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$L_{\kappa\omega}$-equivalence of abelian groups with partial decomposition bases

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Abstract – We consider the class of abelian groups possessing partial decomposition bases in the language $L_{\kappa\omega}$ for uncountable cardinals κ. Jacoby, Leistner, Loth and Strüngmann developed a numerical invariant deduced from the classical global Warfield invariant and proved that if two groups have identical modified Ulm invariants and Warfield invariants up to $\omega\delta$ for some ordinal δ, then they are equivalent in $L^{\delta}_{\infty\omega}$. Subsequently, Jacoby and Loth showed that the converse is true for appropriate δ. In this paper we prove that the modified Warfield invariant up to κ is expressible in $L_{\kappa\omega}$, thus a complete classification theorem in $L_{\kappa\omega}$ is obtained. This generalizes a result of Barwise and Eklof.

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Dedicated to László Fuchs on his 100th birthday

1. Introduction

The classical theorem of Ulm [21] presents invariants that classify countable abelian p-groups up to isomorphism. Hill [3] and Walker [22] were able to extend this result to the class of totally projective p-groups, the largest natural class of abelian p-groups in which the Ulm invariants distinguish between nonisomorphic p-groups. It was Szmielew [20] who first considered abelian groups from a model-theoretic point of

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view. Barwise and Eklof [1] took up Szmielew's approach and extended Ulm's theorem to the classification of arbitrary torsion abelian groups up to equivalence in some infinitary language, e.g., $L_{\infty\omega}$. Ulm's theorem was generalized to Warfield groups by Warfield [23] in the p-local case and by Hunter, Richman [7] and Stanton [19] in the global case. In [8] Jacoby developed the class of abelian groups with partial decomposition bases in order to extend Barwise and Eklof's [1] classification of torsion groups in $L_{\infty\omega}$ to Warfield groups. This class coincides with the class of k-groups defined by Hill and Megibben [4, 5] (see [12, 14]). Using modifications of the Ulm and Warfield invariants, groups with partial decomposition bases have been classified in $L_{\infty\omega}$, or equivalently, up to partial isomorphism ([13, 15]) as well as in $L^{\delta}_{\infty\omega}$ ([9–11]). It follows from [9, Theorem 6.10] (see the first part of Theorem 4.9) that given any cardinal $\kappa > \omega$, if two groups G and H with partial decomposition bases have identical modified Ulm and Warfield invariants up to κ, then G and H are equivalent in $L_{\kappa\omega}$. In this paper, the converse of this result is proved. More specifically, we show that the modified Warfield invariant up to κ is expressible in $L_{\kappa\omega}$ (Theorem 4.10). Therefore two $L_{\kappa\omega}$-equivalent groups with partial decomposition bases have identical modified invariants up to κ. It follows that for groups with partial decomposition bases, their $L_{\kappa\omega}$-equivalence classes are exactly their $L^{\kappa}_{\infty\omega}$-equivalence classes (see Corollary 4.11).

For notation and terminology on abelian groups and on model theory, we may refer to [2, 6, 16, 18].

2. General and algebraic preliminaries

We accept the ZFC axioms of set theory. Throughout, we use $\alpha, \beta, \gamma, \ldots$ to denote ordinals and $\kappa, \lambda, \ldots$ to denote cardinals. The cardinality of a set X is written $|X|$. Recall that an infinite cardinal is *regular* if its cofinality is equal to itself, otherwise it is *singular*. For an ordinal α, let ω_α (or $\aleph_\alpha$) denote the αth infinite cardinal and set $\omega = \omega_0$. Then ω_α is regular if $\alpha = 0$ or α is a successor ordinal (see [17, p. 49]), thus α is a limit ordinal whenever ω_α is singular. If κ is an infinite cardinal, then $\kappa = \omega_\alpha$ for some α (cf. [17, pp. 29–30]).

All groups considered in this paper are abelian, written additively. Let G be a group and p a prime. The torsion part of G is denoted by tG and the p-torsion part is written t_pG. We write $G_p = G \otimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the group of integers localized at p. Notice that $t_pG \cong t(G_p)$. If S is a subset of G, then $\langle S\rangle$ denotes the subgroup of G generated by S. For all ordinals α, subgroups $p^\alpha G$ of G are defined inductively by $p^0G = G$, $p^{\alpha+1}G = p(p^\alpha G)$, and $p^\alpha G = \bigcap_{\beta<\alpha} p^\beta G$ if α is a limit ordinal. Then the *p-height* $|x|_p$ (or $|x|^G_p$) of $x \in G$ is defined to be α if $x \in p^\alpha G \setminus p^{\alpha+1}G$ and $|x|_p = \infty$ if

$x \in p^\infty G := \bigcap_\alpha p^\alpha G$. There is a smallest ordinal τ such that $p^\tau G = p^\infty G$, called the *p-length* of G. It is called the *length* of G if G is a p-group. Let $G[p] = \{x \in G : px = 0\}$. The *p-rank* of G, written $r_p(G)$, is the cardinality of a maximal independent subset of $t_p G$. We have $r_p(G) = r_p(t_p G) = r_p(G[p]) = \dim_{\mathbb{Z}/p\mathbb{Z}}(G[p])$ (cf. [16, Corollary 1.4.3, p. 11]). The *torsion-free rank* of G, written $r_0(G)$, is the cardinality of a maximal independent set of elements of G of infinite order. We set $r_{p,0}(G) = r_p(G) + r_0(G)$ and define $\hat{r}_{p,0}(G) = \min\{\hat{r}_{p,0}(G), \omega\}$. A subgroup H of G is called *p-isotype* if $p^\alpha G \cap H = p^\alpha H$ for all ordinals α, and H is called *isotype* if it is p-isotype for all primes p. For example, tG is isotype and $t_p G$ is p-isotype in G.

For a prime p and an ordinal α, the *Ulm invariants* of G are defined by

$$u_p(\alpha, G) = \dim_{\mathbb{Z}/p\mathbb{Z}}((p^\alpha G)[p]/(p^{\alpha+1} G)[p])$$

with α an ordinal and

$$u_p(\infty, G) = \dim_{\mathbb{Z}/p\mathbb{Z}}(p^\infty G)[p].$$

Ulm [21] proved that these cardinal numbers are isomorphism invariants of countable torsion groups. Barwise and Eklof's [1] modified Ulm invariant is defined by

$$\hat{u}_p(\alpha, G) = \min\{u_p(\alpha, G), \omega\}.$$

By adding the invariant $\hat{u}_p(\infty, G) = \min\{u_p(\infty, G), \omega\}$, these invariants classify all torsion groups in $L_{\infty\omega}$ (see [1]).

By Ord we mean the class of all ordinals and define $\mathrm{Ord}_\infty = \mathrm{Ord} \cup \{\infty\}$. We adopt the convention that $\infty > \alpha$ whenever $\alpha \in \mathrm{Ord}_\infty$ and set $\infty + 1 = \infty$. For simplicity, we may write $\alpha \wedge \beta$ instead of $\min\{\alpha, \beta\}$. An *Ulm sequence* is an increasing sequence $u = (\beta_i)_{i<\omega}$ where each β_i is an ordinal or the symbol ∞. Two Ulm sequences $u = (\beta_i)$ and $v = (\gamma_i)$ are called *equivalent*, written $u \sim v$, if there exist $k, l < \omega$ such that $\beta_{i+k} = \gamma_{i+l}$ for all $i < \omega$. The *p-Ulm sequence of* $x \in G$ is the sequence $\|x\|_p = (|p^i x|_p)_{i<\omega}$.

An *Ulm matrix* is an $\omega \times \omega$-matrix $M = [m_{p,i}]$ indexed over all primes and non-negative integers such that each row $M_p = (m_{p,i})_{i<\omega}$ is an Ulm sequence. For a group G, the *Ulm matrix of* $x \in G$ (written $\|x\|$ or $\|x\|_G$) is defined to be the Ulm matrix $[|p^i x|_p]$ where the rows are the p-Ulm sequences of x with $p \in \mathbb{P}$. If n is a positive integer, we let nM be the Ulm matrix having $m_{p,i+|n|_p}$ as its (p, i) entry, where $|n|_p$ is the p-height of n in $\mathbb{Z}$. If $M = [m_{p,i}]$ and $N = [n_{p,i}]$, we write $M \geq N$ in the case $m_{p,i} \geq n_{p,i}$ for all (p, i). Two Ulm matrices M and N are called *compatible* if there are positive integers m and n such that $mM \geq N$ and $nN \geq M$. This yields an equivalence relation, and the equivalence classes are called *compatibility classes*.

We recall the following definitions "up to α" for a given ordinal α (see [16]): For $\beta, \gamma \in \mathrm{Ord}_\infty$ we define $\beta \geq_\alpha \gamma$ to mean

$$\beta \wedge \alpha \geq \gamma \wedge \alpha.$$

If equality holds, then we write $\beta =_\alpha \gamma$. For Ulm matrices $M = [m_{p,i}]$ and $N = [n_{p,i}]$ define $M \geq_\alpha N$ if $m_{p,i} \geq_\alpha n_{p,i}$ for all p and i. Then M and N are called *α-compatible*, written $M \sim_\alpha N$, if and only if for some m and n, $mM \geq_\alpha N$ and $nN \geq_\alpha M$. Given compatibility classes c and c' of Ulm matrices, we write $c \sim_\alpha c'$ to mean $M \sim_\alpha N$ for some $M \in c$ and $N \in c'$.

For Ulm sequences $u = (\beta_i)$ and $v = (\gamma_i)$ we define $u \geq_\alpha v$ if and only if $\beta_i \geq_\alpha \gamma_i$ for all $i < \omega$. We call u and v *α-equivalent*, written $u \sim_\alpha v$, if for some $m, n < \omega$ we have $\alpha_{i+m} =_\alpha \beta_{i+n}$ for all $i < \omega$. Given two equivalence classes e and e' of Ulm sequences, we write $e \sim_\alpha e'$ if $u \sim_\alpha v$ for some $u \in e$ and $v \in e'$.

Let G be a group. A subset X of G is called a *decomposition set* if

(1) all elements of X are independent and have infinite order;

(2) for each $x = k_1x_1 + \cdots + k_nx_n$ $(k_1, \ldots, k_n \in \mathbb{Z}, x_1, \ldots, x_n \in X)$ and prime p, we have $|x|_p = \min\{|k_ix_i|_p\}$.

Then X is called a *decomposition basis for G* if, in addition, $G/\langle X\rangle$ is torsion. A decomposition basis X for G is called *nice* if for every prime p, $\langle X\rangle_p$ is a p-nice subgroup of G_p, i.e., if every coset $x + \langle X\rangle_p$ $(x \in G_p)$ has an element of maximal p-height. Groups possessing a nice decomposition basis X with simply presented quotient $G/\langle X\rangle$ are called *Warfield groups*. Stanton defined cardinal numbers that, together with the Ulm invariants, form a complete set of isomorphism invariants for Warfield groups (see Hunter–Richman [7], Stanton [19]).

Jacoby [8] generalized the concept of decomposition basis: Given a group G, a system $\mathcal{C}$ is called a *partial decomposition basis for G* if

(1) $\mathcal{C}$ is a nonempty collection of finite subsets of G;

(2) if $X \in \mathcal{C}$, then X is a decomposition set;

(3) if $X \in \mathcal{C}$ and $x \in G$, there is $Y \in \mathcal{C}$ such that $X \subseteq Y$ and $mx \in \langle Y\rangle$ for some positive integer m.

Now let G be a group with partial decomposition basis $\mathcal{C}$, c a compatibility class of Ulm matrices, p a prime and e an equivalence class of Ulm sequences. Jacoby [8] defined $\widehat{w}(c, p, e, G)$ to be the largest integer n, if it exists, such that there are $X \in \mathcal{C}$ and $x_1, \ldots, x_n \in X$ with $\|x_i\| \in c$ and $\|x_i\|_p \in e$ for all $i = 1, \ldots, n$. If no such n exists, put $\widehat{w}(c, p, e, G) = \omega$. Jacoby, Leistner, Loth and Strüngmann [9] generalized this definition up to an ordinal α:

$$\widehat{w}_\alpha(c,p,e,G) = \min\{\sum_{e'\sim_\alpha e, c'\sim_\alpha c} \widehat{w}(c',p,e',G),\omega\}.$$

3. Model-theoretic preliminaries

The infinitary language $L_{\infty\omega}$ is an extension of a language of first-order logic that allows conjunctions and disjunctions over arbitrary sets of formulas and quantifications over finite sets of variables. The symbols of the language include a variable v_α for each ordinal α, and since we will be using this language to discuss abelian groups, we include the binary function symbol $+$, the unary function symbol $-$ and the constant 0. We use these symbols to form atomic formulas in the usual way.

For each ordinal α we define, by induction on α, $L^\alpha_{\infty\omega}$ to be the smallest class Y that contains the atomic formulas and is closed under the following logical operations:

(1) If $\varphi \in Y$ then $\neg\varphi \in Y$;

(2) if Φ is a set, $\Phi \subseteq Y$, then $\bigwedge \Phi$ and $\bigvee \Phi \in Y$;

(3) if $\varphi \in L^\beta_{\infty\omega}$ for some $\beta < \alpha$ and v is a variable, then $\forall v\varphi \in Y$ and $\exists v\varphi \in Y$.

In (2), $\bigwedge \Phi$ (resp. $\bigvee \Phi$) denotes the conjunction (resp. disjunction) of all elements of the set Φ. Notice that Φ can be of any cardinality and that the empty disjunction $\bigvee \emptyset$ is false everywhere (cf. [6, p. 28]). Then let $L_{\infty\omega} = \bigcup_{\alpha\in\mathrm{Ord}} L^\alpha_{\infty\omega}$. An element φ of $L_{\infty\omega}$ is called a *formula* if φ has at most a finite number of free variables. When we write "$\varphi \in L_{\infty\omega}$" it is understood that φ is a formula. *Sentences* are formulas with no free variables. For $\varphi \in L_{\infty\omega}$, the *quantifier rank of* φ is $\mathrm{qr}(\varphi) = \min\{\alpha : \varphi \in L^\alpha_{\infty\omega}\}$.

The set of *subformulas of* φ, written $\mathrm{sub}(\varphi)$, is defined inductively by the following:

(1) If φ is atomic, then $\mathrm{sub}(\varphi) = \{\varphi\}$;

(2) if φ is $\neg\psi$, $\forall v\psi$ or $\exists v\psi$, then $\mathrm{sub}(\varphi) = \mathrm{sub}(\psi) \cup \{\varphi\}$;

(3) if φ is $\bigwedge \Psi$ or $\bigvee \Psi$, then $\mathrm{sub}(\varphi) = \bigcup_{\psi\in\Psi} \mathrm{sub}(\psi) \cup \{\varphi\}$.

A subformula ψ of φ is called *proper* if $\psi \neq \varphi$. Hodges [6] defined the *complexity* of a formula $\varphi \in L_{\infty\omega}$ inductively to be the ordinal $\mathrm{comp}(\varphi)$ such that

$$\mathrm{comp}(\varphi) = \begin{cases} 0 & \text{if } \varphi \text{ is an atomic formula,} \\ \sup\{\mathrm{comp}(\psi)+1 : \psi \text{ is a proper subformula of } \varphi\} & \text{otherwise.} \end{cases}$$

For an infinite cardinal κ, the sublanguage $L_{\kappa\omega}$ of $L_{\infty\omega}$ is defined to be

$$L_{\kappa\omega} = \bigl\{\varphi \in L_{\infty\omega} : |\mathrm{sub}(\varphi)| < \kappa\bigr\}$$

(see [1]). Two groups G and H are called $L^{\alpha}_{\infty\omega}$*-equivalent* (resp. $L_{\kappa\omega}$*-equivalent*), and we write

$$G \equiv_{\alpha} H \quad (\text{resp. } G \equiv_{\kappa\omega} H),$$

if G and H satisfy the same sentences of $L^{\alpha}_{\infty\omega}$ (resp. $L_{\kappa\omega}$).

The following result on $L_{\kappa\omega}$ is stated (without proof) in terms of cardinals $\kappa = \omega_{\alpha}$ in [1, p. 28]:

Proposition 3.1. *Let κ be a an infinite cardinal.*

(1) *If κ is regular, then $L_{\kappa\omega}$ is the smallest class Y of formulas such that*
   - (a) *Y contains all atomic formulas;*
   - (b) *if φ is in Y, then $\neg\varphi$ is in Y;*
   - (c) *if φ is in Y and v is a variable, then $\forall v\varphi$ and $\exists v\varphi$ are in Y;*
   - (d) *if Ψ is a subset of Y and $|\Psi| < \kappa$, then $\bigwedge \Psi$ and $\bigvee \Psi$ are in Y.*

(2) *If $\kappa = \omega_{\alpha}$, where α is a limit ordinal, then $L_{\kappa\omega} = \bigcup_{\omega \le \lambda < \kappa} L_{\lambda\omega}$.*

(3) *If $\varphi \in L_{\kappa\omega}$ then $\mathrm{qr}(\varphi) < \kappa$, i.e., $L_{\kappa\omega} \subseteq \bigcup_{\gamma<\kappa} L^{\gamma}_{\infty\omega}$. Thus $L^{\kappa}_{\infty\omega}$-equivalence implies $L_{\kappa\omega}$-equivalence.*

Proof. (1) It is clear that $L := L_{\kappa\omega}$ satisfies (a)–(c). Now assume that φ equals $\bigwedge \Phi$ or $\bigvee \Phi$, where $\Psi \subseteq L$ and $|\Psi| < \kappa$. Since κ is regular, we have $|\mathrm{sub}(\varphi)| = |\bigcup_{\psi\in\Psi} \mathrm{sub}(\psi) \cup \{\varphi\}| < \kappa$. Thus φ is in L.

To prove that L is the smallest such class, let M be any class of formulas satisfying (a)–(d). To show that $L \subseteq M$ we use induction on the complexity of formulas, so let $\varphi \in L$. If φ is atomic then $\varphi \in M$ by (a). Now assume that φ is not an atomic formula and let $\psi \in L$. If φ is $\neg\psi$, $\forall v\psi$ or $\exists v\psi$, then $\psi \in M$ by the induction hypothesis, thus $\varphi \in M$ by (b) and (c). Finally, assume that φ equals $\bigwedge \Psi$ or $\bigvee \Psi$ with $\Psi \subseteq L$, where $|\Psi| < \kappa$. By the induction hypothesis, $\Psi \subseteq M$, so by (d) we have $\varphi \in M$ as desired.

(2) If $\varphi \in L_{\kappa\omega}$ then $|\mathrm{sub}(\varphi)| < \omega_{\gamma}$ for some $\gamma < \alpha$. Thus $\varphi \in L_{\omega_{\gamma}\omega} \subseteq \bigcup_{\beta<\alpha} L_{\omega_{\beta}\omega} = \bigcup_{\omega\le\lambda<\kappa} L_{\lambda\omega}$.

(3) We write $\kappa = \omega_{\alpha}$ and prove the assertion by transfinite induction on α. First assume that ω_{α} is a regular cardinal; this includes the case $\alpha = 0$. We show that $M := \bigcup_{\gamma<\kappa} L^{\gamma}_{\infty\omega}$ satisfies (1) (a)–(d). Clearly, (a) and (b) are true. To see that M satisfies (c), let $\varphi \in M$. Then $\varphi \in L^{\gamma}_{\infty\omega}$ for some $\gamma < \kappa$, thus $\forall v\varphi$ and $\exists v\varphi$ are in $L^{\gamma+1}_{\infty\omega} \subseteq M$. To show that M satisfies (d), let $\Psi = \{\psi_i : i \in I\} \subseteq M$, where $|I| < \kappa$, say $\psi_i \in L^{\gamma_i}_{\infty\omega}$ with $\gamma_i < \kappa$ for all $i \in I$. Since κ is regular we have $\gamma' := \sup\{\gamma_i : i \in I\} < \kappa$, thus $\psi_i \in L^{\gamma'}_{\infty\omega}$ for all i. Hence both $\bigwedge \Psi$ and $\bigvee \Psi$ are in $L^{\gamma'}_{\infty\omega} \subseteq M$. By (1) we have $L_{\kappa\omega} \subseteq M$.

Now let $\alpha > 0$ and assume the claim holds for all $\beta < \alpha$. If ω_α is regular we are done by the paragraph above. If ω_α is singular then α is a limit ordinal, so by (2) and the induction hypothesis we have $L_{\omega_\alpha\omega} = \bigcup_{\beta<\alpha} L_{\omega_\beta\omega} \subseteq \bigcup_{\beta<\alpha} \bigcup_{\gamma<\omega_\beta} L^{\gamma}_{\infty\omega} = \bigcup_{\gamma<\omega_\alpha} L^{\gamma}_{\infty\omega}$, as desired. ∎

4. $L_{\kappa\omega}$-equivalence of groups with partial decomposition bases

Barwise and Eklof [1] examined torsion groups from the viewpoint of $L_{\infty\omega}$ and showed that key characteristics of p-groups are expressible in $L_{\infty\omega}$ and $L_{\kappa\omega}$, with some proof details omitted that are included here. The proofs apply to arbitrary groups as well.

Lemma 4.1. *Let κ be an infinite cardinal, α an ordinal and p a prime.*

(1) *There is a formula $\theta_\alpha(x)$ that has at most $|\alpha| + 1$ subformulas such that for any group G and any $a \in G$, $G \models \theta_\alpha[a]$ if and only if $a \in p^\alpha G$. More specifically, letting $s_\alpha = |\mathrm{sub}(\theta_\alpha(x))|$, we have $s_0 = 1$, $s_\alpha = 2$ if $0 < \alpha < \omega$ and $s_\alpha = |\alpha|$ if $\alpha \geq \omega$.*

(2) *If $\alpha < \kappa$ then $\theta_\alpha(x)$ is a formula of $L_{\kappa\omega}$.*

Proof. (1) For $\theta_0(x)$, use $x = x$. If $0 < \alpha < \omega$ let $\theta_\alpha(x) = \exists y(x = p^\alpha y)$. To prove that $s_\alpha = |\alpha|$ for all $\alpha \geq \omega$ we use transfinite induction. It is clear that $s_\omega = \aleph_0$ since "$x \in p^\omega G$" is expressible by $\theta_\omega(x) = \bigwedge_{\beta<\omega} \theta_\beta(x)$. Now let $\alpha > \omega$ and assume that $s_\beta = |\beta|$ for all $\omega \leq \beta < \alpha$. If $\alpha - 1$ exists, let $\theta_\alpha(x) = \exists y(\theta_{\alpha-1}(y) \wedge x = py)$ and if α is a limit ordinal, let $\theta_\alpha(x) = \bigwedge_{\beta<\alpha} \theta_\beta(x)$. In either case we conclude that $s_\alpha = |\alpha|$.

(2) follows from (1). ∎

We will adopt the convention of using an algebraic formula in quotation marks to represent the corresponding expression in $L_{\infty\omega}$. For example, by Lemma 4.1 (1), "$x \in p^\alpha G$" is expressible by a formula that has $\leq \lambda := |\alpha| + 1$ subformulas. Alternatively, we may write

$$\text{"}x \in p^\alpha G\text{" is in } L_{\infty\omega} \text{ and has } \leq \lambda \text{ subformulas.}$$

Notice that "$x \in tG$" is in $L_{\omega_1\omega}$ since it is expressible by the formula $\varphi(x) = \bigvee_{0<n<\omega}(nx = 0)$. Likewise, "$x \in t_p G$" and "$x$ has infinite order" are in $L_{\omega_1\omega}$.

Lemma 4.2. *Suppose that H is a subgroup of a group G such that "$x \in H$" is in $L_{\infty\omega}$ and has λ subformulas.*

(1) *If* $a_1, \ldots, a_m \in \mathbb{Z}$ *then* "$a_1x_1 + \cdots + a_mx_m \in H$" *has* $\lambda + 2$ *subformulas.*

(2) "$x_1, \ldots, x_m$ *are independent modulo* H" *has* $\aleph_0 \cdot \lambda$ *subformulas.*

(3) "x *has infinite order modulo* H" *and* "x *has* p*-power order modulo* H" *have* $\aleph_0 \cdot \lambda$ *subformulas, where* p *is a given prime.*

Proof. (1) Suppose that "$x \in H$" is expressed by the formula $\psi(x)$. Then $a_1x_1 + \cdots + a_mx_m \in H$ if and only if $\exists x(\psi(x) \wedge x = a_1x_1 + \cdots + a_mx_m)$.

(2) $x_1, \ldots, x_m$ are independent modulo H if and only if for all $a_1, \ldots, a_m \in \mathbb{Z}$, $a_1x_1 + \cdots + a_mx_m \in H$ implies $a_ix_i \in H$ for all $1 \leq i \leq m$. This may be written as $\bigwedge_{a_1\in\mathbb{Z}} \cdots \bigwedge_{a_m\in\mathbb{Z}}$("$a_1x_1 + \cdots + a_mx_m \in H$" $\to \bigwedge_{i=1}^{m}$ "$a_ix_i \in H$"). Now apply (1).

(3) "x has infinite order modulo H" is true if and only if $\bigwedge_{0<n<\omega} \neg$("$nx \in H$"), and "$x$ has p-power order modulo H" holds exactly if $\bigvee_{n<\omega}$("$p^nx \in H$"). ∎

Lemma 4.3. *Let* G *be a group,* p *a prime,* α *an ordinal and* $m < \omega$. *Then* "$r_p(p^\alpha G) \geq m$" *and* "$u_p(\alpha, G) \geq m$" *are expressible by formulas that have at most* $\aleph_0 \cdot |\alpha|$ *subformulas.*

Proof. We have $r_p(p^\alpha G) \geq m$ if and only if there are m independent elements of p-height $\geq \alpha$, if and only if $G \models \exists x_1 \cdots \exists x_m(\bigwedge_{i=1}^{m}$ "$x_i \in p^\alpha G$" $\wedge$ "$x_1, \ldots, x_m$ are independent" $\wedge$ "$x_1, \ldots, x_m \in t_pG$").

Note that $u_p(\alpha, G) \geq m$ if and only if $G \models \exists x_1 \cdots \exists x_m[\bigwedge_{i=1}^{m}$("$x_i \in p^\alpha G$" $\wedge\ px_i = 0) \wedge$ "$x_1, \ldots, x_m$ are independent modulo $p^{\alpha+1}G$"]. ∎

The following will be needed.

Lemma 4.4. *Let* G *be a group,* p *a prime,* $\tau = \mathrm{length}(t_pG)$ *and* $\nu \geq \tau$. *Then* $u_p(\infty, G) = r_p(p^\nu G)$.

Proof. We have $u_p(\infty, G) = r_p(p^\infty G) = r_p(\bigcap_{\sigma\geq\nu} p^\sigma G \cap t_pG)$, so since t_pG is p-isotype in G we obtain $u_p(\infty, G) = r_p(\bigcap_{\sigma\geq\nu} p^\sigma(t_pG)) = r_p(p^\nu(t_pG)) = r_p(p^\nu G \cap t_pG) = r_p(p^\nu G)$. ∎

The next result was proved for p-groups in [1, Corollary 3.5 (1)⇒(2)].

Theorem 4.5. *Let* G *and* H *be groups such that* $G \equiv_{\kappa\omega} H$, *where* κ *is a cardinal* $> \omega$. *Then we have*

(1) $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ *for all primes* p *and ordinals* $\alpha < \kappa$;

(2) *if* $\mathrm{length}(t(G_p)) < \kappa$, *then* $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$.

PROOF. (1) Let p be a prime and $\alpha < \kappa$. If $\hat{u}_p(\alpha, G) = m < \omega$ then $G \models$ "$u_p(\alpha, G) \geq m$" $\wedge \neg$"$u_p(\alpha, G) \geq m+1$", an expression with $\leq \aleph_0 \cdot |\alpha| < \kappa$ subformulas by Lemma 4.3. Thus H models the same expression, i.e., $\hat{u}_p(\alpha, H) = m$. Now assume that $\hat{u}_p(\alpha, G) = \omega$. Then $G \models \bigwedge_{m<\omega}$ "$u_p(\alpha, G) \geq m$", hence $\hat{u}_p(\alpha, H) = \omega$.

(2) Suppose $t(G_p) \cong t_p G$ has length $\tau < \kappa$. For $\sigma \geq \tau$ we have $(p^\sigma G)[p] = (p^\sigma G \cap t_p G)[p] = (p^\sigma(t_p G))[p] = (p^\tau(t_p G))[p]$, thus $u_p(\sigma, G) = 0$. Then (1) shows that $u_p(\sigma, H) = 0$ for all $\tau \leq \sigma < \kappa$. Assume that $\tau' := \text{length}(t(H_p)) \geq \kappa$. Since κ is a limit ordinal we have $u_p(\sigma, H) \neq 0$ for infinitely many $\tau < \sigma < \kappa$ by [16, Lemma 1.13.7 (2), p. 34], a contradiction. Therefore $\tau' < \kappa$. For $\nu = \max\{\tau, \tau'\}$ we have $u_p(\infty, G) = r_p(p^\nu G)$ and $u_p(\infty, H) = r_p(p^\nu H)$ (Lemma 4.4). Since "$r_p(p^\nu G) \geq m$" has at most $\aleph_0 \cdot |\nu| < \kappa$ subformulas (Lemma 4.3), we conclude that $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$. ∎

We wish to show that the modified Warfield invariant $\widehat{w}_\alpha(c, p, e, G)$ is expressible in $L_{\kappa\omega}$ whenever $\alpha < \kappa$. First we turn to a modification of an invariant defined by Stanton [19], equivalent to $\widehat{w}_\alpha(c, p, e, G)$, which we then prove is expressible in $L_{\kappa\omega}$.

For a group G, a prime p, an Ulm matrix M and an ordinal α, we define the following as in [16]:

$$\begin{aligned} G_\alpha(M) &= \{x \in G : \|x\| \geq_\alpha M\}, \\ G_\alpha(M^*)^\diamond &= \{x \in G_\alpha(M) : \|x\| \nsim_\alpha M \text{ or } x \text{ is torsion}\}, \\ G_\alpha(M_p^*, p)^\diamond &= \{x \in G : \|x\|_p \geq_\alpha M_p \text{ and } |p^i x|_p \neq_\alpha m_{p,i} \text{ for infinitely many } i\}, \\ G_\alpha(M^*, p)^\diamond &= G_\alpha(M^*)^\diamond \cup G_\alpha(M_p^*, p)^\diamond, \\ G_\alpha(M^*, p) &= G_\alpha(M) \cap \langle G_\alpha(M^*, p)^\diamond \rangle. \end{aligned}$$

Then for a compatibility class c of Ulm matrices and an equivalence class e of Ulm sequences, the modified Stanton invariant is defined to be

$$\hat{s}_\alpha(c, p, e, G) = \sup\{\hat{r}_{p,0}(G_\alpha(M)/G_\alpha(M^*, p))\},$$

where the supremum is taken over all Ulm matrices $M \in c$ with $M_p \in e$ (cf. [16]). For groups with partial decomposition bases, this invariant can be expressed as follows.

THEOREM 4.6 (Jacoby–Loth [10, 16]). *Let G be a group with partial decomposition basis, α an ordinal, c a compatibility class of Ulm matrices, p a prime and e an equivalence class of Ulm sequences. Then we have*

(1) $\hat{s}_\alpha(c, p, e, G) = \widehat{w}_\alpha(c, p, e, G)$;

(2) $\hat{s}_\alpha(c, p, e, G) = \hat{r}_{p,0}(G_\alpha(M)/G_\alpha(M^*, p))$ *for all $M \in c$ with $M_p \in e$.*

The next lemma will be useful.

LEMMA 4.7. *Let α be an ordinal and let $\beta, \gamma \in \mathrm{Ord}_\infty$.*

(1) $\beta \geq_\alpha \gamma$ *if and only if* $\beta \geq \gamma \wedge \alpha$.

(2) $\beta \geq_\alpha \gamma$ *and* $\beta \neq_\alpha \gamma$ *if and only if* $\gamma \ngeq_\alpha \beta$.

(3) $\gamma \ngeq_\alpha \beta$ *if and only if* $\beta \geq (\gamma + 1) \wedge \alpha$ *and* $\gamma < \alpha$.

PROOF. (1) If $\beta \geq_\alpha \gamma$ then $\beta \geq \beta \wedge \alpha \geq \gamma \wedge \alpha$ and if $\beta \geq \gamma \wedge \alpha$, then $\beta \wedge \alpha \geq \gamma \wedge \alpha$.

(2) $\beta \wedge \alpha \geq \gamma \wedge \alpha$ and $\beta \wedge \alpha \neq \gamma \wedge \alpha$ if and only if $\beta \wedge \alpha > \gamma \wedge \alpha$.

(3) By (1), $\gamma \ngeq_\alpha \beta$ if and only if $\gamma < \beta \wedge \alpha$. First, assume that $\gamma < \beta \wedge \alpha$. Then $\gamma < \alpha$ and since $\beta \geq \beta \wedge \alpha > \gamma$, we have $\beta \geq \gamma + 1 \geq (\gamma + 1) \wedge \alpha$.

Conversely, assume $\beta \geq (\gamma + 1) \wedge \alpha$ and $\gamma < \alpha$. Then $\gamma + 1 \leq \alpha$, hence $\beta \geq (\gamma + 1) \wedge \alpha = \gamma + 1$. Therefore $\beta \wedge \alpha > \gamma$. ∎

LEMMA 4.8. *Let G be a group, p a prime and α an ordinal. Let $\lambda = \aleph_0 \cdot |\alpha|$.*

(1) *If M is any Ulm matrix, then each of the expressions "$x \in G_\alpha(M)$", "$x \in G_\alpha(M^*)^\diamond$", "$x \in G_\alpha(M_p^*, p)^\diamond$", "$x \in G_\alpha(M^*, p)^\diamond$" and "$x \in G_\alpha(M^*, p)$" can be written as a formula that has $\leq \lambda$ subformulas.*

(2) *If $m < \omega$, then "$\hat{s}_\alpha(c, p, e, G) \geq m$" is expressible by a formula that has $\leq \lambda$ subformulas.*

PROOF. (1.1) We write $M = [m_{q,i}]$. Then Lemma 4.7 (1) shows that $x \in G_\alpha(M)$ if and only if $q^i x \in q^{m_{q,i} \wedge \alpha} G$ for all q and i. By Lemmas 4.1 (1) and 4.2 (1), the expression $\bigwedge_{(q,i) \in \mathbb{P} \times \omega}$ "$q^i x \in q^{m_{q,i} \wedge \alpha} G$" has $\leq \lambda$ subformulas.

(1.2) We have $x \in G_\alpha(M^*)^\diamond$ exactly if $x \in G_\alpha(M)$ and either x is torsion or $\|x\| \sim_\alpha M$. If $x \in G_\alpha(M)$, then $\|x\| \sim_\alpha M$ if and only if $jM \ngeq_\alpha \|x\|$ for every $j < \omega$. By Lemma 4.7 (3), this holds if and only if for every $j < \omega$ there are q and i, for which $|q^i x|_q \geq (m_{q,i+|j|_q} + 1) \wedge \alpha$ and $m_{q,i+|j|_q} < \alpha$. Therefore the statement may be written as "$x \in G_\alpha(M)$" $\wedge$ ("x is torsion" $\vee \bigwedge_{j<\omega} \bigvee_{(q,i) \in X_j}$ "$q^i x \in q^{(m_{q,i+|j|_q}+1) \wedge \alpha} G$"), where $X_j = \{(q,i) \in \mathbb{P} \times \omega : m_{q,i+|j|_q} < \alpha\}$ for $j < \omega$. Since the expressions in quotes have $\leq \lambda$ subformulas and $|X_j| \leq \aleph_0$ for all j, "$x \in G_\alpha(M^*)^\diamond$" has $\leq \lambda$ subformulas.

(1.3) "$x \in G_\alpha(M_p^*, p)^\diamond$" holds if and only if $\|x\|_p \geq_\alpha M_p$ and for every $j < \omega$ there exists an integer $i \geq j$ such that $|p^i x|_p \neq_\alpha m_{p,i}$. By Lemma 4.7 (2),(3) we have $|p^i x|_p \geq_\alpha m_{p,i}$ and $|p^i x|_p \neq_\alpha m_{p,i}$ if and only if $|p^i x|_p \geq (m_{p,i} + 1) \wedge \alpha$ and $m_{p,i} < \alpha$. Therefore the statement is expressible by ($\bigwedge_{i<\omega}$ "$p^i x \in p^{m_{p,i} \wedge \alpha} G$") $\wedge \bigwedge_{j<\omega} \bigvee_{i \in X_j}$ "$p^i x \in p^{(m_{p,i}+1) \wedge \alpha} G$", where $X_j = \{i < \omega : i \geq j$ and $m_{p,i} < \alpha\}$. Again, the formula has $\leq \lambda$ subformulas.

(1.4) By (1.2) and (1.3), "$x \in G_\alpha(M^*, p)^\diamond$" has $\leq \lambda$ subformulas.

(1.5) "$x \in G_\alpha(M^*, p)$" is true if and only if $x \in G_\alpha(M)$ and there are $0 < n < \omega$ and $z_1, \ldots, z_n \in G_\alpha(M^*, p)^\diamond$ such that $x = z_1 + \cdots + z_n$. Thus "$x \in G_\alpha(M^*, p)$" has $\leq \lambda$ subformulas.

(2) We have $\hat{s}_\alpha(c, p, e, G) \geq m$ if and only if for any $M \in c$ with $M_p \in e$, the following is true: $G \models \exists x_1 \cdots \exists x_m[\bigwedge_{i=1}^m$("$x_i \in G_\alpha(M)$" $\wedge$ "x_i has either infinite or p-power order modulo $G_\alpha(M^*, p)$") $\wedge$ "$x_1, \ldots, x_m$ are independent modulo $G_\alpha(M^*, p)$"], an expression that has $\leq \lambda$ subformulas. ■

In $L^\delta_{\infty\omega}$, groups with partial decomposition bases are classified using modified Ulm and Warfield invariants.

Theorem 4.9 (Jacoby–Leistner–Loth–Strüngmann [9], Jacoby–Loth [10]). *Let G and H be groups with partial decomposition bases, and let δ be an ordinal. Suppose*

(1) $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ *for all primes p and $\alpha < \omega\delta$;*

(2) $\widehat{w}_{\omega(\nu+1)}(c, p, e, G) = \widehat{w}_{\omega(\nu+1)}(c, p, e, H)$ *for every compatibility class c of Ulm matrices, prime p, equivalence class e of Ulm sequences, and $\nu < \delta$;*

(3) *if* $\mathrm{length}(t(G_p)) < \omega\delta$, *then* $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$.

Then $G \equiv_\delta H$. If $\delta = \omega\gamma$ and γ is a limit ordinal, then the converse is also true.

If $\kappa > \omega$, then two $L_{\kappa\omega}$-equivalent groups with partial decomposition bases have identical modified Warfield invariants up to κ.

Theorem 4.10. *Let G and H be groups with partial decomposition bases and let κ be a cardinal $> \omega$ such that $G \equiv_{\kappa\omega} H$. If α is an ordinal $< \kappa$, then $\widehat{w}_\alpha(c, p, e, G) = \widehat{w}_\alpha(c, p, e, H)$ for all c, p, e.*

Proof. Let $m < \omega$. Both "$\hat{s}_\alpha(c, p, e, G) = m$" and "$\hat{s}_\alpha(c, p, e, G) = \omega$" are in $L_{\kappa\omega}$ as they have $\leq \aleph_0 \cdot |\alpha| < \kappa$ subformulas by Lemma 4.8 (2), thus $\hat{s}_\alpha(c, p, e, G) = \hat{s}_\alpha(c, p, e, H)$. Therefore, $\widehat{w}_\alpha(c, p, e, G) = \widehat{w}_\alpha(c, p, e, H)$ by Theorem 4.6 (1). ■

Corollary 4.11. *Let G and H be groups with partial decomposition bases and let κ be a cardinal $> \omega$. The following conditions are equivalent:*

(1) $G \equiv_{\kappa\omega} H$;

(2) (a) $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ *for all primes p and ordinals $\alpha < \kappa$;*

(b) $\widehat{w}_{\omega(\nu+1)}(c, p, e, G) = \widehat{w}_{\omega(\nu+1)}(c, p, e, H)$ *for every compatibility class c of Ulm matrices, prime p, equivalence class e of Ulm sequences, and ordinal $\nu < \kappa$;*

(c) *if* length$(t(G_p)) < \kappa$, *then* $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$;

(3) $G \equiv_\kappa H$.

Proof. We have (2) $\Leftrightarrow$ (3) by Theorem 4.9 since $\omega\delta = \delta$ for $\delta = \kappa$. That (1) $\Rightarrow$ (2) follows from Theorems 4.5 and 4.10, and (3) $\Rightarrow$ (1) is immediate by Proposition 3.1 (3). ∎

Corollary 4.12 (Barwise–Eklof [1]). *Let G and H be p-groups and let κ be a cardinal $> \omega$. The following conditions are equivalent:*

(1) $G \equiv_{\kappa\omega} H$;

(2) (a) $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ *for all ordinals* $\alpha < \kappa$;

(b) *if* length$(G) < \kappa$, *then* $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$.

(3) $G \equiv_\kappa H$.

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References

[1] J. Barwise – P. Eklof, Infinitary properties of abelian torsion groups. *Ann. Math. Logic* **2** (1970/71), no. 1, 25–68. Zbl 0222.02014 MR 0279180

[2] L. Fuchs, *Abelian groups*. Springer Monogr. Math., Springer, Cham, 2015. Zbl 1416.20001 MR 3467030

[3] P. Hill, On the classification of abelian groups. *Period. Math. Hungar.* **69** (2014), no. 1, 41–52. Zbl 1349.20042 MR 3269707

[4] P. Hill – C. Megibben, Axiom 3 modules. *Trans. Amer. Math. Soc.* **295** (1986), no. 2, 715–734. Zbl 0597.20048 MR 0833705

[5] P. Hill – C. Megibben, Mixed groups. *Trans. Amer. Math. Soc.* **334** (1992), no. 1, 121–142. Zbl 0798.20050 MR 1116315

[6] W. Hodges, *Model theory*. Encyclopedia Math. Appl. 42, Cambridge University Press, Cambridge, 1993. Zbl 0789.03031 MR 1221741

[7] R. Hunter – F. Richman, Global Warfield groups. *Trans. Amer. Math. Soc.* **266** (1981), no. 2, 555–572. Zbl 0471.20038 MR 0617551

[8] C. C. Jacoby, *The classification in $L_{\infty\omega}$ of groups with partial decomposition bases*. Ph.D. thesis, University of California, Irvine, 1980. MR 2630562

[9] C. Jacoby – K. