability for representable matroids over finite fields if the n bases are chosen independently and uniformly at random.

There are two natural directions for working towards Conjecture 1 in full, as was highlighted by Pokrovskiy [11], and which were the focus of this talk. The first of these is the *packing problem* in this setting: given n bases in a rank-n matroid, how many disjoint transversal bases can we find? The second is the *covering problem* in this setting: given n bases in a rank-n matroid, how many transversal bases do we need to cover all of the elements of these bases?

Our first main result discussed in this talk gives an asymptotically-tight answer to the packing problem.

Theorem 2 (Asymptotic Packing Theorem)**.** *For any $\varepsilon > 0$, the following holds whenever n is sufficiently large with respect to ε. Any collection of n bases of a rank-n matroid has at least $(1-\varepsilon)n$ disjoint transversal bases.*

This improves earlier bounds for the number of transversal bases in this setting, namely a bound of $\Omega(\sqrt{n})$ by Geelen and Webb [7], of $\Omega(n/\log n)$ by Geelen and Dong [4], and of $(1/2-\varepsilon)n$ by Bucić, Kwan, Pokrovskiy, and Sudakov [2].

Our second main result gives an asymptotically-tight answer to the covering problem.

Theorem 3 (Asymptotic Covering Theorem)**.** *For any $\varepsilon > 0$, the following holds whenever n is sufficiently large with respect to ε. Any collection of n bases of a rank-n matroid can be covered by at most $(1+\varepsilon)n$ transversal bases.*

Using topological tools, Aharoni and Berger [1, Assertion 8.11] showed in 2006 that $2n$ transversal bases suffice for a covering in the setting of Theorem 3. The Polymath 12 project [12] later improved this bound slightly, showing that $2n-2$ transversal bases are already sufficient.

To prove Theorems 2 and 3, we modified and extended the methods of [2] and [11], while introducing new randomised approaches to extend families of rainbow independent sets to transversal bases, careful notions of 'density' within subsets of the matroid, and intricate techniques to modify and improve families of disjoint rainbow independent sets.

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Independent sets in Hypergraphs

Jacques Verstraete
(joint work with Jiaxi Nie, Jeroen Schillewaert, Chase Wilson)

Introduction. An *independent set* in a hypergraph is a set of vertices of the hypergraph containing no edges of the hypergraph. The fundamental question is to determine the largest size of an independent set in a uniform hypergraph with a prescribed number of vertices and edges. This question arises naturally in many applications and contains some of the main open problems in Extremal Hypergraph Theory, Ramsey Theory and Algorithms.

By weighting the vertices of a hypergraph independently and randomly with real random variables, it is possible to show that every hypergraph $H=(V,E)$ has an independent set of size at least

$$\sum_{v\in V}\int_0^1 \prod_{v\in e\in E}(1-x^{|e|-1})dx.$$

The above bound is most easily proved by extending the Caro-Wei [3, 18] bound for independent sets in graphs: taking a random ordering of the vertices of the hypergraph, let I be the set of vertices v such that for every edge e containing v, v is not the smallest vertex in e. The above expression is a lower bound for the expected size of I. If the hypergraph is r-uniform and d-regular, then this formula is exactly

$$|V|\cdot\int_0^1(1-x^{r-1})^d dx=\frac{|V|}{\binom{d+1/(r-1)}{d}}.$$

This bound was discovered by Caro and Tuza [4] using a different method, and is exactly best possible in the case of graphs ($r = 2$), but is only tight up to a constant factor for $r \geq 3$ – this is the notoriously difficult *Turán problem for hypergraphs*. The choice of vertices in the independent set can be implemented via the following random process: after weighting the vertices with independent uniform $[0, 1]$ random variables, iteratively select the vertex with largest weight and delete all its neighbours. Then this random process will be equivalent to the randomized greedy algorithm.

Randomized greedy. For $g \geq 2$, a cycle of length g in a hypergraph H is a set of distinct edges $e_0, e_1, \ldots, e_{g-1}$ such that the sets $e_i \cap e_{i+1}$ with subscripts mod g have a system of distinct representatives. The *girth* of a hypergraph containing a cycle is the minimum g such that the hypergraph contains a cycle of length g. We consider the performance of the randomized greedy algorithm on a d-regular r-uniform n-vertex hypergraph of large girth g. To state the results, it is convenient to define the following: Let $u(d, r)$ be the unique positive real number that satisfies the following equation:

$$\sum_{n \geq 0} \binom{n+d-2}{d-2} \frac{u(d,r)^{rn+1}}{rn+1} = 1.$$

Define the quantities

$$\epsilon = \frac{d(d-1)^{\lfloor \frac{g-3}{2} \rfloor}}{r \prod_{k=1}^{\lfloor \frac{g-1}{2} \rfloor} (k + \frac{1}{r})} \quad \text{and} \quad f(d,r) = u(d,r) - \frac{u(d,r)^{r+1}}{r+1}.$$

Then we have the following theorem [12], generalizing results of Gamarnik and Goldberg [8]: For any integers $r \geq 1$, $d \geq 2$ and $g \geq 4$, let G be an $(r+1)$-uniform d-regular hypergraph with n vertices and girth g, let I be the independent set of G generated by the greedy algorithm. Then

$$(f(d,r) - \epsilon)n \leq \mathbb{E}[|I|] \leq (f(d,r) + \epsilon)n,$$

In particular, due to the form of the quantity ϵ, this theorem is effective for fixed d and large g, and as $d \to \infty$,

$$f(d,r) = (1 + o_d(1)) \Big(\frac{\log d}{rd} \Big)^{\frac{1}{r}},$$

and so if g is large enough relative to d, then this implies that any $(r+1)$-uniform d-regular n-vertex hypergraph of large girth contains an independent set of size at least roughly $f(d,r)n$ as $d \to \infty$. It is also possible to show that the size of the independent set generated by the greedy algorithm concentrates around its mean asymptotically almost surely as $n \to \infty$ even for *linear hypergraphs* – those without cycles of length 2 – with bounded degree. It is a notorious open problem to give an efficient algorithm to find, for some $c > 0$, independent sets of size $c \cdot \mathbb{E}[|I|]$ in a d-regular r-uniform hypergraph of large girth, including in random d-regular graphs, where an independent set of size $(2 - o(1))\mathbb{E}[|I|]$ is known to exist.

Uncrowded hypergraphs. Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] gave a technical analysis of a randomized greedy algorithm to show that if an $(r+1)$-uniform hypergraph (V,E) of maximum degree $d \geq 1$ is *uncrowded*, which is to say it has girth at least five, then it contains an independent set of size at least

$$0.98e^{-10/r} \cdot \frac{|V|(\log d)^{1/r}}{d^{1/r}}.$$

This is best possible up to a constant factor: the independence number in a random d-regular n-vertex $(r+1)$-uniform hypergraph is

$$(1+o_d(1)) \cdot \left(1+\frac{1}{r}\right)^{1/r} \cdot \frac{n(\log d)^{1/r}}{d^{1/r}}$$

with high probability. The result of Ajtai et al. has many applications, for instance Komlós, Pintz and Szemerédi [11] used this result to disprove *Heilbronn's Conjecture*. Inspired by the approach of Shearer [16] for graphs, and work of Kostochka, Mubayi and the author for Steiner systems [10], we give a short proof of this theorem. In particular, we prove the following [17]: The independence number of any uncrowded n-vertex triple system of maximum degree $d \geq 1$ is at least

$$\frac{n\sqrt{\log d}}{4\sqrt{d}}.$$

We believe that for any $n, r, d \geq 1$, as $d \to \infty$, any linear $(r+1)$-uniform n-vertex hypergraph with maximum degree d contains an independent set of size at least

$$(1+o_d(1)) \cdot \frac{n(\log d)^{1/r}}{d^{1/r}}.$$

This is known for $r = 1$ (the case of graphs) due to the results of Shearer [16], but is not known for any $r \geq 2$. It may even be that the $1+o_d(1)$ term can be replaced by $(1+o_d(1))(1+1/r)^{1/r}$, but this is wide open even in the case $r = 1$ i.e. the case of graphs. One of the key obstacles for $r \geq 2$ is to determine the expected value of the number elements of the *r-shadow* $\partial H = \{e \subset f : f \in H, |e| = r\}$ which are contained in a randomly uniformly chosen independent set I. We ask [17] whether this expected value is at most roughly $dn^{1-r}\mathbb{E}(|I|^r)$. The following conjecture is raised in [10] in the specific case of Steiner triple systems: Every n-vertex Steiner triple system contains an independent set of size at least $(1+o_n(1))\sqrt{3n\log n}$ as $n \to \infty$.

