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Model Theory and Arithmetic

Organized by
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Abstract. The workshop "Model Theory and Arithmetic" brought together communities working in model theory (a subfield of mathematical logic) and in number theory, in particular arithmetic geometry. There has been a flurry of new ideas and results going back and forth between these areas recently. For example, o-minimality has been fruitfully used in Hodge theory and diophantine geometry, and model theoretic approaches to perfectoid fields and Arakelov geometry are emerging. Oberwolfach provided the ideal setting to gather these two groups in one place in order to facilitate the exchange of ideas between them. Talks covered a diverse range of topics, but common themes were tame geometry (also in non-archimedean settings) and transcendence questions.

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Introduction by the Organizers

The workshop *Model Theory and Arithmetic*, organized by Matthias Aschenbrenner (Vienna), Philipp Habegger (Basel), Annette Huber-Klawitter (Freiburg), and Thomas Scanlon (Berkeley) was held January 18–23, 2026.

There were 48 participants, among them two online participants. We had a total of 18 talks, all in-person, each 60 minutes long. The schedule left ample opportunity to do and talk about mathematics during the breaks. The traditional hike on Wednesday afternoon was the highlight of the social program of the workshop.

The interplay of arithmetic with model theory, a branch of mathematical logic, has a long history, even going back to the foundational work in mathematical logic

due to Gödel, Herbrand, Mal′cev, Presburger, Skolem, Tarski and others in the first half of the last century. These interactions have intensified in the last two decades, and have led to several breakthroughs including advances towards conjectures in diophantine geometry, in algebraic geometry (Hodge theory), functional transcendence, the study of periods, and more. The interaction runs both ways, and for example has led to the development of the model theory of height functions via globally valued fields. The conference provided an opportunity for researchers in different fields to meet and to build on the foundations.

In order to accommodate the mixed background of the audience, the speakers made an extra effort to be understandable to everyone. Several speakers had been asked to give overview style talks.

We come to a short summary of the talks, in chronological order:

Joseph Ayoub opened the workshop with an overview of the Period Conjecture by Grothendieck and Konstevich–Zagier, the motivic Galois group and non-archimedean version. **Michał Szachniewicz** then gave an introduction to Ben Yaacov and Hrushovski's theory of globally valued fields and explained his recent result on the existential closedness of $\overline{\mathbb{Q}}$. Functional transcendence properties and the classification problem for bi-algebraic varieties in the case of Euler's Γ-function were the focus of the talk by **Adele Padgett**. **Martin Orr** finished Monday's session by presenting his work on effective height functions for unlikely intersections using G-functions.

On Tuesday, **Hélène Esnault** discussed the conjectural algebraicity of Simpson's non-abelian Hodge Locus as well as the failure of Grauert's Coherence Theorem in a suitable o-minimal setting. **Immanuel Halupczok** reported on his recent development of h(ensel)-minimal geometry, a non-archimedean variant of o-minimal geometry. An application of h-minimality and the Tate uniformization to just likely intersections appeared in **Sebastian Eterović's** presentation. By work of Brown, periods of certain moduli spaces of genus 0 curves are $\mathbb{Q}[2\pi i]$-linear combinations of multiple zeta values. In this vein, **Riccardo Tosi** considered periods of certain hyperplane arrangements and drew connections to polylogarithms.

On Wednesday we had three talks, in the morning. **Ehud Hrushovski** reported on his work giving a model theoretic account of the problem of expressing the first two terms in the expansion as a power of $q^{\frac{1}{2}}$ of the number of points on algebraic varieties in the finite field $\mathbb{F}_q$. **Katharina Hübner** gave an account of the Berkovich geometry of curves over valued fields of higher rank. **Konstantinos Kartas** spoke about using the model theory of tame valued fields to study more general perfectoid rings proving theorems from "almost mathematics" as on the nose results.

The first talk on Thursday was given by **François Loeser**, who spoke on his proof of two theorems of F.-V. Kuhlmann in valuation theory using the stable completion of a curve over a valued field, which is a model-theoretic version of the Berkovich analytification introduced by him and Hrushovski. Then **Gareth Jones** presented his recent effective results on special points problems in the context of unlikely intersections using powerful uniformity statements about Pfaffian

functions. In the afternoon, **James Freitag** gave an introduction to transcendence results in differential and difference algebra shown via a combination of model theory and algebraic group theory. **Tobias Kaiser** reported on a program to introduce a definable de Rham cohomology and initial successes that involve globally subanalytic functions.

On Friday, **Francesco Gallinaro** began by giving an hitherto unknown simple axiomatization of the complete theory of the real exponential field assuming Schanuel's Conjecture. **Jonathan Kirby** then introduced the classes of exponentially-algebraic functions and presented results investigating when such functions appear as solutions of algebraic differential equations. **Peter Jossen** delivered the final talk on conjectures and outlooks that intertwine transcendence properties of values of G-, E-, and Θ-functions and Borel summation.

The workshop brought together researchers working on different questions and with different methods—all related to model theory and number theory—who would not necessarily meet each other at conferences in their respective fields. On the one hand, this allowed us to take stock of the recent advances on several fronts in both subjects. On the other hand, everyone profited from being exposed to topics outside of their comfort zone, and many new research ideas were generated from bringing together a mix of people with distinct backgrounds. The success of this endeavor gives us hope that this will be the first in a sequence of similar meetings in the future.

Workshop: Model Theory and Arithmetic

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Abstracts

Periods and the Motivic Galois Group

Joseph Ayoub

1. Complex periods

Let $\mathbb{D} = \{z \in \mathbb{C}; |z| \leqslant 1\}$ be the closed unit disc in $\mathbb{C}$. For a nonnegative integer n, let $\mathcal{O}(\mathbb{D}^n)$ be the $\mathbb{C}$-algebra of holomorphic functions which are defined on an open neighbourhood of $\mathbb{D}^n$ in $\mathbb{C}^n$. Equivalently, $\mathcal{O}(\mathbb{D}^n)$ is the algebra of formal power series

$$f = \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n} a_{i_1,\dots,i_n} z_1^{i_1}\cdots z_n^{i_n}$$

such that, for some radius $\rho > 1$, the sequence $|a_{i_1,\dots,i_n}|\rho^{i_1+\cdots+i_n}$ tends to zero when $i_1 + \cdots + i_n$ goes to infinity. We set

$$\mathcal{O}(\mathbb{D}^\infty) = \operatorname*{colim}_{n\in\mathbb{N}} \mathcal{O}(\mathbb{D}^n).$$

The elements of $\mathcal{O}(\mathbb{D}^\infty)$ are formal power series in finitely many variables with coefficients in $\mathbb{C}$ and with radii of convergence > 1.

For $n \geqslant 0$, let $\Omega^\bullet(\mathbb{D}^n)$ be the de Rham complex of holomorphic forms on an open neighbourhood of $\mathbb{D}^n$ in $\mathbb{C}^n$. This complex is concentrated in degrees $0 \leqslant m \leqslant n$ where it is given by

$$\Omega^m(\mathbb{D}^n) = \bigoplus_{I\subset\{1,\dots,n\},\, \#(I)=m} \mathcal{O}(\mathbb{D}^n)\cdot \mathrm{d}z_I,$$

where $\mathrm{d}z_I = \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_m}$ if $I = \{i_1 < \cdots < i_m\}$. We denote by $\Omega^{n-\bullet}(\mathbb{D}^n)$ the connective complex obtained by shifting $\Omega^\bullet(\mathbb{D}^n)$ to the left n times while keeping the differential unchanged. Thus, in homological degree m, this shifted complex is given by $\Omega^{n-m}(\mathbb{D}^n)$. We have a sequence

$$\Omega^{0-\bullet}(\mathbb{D}^0) \xrightarrow{-\wedge \mathrm{d}z_1} \cdots \xrightarrow{-\wedge \mathrm{d}z_n} \Omega^{n-\bullet}(\mathbb{D}^n) \xrightarrow{-\wedge \mathrm{d}z_{n+1}} \Omega^{n+1-\bullet}(\mathbb{D}^{n+1}) \xrightarrow{-\wedge \mathrm{d}z_{n+2}} \cdots$$

and we set

$$\Omega^{\infty-\bullet}(\mathbb{D}^\infty) = \operatorname*{colim}_{n\in\mathbb{N}} \Omega^{n-\bullet}(\mathbb{D}^n).$$

By construction, the complex $\Omega^{\infty-\bullet}(\mathbb{D}^\infty)$ is connective, i.e., concentrated in nonnegative homological degrees. Moreover, for $m \geqslant 0$, we have

$$\Omega^{\infty-m}(\mathbb{D}^\infty) = \bigoplus_{I\subset\mathbb{N}\smallsetminus\{0\},\, \#(I)=m} \mathcal{O}(\mathbb{D}^\infty)\cdot \widehat{\mathrm{d}z_I},$$

where $\widehat{\mathrm{d}z_I}$ is the codegree-m form on $\mathbb{D}^\infty$ obtained by wedging all the $\mathrm{d}z_i$'s, for $i \notin I$. In fact, it is easy to see that the complex $\Omega^{\infty-\bullet}(\mathbb{D}^\infty)$ is acyclic. We are rather concerned with the subcomplex

$$\widetilde{\Omega}^{\infty-\bullet}(\mathbb{D}^\infty) \subset \Omega^{\infty-\bullet}(\mathbb{D}^\infty)$$

given in homological degree m by those codegree-m forms on $\mathbb{D}^\infty$ that vanish when restricted to the hypersurfaces defined by the equations $z_i(1-z_i)=0$ for $i \geqslant 1$. More explicitly, a codegree-m form

$$\sum_{I\subset\mathbb{N}\smallsetminus\{0\},\,\#(I)=m} f_I\cdot\widehat{\mathrm{d}z_I}$$

belongs to $\widetilde{\Omega}^{\infty-m}(\mathbb{D}^\infty)$ if and only if, for every $I\subset\mathbb{N}\smallsetminus\{0\}$ of cardinality m, we have $f_I|_{z_i=0}=f_I|_{z_i=1}=0$ for all $i\in I$.

Lemma 1. *There is a quasi-isomorphism*

$$\widetilde{\Omega}^{\infty-\bullet}(\mathbb{D}^\infty)\xrightarrow{\text{q.-i.}}\mathbb{C}[0],\qquad f\cdot\widehat{\mathrm{d}z_\varnothing}\mapsto\int_{[0,1]^\infty} f. \tag{1}$$

Definition. Let $\mathcal{O}_{\mathrm{alg}}(\mathbb{D}^\infty)\subset\mathcal{O}(\mathbb{D}^\infty)$ be the sub-$\mathbb{Q}$-algebra whose elements are those holomorphic functions which are algebraic over the field of rational functions $\mathbb{Q}(z_1,\dots,z_n,\dots)$. Similarly, let

$$\Omega^{\infty-\bullet}_{\mathrm{alg}}(\mathbb{D}^\infty)\subset\Omega^{\infty-\bullet}(\mathbb{D}^\infty)$$

be the subcomplex given in homological degree m by

$$\Omega^{\infty-m}_{\mathrm{alg}}(\mathbb{D}^\infty)=\bigoplus_{I\subset\mathbb{N}\smallsetminus\{0\},\,\#(I)=m}\mathcal{O}_{\mathrm{alg}}(\mathbb{D}^\infty)\cdot\widehat{\mathrm{d}z_I}.$$

Finally, let

$$\widetilde{\Omega}^{\infty-\bullet}_{\mathrm{alg}}(\mathbb{D}^\infty)\subset\widetilde{\Omega}^{\infty-\bullet}(\mathbb{D}^\infty)$$

be the intersection of $\Omega^{\infty-\bullet}_{\mathrm{alg}}(\mathbb{D}^\infty)$ and $\widetilde{\Omega}^{\infty-\bullet}(\mathbb{D}^\infty)$ in $\Omega^{\infty-\bullet}(\mathbb{D}^\infty)$.

The following conjecture was made in [1]. (See [1, Remarque 2.109 & Conjecture A].)

Conjecture 2. The complex $\Omega^{\infty-\bullet}_{\mathrm{alg}}(\mathbb{D}^\infty)$ is acyclic except in degree zero and the map

$$\mathrm{H}_0(\Omega^{\infty-\bullet}_{\mathrm{alg}}(\mathbb{D}^\infty))\to\mathbb{C},\qquad f\cdot\widehat{\mathrm{d}z_\varnothing}\mapsto\int_{[0,1]^\infty} f, \tag{2}$$

deduced from (1), is injective.

Remark 3. The image of the map (2) is precisely the ring of effective periods. The second assertion in Conjecture 2 is essentially a reformulation of the Kontsevich–Zagier conjecture on periods [6]. (This is not completely trivial and relies on [5].) The first assertion is closely related to the existence of a motivic t-structure.

2. Adic periods

Let K be a field endowed with a nontrivial discrete valuation. Denote by k the residue field of K and by $\widehat{K}$ its completion. Assume also that K has characteristic zero. The two main examples for us are $K=\mathbb{Q}$, with the p-adic valuation, and $K=k(\pi)$, with the π-adic valuation. Let $\mathbb{B}$ be the Tate ball over $\widehat{K}$ considered as a rigid analytic variety. Let $\mathcal{O}(\mathbb{B}^n)$ be the $\widehat{K}$-algebra of power series in n variables which are convergent in an open neighbourhood of $\mathbb{B}^n$. Then, as in the previous

section, we may form the $\widehat{K}$-algebra $\mathcal{O}(\mathbb{B}^\infty)$ and the complexes $\Omega^{\infty-\bullet}(\mathbb{B}^\infty)$ and $\widetilde{\Omega}^{\infty-\bullet}(\mathbb{B}^\infty)$. The former complex is acyclic while the latter is quasi-isomorphic to $\widehat{K}[0]$. Indeed, we have the following analog of Lemma 1.

Lemma 4. *There is a quasi-isomorphism*

$$\widetilde{\Omega}^{\infty-\bullet}(\mathbb{B}^\infty) \xrightarrow{\text{q.-i.}} \widehat{K}[0], \qquad f \cdot \widehat{\mathrm{d}z_\varnothing} \mapsto \int_{[0,1]^\infty} f. \tag{3}$$

Note that the map (3) is given by "formal integration" of power series, i.e., by taking formal primitives and evaluating at 0 and 1.

Definition. Let $\mathcal{O}_{K\text{-alg}}(\mathbb{B}^\infty) \subset \mathcal{O}(\mathbb{B}^\infty)$ be the sub-K-algebra whose elements are those power series which are algebraic over the field of rational functions $K(z_1, \ldots, z_n, \ldots)$. As in Definition 1, this yields subcomplexes $\Omega^{\infty-\bullet}_{K\text{-alg}}(\mathbb{B}^\infty)$ and $\widetilde{\Omega}^{\infty-\bullet}_{K\text{-alg}}(\mathbb{B}^\infty)$ of $\Omega^{\infty-\bullet}(\mathbb{B}^\infty)$.

By analogy with Conjecture 2, we make the following conjecture.

Conjecture 5. Assume that $K = \mathbb{Q}$, with the p-adic valuation, or $K = k(\pi)$, with the π-adic valuation. Then, the complex $\Omega^{\infty-\bullet}_{K\text{-alg}}(\mathbb{B}^\infty)$ is acyclic except in degree zero and the map

$$\mathrm{H}_0(\Omega^{\infty-\bullet}_{K\text{-alg}}(\mathbb{B}^\infty)) \to \widehat{K}, \qquad f \cdot \widehat{\mathrm{d}z_\varnothing} \mapsto \int_{[0,1]^\infty} f, \tag{4}$$

deduced from (3), is injective.

Remark 6. In the p-adic case, Conjecture 5 is wide-open. In the π-adic case, it is essentially a theorem by [3, 4]. More precisely, it is known to hold after stabilisation; see [4, Théorème 1.7].

3. The motivic Galois group

For every field k endowed with a complex embedding $k \hookrightarrow \mathbb{C}$, there is an associated motivic Galois group $\mathcal{G}_{\mathrm{mot}}(k)$. It is a derived affine group scheme, i.e., the spectrum of a connective derived Hopf algebra $\mathcal{H}_{\mathrm{mot}}(k)$. Roughly speaking, $\mathcal{G}_{\mathrm{mot}}(k)$ is the group of autoequivalences of the Betti realisation

$$\mathrm{B}^* : \mathrm{DM}(k; \mathbb{Q}) \to \mathrm{D}(\mathbb{Q}), \qquad \mathrm{M}(X) \mapsto \mathrm{C}_*(X(\mathbb{C})).$$

It turns out that the complex $\widetilde{\Omega}^{\infty-\bullet}(\mathbb{D}^\infty)$ is, up to inverting the formal period corresponding to $2\pi\mathbf{i}$, the Betti realisation of the (big) motive representing de Rham cohomology in $\mathrm{DM}(k; \mathbb{Q})$. This shows that $\widetilde{\Omega}^{\infty-\bullet}(\mathbb{D}^\infty)$ is essentially the algebra of a torsor over $\mathcal{G}_{\mathrm{mot}}(k)$. This perspective is crucial in proving Conjecture 5 in the π-adic case (after stabilisation). In fact, the said conjecture becomes a translation of the property that the derived affine group scheme

$$\mathcal{G}_{\mathrm{rel}}(K/k) \coloneqq \ker\{\mathcal{G}_{\mathrm{mot}}(K) \to \mathcal{G}_{\mathrm{mot}}(k)\}$$

is classical (i.e., its Hopf algebra is concentrated in degree zero) and is a quotient of the pro-algebraic completion of the fundamental group of $\mathbb{C} \smallsetminus k$, considered as a pro-Zariski open. These results are proven in [2, Théorèmes 2.55 & 2.57].

References

[1] J. Ayoub, *L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, I*, J. Reine Angew. Math., **693** (2014), 1–149.

[2] J. Ayoub, *L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, II*, J. Reine Angew. Math., **693** (2014), 151–226.

[3] J. Ayoub, *Une version relative de la conjecture des périodes de Kontsevich–Zagier*, Ann. of Math. (2), **181** (2015), no. 3, 905–992.

[4] J. Ayoub, *La version relative de la conjecture des périodes de Kontsevich–Zagier revisitée*, Tohoku Math. J. (2), **71** (2019), no. 3, 465–485.

[5] U. Choudhury and M. Gallauer, *An isomorphism of motivic Galois groups*, Adv. Math., **313**, (2017), 470–536.

[6] M. Kontsevich and D. Zagier, *Periods*, Mathematics unlimited 2001 and beyond, Springer, Berlin, (2001), 771–808.

Globally Valued Fields and Arakelov Geometry

Michał Szachniewicz

(joint work with Itaí Ben Yaacov, Pablo Destic and Ehud Hrushovski)

The theory of globally valued fields was initiated by Ben Yaacov and Hrushovski in 2010's. These structures generalise global fields and form a first-order class. We survey this theory and conclude with some questions.

