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Noncommutative Harmonic Analysis and Quantum Information

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Abstract. Quantum information theory deals with the way information can be exchanged using the laws of quantum mechanics. The field is relatively young, but has seen an extaordinary rapid development over the past decade. The particular focus of this workshop is the analysis of quantum information theory and its connections to infinite dimensional systems and operator algebras. On the other hand the theory of noncommutative harmonic analysis has established itself further in a parallel development, with successful attempts at generalizing the theory of singular integrals and Fourier multipliers in the noncommutative realm. While there are close ties between the mathematics underpinning noncommutative harmonic analysis and quantum information theory, the two fields have so far remained largely disjoint, despite ample evidence that there could be a fruitful cross pollination between the two. In this workshop we have brought together some of the leading experts in both fields, as well as talented young mathematicians.

Mathematics Subject Classification (2020): 46L51, 22D25, 81P99.

Introduction by the Organizers

The theory of *quantum information* is a young branch of mathematics, physics and computer science that deals with the way information is exchanged using the laws of quantum mechanics. The most fundamental difference with classical information is that the laws governing quantum phenomena are by their very nature ‘noncommutative’ and ‘probabilistic’; the first fundament of which was already laid by Heisenberg in his matrix analysis and uncertainty principle. The language was then mathematically formalized by John von Neumann which turned into the eponymous theory of *von Neumann algebras*, a very active branch of mathematics nowadays.

Quantum information has seen remarkable development since the introduction of Shor's algorithm for factorisation of prime numbers in 1994, and has since then only moved from strength to strength. One of the most exciting recent advances - in both quantum information and von Neumann algebras - was the resolution of Connes' Embedding Problem through a difficult problem in quantum information (Tsirelson's problem) and the theory of nonlocal games. Several of the attendees and talks in this workshop have achieved further advances that build on this resolution; most notably the results that pertain to noncommutative cones, nonlocal games, complexity theory (NPA hierarchy) and analysis of quantum channels. In particular we scheduled two mini-lecture series by Carlos Palazuelos and Melchior Wirth on recent progress on nonlocal games and quantum Markov semigroups making further connections to the geometry of noncommutative spaces and state spaces of quantum systems.

At the same time as the mathematical foundations for quantum mechanics were being laid, harmonic analysis underwent radical changes throughout the 20^{th} century, led by the pioneering efforts of Bourgain, Carleson, Harish-Chandra, Hörmander, Fefferman, and many other talented analysts. Again this theory had a broad impact. Hörmander's analysis of pseudo-differential operators soon led to applications in differential geometry and the Atiyah–Singer index theorem. Bourgain gave several applications to the geometry of Banach spaces and analysis of partial differential equations. The theory deals with 'commutative harmonic analysis' on $\mathbb{R}^n$ or $\mathbb{T}^n$ where a vast theory of convolution operators and the closely related pseudo-differential calculus was established.

Currently there is tremendous interest in developing a complete theory of harmonic analysis in the noncommutative realm. Here Euclidean groups, like $\mathbb{R}^n$, are replaced by non-abelian groups and quantum groups, while their analysis takes place on the associated von Neumann algebras. The urgency to construct such a theory of 'noncommutative harmonic analysis' is kindled by several problems in operator theory, mathematical physics and quantum information. There is strong evidence that tools from harmonic analysis will provide powerful means to attack such problems. As a matter of fact, some of these problems have been resolved recently by some of our participants. Moreover a strategy and pivotal (unresolved) conjecture towards Connes' rigidity conjecture was announced and outlined in the mini-lectures of Javier Parcet during this workshop, supplemented by the talk of Mikael de la Salle.

That the two fields, quantum information and noncommutative harmonic analysis, are so deeply intertwined leads to the *key focus* of our workshop: to elaborate upon this connection and to make explicit the shared foundations and developments in the two fields. The workshop was well attended with over 40 participants with broad representation in geography, age, gender and themes. At the same time it was well rooted in the German academic landscape which houses several institutes that are at the forefront of present day quantum research. The organizers wish to express their gratitude to the Oberwolfach Research Center (MFO) and its staff for their hospitality and truly excellent organisation.

By the organizers,
Michael Brannan, *Martijn Caspers*, *Lyudmyla Turowska*, *Fedor Sukochev*

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Workshop: Noncommutative Harmonic Analysis and Quantum Information

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Abstracts

Non-signalling values of quantum games and Grothendieck's theorem for operator spaces

Carlos Palazuelos

The aim of the presentation is to introduce two-player quantum games from an operator space point of view. Special emphasis will be placed on explaining why the theory of noncommutative L_p-spaces is essential in this context and how these spaces, together with certain tensor norms, provide the appropriate framework for describing several notions of the value of such games. Some explicit examples of games will be presented.

In the last part of the presentation, we will discuss the connection between these games and specific norms in the theory of operator algebras; in particular, with an operator space version of Grothendieck's theorem.

Quantum Markov Semigroups

Melchior Wirth

In this mini-lecture series, I gave a short overview over quantum Markov semigroups with a focus on the structure of their generators. Quantum Markov semigroups originated in mathematical physics as models of the time evolution of certain open quantum system [GKS76, Lin76, AL07] and have since developed into versatile tools in various fields of pure and applied mathematics, including noncommutative analysis, operator algebras and quantum information.

Mathematically, a *quantum Markov semigroup* (QMS) is a family $(\Phi_t)_{t\geq 0}$ of maps on a von Neumann algebra M such that

(i) $\Phi_0 = \mathrm{id}_M$, $\Phi_{s+t} = \Phi_s\Phi_t$ for $s, t \geq 0$,
(ii) Φ_t is normal ucp for all $t \geq 0$,
(iii) $\Phi_t(x) \to x$ as $t \to 0$ in the weak* topology for every $x \in M$.

As a consequence of the semigroup law (i), a QMS is uniquely determined by its derivative at 0, which is called its *generator*. More precisely, the generator is the (possibly unbounded) operator $\mathcal{L}$ on M with domain

$$\mathrm{dom}(\mathcal{L}) = \left\{ x \in M : \lim_{t\to 0} \frac{1}{t}(x - \Phi_t(x)) \text{ exists in weak}^* \text{ topology} \right\},$$

on which it acts as

$$\mathcal{L}(x) = \text{weak}^*\text{-} \lim_{t\to 0} \frac{1}{t}(x - \Phi_t(x)).$$

First results concentrated on QMS with bounded generator, which is equivalent to the QMS being point-norm continuous on M. In this setting, Lindblad [Lin76] and Gorini–Kossakowski–Sudarshan [GKS76] obtained characterizations of generators of QMS on type I factors, before Christensen–Evans [CE79] settled the case of general von Neumann algebras in the following theorem.

Theorem 1 ([CE79, Theorem 3.1]). *A bounded operator $\mathcal{L}$ on M is the generator of a QMS if and only if there exist $k \in M$ and a normal cp map $\Phi\colon M \to M$ such that $\Psi(1) = k + k^*$ and*

$$\mathcal{L}(x) = kx + xk^* - \Psi(x), \quad x \in M.$$

A key point of the proof is that if $\mathcal{L}$ is the generator of a QMS, then the *carré du champ* operator

$$\Gamma\colon M \times M \to M,\, (x,y) \mapsto \frac{1}{2}(\mathcal{L}(x)^*y + x^*\mathcal{L}(y) - \mathcal{L}(x^*y))$$

is completely positive in the sense that $[\Gamma(x_j, x_k)]_{j,k} \in M_n(M)$ is positive for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in M$. Using this fact, one can construct a Hilbert W^*-bimodule F over M and a derivation $\delta\colon M \to F$ such that $\langle \delta(x) | \delta(y) \rangle_F = \Gamma(x,y)$ for $x, y \in M$. To conclude, one uses deep results of Christensen and Sakai that show that certain derivations on von Neumann algebras are inner.

The case of QMS with unbounded generators is technically far more challenging and to date, no complete characterization is known. A particularly interesting case are QMS that satisfy an additional symmetry condition. This is both a mathematically natural assumption and also physically motivated by the study of open quantum systems coupled to an environment in equilibrium.

Given a normal faithful state φ on M, there are various (non-equivalent) symmetry conditions expressed in terms of modular theory. A QMS (Φ_t) on M is called *GNS-symmetric* with respect to φ if

$$\varphi(\Phi_t(x)^*y) = \varphi(x^*\Phi_t(y)), \quad x, y \in M, t \geq 0$$

and *KMS-symmetric* if

$$\langle \Phi_t(x)\Omega_\varphi, \Delta_\varphi^{1/2} y\Omega_\varphi \rangle = \langle x\Omega_\varphi, \Delta_\varphi^{1/2} P_t(y)\Omega_\varphi \rangle, \quad x, y \in M, t \geq 0.$$

Here Ω_φ denotes the cyclic vector in the GNS representation of φ and Δ_φ the associated modular operator. While not apparent from the definition, GNS symmetry is stronger than KMS symmetry and thus easier to handle. In fact, a QMS (Φ_t) is GNS-symmetric if and only if it is KMS-symmetric and commutes with the modular group σ^φ. In particular, both symmetry conditions coincide if τ is a tracial state.

Every KMS-symmetric QMS (Φ_t) can be extended to a strongly continuous symmetric contraction semigroup (T_t) on the GNS Hilbert space $L^2(M, \varphi)$ given by the formula $T_t(\Delta_\varphi^{1/4} x\Omega_\varphi) = \Delta_\varphi^{1/4}\Phi_t(x)\Omega_\varphi$. As such, the generator L of (T_t) is a positive self-adjoint operator on $L^2(M, \varphi)$ and (T_t) can be expressed in terms of L by $T_t = e^{-tL}$ via functional calculus. The operator L is called the L^2-*generator* of (Φ_t). In the following we collect some structural properties of the L^2-generators that amount almost to a full characterization in the GNS-symmetric case.

Proposition 2 ([Wir22, Theorem 6.3]). *Assume that (Φ_t) is GNS-symmetric. The set*

$$\mathcal{C} = \{x \in M \mid x\Omega_\varphi \operatorname{dom}(\Delta_\varphi^n),\, \Delta_\varphi^n(x\Omega_\varphi) \in \operatorname{dom}(L^{1/2}) \cap M\Omega_\varphi \textit{ for all } n \in \mathbb{Z}\}$$

is a σ^φ-invariant weak dense unital $*$-subalgebra of M and $\mathcal{C}\Omega_\varphi$ is a form core for L.*

Theorem 3 ([Wir22, Theorem 6.8])**.** *Assume that (Φ_t) is GNS-symmetric and let $C = \overline{\mathcal{C}}^{\|\cdot\|}$. There exists a Hilbert bimodule $\mathcal{H}$ over C, an anti-unitary involution $\mathcal{J}\colon \mathcal{H} \to \mathcal{H}$, a strongly continuous unitary group $(\mathcal{U}_t)$ on $\mathcal{H}$ and a closable operator $\delta\colon \mathcal{C}\Omega_\varphi \to \mathcal{H}$ satisfying*

(i) $\mathcal{J}(x\xi y) = y^*(\mathcal{J}\xi)x^*$, $\mathcal{U}_t(x\xi y) = \sigma_t^\varphi(x)(\mathcal{U}_t\xi)\sigma_t^\varphi(y)$ *for* $x, y \in C$, $\xi \in \mathcal{H}$, $t \in \mathbb{R}$,
(ii) $\mathcal{U}_t\mathcal{J} = \mathcal{J}\mathcal{U}_t$ *for* $t \in \mathbb{R}$,
(iii) $\delta \circ \Delta_\varphi^{it} = \mathcal{U}_t \circ \delta$, $\delta \circ J_\varphi = \mathcal{J} \circ \delta$,
(iv) $\delta(x\Omega_\varphi y) = x\delta(\Omega_\varphi y) + \delta(x\Omega_\varphi)y$ *for* $x, y \in C$

such that

$$L = \delta^*\overline{\delta}.$$

Theorem 4 ([Wir24, Theorem 4.4])**.** *If $\mathcal{C} \subset M$ is a σ^φ-invariant weak* dense unital $*$-subalgebra of M, the quadruple $(\mathcal{H}, \mathcal{J}, (\mathcal{U}_t), \delta)$ satisfies (i)–(iv) from Theorem 3 and the left and right action of C on $\mathcal{H}$ are weak* continuous, then $\delta^*\overline{\delta}$ is the L^2-generator of a GNS-symmetric QMS.*

In the case when φ is a trace, Proposition 2 was first obtained by [DL92] and Theorems 3, 4 where first proven by Cipriani and Sauvageot[CS03]. In this case, the requirement that the bimodule actions are weak* continuous is not necessary in Theorem 4.

For KMS-symmetric QMS, the picture is much less clear. Even the case of bounded generators is not completely understood in the sense that it is not known whether the maps $x \mapsto kx + xk^*$ and Ψ in the Christensen–Evans theorem can be separately taken to be KMS-symmetric.

Together with Matthijs Vernooij (work in progress), we obtained the following structure result for bounded L^2-generators of KMS-symmetric QMS. This improves upon previous joint work with Vernooij [VW23] in that it fully characterizes KMS-symmetric QMS on matrix algebras, and we conjecture that this continues to hold for arbitrary von Neumann algebras.

Theorem 5. *If (Φ_t) is a KMS-symmetric QMS on M with bounded generator, then there exists an M-M-correspondence $\mathcal{H}$, an anti-unitary involution $\mathcal{J}\colon \mathcal{H} \to \mathcal{H}$ and an operator $\delta\colon M\Omega_\varphi \cap M'\Omega_\varphi \to \mathcal{H}$ such that*

(i) $\mathcal{J}(x\xi y) = y^*(\mathcal{J}\xi)x^*$ *for* $x, y \in M$, $\xi \in \mathcal{H}$, $t \in \mathbb{R}$,
(ii) $\delta \circ J_\varphi = \mathcal{J} \circ \delta$,
(iii) $\delta(x\Omega_\varphi y) = x\delta(\Omega_\varphi y) + \delta(x\Omega_\varphi)y$ *for* $x, y \in M$

such that

$$\langle a, Lb\rangle = \int_{\mathbb{R}} \langle \delta(\Delta_\varphi^{it}a), \delta(\Delta_\varphi^{it}b)\rangle_{\mathcal{H}} \frac{2\,dt}{\cosh(2\pi t)}, \quad a, b \in L^2(M, \varphi).$$

If M is a type I *factor, then δ is bounded.*

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Cones and asymptotic liftings

Tatiana Shulman

Many important properties of C*-algebras are invariant under homotopies. A somewhat surprising example of such a property is quasidiagonality as was discovered by Voiculescu [4]. Recall that a C*-algebra is *quasidiagonal (QD)* if if for each $\epsilon > 0$ and finite $\mathcal{F} \subset A$ there exist $n \in \mathbb{N}$ and a cpc map $\phi : A \to M_n$ such that

$$\|\phi(ab) - \phi(a)\phi(b)\| \le \epsilon,$$
$$\|\phi(a)\| \ge \|a\| - \epsilon,$$

for any $a, b \in \mathcal{F}$. A property closely related with quasidiagonality is the MF-property. A C*-algebra is called *MF (matricial field)* if it embeds into $\prod M_n / \oplus M_n$. Similar notions are defined for traces and play an important role in the classification program ([3]).

Recall that a trace τ on a C*-algebra A is *amenable* if for any $\epsilon > 0$ and a finite $G \subset A$ there is $n \in \mathbb{N}$ and a ccp map $\phi : A \to M_n$ such that

$$\|\phi(ab) - \phi(a)\phi(b)\|_2 < \epsilon \ \text{ and } \ |\tau(a) - tr\phi(a)| < \epsilon,$$

for any $a \in G$.

A trace τ is *quasidiagonal* if for any $\epsilon > 0$ and a finite $G \subset A$ there is $n \in \mathbb{N}$ and a ccp map $\phi : A \to M_n$ such that

$$\|\phi(ab) - \phi(a)\phi(b)\| < \epsilon \ \text{ and } \ |\tau(a) - tr\phi(a)| < \epsilon,$$

for any $a \in G$.

A trace τ is *hyperlinear* if for any $\epsilon > 0$ and a finite $G \subset A$ there is $n \in \mathbb{N}$ and a map $\phi : A \to M_n$ such that

$$\|\phi(ab) - \phi(a)\phi(b)\|_2 < \epsilon \text{ and } |\tau(a) - tr\phi(a)| < \epsilon,$$

for any $a \in G$.