The method of the proof applies to various other open questions on hypergraphs. For instance, it is a consequence of the method of Shearer [16] that if $G = (V,E)$ is a graph of maximum degree d such that every neighborhood induces a graph with bounded chromatic number, then G contains an independent set of size $\Omega(|V|(\log d)/d)$. The *induced link* of a vertex v in a triple system $H = (V,E)$, denoted IL_v, is the set of pairs $\{w,x\}$ such that there exist (not necessarily distinct) edges $e, f, g \in E$ such that $\{v,w\} \subset e$, $\{v,x\} \subset f$ and $\{w,x\} \subset g$. In other words, we take the subgraph of the *shadow graph* ∂H of the triple system H induced by the set of all vertices which are in a common edge with v. We show that if for each $v \in V$, the induced link graph IL_v has bounded chromatic number, then

H contains an independent set of size $\Omega(|V|\sqrt{(\log d)/d})$. It seems likely that in fact H has chromatic number $O(\sqrt{d/\log d})$ – see results of Frieze and Mubayi [6] and Cooper and Mubayi [5] on coloring triangle-free hypergraphs.

Small maximal independent sets. In many applications, one seeks a small *maximal independent set* in a hypergraph i.e. an independent set I such that $I \cup \{v\}$ is no longer an independent set in the hypergraph. Unfortunately, the random process analyzed by Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] in an uncrowded $(r+1)$-uniform hypergraph ends too soon to guarantee that the independent set of size $\Omega(n((\log d)/d)^{1/r})$ produced is maximal. The randomized greedy algorithm was carefully analyzed by Bennett and Bohman [2] for d-regular hypergraphs where $d \geq n^{\epsilon}$ for some $\epsilon > 0$, with some natural additional structural conditions, but they are only able to guarantee that the maximal independent set produced by the algorithm has size $\Omega(n((\log d)/d)^{1/r})$. As an example of an application, *Segre's problem* on *maximal arcs* in projective planes [15] asks for the smallest maximal set of points in $PG(2,q)$ no three of which are collinear – these are *maximal arcs*. If $H = (V,E)$ is the hypergraph consisting of all collinear triples of points in $PG(2,q)$, then we seek a small maximal independent set in H. Kim and Vu [9] used the polynomial concentration method to show that $PG(2,q)$ contains such a set of points of size at most $\sqrt{q}(\log q)^{300}$. Schillewaert and the author [13] used a randomized approach to show that the elliptic quadric contains a *maximal partial ovoid* of size at most $q(\log q)^5$, resolving another long standing open problem in finite geometry.

Unifying these results, we consider maximal independent sets in hypergraphs with conditions on the *codegrees*. Specifically, let us assume for some $r \geq 1$ that $H = (V,E)$ is a hypergraph whose edges have size at most $r+1$ and for each integer $i \in [2, r+1]$, the number of hyperedges of size i containing a vertex of H is some number $d_i \geq 1$. For a set J of vertices and a vertex $v \in V \backslash J$, let $d_i(v,J)$ denote the number of hyperedges $e \in E$ such that $e = J \cup \{x\}$ for some $x \in V$ and there exists an edge $f \in e$ such that $|f| = i$ and $\{v,x\} \subseteq f$. We impose two conditions: we require that there exists $\delta > 0$ and some absolute constant $K > 0$ such that

(1) for any $i \in [2,r]$ and any $j \in [i+1, r+1]$, the number of hyperedges of size j containing a fixed set of i vertices is at most $Kd_j^{(r+1-i)/r-\delta}$ and
(2) for any $i \in [2, r+1]$ and any set $J \subseteq V$, $d_i(v,J) \leq Kd_i^{(1-\delta)/(i-1)}$.

A hypergraph satisfying these two conditions is called *δ-sparse hypergraph* with *degree profile* $\boldsymbol{d} = (d_2, d_3, \ldots, d_{r+1})$. Finally, we define $\rho(\boldsymbol{d}) = \max\{d_i^{1/(i-1)} : 2 \leq i \leq r+1\}$. Then we prove the following: For each $\delta > 0$ and $r \geq 1$, there exists $c_\delta > 0$ such that if H is any n-vertex δ-sparse hypergraph with degree profile $\boldsymbol{d}$, then H contains a maximal independent set of size at most

$$c_\delta \frac{n \log \rho(\boldsymbol{d})}{\rho(\boldsymbol{d})}.$$

It seems reasonable to conjecture that each δ-sparse n-vertex hypergraph with degree profile $\boldsymbol{d}$ contains a maximal independent set of size $O(n(\log \rho(\boldsymbol{d}))^{1/r}/\rho(\boldsymbol{d}))$.

It is straightforward to see that condition (1) on the codegrees is necessary for the theorem to hold. The second condition is also necessary, since the complete bipartite graph $K_{n,n}$ certainly fails condition (2), and every maximal independent set in this graph has size n. To apply this theorem to Segre's problem, it is not hard to see that the hypergraph of collinear triples in $PG(2,q)$ is δ-sparse with $\delta = 1/6$ and degree profile (d_2, d_3) where $d_2 = 0$ and $d_3 = (q+1)\binom{q}{2}$, and therefore according to the theorem, $PG(2,q)$ has a maximal arc of size at most $O(\sqrt{q}\log q)$, improving the main theorem of Kim and Vu [9]. As another example, this theorem shows that *hyperbolic quadrics* $Q^+(2n+1,q)$, *parabolic quadrics* $Q(2n,q)$ and *elliptic quadrics* $Q^-(2n+1,q)$ for odd q all contain maximal partial ovoids of size $O(q\log q)$, improving the many bounds of order roughly q^n for $Q^+(2n+1,q)$ and $Q(2n,q)$ and of order roughly q^{n+1} for $Q^-(2n+1,q)$ in the literature. We conjecture [13] these polar spaces have no maximal partial ovoids of size $O(q)$.

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Bounded diameter monochromatic covers

Alexey Pokrovskiy

This talk was motivated by a conjecture commonly attributed to Ryser. It first appeared in the thesis of Henderson [2]).

Conjecture 1 (Ryser)**.** *Every r-uniform, r-partite hypergraph $\mathcal{H}$ has $\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$.*

Results about Ryser's Conjecture are limited. For $r = 2$ it is equivalent to the famous König-Hall Theorem. For $r = 3$ it was proved by Aharoni using a very surprising proof relying on methods from topology [1]. For $r \geq 4$ it is open in general. However, in the case $\alpha(G) = 1$ (i.e. when G is a complete graph), the conjecture has been proved for $r \leq 5$.

Recall that for a graph G, the independence number of G (denoted by $\alpha(G)$) denotes the size of a maximum set of vertices containing no edges. As observed by Erdős-Gyárfás-Pyber, Ryser's conjecture has an equivalent graph-theoretic version "every r-edge-coloured graph can be covered by $(r-1)\alpha(G)$ monochromatic components". When phrased like this, one can wonder whether more structure can be imposed on these monochromatic components. DeBiasio-Kamel-McCourt-Sheats made the following strengthening of Ryser's Conjecture.

Conjecture 2 (DeBiasio-Kamel-McCourt-Sheats, [4])**.** *For all $\alpha \geq 1$, there exists $d = d(\alpha)$ such that for all $r \geq 2$, if G is a graph with $\alpha(G) = \alpha$, then in every r-coloring of G, there exists $(r-1)\alpha$ monochromatic components of diameter $\leq d$ covering G.*

This conjecture is a variant of an earlier conjecture of Milićević, which said that there a constant $D(r)$ such that every r-edge-coloured complete graph can be covered by $r-1$ monochromatic components of diameter $\leq D(r)$.

This conjecture is known for $(\alpha, r) = (1,2), (1,3), (1,4)$ (where it was proved by Milićević [5]), $(\alpha, r) = (2,2)$, where it was proved by DeBiasio, Kamel, McCourt, Sheats [4], and $r = 2$, $\alpha \in \mathbb{N}$ where it was proved by DeBiasio, Girão, Haxell, Stein [3].

The main result presented was a reduction of these bounded diameter conjectures to Ryser's Conjecture.

