Rend. Sem. Mat. Univ. Padova 156 (2026), 19–44
DOI 10.4171/RSMUP/174

Non-linear microlocal cut-off functors

Bingyu Zhang (*)

Abstract – To any conic closed set of a cotangent bundle, one can associate four functors on the category of sheaves, which are called non-linear microlocal cut-off functors. Here we explain their relation with the microlocal cut-off functor defined by Kashiwara and Schapira, and prove a microlocal cut-off lemma for non-linear microlocal cut-off functors, adapting inputs from symplectic geometry. We also prove two Künneth formulas and a functor classification result for categories of sheaves with microsupport conditions.

Mathematics Subject Classification 2020 – 35A27 (primary); 53D35 (secondary).

Keywords – microlocal cut-off lemmas, sheaf quantization.

Introduction

In the study of microlocal theory of sheaves, a crucial tool is the microlocal cut-off lemma of Kashiwara and Schapira (see [23, Proposition 5.2.3, Lemma 6.1.5], [9] and [13, Chapter III]), which constructs certain functors that enable us to cut off the microsupport of sheaves functorially. On the other hand, in the study of symplectic topology, very similar functors were defined and studied for different purposes [26, 34, 41].

Precisely, for a smooth manifold M, a conic closed set $Z \subset T^*M$, and $U = T^*M \setminus Z$, we will study the *non-linear microlocal cut-off functors* $L_U, L_Z, R_U, R_Z : \mathrm{Sh}(M) \to \mathrm{Sh}(M)$, together with fiber sequences of functors $L_U \to \mathrm{id} \to L_Z$, $R_Z \to \mathrm{id} \to R_U$ defined using adjoint data of the split Verdier sequence $\mathrm{Sh}_Z(M) \to \mathrm{Sh}(M) \to \mathrm{Sh}(M; U)$. See Section 3 for more details.

The main goal of this article is proving a non-linear microlocal cut-off lemma using the wrapping formula of L_Z introduced in [26].

(*) *Indirizzo dell'A.*: Centre for Quantum Mathematics, University of Southern Denmark, Campusvej 55, 5230 Odense M, Denmark; bingyuzhang@imada.sdu.dk

THEOREM A (Non-linear microlocal cut-off lemma, Theorem 3.6 below). *For a conic closed set $Z \subset T^*M$, and $U = T^*M \setminus Z$, we have the following:*

(1) (a) *The morphisms $F \to L_Z(F)$ and $R_Z(F) \to F$ are isomorphisms if and only if* $\mathrm{SS}(F) \subset Z$.

(b) *The morphism $L_U(F) \to F$ is an isomorphism if and only if $F \in {}^{\perp}\mathrm{Sh}_Z(M)$, and the morphism $F \to R_U(F)$ is an isomorphism if and only if $F \in \mathrm{Sh}_Z(M)^{\perp}$.*

(2) (a) *The morphisms $L_U(F) \to F$ and $F \to R_U(F)$ are isomorphisms on U.*

(b) *If $0_M \subset Z$, the morphisms $F \to L_Z(F)$ and $R_Z(F) \to F$ are isomorphisms on* $\mathrm{Int}(Z) \setminus 0_M$. *Equivalently, we have* $\mathrm{SS}(L_U(F)) \cup \mathrm{SS}(R_U(F)) \subset \overline{U} \cup 0_M$.

In fact, all of them except (2) (b) follow directly from the definition. Only (2) (b) of the theorem is a non-trivial fact.

Here we shall explain the relation between non-linear microlocal cut-off functors and microlocal cut-off functors in [23]. We follow the notation of [13, Section III.1] for microlocal cut-off functors.

We take $M = V$ to be a real vector space, $\lambda, \gamma \subset V$ as pointed closed convex cones (we assume further that λ is proper). The comparison is summarized in Table 1, where the first column marks corresponding references, and the last column marks their position in this article. Functors in the same columns are isomorphic with corresponding U and Z.

This article	Z	U	L_U	L_Z	R_U	R_Z	Definition 3.4
[13, Section III.1]	$V \times \lambda^{\circ a}$		P'_λ	Q_λ	Q'_λ	P_λ	Proposition 5.1
[15, Section 3]		$V \times \mathrm{Int}(\gamma^\circ)$	L_γ		R_γ		Example 5.3

TABLE 1. Known microlocal cut-off functors.

Based on this comparison, we see the relation between linear microlocal cut-off lemmas [23, Proposition 5.2.3, Lemma 6.1.5], [15, Propositions 3.17, 3.19, 3.20, 3.21] and our result. However, let us emphasize that our result is not completely irrelevant to those linear cut-off lemmas: In fact, to derive the wrapping formula of L_Z, we somehow use the 1-dimensional linear cut-off lemma for $\lambda = (-\infty, 0]$. Therefore, one could say that Theorem A constructs non-linear microlocal cut-off functors, generalizing the linear microlocal cut-off functor which used the vector space structures, but it also provides a unified statement for all microlocal cut-off functors. For example, in Example 6.3, we will explain how to prove a cut-off lemma for the Tamarkin categories and its equivariant version.

We remark that our approach aims to explain the microlocal cut-off lemmas as globally as possible on T^*M using input from symplectic topology. Therefore, we did not consider the refined microlocal cut-off lemmas (see [23, Propositions 6.1.4, 6.1.6], [9], and [13, Sections III.2, III.3]), which concern more the local behavior of cut-off functors.

We also prove two Künneth formulas in Section 7 (with more precise statements), which remove the isotropic condition of [27, Theorems 1.1, 1.2] via a different approach.

THEOREM B. *For two manifolds M, N, conic closed sets $Z \subset T^*M$ and $X \subset T^*N$, and we set $U = T^*M \setminus Z$ and $V = T^*N \setminus X$. We have*

$$\mathrm{Sh}_{X\times Z}(N \times M) \simeq \mathrm{Sh}_X(N) \otimes \mathrm{Sh}_Z(M),$$
$$\mathrm{Sh}(N; V) \otimes \mathrm{Sh}(M; U) \simeq \mathrm{Sh}(N \times M; V \times U).$$

Combining the dualizability of $\mathrm{Sh}_Z(M)$ and $\mathrm{Sh}(M;U)$ via [28, Remark 3.7], we have the following functor classification result:

$$\begin{aligned} \mathrm{Sh}_{-X\times Z}(N \times M) &\simeq \mathrm{Fun}^L(\mathrm{Sh}_X(N), \\ &\qquad \mathrm{Sh}_Z(M)) \simeq \mathrm{Fun}^R(\mathrm{Sh}_Z(M), \mathrm{Sh}_X(N))^{\mathrm{op}}, \\ \mathrm{Sh}(N \times M; -V \times U) &\simeq \mathrm{Fun}^L(\mathrm{Sh}(N; V), \\ &\qquad \mathrm{Sh}(M; U)) \simeq \mathrm{Fun}^R(\mathrm{Sh}(M; U), \mathrm{Sh}(N; V))^{\mathrm{op}}. \end{aligned} \tag{0.1}$$

Category convention. In this article, a category means an ∞-category. We refer to [11, Chapter 1] for basics about higher algebra, and more details can be found in [30–32].

In this article, we shall consider a fixed presentable stable symmetric monoidal category $(\mathbf{k}, \otimes, 1)$, and we denote by $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ the category of $\mathbf{k}$-linear presentable stable categories and left adjoint $\mathbf{k}$-linear functors. There exists a $\mathbf{k}$-relative Lurie tensor product that forms a closed symmetric monoidal structure on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ such that the category of left adjoint functors Fun^L serves as the internal hom, where we write $\otimes/\mathrm{Fun}^L$ directly instead of $\otimes_{\mathbf{k}}/\mathrm{Fun}^L_{\mathbf{k}}$. We say a $\mathbf{k}$-linear presentable stable category is dualizable if it is a dualizable object in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

For purpose of this paper, we will restrict $(\mathbf{k}, \otimes, 1)$ to be compactly generated, and be locally rigid in the sense of [25, Definition 4.2.1, Corollary 4.3.5]. We refer to [25, Sections 4.2, 4.3, 4.4] and [37] for more discussion on local rigidity. The standard example for $(\mathbf{k}, \otimes, 1)$ would be the category of modules Mod_E over an $\mathbb{E}_\infty$ ring spectrum E. The compactly generation condition guarantees the microlocal sheaf theory with $\mathbf{k}$-coefficient to run without any modification, as in [23]. The local rigidity condition ensures that plain adjoints of $\mathbf{k}$-linear functors are also $\mathbf{k}$-linear ([25, Lemma 4.3.7]). Later, we will discuss in more detail the dependence of our results on the requirement. But let us remark here the following:

⚠ Upon proving [23, Proposition 5.4.13, Theorem 7.2.1] using the so-called Ω-lens definition (i.e. the equivalent of terms (2) and (3) of Lemma 2.2), then all the results of this paper are true if $\mathbf{k}$ is locally rigid but not necessarily compactly generated.

1. Sheaves and integral kernel

For a locally compact Hausdorff topological space X, we set $\mathrm{Sh}(X;\mathcal{C})$ to be the category of $\mathcal{C}$-valued sheaves [30, Definition 7.3.3.1] for a presentable symmetric monoidal category, which is also $\mathcal{C}$-linear. It is explained in [43] that since $\mathcal{C}$ is a symmetric monoidal category, we can define the $\mathcal{C}$-linear 6-functor formalism. We also refer to [39] for the 6-functor formalism. In this paper, we are mostly interested in the case that $\mathcal{C} = \mathbf{k}$, and in this case, we simply denote $\mathrm{Sh}(X) = \mathrm{Sh}(X;\mathbf{k})$. In this case, it is explained in [25, Lemma 4.6.17] that local rigidity can guarantee that the Verdier dual functor behaves as expected.

For a locally closed inclusion $i\colon Z \subset X$ and $F \in \mathrm{Sh}(X)$, we set $F_Z = i_!i^{-1}F$ and $\Gamma_Z F = i_*i^!F$. We denote the constant sheaf $1_X = a^*1$, where a is the constant map $X \to \mathrm{pt}$, and denote $1_Z = (1_X)_Z \in \mathrm{Sh}(X)$ if there is no confusion.

The main reason for using the ∞-category is the following classification of left adjoint functors, which is a combination of a sequence of Lurie's results. We refer to [28, Proposition 3.2] for a proof.

Proposition 1.1. *If H_1 is a locally compact Hausdorff space, then for all topological spaces H_2, we have the equivalence of categories*

$$\mathrm{Sh}(H_1 \times H_2) \simeq \mathrm{Fun}^L(\mathrm{Sh}(H_1), \mathrm{Sh}(H_2))$$

that sends $K \in \mathrm{Sh}(H_1 \times H_2)$ to the convolution functor $\Phi_K = [F \mapsto p_{2!}(K \otimes p_1^* F)]$, where $p_i\colon H_1 \times H_2 \to H_i$ are projections. We call K the kernel of Φ_K.