Let F be a field. A *global height* on F is a collection of functions $h : \mathbb{P}^n(F) \to \mathbb{R}_{\geq 0}$, for every natural $n \geq 1$, satisfying the following axioms.

Height of one:		$h(1:1) = 0$
Invariance:	$\forall x \in \mathbb{P}^n(F),\ \forall \sigma \in \mathrm{Sym}_{n+1},$	$h(\sigma x) = h(x)$
Additivity:	$\forall x \in \mathbb{P}^n(F),\ \forall y \in \mathbb{P}^m(F),$	$h(x \otimes y) = h(x) + h(y)$
Monotonicity:	$\forall x \in \mathbb{P}^n(F),\ \forall y \in \mathbb{P}^m(F),$	$h(x) \leq h(x:y)$
Triangle inequality:	$\forall x, y \in F^n, x + y \neq 0,$	$h(x+y) \leq h(x:y) + h(2:1)$

where $(x_i)_i \otimes (y_j)_j = (x_i y_j)_{i,j}$ is the Segre embedding. A *globally valued field* is a field equipped with a global height. The axioms above together with $h(2:1) \leq e$, for some non-negative real number e, define an unbounded continuous logic theory GVF_e. Number fields with the Weil logarithmic height are models of this theory if $e \geq \log 2$. Function fields of curves provide another examples, with $e = 0$. In fact Weil logarithmic height is the unique global height on $\overline{\mathbb{Q}}$ with $h(2:1) = \log 2$.

To understand global heights better, let us introduce some notation, and assume for simplicity that F is countable:

- M_F is the set of absolute values on F, i.e., functions $|\cdot| : F \to \mathbb{R}_{\geq 0}$ that satisfy $|x| = 0 \iff x = 0, |xy| = |x||y|, |x+y| \leq |x| + |y|$. We equip it with the pointwise convergence topology.
- We define $\mathrm{LDiv}(F)$ to be the set of functions $M_F \to \mathbb{R}$ that are of the form

$$|\cdot| \mapsto \log \max_{i=1,\dots,n} |a_i| - \log \max_{j=1,\dots,m} |b_j|,$$

for some finite subsets $\{a_i\}_{i=1}^n, \{b_j\}_{j=1}^m \subseteq F^\times$. It is an ℓ-group, i.e., an ordered group that is a lattice (there exist minima and maxima of pairs of elements).

- An L^1-lattice is a partially ordered Banach space Γ that is a lattice, such that if $x, y \geq 0$, then $\|x+y\| = \|x\| + \|y\|$. Equivalently, $\Gamma \cong L^1(\Omega)$, for some measure space Ω. For $\gamma \in \Gamma$ we write $\int \gamma := \|\max(\gamma, 0)\| - \|\min(\gamma, 0)\|$.
- An L^1-valuation on F is a function $v : F \to \Gamma \cup \{\infty\}$, where Γ is an L^1-lattice, such that $v(x) = \infty \iff x = 0, v(xy) = v(x) + v(y), v(x+y) \geq \min(v(x), v(y)) + \min(v(2), 0)$ and that satisfies the product formula, i.e., $x \neq 0 \implies \int v(x) = 0$. We define equivalence of L^1-valuations in the same way as the equivalence of ordinary (arbitrary rank) valuations on a field.

With that at hand, we can state the following.

Theorem 1. [1] *There is a bijection between the sets of:*

(1) positive homomorphisms $q : \mathrm{LDiv}(F) \to \mathbb{R}$ that satisfy $q(\log|a|) = 0$ for $a \in F^\times$,
(2) equivalence classes of L^1-valuations on F,
(3) global heights on F.

Furthermore, for every global height on F there exists a (Radon) measure μ on M_F such that the global height is given by

$$h(a_0 : \cdots : a_n) = \int_{v \in M_F} \log \max_{i=0,\ldots,n} |a_i|_v \, \mathrm{d}\mu(v).$$

Let $\mathscr{X}$ be a projective integral scheme over $\mathbb{Z}$. We call such $\mathscr{X}$ arithmetic varieties. An arithmetic divisor on $\mathscr{X}$ is a pair $(\mathscr{D}, g)$ consisting of a Cartier divisor $\mathscr{D}$ on $\mathscr{X}$ together with a Green's function $g : \mathscr{X}(\mathbb{C}) \setminus \mathrm{supp}(\mathscr{D}) \to \mathbb{R}$. Being a Green's function means that for every Zariski open $\mathscr{U} \subseteq \mathscr{X}$, if $\mathscr{D}$ is the divisor of a rational function f on $\mathscr{U}$, then $g + \log|f|$ extends to a continuous function on $\mathscr{U}(\mathbb{C})$. We denote the group of arithmetic divisors on $\mathscr{X}$ by $\mathrm{ADiv}(\mathscr{X})$. An arithmetic divisor is effective if $\mathscr{D}$ is, and $g \geq 0$.

The former theorem allows to connect global heights with Arakelov geometry using the following construction. Assume that F is finitely generated over $\mathbb{Q}$ and let Φ be the category where an object is an arithmetic variety $\mathscr{X}$ together with a morphism $\mathrm{Spec}(F) \to \mathscr{X}$ that is an isomorphism onto the generic point of $\mathscr{X}$. Morphisms in Φ are defined as morphisms between the ambient arithmetic varieties such that the triangle formed when adding two morphisms from $\mathrm{Spec}(F)$ commutes. Let us denote by $\mathrm{ADiv}(F)$ the colimit of $\mathrm{ADiv}(\mathscr{X})$ over the system Φ. It is an ℓ-group with respect to the effective order.

Theorem 2. [3] *There is a unique embedding of ℓ-groups*

$$\mathrm{LDiv}(F) \longrightarrow \mathrm{ADiv}(F)$$

that maps the function $\big(|\cdot| \mapsto -\log|a|\big)$ to $(\mathrm{div}(a), -\log|a|)$. Furthermore, the image is dense with respect to the effective order. Thus, every positive homomorphism $\mathrm{LDiv}(F) \to \mathbb{R}$ has a unique positive homomorphism extension $\mathrm{ADiv}(F) \to \mathbb{R}$.

In the function field case, if $k \subseteq K$ is a finitely generated extension of fields, then global heights on K that are zero on k have the following description.

Theorem 3. [2] *The quotient ℓ-group*

$$\mathrm{LDiv}(K)/\mathrm{LDiv}(k)$$

is isomorphic to the colimit of the groups of Cartier divisors on projective varieties X over k that have their function fields isomorphic to K over k (with the order coming from the effective cones).

Theorem 2 and Theorem 3 are not mere translations as they are key in proving existential closedness of some globally valued fields. A field with a global height (F, h) is existentially closed, if for every variety X over F and every global height h_X on the function field of X that extends h, there is a generic net of points $(x_i)_{i\in I}$ in $X(F)$ such that for every tuple of rational functions $f_0, \dots, f_k$ on X that are not all zero

$$\lim_{i\in I} h\big(f_0(x_i) : \cdots : f_k(x_i)\big) = h_X(f_0 : \cdots : f_k).$$

The main result of [2] is that $\overline{k(t)}$ with the unique global height that is zero on k and satisfies $h(t : 1) = 1$, is existentially closed. Analogously, the main result of [3] says that the field of algebraic numbers $\overline{\mathbb{Q}}$ equipped with the Weil logarithmic height is existentially closed. Naturally, it would be desirable to answer the following.

Question 1. Is the class of existentially closed globally valued fields first order? In other words, does there exist a model companion of the theory GVF_e?

Model companions of classes of fields with an extra structure are often connected to rich underlying geometries. Can one find such a geometry in the case of globally valued fields?

Question 2. Does there exist a geometry of Berkovich type where closed points represent global heights on residue rings of an algebra?

Such a geometry could be useful in studying not only height problems over $\overline{\mathbb{Q}}$, but over number fields as well.

Question 3. What is the asymptotic theory of the class of number fields as globally valued fields? Also, what is the asymptotic theory of all globally valued fields that satisfy the Northcott's principle?

References

[1] Itaï Ben Yaacov, Pablo Destic, Ehud Hrushovski, MichałSzachniewicz, *Globally valued fields: foundations*, preprint 2024, arXiv:2409.04570.

[2] Itaï Ben Yaacov, Ehud Hrushovski, *Globally valued function fields: existential closure*, preprint 2022, arXiv:2212.07269.

[3] MichałSzachniewicz, *Existential closedness of $\overline{\mathbb{Q}}$ as a globally valued field via Arakelov geometry*, preprint 2023, arXiv:2306.06275.

Bialgebraic varieties of the Gamma function

Adele Padgett
(joint work with Sebastian Eterović and Roy Zhao)

Let $V, W \subset \mathbb{C}^n$ be algebraic varieties of positive dimension. Let $U \subset \mathbb{C}$ be a domain and fix a transcendental meromorphic function $f : U \to \mathbb{C}$. Also denote by $f : U^n \to \mathbb{C}^n$ the n-fold function given by $f(z_1, \dots, z_n) = (f(z_1), \dots, f(z_n))$. If $f(W) \subseteq V$, what can we conclude about W and V? Since f is a transcendental function, the image $f(W)$ is usually not contained in a proper algebraic variety, unless W witnesses some special algebraic property of f.

This question has been answered for a wide array of functions: for the complex exponential function [1], the modular j-function [10, 12], more general modular functions [3], and uniformization maps of Shimura varieties [8], among others [2, 7]. Results of this type have seen important applications in problems of unlikely intersections in arithmetic geometry, for example in the proof of the André–Oort conjecture for Shimura varieties [10, 13, 11].

In all these cases, the functions in question are periodic and differentially algebraic, and the proof methods exploit this behavior. One of the most well-known differentially transcendental functions is the Γ function, which satisfies:

(1) For every $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$,

$$\Gamma(z+1) = z\Gamma(z). \tag{1}$$

(2) For every $z \in \mathbb{C}$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{2}$$

(3) For every $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and for every integer $n \geq 2$,

$$\prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) = (2\pi)^{\frac{1}{2}(n-1)} n^{\frac{1}{2}-nz} \Gamma(nz). \tag{3}$$

These functional equations can be used to build many interesting examples of varieties W and V such that $\Gamma(W) \subseteq V$.

For functions which are differentially algebraic and periodic, one can get a long way toward characterizing when $f(W) \subseteq V$ by studying the case when $\dim W = \dim V$: an algebraic variety W of U^n is *bialgebraic for* f if $\dim W = \dim \overline{f(W)}^{\mathrm{Zar}}$. Varieties $W \subset \mathbb{C}^n$ defined by equations of the form $X_i = X_k$ and $X_i = w$ for some $i, k \in \{1, \dots, n\}$ and $w \in \mathbb{C}$ are called *trivially bialgebraic* because they are bialgebraic for any function. In [6], we prove the following:

Theorem 1. *If an algebraic variety W is bialgebraic for Γ, then it is trivially bialgebraic.*

Though the bialgebraic varieties for Γ do not capture the consequences of equations (1), (2), and (3), my collaborators and I expect the proof methods we used to obtain this theorem extend to other interesting cases.

The results for the differentially alglebraic functions listed above were all obtained using either differential algebra or o-minimality (an area of model theory which has been highly successful in solving problems in Diophantine geometry). Recently, L. Di Vizio and F. Pellarin proved a functional transcendence result for Γ using techniques from difference algebra [4]. Also, P. Speissegger and I proved the complex Γ function is *definable* in an o-minimal structure called $\mathbb{R}_{\mathcal{G},\exp}$ when restricted to certain unbounded domains, making the o-minimal approach employed for other functions conceivable [9]. However, there are significant obstructions to using either difference algebra or o-minimality to fully characterize the functional algebraic properties of Γ. My collaborators and I developed two new, separate strategies to prove Theorem 1. Both strategies use asymptotic growth of Γ in different ways, and we are currently working to unify them into a stronger approach.

In our first strategy, we study the points of W where at least one coordinate is a pole of Γ. Most of the work goes into proving the theorem holds if W and V are curves, and the cases when $\dim W = \dim V > 1$ can be proved by induction from this case. After reducing to the case when W is an irreducible plane curve with no constant coordinate, we show that if one coordinate of W is a pole for Γ, then so is the other. Then we use Hilbert's irreducibility theorem to show that W is defined by an equation of the form $aX_1X_2 + bX_1 + cX_2 + d$ for integers a, b, c, and d. Using simple difference algebra it can then be deduced that W must be defined by $X_1 = X_2$. To sum up, if W is not defined by an equation of the form $X_i = w$ for some $w \in \mathbb{C}$, then it the diagonal, so it is trivially bialgebraic. The full argument is detailed in [6].

Our second strategy is essentially valuation-theoretic—it uses *transseries*, which give a very fine measure of asymptotic growth of a large class of real-variable functions, including Γ. Again focusing on the case when W and V are plane curves, the property $\Gamma(W) \subseteq V$ can be translated into a functional equation

$$\Gamma(\alpha(x)) = \beta(\Gamma(x))$$

where α and β are algebraic functions. If α and β are constant, they correspond to constant coordinates of W and V. If they are nonconstant, then one can reduce to the case when the real and imaginary parts of $\Gamma(\alpha(x))$ and $\beta(\Gamma(x))$ have transseries expansions. Germs at $+\infty$ of definable functions in the o-minimal structure $\mathbb{R}_{\mathcal{G},\exp}$ have unique transseries expansions [5], so the terms of the transseries expansions of $\Re\Gamma(\alpha(x))$ and $\Re\beta(\Gamma(x))$ must match. The same holds for $\Im\Gamma(\alpha(x))$ and $\Im\beta(\Gamma(x))$. Comparing the series expansions shows that $\Gamma(\alpha(x)) = \beta(\Gamma(x))$ only if $\alpha(x) = \beta(x) = x$. This corresponds to W and V being diagonals, so as before, W and V are trivially bialgebraic. We expect this transseries strategy to be fairly general—for instance, it can be used to recover the Ax–Lindemann–Weierstrass theorem for exp.

References

[1] J. Ax, *On Schanuel's conjectures*, Ann. of Math. (2) **93**, (1971), 252–268.

[2] D. Blázquez-Sanz, G. Casale, J. Freitag, and J. Nagloo, *A differential approach to Ax-Schanuel, I*, preprint, (2023), arXiv:2102.03384.

[3] G. Casale, J. Freitag, and J. Nagloo, *Ax-Lindemann-Weierstrass with derivatives and the genus 0 Fuchsian groups*, Ann. of Math. (2), **192**(3), (2020), 721–765.
[4] L. Di Vizio and F. Pellarin, *The Carlitz module and a differential Ax-Lindemann-Weierstrass theorem for the Euler Gamma function*, preprint, (2025), arXiv:2508.21237.
[5] L. van den Dries and P. Speissegger, *The field of reals with multisummable series and the exponential function*, Proc. London Math. Soc. (3), **81**(3), (2000), 513–565.
[6] S. Eterović, A. Padgett, and R. Zhao, *Bialgebraic varieties of the Gamma function*, preprint, (2025), arXiv:2509.24982.
[7] Z. Gao and B. Klingler, *The Ax-Schanuel conjecture for variations of mixed Hodge structures*, Math. Ann., **388**(4), (2024), 3847–3895.
[8] N. Mok, J. Pila, and J. Tsimerman, *Ax-Schanuel for Shimura varieties*, Ann. of Math. (2), **189**(3), (2019), 945–978.
[9] A. Padgett and P. Speissegger, *Definability of complex functions in o-minimal structures*, preprint, (2025), arXiv:2506.15119.
[10] J. Pila, *O-minimality and the André-Oort conjecture for* $\mathbb{C}^n$. Ann. of Math. (2), **173**(3), (2011), 1779–1840.
[11] J. Pila, A. N. Shankar, J. Tsimerman, H. Esnault, and M. Groechenig, *Canonical heights on Shimura varieties and the André-Oort conjecture*, preprint, (2021), arXiv:2109.08788.
[12] J. Pila and J. Tsimerman, *Ax-Schanuel for the j-function*, Duke Math. J., **165**(13), (2016), 2587–2605.
[13] J. Tsimerman, *The André-Oort conjecture for* $\mathcal{A}^g$, Ann. of Math. (2), **187**(2), (2018), 379–390.

Explicit height bounds for (very) unlikely intersections using *G*-functions

Martin Orr

The following theorem of Bombieri, Masser and Zannier [4] is one of the foundational theorems in the theory of unlikely intersections.

Theorem 1. *Let* $G = \mathbb{G}_m^n$ *with* $n \geq 2$. *For a positive integer* d, *let* Σ_d *denote the union of all algebraic subgroups of* G *of codimension at least* d. *Let* $C \subset G$ *be a geometrically irreducible algebraic curve, defined over* $\overline{\mathbb{Q}}$ *and not contained in a translate of a proper algebraic subgroup of* $\mathbb{G}_m^n$. *Then:*

(1) the points of $C \cap \Sigma_1$ *have bounded Weil height.*
(2) $C \cap \Sigma_2$ *is finite.*

Part (2) of this theorem is a special case of what later became known as the Zilber–Pink conjecture.

The proof of Theorem 1(1) can in principle be made effective. An explicit bound for the height in Theorem 1(1) was proved by Habbeger [8].

Theorem 2. *In the setting of Theorem 1(1), every point* $s \in c \cap \Sigma_1$ *satisfies*

$$h(s) \leq (2\deg(C))^{600n^7(2n)^{n^2}}(h(C)+1).$$

The method of G-functions gives an alternative approach to proving height bounds such as Theorem 1(1), subject to additional constraints. This method has the benefit that it is in principle effective, although a significant amount of work is required to obtain explicit bounds. The aim of my work in progress is to do this

work and see what explicit bounds one can get, both for Theorem 1 and for the analogous question for abelian varieties.

1. The method of G-functions

The method of G-functions for proving height bounds for "unlikely intersections" was introduced by André in 1989 [1]. It uses the notion of G-functions defined by Siegel, and the "Hasse principle" for relations between values of G-functions due to Bombieri [3].

In recent years, the G-functions method has been applied to unlikely intersections in Shimura varieties, in order to deduce special cases of the Zilber–Pink conjecture for Shimura varieties. Examples of this are due to Binyamini and Masser [2], Daw and myself [6, 7], Papas [9, 10], Urbanik [12] and Youell [15].

Bombieri's theorem gives an explicit height bound, in terms of various quantities associated with the G-functions themselves and the differential equations which they satisfy (for example radii of convergence). While it seems obvious that, when the theorem is applied to unlikely intersections, these quantities should be effectively computable, in fact it requires a substantial amount of work to check the details and especially to obtain explicit bounds. This detailed work on effectivity has not yet been carried out in any of the applications to Shimura varieties, where the focus was mostly on finiteness theorems.

2. Effective bounds for tori

In my work on progress, for the first time, I work out explicit bounds in an application of the G-functions method. The first result is the following.