A trace τ is *MF* if for any $\epsilon > 0$ and a finite $G \subset A$ there is $n \in \mathbb{N}$ and a map $\phi : A \to M_n$ such that

$$\|\phi(ab) - \phi(a)\phi(b)\| < \epsilon \text{ and } |\tau(a) - tr\phi(a)| < \epsilon,$$

for any $a \in G$.

Clearly each MF-trace is hyperlinear and each quasidiagonal trace is amenable. The opposite implications are an open problem.

Though MF-property is closely related with quasidiagonality, it is not known whether it is homotopy invariant. It is also not known whether the property that all amenable (hyperlinear, respectively) traces are quasidiagonal (MF, respectively) is homotopy invariant. In [2] R. Neagu proved that if A is exact, has a faithful trace and is homotopy dominated B and if all amenable traces on B are quasidiagonal, then the same is true for A. In [1] Brown, Carrion and White proved that on any cone C*-algebra all amenable traces are quasidiagonal.

Here we first give new characterizations of all the aforementioned properties in terms of liftings. We say that a $*$-homomorphism $\phi : A \to B/I$ *lifts to a (discrete) asymptotic homomorphism* if there exists an asymptotic homomorphism $f_t : A \to B$, $t \in [0, \infty)$ ($f_n : A \to B$, $n \in \mathbb{N}$ in discrete case) such that $q \circ f_t = \phi$, for each t ($q \circ f_n = \phi$, for each n, respectively).

Theorem 1. *TFAE:*

(i) *A is MF (QD, respectively),*

(ii) *every $*$-homomorphism from A to $B(H)$ lifts to a (cp) discrete asymptotic homomorphism from A to $\mathcal{D}$,*

Theorem 2. *Let τ be a trace on a C*-algebra A. TFAE:*

(i) *τ is an MF (quasidiagonal, respectively) trace,*

(ii) *there exists a $*$-homomorphism $f : A \to \prod M_{m_n} / \oplus_{2,\omega} M_{m_n}$ such that $\tau = trf$ and f lifts to a (cp) discrete asymptotic homomorphism from A to $\prod M_n$,*

(iii) *there exists a discrete asymptotic homomorphism $f^k : A \to \prod M_{m_n} / \oplus_{2,\omega} M_{m_n}$, $k \in \mathbb{N}$, such that $\tau(a) = \lim_{k\to\infty} trf^k(a)$, for any $a \in A$, and f^k lifts asymptotically to a (cp) discrete asymptotic homomorphism from A to $\prod M_{m_n}$.*

Our main result is the following homotopy lifting theorem.

Theorem 3. *Let $\phi : B \to A$ and $\psi : B \to A$ be two homotopic $*$-homomorphisms. Suppose ψ lifts to an asymptotic homomorphism. Then ϕ lifts to an asymptotic homomorphism. Moreover the whole homotopy lifts.*

We also obtain a completely positive version of this theorem. Using these homotopy lifting theorems and the lifting characterization of MF-property etc. above we obtain the following applications.

Theorem 4. *MF-property is homotopy invariant.*

The following theorem covers both Neagu's result and Brown-Carrion-White's result for cones as particular cases.

Theorem 5. *If either A or B is exact, A is homotopy dominated by B and all amenable traces on B are quasidiagonal, then all amenable traces on A are quasidiagonal.*

Theorem 6. *If a C*-algebra A is homotopy dominated by a nuclear C*-algebra B and all hyperlinear traces on B are MF, then all hyperlinear traces on A are MF.*

Theorem 7. *If B is Hilbert-Schmidt stable and A is homotopy dominated by B, then all hyperlinear traces on A are MF, and all amenable traces on A are quasidiagonal.*

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Quantum isometry groups of log-Laplacians on Cuntz–Krieger algebras

Adam Skalski

(joint work with Amaury Freslon and Dimitris Gerontogiannis)

Compact quantum groups appeared in mathematics first in a purely algebraic guise, as deformations of classical Lie groups. Only later they were understood as structures encoding *quantum* symmetries of C*-algebras, following Shuzhou Wang's seminal work [Wan98] (see [Ska17] and references therein). The general idea behind the construction of the quantum symmetry group of a given C*-algebra A, is based on the consideration of the universal (final) object in the category of compact quantum group actions on A, often preserving some additional structure – such as a given state on A.

Several years later Debashish Goswami and his collaborators initiated the study of quantum *isometry* groups (see [Gos09], [BG16]). The starting point for this programme is the fact that a transformation of a classical compact Riemannian manifold X is isometric if and only if it 'commutes' with natural differential operators associated with X, such as the Laplacian or the Dirac operator. Goswami proposed then to consider a C*-algebra A equipped with a spectral triple à la Connes, so in particular with a *Dirac operator* D, and look at the universal compact quantum group action on A 'commuting' in a suitable sense with D.

In [FGS26] we explore the quantum isometries of a Cuntz-Krieger algebra O_A of a finite directed graph with adjacency matrix $A \in M_N(\{0,1\})$, which we throughout assume to be *primitive*. Cuntz-Krieger algebras, introduced in [CK80], form an important class of C*-algebras encoding various aspects of the orbit space of topological Markov chains. Recently, in the article [GGM25], the algebras O_A were equipped with canonical Dirac-type operators. The key tool for this was the so-called log-Laplacian introduced in [GM25], a non-local diffusion operator on the underlying shift space. In the context of Cuntz–Krieger algebras, the construction in [GGM25] proceeds by endowing the associated Deaconu–Renault groupoid Γ_A with an Ahlfors regular structure and then, via additional intermediate steps, defining a Hamiltonian-type operator $D := -\Delta + V$ which yields a spectral triple on O_A. The relevant Hilbert space $L^2(\Gamma_A)$ can be identified with the GNS space $L^2(O_A, \tau)$, where τ is the unique KMS state of the *gauge* action on O_A. The classical isometry group associated with D was shown in [GGM25] to be the compact Lie group $\mathbb{T} \wr \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the automorphism group of A; note that the inclusion $\mathbb{T}^N \subset \mathbb{T} \wr \mathrm{Aut}(A)$ should be viewed as a multiparameter incarnation of the gauge action.

The main result of [FGS26] is the following theorem. For the exact meaning of the D-isometric action we refer to [FGS26]; note however that the action is assumed in particular to preserve the state τ. A *magic unitary* is a unitary matrix whose entries are orthogonal projections (see [Fre23]); by V_A^n we denote the set of multiindices in $\{1, \dots, N\}^n$ with 'transitions' allowed by A.

Theorem 1. *Consider the universal C^*-algebra $C(\mathbb{G}_A^\infty)$ generated by partial isometries $u = (u_{i,j})_{1\le i,j\le N}$, with range projections $p = (p_{i,j})_{1\le i,j\le N}$ and source projections $q = (q_{i,j})_{1\le i,j\le N}$, such that*

1. *p, q are magic unitaries;*
2. *both p and q preserve the* right Perron–Frobenius eigenvector *$\vec{v}$ of A: $p\vec{v} = \vec{v}1 = q\vec{v}$;*
3. *A intertwines p and q: $Ap = qA$;*
4. *for any $n \in \mathbb{N}$ and multiindices $\mathbf{i}, \mathbf{j} \in V_A^n$, the element $u_{\mathbf{i},\mathbf{j}} := u_{i_1,j_1} \dots u_{i_n,j_n}$ is a partial isometry.*

Then $C(\mathbb{G}_A^\infty)$, equipped with a natural coproduct, defines a compact matrix quantum group of Kac type. Moreover the quantum group $\mathbb{G}_A^\infty$ is the quantum isometry group of the spectral triple on O_A associated with D.

In fact the quantum group $\mathbb{G}_A^\infty$ belongs to a whole series (that we call *Ariadne's thread*) of compact quantum groups $\mathbb{G}_A^\ell$, with $\ell \in \mathbb{N} \cup \{\infty\}$, such that $\mathbb{G}_A^\ell \supset \mathbb{G}_A^{\ell'}$ for $\ell \le \ell'$. These are defined by keeping the conditions (1)-(3) above, but requiring only that the operators $u_{\mathbf{i},\mathbf{j}}$ appearing in (4) are partial isometries for all multiindices $\mathbf{i}, \mathbf{j}$ of length at most ℓ.

The properties defining $\mathbb{G}_A^\ell$ simplify in the Cuntz algebra case, i.e. when the matrix $A := \mathbf{1}$ consists only of 1's. Then conditions (2) and (3) in the above theorem are then automatically satisfied. It turns out that each $\mathbb{G}_{\mathbf{1}}^\ell$ is a unitary easy quantum group in the class discovered by Alexander Mang in his thesis [Man22]. Moreover, the quantum groups $\mathbb{G}_{\mathbf{1}}^\ell$ are mutually non-isomorphic for different values of ℓ. On the other hand it can happen that the thread terminates in one step, namely $\mathbb{G}_A^1 = \mathbb{G}_A^\infty$. This is the case for example when A is the adjacency matrix of the Fibonacci graph. There all these quantum groups coincide with $\widehat{\mathbb{F}_N}$ (so we obtain purely the 'quantum gauge action').

The quantum groups $\mathbb{G}_A^\ell$ appear to be closely connected to *quantum automorphism groups of graphs* ([Ban05]), denoted by $\mathrm{QAut}(A)$. Whilst the full relationship between $\mathrm{QAut}(A)$ and $\mathbb{G}_A^\infty$ remains somewhat elusive, we can see for example that $\mathbb{G}_{\mathbf{1}}^\infty$ admits $S_N^+ \times S_N^+$ as a *subquotient*, where S_N^+ denotes the quantum permutation group (which can be viewed as the quantum symmetry group of the complete graph on N vertices).

This leads us to the next main result of this paper, i.e. the fact that, although the action of the classical isometry group on O_A is never ergodic, in the case of the Cuntz algebra the quantum isometry group does act ergodically.

Theorem 2. *The action of $\mathbb{G}_{\mathbf{1}}^\infty$ on the Cuntz algebra O_N is ergodic.*

Thus the quantum isometry groups of Cuntz-Krieger algebras are much larger than their classical counterparts, which should be contrasted with the fact that quantum isometry groups of classical smooth manifolds coincide with their classical versions ([GJ18]). It turns out that the construction described above yields also a compact quantum group acting faithfully and ergodically on the Cantor set. Such quantum actions were considered in the literature as arising from certain inductive limits. Now however, for the first time, we produce an example of an ergodic action of a compact *matrix* quantum group on the Cantor set.

Theorem 3. *The action of $\mathbb{G}_A^\infty$ preserves the path space Σ_A (i.e. a Cantor set). In particular, a certain quotient $\mathbb{H}_{\mathbf{1}}$ of $\mathbb{G}_{\mathbf{1}}^\infty$, which is again a (non-classical) compact quantum group, acts faithfully and ergodically on $\Sigma_{\mathbf{1}}$.*

As mentioned above, the structure of quantum groups $\mathbb{G}_A^\infty$ remains somewhat mysterious. In particular we do not know the answers to the following questions.

Question 1. *The quantum group $\mathbb{G}_A^1$ contains $\mathrm{QAut}(A)$ as a quantum subgroup – but does the same hold for $\mathbb{G}_A^\infty$, when $\mathrm{QAut}(A) \neq \mathrm{Aut}(A)$?*

Question 2. *Can we establish ergodicity of the quantum isometry action in some cases beyond $A = \mathbf{1}$?*

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Correlations over the Cantor space

Alexandros Chatzinikolaou

(joint work with G. Baziotis, I. G. Todorov, L. Turowska)

We study infinite parallel repetition of finite non-local games by modelling it as a single game over Cantor spaces arising from inductive families of finite sets. This framework allows us to treat asymptotic behaviour intrinsically, rather than as a limiting procedure.

A *Cantor correlation* is formulated as a unital completely positive, no-signalling map between function algebras on Cantor spaces. Such correlations are shown to be equivalent to compatible families of finite-level correlations. Using operator-system techniques, we obtain characterisations of spatial, approximate and commuting quantum models via operator-valued information channels, which serve as infinite-dimensional analogues of POVMs. Membership in these classes is detected entirely at the finite levels.

In the graded synchronous setting, we introduce a universal C*-algebra generated by compatible projection-valued measures and prove a tracial characterisation of quantum commuting and quantum approximate Cantor correlations. In the bisynchronous case, this algebra is replaced by an inductive limit of free wreath products, yielding a compact quantum group that can be interpreted as a quantum automorphism group of a rooted tree.

We further establish a correspondence between graded bisynchronous Cantor correlations and factorizable maps on $\mathcal{B}(L^2(\Omega_X))$, implemented by limits of compatible finite-level quantum permutations, which we call *Cantor quantum permutations*. Finally, we extend the operator-algebraic characterisation of perfect quantum strategies in graph isomorphism games to the setting of Cantor graphs.

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Quantum graphs and connectivity

Priyanga Ganesan

Quantum Graphs are a non-commutative generalization of classical graphs that have recently appeared in operator algebras, quantum information theory and non-commutative geometry. In this talk, we discuss a notion of connectivity for quantum graphs using quantum adjacency matrices, which generalizes the classical notion and unifies existing notions in the literature. We show that this new algebraic perspective of connectivity simplifies and leads to quantum analogues of several classical results, including characterizations involving the irreducibility of the adjacency operator and nullity of the quantum graph Laplacian.

Bisynchronous quantum games

Adina Goldberg

We describe how equivalences of quantum sets (finite, discrete quantum spaces) are witnessed by quantum information theoretic properties of a highly symmetric class of quantum input/output games, called bisynchronous quantum games. This talk is an overview of [1, Section 4]. The definitions used here of quantum games and synchronicity are also introduced in [1].

Here, quantum sets are the finite quantum sets of [3, 2], which are finite direct sums of matrix algebras with a distinguished orthonormal basis. The key is that they have both an algebra structure and a compatible Hilbert-space structure.

The usual formulation of nonlocal games (with classical questions and answers) are as 'rule' functions $\lambda : X \times Y \times A \times B \to \{0, 1\}$ on a quadruple of finite sets of questions and answers. These can be equivalently formulated as linear maps

$C(X) \otimes C(Y) \to C(A) \otimes C(B)$ whose matrices are binary in the dual basis. In other words, their matrices are real Schur-product projections.

We quantize nonlocal games by defining a quantum game to be (appropriately analogous to) a 'real Schur projection' $\lambda : \mathcal{X} \otimes \mathcal{Y} \to \mathcal{A} \otimes \mathcal{B}$ between quantum sets. In Carlos Palazuelos' talk on quantum games earlier this week, a partial transpose of λ was introduced as a game tensor.

We then define synchronicity and bisynchronicity of quantum games, generalizing these properties for nonlocal games [4, 5]. Finally, using some terminology from the nonlocal/quantum games literature, we state Corollary 4.9 of [1], which can be summarized as follows.

Let $\mathcal{X}, \mathcal{A}$ be quantum sets, and $\mathcal{X}^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}}$ their opposites. Given a bisynchronous quantum game $\lambda : \mathcal{X} \otimes \mathcal{X}^{\mathrm{op}} \to \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ with the information theoretic property in the left column, then the quantum sets $\mathcal{X}, \mathcal{A}$ are equivalent in the sense presented in the right column.

If λ has a perfect...	Then $\mathcal{X}$ and $\mathcal{A}$ are...
bicorrelation	of equal dimension
bistrategy	of equal length (same # full matrix blocks)
quantum commuting bistrategy	quantum-commuting isomorphic
deterministic bistrategy	isomorphic

For a brief explanation of the result, one must interpret correlations and strategies in the quantum question/answer setting.

From a quantum-information theoretic point of view, correlations are bipartite channels, and bicorrelations are those who are channels in reverse. Strategies can be thought of as entanglement-assisted bipartite channels, and again, bistrategies are also strategies in reverse.

From an operator algebraic point of view, a correlation is a completely positive, trace preserving map $\mathcal{X} \otimes \mathcal{Y} \to \mathcal{A} \otimes \mathcal{B}$ and a strategy is a quantum function as given by [3], which is equivalent to a $*$-homomorphism $\mathcal{A} \otimes \mathcal{B} \to \mathcal{X} \otimes \mathcal{Y} \otimes B(\mathcal{H})$. The 'bi-' prefix indicates that the adjoint also has the named property.

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Highly entangled states from S_N^+

Alexander Wendel

In the 2000's it was shown that random subspaces of fixed dimension of a tensor product space are with high probability highly entangled in the geometric sense. A deterministic example of high relative dimension has been due for quite a time. In 2018 Brannan and Collins [1] constructed a first deterministic example from the representation theory of the orthogonal quantum group O_N^+. I will report on some results of my MSc thesis [2] where I develop some of their results for general free easy quantum groups and finally apply those to construct similar examples for the symmetric quantum group S_N^+.