It is known that taking the right adjoint induces an equivalence of categories

$$\mathrm{Fun}^L(\mathrm{Sh}(H_1), \mathrm{Sh}(H_2)) \simeq \mathrm{Fun}^R(\mathrm{Sh}(H_2), \mathrm{Sh}(H_1))^{\mathrm{op}},$$

where Fun^R stands for right adjoint functors.

However, for any convolution functor $\Phi_K\colon \mathrm{Sh}(H_1) \to \mathrm{Sh}(H_2)$ with $K \in \mathrm{Sh}(H_1 \times H_2)$, there exists an obvious right adjoint functor[1]

$$\Psi_K\colon \mathrm{Sh}(H_2) \to \mathrm{Sh}(H_1), \quad F \mapsto p_{1*}\mathcal{H}om(K, p_2^! F).$$

(1) We call it an nvolution functor since it is a dual of a convolution functor, while coconvolution should be the same as nvolution.

Then we have the following equivalence of categories:

COROLLARY 1.2. *Under the same condition as Proposition 1.1, we have*

$$\mathrm{Sh}(H_1 \times H_2) \simeq \mathrm{Fun}^R(\mathrm{Sh}(H_2), \mathrm{Sh}(H_1))^{\mathrm{op}}, \quad K \mapsto \Psi_K.$$

REMARK 1.3. Proposition 1.1 and Corollary 1.2 are only true on the ∞-category level. In the classical theory of microlocal sheaves, many functors are built as triangulated convolution (nvolution) functors, which do not admit similar results. In particular, we do not know whether the kernel of triangulated convolution (nvolution) functors is unique in general.

2. Microsupport of sheaves

From now on, we assume M is a smooth manifold. Regarding the microlocal theory of sheaves in the ∞-categorical setup, we remark that all arguments of [23] work well provided we have the non-characteristic deformation lemma [23, Proposition 2.7.2] for all sheaves. When $\mathbf{k}$ is compactly generated, the non-characteristic deformation lemma is proven for all hypersheaves ([38]), and then for all sheaves because of hypercompleteness for manifolds ([30, Theorem 7.2.3.6, Corollary 7.2.1.12], and [19, Section 1]).

To any object $F \in \mathrm{Sh}(M)$, one can associate a conic closed set $\mathrm{SS}(F) \subset T^*M$ ([23, Definition 5.1.2]), which satisfies the following triangle inequality: for a fiber sequence $F \to G \to H$, we have $\mathrm{SS}(F) \subset \mathrm{SS}(G) \cup \mathrm{SS}(H)$.

We recall the following definition and result regarding an equivalent definition of microsupport.

DEFINITION 2.1 ([17, Definition 3.1]). Let $\Omega \subset T^*M \setminus 0_M$ be an open conic subset. We call Ω-*lens* a locally closed subset Σ of M with the following properties: $\bar{\Sigma}$ is compact and there exist an open neighborhood U of $\bar{\Sigma}$ and a function $g: U \times [0,1] \to \mathbb{R}$ such that

(1) $dg_t(x) \in \Omega$ for all $(x,t) \in U \times [0,1]$, where $g_t = g|_{U \times \{t\}}$,

(2) $\{g_t < 0\} \subset \{g_{t'} < 0\}$ if $t \leq t'$,

(3) the hypersurfaces $\{g_t = 0\}$ coincide on $U \setminus \bar{\Sigma}$,

(4) $\Sigma = \{g_1 < 0\} \setminus \{g_0 < 0\}$.

LEMMA 2.2 ([17, Lemmas 3.2, 3.3]). *Let $F \in \mathrm{Sh}(M)$ and let $\Omega \subset T^*M \setminus 0_M$ be an open conic subset. Then following are equivalent:*

(1) $\mathrm{SS}(F) \cap \Omega = \varnothing$,

(2) $\mathrm{Hom}(1_\Sigma, F) \simeq 0$ *for any* Ω*-lens* Σ,

(3) $\Gamma(M; 1_\Sigma \otimes F) \simeq 0$ *for any* $(-\Omega)$*-lens* Σ.

Remark 2.3. The proof of this lemma needs the non-characteristic deformation lemma ([23, Proposition 2.7.2], [38]), which is not true if **k** is not compactly generated [10, Remark 4.25].

The lemma motivates a new definition[2] of microsupport using Ω-lens and the equivalent conditions (2) or (3), which then avoids the use of non-characteristic deformation. The definition works very well without **k** being compactly generated, and it is very likely equivalent to the one proposed in [10, Remark 4.24]. One can expect that many results in microlocal sheaf theory will still be true with this definition (in the case that **k** is not compactly generated). Nevertheless, we will not discuss the extension of the definition of microsupport further in this article, and restrict ourselves to the compactly generation case for safety.

The lemma leads to the following property of microsupports, where the microlocal cut-off lemma in [23, Section 5.2] is not involved.

Proposition 2.4 ([23, Exercise V.7]). *For a set of sheaves* $F_\alpha \in \mathrm{Sh}(M)$ *indexed by* $\alpha \in A$*, we have*

$$SS\Big(\prod_\alpha F_\alpha\Big) \cup SS\Big(\bigoplus_\alpha F_\alpha\Big) \subset \overline{\bigcup_\alpha SS(F_\alpha)}.$$

For $X_n \subset X$ with $n \in \mathbb{N}$, we set

$$\begin{aligned}\limsup_n X_n &= \bigcap_{N\geq 1} \overline{\bigcup_{n\geq N} X_n}\\ &= \big\{x : \exists (x_n) \text{ such that } x_n \in X_n \text{ for infinitely many } n,\ x_n \to x\big\},\\ \liminf_n X_n &= \big\{x : \exists (x_n) \text{ such that } x_n \in X_n \text{ for all } n,\ x_n \to x\big\}.\end{aligned}$$

By definition, we have $\liminf_n X_n \subset \limsup_n X_n$. As a corollary of Proposition 2.4, we have the following corollary.

Corollary 2.5 ([17, Proposition 6.26]). *Denote by* $N(\mathbb{N})$ *the nerve of the* 1*-category* $\mathbb{N}$*. For a functor* $N(\mathbb{N}) \to \mathrm{Sh}(X)$*, we have*

$$SS\left(\varinjlim_n F_n\right) \subset \liminf_n SS(F_n).$$

(2) The author learned this idea from Stéphane Guillermou and Marco Volpe.

REMARK 2.6. We can also denote by $\mathbb{N}$ a simplicial set which consists of all vertices, together with the edges that join consecutive integers, and no higher simplexes. Then the natural inclusion of simplicial sets $\mathbb{N} \to N(\mathbb{N})$ is cofinal. Therefore, we can compute the colimit using a mapping telescope construction as in triangulated categories, and the proof of Corollary 2.5 is the same as in the references.

For our later application, we present its proof here.

PROOF OF COROLLARY 2.5. By Remark 2.6, and the fact that all strictly increasing sequences are cofinal in $\mathbb{N}$, we have cofiber sequences for all strictly increasing sequences $\{n_i\}_{i\in\mathbb{N}}$,

$$\bigoplus_i F_{n_i} \to \bigoplus_i F_{n_i} \to \varinjlim_n F_n.$$

We first take $n_i = i + N$ for all $N \geq 1$, and then we have $SS(\varinjlim_n F_n) \subset \limsup_n SS(F_n)$ by Proposition 2.4 and the triangle inequality.

Now take $x \notin \liminf_n SS(F_n)$. By the definition of $\liminf_n$, we can find a strictly increasing sequence n_i such that $x \notin \limsup_i SS(F_{n_i})$. Then we have $x \notin SS(\varinjlim_i F_{n_i}) = SS(\varinjlim_n F_n)$. That is, $SS(\varinjlim_n F_n) \subset \liminf_n SS(F_n)$. ∎

Another application of Lemma 2.2 is the following property, which could be understood as a version of 1-dimensional microlocal cut-off lemma. We denote $i_s\colon M \to M \times \mathbb{R}$, $x \mapsto (x, s)$.

PROPOSITION 2.7. *For $F \in Sh(M \times \mathbb{R})$, if $SS(F) \subset T^*M \times \mathbb{R} \times (-\infty, 0]$, then for $a \leq b$ there exists a morphism $c(a,b)\colon i_a^* F \to i_b^* F$.*

PROOF. Let $\Omega = T^*M \times \mathbb{R} \times (0, \infty)$. For any relatively compact open set $U \subset M$, we take an Ω-lens $\Sigma = U \times [y, z)$ for $y < z$. Therefore, by Lemma 2.2, the following morphism is naturally an equivalence for $x < y < z$:

$$F(U \times (x,z)) \xrightarrow{\simeq} F(U \times (x,y)).$$

For general U, the equivalence is still true: We take an open cover $\{U_\alpha\}_\alpha$ of U, and then we use $F(U \times (y,z)) = \varprojlim_\alpha F(U_\alpha \times (y,z))$ since F is a sheaf.

Then we can define a morphism for open sets $U \subset M$, which is natural with respect to U:

$$F(U \times (a-\varepsilon, a+\varepsilon)) \simeq F(U \times (a-\varepsilon, b+\varepsilon)) \to F(U \times (b-\varepsilon, b+\varepsilon)).$$

Passing to $\varinjlim_{\varepsilon\to 0}$, we define a morphism of presheaves

$$c^{\mathrm{pre}}(a,b)\colon i_a^{*,\mathrm{pre}}F \to i_b^{*,\mathrm{pre}}F,$$

where we denote by $i_s^{*,\mathrm{pre}}$ the presheaf pullback.

Then we define $c(a,b)$ as the sheafification of $c^{\mathrm{pre}}(a,b)$. ■

Lastly, we need the following microsupport estimation of (co)nvolution functors:

PROPOSITION 2.8 ([27, Lemma 4.4, Proposition 4.5]). *Let $K \in \mathrm{Sh}(N \times M)$ with $\mathrm{SS}(K) \subset (-X) \times Z$ for conic closed sets $X \subset T^*N$, $Z \subset T^*M$. Then we have, for $F \in \mathrm{Sh}(N)$,*

$$\mathrm{SS}(\Phi_K(F)) \subset Z, \quad \mathrm{SS}(\Psi_K(F)) \subset X.$$

Let us also give a proof of this proposition using an Ω-lens.