Theorem 3. *Let $G = \mathbb{G}_m^3$.*

Let $C \subset G$ be the intersection of a line in $\mathbb{A}^3$ (defined over a number field K) with $\mathbb{G}_m^3$. Suppose that C contains $s_0 := (1,1,1)$, and that C is not contained in a proper algebraic subgroup of G.

Let $x : \overline{C} \to \mathbb{A}^1$ be an isomorphism such that $x(s_0) = 0$. Let

$$H = \sum_{\alpha} h(x(\alpha)),$$

where the sum is over $\alpha \in \overline{C} \setminus C$.

Then, for every point $s \in C \cap \Sigma_2$, letting $\delta = [K(s) : \mathbb{Q}]$, we have:

$$h(x(s)) < \begin{cases} 2706(H+1) & \text{if } K(s) = \mathbb{Q}, \\ 1323\,\delta^3\big(\log(\delta)+4\big)(H+1) & \text{otherwise.} \end{cases}$$

This may be compared with Theorem 2: in the case $n = 3$, is bound is around $2^{10^{13}}(h(C)+1)$. The bound in Theorem 3 depends on $[K(s) : \mathbb{Q}]$, which Theorem 3 does not. While the two theorems use different notions of height, Theorem 3 gives a better bound for $[K(s) : \mathbb{Q}]$ up to around 10^{12}.

Theorem 3 has several restrictions compared to Theorem 2. The restrictions fundamental to the G-functions method are:

(1) s_0 must be in C;
(2) the height is bounded for $C \cap \Sigma_2$ instead of $C \cap \Sigma_1$;
(3) the bound depends on $[K(s) : \mathbb{Q}]$ (but Theorem 3 is sufficient to deduce finiteness of $C \cap \Sigma_2$ using the Pila–Zannier method).

The following restrictions in Theorem 3 can be removed with additional work (for suitable constants in the bound):

(1) $n = 3$;
(2) C is the intersection of a line with G (instead of an arbitrary algebraic curve).

3. Height bounds for unlikely intersections in abelian varieties

The analogue of Theorem 1 for abelian varieties was proved by Rémond [11]. Explicit bounds have been obtained by Checcoli, Veneziano and Viada, mostly in the case where the abelian variety is a power of an elliptic curve (see, among other papers by these authors, [5, 14, 13]).

In work in progress, I have used the method of G-functions to prove a theorem similar to Theorem 3 for abelian varieties (allowing arbitrary algebraic curves for C) This is subject to some additional restrictions: the ambient abelian variety G should be of the form A^n for some simple abelian variety A and some $n \in \mathbb{Z}_{>0}$, and the curve C is intersected only with algebraic subgroups of large codimension (more precisely of codimension at least $(2g + 1)g$ where $g = \dim(A)$).

References

[1] Y. André, *G-functions and geometry*, Aspects of Mathematics, E13, Frider. Vieweg & Sohn, Braunschweig (1989).
[2] G. Binyamini and D. Masser, *Effective André–Oort for non-compact cures in Hilbert modular varieties*, Comptes Rendus. Mathématique **359**(3) (2021), 313–321.
[3] E. Bombieri, *On G-functions, In: Recent progress in analytic number theory, Vol. 2 (Durham, 1979)*, Academic Press, London-New York (1981), 1–67.
[4] E. Bombieri, D. Masser and U. Zannier, *Intersecting a curve with algebraic subgroups of multiplicative groups*, Int. Math. Res. Not. (1990), 1119–1140.
[5] S. Checcoli and E. Viada, *On the torsion anomalous conjecture in CM abelian varieties*, Pacific J. Math. **271**(2) (2014), 321–345.
[6] C. Daw and M. Orr, *Unlikely intersections with $E \times CM$ curves in A_2*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **22**(4) (2021), 1705–1745.
[7] C. Daw and M. Orr, *The large Galois orbits conjecture under multiplicative degeneration*, preprint 2023, arXiv:2306.13463.
[8] P. Habbeger, *Effective height upper bounds on algebraic tori, In: Around the Zilber-Pink conjecture/Autour de la conjecture de Zilber-Pink*, Panor. Synthèses **52** (2017), 167–242.
[9] G. Papas, *Unlikely intersections in the Torelli locus and the G-functions method*, preprint 2022, arXiv:2201.11240.
[10] G. Papas, *Some cases of the Zilber-Pink conjecture for curves in $\mathcal{A}_g$*, Int. Math. Res. Not. (2024), 4160–4206.
[11] G. Rémond, *Intersection de sous-groupes et de sous-variétés. I*, Math. Ann., **333**(3) (2005), 525–548.
[12] D. Urbanik, *Geometric G-functions and atypicality*, Duke Math. J. **174**(12) (2025), 2425–2512.

[13] E. Viada, *An explicit Manin-Dem'janenko theorem in elliptic curves*, Canad. J. Math. **70**(5) (2018), 1173–1200.
[14] F. Veneziano and E. Viada, *Explicit height bounds for k-rational points on transverse curves in powers of elliptic curves* **315**(2) (2021), 477–503.
[15] Z. Youell, *A height bound for abelian schemes with real* $\times \mathbb{Q}^2$ *multiplication*, Archiv der Mathematik **120** (2023), 381–394.

Algebraic flat connections and o-minimality

Hélène Esnault
(joint work with Moritz Kerz)

1. Motivation

We briefly describe a motivation for our work.[1]

On a smooth complex quasi-projective variety S, let $\mathcal{V} = (\mathbb{V}, (V, \nabla), Q, F^\bullet)$ be a $\mathbb{Z}$-polarized variation of Hodge structure ($\mathbb{Z}$-PVHS) of weight $2w$. Considering V as an algebraic variety over S via the Deligne extension, we know from Cattani–Deligne–Kaplan that fixing a constant $C \in \mathbb{R}_{>0}$, the Hodge locus $\mathcal{H}_C \subset V$ consisting of classes $(\xi_s, s) \in V(\mathbb{C})$ where $s \in S$, $\xi_s \in F^w V_s \cap \mathbb{V}_s$ and $|Q(\xi_s, \xi_s)| < C$ is algebraic. This theorem is a combination of the following facts:

(i) the F-filtration in V is algebraic [Sch73];
(ii) for any simply connected open $U \subset S^{\mathrm{an}}$, definable in the $\mathbb{R}_{\mathrm{an,exp}}$ structure, any flat section $\xi_U : U \to \mathbb{V}|_U \subset V|_U$ is definable;
(iii) for U as in (ii), $\mathcal{I}_{U,C} = \{\xi_U : U \to \mathbb{V}|_U \subset V|_U,\ \varnothing \neq \xi_U(U) \cap F^w|_U \subset V|_U,\ |Q(\xi, \xi)| < C\}$ is finite [CDK95].

From (i) and (iii) one deduces that $\mathrm{im}(\xi_U) \cap F^w \subset V$ is definable. Writing S as a finite union of definable open simply connected subsets U and observing that

$$\mathcal{H}_C \cap V_U = \bigcup_{\xi_U \in \mathcal{I}_{U,C}} \mathrm{im}(\xi_U) \cap F^w$$

is definable by (iii), we deduce that $\mathcal{H}_C$ is definable as well.

Then the GAGA theorem of Peterzil-Starchenko [PS09] applied to $\mathcal{H}_C \subset V$ enables one to conclude.

Simpson in [Sim97] formulated a 'non-abelian' version of this algebraicity theorem as a conjecture: let $X \to S$ be a smooth projective morphism with S as above. Let $M_{\mathrm{dR}}(X/S) \to S$ and $M_{\mathrm{Dol}}(X/S) \to S$ be the relative de Rham and Dolbeault moduli spaces in rank r constructed in *loc. cit.* Those are quasi-projective morphisms. A complex point of $M_{\mathrm{dR}}(X/S)$ is a pair $((V, \nabla)_s, s)$ consisting of a semi-simple flat connection $(V, \nabla)_s$ of rank r on the fiber X_s. Similarly a complex point of $M_{\mathrm{Dol}}(X/S)$ is a pair consisting of a semi-stable Higgs bundle $((E, \theta)_s, s)$

[1] I thank Moritz Kerz for the time we spent together along the years discussing on our mathematics.

on the fibre X_s with vanishing Chern classes. The conjecture then predicts that the complex analytic subloci

$$\mathcal{H}_{\mathrm{dR}}^{\mathrm{nonab}} := \{((V,\nabla)_s, s) \in M_{\mathrm{dR}}(X/S)(\mathbb{C}), \\ ((V,\nabla)_s, s) \text{ underlies a } \mathbb{Z}-\mathrm{PVHS}\} \subset M_{\mathrm{dR}}(X/S)(\mathbb{C})$$

and

$$\mathcal{H}_{\mathrm{Dol}}^{\mathrm{nonab}} := \{((E,\theta)_s, s) \in M_{\mathrm{Dol}}(X/S)(\mathbb{C}), \\ ((E,\theta)_s, s) \text{ underlies a } \mathbb{Z}-\mathrm{PVHS}\} \subset M_{\mathrm{Dol}}(X/S)(\mathbb{C})$$

are algebraic.

The finiteness analogous to (iii) for the non-abelian Hodge locus in $\mathcal{H}_{\mathrm{dR}}^{\mathrm{nonab}} \to S$ and of $\mathcal{H}_{\mathrm{Dol}}^{\mathrm{nonab}} \to S$ is an unsolved conjecture, the study of which has been initiated by Engel-Tayou in a particular case in [ET23].

2. Theorems

2.1. **There is no Definable Grauert Direct Image Theorem.** Fix any o-minimal structure which expands $\mathbb{R}_{\mathrm{an}}$.

Theorem 1 ([EK26])**.** *For $S = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$, $X = Y \times S$ and f the projection of X onto S, where Y is a smooth projective algebraic variety with $H_1(Y,\mathbb{Q}) \neq 0$, there always exists a definable analytic line bundle L on X such that the sheaves $f_*L(n)$ for $n > 0$ cannot all be definably coherent with respect to our fixed o-minimal structure.*

Remark 2. This theorem suggests that the standard approach to prove definability of the isomonodromic deformation of $\big((V,\nabla)_s, s\big) \in \mathcal{H}_{\mathrm{dR}}^{\mathrm{nonab}}$ in $M_{dR}(X/S)$ does not work. Bruno Klingler asked orally whether for S a point, the Simspon correspondence from the Betti to the Dolbeault moduli space might be $\mathbb{R}_{\mathrm{an,exp}}$ definable. We may perhaps ask whether the correct non-abelian analogous to (ii) would predict that the isomonodromic deformation of $\big((E,\theta)_s, s\big) \in \mathcal{H}_{\mathrm{Dol}}^{\mathrm{nonab}}$ in $M_{\mathrm{Dol}}(X/S)$ is $\mathbb{R}_{\mathrm{an,exp}}$ definable.

2.2. **Flat sections of connections.** We now come back to (ii) in the 'abelian' situation. We fix the $\mathbb{R}_{\mathrm{an,exp}}$ structure.

Theorem 3 ([EK25])**.** *The algebraic flat bundle (V,∇) has definable flat sections on all simply connected definable opens $U \subset S^{\mathrm{an}}$ if and only if it is regular singular with unitary monodromy eigenvalues at infinity.*

The 'if' part is due to Bakker–Mullane [BM23] and the 'only if' part refines their Example 3.3.

In particular, both Brieskorn-Grothendieck's theorem according to which local monodromies at infinity of Gauß-Manin systems of a smooth projective morphism are quasi-unipotent and Griffiths' theorem according to which Gauß-Manin connections are regular singular, can be summarized in one property:

> Flat sections on definable simply connected opens $U \subset S^{\mathrm{an}}$ of Gauß-Manin bundles of a complex smooth projective morphism $f: X \to S$ are definable.

This is a consequence of Kronecker's theorem saying that an algebraic integer is a root of unity if and only if it has complex norm 1 in any complex embedding (see e.g. the discussion in Section 2 of [Esn23]).

References

[BBT23] B. Bakker, Y. Brunebarbe, J. Tsimerman, *o-minimal GAGA and a conjecture of Griffiths*, Invent. math. **232** (2023), no.1, 163-228.

[BM23] B. Bakker,S., Mullane, *Definable structures on flat bundles*, Bull. Lond. Math. Soc. **55** (2023), no. 5, 2515–2524.

[CDK95] E. Cattani, P. Deligne, A. Kaplan, *On the locus of Hodge classes*, JAMS **82** (1995) pp. 483–505

[ET23] P. Engel, S. Tayou, *On the non-abelian Hodge locus I*, preprint 2023, `https://arxiv.org/abs/2305.00943`

[Esn23] H. Esnault, *Lectures on Local Systems in Algebraic-Arithmetic Geometry*, Springer Lecture Notes **2337** (2023), i-vii + 96 pp.

[EK25] H. Esnault, M. Kerz, *Algebraic flat connections and o-minimality preprint 2025*, 9 pages, appears in Laumon Volume, Springer Verlag, Simons subseries.

[EK26] H. Esnault, M. Kerz, *There is no Definable Grauert Direct Image Theorem*, preprint 2026, `https://arxiv.org/abs/2601.18711`.

[PS09] Y., Peterzil, S. Starchenko, *Complex analytic geometry and analytic-geometric categories*, J. Reine Angew. Math. **626** (2009), 39–74.

[Sch73] W. Schmid, *Variations of Hodge structure: the singularities of the period mapping*, Invent. math., **22**, 211-319 (1973)

[Sim97] C. Simpson, *The Hodge filtration on nonabelian cohomology*, Algebraic geometry–Santa Cruz 1995, 217–81. Proc. Sympos. Pure Math., **62**, Part 2 American Mathematical Society, Providence, RI, 1997.

H-Minimality: Tame Geometry in Valued fields

Immanuel Halupczok

(joint work with Raf Cluckers, Silvain Rideau-Kikuchi and Floris Vermeulen)

The talk (and this abstract) is intended to be accessible to non-model theorists.

1. Goal

Fix a topological field K; for the moment, think of $K = \mathbb{R}$ or $K = \mathbb{C}$. We want to have a notion of "tame subsets" $X \subset K^n$, meaning that we want them to be suitable for geometric arguments. Once we have such a notion, we also obtain a notion of tame maps for free, namely, $f: X \to Y$ is tame if its graph is tame. What we want to exclude are things like space filling curves, Banach–Tarski like sets, no-where continuous maps, etc.

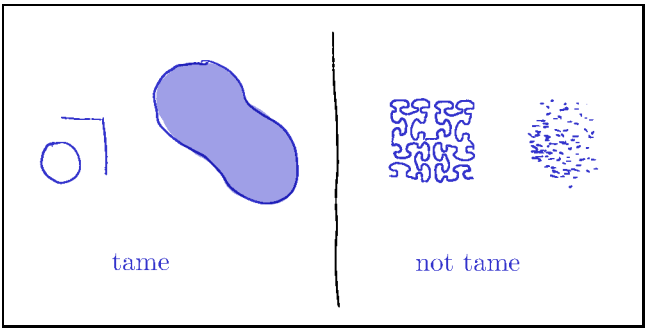

FIGURE 1. Tame and non-tame sets.

Here is a non-exhaustive list of wishes that tame sets and maps should satisfy:

(T1) There is a good notion of dimension of tame subsets (taking natural numbers as values, and satisfying the usual properties of dimension one would expect, like $\dim(X \cup Y) = \max\{\dim X, \dim Y\}$, $\dim(X \times Y) = \dim X + \dim Y$.

(T2) It behaves well with respect to the topology, e.g. $\dim X^{\mathrm{cl}} = \dim X$ and even $\dim \partial X < \dim X$.

(T3) Tame maps are continuous almost everywhere (i.e., everywhere except on a lower-dimensional set), and maybe even C^1.

Additionally, tameness should be closed under various operations, notably:

(C1) If X, Y are tame, then so are $X \times Y$, $X \cup Y$, $X \cap Y$, $K^n \setminus X$, and also $\pi(X)$, where $\pi: K^n \to K^m$ is a projection to a subset of the coordinates.

(C2) If $f: X \to Y$ is tame, then $f(X)$ should be tame as well, and also $f^{-1}(Z)$, for tame $Z \subset Y$.

(C3) Operations involving the topology should also preserve tameness, e.g., X^{cl}, X^{int}, ∂X. And partial derivatives of tame maps should be tame (on the domain where they exist).

Finally, we typically have some sets we are interested in which we would like to be tame. Let us at least impose:

(C4) For every polynomial $f \in \mathbb{Z}[x_1, \ldots, x_m]$, the set $\{x \in K^n \mid f(x) = 0\}$ is tame.

The model theoretic way of phrasing this is: We want to fix a structure on K which extends the field structure, such that the definable sets satisfy (T1)–(T3).

Explanation for non-model theorists: a *structure* determines a notion of *definable subset* of K^n, for each n. By definition, the class of definable subsets of a structure satisfies (C1). That it extends the field structure means that it also satisfies (C4). On can then show that this implies (C2), and, in the settings we will be interested in, also (C3).

Example 1. In the case $K = \mathbb{C}$, the constructible subsets of $\mathbb{C}^n$ satisfy all wishes. (Constructible set means: boolean combination of finitely many algebraic sets.)

Example 2. In the case $K = \mathbb{R}$, the constructible subsets of $\mathbb{R}^n$ satisfy many wishes, but they are not closed under coordinate projections. Instead, one can take the semi-algebraic subsets of $\mathbb{R}^n$, i.e., boolean combinations of sets defined polynomial equations and inequations (where by "inequation", I mean something like $f(x) > 0$).

One can declare any additional sets one is interested in to be definable and then again close under (C1). This yields a structure in the above sense, but in general, it will not satisfy (T1)–(T3). The big question is which sets can be declared to be definable while still preserving (T1)–(T3).

Non-Example 3. In the case $K = \mathbb{Q}$, no tame structure exists. Indeed, the set $\square_{\mathbb{Q}} \subset \mathbb{Q}$ of squares (which is necessarily definable, since it is the projection of $\{(x, y) \mid y - x^2 = 0\}$ to the second coordinate) is at most 1-dimensional, since it is a subset of $\mathbb{Q}$, but its boundary $\partial\square_{\mathbb{Q}}$ is too big to be 0-dimensional.

If K is an ordered field (like $\mathbb{R}$ or $\mathbb{Q}$), then an extremely successful tool to understand tameness is o-minimality. (We consider K as a topological field using the interval topology.)

Definition. Let K be an ordered field. Our structure on K is called *o-minimal* if every definable subset $X \subset K$ is a union of finitely many points and open intervals (where intervals going to $\pm\infty$ are allowed).[1]

Note that this definition only speaks about definable subsets of K itself. Nevertheless, one obtains:

Theorem 4. *Any o-minimal structure is tame in the sense that it satisfies (T1)–(T3).*

Exercise 5. The structure consisting of the semi-algebraic subsets of $\mathbb{R}^n$ is o-minimal.[2]

Example 6. The globally subanalytic subsets of $\mathbb{R}^n$ form an o-minimal structure (called $\mathbb{R}_{\text{an}}$).