Theorem 3. *For fixed r, α the following are equivalent.*

(1) Ryser's Conjecture holds for all r-edge-coloured graphs with $\alpha(G) = \alpha$.
(2) Every r-edge-coloured graph with $\alpha(G) = \alpha$ can be covered by $(r-1)\alpha(G)$ monochromatic components of diameter $\leq 9^{r^{2^{(r+\alpha)^{4r}}}}$.

Previously unknown corollaries of this are that Milićević's Conjecture is true for $r = 5$ and that the DeBiasio-Kamel-McCourt-Sheats conjecture is true for $r = 3$ and any fixed α.

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Subset sums and their applications

David Conlon

(joint work with Jacob Fox, Huy Tuan Pham)

Given a set or a sequence A of integers, its set of subset sums $\Sigma(A)$ is the set of all integers representable as a sum of distinct elements from A. That is,

$$\Sigma(A) = \left\{ \sum_{s \in S} s : S \subseteq A \right\}.$$

A fundamental result of Szemerédi and Vu [5] regarding subset sums says that there is a constant C such that if $A \subset [n]$ with $|A| \geq C\sqrt{n}$, then $\Sigma(A)$ contains an arithmetic progression of length n. In [2], we prove a strengthening of this result, saying that under the same conditions $\Sigma(A)$ contains a homogeneous progression of length n, that is, a progression of the form $kd, (k+1)d, \ldots, \ell d$ for some integers k, ℓ and d. This result has several applications, for instance, allowing us to resolve an old question of Erdős and Graham regarding the maximum size $g(n, m)$ of a subset of $[n]$ which has no subset sum equal to m.

Another paper by Szemerédi and Vu [6] extended their earlier result by showing that for every positive integer k there are positive constants C and c such that if A is a subset of $[n]$ of size $m \geq Cn^{1/k}$, then, for some $d \leq k - 1$, $\Sigma(A)$ contains a proper d-dimensional generalised arithmetic progression P of size at least cm^{d+1}. In [3], we prove a homogeneous strengthening of this result, which also has several applications. In particular, using this result, Pham and Zakharov [4] answered a longstanding problem of Erdős and Straus by showing that the largest non-averaging subset of $[n]$ has order $n^{1/4+o(1)}$. Another application [1], resolving one more problem of Erdős and Graham, says that the number of Egyptian fractions, that is, sums of the form $\sum_{i=1}^{\ell} 1/n_i$, which have all $n_i \leq n$ and equal one is $2^{cn+o(n)}$ for a constant $c \approx 0.91117$.

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The multi-colour size Ramsey number of a path

Julian Sahasrabudhe

(joint work with Csongor Beke and Anqi Li)

For graphs G and H, we write $G \to_r H$ if every r-colouring of G contains a monochromatic copy of H. The *r-colour size Ramsey number* of a graph H, denoted $\widehat{R}_r(H)$, is the smallest number of edges in a graph G for which $G \to_r H$. That is,

$$\widehat{R}_r(H) = \min\{e(G) : G \to_r H\}.$$

This is a natural relative of the classical r-colour Ramsey number of a graph H, denoted $R_r(H)$, which is the minimum number of *vertices* in a graph G for which $G \to_r H$.

The size Ramsey number was originally introduced by Erdős, Faudree, Rousseau and Schelp [6] in the 1970s and has since become a central topic in Ramsey theory. One of the most celebrated results in this line is the 1983 work of Beck [2] who showed, perhaps surprisingly, that the size Ramsey number of a path is as small as it could possibly be: linear in its number of vertices. That is, he showed

$$\widehat{R}_2(P_k) = \Theta(k),$$

thereby settling a question of Erdős.

While Beck's work also implies that $\widehat{R}_r(P_k) = \Theta_r(k)$, it leaves open the interesting question regarding the dependence of $\widehat{R}_r(P_k)$ on the number of colours r, a problem which has gained some traction in the community in recent years. To date the best known bounds are

$$c_1 r^2 k \leqslant \widehat{R}_r(P_k) \leqslant c_2 r^2 (\log r) k,$$

where $c_1, c_2 > 0$, r is fixed and k is sufficiently large. The upper bound is due to Krivelevich [7] and the lower bound is due to Dudek and Prałat [4], though the implicit constant has been improved by Krivelevich [7] and Bal and DeBaisio [1].

In this talk I discuss a recent paper of myself with Beke and Li [3] where we determine $\widehat{R}_r(P_k)$ up to constant factors.

Theorem 1. *For any r and $k \geqslant 100 \log r$, we have*

$$\widehat{R}_r(P_k) = \Theta\left((r^2 \log r)\, k\right).$$

We remark that the best known upper bound was proved in rather elegant fashion, by showing that a random graph supplies the host graph G. Indeed, Krivelevich [7] showed (generalizing Beck) that if $G \sim G\big(Ckr, (\log r)/k\big)$, we have

$$G \to P_k, \tag{1}$$

which high probability, for r fixed and k tending to infinity.

Discussion of proof. By the work of Krivelevich it is enough to prove the lower-bound on $\widehat{R}_r(P_k)$. This means that we need to show that every graph G with $e(G) \leqslant c(\log r)r^2k$ can be coloured so that there are no monochromatic paths of length k.

So let G be a graph on n vertices with

$$e(G) \leqslant \lambda r^2 k \qquad \text{where} \qquad \lambda \leqslant c(\log r),$$

where $c > 0$ is a small constant. For the purposes of illustration we also assume

$$n = rk,$$

for some r. We give no justification for this assumption at the moment other than to note that it falls inline with the upperbound of Krielevich (1). In our sketch here, we will assume for now that the graphs that we encounter are regular - and will discuss later how we force this situation. Thus we assume that the maximum degree satisfies

$$\Delta(G) \approx 2\lambda r.$$

We now describe the main idea behind our colouring procedure in this special case. We will define an iterative procedure where we remove a single colour class at a time, until the graph is trivial. Of course we need to ensure that each colour class does not contain a P_k.

Definition of a colour class. We first partition the graph into two parts $V(G) = A \cup B$, where A, B are of equal size and at least half the edges of G are between A and B. So we assume our graph is approximately λr regular. We now randomly partition $A = A_1 \cup \cdots \cup A_r$ where each has size $< k/2 - 1$ and define B_i to be

$$B_i = \big\{x : |N(x) \cap A_i| \text{ is maximum among all } i\big\}.$$

Our first colour class is simply the bipartite graphs induced between the (A_i, B_i);

$$G[A_1, B_1] \cup \cdots \cup G[A_r, B_r],$$

In particular, note that this graph is P_k-free, since each $|B_i| < k/2$.

Evolution of the degrees. We now turn to understand how much of the graph we remove when we remove each such colour class. It's here that we observe an interesting behavior: as the process runs, we are able to do "better and better". But let's set this up more carefully first.

Now if we fix some $x \in B$ and some $i \in [q]$, we can easily calculate

$$\mathbb{E}\,|N(x) \cap A_i| = d_A(x)/r \approx \lambda.$$

Thus, if we just use this expected degree bound, we see that if our degrees started as λr after one round we have degrees like

$$\lambda r - \lambda = \lambda r\big(1 - r^{-1}\big). \tag{2}$$

So after r steps we would expect $\lambda r\big(1 - r^{-1}\big)^r \approx \lambda r/e$, edges incident to x that have been left uncoloured; clearly not enough.

But here we really want to use that $|N(x) \cap A_i|$ is *not* well concentrated about its mean to obtain a substantial advantage. Indeed, if we fix $x \in B$ we have

$$\mathbb{E}\left|\left\{i : |N(x) \cap A_i| \geqslant t\lambda\right\}\right| \geqslant \binom{d(x)}{\lambda t}\left(\frac{1}{r}\right)^{\lambda t} r \approx \left(\frac{e}{t}\right)^{\lambda t} r.$$

Thus we expect that the number of neighbours of x that get coloured in one round should be like $t\lambda$ where t satisfies

$$\left(\frac{e}{t}\right)^{\lambda t} r \approx 1, \tag{3}$$

since this is exactly the point at which we expect at least one B_i to have at least $t\lambda$ neighbours of x.

Now, by staring at (3), we can see that if $\lambda = c \log r$, where c is a small constant, we can take $t = C$, where $C > 0$ is a *large* constant. Thus we get a large constant improvement over what we saw at (2); we can remove $C\lambda$ neighbours of x in each round.