PROOF. We assume that X and Z contain the zero section; the general case can be proven using $\mathrm{SS}(F) \cap 0_M = \mathrm{supp}(F)$. By Lemma 2.2, we only need to show that for a $-(T^*M \setminus Z)$-lens Σ, we have

$$\Gamma(M, 1_\Sigma \otimes \Phi_K(F)) = 0.$$

By definition of an Ω-lens, the closed set $\overline{\Sigma}$ is compact. Then we can replace $\Gamma = p_{2*}$ by $\Gamma_c = p_{2!}$. Then we use the base change formula and projection formula to see that we have

$$\Gamma(M, 1_\Sigma \otimes \Phi_K(F)) = \Gamma_c(N \times M, (F \boxtimes 1_\Sigma) \otimes K).$$

Notice that $F = \varinjlim_\alpha 1_{W_\alpha}$ for relatively compact open sets W_α by [11, Proposition 5.4.5] and the definition of sheaves. We have

$$\Gamma(M, 1_\Sigma \otimes \Phi_K(F)) = \varinjlim_\alpha \Gamma(N \times M, (1_{W_\alpha \times \Sigma}) \otimes K).$$

Therefore, noticing that

$$W_\alpha \times \Sigma$$

is a $T^*N \times (-(T^*M \setminus Z)) = -T^*N \times (T^*M \setminus Z)$-lens, we can conclude by Lemma 2.2 that

$$\Gamma(N \times M, (1_{W_\alpha \times \Sigma}) \otimes K) = 0$$

since $\mathrm{SS}(K) \subset (-X) \times Z \subset T^*N \times Z$.

The statement about $\Psi_K(F)$ can be proven using Lemma 2.2 (2) and a similar argument. We leave the details to the readers. ■

3. Split Verdier sequence from microlocalization

Throughout this article, we set $Z \subset T^*M$ to be a conic closed set and $U = T^*M \setminus Z$ (which is a conic open set). We set $\mathrm{Sh}_Z(M)$ as the full subcategory of $\mathrm{Sh}(M)$ spanned by sheaves F with $\mathrm{SS}(F) \subset Z$.

Proposition 3.1. *The category* $\mathrm{Sh}_Z(M)$ *is a stable subcategory of* $\mathrm{Sh}(M)$ *closed under small limits and colimits.*

In particular, (1) *the inclusion* $\iota\colon \mathrm{Sh}_Z(M) \to \mathrm{Sh}(M)$ *admits both left and right adjoints, and* (2) $\mathrm{Sh}_Z(M)$ *is* $\mathbf{k}$*-linear.*

Proof. By the triangle inequality of microsupport, we only need to show that $\mathrm{Sh}_Z(M)$ is closed under small products and coproducts, which follows from Proposition 2.4.

The inclusion ι admits both adjoints by the adjoint functor theorem [30, Corollary 5.5.2.9]; $\mathrm{Sh}_Z(M)$ is presentable and $\mathbf{k}$-linear by [36] since ι is reflective and $\mathbf{k}$ is locally rigid. ∎

Remark 3.2. (1) This is the only place in the article that we need to use the local rigidity assumption of $\mathbf{k}$ explicitly.

(2) Another way to formulate Lemma 2.2 is that the left (right) semi-orthogonal complement $\mathrm{Sh}_Z(M)$ is generated by 1_Σ for $T^*M \setminus Z$-lens ($-(T^*M \setminus Z)$-lens) under colimits (limits). This observation was also provided in [22], where condition (3) of Lemma 2.2 was formulated in terms of the dual of 1_Σ. It does not make a significant difference as we can assume the lens Σ is small enough that we can orient a small neighborhood of Σ.

Definition 3.3 ([23, Definition 6.1.1]). For the conic open set $U \subset T^*M$, we set

$$\mathrm{Sh}(M;U) := \mathrm{Sh}(M)/\,\mathrm{Sh}_Z(M).$$

We refer to [6, Section 5], [35, Section 1.3] and [7, Appendix A] for basic results on Verdier quotients of stable categories. Here we say that $\mathcal{C} \xrightarrow{\iota} \mathcal{D} \xrightarrow{j} \mathcal{D}/\mathcal{C}$ is a *split Verdier sequence* if ι admits both left and right adjoints; see [7, Definition A.2.4, Corollary A.2.6]. We denote by ${}^{\perp}\mathcal{C}$ (resp. $\mathcal{C}^{\perp}$) the left (resp. right) semi-orthogonal complement of $\mathcal{C}$ in $\mathcal{D}$, and we can identify ${}^{\perp}\mathcal{C}$ (resp. $\mathcal{C}^{\perp}$) with $\mathcal{D}/\mathcal{C}$ via a left (resp. right) adjoint functor in the split case.

By Proposition 3.1, the inclusion $\iota\colon \mathrm{Sh}_Z(M) \to \mathrm{Sh}(M)$ defines a split Verdier sequence. Precisely, we have the following diagram of functors:

$$\begin{tikzcd} \mathrm{Sh}_Z(M) \arrow[r, "\iota"] & \mathrm{Sh}(M) \arrow[l, bend right, "\iota^*"'] \arrow[l, bend left, "\iota^!"] \arrow[r, "j"] & \mathrm{Sh}(M;U). \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{3.1}$$

Then we have adjunction pairs

$$L_U := j_! j \dashv j_* j =: R_U, \quad L_Z := \iota\iota^* \dashv \iota\iota^! =: R_Z,$$

and the units/counits give us the following fiber sequences of functors on $\mathrm{Sh}(M)$:

$$L_U \to \mathrm{id} \to L_Z, \quad R_Z \to \mathrm{id} \to R_U. \tag{3.2}$$

By Proposition 1.1 and Corollary 1.2, there exists a fiber sequence in $\mathrm{Sh}(M \times M)$,

$$K_U \to 1_{\Delta_M} \to K_Z \tag{3.3}$$

that gives corresponding convolution/nvolution functors

$$L_U = \Phi_{K_U}, \quad L_Z = \Phi_{K_Z}, \quad R_U = \Psi_{K_U}, \quad R_Z = \Psi_{K_Z}$$

and corresponding natural transformations. Therefore, in practice, to construct those functors and the fiber sequences between them, we only need to write down the fiber sequence (3.3).

Definition 3.4. We call the functors L_U, L_Z, R_U, R_Z the *non-linear microlocal cut-off functors*. The corresponding kernels K_U, K_Z are called microlocal cut-off kernels (or microlocal kernels for short).

For any object in $\mathrm{Sh}(M;U)$, the triangle inequality implies that $SS_U([F]) := SS(F) \cap U$ for any representative $F \in \mathrm{Sh}(M)$ is well defined. Then we have $SS_U([F]) = SS(R_U(F)) \cap U = SS(L_U(F)) \cap U$ for all $F \in \mathrm{Sh}(M)$. For conic closed sets $X \supset Z$, we consider the category $\frac{\mathrm{Sh}_X(M)}{\mathrm{Sh}_Z(M)}$ as a full subcategory of $\mathrm{Sh}(M;U)$. Then the essential image of $\frac{\mathrm{Sh}_X(M)}{\mathrm{Sh}_Z(M)}$ in $\mathrm{Sh}(M;U)$ is $\mathrm{Sh}_{X\cap U}(M;U)$, the full subcategory spanned by objects with $SS_U([F]) \subset X \cap U$. We will recall this construction in Section 6.

Example 3.5. Take an open set $W \subset M$, and set $U = T^*W \subset T^*M$ with $Z = T^*M \setminus T^*W$. We naturally identify $\mathrm{Sh}_Z(M)$ with sheaves supported in $M \setminus W$ since $\pi_{T^*M}(SS(F)) = \mathrm{supp}\, F$ for the cotangent projection $\pi_{T^*M}\colon T^*M \to M$, and we can verify that the restriction functor $(\bullet)|_W : \mathrm{Sh}(M) \to \mathrm{Sh}(W)$ exhibits $\mathrm{Sh}(W)$ as the Verdier quotient $\mathrm{Sh}(M; T^*W)$. Then we have

$$L_U(F) = F_W, \quad L_Z(F) = F_{M\setminus W}, \quad R_U(F) = \Gamma_W(F), \quad R_Z(F) = \Gamma_{M\setminus W}(F),$$

and can take microlocal kernels as the following:

$$K_U = 1_{\Delta_W} \to 1_{\Delta_M} \to K_Z = 1_{\Delta_{M\setminus W}}.$$

Therefore, the fiber sequences (3.2) are exactly the excision sequence for sheaves. This example also motivates the notation of functors in equation (3.1).

At the end of this section we state the main result of this article. Recall that we say a morphism $f : F \to G$ in $\mathrm{Sh}(M)$ is an isomorphism on an open set U if $\mathrm{SS}(\mathrm{cofib}(f)) \cap U = \varnothing$, or equivalently, f is an isomorphism in $\mathrm{Sh}(M; U)$ (cf. [23, Definition 6.1.1]).

THEOREM 3.6. *For a conic closed set $Z \subset T^*M$, and $U = T^*M \setminus Z$, we have the following:*

(1) (a) *The morphisms $F \to L_Z(F)$ and $R_Z(F) \to F$ are isomorphisms if and only if $\mathrm{SS}(F) \subset Z$.*

(b) *The morphism $L_U(F) \to F$ is an isomorphism if and only if $F \in {}^{\perp}\mathrm{Sh}_Z(M)$, and the morphism $F \to R_U(F)$ is an isomorphism if and only if $F \in \mathrm{Sh}_Z(M)^{\perp}$.*

(2) (a) *The morphisms $L_U(F) \to F$ and $F \to R_U(F)$ are isomorphisms on U.*

(b) *If $0_M \subset Z$, the morphisms $F \to L_Z(F)$ and $R_Z(F) \to F$ are isomorphisms on $\mathrm{Int}(Z) \setminus 0_M$. Equivalently, we have $\mathrm{SS}(L_U(F)) \cup \mathrm{SS}(R_U(F)) \subset \overline{U} \cup 0_M$.*

PROOF. Items (1) (a) and (2) (a) follow from the definitions of L_Z and L_U and fiber sequences (3.2). Item (1) (b) follows from [7, Corollary A.2.8]. For (2) (b), we will prove in Proposition 4.5 that $\mathrm{SS}(K_U) \subset (-\overline{U}) \times \overline{U} \cup 0_{M\times M}$ under the condition $0_M \subset Z$. Therefore, $\mathrm{SS}(L_U(F)) \cup \mathrm{SS}(R_U(F)) \subset \overline{U} \cup 0_M$ follows from Proposition 2.8. ∎

4. Wrapping formula of non-linear microlocal cut-off functors

In this section we present an explicit formula for microlocal kernels using Guillermou–Kashiwara–Schapira sheaf quantization [14]. We recall their results here.

Let $\dot{T}^*M$ be the complement of the zero section in T^*M, and, for a subset $A \subset T^*M$, we set $\dot{A} = A \cap \dot{T}^*M$. In particular, we have the notion of $\dot{\mathrm{SS}}(F)$ for $F \in \mathrm{Sh}(M)$.