Example 7. A famous theorem of Wilkie says: If, to the subanalytic sets, we add the graph of the exponential $x \mapsto e^x$ and then again close under (C1), the resulting structure (called $\mathbb{R}_{\text{an,exp}}$) is still o-minimal.

As Example 3 shows, there can be no o-minimal structure on $\mathbb{Q}$. More precisely, one can show that o-minimality implies that K is real closed.

Building on tameness results like Theorem 4, various "big results" have been obtained for sets definable in o-minimal structures, e.g. Pila–Wilkie Point Counting and the existence of Whitney-stratifications.

[1] Model theorists may have noticed that here, I am using the term "definable" to mean $\varnothing$-definable. While usually, the o-minimality condition is also imposed for sets definable with parameters, it has been pointed out to me after the talk that in this context, this nevertheless yields the right notion of o-minimality, due to the existence of prime models.

[2] It is *not* part of the exercise that the semi-algebraic subsets of $\mathbb{R}^n$ form a structure; this is Tarski's Theorem, which is far too difficult to be an exercise.

The goal of this talk is to present an analogue of o-minimality for valued fields, which also implies tameness and from which one can also obtain some "big theorems".

2. H-minimality

Let us from now on assume that K is a valued field and that the valuation is non-trivial. We fix some notation.

Notation 8. We write $v\colon K \to \Gamma \cup \{\infty\}$ for the valuation (where Γ is an ordered abelian group, e.g. $\Gamma = \mathbb{Z}$). The valuation ring is denoted by $\mathcal{O}_K$.

We will need to impose that K is of characteristic 0. (The residue characteristic can be arbitrary.)

Example 9. Some fields we might be interested in: $K = \mathbb{Q}_p$, $K = k((t))$ (for any field k of characteristic 0), the field $K = \bigcup_n k((t^{1/n}))$ of Puiseux series, the field $K = k((t^\Gamma))$ of Hahn series (where Γ is any non-trivial ordered abelian group).

We would like to be allowed to use the valuation to construct our definable (and hopefully tame) sets. We therefore from now on additionally impose:

(C5) The valuation ring $\mathcal{O}_K$ is definable.

In model theoretic terms, this means that our structure is now supposed to expand the structure of *valued* fields.

Like o-minimality, h-minimality is a condition on a structure on K, expressed in terms of the definable subsets of K itself, and as in the case of o-minimality, that condition implies tameness also for definable subsets of K^n:

Theorem 10 (Cluckers–H–Rideau-Kikuchi–Vermeulen)**.** *Any h-minimal structure is tame in the sense that it satisfies (T1)–(T3).*

The notion of h-minimality is intended to be applied to a field K which is henselian, meaning that it satisfies the conclusion of Hensel's Lemma, like all valued fields from Example 9. (That's what the "h" in h-minimality stands for.) Indeed, in a similar way as o-minimality implies that K is real closed, h-minimality essentially[3] implies that K is henselian.

And as for o-minimality, once tameness is established, h-minimality implies some big results like some form of Pila–Wilkie Point Counting (this is work in progress) and the existence of some kind of stratifications. One also obtains that motivic integration works. (Some of these results need additional assumptions on the residue field or on the value group. The notion of h-minimality itself intentionally makes no assumptions on those.)

Of course all of this is interesting only if h-minimal structures exist. Indeed:

Example 11. The valued field structure of any henselian valued field of characteristic 0 is h-minimal.

[3]To be precise, for this to be true in mixed characteristic, one needs to impose finite ramification.

Example 12. In $\mathbb{Q}_p$ or $k((t))$, one can add all analytic functions on cartesian powers of the valuation ring to the structure; the result is still h-minimal.

Example 13. Take an o-minimal structure on $\mathbb{R}$, take a non-standard model K of that (which is still an o-minimal structure), and additional equip K with its natural valuation. If the exponential was *not* definable in $\mathbb{R}$, then the resulting structure on K is h-minimal.[4]

More examples are given in [1].

3. Definition of h-minimality: intuition

To define h-minimality, the idea is to impose, as in o-minimality, how definable subsets of K itself are allowed to look like. To get an intuition, let us take $K = k((t))$ as an example and consider the set $\square_K$ of squares in that field K. To describe that set, we first introduce the notion of leading terms of elements of a valued K:

Definition. Given a valued field K, we set $\mathrm{RV} := K^\times/(1+\mathcal{M}_K) \cup \{0\}$, where $\mathcal{M}_K$ is the maximal ideal of the valuation ring $\mathcal{O}_K$, and we define $\mathrm{rv}\colon K \to \mathrm{RV}$ to be the quotient map. We call $\mathrm{rv}(x)$ the *leading term* of x.

Exercise 14. Check that in the case $K = k((t))$, we have $\mathrm{rv}(x) = \mathrm{rv}(x')$ if and only if x and x' have the same leading term in the classical sense, i.e., if $x = \sum_{i\geq N} a_i t^i$ and $x' = \sum_{i\geq N} a'_i t^i$ with $a_N = a'_N \neq 0$.

Clearly, if $x \in K = k((t))$ is a square, then so is its leading term $a_N t^N$, i.e., N is even and a_N is a square in k. Using Hensel's lemma, one gets that this is an if and only if condition. Given that we might take k to be $\mathbb{Q}$, Example 3 seems to imply that $\square_K$ cannot be tame in general. However, we are saved by the fact that each square a in k and each even integer N yields an entire clopen ball in $\square_K$, namely $at^N + t^{N+1}k[[t]]$. Since $K^\times$ is the disjoint union of the clopen balls $\mathrm{rv}^{-1}(\xi)$, where ξ runs over $\mathrm{RV}\setminus\{0\}$, the topological closure of a completely arbitrary union Y of such balls at most contains $\{0\}$ as an additional point. This means that the evilness argument from Example 3 does not apply to any such Y and in particular not to $\square_K$.

Clearly, in our h-minimal structure, we will also have translates of sets like $\square_K$. To get an idea of how h-minimality could now be defined, let us have another look at the o-minimal setting. Recall that there, a definable subset $X \subset K$ is required to be a union of finitely many points and intervals. If we take $C \subset K$ to be the set of all the individual points and all endpoints of the intervals, then whether an element $x \in K$ lies in X only depends on the position of x relative to the elements of C. In other words, o-minimality can also be defined as follows:

[4]To be precise, there exist different variants of h-minimality. Example 12 might only be 1-h-minimal, whereas the other ones satisfy the stronger condition of ω-h-minimality.

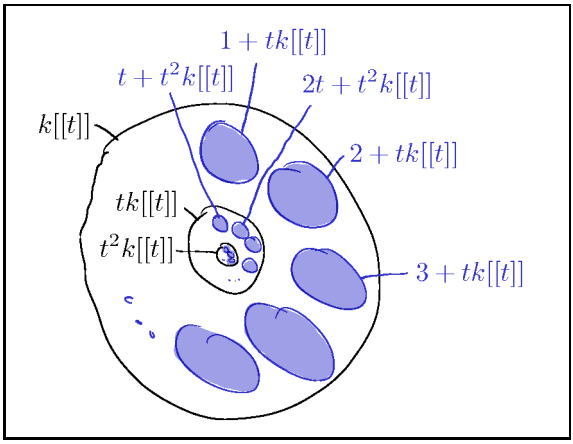

FIGURE 2. The blue balls are the fibers of rv. Any union of blue balls is tame.

Remark 15. A structure on an ordered field K is o-minimal if for every definable set $X \subset K$, there exists a finite set $C \subset K$ such that whether an element $x \in K$ lies in X is determined by the tuple of signs $(\mathrm{sgn}(x-c))_{c\in C}$ (where $\mathrm{sgn}(y)$ is 1, −1, or 0, depending on whether y is positive, negative or zero).

This yields a natural candidate for a definition of h-minimality: just replace sgn by rv:

Not-really-Definition 16. A structure on a valued field K is h-minimal if for every definable set $X \subset K$, there exists a finite set $C \subset K$ such that whether an element $x \in K$ lies in X is determined by the tuple of leading terms $(\mathrm{rv}(x-c))_{c\in C}$.

Unfortunately, this definition does not seem to imply nice tameness results yet. The problem is the following: In o-minimality, to obtain tameness results about definable sets $X \subset K^n$ for $n \geq 2$, one applies the o-minimality condition to fibers of X. Suppose, for example, that we obtain that each fiber X_a is an interval, for a in some subset $Y \subset K^{n-1}$. Then, in our structure, we can define the functions $f, g: Y \to K$ bounding X from below and from above, meaning that we can apply other tameness results to those f and g as ingredients to deduce tameness of X.

This technique does not work in h-minimality if we use Definition 16: Given some definable $X \subset K^n$, even if for each $a \in K^{n-1}$ we find a single element $f(a) \in K$ such that the set $C_a := \{f(a)\}$ witnesses the h-minimality condition for the fiber X_a, we do not get for free that the function f is definable in our structure. The solution to this problem is to impose, in the definition of h-minimality, that "things work well in families". Note that the precise condition one needs is stronger (and more technical) than what the above example suggests, and giving it in full detail

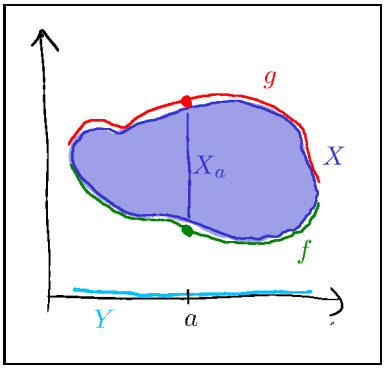

FIGURE 3. Applying o-minimality to fibers yields definable functions f and g.

goes beyond the scope of this introductory talk.[5] Readers interested in the full definition can look into the original articles [1, 2]. (I recommend looking into [1] first, which is about the equi-characteristic 0 case. The mixed characteristic version is even more technical.)

REFERENCES

[1] R. Cluckers, I. Halupczok and S. Rideau-Kikuchi, *Hensel minimality I*, Forum Math. Pi, 10:Paper No. e11, 68 (2022).
[2] R. Cluckers, I. Halupczok, S. Rideau-Kikuchi and F. Vermeulen, *Hensel minimality II: Mixed characteristic and a diophantine application*, Forum Math. Sigma, 11:Paper No. e89, 33 (2023).

Hensel minimality, p-adic exponentiation and Tate uniformization

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(joint work with Floris Vermeulen)

O-minimality is a framework coming from model theory in which one can carry out analytic arguments with guaranteed tameness. By "analytic arguments" we specifically mean arguments from real or complex analysis. The success of o-minimality in important arithmetic geometry problems is now well-established, see for example [16, 17, 20] for applications to the André–Oort conjecture, [1, 13, 18] for results in functions transcendence, and [7] for results regarding the Zilber–Pink conjecture. Various attempts have been made to come up with a similarly tame framework that supports non-Archimedean analytic arguments, such as P-minimality [10] or C-minimality [9, 12]. We will work with the more recent notion of *1-h-minimality* [5, 6], which applies more generally than the previous notions.

[5]There even exist different variants of the definition. Everything in this talk except Example 12 is true both of the two most important variants, namely 1-h-minimality and ω-h-minimality.

Our aim is to show how 1-h-minimality can be used in analogy to various applications of o-minimality. For this, we study some non-Archimedean analogs of problems around transcendental number theory, unlikely intersections, and differential fields.

Most analytic applications of o-minimality start by considering the expansion of the real field $\mathbb{R}$ by a collection of analytic functions $\mathcal{F}$ so that the resulting structure is o-minimal. Similarly, we begin by considering expansions of $\mathbb{C}_p$ by certain collections of analytic functions. Inspired by [21], we first study the existence of non-trivial field derivations $\delta : \mathbb{C}_p \to \mathbb{C}_p$ that respect a collection of functions $\mathcal{F}$ such that the structure $(\mathbb{C}_p, \mathcal{F})$ is 1-h-minimal.

Theorem 1 (Existence of non-trivial derivations)**.** *Let $\mathcal{T}$ be a 1-h-minimal theory and let K be a model of $\mathcal{T}$. Then there is a collection of commuting derivations $\Delta = \{\partial_\tau\}_{\tau\in\mathcal{B}}$ on K with the following properties:*

(1) If $U \subset K^n$ is open and $h : U \to K$ is a $\varnothing$-definable C^1 function, then for every $a = (a_1, \ldots, a_n)$ and every $\partial_\tau \in \Delta$ we have

$$\partial_\tau(h(a)) = \sum_{i=1}^{n} (\partial_i h)(a)\partial_\tau(a_i),$$

where $\partial_i h$ denotes the partial derivative of h with respect to the ith coordinate.

(2) We have $\bigcap_{\tau\in\mathcal{B}} \ker \partial_\tau = \mathrm{acl}(\varnothing)$, where acl *denotes the model-theoretic algebraic closure operator in K.*

A few obstructions need to be sorted in the proof of this theorem, one having to do with choosing appropriate reducts of 1-h-minimal theories preserving 1-h-minimality and having a countable language (in general, reducts do not preserve 1-h-minimality, see [6, Remark 5.1.2]), but the main one being that there is no non-Archimedean analog of Gabrielov's theorem showing that projections of analytic sets remain analytic.

Next, we focus on two specific p-adically analytic functions, which we show are part of 1-h-minimal theories (even in a countable language). One of the most famous transcendental functions known to be part of an o-minimal expansion of $\mathbb{R}$ is the real exponential function, so the first function we will focus on is p-adic exponentiation. We denote by $\mathbb{D}_p \subset \mathbb{C}_p$ the open ball of valuative radius $1/(p-1)$ around zero, and let $\exp : \mathbb{D}_p \to 1 + \mathbb{D}_p$ be the p-adic exponential map.

Using that p-adic exponentiation is part of a 1-h-minimal expansion of $\mathbb{C}_p$, we show that the p-adic version of Schanuel's conjecture implies a uniform version of itself (the analogous result for real exponentiation is due to Kirby and Zilber [11]).

Theorem 2. *Let $V \subseteq \mathbb{C}_p^n \times (\mathbb{C}_p^\times)^n$ be an irreducible algebraic variety defined over $\overline{\mathbb{Q}}$ with $\dim V < n$. Then the p-adic Schanuel conjecture implies that there exist finitely many proper $\mathbb{Q}$-linear subspaces $L_1, \ldots, L_m \subset \mathbb{C}_p^n$ such that if $(\mathbf{z}, \exp(\mathbf{z})) \in V(\mathbb{C}_p)$, then $\mathbf{z} \in L_i$, for some $i \in \{1, \ldots, n\}$.*

Many applications of o-minimality to problems in arithmetic geometry involve periodic holomorphic functions, such as complex exponentiation, the exponential

maps of abelian varieties, or automorphic functions of Shimura varieties. Some key applications (like the o-minimal proofs of Ax–Schanuel) exploit the fact that these functions are not definable in an o-minimal structure (because of periodicity), but their restrictions to a fundamental domain are definable, and one can move between fundamental domains in an algebraic (hence definable) way.

Since exp is injective on $\mathbb{D}_p$ and has no natural extension to all of $\mathbb{C}_p$, we move our attention to our second function of interest: Tate's uniformization map of elliptic curves. Let $v : \mathbb{C}_p \to \mathbb{Q}$ denote the valuation map. For $q \in \mathbb{C}_p^\times$ with $v(q) > 0$, we denote by $\phi_q : \mathbb{C}_p^\times \to \mathbb{C}_p^\times/q^{\mathbb{Z}} \cong E_q$ Tate's uniformization map. We establish a result on the p-adic density of likely intersections for powers of elliptic curves, analogous to the work of [8].

Theorem 3. *Let $V \subseteq E_q^n$ be an irreducible subvariety. Suppose S is an algebraic subgroup of E_q^n such that for every algebraic subgroup $T \subseteq E_q^n$ the quotient map $\psi : E_q^n \to E_q^n/T$ satisfies $\dim \psi(V) + \dim \psi(S) \geq n - \dim T$. Then for every p-adically dense subset $A \subseteq E_q^n(\mathbb{C}_p)$,*

$$\bigcup_{\mathbf{g} \in A} V \cap (\mathbf{g} + S)$$

is p-adically dense in V.

The reader familiar with applications of o-minimality to problems in arithmetic geometry is probably aware of at least two major results in o-minimality: the Pila–Wilkie point-counting theorem [19] and the Definable Chow theorem (also known as the o-minimal GAGA) [15, Theorem 5.1]. These results do not appear in the theorems stated above, but there are non-Archimedean analogs and applications of both using earlier notions of tame p-adic analytic geometry [2, 3, 4, 14]. We believe our results can be combined with those in the works cited above to lead to further applications of 1-h-minimality to problems of unlikely intersections in the non-Archimedean setting.

References

[1] B. Bakker and J. Tsimerman, *The Ax-Schanuel conjecture for variations of Hodge structures*, Invent. Math., **217**(1) (2019), 77–94.

[2] V. Cantoral Farfán, K. H. Nguyen, M. Stout and F. Vermeulen, *A Pila-Wilkie theorem for Hensel minimal curves*, Model Theory, **3**(1) (2024), 119–145.

[3] A. Chambert-Loir and F. Loeser, *A nonarchimedean Ax-Lindemann theorem*, Algebra and Number Theory, **11**(9) (2017), 1967–1999.

[4] R. Cluckers, G. Comte and F. Loeser, *Non-Archimedean Yomdin-Gromov parametrizations and points of bounded height*, Forum Math. Pi, 3:e5 **60** (2015).

[5] R. Cluckers, I. Halupczok and S. Rideau-Kikuchi, *Hensel minimality I*, Forum Math. Pi, 10:Paper No. e11, 68 (2022).

[6] R. Cluckers, I. Halupczok, S. Rideau-Kikuchi and F. Vermeulen, *Hensel minimality II: Mixed characteristic and a diophantine application*, Forum Math. Sigma, 11:Paper No. e89, 33 (2023).

[7] C. Daw and J. Ren, *Applications of the hyperbolic Ax-Schanuel conjecture*, Compos. Math., **154**(9) (2018), 1843–1888.

[8] S. Eterović and T. Scanlon, *Likely intersections*, Forum Math. Sigma, **13**:Paper No. e199 (2025).