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Sofic Groups and Quantum Information

Aareyan Manzoor

Definition. A group Γ is said to be **hyperlinear** (resp. **sofic**) if for each finite subset $F \subset \Gamma$ and $\epsilon > 0$, there is a map $\phi : \Gamma \to U(n)$ (resp. $\phi : \Gamma \to S_n \subset U(n)$) such that:

(1) $\|\phi(xy) - \phi(x)\phi(y)\|_2 \leq \epsilon$ whenever $x, y, xy \in F$;
(2) $\|\phi(x) - 1\|_2 \geq 1 - \epsilon$ whenever $x \in F$ is not the identity.

Here $\| - \|_2$ is the normalized Hilbert-Schmidt norm.

One should read the definition of hyperlinear (resp. sofic) groups as "approximate homomorphisms into unitaries (resp. permutations) separate points of Γ". In this sense, hyperlinearity is an approximation by unitary groups while soficity is an approximation by finite groups.

One can often prove things for sofic groups by first proving them for finite groups and then lifting them up. Some such properties include: Lück's determinant conjecture [Lüc94] (important for L^2-homology and entropy theory), the algebraic eigenvalue conjecture in functional analysis [Tho08], and the Kervaire–Laudenbach conjecture in group theory [NT22]. Entropy can also be defined for actions of sofic groups [Bow10], leading to dynamical properties like Gottschalk's Surjunctivity Conjecture being satisfied by these groups.

A non-sofic group is not known yet, so to even begin to tackle any of these properties, one needs to first resolve the following problem:

Question. Is there a non-sofic group?

The answer is expected to be yes, and my talk discussed some partial results toward this and a potential approach to the problem.

If one could find a non-hyperlinear group, it would also be a non-sofic group. While this is a priori a harder problem, we have more tools to study hyperlinearity. Indeed, hyperlinearity can be detected on the group von Neumann algebra $L(\Gamma)$ of Γ, and so techniques from tracial von Neumann algebras can be applied to study hyperlinearity.

It turns out hyperlinearity can be extended to all tracial von Neumann algebras (where it is also often called Connes embeddability). In this setting we have:

Theorem 1 (MIP* = RE [JNVWY20])**.** *There is a non-hyperlinear tracial von Neumann algebra.*

This was known as the Connes embedding problem. The amazing fact about this paper is that it is a result of quantum complexity theory and not operator algebras.

Narrowing this down to a group von Neumann algebra would yield a non-sofic group. While we do not have this yet, we do have a non-hyperlinear group action:

Theorem 2 ([Man25A])**.** *There is an ergodic probability measure preserving action α of some free group $\mathbb{F}_n$ on a standard probability space (X, μ) such that the corresponding equivalence relation algebra $L(R_\alpha)$ is not hyperlinear.*

This in particular gives a simpler disproof of the Aldous–Lyons conjecture, which stated that unimodular random rooted graphs are limits of finite random graphs [AL07]. A disproof using an uncomputability result called TailoredMIP* = RE already existed [BCLV'24, BCV'25], but the operator algebraic argument simplifies the reduction of Aldous–Lyons to TailoredMIP* = RE signficantly.

The method for this involved the following observation: each separable tracial von Neumann algebra arises from a trace (i.e. a positive definite conjugation-invariant function) on $\mathbb{F}_\infty$, the free group on countably many generators. This is by taking the Gelfand–Naimark–Segal (GNS) representation of the trace and then the von Neumann algebra generated by the image. Hyperlinear algebras correspond to traces that are limits of finite-dimensional traces. Thus, exhibiting a trace on $\mathbb{F}_\infty$ which is not a limit of finite-dimensional traces would negatively resolve Connes embeddability. The uncomputability result MIP* = RE allows one to do precisely this.

An invariant random subgroup (IRS) on a group Γ is a conjugation-invariant probability measure on the space of subgroups of Γ. Since Dirac measures on normal subgroups are IRS, the notion of an IRS generalizes the notion of a normal subgroup. Hence the notion of an IRS of a free group generalizes the notion of a group. One can generalize the notion of sofic groups to co-sofic IRS on free groups, and the Aldous–Lyons conjecture is equivalent to saying all IRS on free groups are co-sofic.

The theorem above was proven by finding a class of traces that correspond to IRS and that admits a computability result analogous to MIP* = RE. For an IRS

H on Γ, this trace is

$$\tau_H(g) = \mathbb{P}(g \in H).$$

This then showed the existence of a non-hyperlinear algebra corresponding to one of these IRS, which naturally embeds into an equivalence relation algebra.

Finding suitable classes of traces and using computability on those classes has led to a lot of recent activity:

Theorem 3 ([Man25B])**.** *There is an IRS H of a free group that is non-hyperlinear but satisfies Lück's determinant conjecture.*

Theorem 4 ([Aru-Man25])**.** *If S is a locally universal tracial von Neumann algebra (i.e. approximate homomorphisms into it separate points for every separable tracial von Neumann algebra), then S has no computable presentations.*

Theorem 5 ([Bow-Cha25])**.** *There is an IRS H of a free group that is non-hyperlinear but satisfies Gottschalk's Surjunctivity Conjecture.*

In particular, since we do have non-co-sofic IRS on free groups, we can study the properties mentioned previously in this setting.

This leads to a proposed approach to finding a non-sofic group:

(1) Find a set of traces C on free groups so that the corresponding algebras are subalgebras of (ultraproducts of) group algebras.
(2) This set should be large enough to expect an uncomputability result. A good litmus test is whether this set contains a Poulsen simplex. This is true for both the set of all traces $T(\mathbb{F}_n)$ and the set of IRS traces $T_{\mathrm{IRS}}(\mathbb{F}_n)$.
(3) This set should be explicit; otherwise none of the computability tools will apply.

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Tensor products of convex compact sets and entanglement in C^*-algebras

Mikael Rørdam
(joint work with Magdalena Musat)

The Namioka–Phelps tensor products of two compact convex sets K_1 and K_2 are defined to be the state spaces:

$$\begin{aligned} K_1 \otimes_* K_2 &= S(A(K_1) \otimes A(K_2), A(K_1)^+ \widehat{\otimes} A(K_2)^+, 1 \otimes 1), \\ K_1 \otimes^* K_2 &= S(A(K_1) \otimes A(K_2), A(K_1)^+ \otimes A(K_2)^+, 1 \otimes 1), \end{aligned}$$

where $A(K_j)$ denotes the real vector space of continuous affine functions on K_j, where $A(K_1)^+ \otimes A(K_2)^+$ is the cone in $A(K_1) \otimes A(K_2)$ generated by $f_1 \otimes f_2$, with $f_j \in A(K_j)^+$, and where $A(K_1)^+ \widehat{\otimes} A(K_2)^+$ is the set of functions f in $A(K_1) \otimes A(K_2)$ for which $\langle f, \varphi_1 \otimes \varphi_2 \rangle \geq 0$, for all $\varphi_j \in (A(K_j)^*)^+$.

It is easy to see that $K_1 \otimes_* K_2 \subseteq K_1 \otimes^* K_2$, and that $K_1 \otimes_* K_2$ and $K_1 \otimes^* K_2$ have the same (finite) dimension, when both K_1 and K_2 are finite dimensional. Namioka and Phelps prove in [3] that $K_1 \otimes_* K_2 = K_1 \otimes^* K_2$ if one of K_1 and K_2 is a Choquet simplex, and that $K_1 \otimes_* \square = K_1 \otimes^* \square$ if and only if K_1 is a Choquet simplex, and where $\square$ denotes the solid 2-dimensional square. It is left open if $K_1 \otimes_* K_2 \neq K_1 \otimes^* K_2$ in general when neither K_1 or K_2 are Choquet simplexes. This is known as Barker's conjecture, and it was confirmed in [1] in the case where both K_1 and K_2 are finite dimensional.

We study the Namioka–Phelps tensor products for compact convex sets arising as the state spaces, respectively, the set of tracial states, on unital C^*-algebras. Specifically, if $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras, then it is shown that

$$S(\mathcal{A}) \otimes_* S(\mathcal{B}) = S_*(\mathcal{A} \otimes \mathcal{B}),$$

where the latter set denotes the set of *separable states* on $\mathcal{A} \otimes \mathcal{B}$, i.e., the weak* closure of the convex hull of product states $\varphi_{\mathcal{A}} \otimes \varphi_{\mathcal{B}}$, with $\varphi_{\mathcal{A}} \in S(\mathcal{A})$ and $\varphi_{\mathcal{B}} \in S(\mathcal{B})$. In general,

$$S(\mathcal{A}) \otimes_* S(\mathcal{B}) = S_*(\mathcal{A} \otimes \mathcal{B}) \subseteq S(\mathcal{A} \otimes \mathcal{B}) \subseteq S(\mathcal{A} \otimes_{\max} \mathcal{B}) \subseteq S(\mathcal{A}) \otimes^* S(\mathcal{B}), \tag{1}$$

with strict inequalites at the first and last inclusion whenever both $\mathcal{A}$ and $\mathcal{B}$ are non-commutative; and equalities everywhere when one of $\mathcal{A}$ and $\mathcal{B}$ is commutative. It is well-known, and also follows from the result above, that $S(\mathcal{A})$ is a Choquet simplex if and only if $\mathcal{A}$ is commutative. Combining these facts, one sees that Barker's conjecture holds when K_1 and K_2 above are state spaces of C^*-algebras.

While the minimal Namioka–Phelps tensor product $S(\mathcal{A}) \otimes_* S(\mathcal{B})$ is well understood, the maximal one $S(\mathcal{A}) \otimes^* S(\mathcal{B})$ is more elusive. The relative size of the last inclusion in (1) can be described via the quantity

$$\kappa(\mathcal{A}, \mathcal{B}) = \sup\{\|\rho\|_{\max} : \rho \in S(\mathcal{A}) \otimes^* S(\mathcal{B})\},$$

where $\|\rho\|_{\max}$ is the norm of ρ with respect to the max-tensor norm on (the algebraic tensor product) $\mathcal{A} \odot \mathcal{B}$. We have $S(\mathcal{A} \otimes_{\max} \mathcal{B}) = S(\mathcal{A}) \otimes^* S(\mathcal{B})$ if and only if $\kappa(\mathcal{A}, \mathcal{B}) = 1$, which is shown to happen precisely when one of $\mathcal{A}$ and $\mathcal{B}$ is commutative. We show that

$$\begin{aligned} &S(M_n(\mathbb{C})) \otimes^* S(M_m(\mathbb{C})) \\ &\qquad = \big\{\rho \circ (\Phi \otimes \mathrm{id}_m) : \Phi \in \mathrm{UPos}(n, m),\ \rho \in S(M_m(\mathbb{C}) \otimes M_m(\mathbb{C}))\big\}, \end{aligned}$$

where $\mathrm{UPos}(n, m)$ denotes the set of unital positive linear maps $\Phi\colon M_n(\mathbb{C}) \to M_m(\mathbb{C})$, and that

$$\kappa(M_n(\mathbb{C}), M_m(\mathbb{C})) = \min\{n, m\}.$$

It seems plausible that one can describe $S(\mathcal{A}) \otimes^* S(\mathcal{B})$ in terms of positive maps for general C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and, further, it seems plausible that

$$\kappa(\mathcal{A}, \mathcal{B}) = \min\{\mathrm{rank}(\mathcal{A}), \mathrm{rank}(\mathcal{B})\},$$

where the rank of a C^*-algebra is the supremum of those n in $\mathbb{N} \cup \{\infty\}$ for which the C^*-algebra has an irreducible representation on a Hilbert space of dimension n. We show that $\kappa(\mathcal{A}, \mathcal{B}) \geq \min\{\mathrm{rank}(\mathcal{A}), \mathrm{rank}(\mathcal{B})\}$ holds.

Again, with $\mathcal{A}$ and $\mathcal{B}$ being unital C^*-algebras, and $T(\mathcal{A})$ and $T(\mathcal{B})$ denoting the (Choquet simplex) of tracial states on $\mathcal{A}$ and $\mathcal{B}$, respectively, we get

$$T(\mathcal{A}) \otimes_* T(\mathcal{B}) = T_*(\mathcal{A} \otimes \mathcal{B}) = T(\mathcal{A} \otimes \mathcal{B}) = T(\mathcal{A} \otimes_{\max} \mathcal{B}) = T(\mathcal{A}) \otimes^* T(\mathcal{B}), \tag{2}$$

where the equality $T(\mathcal{A}) \otimes_* T(\mathcal{B}) = T(\mathcal{A}) \otimes^* T(\mathcal{B})$ follows from Namioka–Phelps results, since (both) $T(\mathcal{A})$ and $T(\mathcal{B})$ are Choquet simplexes. We denote the common tensor product by $T(\mathcal{A}) \otimes T(\mathcal{B})$. We infer from the equalities in (2) that traces cannot be entangled!

We show that if S_1 and S_2 are Choquet simplexes, then $S_1 \otimes S_2$ is the Poulsen simplex if and only if both S_1 and S_2 are the Poulsen simplex (or one is trivial and the other is the Poulsen simplex). It follows that $T(\mathcal{A} \otimes \mathcal{B})$ is the Poulsen simplex if and only if $T(\mathcal{A})$ and $T(\mathcal{B})$ both are the Poulsen simplex (or one of them is and the other is trivial).

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Quantum Cuntz-Krieger Algebras of some Quantum Cayley Graphs

Luca Junk

Classical Cuntz-Krieger algebras of graphs are an important class of C^*-algebras providing us with many interesting examples. Their structure is well understood and largely determined by the underlying graph. In [1, 2] the Cuntz-Krieger construction was extended to quantum graphs with the goal of producing new C^*-algebras which can be studied via the underlying quantum graph. Unfortunately, not much is known about these algebras until now – except for the cases of the trivial and complete quantum graph – motivating the study of more interesting examples. We look at the quantum Cayley graphs (as defined in [3]) of the Kac-Paljutkin quantum group and compute the associated quantum Cuntz-Krieger algebras.

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Quantum advantage in Feige's game

Simon Schmidt

(joint work with Sigurd A. L. Storgaard, Michael Walter, Yuming Zhao)

A *nonlocal game* consists of two (or more) cooperative players, often called Alice and Bob, that interact with a referee. In the game, the referee sends a question to each player, for which each player gives an answer. Based on the tuple of questions and answers, the referee decides if the players win or lose. Communication is not permitted between Alice and Bob, hence each player has no information about the questions given to the other players, nor do they know the answers provided to the referee by the other players. Nevertheless, the description of the game is known to the players ahead of time, allowing them to agree on a strategy beforehand and maximize their probability of winning the game. The *classical value* $\omega_c(G)$ of a nonlocal game G is the maximum winning probability of classical players, and the *quantum value* $\omega_q(G)$ denotes the maximum winning probability of quantum players sharing a (finite) amount of quantum resources (such as entangled quantum states). Furthermore, the *non-signalling value* $\omega_{ns}(G)$ is the maximum winning probability if one considers probability distributions such that the marginal distribution of either player's answers must be independent of the other player's question, without requiring the probability distribution to be physically realizable.

Feige's game. In this talk, we look at a game first studied by Feige [1], where it was considered as counterexample in parallel repetition since its value does not decrease when played twice in parallel. The game is defined as follows. Alice and Bob both receive a bit as question (with uniform probability), and they are each allowed to choose an answer from the set $\{0, 1, \perp\}$. To win the game, exactly one of the players has to answer with $\perp$, while the other player's answer has to agree with the question the player who answered $\perp$ received. More formally, the game G_F [1] is defined via

$$X = Y = \{0, 1\}, \quad A = B = \{0, 1, \perp\}, \quad \pi(x, y) = \tfrac{1}{4} \quad \text{for all } (x, y)$$

and has the predicate

$$V(a, b, x, y) = \begin{cases} 1 & (a, b) = (\perp, x) \text{ or } (a, b) = (y, \perp), \\ 0 & \text{otherwise.} \end{cases}$$

It has been conjectured in the literature that $\omega_q(G_F) = \omega_c(G_F)$, but so far no attempts for computing the quantum value of the game were made. Since there is no quantum advantage for other variants of ”guess your neighbors input” games, it sounds plausible that there is also no quantum advantage in Feige's game. We show that, surprisingly, there exists a quantum strategy for Feige's game that exceeds the classical value and we compute the quantum value.