Let $(I, 0)$ be a pointed interval. Consider a C^∞ conic symplectic isotopy

$$\phi : I \times \dot{T}^*M \to \dot{T}^*M$$

which is the identity at $0 \in I$. Such an isotopy is always the Hamiltonian flow for a unique conic function $H\colon I \times \dot{T}^*M \to \mathbb{R}$ and we set $\phi = \phi_H$ when we emphasize the Hamiltonian functions. At fixed $z \in I$, we have the graph of ϕ_z:

$$\begin{aligned} \Lambda_{\phi_z} &:= \{((q,-p), \phi_z(q,p)) : (q,p) \in \dot{T}^*M\} \\ &\subset \dot{T}^*M \times \dot{T}^*M \subset \dot{T}^*(M \times M). \end{aligned} \tag{4.1}$$

As for any Hamiltonian isotopy, we may consider the Lagrangian graph, which by definition is a Lagrangian subset $\Lambda_\phi \subset T^*I \times \dot{T}^*(M \times M)$ with the property that $\Lambda_{\phi_{z_0}}$ is the symplectic reduction of Λ_ϕ along $\{z = z_0\}$. It is given by the formula

$$\Lambda_\phi := \big\{\big(z, -H(z, \phi_z(q,p)), (q,-p), \phi_z(q,p)\big) : z \in I,\ (q,p) \in \dot{T}^*M\big\}.$$

Theorem 4.1 ([14, Theorem 3.7, Proposition 4.8]). *For ϕ as above, there is a sheaf $K = K(\phi) \in \mathrm{Sh}(I \times M^2)$ such that $\dot{\mathrm{SS}}(K) \subset \Lambda_\phi$ and $K|_{\{0\}\times M^2} \cong 1_{\Delta_M}$. The pair $(K, K|_{\{0\}\times M^2} \cong 1_{\Delta_M})$ is unique up to a unique isomorphism.*

Moreover, for isotopies $\phi_H, \phi_{H'}$ with $H' \le H$, there is a map $K(\phi_{H'})|_1 \to K(\phi_H)|_1$. In particular, when $H \ge 0$, then there is a map $1_{\Delta_M} \to K(\phi_H)|_1$.

Remark 4.2. Here we make some remarks on the proof of the theorem:

(1) In the proof of Theorem 4.1, the requirements for **k** contribute to different parts of the proof.

As we explained at the beginning of Section 1, we need local rigidity if we want to use the Verdier dual. It basically shows up everywhere in the proof.

However, it is not completely clear whether compact generation is essential. The main role of compact generation here is to guarantee that we can use arguments in [23] to prove the following type of microsupport estimations: box product, pullback and pushforward. The proofs of the box product estimation and pullback by submersion estimation in [23, Propositions 5.4.1, 5.4.5] essentially use the Ω-lens definition, while it is direct to prove the pushforward estimation [23, Proposition 5.4.4] using the Ω-lens definition. However, we do not know whether pullback by embedding can be proven using the Ω-lens definition directly. If one can do this, the GKS theorem is true in the case that **k** is locally rigid but not necessarily compactly generated.

(2) The existence does not use the linear cut-off lemmas. The continuation map can be derived directly from equation (4.1) and Proposition 2.7, which is a version of the 1-dimensional linear cut-off lemma, upon which we have the existence.

Motivated by ideas of [12, 33], it was shown in [26] that for any closed set $Z^\infty \subset S^*M$ and the conic closed set $Z = \mathbb{R}_{>0}Z^\infty \cup 0_M \subset T^*M$, the left and right adjoint inclusion $\iota\colon \mathrm{Sh}_Z(M) \to \mathrm{Sh}(M)$ can be computed by “wrapping”. More precisely, we have the following theorem.

THEOREM 4.3 ([26, Theorem 1.2]). *If H_n is*[3] *any increasing sequence of positive compactly supported Hamiltonians supported on $S^*M \setminus Z^\infty$ such that $H_n \uparrow \infty$ pointwise in $S^*M \setminus Z^\infty$, then the adjoints of ι can be computed by*

$$\iota^* F = \varinjlim \Phi_{K(\phi_{H_n})|_1}(F), \quad \iota^! F = \varprojlim \Phi_{K(\phi_{-H_n})|_1}(F).$$

Moreover, the continuation maps $F \to \Phi_{K(\phi_{H_n})|_1}(F)$ (resp. $\Phi_{K(\phi_{-H_n})|_1}(F) \to F$) induced by $1_{\Delta_M} \to K(\phi_{H_n})|_1$ (resp. $K(\phi_{-H_n})|_1 \to 1_{\Delta_M}$) from positivity of H_n give the unit (resp. counit) after taking the colimit (resp. limit).

REMARK 4.4. Here we make some remarks on the theorem:

(1) In fact, as explained in [28, Remark 6.5], the colimit in [26] is taken over an ∞-categorical "wrapping category", but one can compute the colimit by a cofinal sequence as explained in [26, Lemma 3.31].

(2) As we note in Remark 4.2, we may think of the wrapping formula as a cut-off result derived from the 1-dimensional linear cut-off lemma.

(3) In [26], the author requires $\mathbf{k}$ to be compactly generated and rigid to ensure the present proofs for the existence and uniqueness parts of the GKS theorem work. We have constructed the continuation map using the Ω-lens definition, which only requires that $\mathbf{k}$ be dualizable.

Although the argument of [26, Theorem 1.2] was written using [23, Definition 5.1.2], but can be written in terms of the Ω-lens.

(4) It is clear that we also have $\iota^! F = \varprojlim \Psi_{K(\phi_{H_n})|_1}(F)$ via nvolution functors. The formula coincides with the above formula using the limit of convolutions since $\Psi_{K(\phi_{H_n})|_1} \simeq \Phi_{K(\phi_{-H_n})|_1}$ by [14, Proposition 3.2 (ii)].

For $H \geq 0$, we set $K^\circ(\phi_{H_n}) = \mathrm{cofib}(1_{I\times\Delta_M} \to K(\phi_{H_n}))$.

Composing with the natural inclusion ι and combining with [26, Proposition 3.5], one can see that the fiber sequence (3.3) can be taken as

$$K_U = \varinjlim (K^\circ(\phi_{H_n})|_1) \to 1_{\Delta_M} \to K_Z = \varinjlim (K(\phi_{H_n})|_1). \tag{4.2}$$

To complete the proof of Theorem 3.6 (2) (b), we prove the following microsupport estimation of $K^\circ(\phi_H)|_1$. We also notice that the requirement that $0_M \subset Z$ comes from the wrapping formula.

(3) Here we do not distinguish between a function on S^*M and its conic lifting on $\dot{T}^*M$.

Proposition 4.5. *For a Hamiltonian function H supported in $U^\infty = U/\mathbb{R}_{>0}$, we have*

$$\dot{\mathrm{SS}}(K^\circ(\phi_H)|_1) \subset (-U) \times U.$$

In particular, we have

$$\dot{\mathrm{SS}}(K_U) \subset (-\bar{U}) \times \bar{U}.$$

Proof. By the triangle inequality, we have that

$$\dot{\mathrm{SS}}(K^\circ(\phi_H)|_1) \subset \Lambda_{\phi_{H,1}} \cup \Lambda_{\mathrm{id}}.$$

Since $\phi_{H,1}$ is compactly supported, there exists a maximal conic open set $W \subset T^*M$ at infinity such that $T^*M \setminus U \subset W$ such that $H|_W = 0$. Therefore, we have

$$\dot{\mathrm{SS}}(K^\circ(\phi_H)|_1) \setminus (-W^c) \times W^c \subset \big\{(q, -p, q, p) : (q, p) \in W\big\}.$$

Now we want to prove that the right-hand side of the above is not in $\dot{\mathrm{SS}}(K^\circ(\phi_H)|_1)$. To see this, we notice that $K^\circ(\phi_H)|_1$ acts as a quantized contact transform in the sense of [23, Theorem 7.2.1]. In particular, the action of $K^\circ(\phi_H)|_1$ on a microstalk can be given by geometric action given by $(\phi_H^1)_*$ on sheaves on a cotangent bundle. Therefore, as $H|_W = 0$, the monotonicity morphism $1_{\Delta_M} \to K(\phi_H)|_1$ induces the identity map on microstalks at $(q, -p, q, p)$ for $(q, p) \in W$.

Then we have that the microstalk of $K^\circ(\phi_H)|_1$ at $(q, -p, q, p)$ is zero for $(q, p) \in W$. This implies that

$$\dot{\mathrm{SS}}(K^\circ(\phi_H)|_1) \subset (-W^c) \times W^c \subset (-U) \times U.$$

The second statement follows from Corollary 2.5 and the wrapping formula (4.2). ∎

Using the wrapping formula, we can also prove the following microsupport estimation.

Proposition 4.6. *If Z contains the zero section, then for any increasing sequence of positive compactly supported Hamiltonians H_n supported on $S^*M \setminus Z^\infty$ such that $H_n \uparrow \infty$ pointwise in $S^*M \setminus Z^\infty$, we have*

$$\mathrm{SS}(L_Z(F)) \subset \big\{x \colon \forall n \in \mathbb{N},\ \exists x_n \in \dot{\mathrm{SS}}(F) \text{ such that } \phi_{H_n,1}(x_n) \to x\big\} \cup 0_M,$$
$$\mathrm{SS}(R_Z(F)) \subset \big\{x \colon \forall n \in \mathbb{N},\ \exists x_n \in \dot{\mathrm{SS}}(F) \text{ such that } \phi_{-H_n,1}(x_n) \to x\big\} \cup 0_M.$$

Proof. By [14, equation (1.12)], we have the microsupport estimation [14, equation (4.4)]

$$\dot{\mathrm{SS}}(K(\phi)|_z \circ F) = \phi_z(\dot{\mathrm{SS}}(F)). \tag{4.3}$$

For L_Z, the estimation follows immediately from Corollary 2.5 and equation (4.3) using the wrapping formula Theorem 4.3.

For the statement for R_Z, we cannot use Corollary 2.5 directly. Notice that the proof of Corollary 2.5 only needs the fact that, for abstract reasons, for all strictly increasing sequences $\{n_i\}_{i\in\mathbb{N}}$, we have $\varinjlim_i F_{n_i} \simeq \varinjlim_n F_n$. In general, for sequences $\{n_i\}_{i\in\mathbb{N}}$ as before, we do not have $\varprojlim_i F_{n_i} \not\simeq \varprojlim_n F_n$ for abstract reasons. Nevertheless, for the given H_n, functions $\{H_{n_i}\}_{i\in\mathbb{N}}$ still go to infinity on $S^*M \setminus Z^\infty$ for any strictly increasing subsequence $\{n_i\}_{i\in\mathbb{N}}$. Therefore, Theorem 4.3 implies, for all strictly increasing sequences $\{n_i\}_{i\in\mathbb{N}}$, that

$$R_Z(F) \simeq \varprojlim_i \Phi_{K(\phi_{-H_{n_i}})|_1}(F).$$

Then we can adapt the argument of Corollary 2.5 for the specific limit to show that

$$\mathrm{SS}(R_Z(F)) \subset \liminf_n \mathrm{SS}(\Phi_{K(\phi_{-H_n})|_1}(F)).$$

The required microsupport estimation follows from equation (4.3). ∎

Remark 4.7. Suppose we did not know the formula for the adjoint functor $L_Z(F)$. The proposition still shows a singular support estimation on $\varinjlim \Phi_{K(\phi_{H_n})|_1}(F)$. It is possible to use this singular support estimation to show that $\varinjlim \Phi_{K(\phi_{H_n})|_1}(F) \subset Z$. Subsequently, one can conclude that $\varinjlim \Phi_{K(\phi_{H_n})|_1}$ is indeed the left adjoint L_Z. This offers us a new proof of the wrapping formula. We leave the details to the readers.