Thus if we repeat this step $r/2$ times, the number of uncoloured neighbours of x should be about

$$\lambda r(1 - C/r)^{r/2} \approx \lambda r e^{-C/2}.$$

Thus our λ has effectively dropped down to

$$\lambda \quad \longrightarrow \quad \lambda e^{-C/2}.$$

So if we repeat our colouring procedure with this new λ for $r/4$ further steps, our degree drops to

$$\lambda r e^{-C/2} \quad \longrightarrow \quad \lambda r e^{-e^{C/2}/2}.$$

Thus we see that our procedure "speeds up" as we run it. Thus very rapidly our process results in a rather trivial graph that we can colour by other means.

Cheats and lies. There are at least two rather large "cheats" in the above discussion. The first is that we made the rather large assumption that $n = rk$. In the general case, we first apply a different colouring strategy, to reduce it to the case $n = O(r(\log r)k)$.

The second major "cheat" in the above discussion is that we assumed that our graph is regular (or approximately regular) all the way through the process, which may not be the case. To correct this, we reserve a constant proportion of our colours to colour edges incident with vertices of "high" degree and "low" degree. We also choose a more clever bipartition than sketched above - so that the degrees are closer to regular. We refer the interested reader to our paper [3] for details.

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Local Removal Lemmas

Fan Wei

(joint work with Jacob Fox, Hung Le, Ruilin Shi)

The graph removal lemma is among the central results in extremal combinatorics and has broad applications, including to extremal problems for graphs and hypergraphs, additive combinatorics, discrete geometry, and theoretical computer science. It qualitatively says that if a graph has relatively few copies of a fixed graph H, then it is close to a H-free graph.

It is natural to study removal lemmas under a stronger metric. The usual edit distance is equivalent to the normalized L_1 length of the degree sequence of the removed spanning subgraph. It is also natural to consider the L_p distance for any finite $p > 0$, but any of these metrics are equivalent, meaning if one is small ($o(1)$) then the others are small, and so are not that interesting for further study. The L_∞ *distance* is the normalized maximum degree in the subgraph of the removed edge. Two graphs G_1, G_2 on the same vertex set V have L_∞ *distance at most* ε if the graph on V with edge set $E(G_1)\Delta E(G_2)$ has maximum degree at most $\varepsilon|V|$. Thus the L_∞ distance is at least the corresponding L_1 distance. Hence, the L_∞ distance is a stronger metric than the edit distance.

It is natural to ask the following question.

> Does a variant of the graph removal lemma hold under the L_∞ distance?

To see why this is a natural notion, recall in the usual graph removal lemma, we bound the total number of edges removed, which is equivalent to bounding the average degree of the spanning subgraph of the removed edges. In contrast, here we are asking to bound the maximum degree of the spanning subgraph of the removed edges.

The following theorem resolves the question for cliques.

Theorem 1 (Local clique removal lemma)**.** *For any $\varepsilon > 0$ and integer $k \geq 2$, there is $\delta = \delta(\varepsilon, k) > 0$ such that every graph on n vertices has a vertex in δn^{k-1} copies of K_k or can be made K_k-free by deleting the edges of a subgraph of maximum degree at most εn.*

We further characterize graphs which are $o(1)$-close to being K_k-free under L_∞. We have the following converse to the Theorem above.

Proposition 2. *Let G be a graph on n vertices which has a vertex v in αn^{k-1} copies of K_k. If we delete the edges of a subgraph of G of maximum degree less than $2\alpha n/3$, then at least one copy of K_k containing v remains.*

To establish this simple proposition, observe that each edge containing v is in at most $\binom{n-2}{k-2} < n^{k-2}$ copies of K_k containing v. For $k > 2$, each edge not containing v is in at most $\binom{n-3}{k-3} \leq n^{k-3}$ copies of K_k containing v. In total, the number of copies of K_k containing v in G that contain at least one edge from a subgraph of G of maximum degree less than $2\alpha n/3$ is less than $(2\alpha n/3)n^{k-2} + (\alpha n^2/3)n^{k-3} = \alpha n^{k-1}$, where $2\alpha n/3$ is an upper bound on the number of edges incident to v in this subgraph of G and $\alpha n^2/3 = n(2\alpha n/3)/2$ is an upper bound on the number of edges in this subgraph of G. Hence, at least one additional copy of K_k containing v remains after deleting this subgraph.

A particular example of a graph with less than n^{k-1} total copies of K_k which is not close to being K_k-free under the L_∞ distance is a complete k-partite graph on n vertices with one part of size one and the other parts are of size as equal as possible.

We have the following corollary which characterizes graphs that are $o(1)$-close to being K_k-free under the L_∞ distance.

Corollary 3. *A graph on n vertices is $o(1)$-close to be K_k-free under the L_∞ distance if and only if every vertex is in $o(n^{k-1})$ copies of K_k.*

The statement of the local removal lemma is more involved for general graphs than for cliques. Recall Theorem 1 states that a graph on n vertices is $o(1)$-close to being K_k-free under L_∞ if and only if every vertex is in $o(n^{k-1})$ copies of K_k. However, for most graphs H on k vertices, it is not true that a graph on n vertices is $o(1)$-close to being H-free under L_∞ if and only if every vertex is in $o(n^{k-1})$ copies of H. Here we want to emphasize that a copy of H in G is as a subgraph, where the mapping of $V(H)$ to $V(G)$ is injective.

A simple example illustrating this point is $H = K_{2,2}$. For any $0 < \varepsilon < 1/2$ and sufficiently large integer n, consider the host graph $K_{2,n-2}$ on n vertices, and let A be the bipartite class with $n-2$ vertices and B the the one with two vertices. In any $K_{2,2}$-free subgraph of $K_{2,n-2}$, at most one vertex in A can be adjacent to both vertices in B. Thus at least $|A| - 1$ edges must be removed, implying that the degree of some vertex in B drops by at least $(|A| - 1)/2 > \varepsilon n$. Therefore $K_{2,n-2}$ is ε-far from being $K_{2,2}$-free under L_∞, while there are only $O(n^2)$ copies of $K_{2,2}$ in $K_{2,n-2}$. Similarly, if H is the complete bipartite graph $K_{a,b}$ with $a \leq b$, if $\varepsilon < 1/a$ and n is sufficiently large, the host graph $K_{a,n-a}$ is ε-far from being $K_{a,b}$-free under L_∞, while there are only $O(n^b)$ copies of $K_{a,b}$ in $K_{a,n-a}$.

It appears that the correct form of the local removal lemma is that for any H, there is an $a(H)$ such that a graph G on n vertices is $o(1)$-close from being H-free under L_∞ if and only if every subset of $a(H)$ vertices of G extends to $o(n^{v(H)-a(H)})$ copies of H. The following definition gives what we conjecture is the correct function $a(H)$ for such a local graph removal lemma.

Definition 4. Let H be a graph. Let $a(H)$ be the size of the maximum independent subset A in H such that for every homomorphism ρ from H to itself, if every vertex in A has at most one pre-image under ρ, then ρ has exactly $|A|$ vertices mapping to elements of A.

It is not difficult to show that $a(K_k) = 1$ for every complete graph K_k. We also have $a(K_{a,b}) = \min(a, b)$ for every complete bipartite graph $K_{a,b}$. In general for a connected bipartite graph H, the value $a(H)$ is the size of one of its bipartite classes.

The proof of Theorem 1 extends to the following result. Note that there are graphs besides cliques with $a(H) = 1$. Examples include when H is a book graph on k vertices, which for some $1 \le t \le k-2$ consists of a clique on t vertices joined completely to an independent set on $r - t$ vertices.

Theorem 5. *Let H be a graph with k vertices and $a(H) = 1$. For each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, H) > 0$ such that every graph on n vertices in which each vertex is in less than δn^{k-1} copies of H can be made H-free by deleting edges, decreasing the degree of each vertex by less than εn.*

Our next result is the local removal lemma for bipartite graphs.

Theorem 6. *Let H be a connected bipartite graph on k vertices. For each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, H) > 0$ such that every graph on n vertices can be made H-free by deleting edges, decreasing the degree of each vertex by at most εn, or, there are $a(H)$ vertices in at least $\delta n^{k-a(H)}$ copies of H. Furthermore, we may take $\delta = \delta(\varepsilon, H) > 0$ to be a polynomial in ε.*

We would like to emphasize that when H is bipartite, our bound on $\delta(\varepsilon, H)$ in the local removal lemma is polynomial in ε. In contrast, for cliques, we have shown (Theorem 1) that $\delta(\varepsilon, K_k)$ with $k \ge 3$ in the local removal lemma cannot be taken as a polynomial in ε.