Theorem 1. *It holds $\omega_q(G_F) = \frac{9}{16} > \frac{1}{2} = \omega_c(G_F)$ for Feige's game G_F.*

Interestingly, the players use operators acting on $\mathbb{C}^3$ in the optimal quantum strategy and they share the maximally entangled state $|\psi_3\rangle = \sum_{i=0}^{2} |i\rangle \otimes |i\rangle$. The strategy that achieves a winning probability of $\frac{9}{16}$ is constructed from the operators

$$\tilde{Z} = Z \oplus 1 \quad \text{and} \quad \tilde{X} = X \oplus 1 \quad \text{on} \quad \mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C},$$

where X and Z denote the usual Pauli matrices. The projective measurements are given by

$$P_a^x = \tilde{X}\tilde{Z}^x\tilde{X} |\tilde{a}\rangle \langle\tilde{a}| \tilde{X}\tilde{Z}^x\tilde{X}, \quad a \in \{0, 1, \perp\} \quad \text{and} \quad Q_b^y = \tilde{X}P_b^y\tilde{X}, \quad b \in \{0, 1, \perp\},$$

for a carefully chosen basis $\{|\tilde{a}\rangle\}$ of $\mathbb{C}^3$. To obtain this basis, we crucially use that $\tilde{Z}$ and $\tilde{X}$ are two non-commuting observables such that $\tilde{Z}$ and $\tilde{X}\tilde{Z}\tilde{X}$ commute.

To prove that the quantum value of G_F is at most $\frac{9}{16}$, we find a *sums of square decomposition* of the game polynomial of Feige's game. More precisely, we obtain a vector of formal variables $V = V(P_a^x, Q_b^y)$ and a positive real matrix Y such that $V^*YV = \frac{9}{16}I - p(P_a^x, Q_b^y)$, which implies $\omega_q(G_F) \leq \frac{9}{16}$.

Parallel repetition. The original motivation of Feige was to give a counterexample in parallel repetition as its value does not decrease when played twice in parallel. In the *n-fold parallel repetition* $G^{\times n}$ of a game G, the players receive tuples of questions $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ from the original game and they answer with tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. Alice and Bob win in $G^{\times n}$ if the questions and answers x_i, y_i, a_i, b_i win the game G for each $i \in \{1, \ldots, n\}$. We study the behaviour of the game with respect to parallel repetition for the classical, quantum and non-signalling value. The results are summarized in 1.

From the previous discussion, we know that the classical value does not agree with the quantum value for the original game, and we show that the non-signalling

	$n=1$	$n=2$	$n=3$	n even
ω_c	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbf{\frac{5}{16}}$	$\frac{1}{2^{n/2}}$
ω_q	$\mathbf{\frac{9}{16}}$	$\mathbf{\frac{1}{2}}$	?	$\mathbf{\frac{1}{2^{n/2}}}$
ω_{ns}	$\mathbf{\frac{2}{3}}$	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{3}}$	$\mathbf{\frac{1}{2^{n/2}}}$

Table 1. Summary of values of the n-fold parallel repetition of Feige's game.

value is even higher. Furthermore, we get that at least the classical and non-signalling value are different for $n = 3$, despite this not being the case for $n = 2$. We do not know if there is quantum advantage for $n = 3$.

Question. Does the quantum value exceed the classical value for the 3-fold parallel repetition of Feige's game?

We are not aware of a game with quantum advantage, for which the classical and quantum value agree for the 2-fold parallel repetition, but then they separate again for $n = 3$. As we can see in 1 this happens for the classical and the non-signalling value of Feige's game. Note that the quantum strategy consisting of the optimal quantum strategy for one game and the optimal classical strategy for the 2-fold parallel repetition performs worse than the optimal classical strategy for $n = 3$, since $\frac{9}{16} \cdot \frac{1}{2} = \frac{9}{32} < \frac{5}{16}$. Therefore, there is no natural candidate of a quantum strategy that yields quantum advantage for the 3-fold parallel repetition.

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A quantum version of Otto's theorem

Christoph Richter

Over the past three decades, the space of probability measures with finite second moment, endowed with the quadratic Wasserstein distance, has emerged as a fundamental geometric framework for dissipative evolution equations. A cornerstone of this theory is the observation, initiated by Jordan, Kinderlehrer, and Otto, [1] that a large class of diffusion-type partial differential equations can be interpreted as gradient flows of suitable energy functionals on the Wasserstein space. This perspective has reshaped the analysis of nonlinear partial differential equations by revealing an underlying variational and gradient structure.

At the heart of this development lies the Jordan–Kinderlehrer–Otto (JKO) scheme [1], a variational time discretization of gradient flows. Given an energy functional on the space of probability measures, the JKO scheme constructs a discrete trajectory by iteratively minimizing the sum of the energy and a quadratic Wasserstein distance term. In the vanishing time-step limit, this minimizing-movement procedure converges to the continuous-time evolution governed by the associated Wasserstein gradient flow. Prominent examples include the Fokker–Planck equation and related diffusion equations, which arise as gradient flows of the free energy functional.

The geometric meaning of the quadratic Wasserstein distance is clarified by the Benamou–Brenier formulation [3], which expresses the Wasserstein metric as the minimal kinetic energy required to transport one probability distribution into another along curves satisfying the continuity equation. This dynamic formulation identifies Wasserstein geodesics with optimal transport paths and provides a variational characterization of the distance that is central to the JKO scheme and the subsequent geometric interpretation.

Building on these insights, Felix Otto introduced a formal Riemannian calculus on the space of probability measures [4]. In this framework, tangent vectors are represented by velocity fields solving the continuity equation, and the Riemannian metric is defined as a weighted L^2-inner product with respect to the underlying probability density. Within this formalism, the quadratic Wasserstein distance appears as the induced geodesic distance, and many diffusion-type PDEs can be interpreted as Wasserstein gradient flows of suitable entropy or free energy functionals. The formal nature of this Riemannian interpretation was subsequently justified by the metric theory of gradient flows developed by Ambrosio, Gigli, and Savaré [5].

An important finite-dimensional instance of a quantum analogue of Otto's theorem is provided by the work of Carlen and Maas [7] in the setting of finite-dimensional von Neumann algebras. In their setting, they introduce a noncommutative transport metric on the space of density matrices and show that, for GNS-symmetric quantum Markov semigroups, the associated quantum master equation can be interpreted as the gradient flow of the quantum relative entropy with respect to this metric. This result provides a rigorous realization of the Otto calculus in the finite-dimensional noncommutative setting and serves as a key motivation for extending the Wasserstein gradient flow framework to infinite-dimensional von Neumann algebras.

In this talk, we outline an approach towards a possible quantum analogue of Otto's classical theorem. This approach is based on a result by Wirth [2], which identifies gradient flow curves of the von Neumann entropy with tracially symmetric quantum Markov semigroups (QMS) on tracial von Neumann algebras. The goal of this project is to investigate whether one can construct a noncommutative transport metric on the state space of a general von Neumann algebra such that a given GNS-symmetric QMS arises as the gradient flow of the relative entropy. We briefly discuss a reduction method, originally introduced by Haagerup in an unpublished manuscript and later rigorously developed in [6], which provides a possible route towards obtaining such a transport metric in the GNS-symmetric case.

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Schur multipliers in harmonic analysis and operator algebras

Javier Parcet

Schur multipliers are basic linear maps on matrix algebras acting by entrywise multiplication with a row×column function M, called symbol. Their close (albeit still intriguing) connection with Fourier multipliers establishes a powerful bridge between harmonic analysis and operator algebras. In this talk, I will survey their growing impact over the past 15 years. Particular attention will be drawn to recent bounds on Schatten p-classes, with far-reaching applications in harmonic analysis on group von Neumann algebras and operator rigidity phenomena for higher-rank Lie groups and lattices. Key novelties arise from new insights into nonToeplitz Schur multipliers and unprecedented connections with highly singular operators from Euclidean harmonic analysis.

After a brief overview of L_p-bounds for Euclidean Fourier multipliers, we shall explore the so-called Fourier-Schur transference [1, 7] which show that Fourier L_p-multipliers can be frequently regarded as Toeplitz like Schur multipliers. In the more general nonToeplitz setting, we shall explain a recent simple criterion for the boundedness of Schur multipliers S_M on Schatten p-classes [4], which solves a conjecture proposed by Mikael de la Salle. Given $1 < p < \infty$, a simple form this result reads for $\mathbf{R}^n \times \mathbf{R}^n$ matrices as follows

$$\big\| S_M \colon S_p \to S_p \big\|_{\mathrm{cb}} \le C_n \frac{p^2}{p-1} \sum_{|\gamma| \le [\frac{n}{2}]+1} \Big\| |x-y|^{|\gamma|} \Big\{ \big| \partial_x^\gamma M(x,y) \big| + \big| \partial_y^\gamma M(x,y) \big| \Big\} \Big\|_\infty .$$

In this form, this can be regarded as a full matrix (nonToeplitz/nontrigonometric) amplification of the Hörmander-Mikhlin multiplier theorem, which admits lower fractional differentiability orders $\sigma > \frac{n}{2}$ as well. It implies Arazy's conjecture for divided differences [11] and several strengthenings [2, 4]. New Littlewood-Paley inequalities for matrices and strong applications in harmonic analysis for high rank simple Lie group algebras [5, 9] follow from it, see also [3] for a nonToeplitz extension of the Marcinkiewicz multiplier theorem.

Next, we consider idempotent Schur multipliers, whose symbols are indicator functions of smooth Euclidean domains. Given $1 < p \neq 2 < \infty$, we will give a local characterization (under some mild transversality condition) for the boundedness on Schatten p-classes of Schur idempotents in terms of a lax notion of boundary flatness. We prove in particular that all Schur idempotents are modeled on a single fundamental example: the triangular projection. As an application, we fully characterize the local L_p-boundedness of smooth Fourier idempotents on connected Lie groups. They are all modeled on one of three fundamental examples: the classical Hilbert transform, and two new examples of Hilbert transforms that we call affine and projective. Both of these results [10] are vast noncommutative generalizations of Fefferman's celebrated ball multiplier theorem. They confirm the intuition that Schur multipliers share profound similarities with Euclidean Fourier multipliers (even in the lack of a Fourier transform connection) and complete, for Lie groups, a longstanding search of Fourier L_p-idempotents.

To conclude, I will relate these and other related results with a longstanding attempt to address Connes' rigidity conjecture on the pairwise nonisomorphy of the group von Neumann algebras associated to $PSL_n(\mathbf{Z})$ for $n \geq 3$ and other families of higher rank simple lattices. In doing so, I will review Lafforgue-de la Salle's theorem [6] and explain some partial results towards the establishment of certain approximation properties which only hold for small enough ranks. The survey paper [8] exposes all these results in an historical framework.

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On the best constants of Schur multipliers of higher order divided differences

Jesse Reimann
(joint work with Martijn Caspers)

Schur multipliers arise as a generalisation of the componentwise multiplication of matrices, i.e. $A * B := (a_{ij}b_{ij})_{ij}$ for $A, B \in M_n(\mathbb{C})$. This concept can be extended to operators in *Schatten classes* $S^p := \{x \in B(H) \mid \mathrm{Tr}(|x|^p) < \infty\}$ as follows.

Given a compact operator x on a separable Hilbert space H, and a countable set of orthogonal projections $(p_\lambda)_{\lambda\in\mathbb{Z}}$ on H such that $\sum_\lambda p_\lambda = 1$ in the strong operator topology, we define a discrete Schur multiplier with symbol $m : \mathbb{Z}^2 \to \mathbb{C}$ as

$$M_m(x) = \sum_{\lambda_0,\lambda_1\in\mathbb{Z}} m(\lambda_0, \lambda_1)p_{\lambda_0}xp_{\lambda_1}.$$

A *multilinear Schur multiplier* with symbol $m : \mathbb{Z}^{n+1} \to \mathbb{C}$ can similarly be defined in the discrete setting as

$$M_m : (x_1, \dots, x_n) \mapsto \sum_{\lambda_0,\dots,\lambda_n\in\mathbb{Z}} m(\lambda_0, \dots, \lambda_n)p_{\lambda_0}x_1p_{\lambda_1}x_2 \cdots x_np_{\lambda_n}.$$

Such operators arise naturally in many areas of functional analysis. In particular, they can be used to calculate operator derivatives and Taylor remainders in functional calculus expressions.

One celebrated result in this direction [PSS13] is the existence of higher order spectral shift functions $\eta_{n,H_0,V}$ such that for H_0, V self-adjoint with $V \in S^n$ and $f \in C^n(\mathbb{R})$ sufficietly well-behaved one has

$$\mathrm{Tr}\left(f(H_0 + V) - \sum_{k=0}^{n-1}\frac{1}{k!}\frac{d^k}{dt^k}f(H_0 + tV)\bigg|_{t=0}\right) = \int_{\mathbb{R}} f^{(n)}(t)\eta_{n,H_0,V}(t)dt.$$

Key to this proof is the boundedness of the multilinear Schur multipliers of higher order divided differences as maps $S^{p_1} \times \ldots \times S^{p_n} \to S^p$ for $1 < p^{-1} = \sum_j p_j^{-1} < \infty$ and $f \in C^n(\mathbb{R})$ with $\|f^{(n)}\|_\infty < \infty$. The kth order *divided difference* of $f \in C^n(\mathbb{R})$, $k \leq n$, is defined inductively as

$$f^{[1]}(\lambda_0, \lambda_1) = \begin{cases} \frac{f(\lambda_0)-f(\lambda_1)}{\lambda_0-\lambda_1}, & \lambda_0 \neq \lambda_1, \\ f'(\lambda_0), & \lambda_0 = \lambda_1, \end{cases},$$

$$f^{[k]}(\lambda_0, \dots, \lambda_k) = \begin{cases} \frac{f^{[k-1]}(\lambda_0,\dots,\lambda_{k-1})-f^{[k-1]}(\lambda_1,\dots,\lambda_k)}{\lambda_0-\lambda_k}, & \lambda_0 \neq \lambda_k, \\ \frac{1}{k!}f^{(k)}(\lambda_0), & \lambda_0 = \ldots = \lambda_k. \end{cases}$$

Note that divided differences are invariant under permutation of variables.

It is known [PSST17] that the range $1 < p < \infty$ is optimal, as there exist functions $f \in C^n(\mathbb{R})$ such that the Schur multiplier of divided differences $M_{f^{[n]}}$ is not a bounded map from $S^n \times \ldots \times S^n$ into S^1. It is therefore natural to ask about the asymptotic behaviour of the norm as $p \to 1, \infty$.

Our main result in this direction is the following theorem [CR25].

Theorem 1. *Given $f \in C^n(\mathbb{R})$ with $\|f^{(n)}\|_\infty < \infty$, the norm of the associated Schur multiplier of divided differences $M_{f^{[n]}}$ on Schatten classes grows in p as*

$$\|M_{f^{[n]}} : S^{np} \times \ldots \times S^{np} \to S^p\| \lesssim p^* p^n \|f^{(n)}\|_\infty,$$

where $p^* = p/(p-1)$ describes the asymptotic behaviour as $p \to 1$.