Lastly, we write down the following limit formula.

Proposition 4.8. *For an increasing sequence of conic open sets U_n with $U = \bigcup_n U_n$, we set $Z_n = T^*M \setminus U_n$. Then there exists a colimit of fiber sequences in* $\mathrm{Sh}(M \times M)$*:*

$$K_U \to 1_{\Delta_M} \to K_Z \simeq \varinjlim [K_{U_n} \to 1_{\Delta_M} \to K_{Z_n}].$$

Proof. In [44, Appendix], we prove a version of the proposition for triangulated Tamarkin categories. The construction therein relies on [45, Proposition 2.4], which we can prove directly here: Let $X \subset Y \subset T^*M$ be two conic closed sets. We consider the inclusions $\iota_{XY}: \mathrm{Sh}_X(M) \to \mathrm{Sh}_Y(M)$. In particular, we denote $\iota_X = \iota_{X\,T^*M}$. The same argument as Proposition 3.1 shows that ι_{XY} has a left adjoint ι_{XY}^*. Moreover, as $\iota_X = \iota_Y \iota_{XY}$ by definition, we have $\iota_X^* = \iota_{XY}^* \iota_Y^*$. Therefore, the unit of $(\iota_{XY}^*, \iota_{XY})$ constructs a natural transform

$$L_X = \iota_X \iota_X^* = \iota_Y \iota_{XY} \iota_{XY}^* \iota_Y^* \leftarrow \iota_Y \mathrm{id}_{\mathrm{Sh}_Y(M)} \iota_Y^* = L_Y.$$

Similarly, we can construct a morphism $L_{T^*M\setminus Y} \to L_{T^*M\setminus X}$, and a morphism of a fiber sequence

$$\begin{tikzcd}
L_{T^*M\setminus Y} \arrow[r] \arrow[d] & \mathrm{id}_{\mathrm{Sh}(M)} \arrow[r] \arrow[d, "="] & L_Y \arrow[d] \\
L_{T^*M\setminus X} \arrow[r] & \mathrm{id}_{\mathrm{Sh}(M)} \arrow[r] & L_X.
\end{tikzcd}$$

Replacing functors in the commutative diagram by their kernels, we obtain [45, Proposition 2.4] in the ∞-categorical setting.

Next we consider the decreasing sequence $\{Z_n\}_n$ and we set $Z = Z_\infty$. For any $1 \le n < m \le \infty$, it is clear that $\iota_{Z_m Z_n} = \iota_{Z_m Z_{m-1}} \cdots \iota_{Z_{n+1} Z_n}$. Here we notice that for finite m, $n \le n+1 \le \cdots \le m-1 \le m$ defines an $(n-m)$-simplex in $N(\mathbb{N})$. Therefore, taking adjoints and considering units of adjoints as above gives a homotopy coherent diagram $N(\mathbb{N}) \to \mathrm{Fun}^L(\mathrm{Sh}(M), \mathrm{Sh}(M))$, $n \mapsto L_{Z_n}$. When taking $m = \infty$, the construction gives a morphism $\varinjlim L_{Z_n} \to L_{Z_\infty} = L_Z$.

Lastly, we prove that the morphism $\varinjlim L_{Z_n} \to L_Z$ is an equivalence by showing that $SS(F) \subset Z$ if and only if $F \xrightarrow{\simeq} \varinjlim L_{Z_n}(F)$: If $SS(F) \subset Z$, then all morphisms $F \to L_{Z_n}(F)$ are equivalences since $Z \subset Z_n$, so we have $F \xrightarrow{\simeq} \varinjlim L_{Z_n}(F)$. Conversely, if $F \xrightarrow{\simeq} \varinjlim L_{Z_n}(F)$, then Corollary 2.5 shows that

$$SS(F) \subset \bigcap_N \overline{\bigcup_{n\ge N} SS(L_{Z_n}(F))} \subset \bigcap_N \overline{\bigcup_{n\ge N} Z_n} = \bigcap_N Z_N = Z.$$

Passing to kernels, we have a homotopy coherent diagram $N(\mathbb{N}) \to \mathrm{Sh}(M \times M)$, $n \mapsto K_{Z_n}$, and an equivalence $\varinjlim K_{Z_n} \xrightarrow{\simeq} K_Z$. For K_U, we notice that $K_U = \mathrm{cofib}(1_{\Delta_M} \to K_Z)$ and then we use that colimits are commutative. ■

5. Kashiwara–Schapira microlocal cut-off functors

In this section we study the microlocal cut-off functors defined by Kashiwara and Schapira. We will follow the notation and formulation of [13, Chapter III].

Let $M = V$ be a real vector space of dimension n, and we naturally identify $T^*V = V \times V^*$. A subset $\gamma \subset V$ is a cone if $\mathbb{R}_{>0}\gamma = \{xv : x > 0,\ v \in \gamma\} \subset \gamma$, and we say γ is pointed if $0 \in \gamma$. We set $\gamma^a = -\gamma$. We say a cone γ is convex/closed if it is a convex/closed set, and is proper if $\gamma \cap \gamma^a = \{0\}$ (equivalently, γ contains no line). We define the dual cone $\gamma^\circ \subset V^*$ and $\tilde\gamma \subset V \times V$ as

$$\gamma^\circ = \big\{l \in V^* : l(v) \ge 0, \forall v \in \gamma\big\}, \quad \tilde\gamma = \big\{(x,y) \in V \times V : x - y \in \gamma\big\}.$$

For a pointed closed cone $\gamma \subset V$, we define four functors

$$
\begin{aligned}
&P_\gamma\colon \mathrm{Sh}(V) \to \mathrm{Sh}(V), \quad F \mapsto q_{2*}(1_{\tilde\gamma} \otimes q_1^* F),\\
&Q_\gamma\colon \mathrm{Sh}(V) \to \mathrm{Sh}(V), \quad F \mapsto q_{2!}(\mathcal{H}om(1_{\tilde\gamma^a}, q_1^! F)),\\
&P'_\gamma\colon \mathrm{Sh}(V) \to \mathrm{Sh}(V), \quad F \mapsto q_{2!}(\mathcal{H}om(1_{\tilde\gamma^a \setminus \Delta_V}[1], q_1^! F)),\\
&Q'_\gamma\colon \mathrm{Sh}(V) \to \mathrm{Sh}(V), \quad F \mapsto q_{2*}(1_{\tilde\gamma \setminus \Delta_V}[1] \otimes q_1^* F).
\end{aligned} \tag{5.1}
$$

For $\gamma = \{0\}$, we have $\tilde{\{0\}} = \Delta_V$ and $P_{\{0\}}(F) \simeq Q_{\{0\}}(F) \simeq F$. Using the fiber sequence $1_{\tilde\gamma} \to 1_{\Delta_V} \to 1_{\tilde\gamma\setminus\Delta_V}[1]$ (and the same with γ^a), we obtain the fiber sequences of functors

$$P'_\gamma \to \mathrm{id} \to Q_\gamma, \quad P_\gamma \to \mathrm{id} \to Q'_\gamma.$$

Proposition 5.1. *Let γ be a pointed closed convex cone. For $Z = V \times \gamma^{\circ a}$ and $U = T^*M \setminus Z$, we have the isomorphisms of fiber sequences*

$$[P'_\gamma \to \mathrm{id} \to Q_\gamma] \simeq [L_U \to \mathrm{id} \to L_Z], \quad [P_\gamma \to \mathrm{id} \to Q'_\gamma] \simeq [R_Z \to \mathrm{id} \to R_U].$$

Remark 5.2. Originally, [13, 23] define triangulated functors P_γ, Q_γ, P'_γ, Q'_γ. Here we use the same formula to define those functors on ∞-level, so they automatically descend to corresponding triangulated functors.

Proof of Proposition 5.1. It is explained in [13, Remark III.1.9] that, for the inclusion $\iota\colon \mathrm{Sh}_{V\times\gamma^{\circ a}}(V) \to \mathrm{Sh}(V)$, we have that $Q_\gamma = \iota\iota^*$ and $P_\gamma = \iota\iota^!$; and the natural transformations induced by $1_{\Delta_V} \to 1_{\tilde\gamma\setminus\Delta_V}[1]$ and $1_{\tilde\gamma^a} \to 1_{\Delta_V}$ are the unit/counit of the corresponding adjunctions. ∎

By virtue of Section 3, the existence of the cut-off functor appears as a purely categorical result. However, the advantage of equation (5.1) is that the kernel is explicitly written (also no limit!). It is explained in [13, equation (III.1.5)] that one can write down kernels of P'_γ and Q_γ as Proposition 1.1 predicted, such that the fiber sequence $K_U \to 1_{\Delta_V} \to K_Z$ is given by the fiber sequence

$$K_U = \mathrm{D}'(1_{\tilde\gamma^a\setminus\Delta_V})[n-1] \to 1_{\Delta_V} \to K_Z = \mathrm{D}'(1_{\tilde\gamma^a})[n],$$

where $\mathrm{D}'(F) = \mathcal{H}om(F, 1_{V^2})$.

Example 5.3. We consider a variant. Let $M = V$ be a real vector space. Take a pointed closed convex proper cone $\gamma \subset V$, set $U = V \times \mathrm{Int}(\gamma^\circ)$ and $Z = V \times (V^* \setminus \mathrm{Int}(\gamma^\circ))$. The microlocal cut-off functors for U are studied by Tamarkin [41] and Guillermou–Schapira [15] (where L_U, R_U are called L_γ, R_γ respectively).

The fiber sequence of kernels is given by

$$K_U = 1_{\tilde{\gamma}} \to 1_{\Delta_V} \to K_V = 1_{\tilde{\gamma}\setminus\Delta_V}[1].$$

By Proposition 5.1, if we assume there exists a pointed closed convex cone λ such that $V^* \setminus \mathrm{Int}(\gamma^\circ) = \lambda^{\circ a}$, we have $L_U = P'_\lambda = L_\gamma$, $R_U = Q'_\lambda = R_\gamma$. Notice that we can show by either computation of a dual functor D' or abstract uniqueness for the kernel that $K_U = 1_{\tilde{\gamma}} \simeq \mathrm{D}'(1_{\tilde{\lambda}^a\setminus\Delta_V})[n-1]$.