[9] D. Haskell and D. Macpherson, *Cell decompositions of C-minimal structures*, Ann. Pure Appl. Logic, **66**(2) (1994), 113–162.
[10] D. Haskell and D. Macpherson, *A version of o-minimality for the p-adics*, J. Symbolic Logic, **62**(4) (1997), 1075–1092.
[11] J. Kirby and B. Zilber, *The uniform Schanuel conjecture over the real numbers*, Bull. London Math. Soc. **38**(4) (2006), 568–570.
[12] D. Macpherson and C. Steinhorn, *On variants of o-minimality*, Ann. Pure Appl. Logic, **79**(2) (1996), 165–209.
[13] N. Mok, J. Pila, J. Tsimerman, *Ax-Schanuel for Shimura varieties*, Ann. of Math. (2), **189**(3) (2019), 945–978.
[14] A. Oswal, *A non-archimedean definable Chow theorem*, J. Eur. Math. Soc. (JEMS), **27**(4) (2025), 1407–1446.
[15] Y. Peterzil and S. Starchenko, *Complex analytic geometry in a nonstandard setting*, in *Model theory with applications to algebra and analysis. Vol. 1*, volume 39 London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge (2008), 117–165.
[16] J. Pila, *O-minimality and the André-Oort conjecture for* $\mathbb{C}^n$, Ann. of Math. (2) **173**(3) (2011), 1779–1840.
[17] J. Pila, A. N. Shankar, J. Tsimerman, Hélène Esnault and Michael Groechenig, *Canonical heights on Shimura varieties and the André–Oort conjecture*, preprint 2021, arxiv:2109.08788.
[18] J. Pila and J. Tsimerman, *Ax-Schanuel for the j-function*, Duke Math. J., **165**(13) (2016), 2587–2605.
[19] J. Pila and A. J. Wilkie, *The rational points of a definable set*, Duke Math. J., **133**(3) (2006), 591–616.
[20] J. Tsimerman, *The André–Oort conjecture for* $\mathcal{A}_g$, Ann. of Math. (2), **187**(2) (2018), 379–390.
[21] A. J. Wilkie, *Some local definability theory for holomorphic functions*, in *Model theory with applications to algebra and analysis. Vol. I*, volume 39 London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge (2008), 197–213.

Periods of hyperplane arrangements and multiple polylogarithms

Riccardo Tosi

An elementary criterion for proving that a number $\alpha \in \mathbb{R}$ is irrational is the following. Suppose that there are two sequences $\{a_n\}_n$, $\{b_n\}_n$ of integers such that $a_n\alpha + b_n \neq 0$ for all n and $a_n\alpha + b_n \to 0$ as $n \to \infty$. Then α is irrational, for, if it were not, after clearing out denominators one would get a sequence of non-zero integers that tends to zero. This leads to the question: how can we construct such linear combinations of α and 1 with integer coefficients?

If α is a *period* [7], that is, the integral of an algebraic differential form over a semi-algebraic domain, one hopes to exploit geometric tools to produce such linear combinations. This is the case for the values $\zeta(\ell)$ of the Riemann zeta function at positive integers $\ell \geq 2$. For example, the integral representation

$$\zeta(2) = \int_{0\leq t_1\leq t_2\leq 1} \frac{1}{1-t_1}\frac{1}{t_2}\,\mathrm{d}t_1\,\mathrm{d}t_2$$

can be used to realize $\zeta(2)$ as a period of an algebraic variety.

For $\zeta(\ell)$, we consider the moduli space of smooth curves of genus zero with $s+3$ marked points:

$$\mathfrak{M}_{0,\ell+3} = \operatorname{Spec} \mathbb{Q}[t_1, \dots, t_\ell]\left[\frac{1}{t_i}, \frac{1}{1-t_i}, \frac{1}{t_i - t_j} \,\middle|\, i, j = 1, \dots, \ell,\ i \neq j\right].$$

There is a canonical compactification $\overline{\mathfrak{M}}_{0,\ell+3}$ of $\mathfrak{M}_{0,\ell+3}$ by a simple normal crossing divisor. Goncharov and Manin [6] first showed that $\zeta(\ell)$ is a period of $\overline{\mathfrak{M}}_{0,\ell+3}$ relative to the boundary divisor.

In some sense, all periods of these varieties are related to zeta values:

Theorem 1 (Brown [1])**.** *The periods of $\overline{\mathfrak{M}}_{0,\ell+3}$ relative to the boundary divisor are $\mathbb{Q}[2\pi i]$-linear combinations of multiple zeta values of weight at most ℓ.*

Multiple zeta values [3] are numbers of the form

$$\zeta(k_1, \dots, k_r) = \sum_{1 \leq n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}}$$

for positive integers k_i with $k_r \geq 2$, and their weight is $k_1 + \cdots + k_r$. The $\mathbb{Q}$-vector space they generate is in fact a $\mathbb{Q}$-algebra, so it contains all products of single zeta values $\zeta(\ell_1)\zeta(\ell_2)$.

Brown's proof yields an effective method to compute period integrals over $\overline{\mathfrak{M}}_{0,\ell+3}$ and was later implemented into an actual algorithm by Panzer [8]. This result gives a natural source of linear combinations of zeta values; a programmatic study of these integrals for applications to irrationality proofs started in [2].

The varieties $\mathfrak{M}_{0,\ell+3}$ are complements in the affine space of a family of hyperplanes, so it is natural to ask whether Brown's work can be extended to compute periods associated with more general hyperplane arrangements and what numbers appear as periods thereof.

Given a hyperplane arrangement $\mathcal{A}$ defined over a subfield k of $\mathbb{C}$, let $Y_{\mathcal{A}}$ be its complement in the affine space. De Concini and Procesi [4] have constructed an explicit compactification $\overline{Y}_{\mathcal{A}}$ of $Y_{\mathcal{A}}$ via a simple normal crossings divisor.

Theorem 2 (T. [9])**.** *Suppose that $\mathcal{A}$ has enough modular elements. Then period integrals of $\overline{Y}_{\mathcal{A}}$ relative to the boundary divisor $\overline{Y}_{\mathcal{A}} \smallsetminus Y_{\mathcal{A}}$ are $k[2\pi i]$-linear combinations of multiple polylogarithms of weight at most $\dim Y_{\mathcal{A}}$ evaluated at a specific finite set of points of k. This set depends only on the one-dimensional hyperplane arrangements obtained from $\mathcal{A}$ via iterated restrictions and deletions.*

The condition for $\mathcal{A}$ to have enough modular elements means that the modular lines in the intersection lattice of $\mathcal{A}$ generate the ambient vector space of $\mathcal{A}$. Geometrically, it ensures the existence of retractions from $\overline{Y}_{\mathcal{A}}$ onto each irreducible component of the boundary divisor $\overline{Y}_{\mathcal{A}} \smallsetminus Y_{\mathcal{A}}$. This condition depends only on the combinatorics of the arrangement $\mathcal{A}$ and is strictly stronger than being fiber type.

To exemplify what numbers appear as periods of these arrangements, we focus on the reflection arrangement of the full monomial group:

$$Y_{\mathcal{A}} = \operatorname{Spec} \mathbb{Q}(\mu)[t_1, \dots, t_\ell]\left[\frac{1}{t_i}, \frac{1}{\mu^h - t_i}, \frac{1}{t_i - \mu^h t_j} \;\middle|\; i \neq j,\, h = 1, \dots, q\right],$$

where μ is a primitive q-th roots of unity. Theorem 2 implies that the periods of $\overline{Y}_{\mathcal{A}}$ relative to the boundary divisor are $\mathbb{Q}(\mu)[2\pi i]$-linear combinations of values of multiple polylogarithms at q-th roots of unity, i.e. numbers of the form

$$\sum_{1 \leq n_1 < \cdots < n_r} \frac{\mu^{h_1 n_1} \dots \mu^{h_r n_r}}{n_1^{k_1} \dots n_r^{k_r}}$$

for some $h_i \geq 1$, $k_i \geq 1$, $k_r \geq 2$. These numbers include special values of the β-function, such as Catalan's constant, and of Dirichlet L-functions in general.

It would be interesting to translate Brown's program [2] for irrationality proofs to this more general setup. Rational linear combinations of the above polylogarithmic values have appeared in [5] by means of hypergeometric series, and Theorem 2 may give a geometric framework to handle such linear forms.

From a motivic point of view, one could investigate the connection between arrangements with enough modular elements and the subcategory of mixed Tate motives generated by the pro-unipotent completion of the fundamental groupoid of $\mathbb{P}^1$ with 0, ∞ and roots of unity removed.

References

[1] F. C. S. Brown, *Multiple zeta values and periods of moduli spaces* $\overline{\mathfrak{M}}_{0,n}$, Ann. Sci. Éc. Norm. Supér. (4), **42**(3) (2009), 371–489.

[2] F. C. S. Brown, *Irrationality proofs for zeta values, moduli spaces and dinner parties*, Mosc. J. Comb. Number Theory **6** (2016), no. 2-3, 102–165.

[3] J.I. Burgos Gil and J. Fresán, *Multiple zeta values: from numbers to motives*, Clay Mathematics Proceedings. To appear.

[4] C. De Concini and C. Procesi, *Wonderful models of subspace arrangements*, Selecta Math. (N.S.), **1**(3) (1995), 459–494.

[5] S. Fischler, J. Sprang, and W. Zudilin, *Many odd zeta values are irrational*, Compos. Math., **155**(5) (2019), 938–952.

[6] A.B. Goncharov and Yu. I. Manin, *Multiple ζ-motives and moduli spaces* $\overline{\mathcal{M}}_{0,n}$, Compos. Math., **140**(1) (2004), 1-14.

[7] A. Huber and S. Müller-Stach, *Periods and Nori motives*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, Springer, **65** (2017). With contributions by Benjamin Friedrich and Jonas von Wangenheim.

[8] E. Panzer, *Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals*, Comput. Phys. Commun., **188** (2015), 148–166.

[9] R. Tosi, *Periods of hyperplane arrangements and multiple polylogarithms*, (2025), doi: 10.17185/duepublico/84492.

Finite fields with monodromy: the search for imaginaries

Ehud Hrushovski
(joint work with Will Johnson)

This is a report on work in progress, jointly with Will Johnson. Any mistakes introduced in the talk are mine.

1. Introduction

Ax in [1] determined the theory of finite fields, asymptotically as the field grows larger. His theorem applied to the ring language. But algebraic geometry knows additional structures on finite fields, developed by Weil, Grothendieck, Deligne towards the Weil conjectures but of significance in many other directions. We will study varieties over finite fields enriched with the complex-valued 'trace of Frobenius' map associated to l-adic local systems, described in more detail below.

We work in the continuous logic of [2], with trivial metric. Formulas in this logic are real-valued; the basic connectives and quantifiers are scalar multiplication, addition, multiplication and the supremum operator $x \mapsto \sup_y \phi(x, y)$.

The results can be viewed as an archimedean counterpart to [8], where the size of a finite set is shown to be definable following a milder expansion of the language.

They can also be considered as a refinement of [3]. In [3], the (archimedean) asymptotic cardinality of a definable set X over $\mathbb{F}_q$ was studied; this is a priori a real number $c_1(X) = \lim_q |X|/q^{\dim(X)}$ that depends on how p approaches infinity, but is shown to be a rational number, and to vary through only finitely many values as X varies in a family. The theory we consider is able to define the second term c_2; so that for nonstandard q, $|X(\mathbb{F}_q)| = c_1(X)q^{dim(X)} + c_2(X)q^{\dim(X)-1/2} + o(q^{\dim(X)-1/2})$. Here c_2 is genuinely a real number, varying continuously (as X varies) through a range described by Deligne's equidistribution theorem ([4] 3.5.)

This answers positively the question raised in the last lines of [6], on the existence of a simple unstable theory where c_2 is definable.

Finally, our results can be regarded as a continuous-logic realization of Grothendieck's sheaf-function correspondence. Given a $\mathbb{Q}_l^a$- local system L on a variety U over a finite field k, Grothendieck defines a function

$$\alpha_L(u) = tr(\alpha_u, L)$$

Initially this function takes values in $\mathbb{Q}_l^a$. But following Deligne, we fix an embedding ι of $\mathbb{Q}_l^a$ in $\mathbb{C}$, thus obtaining a complex-valued function. Certain numerical operations on the functions α_L can be carried out geometrically; thus $\alpha_{L\otimes L'} = \alpha_L\alpha_{L'}$, $\alpha_{L\oplus L'} = \alpha_L + \alpha_{L'}$. Grothendieck also showed that given a morphism $f : X \to U$ of relative dimension d, the 'integration on fibers' operator $x \mapsto q^{-d}\sum_{x\in X(k), f(x)=u} \alpha_L(x)$ comes from a geometric operation on constructible sheaves. (This is an 'expectation operator' in the sense of [7], §3.) Quantifier-elimination for the class of enriched finite fields amounts geometrically to a new approximate operation on virtual constructible sheaves; for any virtual constructible

sheaf L on X and $\epsilon > 0$, there exists a virtual constructible sheaf L' on U such that $|\sup_{x\in X(k), f(x)=u} \alpha_L(x) - \alpha_{L'}(u)| < \epsilon$.

In fact we do not work directly with the functions α_L on $X(k)$, but rather with functions taking values in conjugacy classes of compact Lie groups.

2. The language

Let $\mathbb{F}_p$ be a finite field, and X a normal, connected variety over $\mathbb{F}_p$. Let $\pi_1(X)$ denote the (arithmetic) étale fundamental group of X; we treat $\pi_1(X)$ as an object in the category of groups with homomorphisms up to conjugation in the target, so a choice of base point is not required. The set of conjugacy classes of a group G is denoted $G^\sharp$. We have $\pi_1(X) = \mathrm{Aut}(\mathbb{F}_p(X)^{unram}/k(X))$. When k is the fixed field of an automorphism σ of k^{alg}, we have $\sigma \in \pi^1(k)$, and functoriality of π_1 yields a map $\alpha = \pi_1(\sigma) : X(k) = Hom(k, X) \to \pi_1(X)^\sharp$, assigning a conjugacy class in $\pi_1(X)$ for any rational point.

A *local system* on X is a continuous homomorphism $\rho : \pi_1(X) \to GL_n(\mathbb{Q}_l^a)$. The Zariski closure G of the image of ρ is called the *monodromy group* of (X, ρ). Composing with the map α above, we obtain a canonical map $X(k) \to G(\mathbb{Q}_l^a)$. At this point we follow Deligne to move to the archimedean world, using an embedding $\iota : \mathbb{Q}_l^a \to \mathbb{C}$. By composition with ι we obtain a map $X(k) \to G(\mathbb{C})^\sharp$. Assuming that ρ has pure weight 0 is equivalent to saying that (after semisimplification) the conjugacy class in $G(\mathbb{C})^\sharp$ in fact lies in $cG^\sharp$, where cG is a maximal compact subgroup of $G(\mathbb{C})$. This is the map that we seek to axiomatize. Thus for each triple $\psi_{X,\rho,G}$ as above we have a symbol $\psi_{X,\rho,G}$ for a function on X taking values in $cG^\sharp$.

This language appears vast at a first sight. I am grateful to Hélène Esnault for pointing out that this impression is largely illusory; for example if X is a smooth projective variety, and n is given, up to scalar multiplication by functions constant on X, only finitely many functions $\psi_{L,\rho,G}$ with $G \leq GL_n$ actually appear. See [5].

Let PF_{mon} be the limit theory of finite fields enriched in this way. Then:

(1) The normalized counting measure is definable; and PF_{mon} admits elimination of expectation quantifiers. PF_{mon} is a simple unstable theory in the sense of Shelah, of SU-rank 1.

(2) (Next to leading term:) For any family $(X_u)_{u\in U}$ of definable sets of dimension d, PF_{mon} admits definable real-valued functions $c_1 : U \to \mathbb{Q}$, $c_2 : U \to \mathbb{R}$ s.t.

$$|X_u(\mathbb{F}_q)| = c_1 q^d + c_2 q^{d-1/2} + O(q^{d-1})$$

(3) PF_{mon} admits quantifier elimination: every definable function $U \to \mathbb{R}$ is uniformly approximated, on each piece of a constructible partition, by a linear combination of α_L 's.

(4) Definable *sets* remain the same as in Ax's theory of pseudofinite fields.

(5) In fact, PF_{mon} is the model completion of a natural universal theory expressing functoriality of $\psi_{X,\rho,G}$ in X and in G, and describing explicitly the map ψ in the case of finite G.

The proof of the existence of a model completion in (5) is model-theoretic; the proof that ultraproducts of enriched finite fields satisfy the axioms follows Deligne's proof of his equidistribution theorem.

3. Imaginaries originating in compact groups

The model theoretic framework leads to a natural question: classify the imaginary sorts.

Let C be a compact group, $j : C \to F$. Let C_n be the n-fold fiber product of C over F. Let $Y_n = C_n^\sharp$. Let $E \subset Y_2$ be a closed subset satisfying: $\pi_{xz}(\pi_{xy}^{-1}(E) \wedge \pi_{yx}^{-1}(E)) \subseteq E$.

Let T be a (continuous logic) theory with a definable $\psi : X^2 \to Y_2$. Then the pullback $\psi^* E$ of E is a $\bigwedge$-definable equivalence relation on X. The corresponding imaginary elements be called a compact simplicial imaginary. When C is a compact Lie group, $\psi^* E$ has finitely many classes on each 1-type.

Returning to the asymptotic theory of finite fields with monodromy, we show that any imaginary of $M \models PF_m$ is algebraic (in the sense of continuous logic) over a tuple of 'real' elements defined over the given imaginary. In the talk, a reduction of imaginaries to compact simplicial imaginaries was sketched, pursuing 2- and 3-amalgamation through a series of categories, from varieties, their étale fundamental groups, representations in algebraic groups and in compact groups, and finally the above category of conjugacy classees in compact groups. An example was given that seems plausibly to give a new imaginary.

References

[1] James Ax, *The Elementary Theory of Finite Fields*, Annals of Mathematics, Second Series, Vol. 88, No. 2 (Sep., 1968), pp. 239–271.

[2] I. Ben Yaacov, A. Berenstein, C. Ward Henson, and A. Usvyatsov, *Model Theory for Metric Structures*, in *Model Theory with Applications to Algebra and Analysis*, Vol. II, Lecture Notes series of the London Mathematical Society, No. 350, Cambridge University Press, 2008, 315–427.

[3] Z. Chatzidakis, L. van den Dries and A. Macintyre, *Definable sets over finite fields*, Journal für die reine und angewandte Mathematik **427** (1992), 107–135.

[4] P. Deligne, *La Conjecture de Weil. II*, Publications Mathématiques de L'Institut des Hautes Scientifiques **52**, 137—252, §3.5 (1980).

[5] H. Esnault, M. Kerz: *A finiteness theorem for Galois representations of function fields over finite fields (after Deligne)*, Acta Mathematica Vietnamica **37** 4 (2012), 531—562.

[6] E. Hrushovski, *Ax's theorem with an additive character*, EMS Surveys in Mathematical Sciences Volume 8, Number 1/2, 2021, pp. 179–216, `https://ems.press/journals/emss/issues/1652`.

[7] E. Hrushovski, *Approximate equivalence relations*, Model Theory, Vol. 3, No. 2, 2024, §3.

[8] W. Johnson, *Counting mod n in pseudofinite fields*, Israel Journal of Mathematics **247** (2022) pp. 697—739.