Note that commonly used techniques for Schur multipliers cannot be directly applied to our Schur multiplier of higher order divided differences – the multilinear *transference* technique [NeRi11, CaSa15, CKV22] only applies to Schur multipliers with Toeplitz symbols (i.e. $m(\lambda_0, \ldots, \lambda_n) = \tilde{m}(\lambda_0 - \lambda_1, \ldots, \lambda_{n-1} - \lambda_n)$), and recent sufficient boundedness criteria such as the HMS-theorem [CGPT23], which (specialised to our setting) states that

$$\|M_m : S^p \to S^p\| \lesssim pp^*(\|m\|_\infty + \|(\lambda, \mu) \mapsto |\lambda - \mu|(|\partial_\lambda m(\lambda, \mu)| + |\partial_\mu m(\lambda, \mu)|)\|_\infty),$$

only apply to *linear* Schur multipliers. We therefore decompose our multilinear Schur multiplier into linear ones, using that if

$$m(\lambda_0, \ldots, \lambda_n) = m_1(\lambda_0, \ldots, \lambda_j, \lambda_{j+k}, \ldots, \lambda_n) m_2(\lambda_j, \ldots, \lambda_{j+k}),$$

then the multilinear Schur multiplier M_m can be decomposed as

$$M_m(x_1, \ldots, x_n) = M_{m_1}(x_1, \ldots, x_j, M_{m_2}(x_{j+1}, \ldots, x_{j+k}), x_{j+k+1}, \ldots, x_n).$$

For divided differences, we make use of the well-known decomposition formula

$$f^{[n]}(\lambda_0, \ldots, \lambda_n) = \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2} f^{[n]}(\lambda_0, \lambda_2, \lambda_2, \lambda_3, \ldots, \lambda_n) + \frac{\lambda_2 - \lambda_1}{\lambda_0 - \lambda_1} f^{[n]}(\lambda_1, \lambda_2, \lambda_2, \lambda_3, \ldots, \lambda_n),$$

where we note that due to the permutation invariance of the divided differences, this formula can be applied to any (pairwise distinct) $\lambda_i, \lambda_j, \lambda_k$. If applied repeatedly in a strategic manner, this formula eventually yields a linear combination of terms of the form

$$H_{i,j,k}(\lambda_0, \ldots, \lambda_n) f^{[n]}(\underbrace{\lambda_i, \ldots, \lambda_i}_{k \text{ times}}, \lambda_j, \ldots, \lambda_j), \tag{1}$$

where $H_{i,j,k}(\lambda_0, \ldots, \lambda_n) = \tilde{H}_{i,j,k}(\lambda_{k_1} - \lambda_{k_2}, \ldots, \lambda_{k_{n-2}} - \lambda_{k_{n-1}}, \lambda_i - \lambda_j)$ consists of the Toeplitz-form fractions resulting from applying the decomposition formula and some partition of unity terms introduced to guarantee boundedness of the fractions. Moreover, the divided difference now only depends on two pairwise distinct variables, thus yielding a *linear* Schur multiplier.

We furthermore extend a result of [PoSu11] and show that under some assumptions on the support, smooth homogeneous functions $\tilde{H}(\xi_1, \ldots, \xi_{n-1})$ can be expressed as linear combinations of terms of the form

$$\int_{\mathbb{R}^{n-1}} g(s_1, \ldots, s_{n-1}) |\xi_1|^{-is_1} |\xi_2|^{i(s_1-s_2)} \cdots |\xi_{n-1}|^{is_{n-1}} ds,$$

where g is a Schwartz function. As the $\tilde{H}_{i,j,k}$ in (1) are of the required form, this can be used to decompose our Schur multipliers of divided differences into linear combinations of compositions of n *linear* Schur multipliers, of which $n-1$ multipliers have symbols of the form $(\lambda, \mu) \mapsto |\lambda - \mu|^{is}$, and one multiplier has symbol $(\lambda, \mu) \mapsto |\lambda - \mu|^{is} f^{[n]}(\lambda, \ldots, \lambda, \mu, \ldots, \mu)$. The boundedness of our Schur multiplier of divided differences can hence be shown with n applications of the HMS-theorem, yielding the asymptotics described above (where we note that $(np)^* \to n/(n-1)$ as $p \to 1$).

Towards a lower bound, we show that by restricting the divided differences of the *generalised absolute value function* $a_n(\lambda) := \lambda|\lambda|^{n-1}$ to certain discrete grids, such that they approximate triangular truncation operators $T^{\pm}$. For such operators it is known that $\|T^{\pm} : S^p \to S^p\|, \|T^+ - T^- : S^p \to S^p\| \gtrsim pp^*$, $1 < p < \infty$. Using this, we can show

$$p^* p^{\min(2,n)} \lesssim \|M_{a_n^{[n]}} : S^{np} \times \ldots \times S^{np} \to S^p\|$$

for $1 < p < \infty$.

We believe that the upper bound of order $O(p^n)$ as $p \to \infty$ is sharp for all $n \in \mathbb{N}$. While it is possible to use Schur multipliers of nth order divided differences of generalised absolute value functions to approximate k-linear operators composed of triangular truncations for any $k \leq n$, it remains to show for $n > 2$ that the operators obtained in this manner are indeed bounded from below by a constant of order p^n for large p.

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Games, Equilibria and Correlations

Andreas Winter

1. Games. Since the works of von Neumann [9], von Neumann and Morgenstern [10], and Nash [7], game theory became a mathematical discipline with many connections to analysis, linear, convex and non-convex optimisation, and recently even quantum information the study of quantum entanglement, cf. [1]. Note that this setting is broader than that of combinatorial game theory [4], which treats games such as chess or tic-tac-toe, where each player has a deterministic optimal strategy.

Within this classical framework, we follow Harsanyi's [6] and Bell's [3] model of *Bayesian games* (aka games of incomplete information). Such a game is defined by the following data:

- n players $i \in [n] = \{1, \dots, n\}$;
- the set of type profiles $\mathcal{T} := \mathcal{T}_1 \times \mathcal{T}_2 \times \cdots \times \mathcal{T}_n$;
- the set of action profiles $\mathcal{A} := \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_n$;
- a prior probability distribution p on the type profiles;
- payoff functions $u_i : \mathcal{T} \times \mathcal{A} \to \mathbb{R}$, $i \in [n]$.

Then, the joint strategies of the players determine a *behaviour* $Q(a|t)$, i.e. a conditional probability distribution over action profiles $a = a_1 \dots a_n$ conditioned on type profiles t. This makes types T and actions A into jointly distributed random variables, $\Pr\{T = t, A = a\} = p(t)Q(a|t)$, and player i's objective is to choose a strategy that maximises their expected payoff $\mathbb{E}u_i(T, A)$.

In the model of von Neumann-Morgenstern-Nash, this is a game in extensive form; its normal or strategic form is given by considering the function spaces $\mathcal{F}_i = \mathcal{A}_i^{\mathcal{T}_i}$ as the fundamental sets of strategies; the payoff functions can be read as functions on $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_n$. For example, then, Nash's theorem applies and guarantees the existence of a Nash equilibrium.

2. Equilibria. Following Aumann, a *classically correlated equilibrium* of the Bayesian game G is a collection of jointly distributed random variables $F_i \in \mathcal{F}_i = \mathcal{A}_i^{\mathcal{T}_i}$, $i \in [n]$, such that for every player j and every Markov chain F'_j—F_j—$F_{[n]\setminus j}$, it holds $\mathbb{E}u_j(T, A) \geq \mathbb{E}u_j(T, A'_j A_{[n]\setminus j})$. In other words, each player fares best when they follow the advice F_j, assuming all other players follow theirs. A *Nash equilibrium* arises as the special case of independent random variables $F_i \in \mathcal{F}_i$.

The behaviours Q of correlated equilibria form a convex polytope, the intersection of which with product behaviours $Q(a|t) = Q_1(a_1|t_1) \cdots Q_n(a_n|t_n)$ correspond to the Nash equilibria.

Finally, a *quantum correlated equilibrium* is given by a quantum state ρ on a suitable tensor product Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ and POVMs $M^{t_i} = (M^{t_i}_{a_i} :$

$a_i \in \mathcal{A}_i$) for player i, $t_i \in \mathcal{T}_i$, such that again $\mathbb{E}u_j(T, A) \geq \mathbb{E}u_j(T, A'_j A_{[n]\setminus j})$ holds for the advised behaviour

$$Q(a|t) = \operatorname{Tr} \rho(M_{a_1}^{t_1} \otimes \cdots \otimes M_{a_j}^{t_j} \otimes \cdots \otimes M_{a_n}^{t_n}),$$

and any unilateral alternative

$$Q'(a'_j a_{[n]\setminus j}|t) = \operatorname{Tr} \rho(M_{a_1}^{t_1} \otimes \cdots \otimes M_{a'_j}^{t_j\prime} \otimes \cdots \otimes M_{a_n}^{t_n}).$$

The behaviours of quantum correlated equilibria, too, form a convex set, but generally not a closed one and not at all a polytope.

To compare these different sets of equilibria we can study linear functions, and specifically we look at separating them by means of the *social welfare* $s = \frac{1}{n}\sum_{i=1}^n u_i$ and its expectation $\mathbb{E}s$ at an equilibrium.

For example, when all $u_j \equiv u$ are identical, so the players want to cooperate, Bell inequalities [3, 5] show that there is a quantum advantage to the social payoff. What this really shows is that the set of quantum behaviours is strictly larger than that of classical ("local") behaviours.

3. Quantum correlated equilibria: entangled and separable states. As reviewed in [1], since 2005 (La Mura) and 2015 (Pappa et al.) multiple examples of quantum advantage of social welfare in equilibria were exhibited in competitive games. All those until now were of the kind that the types and actions are those of a Bell inequality, the payoffs are such that the optimal quantum strategy of the Bell inequality is a quantum correlated equilibrium and the social welfare is the original Bell observable, so that the maximum social welfare over classically correlated equilibria is upper-bounded by the maximum over all local behaviours.

By modifying the nonlocal game via a mechanism [11] we go beyond this paradigm, as shown on the example of the CHSH game [5]: we define the two-player game $G = \mathrm{CHSH}^\Lambda$ for a parameter $\Lambda > 0$, with types $\{*\}$ and $\mathcal{Y} = \{0, 1\}$, and action sets $\mathcal{A} = \{0, 1\}^2 = \mathcal{B}$ whose elements are denoted (x, a) and $(b, \hat{x})$ for Alice and Bob, resp. The payoff functions are

$$u_1 := \delta_{a\oplus b, xy} - 2\Lambda(2\delta_{x,\hat{x}} - 1), \quad u_2 := \delta_{a\oplus b, xy} + \Lambda(2\delta_{x,\hat{x}} - 1).$$

It can be shown [11] that the advice consisting of uniformly random x and $\hat{x}$, as well as a two-qubit maximally entangled state and the CHSH measurements yields a quantum correlated equilibrium with $\mathbb{E}u_1 = \mathbb{E}u_2 = \mathbb{E}s = \cos^2 \pi/8 \approx 0.85$. On the other hand, for sufficiently large Λ every classically correlated equilibrium of G has $\mathbb{E}s \leq 3/4$.

What is more, for the purpose of the quantum advice, the measurement of Alice can be performed already, since there is only the trivial type. This shows that, with the post-measurement states $|\varphi_{xa}\rangle$,

$$\omega := \frac{1}{8}\sum_{a,x,\hat{x}} |x, a\rangle\langle x, a|^A \otimes (|\varphi_{xa}\rangle\langle\varphi_{xa}| \otimes |\hat{x}\rangle\langle\hat{x}|)^B$$

and the CHSH measurements for Bob is a quantum correlated equilibrium with the same payoffs as before.

4. Open question: semidefinite relaxation of equilibrium behaviours? For cooperative games, the hierarchy of semidefinite programs (SDPs) described in [8] provides a natural and systematic sequence of upper bounds on the quantum value of a given nonlocal game. It always converges to the maximum value of the game over commuting strategies, and a problem in general is that this is different from the supremum over tensor product strategies.

To start, in the tensor product setting, we can use SDP duality to express the local maximum condition of a quantum correlated equilibrium as follows: in addition to the state ρ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ and POVMs $M^{t_i} = (M^{t_i}_{a_i} : a_i \in \mathcal{A}_i)$ for player i, $t_i \in \mathcal{T}_i$, we require matrices $W^{(i)}_{t_i}$ acting on $\mathcal{H}_j$ such that for all j, a_j and t_j,

$$W^{(j)}_{t_j} \geq V^{(j)}_{a_j t_j} := \sum_{t_{[n]\setminus j}, a_{[n]\setminus j}} p(t) u_j(t,a) \mathrm{Tr}_{[n]\setminus j}\, \rho(M^{t_1}_{a_1} \otimes \cdots \otimes \mathbb{1}_{\mathcal{H}_j} \otimes \cdots \otimes M^{t_n}_{a_n}),$$

and for all j and t_j,

$$\sum_{t_{[n]\setminus j}, a} p(t) u_j(t,a) \mathrm{Tr}\, \rho(M^{t_1}_{a_1} \otimes \cdots \otimes M^{t_j}_{a_j} \otimes \cdots \otimes M^{t_n}_{a_n}) = \mathrm{Tr}\, W^{(j)}_{t_j}.$$

Then, maximising the social welfare over the set of solutions of these inequalities is already a polynomial optimisation problem in noncommutative variables, but it still has tensor products built in. The remaining difficulty, so that general methods of SDP relaxation of noncommutative polynomial optimisation apply, is to replace the explicit tensor product by some kind of commutation condiition.

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Proper cocycles, measure equivalence and L_p-Fourier multipliers

Simeng Wang
(joint work with Runlian Xia, Gan Yao)

A basic ingredient in the seminal work [5] on the isomorphism problem for group von Neumann algebras is a transference method that lifts Fourier multipliers from a lattice to its ambient group. Inspired by this work, Cowling and Zimmer [6] and Jolissaint [11] introduced an analogous transference method for proper cocycles arising from group actions on probability spaces, thereby showing that weak amenability—and more generally other approximation properties—remains invariant under measure equivalence (see also [12]). The corresponding L_p theory of the aforementioned transference method arises naturally in the study of noncommutative L_p-spaces and the Fourier multipliers acting on them, in particular for noncommutative L_p-spaces over higher rank lattices. Moreover, as remarked by [2, Appendix B], the extension of Fourier multipliers from lattices to ambient groups may be viewed as a noncommutative counterpart of Jodeit's theorem, which in the classical setting addresses the extension from $\mathbb{Z}^d$ to $\mathbb{R}^d$. Our recent results establish the aforementioned transference theory in the setting of Fourier multipliers on noncommutative L_p-spaces over groups, introducing new examples of completely bounded Fourier multipliers arising from measure equivalence, lattices, and Hilbert transforms.

In the following let G be a unimodular locally compact group and $\mathcal{L}(G)$ its group von Neumann algebra generated by the left regular representation λ. Let τ be the canonical trace on the von Neumann algebra $\mathcal{L}(G)$. The associated noncommutative L_p-spaces $L_p(\mathcal{L}(G))$ with respect to τ naturally generalize classical L_p-spaces on the Pontryagin dual group. Given $m \in L_\infty(G)$, the Fourier multiplier with symbol m is defined as the (potentially unbounded) linear operator $T_m : \lambda(C_c(G)) \subset \mathcal{L}(G) \to \mathcal{L}(G)$ given by

$$T_m(\lambda(f)) = \int_G m(g)f(g)\lambda_g \mathrm{d}g.$$

Consider a measure-preserving action of G on a finite measure space (X, μ), and let H be another locally compact group. A Borel map $\alpha : G \times X \to H$ is called a *Borel cocycle* if it satisfies

$$\alpha(g_1g_2, x) = \alpha(g_1, g_2x)\alpha(g_2, x). \tag{1}$$

Haagerup [5, 9] and Jolissaint [11, 12] observed a close connection between Fourier multipliers on H and those on G. More precisely, given a symbol $m \in L_\infty(H)$, one may define an associated symbol $\widetilde{m} \in L_\infty(G)$ by

$$\widetilde{m}(g) = \frac{1}{\mu(X)} \int_X m(\alpha(g, x))\mathrm{d}\mu(x), \quad g \in G. \tag{2}$$

It has been shown in [11, 12] that $\widetilde{m}$ is a completely bounded (resp. completely positive) Fourier multiplier on $\mathcal{L}(G)$ if m is so on $\mathcal{L}(H)$ (see also [5, 9] for partial results for lattices); moreover, this transference keeps certain geometric or analytic properties when α is proper. The following theorem establishes analogous results for Fourier multipliers on the associated noncommutative L_p-spaces.

Theorem 1. *Let $1 \le p \le \infty$. Let H be a discrete group, G be a unimodular locally compact second countable group and (X, μ) be a finite measure space. Let θ be a μ-preserving action of G on (X, μ). Assume that the crossed product $L_\infty(X) \rtimes_\theta G$ is QWEP. Let $\alpha : G \times X \to H$ and $m, \widetilde{m}$ be as above. Then we have*

$$\|T_{\widetilde{m}} : L_p(\mathcal{L}(G)) \to L_p(\mathcal{L}(G))\|_{\mathrm{cb}} \le \|T_m : L_p(\mathcal{L}(H)) \to L_p(\mathcal{L}(H))\|_{\mathrm{cb}}.$$

This approach leads to several applications. It is well-known that the above cocycle exists when G and H are measure equivalent. It has been shown in [12] that weak amenability is an invariant under measure equivalence. Now we may establish an analogous result in the L_p-setting. For $1 \le p \le \infty$, we say that G is *L_p-weakly amenable* if G admits a sequence of symbols $(m_n)_{n\ge 0} \subset A(G)$ such that $m_n \to 1$ uniformly on compact sets and

$$\sup_n \|T_{m_n} : L_p(\mathcal{L}(G)) \to L_p(\mathcal{L}(G))\|_{\mathrm{cb}} < \infty,$$

where $A(G)$ denotes the Fourier algebra of G. The smallest possible supremum of the above quantity, for which such a sequence (m_n) exists, is denoted by $\Lambda_{p,\mathrm{cb}}(G)$, which serves as the L_p-analogue of the celebrated Cowling–Haagerup constant.