6. Results for pairs

In this section we consider the following construction: Take two conic closed sets $Z \subset X \subset T^*M$. Let $U = T^*M \setminus Z$ and $V = T^*M \setminus X$; then $V \subset U$. Recall that $\mathrm{Sh}_{X\cap U}(M;U) = \frac{\mathrm{Sh}_X(M)}{\mathrm{Sh}_Z(M)}$.

We consider the following 9-diagram of $\mathbf{k}$-linear categories:

$$\begin{tikzcd}
\mathrm{Sh}_Z(M) \arrow[r, hook] \arrow[d, equal] & \mathrm{Sh}_X(M) \arrow[r, two heads] \arrow[d, hook] & \mathrm{Sh}_{X\cap U}(M;U) \arrow[d, hook] \\
\mathrm{Sh}_Z(M) \arrow[r, hook] \arrow[d, two heads] & \mathrm{Sh}(M) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Sh}(M;U) \arrow[d, two heads] \\
0 \arrow[r, hook] & \mathrm{Sh}(M;V) \arrow[r, "\simeq"] & \mathrm{Sh}(M;U)\big/\mathrm{Sh}_{X\cap U}(M;U).
\end{tikzcd} \tag{6.1}$$

In this diagram, by the same argument as in Proposition 3.1, we have that all vertical and horizontal sequences are split Verdier sequences in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}(\mathbf{k})$. We say a morphism $f\colon F \to G$ in $\mathrm{Sh}(M;U)$ is an isomorphism on an open set $V \subset U$ if f is an isomorphism in $\mathrm{Sh}(M;V)$. Because $\mathrm{Sh}_{X\cap U}(M;U)$ is a thick subcategory of $\mathrm{Sh}(M;U)$, we know that f is an isomorphism on V if and only if $SS_U(\mathrm{cofib}(f)) \cap V = \varnothing$.

Here we write down the adjoint functors in the last column precisely:

$$\begin{tikzcd}
\mathrm{Sh}_{X\cap U}(M;V) \arrow[r, "\iota"] & \mathrm{Sh}(M;U) \arrow[r, "j"] \arrow[l, bend right, "\iota^*"'] \arrow[l, bend left, "\iota^!"] & \mathrm{Sh}(M;V). \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

Then we have adjunction pairs

$$L_{(U,V)} := j_!j \dashv j_*j =: R_{(U,V)}, \quad L_{(Z,X)} := \iota\iota^* \dashv \iota\iota^! =: R_{(Z,X)},$$

and the units/counits give us following fiber sequences of functors on $\mathrm{Sh}(M;U)$:

$$L_{(U,V)} \to \mathrm{id} \to L_{(Z,X)}, \quad R_{(Z,X)} \to \mathrm{id} \to R_{(U,V)}. \tag{6.2}$$

Remark 6.1. By equation (0.1), we can also represent functors in equation (6.2) by integral kernels $K_{(Z,X)}, K_{(U,V)} \in \mathrm{Sh}(M \times M; (-U) \times U)$ and they are images of K_X, K_V under the quotient functor $\mathrm{Sh}(M \times M) \to \mathrm{Sh}(M \times M; (-U) \times U)$. The relative version of microsupport estimation Proposition 4.5 is also true.

Therefore, we can state the following version of the cut-off lemma.

Theorem 6.2. *For conic closed sets $Z \subset X \subset T^*M$, and $U = T^*M \setminus Z$ and $V = T^*M \setminus X$, we have the following:*

(1) (a) *The morphisms $F \to L_{(Z,X)}(F)$ and $R_{(Z,X)}(F) \to F$ are isomorphisms if and only if $\mathrm{SS}_U(F) \subset X \cap U$.*

(b) *The morphism $L_{(U,V)}(F) \to F$ is an isomorphism if and only if $F \in {}^{\perp}\mathrm{Sh}_{X \cap U}(M; U)$, and the morphism $F \to R_{(U,V)}(F)$ is an isomorphism if and only if $F \in \mathrm{Sh}_{X \cap U}(M; U)^{\perp}$.*

(2) (a) *The morphisms $L_{(U,V)}(F) \to F$ and $F \to R_{(U,V)}(F)$ are isomorphisms on $U \cap V$.*

(b) *If $0_M \subset Z$, the morphisms $F \to L_{(Z,X)}(F)$ and $R_{(Z,X)}(F) \to F$ are isomorphisms on $\mathrm{Int}(X) \cap U$. Equivalently, we have $\mathrm{SS}_U(L_{(U,V)}(F)) \cup \mathrm{SS}_U(R_{(U,V)}(F)) \subset \overline{V} \cap U$. Here $\overline{V}$ denotes the closure of V in T^*M.*

Proof. Here we notice that $\mathrm{SS}_U([F]) = \mathrm{SS}(R_U(F)) \cap U = \mathrm{SS}(L_U(F)) \cap U$ for $[F] \in \mathrm{Sh}(M; U)$. Then we apply the absolute version of Theorem 3.6 to conclude the result. ∎

Example 6.3. We consider the manifold $M \times \mathbb{R}_t$. We take a closed set $\mathcal{Z} \subset J^1M = T^*M \times \mathbb{R}$ and $\mathcal{U} = J^1M \setminus \mathcal{Z}$. We consider their conification in $T^*(M \times \mathbb{R}_t) = T^*M \times \mathbb{R}_t \times \mathbb{R}_\tau$. We set

$$\begin{aligned}
\Omega_{>0} &= \{\tau > 0\}, \quad \Omega_- = \{\tau \le 0\},\\
U &= \{(q, \tau p, t, \tau) : (q, p, t) \in \mathcal{U},\ \tau > 0\},\\
Z_{>0} &= \{(q, \tau p, t, \tau) : (q, p, t) \in \mathcal{Z},\ \tau > 0\},\\
Z &= Z_{>0} \cup \Omega_-.
\end{aligned}$$

Then $\Omega_- \subset Z$ are two closed conic sets. In this case, the category

$$\mathrm{Sh}(M \times \mathbb{R}_t; U) \simeq \mathrm{Sh}(M \times \mathbb{R}_t; \Omega_{>0}) / \mathrm{Sh}_{Z_{>0}}(M \times \mathbb{R}_t; \Omega_{>0})$$

is useful for contact topology of J^1M, since $\mathbb{R}_{>0}$ acts on $\Omega_{>0}$ freely, with the quotient $J^1M = T^*M \times \mathbb{R}_t$.

If we take $\mathcal{U} = U_0 \times \mathbb{R}$ for an open set $U_0 \subset T^*M$ and $\mathcal{Z} = Z_0 \times \mathbb{R}$ for the closed set $Z_0 = T^*M \setminus U_0$, the category $\mathrm{Sh}(M \times \mathbb{R}_t; U)$ is exactly the so-called Tamarkin category $\mathcal{T}(U_0)$ (cf. [28]). Tamarkin categories are first defined in [41], and then studied in [1,2,4,5,8,16,20,42,46]. In this case, U admits an $\mathbb{R}$-action by translation along $\mathbb{R}_t$. Therefore, there exists a $\mathcal{T} := \mathcal{T}(\mathrm{pt})$ (which is a symmetric monoidal category [15,28]) action on $\mathcal{T}(U_0)$ that helps us understand the action filtration from symplectic geometry. The functors

$$L_{U_0}^{\mathcal{T}} := L_{(\Omega_{>0},U)}, \quad L_{Z_0}^{\mathcal{T}} := L_{(Z,\Omega_-)}$$

are actually $\mathcal{T}$-linear. By the $\mathcal{T}$-linear left adjoint functors classification (see [28, Proposition 5.12], which is an enriched version of Remark 6.1), there is a fiber sequence $K_{U_0}^{\mathcal{T}} \to 1_{\Delta_M}^{\mathcal{T}} \to K_{Z_0}^{\mathcal{T}}$ in $\mathcal{T}(T^*(M \times M)) = \mathrm{Sh}(M \times M; \mathcal{T})$ with

$$L_{U_0}^{\mathcal{T}} = \Phi_{K_{U_0}^{\mathcal{T}}}^{\mathcal{T}}, \quad L_{Z_0}^{\mathcal{T}} = \Phi_{K_{Z_0}^{\mathcal{T}}}^{\mathcal{T}},$$

where $\Phi^{\mathcal{T}}$ means the convolution functor defined using $\mathcal{T}$-linear 6-operators. These functors are studied in [8,28,40,44,45], where the corresponding kernels are denoted by $K_{U_0}^{\mathcal{T}} = P_{U_0}$, $K_{Z_0}^{\mathcal{T}} = Q_{U_0}$.

Here we can apply Theorem 6.2 to Tamarkin categories, regarded as **k**-linear categories $\mathrm{Sh}(M \times \mathbb{R}_t; U)$, to obtain an microsupport estimation $SS_{\Omega_{>0}}(L_{(\Omega_{>0},U)}(F)) \subset \bar{U} \cap \Omega_{>0}$. Then we obtain an estimation for reduced microsupport:

$$\begin{aligned} \mathrm{RS}(L_{U_0}^{\mathcal{T}}(F)) &= \mathrm{RS}(L_{(\Omega_{>0},U)}(F)) \\ &= \left\{(q,p) : (q,p,t,1) \in SS_{\Omega_{>0}}(L_{(\Omega_{>0},U)}(F))\right\} \subset \bar{U}_0. \end{aligned}$$

In general, if the open set $\mathcal{U} \subset J^1M$ admits a $\mathbb{G}$-action for a subgroup $\mathbb{G} \subset \mathbb{R}$ via translation, Asano, Ike and Kuwagaki introduce an equivariant version of categories; see [3,21,29]: We consider $\mathbb{G}$ as a discrete group by forgetting its topology, in this case, the group homomorphism $\mathbb{G} \to \mathbb{R}_t$ is still continuous. Then we consider the $\mathbb{G}$-action on $\mathrm{Sh}(M \times \mathbb{R}; \Omega_+)$ via translation, and define $\mathrm{Sh}(M \times \mathbb{R}; \Omega_+)$,

$$\mathrm{Sh}^{\mathbb{G}}(M \times \mathbb{R}; \Omega_+) := (\mathrm{Sh}(M \times \mathbb{R}; \Omega_+))^{\mathbb{G}} := \varprojlim_{\mathbb{G}} \mathrm{Sh}(M \times \mathbb{R}; \Omega_+).$$

There exists a forgetful functor $\mathfrak{f}\colon \mathrm{Sh}^{\mathbb{G}}(M \times \mathbb{R}; \Omega_+) \to \mathrm{Sh}(M \times \mathbb{R}; \Omega_+)$, and the notion $SS_{\Omega_{>0}}(F) := SS_{\Omega_{>0}}(\mathfrak{f}(F))$ is well defined. Therefore, we can define categories

$$\mathrm{Sh}^{\mathbb{G}}(M \times \mathbb{R}_t; U), \quad \mathrm{Sh}^{\mathbb{G}}_{Z_{>0}}(M \times \mathbb{R}_t; \Omega_{>0}).$$

The cut-off functor and cut-off lemma are still true and can be proven by the following ingredients: The forgetful functor $\mathfrak{f}$ admits both left and right adjoint, and $\mathrm{Sh}(N \times M \times$

$\mathbb{R};\Omega_+)$ acts on $\mathrm{Sh}^{\mathbb{G}}(M\times\mathbb{R};\Omega_+)$ via a convolution and $\mathfrak{f}^L$. Then we can construct the wrapping formula via the action and show all results of this article directly and carefully (especially on the categorical aspect rather than the microlocal aspect). We also leave the details to the readers as it is a little beyond the purpose of this article.