Non-archimedean curves over valued fields of higher rank

Katharina Hübner
(joint work with Michael Temkin)

I would like to report on joint work with Michael Temkin on the structure of non-archimedean curves over valued fields that are not necessarily of rank 1. We consider a complete microbial valued field (k, k^+) (microbial meaning that there is a valuation ring R of rank 1 with $k^+ \subseteq R \subseteq k$). For the sake of simplicity of the exposition we assume that (k, k^+) is algebraically closed. We set $S = \mathrm{Spa}(k, k^+)$ and denote by Γ the value group of (k, k^+). The points of S are totally ordered by specialization and correspond to the valuation rings R with $k^+ \subseteq R \subseteq k^\circ$.

We work in the category of adic spaces over S. So by a curve over k we mean a morphism $C \to S$ of adic spaces locally of topologically finite type of relative dimension 1. For a point $s \in S$ we denote by C_s the fiber of $C \to S$ over s. In this setting one of the main results from our preprint [1] reads as follows.

Theorem 1. *For a smooth curve C over S and a point $s \in S$ the following hold.*

(1) (C_s, Γ_{C_s}) is a Γ-graph with pinched leaves, whose vertices are the divisorial points, whose edges are the cut and infinitesimal points, and whose pinched leaves are the unibranch and classical points.

(2) For any finite set of classical and divisorial points $V_0 \subset C_s$, there is a triangulation V containing V_0.

The theorem claims that the fiber C_s is a graph in a certain sense. We know this sort of statement from the theory of Berkovich spaces, where Berkovich curves are infinite real metric graphs and we can construct skeleta and triangulations as developed in [2]. Our theorem now claims a similar structure for the corresponding adic space and can even describe curves over valued fields of higher rank. However, the edges of this graph are not parametrized by the real numbers as for Berkovich curves but by a different ordered set.

We will spend the rest of this exposition explaining in what way C_s can be interpreted as a graph. Here we will ignore the additional structure like the sheaf Γ_{C_s} and pinched leaves. Our only goal is to illustrate in what way the topological space C_s can be viewed as a graph.

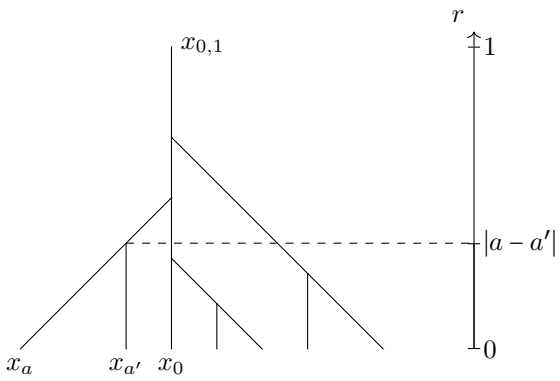
This structure can be seen already for the unit disc, so let us discuss the unit disc over $\mathbb{C}_p$:

$$C = \mathbb{D} = \mathrm{Spa}(\mathbb{C}_p\langle T\rangle).$$

Here the underlying set of the adic spectrum Spa is the set of equivalence classes of continuous valuations $\mathbb{C}_p\langle T\rangle \to \Gamma \cup \{0\}$ for a totally ordered group Γ (that varies with the points of Spa). For instance, it contains the set of classical points x_a that are obtained from an element $a \in \mathbb{C}_p$ with $|a| \leq 1$ in the following way:

$$x_a : \mathbb{C}_p\langle T\rangle \xrightarrow{T \mapsto a} \mathbb{C}_p \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}.$$

The unit disc $\mathbb{D}$ is often depicted as in the picture below:



The points at the bottom are classical points and the point $x_{0,1}$ is the Gauß point defined by

$$\begin{aligned} x_{0,1} : \mathbb{C}_p\langle T\rangle &\longrightarrow \mathbb{R}_{\geq 0} \\ \sum_{i=0}^{\infty} a_i T^i &\longmapsto \max_i |a_i|. \end{aligned}$$

On the line connecting the Gauß point with the classical point x_0 we have points of the form $x_{0,r}$ for $0 \leq r \leq 1$ defined by

$$x_{0,r} : \mathbb{C}_p\langle T\rangle \longrightarrow \mathbb{R}_{\geq 0} \tag{1}$$

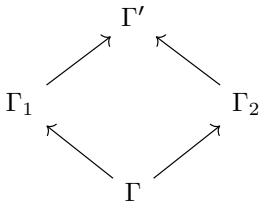
$$\sum_{i=0}^{\infty} a_i T^i \longmapsto \max_i |a_i| r^i. \tag{2}$$

Now one could get the impression that $\mathbb{D}$ is indeed a graph whose edges are parametrized by the real numbers. However, this is not the case for a number of reason. Firstly, there are additional points on the connecting line for each point $x_{0,r}$ with $r \in |\mathbb{C}_p^\times|$ that are described by the same formula as in (1) but with a radius that is infinitesimally bigger/smaller than r. Secondly, the induced topology from $\mathbb{D}$ on the connecting line is not the topology of $\mathbb{R}$. It is the topology generated by the intervals $[x_{0,r}, x_{0,s}]$ for $r, s \in |\mathbb{C}_p^\times|$.

In order to axiomatize this structure, we introduce the notion of rangers:

Definition. Let Γ be a totally ordered abelian group. A **ranger** of Γ is a pair (Γ', γ') consisting of a totally ordered abelian group $\Gamma' \supseteq \Gamma$ and an element $\gamma' \in \Gamma'$.

Two rangers (Γ_1, γ_1) and (Γ_2, γ_2) of Γ are **equivalent** if there exists a commutative diagram of totally ordered abelian groups



such that γ_1 and γ_2 map to the same $\gamma' \in \Gamma'$.

The set of equivalence classes of rangers is denoted by

$$\mathcal{R}_\Gamma = \{\text{rangers of } \Gamma\}/\sim .$$

We note that $\gamma \mapsto (\Gamma, \gamma)$ defines an injective map $\Gamma \to \mathcal{R}_\Gamma$. The set $\mathcal{R}_\Gamma$ is not a group but it is totally ordered. We equip it with the topology generated by the intervals of the form $[a, b]$, $[0, a]$, and $[b, \infty]$ for $a, b \in \Gamma$. In particular, every element of Γ is an open point in $\mathcal{R}_\Gamma$ and all other points are closed. In this way the set of rangers $\mathcal{R}_\Gamma$ becomes a connected, quasicompact topological space with Γ as a dense subset.

We consider the interval $[0, 1] \subseteq \mathcal{R}_\Gamma$. Then we obtain a homeomorphism from $[0, 1]$ to the connecting line from x_0 to $x_{0,1}$ by sending $r \in [0, 1]$ to the valuation

$$\begin{aligned} x_r \colon \mathbb{C}_p\langle T\rangle &\longrightarrow \Gamma_r \cup \{0\} \\ \sum_{i=0}^{\infty} a_i T_i &\longmapsto \max_i |a_i| \gamma_r^i, \end{aligned}$$

where the ranger (Γ_r, γ_r) is a representative of r.

This illustrates that the edges of the graph C_s are parametrized by the rangers $\mathcal{R}_\Gamma$.

References

[1] K. Hübner and M. Temkin, *Adic curves: stable reduction, skeletons and metric structure*, preprint 2024, arXiv:2406.07414.

[2] V. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Providence, RI: American Mathematical Society, **33** (1990), 169 p.

Taming perfectoid rings

Konstantinos Kartas
(joint work with Franziska Jahnke)

We aim to identify a class of Banach rings for which a certain fundamental result about perfectoid rings, true only in an "almost sense", becomes genuinely true.

1. Almost purity

A non-archimedean Banach algebra is called *uniform* if the subset of power-bounded elements $A^\circ \subseteq A$ is bounded. Equivalently, the norm of A is equivalent to a power-multiplicative norm, namely the spectral norm $\|\cdot\|_{\mathrm{sp}}$.

Definition (Scholze)**.** Let K be a perfectoid field of residue characteristic p. A perfectoid K-algebra is a uniform Banach K-algebra A such that the Frobenius morphism $\Phi : A^\circ/(p) \to A^\circ/(p)$ is surjective.

Perhaps the most foundational result in perfectoid geometry is the almost purity theorem, due to Faltings [1] in a special case, and Scholze [7] and Kedlaya-Liu [4] in full generality:

Theorem 1 (Almost purity)**.** *Let A be a perfectoid K-algebra and $A \to B$ be finite étale. Then:*

(1) B is a perfectoid K-algebra.
(2) $A^\circ \to B^\circ$ is almost finite étale.

Here, "almost" is meant in the technical sense of Faltings' almost mathematics [1], a "rough" version of ring theory in which certain small errors are systematically ignored. The idea is that étaleness at the generic fibre does not quite extend integrally, but it comes very close to doing so.

Our goal is to identify a class of Banach rings for which étaleness at the generic fibre actually extends integrally, i.e.,

$$\text{if } A \to B \text{ is finite étale, then } A^\circ \to B^\circ \text{ is finite étale}$$

We then aim to show that almost purity can be recovered using ultraproducts. This strategy already works for perfectoid fields and yields an alternative proof of almost purity in that case [3].

2. Tame rings

Let K be a complete tame normed field of residue characteristic p in the sense of Kuhlmann [5]. As with fields, we say that a normed algebra $(A, \|\cdot\|)$ is *algebraically maximal* if every pseudo-Cauchy sequence of algebraic type has a pseudo-limit. This notion can be thought of as an algebraic counterpart of spherical completeness.

Definition. A tame K-algebra is uniform Banach K-algebra such that:

(1) $(A, \|\cdot\|_{\mathrm{sp}})$ is algebraically maximal.

(2) For every $a \in A$, there exists $b \in A$ such that

$$\|a - b^p\|_{\mathrm{sp}} < \|a\|_{\mathrm{sp}}$$

Remark 2. The terminology here is provisional; a better name for this notion might be *weakly tame* for reasons which will be explained later.

Tame algebras satisfy a version of almost purity without "almost". However, we have to restrict to Galois p-extensions.

Theorem 3. *Let A be a tame K-algebra and $A \to B$ be G-Galois, where G is a finite p-group. Then:*

(1) B is a tame K-algebra.
(2) $A^\circ \to B^\circ$ is G-Galois.

If $A \to B$ is only assumed finite étale, then $A^\circ \to B^\circ$ need not be finite étale (this already fails for fields). We do not know if the above statement is true for étale extensions of p-power degree which are not necessarily Galois.

3. Taming perfectoid rings

Tame algebras are closely related to perfectoid algebras.

Proposition 4. *Let A be a tame K-algebra. Then A°/pA° is semi-perfect. In particular, A is a perfectoid algebra.*

Perfectoid fields (and algebras) are not necessarily tame but we can nevertheless "tame" them using ultraproducts.

Theorem 5. *Let K be a complete non-archimedean field and A be a Banach K-algebra. For any non-principal ultrafilter on $\mathbb{N}$, the following are equivalent:*

(1) A is a perfectoid K-algebra.
(2) $A^{\mathcal{U}}$ is a tame $K^{\mathcal{U}}$-algebra.

Remark 6. We view Banach K-algebras as two-sorted structures in continuous logic, following Rideau-Kikuchi, Scanlon and Simon [6], and interpret the above ultrapowers in this framework. Both $K^{\mathcal{U}}$ and $A^{\mathcal{U}}$ are equipped with their real-valued norms.

4. Beyond (weakly) tame algebras

Ultraproducts of perfectoid algebras satisfy many more properties beyond (weak) tameness. For example:

Proposition 7 (Scholze)**.** *Let A be a non-principal ultraproduct of perfectoid K-algebras. Then $A^\times/A^{\circ\times}$ is uniquely divisible.*

This implies that every *global* Kummer étale extension of A of degree prime to p comes from a Kummer étale extension of A°. This is a special case of the general statement we are after. However, in order to eventually obtain a non-standard proof of almost purity, it seems crucial to also understand local properties of these ultraproducts.

Theorem 8. *Let A be a non-principal ultraproduct of perfectoid K-algebras. Let $X = Spa(A, A^\circ)$ and $x \in X$. Then $\kappa(x)$ is a tame field with divisible value group.*

This is surprising since $\kappa(x)$ does not inherit saturation from A (heuristically, ultraproducts do not commute with localization), whereas saturation was a key ingredient in our proof that ultrapowers of perfectoid fields are tame.

References

[1] G. Faltings, *Almost étale extensions*, in: Cohomologies p-adiques et applications arithmétiques (II), Astérisque **279** (2002), 185–270.

[2] O. Gabber and L. Ramero, *Almost ring theory*, Lecture Notes in Mathematics **1800**, Springer-Verlag, Berlin, 2003.

[3] F. Jahnke and K. Kartas, *Beyond the Fontaine–Wintenberger theorem*, J. Amer. Math. Soc. **38** (2025), no. 4, 997–1047.

[4] K. S. Kedlaya and R. Liu, *Relative p-adic Hodge theory: Foundations*, Astérisque **371** (2015).

[5] F.-V. Kuhlmann, *The algebra and model theory of tame valued fields*, J. Reine Angew. Math. **719** (2016), 1–43.

[6] S. Rideau-Kikuchi, T. Scanlon, and P. Simon, *The tilting equivalence as a bi-interpretation*, preprint, 2025, arXiv:2505.01321.

[7] P. Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313.

Some applications of the stable completion of curves to the theory of valued fields

François Loeser

(joint work with Antoine Ducros, Ehud Hrushovski and Jinhe Ye)

During the last decade, several extensions of the Ax-Kochen-Ershov principle have been obtained in various settings: for tame fields by F.-V. Kuhlmann [8], for Cohen rings by S. Anscombe and F. Jahnke [1], for deeply ramified and perfectoid fields by F. Jahnke and K. Kartas [6]. These results are all proved by using an embedding lemma which appeared first in [8]. This lemma relies on two results by F.-V. Kuhlmann, namely the generalized stability theorem from [7] and the henselian rationality theorem from [9]. The goal of this talk is to report on model theoretic proofs for these two results, using the definability of the stable completion of curves over valued fields. They are respectively joint work with A. Ducros, E. Hrushovski and J. Ye (for the generalized stability theorem) and with A. Ducros (for the henselian rationality theorem).

We shall use the multiplicative notation for valued fields: a valuation on a field K is a map $|\cdot|: K \to G_0 = G \cup \{0\}$ where G is a multiplicative ordered abelian group. We will use the multi-sorted language of valued fields introduced in [4]. In addition to the three standard sorts (the valued field itself, the residue field and the value group), it contains geometric sorts S_n and T_n, for n running over the positive integers. Adjoining these geometric sorts ensures that the theory ACVF of algebraically closed non-trivially valued fields eliminates imaginaries [4]. We shall write Γ for the value group sort and RV for the leading term sort. If X is an

algebraic variety over a valued field K it is proved in [5] that its stable completion $\widehat{X}$ is (represented by) a strictly pro-definable set in this language. When X is a curve, it is proved in [5] that $\widehat{X}$ is in fact definable over K.

1. The generalized stability theorem

Recall that a valued field is called defectless (or stable) if all its finite extensions are defectless (i.e. satisfy the relation $\sum e_i f_i = n$). The generalized stability theorem of [7] states that an Abhyankar extension of a defectless field is again defectless. It can be restated as follows:

Theorem 1. *Let K be a defectless valued field and H be an ordered abelian group containing $|K^\times|$. Fix $\gamma = (\gamma_1, \cdots, \gamma_n)$ in H^n and endow the field $K(T) = K(T_1, \cdots, T_n)$ with the Gauss norm $|\cdot|_\gamma$ associated with γ. Then $(K(T), |\cdot|_\gamma)$ is stable.*

I will now sketch the model-theoretic proof of this result given in the joint work [3] with A. Ducros, E. Hrushovski and J. Ye.

Step 1: One easily reduces to the case when $n = 1$ and K is a model of ACVF.

Step 2: Assume now that $n = 1$ and K is a model of ACVF. One reduces to the case when $\gamma \in |K^\times|$ by using the following result from [4]: if φ is a definable map from Γ to the set of lattices in K^m, then there exists finitely many bases of K^m such that for any γ, $\varphi(\gamma)$ is diagonalizable in one of these bases.

Step 3: Fix a finite extension F of $K(T)$. We claim there exists a K-definable subset D of Γ such that for any elementary extension $(L, |\cdot|_L)$ of $(K, |\cdot|)$, for any $h \in |L^\times|_L$, the extension $F \otimes_{K(T)} L(T)$ of $(L(T), |\cdot|_{L,h})$ is defectless if and only if $h \in D(L)$ (in short, "the defectless locus is definable").

Take now X a projective model of F and let $p : X \to \mathbb{P}^1_K$ corresponding to $K(T) \hookrightarrow F$. Let $\Sigma \subset \mathbb{P}^1_K$ be the image of Γ under the definable map sending γ to the type associated to $|\cdot|_\gamma$. Denote by Δ the preimage of Σ under the extension of p to a morphism $\widehat{X} \to \widehat{\mathbb{P}^1_K}$. It is Γ-internal, that is, definably isomorphic to a definable subset of some Γ^n. In particular it is a finite union of points and intervals. By Riemann-Roch there is a line bundle $\mathcal{L}$ on X such that the set of ratios s/t of global sections of $\mathcal{L}$ generate multiplicatively (universally) $F^\times$ (this is the key ingredient in proving the definability of $\widehat{X}$ for curves). The existence of the definable set D follows from the fact that the ramification indices may be read out from the slopes of the valuations of quotients s/t on intervals of Δ. One easily proves that $\gamma \in D$ iff $\gamma^+ \in D$ iff $\gamma^- \in D$, where γ^+ and γ^- denote respectively a point infinitely close to the right and to the left of γ. It follows that the definable set D is open and closed. Take $(L, |\cdot|_L)$ an elementary extension of K such that there exists $h \in |L^\times|_L$ larger than any element of $|K^\times|$. One checks that $h \in D(L)$ yielding that D is non-empty. It follows that $D = \Gamma$ since Γ is (definably) connected. □

2. Henselian rationality

Let K be a henselian valued field. One says that K is tame if for any finite valued field extension L of K the following holds: the residue field extension is separable, the ramification index is prime to the residue characteristic, and the extension is defectless. The henselian rationality theorem of F.-V. Kuhlmann [8] is the following result:

Theorem 2. *Let $(K, |\cdot|)$ be a tame henselian field and let $(F, |\cdot|_F)$ be a finitely generated extension of transcendence degree 1. Assume this extension is immediate. Then there exists $t \in F$, transcendental over K, such that $K(t) \hookrightarrow F$ induces an isomorphism between the corresponding henselizations $K(t)^h \simeq F^h$.*

In work in progress with A. Ducros, we provide a purely model-theoretic proof of this result, using again the definability of $\widehat{X}$ for X a curve. Let us sketch our proof in the key case when K is a model of ACVF. Let X be a curve over K. We construct a definable strong deformation retraction $h : I \times \widehat{X} \to \widehat{X}$ with I a closed interval in Γ, with final image Σ a Γ-internal subset of $\widehat{X}$ satisfying some very strong properties (we call such a Σ a strongly admissible skeleton). In particular $\widehat{X} \setminus \Sigma$ is a disjoint union of open discs. We now consider the model theoretic analogue $\widetilde{X}$ of the Huber analytification of X: it represents the functor assigning to a model M of ACVF containing K the set of bounded M-definable types on X. It is proved in [2] that for X of any dimension $\widetilde{X}$ is strictly pro-definable. Let $x \in \widehat{X}$. If $x \in \Sigma$ we set $\hat{r}(x) = x$. If $x \notin \Sigma$, we define $\hat{r}(x)$ as the point in $\widetilde{X}$ that one gets if one stops the homotopy "infinitely close to final time". We write r for the restriction of $\hat{r}$ to X and set $\Sigma_{\mathrm{RV}} = r(X)$. We show that when Σ is a strongly admissible skeleton, Σ_{RV} is an RV-internal subset of $\widetilde{X}$ (that is, definably isomorphic to a definable subset of some RV^m), and for any $x \in X \setminus \Sigma$, the fiber $r^{-1}(r(x))$ is $K\mathrm{RV}(K(x))$-isomorphic to an open disc of $\mathbb{A}^1$ via an étale morphism defined on a Zariski neighborhood of x. To conclude the proof, let X be a projective model of F and let $x \in X$ be a realization of the type of the immediate valuation $|\cdot|_F$. Since the extension is immediate, $\mathrm{RV}(K(x)) = \mathrm{RV}(K)$, hence $r^{-1}(r(x))$ is K-isomorphic to an open disc of $\mathbb{A}^1$ via an étale morphism defined on a Zariski neighborhood of x, from which the statement follows. □

The existence of strongly admissible skeleta for curves have some other notable consequences, in particular, for a curve X we obtain

- the existence of definable triangulations of $\widehat{X}$;
- a proof of the fact that $\widetilde{X}$ is definable;
- a model-theoretic proof of the semi-stable reduction theorem for curves.