Corollary 2. *Let $1 \le p \le \infty$. Let Γ and Δ be two discrete groups that are measure equivalent with a measure equivalence coupling (Σ, σ). Assume that $L_\infty(\Sigma/\Delta) \rtimes \Gamma$ is QWEP. Then Γ is L_p-weakly amenable if and only if Δ is L_p-weakly amenable, and in this case, $\Lambda_{p,\mathrm{cb}}(\Gamma) = \Lambda_{p,\mathrm{cb}}(\Delta)$.*

The classical Jodeit theorem asserts that any L_p-bounded Fourier multiplier on $\mathbb{Z}^n$ is the restriction of an L_p-bounded Fourier multiplier on $\mathbb{R}^n$. More precisely, given an L_p-bounded Fourier multiplier $m \in \ell_\infty(\mathbb{Z}^n)$, which can be viewed as a discrete measure on $\mathbb{R}^n$, the piecewise linear extension

$$\widetilde{m} = \chi_{[-1/2,1/2]} * m * \chi_{[-1/2,1/2]}$$

defines an L_p-bounded Fourier multiplier on $\mathbb{R}^n$. The following provides a noncommutative analogue of Jodiet's theorem; see [7, 4] for partial results in the case of abelian groups, and [13, 2, 1] for related results concerning Schur multipliers and amenable groups.

Corollary 3. *Let $\Gamma \subset G$ be a hyperlinear lattice in a locally compact second countable group G with a fundamental domain Ω. Let $m \in \ell(\Gamma)$ and $\widetilde{m} = \chi_\Omega * m * \chi_\Omega$, and $1 \le p \le \infty$. Then*

$$\|T_{\widetilde{m}} : L_p(\mathcal{L}(G)) \to L_p(\mathcal{L}(G))\|_{\mathrm{cb}} \le \|T_m : L_p(\mathcal{L}(\Gamma)) \to L_p(\mathcal{L}(\Gamma))\|_{\mathrm{cb}}. \tag{3}$$

As in [5], this leads to the following important observation concerning L_p-weak amenability.

Corollary 4. *Let $\Gamma \subset G$ be a hyperlinear lattice in a locally compact second countable group G, and $1 \le p \le \infty$. If Γ is L_p-weakly amenable, then G is L_p-weakly amenable, and in this case, $\Lambda_{p,\mathrm{cb}}(G) \le \Lambda_{p,\mathrm{cb}}(\Gamma)$.*

The converse of the statement corresponds to a form of noncommutative de Leeuw's restriction theorem, which remains open except in certain local or asymptotically invariant settings (see [2, 3]).

Another application goes to the study of Hilbert transforms on $SL_2(\mathbb{R})$. A canonical Hilbert transform on $SL_2(\mathbb{Z})$ is studied in [8]:

$$m(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \mathrm{sgn}(ac+bd), \quad \text{for} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \tag{4}$$

and is shown to define a completely bounded Fourier multiplier on $L_p(\mathcal{L}(SL_2(\mathbb{Z})))$ for every $1 < p < \infty$. However, on $SL_2(\mathbb{R})$ such discontinuous symbols fail to give L_p-bounded multipliers [14]. To obtain a continuous analogue, we may apply Corollary 3 to extend the $SL_2(\mathbb{Z})$ Hilbert transform to $SL_2(\mathbb{R})$, producing a completely bounded Fourier multiplier $\widetilde{m}$ on $L_p(\mathcal{L}(SL_2(\mathbb{R})))$. Interestingly, due to the non-uniform geometry of the lattice inclusion $SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R})$, the resulting symbol may exhibit improved decay properties compared to the Euclidean setting—such as the Hörmander–Mikhlin-type behavior.

Our method also yields a new and simple proof of the main result in [10] regarding the pointwise convergence of noncommutative Fourier series on amenable groups, refining the associated estimate for maximal inequalities.

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Kakeya conjecture and High rank lattice von Neumann algebras

Mikael de la Salle

The aim of my talk was to present a connection between two different open problems: the Kakeya conjecture in real/harmonic analysis, and Connes' rigidity conjecture in the theory of von Neumann algebras.

Connes rigidity conjecture and approximation properties. Given a discrete group Γ, its von Neumann algebra $\mathcal{L}\Gamma$ is the algebra of bounded convolution operators on $\ell_2(\Gamma)$. That is the set of all bounded linear operators $a : \ell_2(\Gamma) \to \ell_2(\Gamma)$ such that there is $f : \Gamma \to \mathbf{C}$ such that $a\xi(s) = \sum_{t\in\Gamma} f(t)\xi(t^{-1}s)$ for every (say finitely supported) $\xi : \Gamma \to \mathbf{C}$. A particular case of Connes' rigidity conjecture predicts that $\mathcal{L}\mathrm{PSL}_n(\mathbf{Z})$ and $\mathcal{L}\mathrm{PSL}_m(\mathbf{Z})$ are not isomorphic whenever $n < m$. It is wide open, except when $n = 2$ where at least two von Neumann algebras invariants distinguish rank 1 from higher rank: property (T) [3] and weak amenability [5, 7, 2].

In ancient works with Vincent Lafforgue [12] and Tim de Laat [10], I had proven that if $|\frac{1}{p} - \frac{1}{2}| > \frac{1}{2d}$, then the non-commutative L_p space of $\mathcal{L}\mathrm{PSL}_{2d-1}(\mathbf{Z})$ does not have the completely bounded approximation property. The condition on p was essential in the proofs. The question whether it is also essential for the statement is therefore natural. Its importance is that a positive answer would distinguish the von Neumann algebras $\mathcal{L}\mathrm{PSL}_n(\mathbf{Z})$ and $\mathcal{L}\mathrm{PSL}_m(\mathbf{Z})$ whenever $|n - m| \geq 2$, by the invariant of a von Neumann algebra $\mathcal{M}$ equal to the set of values of p such that $L_p(\mathcal{M})$ has the completely bounded approximation property.

Question 1. *Let $d \geq 2$. Does $L_p(\mathcal{L}\mathrm{PSL}_{2d-1}(\mathbf{Z}))$ have the completely bounded approximation property for some $p \neq 2$?*

Kakeya conjecture. A subset of the Euclidean space $\mathbf{R}^d$ is a *Kakeya set* if it contains a unit segment in every direction. Besicovich showed that for $d \geq 2$, Kakeya sets exist, that are small in the sense that they have Lebesgue measure 0. But the Kakeya conjecture predicts that they are still quite large in the sense that their dimension is d. They are various precise conjectures depending on what sense we mean dimension: Minkowski upper or lower, Hausdorff... They can all be expressed combinatorically in terms of configuration of tubes in $\mathbf{R}^d$. The precise form of the conjecture we will work with is not exactly one of the usual ones, but it is in the same spirit. It is the following. A δ-tube R in $\mathbf{R}^d$ is the image of $[0, 1] \times B_{\mathbf{R}^{d-1}}(0, \delta)$ by an affine isometry of $\mathbf{R}^d$; in that case we note $\overline{R}$ the image of $[2, 3] \times B_{\mathbf{R}^{d-1}}(0, \delta)$. Define $f_d(\delta)$ as the infimum of the real numbers $\varepsilon > 0$ such that there is a finite family of δ-tubes $\{R_j\}$ such that the $\overline{R}_j$ are pairwise disjoint but

$$\frac{\left|\bigcup_j R_j\right|}{\sum_j |R_j|} \leq \varepsilon.$$

Conjecture 1. *For every $\varepsilon > 0$, $\sup_{\delta>0} f_d(\delta)\delta^{-\varepsilon} = \infty$.*

The Kakeya conjecture is a classical theorem in dimension 2 [4], and more precisely Keich's construction [8] of Besicovitch sets gives that

$$\frac{c}{|\log \delta|} \leq f_2(\delta) \leq \frac{C}{|\log \delta|}$$

In dimension 3, a solution has recently been given in a celebrated breakthrough by Wang and Zahl [17, Theorem 1.12]; in higher dimension it remains open.

The main theorem that I presented in my talk explains why Question 1 is difficult.

Theorem 2. [16] *Let $d \geq 2$. If Question 1 has a positive answer, then Conjecture 1 holds.*

The proof has several steps. The first improves on Lafforgue's ideas that I call rank 0 reduction in [15], developpped in [11, 12, 10]. As a conclusion of this first step, a natural question occurs, a spherical analogue of the following question in Euclidean Harmonic analysis: does a radially symmetric Fourier multiplier that is bounded on $L_p(\mathbf{R}^d)$ for some $p \neq 2$ necessarily have a continuous symbol? We address this question, following Fefferman [6], and then adapt arguments to the spherical setting using ideas from [13, 1, 14].

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Tensor Products of Convex Cones with Applications to Generalized Probabilistic Theories

Tim Netzer
(joint work with Felix Campidell, Markus Dannemüller)

Algebraic tensor products of vector spaces are classical and well-studied constructions that appear throughout pure and applied mathematics. However, when the spaces involved carry additional structure and one wishes the tensor product to reflect that structure, the situation can become considerably more subtle. In many settings, there is no longer a unique natural tensor product. This phenomenon arises, for instance, for normed spaces, operator spaces, ordered spaces, and operator systems. A common feature across all these settings is the existence of both a smallest and a largest such tensor product. These extremal tensor products are typically well understood and enjoy additional desirable properties such as symmetry, associativity, and functoriality. Moreover, they are dual to each other in the sense that a given duality interchanges the two constructions.

In theoretical quantum physics, the minimal tensor product of cones of positive semidefinite matrices gives rise to the so-called separable bipartite states, while the maximal tensor product consists of the bipartite block-positive matrices, which are also in one-to-one correspondence with positive maps. A crucial observation is that the most important convex cone used to describe bipartite quantum systems is neither of these two. Instead, it is the cone of all positive semidefinite matrices on the tensor product space, which contains the bipartite states. This cone lies exactly between the minimal and maximal tensor products in the sense that it is self-dual, whereas the minimal and maximal tensor product cones are dual to each other.

Generalized probabilistic theories in quantum physics replace psd-cones by more general convex cones in order to investigate which features of psd-cones are essential for the physical theory. In these frameworks, however, the bipartite states are typically taken from the maximal tensor product, largely because no alternative tensor product with comparable properties is known for general convex cones. In this respect, such theories do not fully generalize standard quantum mechanics, which uses the intermediate tensor product as its state space.

In this talk we explain that there exists a self-dual, functorial tensor product on the category of all proper finite-dimensional convex cones. An analogous result is established for finite-dimensional abstract operator systems. This provides a new tool for generalized probabilistic theories that is much more in line with the usual formulation of quantum physics.

In standard quantum theory, purification plays a central conceptual role, expressing the idea that every mixed state arises as the marginal of a pure state on a larger system. Extending this principle to more general convex cones and tensor products is an active area of research, and several recent works investigate which structural assumptions on the cones and on the tensor product are necessary for purification to hold, and to what extent it can be recovered beyond the setting of psd-cones. The effect that different tensor products have on this question has hardly been studied yet. We also report on ongoing work aimed at developing a notion of purification within our new framework.

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The limit of the NPA hierarchy is not always attained

William Slofstra

The NPA hierarchy (named after Navascues, Pironio and Acin) gives a sequence of upper bounds on the commuting operator value of a game, which converge to the game value. It is natural to wonder if this limit is always attained, but surprisingly hard to find an explicit counterexample. We show that the decision problem "Given $\mathcal{G}$, is $\omega_{co}(\mathcal{G}) > 1/2$?" is RE hard. This shows that the limit is not always attained. Based on joint work with Fanizza, Kroell, Mehta, Paddock, Rochette, and Zhao.

Navier-Stokes equations on quantum Euclidean spaces

Guixiang Hong

(joint work with Deyu Chen, Liang Wang, Wenhua Wang)

The quantum Euclidean space $\mathbb{R}^d_\theta$ with θ being a $d \times d$ antisymmetric matrix is a generalization of both the classical Euclidean space $L_\infty(\mathbb{R}^d)$ and the Moyal plane when $\theta = \hbar \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ with the Planck constant $\hbar$, $\mathbb{R}^d_\theta$, which plays an important role in quantum mechanics. There exists a canonical trace τ_θ on it, with which one may construct noncommutative L_p-spaces. For $1 \leq k \leq d$, there is an intrinsic definition of the partial derivative ∂_k on this object with the help of a Fourier-like expansion formula, so many notions such as partial differential equations (abbreviated as PDEs), Fourier multipliers, function spaces can be naturally defined. As a mathematical object, they are standard model examples of the noncompact manifolds in Connes' noncommutative geometry theory; motivated by the noncommutative martingale theory and harmonic analysis, there have appeared several fundamental works on harmonic analysis over quantum Euclidean spaces since the seminal one [5] in 2013 on quantum tori (see also [18]). More background and references can be found in the preprint [4].

Inspired by the study of condensed matter physics, nuclear physics, and quantum chemistry etc., several PDEs on the Moyal plane such as the Bogoliubov-de Gennes equations, the generalized Hartree-Fock equations, and the Hartree-Fock-Bogoliubov equations etc. have naturally appeared in the literature as the mean-field limit of many body Schrödinger equation for wave functions [1, 6, 7, 8]. Besides, with different motivations, several other PDEs have also been investigated in the noncommutative setting [3, 16, 13, 2] and the notable one [17] by Voiculescu.

However, as far as the authors know, almost all the previous investigations on the above mentioned noncommutative PDEs are essentially restricted to L_p for $p = 1, 2, \infty$ by exploiting the Hilbert space structure L_2 or the algebraic structure L_∞, except the ones [10, 9, 11] vey related to noncommutative harmonic analysis. The newly emerging but powerful real analysis over other noncommutative L_p-spaces seems to have been ignored by the noncommutative PDE community.

Motivated by all this, we have a project which aims at exploring real analysis over noncommutative L_p-spaces to study noncommutative PDEs, and then finding applications to the theory of mean filed limit and semiclassical limit, and thus facilitating the understanding of condensed matter physics and quantum chemistry etc.. In the present talk, we are restricted to considering one of the most fundamental nonlinear PDEs–Navier-Stokes equation on quantum Euclidean spaces:

$$\begin{cases} \partial_t u - \Delta_\theta u + A(u) + \nabla_\theta p = 0; \\ \mathbf{div}\, u = 0; \\ u(0) = u_0, \end{cases} \tag{1}$$

where $\Delta_\theta = \sum_{i=j}^d \partial_j^2, \nabla_\theta = (\partial_1, \ldots, \partial_d)$, $\mathbf{div}\, u = \partial_1 u_1 + \cdots + \partial_d u_d$, $u = (u_1, \ldots, u_d)$, p are unknown operator-valued functions from $[0, T)$ to operators, $T > 0$, and $A(u)$ denotes the nonlinear term

$$A(u) := u \cdot \nabla_\theta u := \left\{ \sum_{j=1}^d u_j(\partial_j u_k) \right\}_{k=1}^d,$$

whenever the operations are allowed.

We concern about the well-posedness of (1) and deeply exploit various techniques from noncommutative harmonic analysis and function spaces to get a complete quantum analogues of the classical Ladyzhenskaya and Kato's results [14, 12].

In what follows, let $\dot{H}^1(\mathbb{R}^d_\theta)$ denote the homogeneous Sobolev space and $[L_p(\mathbb{R}^d_\theta)]^d_0$ $(1 \le p \le \infty)$ be the set of tuples of divergence free elements.

Theorem 1 *Let $d \ge 2$ and $u_0 \in [L_d(\mathbb{R}^d_\theta)]^d_0$. Then we have the following conclusions:*

(i) *There exists a maximal time $T_{u_0} > 0$ such that the Navier-Stokes equation (1) exists a unique smooth solution*

$$u \in C([0, T_{u_0}); [L_d(\mathbb{R}^d_\theta)]^d_0) \cap L^{\mathrm{loc}}_{d+2}([0, T_{u_0}); [L_{d+2}(\mathbb{R}^d_\theta)]^d_0);$$

if $T_{u_0} < \infty$, *then we have* $\|u\|_{L_{d+2}([0,T_{u_0});[L_{d+2}(\mathbb{R}^d_\theta)]^d)} = \infty$. *Moreover, if* $\|u_0\|_{[L_d(\mathbb{R}^d_\theta)]^d}$ *is sufficiently small, then* $T_{u_0} = \infty$.