7. Künneth formula

Here we present some computations of Lurie tensor products, which generalize the result of [27, Theorem 1.2], where an isotropic condition is needed. We take manifolds N, M, conic closed sets $Z\subset T^*M$ and $X\subset T^*N$, and we set $U=T^*M\setminus Z$ and $V=T^*N\setminus X$.

To start with, we recall that $\mathcal{C}\in\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is dualizable if it is a dualizable object with respect to the symmetric monoidal structure defined by the Lurie tensor product, and its dual is denoted by $\mathcal{C}^\vee$. Then we recall the following.

Lemma 7.1 ([28, Remark 3.7]). *The category* $\mathrm{Sh}_Z(M)$ *is dualizable with dual* $\mathrm{Sh}_{-Z}(M)$. *The category* $\mathrm{Sh}(M;U)$ *is dualizable with the dual* $\mathrm{Sh}(M;-U)$.

Remark 7.2. In fact, Proposition 3.1 is enough to guarantee that $\mathrm{Sh}_Z(M)$ and $\mathrm{Sh}(M;U)$ are dualizable as stable linear categories (cf. [10, Proposition 1.17.]), and local rigidity of $\mathbf{k}$ ensures that the dual is actually a $\mathbf{k}$-linear dual [25, Proposition 4.3.3]. The nature of the lemma here is that we identify the dual of $\mathrm{Sh}_Z(M)$ and $\mathrm{Sh}(M;U)$.

Remark 7.3. In the following propositions, the box tensor $\boxtimes$ of kernels should be understood after identifying $(N\times M)^2\simeq N^2\times M^2$ via $((n_1,m_1),(n_2,m_2))\mapsto(n_1,n_2,m_1,m_2)$.

Theorem 7.4. *We have that* $K_X\boxtimes K_Z\simeq K_{X\times Z}$*, and* $\mathrm{Sh}_X(N)\otimes\mathrm{Sh}_Z(M)\simeq\mathrm{Sh}_{X\times Z}(N\times M)$.

Proof. We first assume that both X and Z contain the zero section.

For the first statement, we set $K:=K_X\boxtimes K_Z$. It is clear that the morphism $1_{\Delta_N}\boxtimes 1_{\Delta_M}=1_{\Delta_{N\times M}}\to K$ induces an equivalence $\Phi_K\xrightarrow{\simeq}\Phi_K\circ\Phi_K$. Henceforth, Φ_K is a projector in the sense of [24, Definition 4.1.1], whose essential image is given by the full subcategory spanned by F with $F\xrightarrow{\simeq}\Phi_K(F)$ (cf. [24, Proposition 4.1.3]).

We are going to prove that $\Phi_K=\Phi_{K_{X\times Z}}$ now. Because $\Phi_{K_{X\times Z}}$ is the projector whose essential image is $\{F\colon F\xrightarrow{\simeq}\Phi_{K_{X\times Z}}(F)\}=\mathrm{Sh}_{X\times Z}(N\times M)$ by its definition, it remains to show that $F\xrightarrow{\simeq}\Phi_K(F)$ if and only if $F\in\mathrm{Sh}_{X\times Z}(N\times M)$.

For $F \in \mathrm{Sh}(N \times M)$ with $\Phi_K(F) \simeq F$, we can write $F = \varinjlim F_\alpha \boxtimes G_\alpha$, and then we have $F \simeq \varinjlim \Phi_{K_X}(F_\alpha) \boxtimes \Phi_{K_Z}(G_\alpha)$. So $SS(F) = SS(\varinjlim \Phi_{K_X}(F_\alpha) \boxtimes \Phi_{K_Z}(G_\alpha)) \subset \overline{SS(\Phi_{K_X}(F_\alpha)) \times SS(\Phi_{K_Z}(G_\alpha))} \subset X \times Z$.

Conversely, we need to show that K fixes $F \in \mathrm{Sh}_{X \times Z}(N \times M)$. We use the wrapping formula Theorem 4.3, and adapt the argument of [1, Lemma 4]. For cofinal sequences of conic non-negative Hamiltonian functions $H_{\lambda_1(n_1)}$ supported in U and $H_{\lambda_2(n_2)}$ supported in V with index sequences $\lambda_i \colon \mathbb{N} \to \mathbb{N}$, we have that $K \simeq \varinjlim_{(n_1,n_2)} K(\phi_{H_{\lambda_1(n_1)}})|_1 \boxtimes K(\phi_{H_{\lambda_2(n_2)}})|_1$. Now consider the 1-parameter version of the GKS quantization $K(\phi_{H_{\lambda_1(n_1)}})$ and $K(\phi_{H_{\lambda_2}(n_2)})$ as given in Theorem 4.1. By a parameter version of (4.3) (precisely, [14, equation (4.3)]), it is direct to see that for $W = \Phi_{K(\phi_{H_{\lambda_1(n_1)}}) \boxtimes K(\phi_{H_{\lambda_2(n_2)}})}(F) \in \mathrm{Sh}(N \times \mathbb{R} \times M \times \mathbb{R})$, we have

$$\begin{aligned} SS(W) \subset \big\{(q_1, z_1, p_1, \zeta_1, q_2, z_2, p_2, \zeta_2) : \exists (q_1', p_1', q_2', p_2') \in SS(F), \\ (q_i, p_i) = \phi_{z_i}^{H_{\lambda_i(n_i)}}(q_i', p_i'), \\ \zeta_i = -H_{\lambda_i(n_i)}(q_i', p_i')\big\}. \end{aligned}$$

Now, since $SS(F) \subset X \times Z$, we have that $SS(W) \subset T^*(N \times M) \times 0_{\mathbb{R}^2}$. Then by [23, Proposition 5.4.5], we have

$$F = W|_{z_1=z_2=0} \simeq W|_{z_1=z_2=1} = \Phi_{K(\phi_{H_{\lambda_1(n_1)}})|_1 \boxtimes K(\phi_{H_{\lambda_2(n_2)}})|_1}(F).$$

The isomorphism commutes with the continuation map given by non-negativity of $H_{\lambda_i(n_i)}$ (see Theorem 4.1). Then we take the limit with respect to (n_1, n_2) to see that $\Phi_K(F) \simeq F$.

Consequently, we have $K_{X \times Z} \simeq K = K_X \boxtimes K_Z$ by uniqueness of the kernel.

For the second statement, under the natural identification $\mathrm{Sh}(N) \otimes \mathrm{Sh}(M) = \mathrm{Sh}(N \times M)$ (see [43, Proposition 2.30]), it remains to verify that the essential image of the functor

$$\mathrm{Sh}_X(N) \otimes \mathrm{Sh}_Z(M) \to \mathrm{Sh}(N) \otimes \mathrm{Sh}(M) = \mathrm{Sh}(N \times M), \quad F \otimes G \mapsto F \boxtimes G$$

is $\mathrm{Sh}_{X \times Z}(N \times M)$. In fact, any object F in $\mathrm{Sh}_{X \times Z}(N \times M)$ can be written as a colimit of some $F = \varinjlim F_\alpha \boxtimes G_\alpha \in \mathrm{Sh}(N \times M)$. We should show that the F_α can be chosen in $\mathrm{Sh}_X(N)$ and G_α can be chosen to be in $\mathrm{Sh}_Z(M)$. By definition of $K_{X \times Z}$, we have $F \simeq \Phi_{K_{X \times Z}}(F)$. As we have already proved that $K_{X \times Z} \simeq K_X \boxtimes K_Z$, we have

$$F \simeq \Phi_{K_{X \times Z}}(\varinjlim F_\alpha \boxtimes G_\alpha) \simeq \varinjlim \Phi_{K_X \boxtimes K_Z}(F_\alpha \boxtimes G_\alpha) = \varinjlim \Phi_{K_X}(F_\alpha) \boxtimes \Phi_{K_Z}(G_\alpha).$$

Then by replacing F_α and G_α by $\Phi_{K_X}(F_\alpha)$ and $\Phi_{K_Z}(G_\alpha)$, we can conclude that $F \in \mathrm{Sh}_X(N) \otimes \mathrm{Sh}_Z(M)$.

Now we remove the assumption that X, Z contain the zero section. In this case, we set $Z_0 = Z \cap 0_M$ and $\widetilde{Z} = Z \cup 0_M$ (similarly for X_0 and $\widetilde{X}$). Then we use the result for $\widetilde{X}$ and $\widetilde{Z}$, which contain zero sections, from above, and we conclude by noticing that $\mathrm{SS}(F) \subset Z$ if and only if $\mathrm{SS}(F) \subset \widetilde{Z}$ and $\operatorname{supp} F \subset Z_0$, and $F_n \otimes G_m \simeq (F \boxtimes G)_{(n,m)}$ for $(n,m) \in N \times M$. ■

Corollary 7.5. *We have an equivalence of fiber sequences of categories*

$$\begin{aligned}\mathrm{Sh}_{X\times Z}(N \times M) &\to \mathrm{Sh}_{X\times T^*M}(N \times M) \to \mathrm{Sh}_{X\times U}(N \times M; T^*N \times U)\\ &\simeq \mathrm{Sh}_X(N) \otimes (\mathrm{Sh}_Z(M) \to \mathrm{Sh}(M) \to \mathrm{Sh}(M;U)).\end{aligned}$$

Proof. The functor $- \mapsto \mathrm{Sh}_X(N) \otimes -$ preserves colimits and full faithfulness. This can be seen from two different points of view, one from dualizability of $\mathrm{Sh}_X(N)$ and [10, Theorem 2.2], the other one from [18, Corollary 2.29] since in the stable setting, the notions of recollement pair and split Verdier sequence are equivalent [7, Proposition A.2.11] (in this approach we do not need dualizability). Therefore, we have the equivalence of Verdier sequences

$$\begin{aligned}&\mathrm{Sh}_X(N) \otimes (\mathrm{Sh}_Z(M) \to \mathrm{Sh}(M) \to \mathrm{Sh}(M;U))\\ &\quad\simeq \mathrm{Sh}_X(N) \otimes \mathrm{Sh}_Z(M) \to \mathrm{Sh}_X(N) \otimes \mathrm{Sh}(M) \to \mathrm{Sh}_X(N) \otimes \mathrm{Sh}(M;U). \quad ■\end{aligned}$$

Theorem 7.6. *We have an equivalence*

$$\mathrm{Sh}(N;V) \otimes \mathrm{Sh}(M;U) \simeq \mathrm{Sh}(N \times M; V \times U).$$

In particular, we have that $K_V \boxtimes K_U \simeq K_{V\times U}$.