We show that any connected graph H satisfies $a(H) \ge 1$. We also have the following corollary.

Corollary 7. *Let H be a connected graph on k vertices and $\varepsilon > 0$ sufficiently small in terms of k. For any sufficiently large n, there is a graph G on n vertices such that G is ε-far with respect to L_∞ from being H-free, and the number of copies of H in G is $O(n^{|V(H)|-1})$.*

In general, we have the following conjecture.

Conjecture 8. *Let H be a fixed graph on k vertices. A graph G on n vertices is $o(1)$-close to being H-free under the L_∞ distance if and only if, for every $a(H)$*

vertices of G, the number of copies of H in which these $a(H)$ vertices form an independent set of the copy of H is $o(n^{k-a(H)})$.

We are able to confirm Conjecture 8 for some other graphs, including odd cycles as shown by the following theorem. Note that for an odd cycle H on k vertices, the value of $a(H)$ is the independence number of H, which is $(k-1)/2$. This involves some more involved embedding lemma and ideas from strong regularity lemma.

The above work is joint with Jacob Fox.

We have also investigated the L_∞ removal lemma for permutations, where we were able to give the full characterization, with the correct degree of polynomials for the quantitative bounds. This is joint with Hung Le and Ruilin Shi.

Ramsey Cayley graphs over non-abelian groups

Carl Schildkraut

Ramsey's theorem is one of the foundational results in extremal graph theory, and there is a great amount of literature on its variations, extensions, and quantitative aspects. The following definition is inspired by its quantitative aspects.

Definition 1. Let $C > 0$. A graph on n vertices is *C-Ramsey* if its clique number and independence number are both at most $C \log_2 n$.

The proof of Ramsey's theorem by Erdős and Szekeres [7] yields that no graph is C-Ramsey for $C < 1/2$, while Erdős [6] showed 12 years later, in one of the first applications of the probabilistic method, that a random n-vertex graph is 2-Ramsey with high probability. However, we do not yet know any explicit constructions of C-Ramsey graphs for any constant C.

To this end, it is natural to ask where C-Ramsey graphs can be found, and what properties they can have. The following conjecture along these lines is due to Noga Alon:

Conjecture 2 (Alon [1, 2])**.** *There exists some absolute constant $C > 0$ for which every finite group G admits a C-Ramsey Cayley*[1] *graph.*

The first positive result in the line of this conjecture was given by Green [8], who showed that Conjecture 2 holds for cyclic groups. Recently, a significant advance was made by Conlon, Fox, Pham, and Yepremyan:

Theorem 3 (Conlon–Fox–Pham–Yepremyan [5])**.** *Conjecture 2 holds for the family of abelian groups whose orders are relatively prime to 6. Moreover, Conjecture 2 holds for all abelian groups of almost all*[2] *orders.*

Our main result is an extension of Theorem 3 to not-necessarily-abelian groups.

[1] Given a group G and a subset $S \subset G \setminus \{1\}$ which is closed under taking inverses, the *Cayley graph* $\mathrm{Cay}(G, S)$ is the graph with vertex set G and edge set $\{(x, y) : xy^{-1} \in S\}$. Such a graph is always vertex-transitive, and always contains G as a subgroup of its automorphism group.

[2] In the sense of natural density.

Theorem 4. *Conjecture 2 holds for the family of groups whose orders are relatively prime to* 6*. Moreover, Conjecture 2 holds for all groups of almost all orders.*

By the Feit–Thompson theorem, the first statement requires only the consideration of solvable groups. The proof of the second statement requires consideration of all groups, and the additional work required is much greater than for the corresponding statement in Theorem 3. (While a finite abelian group whose order has few factors of 2 and 3 has a large subgroup whose order is relatively prime to 6, the same need not hold for non-abelian groups.) We will focus only on solvable groups for the remainder of the discussion, where we outline the modifications we make to the strategy of [5] to attain Theorem 4.

The Conlon–Fox–Pham–Yepremyan strategy. Let G be a (not necessarily abelian) group. The probability that a set $A \subset G$ is a clique or independent set in a uniformly random Cayley graph on G is at most $2^{-(|AA^{-1}|-3)/2}$, where we write

$$AA^{-1} := \{xy^{-1} : x, y \in A\}.$$

Conlon, Fox, Pham, and Yepremyan first show that, in a precise sense, the only sets of concern are those sets A for which $|AA^{-1}|$ is at most some constant times $|A|\log|A|$. They then modify the distribution on Cayley graphs slightly, in such a way that these "sets with small doubling" are *deterministically excluded* from being cliques or independent sets. The following simple lemma is the key to this deterministic exclusion, and the only point in the argument of [5] that requires the ambient group to be abelian.

Lemma 5 ([5, Lemma 4.5])**.** *Let G be an abelian group and $A \subset G$. Suppose that $|AA^{-1}| \le |A|^{1+\frac{1}{144}}$. Then there exists some $x \in G \setminus \{1\}$ such that $x, x^2, x^4, x^8 \in AA^{-1}$.*

A central lemma in our work, from which Theorem 4 follows in combination with the tools of [5], is the following.

Lemma 6. *Let G be a solvable group and let $A \subset G$. Suppose that*

$$|AA^{-1}| \le |A| \cdot \exp\left(\Omega(\log^{1/12}|A|)\right).$$

Then there exists some $x \in G \setminus \{1\}$ such that $x, x^2, x^4, x^8 \in AA^{-1}$.

Non-abelian–to–abelian tools for dense sets. Many tools are used in additive combinatorics in the abelian setting, and only some transfer over to non-abelian settings. (In the case of the proof of Lemma 5, the failure in the non-abelian setting is quite basic: the proof directly uses the commutativity of the group operation to set up an application of the pigeonhole principle.) One trick, due originally to Bergelson, McCutcheon, and Zhang [4], to get around such issues in some settings is to find an abelian subgroup of the ambient non-abelian group and to work only within its cosets. This is best accomplished by the following (asymptotically tight) theorem of Pyber.

Theorem 7 (Pyber [10])**.** *For some constant $c > 0$, every finite group G has an abelian subgroup of order at least* $\exp(c\log^{1/2}|G|)$.

Theorem 7 is quantitatively quite strong, and has been used to great effect to show that dense subsets of arbitrary finite groups contain various patterns (see [3, 4, 12, 13] for a few examples). For example, it can be used to provide upper bounds on the size of a subset of a non-abelian group avoiding three-term arithmetic progressions of nearly the same quantitative strength as we have for Roth's theorem (the same statement in cyclic groups). The main difference between such applications and Lemma 6 is the size of the subset of the group in question. If $A \subset G$ is quite dense in G, and $H \leq G$ is a subgroup, then there exists some coset gH of H for which $A \cap gH$ is quite dense in gH, while if A is sparse then there may not even be a single coset which contains more than one element of A.

A new non-abelian–to–abelian tool for sparse sets. In the setting of Lemma 6, the set A is not dense in the ambient group, but $|AA^{-1}|/|A|$ is quite small. This property ought to mean that A behaves somewhat like a coset of a subgroup; indeed, sets with this property are closely related, by a result of Tao [14, Proposition 4.5], to so-called *approximate groups*. In the abelian setting, such sets can be transformed into dense sets in some other ambient group via Ruzsa's proof of Freiman's theorem. Such transference fails in non-abelian groups [9], and so we must look elsewhere.

To this end, we prove for a version of Pyber's theorem (stated here only for solvable groups) wherein the abelian subgroup $H \leq G$ is not simply large but "large from the view of A."

Theorem 8. *There exists an absolute constant $c > 0$ for which the following holds. Let G be a solvable group and let $A \subset G$. Suppose that $|AA^{-1}| \leq K|A|$. Then there exists some abelian subgroup $H \leq G$ and some $g \in G$ for which*

$$|A \cap gH| \geq |A| \cdot \exp\left(c\frac{\log^{1/6}|A|}{\log 2K}\right).$$

Lemma 6 (and thus Theorem 4) then follows directly from Theorem 8 and Lemma 5.

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The generalised Erdős-Rothschild problem

Katherine Staden
(joint work with Pranshu Gupta, Yanitsa Pehova, Emil Powierski)

The Erdős-Rothschild problem from 1974 [5] asks for the maximum number of s-edge colourings in an n-vertex graph which avoid a monochromatic copy of K_k, given positive integers n, s, k. We systematically study the generalisation of this problem to a given forbidden family of colourings of K_k.

Given a graph G, and a partition P of the edges of K_k which we call a *pattern*, let $F(G; P, s)$ be the number of s-edge-colourings of G such that there is no copy of K_k in which the edge-partition given by the colour classes is isomorphic to P. The original 'monochromatic' problem is the case when the pattern P is the partition into one part. We are interested in the *generalised Erdős-Rothschild problem*, that is, determining

$$F(n; P, s) := \max_{v(G)=n} F(G; P, s)$$

in the case when n is large.

This problem typically exhibits a dichotomy whereby for some values of s, the extremal graph is the 'trivial' one, namely the Turán graph on $k-1$ parts, with no copies of K_k; while for others, we know the extremal graph has many copies of K_k but determining it becomes much harder. There are very few results, asymptotic or exact, which are known. For the monochromatic problem, the only pairs (k, s) where there is a solution are $(k, 2)$, $(k, 3)$, which are known to be the only values with trivial solution; and $(3, 4)$, $(3, 5)$ (asymptotic only), $(3, 6)$, $(3, 7)$ and $(4, 4)$ [1, 4, 9, 11, 12]. The 'rainbow triangle' problem where $k = 3$ and P has three parts each containing one edge, has been solved completely: for $s = 3$ the extremal graph is a complete graph while for $s \geq 4$ it is the trivial one [2]. There are a handful of results for other patterns.

We show there is a close link between the Erdős-Rothschild problem and the following optimisation problem.

Optimisation problem. Let $k \geq 3$ be an integer, let P be a pattern of K_k and let $s \geq 2$ be a number of colours. Maximise

$$q(\phi, \boldsymbol{\alpha}) := 2 \sum_{1 \leq i < j \leq r} \alpha_i \alpha_j \log |\phi(\{i,j\})|$$

subject to $(r, \phi, \boldsymbol{\alpha})$ being *feasible*, that is

- $r \in \mathbb{N}$,
- $\phi \in \Phi_{P,s}(r)$ where $\Phi_{P,s}(r)$ is the set of functions ϕ which map every pair in $\{1, \ldots, r\}$ to a subset of $\{1, \ldots, s\}$ such that for any k-subset A of $\{1, \ldots, r\}$ and any colour $c_{ij} \in \phi(\{i,j\})$ for each distinct pair $i, j \in A$, in the resulting s-edge-coloured K_k induced on A, the edge-partition given by the colour classes is not isomorphic to P,
- $\boldsymbol{\alpha} \in \Delta^r$ where Δ^r is the set of $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ where each $\alpha_i \geq 0$ and $\alpha_1 + \ldots + \alpha_r = 1$.

Let $Q(P,s)$ be the supremum. Let $\mathrm{OPT}^*(P,s)$ be the set of *optimal solutions* $(r, \phi, \boldsymbol{\alpha})$ (that is, the set of feasible $(r, \phi, \boldsymbol{\alpha})$ which attain the maximum) such that every $\phi(\{i,j\})$ has size at least two, and every α_i is non-zero.

We show that, if the set $\mathrm{OPT}^*(P,s)$ of optimal solutions satisfies certain properties, then it suffices to determine this set to solve the corresponding Erdős-Rothschild problem. To avoid technicalities, we state this here for the monochromatic problem only. This result was originally proved in [9], using 'stability' results from [8, 10].

We say that $\mathrm{OPT}^*(P,s)$ has the *extension property* if the following holds. Suppose $(r, \phi, \boldsymbol{\alpha}) \in \mathrm{OPT}^*(P,s)$, and there is $\psi \in \Phi_{P,s}(r+1)$ which extends ϕ so that $\sum_{i \in \{1,\ldots,r\}} \alpha_i \log |\psi(\{i, r+1\})| = Q(P,s)$, where the sum is only taken over those i for which $\psi(\{i, r+1\})$ is non-empty. Then there is some $i^* \in \{1, \ldots, r\}$ such that $\psi(\{i, r+1\}) = \phi(\{i, i^*\})$ whenever $i \neq i^*$, and $\psi(\{i^*, r+1\}) = \emptyset$. In other words, if a zero part connects optimally, then it must be a clone of an existing part i^*.

Theorem 1 (Exact result for the monochromatic Erdős-Rothschild problem [9])**.** *Let $k \geq 3$ be an integer and let P be the monochromatic pattern on K_k. Suppose that $OPT^*(P,s)$ has the extension property. Then, whenever n is sufficiently large, the following holds for every n-vertex extremal graph G. There is an integer r^* and $\boldsymbol{\alpha}^* \in \Delta^{r^*}$ such that G is a complete r^*-partite graph whose i-th part has size $(\alpha_i^* + o(1))n$ for all i, and for $(1 - o(1))F(G;P,s)$ valid colourings of G there is ϕ^* such that $(r^*, \phi^*, \boldsymbol{\alpha}^*) \in OPT^*(P,s)$ such that every edge between the i-th and j-th parts has a colour in $\phi^*(ij)$, for all pairs i, j.*

This result is useful if we can determine $\mathrm{OPT}^*(P,s)$. If we know that there is some $R \in \mathbb{N}$ such that every optimal r is at most R, then determining $\mathrm{OPT}^*(P,s)$ is a finite problem, via the method of Lagrange multipliers. There is such an R for the monochromatic problem, by Ramsey's theorem, but there is e.g. no such R for the rainbow triangle problem with $s = 3$ colours. There is also such an R for any s and the *dichromatic triangle* pattern $P = K_3^{(2)}$ which is the unique edge-partition

of K_3 into two sets. We essentially solve the optimisation problem completely in this case, which, via our general result, leads to the following resolution of the dichromatic triangle case of the generalised Erdős-Rothschild problem. The cases $2 \leq s \leq 26$ were previously solved by Hoppen, Lefmann and Schmidt [7], who showed that the solution is trivial if and only if s is in this range.

Theorem 2 ([6])**.** *Let $s \geq 2$ be an integer. For sufficiently large n, every n-vertex graph G which attains $F(G; K_3^{(2)}, s) = F(n; K_3^{(2)}, s)$ is the r-partite Turán graph $T_r(n)$ and*

$$F(n; (K_3^{(2)}, s)) = (C + o(1)) \cdot e^{\frac{r}{r-1} g(s) e(T_r(n))},$$

where $r \in \{p, p+2\}$ for p the largest even integer such that $(p-1)e^p \leq s$, and $g(s) = \max_{even\ r} g_s(r)$ where

$$g_s(r) = \frac{r-1-(s-(r-1)z)}{r} \log(z) + \frac{s-(r-1)z}{r} \log(z+1) \quad \text{where } z = \left\lfloor \frac{s}{r-1} \right\rfloor,$$

and C depends only on s, r and on $n \pmod r$.

A consequence of this theorem is that for every non-monochromatic colour pattern, every extremal graph is complete partite. This was previously shown for most other patterns by Benevides, Hoppen and Sampaio [3].

The results presented here may be found in [6].

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Is there Geometry in Totally Discrete Spaces?

Nati Linial
(joint work with Daniel Cizma, Maria Chudnovsky)

Let $X = [n]$ be a finite set of *points*. An *xy-path* P_{xy} is a sequence of points that starts at $x \in X$ and ends at $y \in X$. We always take P_{yx} to be the reverse of P_{xy}. Our minimalist notion of a geometry is a *path system* $\mathcal{P}$ on X, i.e., a collection of *xy-paths* P_{xy}, one per each pair of points $x, y \in X$.

Path systems are closely related to graphs. For $\mathcal{P}$ as above, let $G = (X, E)$ be the graph where $xy \in E$ exactly if P_{xy} is a single-hop path. Also, given a graph $H = (V, F)$ it is of interest to investigate path systems on V in which all single-hop paths are edges in F.

A path system $\mathcal{P}$ that is closed under intersection is said to be *consistent*. Equivalently: $\mathcal{P}$ is consistent, if for every path P_{xy} in $\mathcal{P}$ and any two points a and b on P_{xy}, the subpath of P_{xy} that connects a and b coincides with P_{ab}.

It is easy to construct consistent path systems on a graph $G = (V, E)$. Assign positive lengths to the edges $w : E \to \mathbb{R}_{>0}$. With almost certainty the w-shortest xy path is unique and we let it be P_{xy}. Such a path system is said to be *strictly metric*. If some w-shortest paths are not unique, we can still choose between them in a consistent way, and get a consistent *metric* path system. The following question suggests itself:

Basic Question: Is every consistent path system metric (or even strictly metric)?

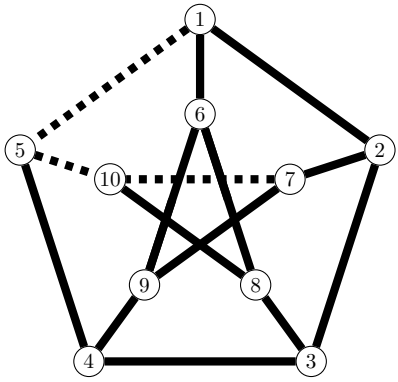
Answer: No. Figure 1a exhibits a non-metrizable path system in the Petersen graph Π. If $uv \in E(\Pi)$, then the path P_{uv} comprises the single edge uv.
If $xy \notin E(\Pi)$, then x and y have a unique common neighbor z, and in most cases we let $P_{x,y} = (x, z, y)$. There are 5 exceptional pairs, namely $P_{1,7} = 1, 5, 10, 7$ (see Figure 1a), and its shifts $P_{2,8} = 2, 1, 6, 8$; $P_{3,9} = 3, 2, 7, 9$; $P_{4,10} = 4, 3, 8, 10$; $P_{5,6} = 5, 4, 9, 6$. This path system is easily seen to be consistent. It is nonmetrizable, since there exists no corresponding weight function w, as Figure 1b shows.

We call a graph G *metrizable* (resp. strictly metrizable) if **Basic Question** is answered positively for every consistent path system on G.

Closure Properties:

- The class of metrizable graphs is closed under topological minors, but is not minor-closed.
- The class of strictly metrizable graphs is minor-closed.

Computational Complexity: It is possible to determine in polynomial time whether a given graph is metrizable resp. strictly metrizable or not.



(A) Non-metrizable path system in the Petersen Graph.

$$w_{1,2} + w_{1,6} + w_{6,8} \leq w_{2,3} + w_{3,8}$$
$$w_{2,3} + w_{2,7} + w_{7,9} \leq w_{3,4} + w_{4,9}$$
$$w_{3,4} + w_{3,8} + w_{8,10} \leq w_{4,5} + w_{5,10}$$
$$w_{4,5} + w_{4,9} + w_{6,9} \leq w_{1,5} + w_{1,6}$$
$$w_{1,5} + w_{5,10} + w_{7,10} \leq w_{1,2} + w_{2,7}$$
which implies
$$w_{6,8}+w_{7,9}+w_{8,10}+w_{6,9}+w_{7,10} \leq 0.$$

(B) A proof that this path system is not metric.

FIGURE 1. An example of a non-metric consistent path system.

The proof in Figure 1b that the Petersen Graph is non-metrizable is the a prototype of a proof that these problems are in *co-NP*, but moreover, they are in *P*.

Open Problem: Find a truly efficient algorithm to decide graph metrizability.

Specifics:

- All outerplanar graphs are metrizable.
- Every 2-connected metrizable graph on 11 or more vertices can be made outerplanar by removing at most one vertex.
- We *conjecture* that a graph is not strictly metrizable iff it has a minor that is a member of a specific list of 6 small graphs (of 8 vertices or less).

Enumeration:

- $n^{n^2(1/2-o(1))} \leq |\mathcal{M}_n| \leq |\mathcal{C}_n| \leq n^{n^2/2}$ are the numbers of n-point metric, resp. consistent path systems.
- $|\mathcal{SM}_n| = 2^{\Theta(n^2)}$ is the number of n-point strictly metric path systems.

Approximation: A given path system $\mathcal{P}$ is metric if a corresponding system of linear inequalities as e.g., in Figure 1b, is feasible. If it is infeasible, but is made feasible by multiplying the "larger than" side by some factor $\alpha > 1$, we say that $\mathcal{P}$ is α-metric. The infimal α for which $\mathcal{P}$ is α-metric is denoted by $\Delta(\mathcal{P})$.

We prove: There exist n-point consistent path systems $\mathcal{P}$ with $\Delta(\mathcal{P}) \geq n^{\frac{1}{2}-o(1)}$.

Open Problem: How large can $\Delta(\mathcal{P})$ be for a consistent n-point path system $\mathcal{P}$?

Group-invariance: There are additional phenomena to consider when G is the Cayley graph of a finite group Γ. In particular, we study Γ-*invariant* path systems $\mathcal{P}$ on G, where $P_{gx,gy} = gP_{x,y}$ for every x, y and g in Γ.

More open questions: The papers in the bibliography are replete with open questions, but let us point out one which is of particular interest:

A path system in a graph $G = (V, E)$ is called *neighborly* if it uses all of G's edges, i.e., $P_{xy} = (x, y)$ whenever $xy \in E$. We call G *neighborly metrizable* if every consistent neighborly path system in G is metric. There exist neighborly metrizable graphs which are not metrizable, but a good description of all neighborly metrizable graphs is still unknown.

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Problem Session

Nati Linial (Chair)

János Pach

The first problem is posed jointly with Erdős.

Question 1 (Erdős–Pach)**.** *Given a set $\mathcal{P}$ of $n > 4$ distinct points in the plane, can one always find two distances that each occur as distances between pairs of points in $\mathcal{P}$ at most n times?*

It is a result of Hopf and Pannwitz [2] that, for any n distinct points in the plane, the *diameter* — that is, the largest distance between any two points — occurs at most n times. So, this question asks whether there is some distance other than the diameter with this same property. The condition $n > 4$ is seen to be necessary by considering two equilateral triangles joined on a side.

If the points $\mathcal{P}$ are in convex position, Question 1 was answered positively by Clemen, Dumitrescu, and Liu [1].

The second problem concerns the crossing number and a variant thereof.

Definition 2. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of edge crossings over all drawings of G in $\mathbb{R}^2$. The *monotone crossing number* $\text{mon-cr}(G)$ is defined the same way, except that each edge must be drawn to be *x-monotone*: each vertical line may intersect each edge at most once.

Question 3. *Does there exist an absolute constant C such that, for every graph G, the inequality*

$$\text{mon-cr}(G) \leq C \cdot \text{cr}(G)$$

holds?

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Alexey Pokrovskiy

A graph is *claw-free* if it has no claw (i.e., $K_{1,3}$) as an induced subgraph, and *Hamiltonian* if it possesses a cycle which passes through all of its vertices. The Matthews–Sumner conjecture [1] posits that every 4-vertex-connected claw-free graph is Hamiltonian. This inspires the following conjecture.

Conjecture 4. *There exists a function f such that every $f(k)$-vertex-connected, induced-$K_{1,k}$-free graph is Hamiltonian.*

In this phrasing, the Matthews–Sumner conjecture asks whether $f(3) \leq 4$. Ryjáček has proven that $f(3)$ exists and is at most 7.

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Matthew Kwan

Conjecture 5 (Grebennikov–Kwan [3])**.** *Let $\mathcal{A} \subseteq 2^{[n]}$ be a family of sets. Let $[n]_p$ be a p-random subset of $[n]$. Let $\ell \geq 2$. Then*

$$\Pr\left[[n]_p \text{ contains exactly } \ell \text{ elements of } \mathcal{A}\right] \leq \frac{2}{e^2} + o_{p\to 0}(1).$$

The constant $2/e^2$ appears as the maximum probability that any Poisson distribution takes any particular value which is at least 2, and cannot be improved for this reason. If the condition "$\ell \geq 2$" is weakened to "$\ell \geq 1$," then the probability is at most $1/e + o_{p\to 0}(1)$, as proven by Fox, Kwan, and Sauermann [1].

Conjecture 5 arises from questions of graph inducibility. Together with results in [1], it would imply that for any $\varepsilon > 0$ there are only finitely many graphs except cliques, stars, single edges, and complements of such graphs which have inducibility greater than $2/e^2 + \varepsilon$. This was conjectured by Ueltzen [6], and would extend the large inducibility theorem proved in [2, 4, 5] (which gives a bound of $1/e + \varepsilon$ for all but finitely many graphs except cliques and their complements).

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Noga Alon

Definition 6. A set in a metric space X is *equilateral* if all distances between pairs are the same. Write $e(X) = \sup\{|A| : A \text{ equilateral}\}$.

There are many conjectures about $e(X)$ for normed spaces X, and very little is known. We do not even know $e(\ell_1^n)$.

Kravitz and Steinerberger suggest consideration of the following family of metric spaces. Write T_p^n for the n-torus $[0,1)^n$ equipped with the ℓ_p distance

$$d(a,b) = \left(\sum_{i=1}^{n} \min(|a_i - b_i|, 1 - |a_i - b_i|)^p \right)^{1/p}.$$

Question 7. *Is it true that, for every finite $p \geq 1$, we have $e(T_p^n) \leq C_p n$? Can it be true that moreover, if $2n+1$ is a prime, then $e(T_p^n) = 2n+1$?*

One always has $e(T_p^n) \geq 2n$; this is given by the set $\{\pm\alpha e_1, \ldots, \pm\alpha e_n\}$ where $e_1, \ldots, e_n$ are the elementary vectors and α is chosen appropriately. If $2n+1$ is prime, then in fact $e(T_p^n) \geq 2n+1$. For $p = \infty$ we have $e(T_\infty^n) = 3^n$. The best upper bound known for fixed $p < \infty$ is $C_p n^{2+2/\lfloor p \rfloor}$.

Hans Yu

The following question is motivated by incidence geometry.

Question 8. *Let $r \geq 3$ be an integer. Let H be a 3-colored $(r-1)$-uniform hypergraph with R red edges, G green edges, and B blue edges. What is the maximum number of vertex sets S of size r such that $H[S]$ contains at least one edge of each color?*

The answer to this question is monotonically non-decreasing in r.

Theorem 9. *For any r, the answer to Question 8 is at most $\sqrt{6RGB}$ for all R, G, and B. For $r = 3$, this can be improved to $\sqrt{2RGB}$.*

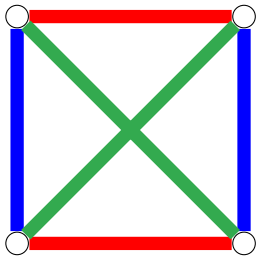

FIGURE 1. A blowup of the proper 3-coloring of K_4.

For $r = 3$, the bound $\sqrt{2RGB}$ is asymptotically tight when $R = G = B$. This is attained by a blowup of a proper 3-coloring of K_4 (see Figure 1).

Although this statement is purely combinatorial, the only known proof for $r > 3$ uses the polynomial method. It is believed that an upper bound of $\sqrt{2RGB}$ holds for all r.

Carl Schildkraut

Let d be a fixed positive integer and w be a fixed word. Let n be a large positive integer. Given a way to fill an $n \times \cdots \times n$ grid with letters, we count the number of times w appears contiguously, in a line, reading in the "word search directions" $\{-1, 0, 1\}^d \setminus \{(0, \ldots, 0)\}$ (see Figure 2 for an example when $d = 2$). The maximum number of such occurrences over all grids behaves as $C_d(w)n^d - O(n^{d-1})$ as n grows; the main problem is to determine $C_d(w)$ given d and w.

Despite the simplicity of this definition, rather little is known. The setup was first studied by Patchell and Spiro [3], who focused mostly on the case $n = \text{len}(w)$. Later, Alon and Kravitz [1] determined $C_d(w)$ for all words with distinct letters. Halberstam and Schildkraut [2] determined $C_d(w)$ for various other pairs (d, w). In particular, they computed that $C_2(\mathsf{ABB}) = 2$, but the proof generalizes neither to higher d nor longer w. Some specific questions are as follows.

Question 10. *Is it true for all d and w that $C_d(w)$ is always attained in the limit by some periodic construction?*

Question 11. *What is $C_d(\mathsf{ABB})$ for $d > 2$? What is $C_2(\mathsf{ABBB})$?*

It is believed that $C_2(\mathsf{ABBB}) = 8/5$. The construction which attains $8/5$ is related to the storied n-queens problem: one places A at locations of five non-attacking queens in $(\mathbb{Z}/5\mathbb{Z})^2$ and B at all other locations. (See Figure 3.)

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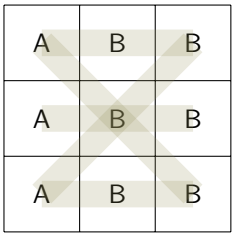

FIGURE 2. Five occurrences of the word ABB in a 3×3 grid. Repeating this grid in both directions shows $C_2(\mathsf{ABB}) \geq 2$.

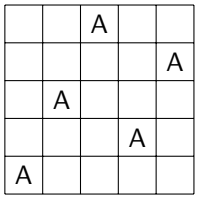

FIGURE 3. The conjectured optimal construction for $C_2(\mathsf{ABBB})$.

MARIA CHUDNOVSKY

For a graph H, write sub(1-sub-H) for the set of subdivisions of the 1-subdivision of H; equivalently, of subdivisions of H in which each edge must be subdivided at least once.

Theorem 12 (Kühn–Osthus [1])**.** *For every positive integer t, there exists some $C > 0$ such that the following holds:*

Let G be a graph which is induced-H-free for every $H \in \{K_t, K_{t,t}\} \cup \text{sub}(1\text{-sub-}K_t)$. Then G has a vertex with degree at most C.

One can read the conclusion, combined with the fact that the family of induced-$\mathcal{H}$-free graphs is closed under vertex deletion, as saying that such graphs have bounded degeneracy.

The following question asks whether a similar (appropriately modified) conclusion holds if the condition on clique number is removed.

Question 13. *For every positive integer t, does there exist $C > 0$ such that the following holds?*

Let G be a graph which is induced-$K_{t,t}$ free and induced sub(1-sub-K_t)*-free. Then G has some vertex v for which $\alpha(N(v)) \leq C$.*

This conclusion can be considered as an independence-number version of degeneracy. This is related to the notion of tree-independence, and Question 13 could be useful to prove statements about tree-independence in settings where one might use Theorem 12 to prove statements about tree-width.

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[1] D. Kühn and D. Osthus, Induced subdivisions in $K_{s,s}$-free graphs of large average degree, *Combinatorica* **24** (2004), no. 2, 287–304.

David Conlon

Question 14. *Let $g(n)$ be the smallest number of colors necessary to edge-color K_n such that there are no two vertex-disjoint color-isomorphic triangles. Estimate $g(n)$.*

It is not hard to show that $c\sqrt{n} \leq g(n) \leq n-4$ for some $c > 0$. It is conjectured that $g(n)$ grows linearly in n and it may even be that $g(n) = n - O(1)$. Conlon, Fox and Pham have shown that $g(n) \geq cn^{1/2+\varepsilon}$ for some $\varepsilon > 0$.

Jacques Verstraete

An (n, d, λ)-graph is a d-regular n-vertex graph whose adjacency matrix has, besides d, all eigenvalues in $[-\lambda, \lambda]$.

Theorem 15 (Alon–Rödl [2])**.** *If G is an (n, d, λ)-graph with $\lambda > (\log n)^2$, then the number of independent sets in G with size $t = \frac{2n(\log n)^2}{d+1}$ is at most*

$$\left(\frac{4e^2\lambda}{(\log n)^2}\right)^t.$$

Mubayi and Verstraete [3] noticed that Theorem 15 gives, via sampling from an appropriate C_4-free graph, a lower bound of $\Omega(t^{3/2}/\log t)$ on the cycle-complete Ramsey number $r(C_4, t)$, which matches the result of the C_4-free process.

The following question asks for an improvement on Theorem 15.

Question 16. *Does there exist a constant $C > 0$ such that the following holds?*

If G is an (n, d, λ)-graph with $\lambda > (\log n)^2$, then the number of independent sets in G with size $t = \frac{Cn \log n}{d+1}$ is at most

$$\left(\frac{4e^2\lambda}{\log n}\right)^t.$$

A positive answer to Question 16 would give a quick proof that $R(C_3, t) \geq \Omega(t^2/\log t)$ by sampling from optimally pseudorandom triangle-free graphs constructed by Alon [1], and improve by logarithms some of the even cycle-complete Ramsey numbers. Moreover, if true it is nearly tight: the example of random d-regular graphs shows that C must be at least 2.

The bound in Question 16 certainly holds true for random d-regular graphs. It's worth asking as well whether it can be proven for specific (n, d, λ)-graphs.

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