References

[1] S. Anscombe and F. Jahnke, *The model theory of Cohen rings*, Confluentes Math. **14** (2022), 1–28.
[2] P. Cubides Kovacsics, M. Hils and J. Ye, *Beautiful pairs*, preprint 2021, arXiv:2112.00651.
[3] A. Ducros, E. Hrushovski, F. Loeser and J. Ye, *Tropical functions on a skeleton*, J. Éc. polytech. Math. **11** (2024), 613–654.

[4] D. Haskell, E. Hrushovski and D. Macpherson, *Definable sets in algebraically closed valued fields: elimination of imaginaries*, J. reine angew. Math. **597** (2006), 175–236.
[5] E. Hrushovski and F. Loeser, *Non-archimedean tame topology and stably dominated types*, Annals of Math. Studies **192**, Princeton University Press, Princeton, NJ, 2016.
[6] F. Jahnke and K. Kartas, *Beyond the Fontaine-Wintenberger theorem*, J. Amer. Math. Soc. **38** (2025), 997–1047.
[7] F.-V Kuhlmann, *Elimination of ramification I: The generalized stability theorem*, Trans. Amer. Math. Soc. **362** (2010), 5697–5727.
[8] F.-V Kuhlmann, *The algebra and model theory of tame valued fields*, J. Reine Angew. Math. **719** (2016), 1–43.
[9] F.-V Kuhlmann, *Elimination of ramification II: Henselian rationality*, Israel J. Math. **234** (2019), 927–958.

Some effective special point results

Gareth Jones
(joint work with Gal Binyamini, Harry Schmidt, and Margaret Thomas)

We prove [1] a uniform form of the Manin-Mumford conjecture (a theorem of Raynaud) for products of g elliptic curves with complex multiplication. Suppose that A is a product of g such curves, and let K be the field obtained by adjoining the j-invariants of the factors of A to $\mathbb{Q}$. Suppose that each factor of A is given by a Weierstrass equation over K. Here is an illustrative special case of our result.

Theorem 1. *Let g be a positive integer and let ϵ be a positive real number. There exist effectively computable positive real numbers c and m with the following property. Suppose that A and K are as above. Suppose that $V \subseteq A$ is an irreducible subvariety defined over a number field L extending K, and suppose further that V does not contain a torsion translate of a positive dimensional abelian subvariety of A. If $P \in V(\mathbb{C})$ is a torsion point of exact order N, then*

$$N \leq c[L:K]^{1+\epsilon} \deg(V)^m.$$

In the case that V does contain a torsion translate of an abelian subvariety of positive dimension, we also obtain a bound on the complexity of the translate.

This result is less uniform than other recent results. For example, Gao, Ge, and Kühne [2] prove a uniform form of Mordell-Lang (much stronger than Manin-Mumford), for abelian varieties in general. But our results give strong bounds on the order of torsion, rather than the number of points. This feature can be applied to prove some effective results of Andrè-Oort type for certain mixed Shimura varieties.

We introduce some further terminology. Let $\mathcal{E}^{(g)}$ be the g-fold fibre power of the Legendre family $Y^2 = X(X-1)(X-\lambda)$, with $\lambda \neq 0, 1$. Call a point $P \in \mathcal{E}^{(g)}(\mathbb{C})$ *special* if it is a torsion point in its fibre and this fibre has complex multiplication. A variety $\mathcal{S} \subseteq \mathcal{E}^{(g)}$ is called *special* if either $\mathcal{S}$ is a component of an algebraic subgroup of $\mathcal{E}^{(g)}_\lambda$, where $\lambda \in \mathbb{C} \setminus \{0,1\}$ is such that $\mathcal{E}^{(g)}_\lambda$ has complex multiplication, or $\mathcal{S}$ is an irreducible component of a flat subgroup scheme of $\mathcal{E}^{(g)}$. We define

the complexity of a special point P to be the maximum of the order of P and the discriminant of the endomorphism ring of the elliptic curve $\mathcal{E}_{\pi(P)}$.

Theorem 2. *Let g be a positive integer. There exist effectively computable positive reals c and m depending only on g with the following property. Suppose that V is an irreducible subvariety of $\mathcal{E}^{(g)}$ defined over a number field K. Suppose further that V does not contain a special subvariety of positive dimension. If $P \in V(\mathbb{C})$ is special, then the complexity of P is at most*

$$\exp\left(c\left([K:\mathbb{Q}]\deg(V)\right)^m\right).$$

As before, if V does contain positive dimensional special subvarieties, we also bound the complexity of these specials.

This gives an effective form of a theorem due to Habegger [3]. In addition, our bounds do not depend the height of the variety.

In work in progress with Schmidt, we extend some of the above to the setting of multiplicative extensions of powers of elliptic curves. Here, rather than a uniform Manin-Mumford for special extensions, it turns out to be more natural to prove an effective André-Oort result for the mixed Shimura variety parameterising points on extensions of A by $\mathbb{G}_m^n$, where, as above A is a product of CM elliptic curves. At present we can prove analogues of the above results, but we have not yet handled positive dimensional special subvarieties.

References

[1] G. Binyamini, G. Jones, H. Schmidt and M. Thomas *An effective Pila–Wilkie theorem for sets definable using Pfaffian functions, with some diophantine applications*, J. Eur. Math. Soc. (JEMS), To appear.

[2] Z. Gao, T. Ge and L. Kühne *The Uniform Mordell-Lang Conjecture*, preprint 2021, arXiv:2105.15085.

[3] P. Habegger *Special points on fibered powers of elliptic surfaces*, J. Reine Angew. Math. **685** (2013), 143–179.

Dynamics, periodic subvarieties, and the degree of nonminimality

James Freitag

In [10], the author and Moosa introduced the notion of the degree of nonminimality of the a stationary finite rank type p in an ω-stable theory, denoted $\mathrm{nmdeg}(p)$. When the type p is itself minimal, we write $\mathrm{nmdeg}(p) = \infty$. The quantity is intended to measure, in a precise way, how far the type is from being minimal via tracking the number of realizations of the type required to witness that the type has a nonalgebraic forking extension. The central goal was, in a variety of situations, to bound how large the $\mathrm{nmdeg}(p)$ could be before it is forced to be ∞ and the type is minimal.

The motivation for the development of the notion came from algebraic differential equations. As of around 2010, the situation in the field was fairly striking: only a few equations had been shown to be minimal (a complete list from an even

later date is given in [2]), but a well-known conjecture of Poizat predicted that almost every nonlinear equation should be minimal. Eventually, powerful bounds in this setting were established in [9], and many new classes of equations were shown to be minimal. For instance, see [3] or [4]. These developments eventually lead to the resolution of Poizat's conjecture [2, 5]. Subsequently, notions around the degree of nonminimality were generalized to the setting of simple theories, where they played a role in the resolution of Koponen's conjecture [1].

In the setting of algebraic difference equations (or even the more particular setting of algebraic dynamics), basically the situation as of now is like that of a decade and a half ago for differential equations: it is believed that most higher order nonlinear difference equations are minimal, but very few examples are known. The goal of this work is the development of the degree of nonminimality in the setting of difference equations and the special case of algebraic dynamical systems. We hope for similar results as in the differential case, eventually leading to wide ranges of application. For instance, one hopes to establish analogs of Poizat's conjecture in this setting.

Let $K \subset \mathcal{U} \models ACFA$ be an embedding of difference fields with distinguished automorphism σ. Given a variety V with a $\phi : V \to V^\sigma$ a rational map over K. Associated with this discrete dynamical system are the solutions of a difference equation $(V,\phi)^\sharp = \{x \in V(\mathcal{U}) \,|\, \sigma(x) = \phi(x)\}$. Not every difference equation can be written in this form (using certain invariants, one can show there are difference equations whose solution set is orthogonal to any equation of this form). For difference equations of this form, [6] prove that the degree of nonminimality of a generic solution is at most $\dim(V) + 3$. When K is contained in the fixed field of σ, a better bound of 2 is proved in [6] and [7]. We conjecture that one can, in general replace $\dim(V) + 3$ by 3.

When (V,ϕ) are given as above, algebraic relations between generic solutions of the difference equation correspond to invariant subvarieties of the dynamical system, several analogs of the results of [8] have been established. In [6], it is shown that if there are algebraic relations between the σ-disjoint solutions of such a difference equation, then already there must be relations between at most $\dim(V)+4$ such solutions. In [7], any distinct solutions are considered, and the stronger bound of 4 is established. The relationship between the two results is interesting—the bound of [7] is only non-trivial in the case that there are no solutions which are σ-transforms of each other. Thus, the result applies non-trivially to a smaller class of difference equations. However, this class still contains, for instance, the situation when the field of definition of V contains any element which is not periodic with respect to σ. It would be natural to ask for a result which applies nontrivially to the general class of σ-varieties as the result of [6], but with the stronger bound of [7].

References

[1] John Baldwin, James Freitag, and Scott Mutchnik. *Simple Homogeneous Structures and Indiscernible Sequence Invariants.* arXiv:2405.08211 (2024).

[2] Matthew DeVilbiss and James Freitag. *Generic differential equations are strongly minimal.* Compositio Mathematica 159.7 (2023): 1387–1412.
[3] Yutong Duan and Joel Nagloo. *Algebraic independence of the solutions of the classical Lotka–Volterra system.* Annals of Pure and Applied Logic 177.5 (2026): 103707.
[4] Christine Eagles and Léo Jimenez. *Internality of autonomous algebraic differential equations.* arXiv:2409.01863 (2024).
[5] Jaoui, Rémi. *On the density of strongly minimal algebraic vector fields.* arXiv:2301.06362 (2023), to appear in *Annales scientifiques de l'École normale supérieure.*
[6] Moshe Kamensky and Rahim Moosa. *A transformal transcendence result for algebraic difference equations.* arXiv:2510.16314 (2025).
[7] James Freitag. *When any four solutions are independent.* arXiv:2510.08387 (2025).
[8] James Freitag, Rémi Jaoui, and Rahim Moosa. *When any three solutions are independent.* Inventiones mathematicae 230.3 (2022): 1249–1265.
[9] James Freitag, Rémi Jaoui, and Rahim Moosa. *The degree of nonminimality is at most 2.* Journal of Mathematical Logic 23.03 (2023): 2250031.
[10] James Freitag and Rahim Moosa. *Bounding nonminimality and a conjecture of Borovik–Cherlin.* Journal of the European Mathematical Society 27.2 (2023): 589-613.

De Rham Theorem in the Globally Subanalytic Setting

Tobias Kaiser

(joint work with Annette Huber-Klawitter and Abhishek Oswal)

We are interested in homology and cohomology theories in the tame setting of o-minimal structures. O-minimal singular homology and cohomology have been successfully developed, first by Delfs and Knebusch [5] in the semialgebraic case and then by Edmundo and Woerheide [6] in the general case of o-minimal expansions of a real closed field. What has been missing so far is an o-minimal version of de Rham cohomology, more precisely of the de Rham theorem comparing definable de Rham with o-minimal singular cohomology. Note that Bianconi and Figueiredo have previously introduced in their preliminary note [3], a version of definable de Rham cohomology groups. In [7], Huber has studied the period pairing for manifolds definable in an o-minimal structure, analyzing which simplices can be used for integration. But on the side of differential forms an o-minimal version leading to a definable de Rham theorem did not exist so far. This is an essential input in any generalisation of Hodge theory or the period conjecture from the algebraic to the more general tame setting. For instance, in [1], the authors develop the theory of coherent sheaves on definable complex analytic spaces and establish a definable version of GAGA for coherent sheaves but not for cohomology. Here, we treat a first paradigmatic case.

We focus on the globally subanalytic setting, i.e. on the o-minimal structure $\mathbb{R}_{\mathrm{an}}$ of the real field with restricted analytic functions. Given a globally subanalytic C^ω-manifold that is compact every real analytic function is globally subanalytic. Since in the C^ω-setting the classical de Rham cohomology groups are generated by real analytic differential forms (see for example Beretta [2]) we have immediately a definable version of the de Rham theorem. But in the non-compact case the situation is completely different. The straightforward definition using definable

forms does not give rise to a good de Rham cohomology theory, mainly due to the failure of the Poincaré lemma. Integration arguments are no longer available in the o-minimal setting.

However in the globally subanalytic setting we are able to remedy the situation. Here one can consider the class of constructible functions as introduced by Raf Cluckers and Dan Miller [4]. Constructible functions are finite sums of finite products of globally subanalytic functions and the logarithm of positive such functions. They are definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$ of the real field with restricted analytic functions and exponentiation (see van den Dries and Miller [10] for the structures $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\text{an,exp}}$). The main property of this class is that it is closed under parametric integration.

Constructible functions of class C^∞ are real analytic by [8]. The reasoning of [2] involves deep results from cohomology of coherent sheaves on Stein complex manifolds which are so far not available in the definable or constructible setting. But in the case of globally subanalytic C^p-manifolds where $0 < p < \infty$ we are able to establish a full constructible de Rham theorem. Given $0 \leq q < p$ we define constructible differential forms of regularity q and the corresponding de Rham cohomology groups. Note that this includes the case of C^1-manifolds: By the tame geometric properties of o-minimal structures we are able to establish a version of the de Rham complex leading to the de Rham theorem also for those.

The definition of constructible differential forms is subtle. We construct them similarly to the definition of regulous functions (see Kucharz and Kurdyka [9]). We indicate the case $q = 0$: A constructible differential form is C^1 on a dense open globally subanalytic set. We consider now the vector space of the continuous constructible differential forms such that the Cartan derivative on such an open set extends to a continuous constructible differential form on the whole manifold. This defines the constructible de Rham complex and the constructible de Rham cohomology groups. It is a consequence of the tame properties of o-minimal structures that they are indeed functorial for globally subanalytic C^1-maps.

Our main result is the de Rham theorem for these constructible de Rham cohomology groups: they are canonically isomorphic to the singular cohomology groups (and if the given manifold is C^ω therefore to the classical de Rham cohomology groups). Following closely the arguments in the classical case, we give two versions of proof, one fairly explicit depending on Stokes' Theorem and a sheaf theoretic one. In both cases the ingredients are the same: the Mayer-Vietoris property and homotopy invariance. The Mayer-Vietoris property follows in a straight-forward way from the existence of globally subanalytic partitions of unity in the C^p-case for $p < \infty$. It is this part of the argument that breaks down for the C^ω-versions of constructible de Rham cohomology. Homotopy invariance turns out to be more subtle. As for functoriality, tameness is used heavily.

References

[1] B. Bakker, Y. Brunebarbe and J. Tsimerman, *o-minimal GAGA and a conjecture of Griffiths*, Invent. Math., **232**(1) (2023), 163–228.

[2] L. Beretta, *An analytic version of the de rham theorem*, Ann. Uni. Ferrara Sez. VII (N.S.), **40** (1994), 23–30.
[3] R. Bianconi and R. Figueiredo, *O-minimal de Rham cohomology*, preprint 2019, arXiv:1904.05485.
[4] R. Cluckers and D. J. Miller, *Stability under integration of sums of products of real globally subanalytic functions and their logarithms*, Duke Math. J., **156**(2) (2011), 311–348.
[5] H. Delfs and M. Knebusch, *On the homology of algebraic varieties over real closed fields*, J. Reine Angew. Math., **335** (1982), 122–163.
[6] M. J. Edmundo and A. Woerheide. *Comparison theorems for o-minimal singular (co)homology*, Trans. Amer. Math. Soc., **360**(9) (2008), 4889–4912.
[7] A. Huber, *The period isomorphism in the tame geometry*, Math. Nachr., **297**(4) (2024), 1230–1247. With an appendix joint with Johan Commelin.
[8] T. Kaiser and A. Opris, *Differentiability properties of log-analytic functions*, Rocky Mountain J. Math., **52**(4) (2022), 1423–1443.
[9] W. Kucharz and K. Kurdyka, *From continuous rational to regulous functions*. In Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, pages 719–747. World Sci. Publ., Hackensack, NJ, 2018.
[10] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J., **84**(2) (1996), 497–540.

Exponentially algebraic differential equations

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(joint work with Gareth Jones and Rémi Jaoui)

1. Definitions of exponential-algebraicity

We give four definitions of what it means for a holomorphic function, of several complex variables, to be exponentially algebraic. It is work in progress to prove that the four definitions agree, but a theorem that they agree at least for germs at sufficiently generic points.

Let $U \subseteq \mathbb{C}^r$ be a connected open domain, and let $\mathrm{Mer}(U)$ and $\mathrm{Hol}(U)$ denote the field of meromorphic functions and its subring of holomorphic functions respectively. We write $\mathbf{t}$ for the coordinate functions on $\mathbb{C}^r$ and so $\mathbb{C}(\mathbf{t})$ is the field of rational functions on U.

For $g_1, \dots, g_n \in \mathrm{Hol}(U)$ define

$$\delta(g_1, \dots, g_n/\mathbb{C}(\mathbf{t})) := \mathrm{td}(g_1, \dots, g_n, e^{g_1}, \dots, e^{g_n}/\mathbb{C}(\mathbf{t})) - \mathrm{ldim}_{\mathbb{Q}}(g_1, \dots, g_n/\mathbb{C})$$

Theorem 1 ([1], known as "Ax-Schanuel"). *We always have* $\delta(g_1, \dots, g_n/\mathbb{C}(\mathbf{t})) \geqslant 0$.

Definition (Using exponential algebra). $f \in \mathrm{Mer}(U)$ is *exponentially algebraic* if there are $n \in \mathbb{N}_0$, $U_0 \subseteq U$ and $g_1, \dots, g_n \in \mathrm{Hol}(U_0)$ such that $\delta(g_1, \dots, g_n/\mathbb{C}(\mathbf{t})) = 0$ and f is field-theoretically algebraic over $\mathbb{C}(\mathbf{t}, g_1, e^{g_1}, \dots, g_n, e^{g_n})$.

Definition (Using o-minimality). $f \in \mathrm{Mer}(U)$ is *exponentially algebraic* if it is locally definable in the expansion of the real field by the restriction of the complex exponential function to compact boxes. By *locally definable* we mean that for

all $a \in U$ there is a neighbourhood of a such that the restriction of f to that neighbourhood is definable.

Definition (Inductive)**.** We say that $f \in \mathrm{Mer}(U)$ is *elementary* in the sense of Liouville if it can be obtained recursively from functions in $\mathbb{C}(\mathbf{t})$ by

1) Composition with rational functions;
2) Composition by exp;
3) Composition by (a branch of) log;
4) Composition with an algebraic function (an implicit function of a polynomial).

We say f is *exponentially algebraic* if it can be obtained recursively from $1), 2), 3)$ and

4′) Taking implicit functions (of previously obtained functions).

For example, Lambert's W-function has the implicit definition $W(z)e^{W(z)} = z$.

Definition (Using differential equations)**.** By the *exponential differential equation* we mean the differential equation $\frac{Dy}{y} = Dx$, which is satisfied when $(x, y) = (f, ce^f)$ for some function f and some non-zero constant c.

The idea of this definition is that f is exponentially algebraic if it satisfies a differential equation which can be built up only from instances of the exponential differential equation.

Precisely, we extend $\mathrm{Mer}(U)$ to a differentially closed field F with no new constants, and then consider the reduct where we forget the derivations but keep the predicate corresponding to the exponential differential equation, see [3]. Then f is *exponentially algebraic* if in the sense of this reduct, the type of f has finite Morley rank.

2. Theorems and examples

This section is based on [2]

Theorem 2 (A)**.** *Let*

$$F(y, y', \ldots, y^{(n)}) = 0 \tag{1}$$

be an algebraic differential equation whose coefficients are Liouville-elementary functions. Assume the equation is internal to the constants*, so the solutions are parametrized by the constant field in some algebraic way. Then every exponentially algebraic solution is actually Liouville-elementary.*

Theorem 3 (B)**.** *Let $\mathbb{C}(\mathbf{t}) \subseteq k \subseteq K$ be a tower of differential fields, all with constants $\mathbb{C}$, with K differentially closed and with k self-sufficient in K, that is, for any $k \subseteq L \subseteq K$ with L finitely generated over k we have $\delta(L/k) \geq 0$. Suppose that $f \in K$ satisfies equation (1) with coefficients in k, and the f is exponentially algebraic over k, in K. Then the differential Galois group (binding group) of f over k is linear and abelian-by-finite. Hence, by Chevalley, up to finite index it is of the form $\mathbb{G}_{\mathrm{a}}^u \times \mathbb{G}_{\mathrm{m}}^v$ for some $u, v \in \mathbb{N}$.*

Corollary 4 (Exponential-algebraic independence)**.** *The four functions* $\int \frac{W(z)}{z^2}dz$*, the Airy function* $\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos(t^3/3+2t)dt$*, the error function* $\mathrm{erf}(z) = \int e^{-z^2}dz$*, and the elliptic integral* $\mathrm{sn}(z) = \int \frac{dz}{z^3+az+b}$ *are exponentially transcendental, and even exponentially-algebraically independent from each other.*

Corollary 5 (Algebraic independence of exponentially-algebraic functions)**.** *Any distinct solutions* $W_1(z), \ldots, W_n(z)$ *of either Lambert's equation* $we^w = z$ *or of the Kepler equation* $w - \sin(w) = z$ *are algebraically independent over* $\mathbb{C}(z)$*.*

References

[1] J. Ax, *On Schanuel's conjectures*, Ann. of Math. 2, **93**(2) (1971), 252–278.
[2] R. Jaoui and J. Kirby, *Integration in finite terms and exponentially algebraic functions*, preprint 2025, arxiv:2510.26248.
[3] J. Kirby, *Blurred complex exponentiation*, Selecta Math. (N.S.), **25**(2) (2019), paper no.:72, 15.

On the elementary theory of the real exponential field

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(joint work with Alessandro Berarducci)

Tarski's seminal paper on the model theory of fields [8] contains a proof that the first-order theory of the real field is decidable, that is, there is an algorithm that given a sentence ϕ in the language of ordered rings determines whether ϕ holds in $\mathbb{R}$. At the end of the paper, Tarski asks whether the result can be extended to the real exponential field $\mathbb{R}_{\exp}$: whether a similar algorithm exists for formulas in the language of ordered exponential rings. A famous theorem of Macintyre and Wilkie gives a conditional answer to this question [5]. Recall that *Schanuel's Conjecture for* $\mathbb{R}$, which I will abbreviate here by $(\mathrm{SC}_{\mathbb{R}})$, is the statement that given $\mathbb{Q}$-linearly independent real numbers $x_1, \ldots, x_n$ the transcendence degree of $(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$ is at least n. Like many transcendence statements, $(\mathrm{SC}_{\mathbb{R}})$ is an open problem, generally considered out of reach.

Theorem 1 (Macintyre-Wilkie)**.** $(SC_{\mathbb{R}})$ *implies that the theory of the real exponential field is decidable.*

Building on Tarski's work on the real field, the notion of *o-minimal structure* was introduced in the 1980s. In one of the first papers on the subject, Pillay and Steinhorn [6] prove the following.

Theorem 2 (Pillay-Steinhorn)**.** *Let* $(R, <)$ *be an ordered field. The following are equivalent:*

(1) R is a real closed field.
(2) R is o-minimal: every definable subset of R is a finite union of points and intervals.
(3) R is definably complete: every definable subset of R which has an upper bound has a least upper bound.

It is then natural to ask whether Theorem 2 remains true for ordered exponential fields. Let me clarify that here by *exponential field* I mean an ordered field R equipped with a function $\exp : R \to R$ which is everywhere continuous and differentiable and satisfies $\exp' = \exp$ and $\exp(0) = 1$.

Question 4. Let R be an ordered exponential field. Are the following equivalent?

(1) R has the same first-order theory as $\mathbb{R}_{\exp}$.
(2) R is o-minimal.
(3) R is definably complete.

Some partial answers to this question are known already. That (1) implies (2) is Wilkie's major theorem on the o-minimality of the real exponential field [9]. Every o-minimal structure is definably complete, so (2) implies (3). That (3) implies (2) follows from work of Fornasiero–Servi [2] and Hieronymi [3] on definably complete expansions of fields. Our main theorem is a conditional result on the remaining implication.

Theorem 3. ($SC_{\mathbb{R}}$) *implies that every definably complete (equivalently, every o-minimal) exponential field has the same first-order theory as* $\mathbb{R}_{\exp}$.

It is easy to write down a first-order axiom scheme which axiomatizes definable completeness, so Theorem 3 can be equivalently rephrased by asserting that, under ($\mathrm{SC}_{\mathbb{R}}$), the first-order theory of definably complete ordered exponential fields is complete.

To prove Theorem 3, we work with *restricted exponential fields*. A restricted exponential field is an ordered field R equipped with a function $\exp_| : (-1,1) \to R$ which is continuous and differentiable on its domain and satisfies $\exp_|' = \exp_|$ on $(-1,1)$ and $\exp_|(0) = 1$. We denote the real restricted exponential field, $\mathbb{R}$ equipped with the restriction of the exponential to $(-1,1)$, by $\mathbb{R}_{\exp_|}$.

A result of Ressayre [7] states that if an ordered exponential field $(R, \exp)$ satisfies some axioms which broadly speaking express the fact that the exponential grows faster than any polynomial, and the corresponding restricted exponential field $(R, \exp_{|(-1,1)})$ is a model of the first-order theory of $\mathbb{R}_{\exp_|}$, then $(R, \exp)$ is a model of the first-order theory of $\mathbb{R}_{\exp}$. The growth axioms follow from the differential equation and definable completeness, so we do not need to worry about them.

Theorem 4. *The theory of definably complete restricted exponential fields is model complete*

Model completeness is an important model-theoretic property. For the reader who is not familiar with it, let me mention that a consequence of model completeness is that if $M \subseteq N$ are definably complete restricted exponential fields, then they have the same first-order theory. Theorem 4 is our main unconditional result, as it does not use ($\mathrm{SC}_{\mathbb{R}}$).

Theorem 5. ($SC_{\mathbb{R}}$) *implies that there is a definably complete restricted exponential subfield of* $\mathbb{R}_{\exp_|}$ *which embeds into any definably complete restricted exponential field.*

Model theorists will guess that the subfield of $\mathbb{R}_{\exp_|}$ in Theorem 5 is the prime model of the theory of $\mathbb{R}_{\exp_|}$, or equivalently the definable closure of the empty set. An analogous result in the unrestricted case was proved by Krapp [4].

Once we have used Ressayre's work to reduce our main question to a question about restricted exponential fields, Theorem 3 follows easily from Theorems 4 and 5: given any model $(M, \exp)$ of the theory of definably complete exponential fields, by Theorem 5 the restricted exponential field $(M, \exp_|)$ contains a submodel which has the same first-order theory as $\mathbb{R}_{\exp_|}$, and their first-order theories must then coincide by Theorem 4.

I conclude with an open question. Our proof of model completeness uses in a crucial way the fact that certain functions definable in definably complete restricted exponential fields are *polynomially bounded*. Recall that if M is an o-minimal structure, a definable function $f : M \to M$ is polynomially bounded if there is $n \in \mathbb{N}$ such that for all sufficiently large $t \in M$ we have $|f(t)| < t^n$; the structure M is polynomially bounded if every definable function $M \to M$ is polynomially bounded.

It follows from the theory of real subanalytic sets that the real restricted exponential field $\mathbb{R}_{\exp_|}$ is polynomially bounded; this was an important ingredient in Wilkie's proof of the model completeness and o-minimality of the theory of $\mathbb{R}_{\exp}$. We do not know whether every definably complete restricted exponential field is polynomially bounded, and in our proof we have to use a functional transcendence result due to Ax [1] to obtain the polynomial bounds on the specific definable functions we work with. Our results imply that under $(\mathrm{SC}_{\mathbb{R}})$ every definably complete restricted exponential field is polynomially bounded, but it seems like this should not depend on Schanuel.

Question 5. Is every definably complete restricted exponential field polynomially bounded?

References

[1] J. Ax, *On Schanuel's conjectures*, Ann. of Math. 2, **93**(2) (1971), 252–278.

[2] A. Fornasiero and T. Servi, *Definably complete Baire structures*, Fund. Math., **209**(3) (2010), 215–241.

[3] P. Hieronymi, An analogue of the Baire category theorem, J. Symb. Log., **78**(1) (2011), 207–213.

[4] L. S. Krapp, *Embedding the prime model of real exponentiation into o-minimal exponential fields*, Bull. Lon. Math. Soc., **56**(3) (2023), 907–913.

[5] A. Macintyre and A. J. Wilkie, *On the decidability of the real exponential field*, in: Kreiseliana: About and around Georg Kreisel (ed. P. Odifreddi), A K Peters (1996), 441–467.

[6] A. Pillay and C. Steinhorn, *Definable sets in ordered structures. I*, Trans. Amer. Math. Soc., **295**(2) (1986), 565–592.

[7] J.-P. Ressayre, *Integer parts of real closed exponential fields*, extended abstract in Arithmetic, proof theory, and computational complexity, (ed. P. Clote and J. Krajicek) Oxford University Press (1993), pp. 278–288.

[8] A. Tarski, *A decision method for elementary algebra and geometry*, prepared for publication with the assistance of J. C. C. McKinsey. University of California Press (1951).

[9] A. J. Wilkie, *Model Completeness Results for Expansions of the Ordered Field of Real Numbers by Restricted Pfaffian Functions and the Exponential Function*, J. Amer. Math. Soc., **9**(4) (1996), 1051–1094.

Transcendence questions for asymptotic series

Peter Jossen

It is a recurring phantasy, as old as the invention of p-adic numbers itself, to somehow define p-adic avatars of interesting complex numbers, and then deduce their irrationality or transcendence by p-adic means. Naively, to define such a p-adic avatar of a real number of interest, one could try to represent it by a nice series of rational numbers, and sum the same series in $\mathbb{Q}_p$. One must however be restrictive in the type of series; given an element in every completion of $\mathbb{Q}$ there exists a series of rational numbers that converges to these given elements in each completion with but one exception. A series that converges in all completions must be a finite sum. One also realises quickly that the nicest series representations of interesting real numbers, such as for example

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \qquad\qquad e = \sum_{n=0}^{\infty}\frac{1}{n!}$$

diverge p-adically for all p. In contrast, certain series that diverge in the real numbers such as

$$\sum_{n=0}^{\infty}(-1)^n n!$$

converge in $\mathbb{Q}_p$ for all primes p, and one suspects that the p-adic sum is indeed transcendental for every p. In 1760, Euler studied this diverging series and attributed to it the real value

$$\delta = \int_0^{\infty}\frac{e^{-z}}{z+1}dz = 0.596347362323194074 34\ldots$$

by a process that we now call Borel summation. The number δ is called the *Gompertz constant*. It is clear that δ is an exponential period, and the period conjecture stipulates that δ is transcendental.

In my talk, I consider a special kind of Borel-summable series where it seems that transcendence or rationality of the Borel summation is matched exactly by transcendence or rationality of the converging p-adic sum. The series in question are evaluations of so-called $\mathfrak{Z}$-functions, closely related to G functions [1, 2, 3]. For every G-function $G(z)$, say with rational coefficients, there is a corresponding $\mathfrak{Z}$-function $\mathfrak{f}(z)$ related to each other by

$$G(z) = \sum_{n=0}^{\infty} a_n z^n \qquad\qquad \mathfrak{f}(z) = \sum_{n=0}^{\infty} a_n n! z^n$$

and the diverging series we wish to evaluate is that for $\mathfrak{f}(1)$. In the complex numbers we may follow Euler by setting

$$\mathfrak{f}(1)_\gamma = \int_\gamma G(z)e^{-z}dz \tag{1}$$

where $\gamma : (0, \infty) \to \mathbb{C}$ is a path with limits $\gamma(0) = 0$ and $\gamma(+\infty) = +\infty$ and which avoids the finitely many singularities of a differential equation solved by G, so that we may analytically continue G along γ. The integral converges, but depends on the choice of γ. One expects however that if $\mathfrak{f}(1)_\gamma$ is rational then any other integration path yields the same rational number. The series

$$\mathfrak{f}(1)_p = \sum_{n=0}^{\infty} a_n n! \tag{2}$$

in $\mathbb{Q}_p$ may diverge for some primes. It is known that (2) converges for infinitely many primes, and if G comes from geometry, then this series converges for all but finitely many p. A simple computation shows that if $G(z) = H(z) - H'(z)$ holds for some G-function $H(z)$, then the quantities (1) and (2) are all equal to the rational number $H(0)$. Expanding on a conjecture of Ferguson [5], we may ask whether it is true that $G(z)$ is of this form if any of the quantities (1) for one path γ or (2) for one prime p is rational.

Experimenting with concrete examples, several equidistribution phenomena can be observed in this context. Let $\mathfrak{f}(z)$ be an $\mathfrak{Diff}$-function as above with corresponding G-function $G(z)$. If G comes from geometry, which is of course the case in all examples, we can prove that the valuation $v_p(\mathfrak{f}(1)_p)$ is bounded below independently of p. We may thus write

$$\mathfrak{f}(1)_p = \sum_{m \geq M} c_p(m) p^{-m}$$

with $c_p(m) \in \{0, 1, \ldots, p-1\}$ and some fixed integer M. Pick any $m \geq M$. A first observation is the following. As p varies, the distribution of the points $p^{-1}c_p(m)$ on the unit interval[1] seems to be composed of a uniform part, and a discrete part supported at rational points. Concretely, that means that there exist rational numbers $t_1, \ldots, t_d \in [0, 1)$ and positive rational numbers $w_0, w_1, \ldots, w_d$ with $w_0 + w_1 + \cdots + w_d = 1$, all depending on $\mathfrak{f}$ and m, such that for every continuous function $\varphi : [0, 1] \to \mathbb{C}$, the equality

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} \varphi\Big(\frac{c_p(m)}{p}\Big) = w_0 \int_0^1 \varphi(t) dt + \sum_{i=1}^{d} w_i \varphi(t_i)$$

holds. Trigonometric polynomials are dense in continuous functions, so it suffices to test this for $\varphi(t) = e^{2\pi i n t}$ with $n \in \mathbb{Z}$. The second observation is that the quantities (1) and (2) are rational if and only if the distribution is discrete, meaning $w_0 = 0$, for all $m \geq M$. If (1) and (2) are not rational, then $w_0 = 1$ for sufficiently big m. Several folklore conjectures about the apparent randomness of expressions involving factorials modulo p can be recovered from this. For instance, equidistribution for the G-function $z^{-1}\mathrm{Li}_2(z)$ and $m = -1$ has to do with equidistribution of Wilson quotients which is at the base of the heuristic that there exist very few but still infinitely many Wilson primes; there should be about $\log\log(x)$ below x, but so far only three have been found.

[1] or on the circle $\mathbb{R}/\mathbb{Z}$, we have secretly chosen an additive character on $\mathbb{F}_p$.

References

[1] Y. André, *G-functions and geometry*, Aspects of Math., E13, Vieweg & Sohn, Braunschweig, 1989.

[2] Y. André, *Séries Gevrey de type arithmétique, I. Théorèmes de pureté et de dualité*, Ann. of math. vol. 151 no. 2, p. 705-740, 2000

[3] B. Dwork, G. Gerotto and F. Sullivan *An introduction to G-functions*, Ann of math. studies 133, Princeton U. Press, New Jersey, 1994.

[4] L. Euler *De series divergentibus*, Novi Commentarii acad. sci. Petrop., Vol 5, pp. 205-237, Transl. in J. Number Th. vol. 229 p. 168–178, 2021

[5] T. Ferguson *Algebraic properties of* $\mathfrak{Z}$*-functions*, J. Number Th. vol. 229 p. 168–178, 2021

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