Proof. We fill the 9-diagram in equation (6.1) by setting the manifold by $N \times M$, two conic closed sets are $T^*N \times Z \subset T^*N \times Z \cup X \times T^*M$. Therefore, we have the following equivalence:

$$\mathrm{Sh}(N \times M; V \times U) \simeq \mathrm{Sh}(N \times M; T^*M \times U)/\,\mathrm{Sh}_{X\times U}(N \times M; T^*M \times U).$$

Now we apply Corollary 7.5 twice to see that

$$\begin{aligned}&[\mathrm{Sh}_{X\times U}(N \times M; T^*N \times U) \hookrightarrow \mathrm{Sh}(N \times M; T^*N \times U)]\\ &\quad= [\mathrm{Sh}_X(N) \hookrightarrow \mathrm{Sh}(N)] \otimes \mathrm{Sh}(M;U).\end{aligned}$$

Consequently, we have

$$\mathrm{Sh}(N \times M; T^*N \times U)/\,\mathrm{Sh}_{X\times U}(N \times M; T^*N \times U) \simeq \mathrm{Sh}(N;V) \otimes \mathrm{Sh}(M;U). \quad ■$$

Acknowledgments – This work is inspired by a discussion with Stéphane Guillermou and Vivek Shende's interests on microlocal cut-off functors. The author thanks them for their helpful discussion and encouragement. We also thank for Alexander I. Efimov, Xin Jin, Christopher Kuo, Wenyuan Li, Pierre Schapira and Matthias Scharitzer for helpful discussion. Thanks to the anonymous referees and their helpful comments.

Funding – This work was supported by the Novo Nordisk Foundation grant NNF20OC0066298 and VILLUM FONDEN, VILLUM Investigator grant 37814.

References

[1] T. Asano – S. Guillermou – V. Humilière – Y. Ike – C. Viterbo, The γ-support as a micro-support. *C. R. Math. Acad. Sci. Paris* **361** (2023), 1333–1340. Zbl 1535.53082 MR 4683312

[2] T. Asano – Y. Ike, Persistence-like distance on Tamarkin's category and symplectic displacement energy. *J. Symplectic Geom.* **18** (2020), no. 3, 613–649. Zbl 1473.18012 MR 4142482

[3] T. Asano – Y. Ike, Sheaf quantization and intersection of rational Lagrangian immersions. *Ann. Inst. Fourier (Grenoble)* **73** (2023), no. 4, 1533–1587. Zbl 1523.53089 MR 4613614

[4] T. Asano – Y. Ike, Completeness of derived interleaving distances and sheaf quantization of non-smooth objects. *Math. Ann.* **390** (2024), no. 2, 2991–3037. Zbl 07932374 MR 4801846

[5] T. Asano – Y. Ike – W. Li, Lagrangian cobordism and shadow distance in Tamarkin category. 2023, arXiv:2312.14429v1.

[6] A. J. Blumberg – D. Gepner – G. Tabuada, A universal characterization of higher algebraic K-theory. *Geom. Topol.* **17** (2013), no. 2, 733–838. Zbl 1267.19001 MR 3070515

[7] B. Calmès – E. Dotto – Y. Harpaz – F. Hebestreit – M. Land – K. Moi – D. Nardin – T. Nikolaus – W. Steimle, Hermitian K-theory for stable ∞-categories II: Cobordism categories and additivity, [v1] 2020, [v4] 2023, arXiv:2009.07224v4.

[8] S.-F. Chiu, Nonsqueezing property of contact balls. *Duke Math. J.* **166** (2017), no. 4, 605–655. Zbl 1372.53087 MR 3619302

[9] A. D'Agnolo, On the microlocal cut-off of sheaves. *Topol. Methods Nonlinear Anal.* **8** (1996), no. 1, 161–167. Zbl 0901.58070 MR 1485761

[10] A. I. Efimov, K-theory and localizing invariants of large categories. 2024, arXiv:2405.12169v2.

[11] D. Gaitsgory – N. Rozenblyum, *A study in derived algebraic geometry. Vol. I. Correspondences and duality*. Math. Surveys Monogr. 221, American Mathematical Society, Providence, RI, 2017. Zbl 1408.14001 MR 3701352

[12] S. Ganatra – J. Pardon – V. Shende, Microlocal Morse theory of wrapped Fukaya categories. *Ann. of Math. (2)* **199** (2024), no. 3, 943–1042. MR 4740209

[13] S. Guillermou, Sheaves and symplectic geometry of cotangent bundles. *Astérisque* (2023), no. 440, x+274. Zbl 1530.53003 MR 4612528

[14] S. Guillermou – M. Kashiwara – P. Schapira, Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems. *Duke Math. J.* **161** (2012), no. 2, 201–245. Zbl 1242.53108 MR 2876930

[15] S. Guillermou – P. Schapira, Microlocal theory of sheaves and Tamarkin's non displaceability theorem. In *Homological mirror symmetry and tropical geometry*, pp. 43–85, Lect. Notes Unione Mat. Ital. 15, Springer, Cham, 2014. Zbl 1319.32006 MR 3330785

[16] S. Guillermou – N. Vichery, Viterbo's spectral bound conjecture for homogeneous spaces. 2022, arXiv:2203.13700v1.

[17] S. Guillermou – C. Viterbo, The singular support of sheaves is γ-coisotropic. *Geom. Funct. Anal.* **34** (2024), no. 4, 1052–1113. Zbl 07890731 MR 4768582

[18] P. J. Haine, From nonabelian basechange to basechange with coefficients. [v1] 2021, [v3] 2022, arXiv:2108.03545v3.

[19] P. J. Haine – M. Porta – J.-B. Teyssier, The homotopy-invariance of constructible sheaves. *Homology Homotopy Appl.* **25** (2023), no. 2, 97–128. Zbl 1528.32046 MR 4651175

[20] Y. Ike, Compact exact Lagrangian intersections in cotangent bundles via sheaf quantization. *Publ. Res. Inst. Math. Sci.* **55** (2019), no. 4, 737–778. Zbl 1429.53098 MR 4024997

[21] Y. Ike – T. Kuwagaki, Microlocal categories over Novikov rings. [v1] 2023, [v2] 2024, arXiv:2307.01561v2.

[22] X. Jin – D. Treumann, Brane structures in microlocal sheaf theory. *J. Topol.* **17** (2024), no. 1, article no. e12325. Zbl 07824479 MR 4821225

[23] M. Kashiwara – P. Schapira, *Sheaves on manifolds*. Grundlehren Math. Wiss. 292, Springer, Berlin, 1990. Zbl 0709.18001 MR 1074006

[24] M. Kashiwara – P. Schapira, *Categories and sheaves*. Grundlehren Math. Wiss. 332, Springer, Berlin, 2006. Zbl 1118.18001 MR 2182076

[25] A. Krause – T. Nikolaus – P. Pützstück, Sheaves on manifolds. Lecture notes, 2024, https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/sheaves-on-manifolds.pdf visited on 3 January 2025.

[26] C. Kuo, Wrapped sheaves. *Adv. Math.* **415** (2023), article no. 108882. Zbl 1506.18014 MR 4543073

[27] C. Kuo – W. Li, Duality, Künneth formulae, and integral transforms in microlocal geometry. 2024, arXiv:2405.15211v2.

[28] C. Kuo – V. Shende – B. Zhang, On the Hochschild cohomology of Tamarkin categories. [v1] 2023, [v2] 2024, arXiv:2312.11447v2.

[29] T. Kuwagaki, Sheaf quantization from exact WKB analysis. *J. Reine Angew. Math.* **814** (2024), 165–203. MR 4793343

[30] J. Lurie, *Higher topos theory*. Ann. of Math. Stud. 170, Princeton University Press, Princeton, NJ, 2009. Zbl 1175.18001 MR 2522659

[31] J. Lurie, Higher algebra. 2017, https://www.math.ias.edu/~lurie/papers/HA.pdf visited on 3 January 2025.

[32] J. Lurie, Spectral algebraic geometry. 2018, https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf visited on 3 January 2025.

[33] D. Nadler, Wrapped microlocal sheaves on pairs of pants. 2016, arXiv:1604.00114v1.

[34] D. Nadler – V. Shende, Sheaf quantization in Weinstein symplectic manifolds. [v1] 2019, [v3] 2022, arXiv:2007.10154v3.

[35] T. Nikolaus – P. Scholze, On topological cyclic homology. *Acta Math.* **221** (2018), no. 2, 203–409. Zbl 1457.19007 MR 3904731

[36] S. Ragimov – T. M. Schlank, The ∞-categorical reflection theorem and applications. 2022, arXiv:2207.09244v1.

[37] M. Ramzi, Locally rigid ∞-categories. 2024, arXiv:2410.21524v1.

[38] M. Robalo – P. Schapira, A lemma for microlocal sheaf theory in the ∞-categorical setting. *Publ. Res. Inst. Math. Sci.* **54** (2018), no. 2, 379–391. Zbl 1403.35016 MR 3784874

[39] P. Scholze, Six-functor formalisms. Lecture notes, 2022, https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf visited on 3 January 2025.

[40] D. Tamarkin, Microlocal category. 2015, arXiv:1511.08961v1.

[41] D. Tamarkin, Microlocal condition for non-displaceability. In *Algebraic and analytic microlocal analysis*, pp. 99–223, Springer Proc. Math. Stat. 269, Springer, Cham, 2018. Zbl 1416.35019 MR 3903318

[42] N. Vichery, *Homogénéisation symplectique et applications de la théorie des faisceaux à la topologie symplectique*. Ph.D. thesis, École polytechnique, 2013, https://tel.archives-ouvertes.fr/pastel-00780016/ visited on 3 January 2025.

[43] M. Volpe, The six operations in topology. [v1] 2021, [v2] 2023, arXiv:2110.10212v2.

[44] B. Zhang, Idempotence of microlocal kernels and S^1-equivariant Chiu–Tamarkin invariant. 2023, arXiv:2306.12316v2.

[45] B. Zhang, Capacities from the Chiu–Tamarkin complex. *J. Symplectic Geom.* **22** (2024), no. 3, 441–524. Zbl 07946770 MR 4819505

[46] J. Zhang, *Quantitative Tamarkin theory*. CRM Short Courses, Springer, Cham, 2020. Zbl 1456.53008 MR 4211770

Manoscritto pervenuto in redazione l'8 agosto 2024.