(ii) *If* $u_0 \in [L_2(\mathbb{R}^d_\theta)]^d_0 \cap [L_d(\mathbb{R}^d_\theta)]^d_0$, *then the solution obtained in* (i) *satisfies*

$$u \in C([0,T_{u_0});[L_2(\mathbb{R}^d_\theta)]^d_0) \cap L_2^{\rm loc}([0,T_{u_0});[\dot{H}^1(\mathbb{R}^d_\theta)]^d_0).$$

Due to the noncommutativity, the nonlinear term $A(u)$ in (1) cannot ensure an energy identity and thus a good global well-posedness theory even if the initial datum u_0 is self-adjoint. Therefore, we will consider the Navier-Stokes equation with nonlinear term of symmetric form $S(u)$:

$$\begin{cases} \partial_t u - \Delta_\theta u + S(u) + \nabla_\theta p = 0; \\ \mathbf{div}\, u = 0; \\ u(0) = u_0, \end{cases} \tag{2}$$

where

$$S(u) := \frac{1}{2}\left[u \cdot (\nabla_\theta u) + ((\nabla_\theta u)^T \cdot u^T)^T\right] := \left\{\frac{1}{2}\sum_{j=1}^d u_j(\partial_j u_k) + (\partial_j u_k)u_j\right\}_{k=1}^d.$$

Here, for any vector or matrix Λ, Λ^T denotes the transpose of Λ.

Theorem 2 *Let* $d \geq 2$ *and* $u_0 \in [L_d(\mathbb{R}^d_\theta)]^d_0$. *Then we have the following conclusions:*

(i) *There exists a maximal time* $T_{u_0} > 0$ *such that the Navier-Stokes equation* (2) *exists a unique smooth solution*

$$u \in C([0,T_{u_0});[L_d(\mathbb{R}^d_\theta)]^d_0) \cap L_{d+2}^{\rm loc}([0,T_{u_0});[L_{d+2}(\mathbb{R}^d_\theta)]^d_0);$$

if $T_{u_0} < \infty$, *then we have* $\|u\|_{L_{d+2}([0,T_{u_0});[L_{d+2}(\mathbb{R}^d_\theta)]^d)} = \infty$. *Moreover, if* $\|u_0\|_{[L_d(\mathbb{R}^d_\theta)]^d}$ *is sufficiently small, then* $T_{u_0} = \infty$.

(ii) *If* $u_0 \in [L_2(\mathbb{R}^d_\theta)]^d_0 \cap [L_d(\mathbb{R}^d_\theta)]^d_0$, *then the solution in (i) satisfies*

$$u \in C([0,T_{u_0});[L_2(\mathbb{R}^d_\theta)]^d_0) \cap L_2^{\rm loc}([0,T_{u_0});[\dot{H}^1(\mathbb{R}^d_\theta)]^d_0).$$

Moreover, if u_0 *is self-adjoint, then the solution satisfies the energy identity*

$$\frac{1}{2}\|u(t)\|^2_{[L_2(\mathbb{R}^d_\theta)]^d} + \int_0^t \|\nabla_\theta u(s)\|^2_{[L_2(\mathbb{R}^d_\theta)]^{d^2}}\,ds = \frac{1}{2}\|u_0\|^2_{[L_2(\mathbb{R}^d_\theta)]^d},\ \forall\, 0 < t < T_{u_0}. \tag{3}$$

(iii) *If* $d = 2$ *and* u_0 *is self-adjoint, then the Navier-Stokes equation* (2) *is globally well-posed in* $C([0,\infty);[L_2(\mathbb{R}^d_\theta)]^2_0) \cap L_2([0,\infty);[\dot{H}^1(\mathbb{R}^d_\theta)]^2_0)$, *and the solution* u *is smooth and satisfies the energy identity*

$$\frac{1}{2}\|u(t)\|^2_{[L_2(\mathbb{R}^d)]^2} + \int_0^t \|\nabla_\theta u(s)\|^2_{[L_2(\mathbb{R}^d_\theta)]^4}\,ds = \frac{1}{2}\|u_0\|^2_{[L_2(\mathbb{R}^d_\theta)]^2},\ \forall\, 0 < t < \infty. \tag{4}$$

Remark

(i) When $\theta = 0$, Theorem 2 recovers the classical results of Ladyzhenskaya [14] and Kato [12].

(ii) With the asymmetric nonlinear term $A(u)$, we establish the local well-posedness theory of (1) in Theorem 1, and do not know how to achieve the global well-posedness theory. In the current paper, we only consider the global well-posedness of equation (2) with symmetric nonlinear term. Let us mention that in Theorem 1, we do not require the self-adjointness of u_0.

Since in the quantum setting the notion "point" is not available anymore, one cannot formally put forward the Navier-Stokes equations (1) (2) in the same way as in classical case. We rigorously formulate the differential equation in [4, (3.1)] with the help of the class of quantum tempered distribution and noncommutative L_p-spaces, and establish its equivalence with the integral form so that we are reduced to focusing on the mild solutions of (1) (2) (see [4, Lemma 3.6]). We also introduce the notion of critical spaces in [4, Section 3], which is non-trivial in the setting of quantum Euclidean spaces.

As in the special case $\theta = 0$, our approach to finding the mild soutions will mainly rely on the contraction mapping principle ([4, Sections 7 and 8]). In order to apply this principle, we establish several sharp time-space estimates for both the linear and nonlinear terms in [4, Section 6]. The sharp time-space estimates in turn involve the sharp (L_r, L_p) estimates of the quantum heat semigroup ([4, Proposition 6.1]), Besov spaces, and various sharp embeddings between Besov spaces and Sobolev spaces ([4, Section 5]). Due to the noncommutativity and the lack of the notion of "points", there need several new techniques to achieve these sharp estimates; among them, a notable one is a transference principle ([4, Theorem 4.1]) which will not only show the L_p-boundedness of the Leray projection—[4, Lemma 7.2] but also yield the sharp embedding—[4, Lemma 5.10].

It should be pointed out that McDonald [15] studied the well-posedness of the equations (2) when $\det(\theta) \neq 0$, the initial datum u_0 is self-adjoint and belongs to the noncommutative Sobolev space $H^2(\mathbb{R}^d_\theta)$ that does not admit any scaling symmetry; one key fact used by him is that when $\det(\theta) \neq 0$, the resulting noncommutative L_p-spaces are the Schatten classes, and thus the Sobolev embeddings trivially hold but with the constant depending on θ. Our Theorem 2 goes further beyond McDonald's result since we work with $u_0 \in L_d(\mathbb{R}^d_\theta)$—the critical space; moreover, our techniques, which are independent of θ, allow to derive the semiclassical limit of quantum Navier-Stokes equations, which reveals below the relationship between the solutions of quantum Navier-Stokes equations and those of classical ones.

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Connes' Integral Formula from 1915

Eva-Maria Hekkelman

(joint work with Edward McDonald)

In the early days of functional analysis, Szegő proved a theorem to compute determinants of Toeplitz matrices [7]. Namely, for $w \in C^1(\mathbb{T})$ positive, with Fourier coefficients

$$w_n := \frac{1}{2\pi}\int_0^{2\pi} w(\theta)e^{-in\theta}\,d\theta, \tag{1}$$

define the Toeplitz matrices

$$T_n := \begin{pmatrix} w_0 & w_1 & w_2 & \cdots & w_n \\ w_{-1} & w_0 & w_1 & \cdots & w_{n-1} \\ w_{-2} & w_{-1} & w_0 & \cdots & w_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{-n} & w_{-n+1} & w_{-n+2} & \cdots & w_0 \end{pmatrix}.$$

For these matrices, Szegő showed that

$$\lim_{n\to\infty} (\det T_n)^{\frac{1}{n}} = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log(w(\theta))\, d\theta\right). \tag{2}$$

In joint work with Edward McDonald we highlight a connection between this theorem and Connes' Trace Theorem [2]. A special case of this latter theorem states that for a d-dimensional compact orientable Riemannian manifold (M, g), for any Dixmier trace Tr_ω on the weak trace-class $\mathcal{L}_{1,\infty}$,

$$\mathrm{Tr}_\omega(M_f(1-\Delta_g)^{-\frac{d}{2}}) = C_d \int_M f\, d\nu_g, \quad f \in C^\infty(M), \tag{3}$$

where Δ_g is the Laplace–Beltrami operator, ν_g the Riemannian volume form, and C_d a constant depending on the dimension d. This theorem is important in noncommutative geometry for providing the philosophical basis for defining the 'noncommutative integral' for spectral triples via the left-hand side of (3).

While these two results might look quite different, the connection can be made as follows. First, by recognising that the Toeplitz matrices (1) are in fact given by compressions of the multiplication operator M_w on $L_2(\mathbb{S}^1)$,

$$T_n = P_n M_w P_n,$$

where P_n is the projection onto the first $n+1$ vectors in the standard basis of $L_2(\mathbb{S}^1)$, Szegő's formula (2) can be rewritten as

$$\lim_{n\to\infty} \frac{1}{n+1} \mathrm{Tr}(\log(P_n M_w P_n)) = \frac{1}{2\pi}\int_0^{2\pi} \log(w(\theta))\, d\theta.$$

Much later, Widom proved a generalisation of Szegő's theorem for compact Riemannian manifolds [8]. Writing now $P_n := \chi_{[0,\lambda_n]}(\Delta_g)$, where λ_n is the n-th largest eigenvalue of Δ_g, Widom showed that

$$\lim_{n\to\infty} \frac{\mathrm{Tr}(f(P_n M_f P_n))}{\mathrm{Tr}(P_n)} = \frac{1}{\mathrm{vol}(M)} \int_M f\, d\nu_g, \quad f \in C^\infty(\mathbb{R};\mathbb{R}). \tag{4}$$

The result linking these together is the following theorem, joint work with Edward McDonald [5]. For any $A \in B(\mathcal{H})$, D self-adjoint with compact resolvent and satisfying Weyl's law $\lambda_k := \lambda(k, |D|) \sim Ck^{\frac{1}{d}}$, writing $P_n := \chi_{[-\lambda_n,\lambda_n]}(D)$, then for all extended limits $\omega \in \ell_\infty(\mathbb{N})^*$ and associated Dixmier traces Tr_ω,

$$\frac{\mathrm{Tr}_\omega(A(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})} = \omega \circ M\left(\frac{\mathrm{Tr}(P_n A P_n)}{\mathrm{Tr}(P_n)}\right). \tag{5}$$

Here, $M : \ell_\infty(\mathbb{N}) \to \ell_\infty(\mathbb{N})$ is the logarithmic averaging defined by $M(x)_n = \frac{1}{\log(n+2)} \sum_{k=0}^{n} \frac{x_k}{k}$. If furthermore $[D, A]$ extends to a bounded operator, then for all Dixmier traces Tr_ω,

$$\text{(6)} \qquad \frac{\mathrm{Tr}_\omega(f(A)(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})} = \omega \circ M\left(\frac{\mathrm{Tr}(f(P_n A P_n))}{\mathrm{Tr}(P_n)}\right), \quad f \in C(\mathbb{R}).$$

This provides Szegő's formula in a non-commutative setting.

Both equations (5) and (6) in fact fit neatly into Connes–van Suijlekom paradigm of spectrally truncated spectral triples [4], where these formulas show an approximation of the noncommutative integral on these truncated triples.

In the final part of this talk, a connection with Quantum Ergodicity was discussed. For a compact Riemannian manifold M and a positive self-adjoint operator Δ on $L_2(M)$ with compact resolvent, Δ is said to be quantum ergodic if for every orthonormal basis $\{e_n\}_{n=0}^\infty$ of $L_2(M)$ consisting of eigenfunctions of Δ, there exists a density one subsequence $J \subseteq \mathbb{N}$ such that

$$\text{(7)} \qquad \lim_{J \ni j \to \infty} \langle e_j, A e_j \rangle_{L_2(M)} \to \frac{1}{\mathrm{vol}(S^*M)} \int_{S^*M} \sigma_0(A)\, d\mu, \quad A \in \Psi^0(M),$$

where ν is a probability measure on M. In this context, a density one subsequence means that

$$\frac{\#(J \cap \{0, \dots, n\})}{n+1} \to 1, \quad n \to \infty.$$

In particular, taking $A = M_f$ in equation (7), quantum ergodicity implies that the measures

$$\lim_{J \ni j \to \infty} |e_j|^2\, d\nu_g = \frac{1}{\mathrm{vol}(M)}\, d\nu_g,$$

in the weak*-topology. This can intuitively be understood as the eigenfunctions of Δ being 'smeared out uniformly' over M for high eigenvalues.

The fundamental theorem of Quantum Ergodicity due to Shnirelman, Zelditch, and Colin de Verdière [6, 9, 1] states that for a compact Riemannian manifold M, if the geodesic flow on M is ergodic, then the Laplace–Beltrami operator Δ_g is quantum ergodic. The geodesic flow is defined on S^*M (vectors in the cotangent bundle T^*M of length 1) in the usual way via geodesics.

Since Alain Connes defined a noncommutative analogue of the cosphere bundle S^*M for spectral triples [3] including an analogue of geodesic flow, one can wonder if a similar Quantum Ergodicity result holds for spectral triples. In joint work with Edward McDonald, we showed that the answer is affirmative [5].

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple. Let $\sigma_t : B(\mathcal{H}) \to B(\mathcal{H})$ be defined by $A \mapsto e^{it|D|} A e^{-it|D|}$. Then define the C^*-algebra

$$S^*\mathcal{A} := C^*\left(\bigcup_{t \in \mathbb{R}} \sigma_t(\mathcal{A}) + K(\mathcal{H})\right) \Big/ K(\mathcal{H}),$$

where σ_t descends to an action of $\mathbb{R}$ on $S^*\mathcal{A}$. If $(\mathcal{A}, \mathcal{H}, D) \simeq (C^\infty(M), L_2(S), D_S)$, then $S^*\mathcal{A} \simeq C(S^*M)$ and σ_t is the geodesic flow. Ergodicity of the geodesic flow on this C^*-algebra can be defined in analogy to the classical case in a straightforward manner, via a noncommutative L_2-space. The result we claim is then the following.

Let $(\mathcal{A}, \mathcal{H}, D)$ be a d-summable regular spectral triple where D satisfies Weyl's law, and with local Weyl laws. If the geodesic flow on $(\mathcal{A}, \mathcal{H}, D)$ is ergodic, then D is quantum ergodic. That is, for every basis $\{e_n\}_{n=0}^{\infty}$ of $\mathcal{H}$ consisting of eigenvectors of D, there exists a density one subset $J \subseteq \mathbb{N}$ such that

$$\lim_{J \ni j \to \infty} \langle e_j, a e_j \rangle = \frac{\mathrm{Tr}_\omega(a(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})}, \quad a \in \mathcal{A}.$$

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A Quantum Fujita theorem

Edward McDonald

This talk is based on joint work with M. Ruzhansky, S. Shaimardan and K. Tulenov. The associated paper is available on the arXiv [9]

The classical Fujita theorem concerns the partial differential equation on $\mathbb{R}^d$, $d \geq 1$ given by

$$\partial_t u = \Delta u + u^p, \quad u|_{t=0} = u_0 \tag{1}$$

where u_0 is a given non-negative function. The interest in this equation comes from the opposing behaviour of the two terms on the right hand side. One the one hand, the heat flow solving $\partial_t u = \Delta u$ has as solution $u(t) = \exp(t\Delta)u_0$, an ultracontractive semigroup which is infinitely smoothing. On the other hand, the nonlinear equation $\partial_t u = u^p$, has a "sharpening" effect and in fact blows up in finite time, the solution is given by

$$u(x,t) = (u_0(x)^{1-p} - (p-1)t)^{-\frac{1}{p-1}}, \quad 0 < t < u_0(x).$$

Fujita's theorem answers the question of which of these two effects is more powerful when one is pitted against the other. Let $p > 1$, and let u be a solution to (1) with initial condition $0 \leq u_0 \in L_\infty(\mathbb{R}^d)$ and maximal time of existence $T_{\max}(u_0) \in (0, \infty]$. Let

$$p_F := 1 + \frac{2}{d}.$$

Fujita's 1966 theorem [3] states the following:

(1) If $1 < p \leq p_F$, then (1) is ill-posed for all nonzero initial data in the sense that $T_{\max}(u_0) < \infty$ for all $u_0 \neq 0$.
(2) If $p > p_F$, then (1) is well-posed for sufficiently small initial data in the sense that there exists $\varepsilon_p > 0$ such that if

$$\|u_0\|_1 + \|u_0\|_\infty < \varepsilon_p$$

then $T_{\max}(u_0) = \infty$.

Numerous proofs of Fujita's theorem, and many extensions to other spaces, other operators, other nonlinearities, etc. are known. For details see the book [12].

It is important for most proofs that the function $u \mapsto u^p$ is monotone. For $p > 1$, the power function is not monotone for operators and this caused us to wonder whether Fujita's theorem still holds for some operator-valued situation.

A natural test case is the noncommutative Euclidean space $L_\infty(\mathbb{R}^d_\theta)$. This is a von Neumann algebra which can be defined pragmatically as a twisted left-regular representation of $\mathbb{R}^d$. Namely, let θ be a $d \times d$ antisymmetric real matrix, and let $\lambda_\theta(t)$ be the unitary operator on $L_2(\mathbb{R}^d)$ given by

$$\lambda_\theta(t)\xi(r) = \exp(i(t,r))\xi(r - \frac{1}{2}\theta t), \quad t, r \in \mathbb{R}^d,\ \xi \in L_2(\mathbb{R}).$$

We define

$$L_\infty(\mathbb{R}^d_\theta) := \{\lambda_\theta(t)\}''_{t \in \mathbb{R}^d}.$$

That is, the weak closure or the double commutant of the set of all $\lambda_\theta(t)$. Heuristically, $\mathbb{R}^d_\theta$ is supposed to be a version of $\mathbb{R}^d$ where the coordinate functions $x_1, \ldots, x_d$ do not commute but instead obey

$$[x_j, x_k] = i\theta_{j,k}, \quad 1 \leq j, k \leq d.$$

We are supposed to interpret

$$\lambda_\theta(t) = \exp(it_1 x_1 + \cdots + it_d x_d).$$

Other function spaces such as $C_0(\mathbb{R}^d_\theta)$, the Schwartz space $\mathcal{S}(\mathbb{R}^d_\theta)$ and L_p spaces $L_p(\mathbb{R}^d_\theta)$ can be defined in terms of $L_\infty(\mathbb{R}^d_\theta)$. This noncommutative space is sometimes called the Moyal d-space due to Moyal's work on quantum statistical mechanics [11]. Associated to this space we have derivatives $\partial_1, \ldots, \partial_d$ and a Laplacian $\Delta = \sum_{j=1}^d \partial_j^2$.

Many tools of harmonic analysis have been developed for $\mathbb{R}^d_\theta$. A limited selection of references for this topic [1, 2, 4, 6, 5, 7, 8, 10]. It is therefore a natural testing ground for noncommutative harmonic analysis.

The main result of our paper [9] is that Fujita's theorem is still true for $\mathbb{R}^d_\theta$, with the same critical exponent p_F. At some level this is a surprise because p_F can be thought of as resulting from the self-similarity of (1) under rescaling which does not have an analogue in $\mathbb{R}^d_\theta$. The main obstacle was the non-monotonicity of the map $u \mapsto u^p$ for $p > 1$. We were able to overcome this problem with the following trick: Suppose that $p > 1$, A and B are two operators such that $A^{1/p} \geq B$, and we would like to deduce that $A \geq B^p$. This is not possible, but when $p \leq 2$, the power function is operator convex and hence for any state φ we have

$$\varphi(A) \geq \varphi(A^{1/p})^p \geq \varphi(B)^p.$$

This is weaker than $A \geq B^p$, but turned out to be enough for our purposes.

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Semicommutative Covering Lemmas

Léonard Cadilhac

In commutative harmonic analysis, a standard and convenient way to prove geometric maximal inequalities is to use covering lemmas. The most well-known example is given by the Hardy-Littlewood maximal function: let $f \in L_1(\mathbb{R}^d)$ and $r > 0$, define

$$A_r f(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f \quad \text{and} \quad A^* f = \sup_{r>0} |A_r f|.$$

The maximal operator A^* is bounded on L_p for any $p > 1$ and is of weak type $(1, 1)$. This is a fundamental result as it is essentially equivalent to the Lebesgue Differentiation Theorem. To understand how it reduces to a geometric argument, we have to take a closer look at the weak $(1, 1)$ inequality and realize that it is directly implied by the following lemma: let $E \subset \mathbb{R}^d$ be covered by a family of balls, $E \subset \bigcup_{i\in I} B(x_i, r_i)$, then we can find $J \subset I$, such that:

(1) $E \subset \bigcup_{j\in J} B(x_j, r_j)$,
(2) $\forall i \neq j \in J,\ B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$.

In other words, if we can lose a multiplicative constant on the radii, any covering by Euclidean balls admits a morally disjoint subcover.

So far, this method was assumed to be unsuited to noncommutative generalizations as it does not translate well to the lattice of projections of a von Neumann algebra. Indeed, simple counterexamples show that the lemma above cannot be exported to the quantum setting by replacing the inclusion of sets by the inequality of projections and the union of sets by the union of projections. This has motivated noncommutative harmonic analysts to develop other strategies, often aimed at reducing maximal inequalities to Doob's inequality for martingales [3], an approach already known to be powerful in the classical setting, especially to prove variational or dimension free inequalities [1, 5].

The aim of this talk is to investigate more direct covering type arguments in the semicommutative setting. Though a naive approach to "covering" conditions cannot work, by taking inspiration from the statements of noncommutative maximal inequalities found in the 70's by Cuculescu [4] and Yeadon [2], it is possible to formulate adequate analogues to several geometric covering lemmas. Variants of the Vitali, Besicovitch and Ornstein-Weiss lemmas, and applications to maximal inequalities that escaped the previously used martingale based approach will be presented in this talk. We obtain for example an operator-valued Hardy-Littlewood maximal inequality for any Borel measure and a maximal inequality for tempered sequences generalizing [6]. In both of those cases, it is not known whether a martingale approach could be used but the more direct method works. A way to interpret this is that those settings lack a form of global regularity that would help to define a filtration but are still good enough for "local" geometric arguments.

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Optimal Khintchine Inequality and Zygmund's Z_2-Property

Tao Mei

This is a joint work with Chian Yeong Chuah and Zhen-Chuan Liu. We study Lust-Piquard/Pisier's operator Khintchine inequality ([4]) that provides an equivalence between the L_1 and L_2 norms of the functions spanned by a given orthonormal sequence. The main result is a loosing equivalence between the so-called Z_2 property for the underlying orthonormal sequence and the validity of the Khintchine inequality with the optimal constant. In particular, for L_2-functions on the torus spanned by the sequence $\{e^{ik\theta}, k \in E\}$ with E a set of integers, we show that E must have a finite Z_2 constant if Lust-Piquard/Pisier's operator Khintchine inequality holds with an equivalent constant close enough to the optimal constant $\sqrt{2}$. On the other hand, we show that, given an orthonormal sequence with a finite Z_2 constant, then Lust-Piquard/Pisier's operator Khintchine inequality holds for all functions spanned by this orthogonal sequence with an equivalent constant $\sqrt{1+Z_2}$. Some of our arguments crucially rely on previous works by Haagerup/Musat [2].

The classical Khintchine inequality says that the the p-norm ($p \in (0,\infty)$)and the 2-norm of the linear combinations of Rademacher functions are equivalent, i.e., there exist constants $\alpha_p, \beta_p > 0$ such that

$$\text{(Kp)} \qquad \beta_p^{-1}\left(\int_0^1 \left|\sum_{k=1}^n c_k\epsilon_k\right|^p dt\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^n |c_k|^2\right)^{\frac{1}{2}} \leq \alpha_p\left(\int_0^1 \left|\sum_{k=1}^n c_k\epsilon_k\right|^p dt\right)^{\frac{1}{p}}$$

for any $n \in \mathbb{N}$ and $c_k \in \mathbb{C}, k = 1, \cdots, n$. Here $\epsilon_k = sign\left[\sin(2^k\pi t)\right], k \in \mathbb{N}, t \in [0,1]$ is a sequence of Radamacher functions. (Kp) is satisfied by a large class of symmetric orthonormal sequences, including $\epsilon_k(t) = e^{i2^k 2\pi t}$ or i. i. d. Gaussian random variables. We are interested in the connections between the algebraic/combinatoric properties of the sequence $\{\epsilon_k\}$ and the validity of the analytic inequality (Kp).

In the case that $\{\epsilon_k\}$ belongs to the family of characters of the integer group $\mathbb{Z}$, Zygmund and Rudin observed that (Kp) holds for $p = 4$ (hence for all $p < 4$) if $\{\epsilon_k\}$ has the so called Z_2 property that

$$Z_2(\{\epsilon_k\}) := \sup_{k\neq j\in\mathbb{N}} \#\{(k',j') \in \mathbb{N}\times\mathbb{N}; \epsilon_{k'}\epsilon_{j'}^{-1} = \epsilon_k\epsilon_j^{-1}\} < \infty.$$

For example, $Z_2(\{\epsilon_k\}) = 1$ for the sequence $\epsilon_k(t) = e^{i2^k2\pi t}$. Moreover, elementary calculations imply that $\beta_4 \leq (Z_2(\{\epsilon_k\}) + 1)^{\frac{1}{4}}$, and $\beta_4 = 2^{\frac{1}{4}}$ iff $Z_2(\{\epsilon_k\}) = 1$ for ϵ_k being characters of $\mathbb{Z}$. Hölder's inequality then implies that $\alpha_1 \leq \beta_4^2 \leq (Z_2(\{\epsilon_k\})+1)^{\frac{1}{2}}$. We generalize the definition of $Z_2(\{\epsilon_k\})$ to orthonormal sequences. For a class of unitary sequences $\{\epsilon_k\}$, we prove an almost equivalent relation between the Z_2-property of $\{\epsilon_k\}$ and the validity of the optimal inequality (K1) for arbitrary matrix (operator)-valued coefficients.

The operator-valued analogue of (Kp) has been intensively studied during the last a few decades ([3, 4, 5, 6]) and has wide applications in noncommutative

analysis and quantum information theory. Lust-Piquard [3] first formulated and proved Khintchine's inequalities for the (operator) matrix-valued coefficients in the case $p > 1$ and later with Pisier [4] proved the case for $p = 1$. Pisier/Ricard ([5]) finally settled the operator Khintchine inequality for the $p < 1$ case. One version of Lust-Piquard/Pisier's *operator-valued Khintchine inequalities* for the Radamacher sequence ϵ_k says: for $d, n \in \mathbb{N}$ and $C_1, \cdots, C_n \in M_d(\mathbb{C})$,
(OKp)

$$B_p^{-1}\left[\int_0^1 tr\left(\left|\sum_{k=1}^n C_k\epsilon_k\right|^p\right)dt\right]^{\frac{1}{p}} \leq \|(C_k)_{k=1}^n\|_{S_p(\ell^2)} \leq A_p\left[\int_0^1 tr\left(\left|\sum_{k=1}^n C_k\epsilon_k\right|^p\right)dt\right]^{\frac{1}{p}}.$$

with constants A_p, B_p depending only on p.

The Khintchine inequality for operator valued coefficients and its connection to the Z_2 property are more subtle. For example, the implication of (Kp)$\Rightarrow$ (Kq) for $0 < q < p < \infty$ is fairly easy for scalar-valued coefficients c_k, but is highly nontrivial for operator-valued c_k, and the forth mentioned inequality $\alpha_1 \leq \beta_4^2$ seems not true any more when considering operator-valued coefficients. One obstacle is the α-Hölder continuity of the Mazur map $x \mapsto x|x|^{\frac{p-q}{q}}$ from L^p to L^q for operator-valued x.

Given an orthonormal sequence $\{\epsilon_k\}$ and $p < \infty$, we say (OKp) holds for $\{\epsilon_k\}$ if it holds for arbitrary matrix-valued coefficient C_k with absolute constants A_p and B_p. Note, the correct "2"-norm of the (operator)-valued C_k in the formulation of (OKp) has to be the operator-space norm $S_p(\ell_2)$. We refer readers to [7, Section 2.4] for a detailed definition of norm $\|\cdot\|_{S_1^n(\ell_2^{rc}))}$.

It is well known that Khintchine's inequality for a smaller value of p (e.g. $p = 1$) is strictly weaker than the one for a larger value $q > 2$ (e.g. $q = 4$). This is due to Bourgain's famous work on Λ_p-sets and Harcharras' work on the complete Λ_p sets. However, the procedure of deducing Khintchine's inequalities for smaller p from a larger q is much more complicated when C_k's are operator (matrix)-valued. This also makes it very hard to track the precise value of the constants A_1 in a process of deducing it from B_4([5],[6]).

In a recent work with Chuah and Liu ([7]), we prove an analogue of the relation

$$A_1 \leq (Z_2(\{\epsilon_k\}) + 1)^{\frac{1}{2}}$$

for general orthonormal sequences $\{\epsilon_k\}$ and operator-valued coefficients following the strategy of Haagerup and Musat ([2]). Moreover, we discovered a backward relation for a class of unitary sequences $\{\epsilon_k\}$ that, if (OK1) holds for $\{\epsilon_k\}$ with a constant A_1 close enough to $\sqrt{2}$, then $Z_2(\{\epsilon_k\}) < \infty$, and the operator-valued Khintchine inequality (OKp) holds for all $p \leq 4$.

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What are quantum measurable spaces?

Tobias Fritz
(joint work with Antonio Lorenzin)

It is often said that von Neumann algebras are the right setting for noncommutative measure theory. Although this makes sense, I will point out that we do not have a good notion of noncommutative *measurable* space. Having such a notion would be relevant for infinite-dimensional quantum information theory in at least the following ways:

- There are two notions of measurement in quantum theory: POVMs on the one hand and quantum-to-classical channels on the other hand. These two notions *should* be equivalent. If one assumes that classical probability theory is the right way to model classical systems, then this correspondence does *not* hold if one takes a quantum system to be modelled by any C*-algebra, but only under certain restrictions.
- The category of classical systems and channels should be equivalent to the category of (nice) measurable spaces and Markov kernels. On the other hand, it should be possible to define a classical system as a quantum system whose algebra of observables is commutative. Therefore the category of (nice) measurable spaces and Markov kernels should be equivalent to the category of commutative (nice) quantum measurable spaces.
- A categorical approach to quantum information processing can be quite powerful [2], but it has only been developed for finite-dimensional systems so far. It is clear that an infinite-dimensional generalization will be quite sensitive to the technical details of what kinds of algebras are considered, and with what kind of maps and what kind of tensor product.

In [1], we have investigated possible definitions of quantum measurable spaces based on σC*-algebras, monotone σ-complete C*-algebras. This class itself turns out to be too large, because there still are σC*-algebras whose self-adjoint elements are not separated by σ-states. I will sketch an improved candidate definition in terms of Pedersen–Baire envelopes [3, Definition 2.2.23]. However, I would prefer to have a definition in terms of an additional completeness or projectivity property for the σC*-algebras under consideration and welcome any suggestions in this direction.

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From Grothendieck to Shor

Marius Junge

(joint work with Carlos Palazuelos and Roy Araiza)

Grothendieck inequalities have a long tradition in Functional Analysis, in particular Banach space Theory and Operator Algebra Theory. Indeed, Effros choice of the term nuclear C^*-algebras is inspired by Grothendieck's work. Pisier generalized Grothendieck's fundamental theorem to bilinear forms on C^*-algebras (see [Pis86]). This generalized Grothendieck inequality found applications in the theory of bounded Cohomology, tensor norms on C^*-algebras and, thanks to the correct operator space version of the inequality proved by Pisier and Shlyakhtenko, in quantum information theory. Pisier and Shlyahktanko [PS02] re-emphasized Blecher's conjecture (see [BL91])which corresponds to a matrix or trilinear version of Grothendieck's inequality. The tri-linear version was disproved by Briet and Palazuelos [BP19]. In this talk we disprove the matrix (functiorial) version

$$K(\ell_2) \otimes_{min} \ell_1(I) \otimes_{\min} \ell_1(J) \neq K(\ell_2) \otimes (\ell_1(I) \otimes_h \ell_1(J) + \ell_1(J) \otimes \ell_1(I)) .$$

Here h denoted the Haagerup tensor product and the non-symmetric summ corresponds to factorization through row and column. The symmetrized sum is embedded in operator algebra free product [OP99].

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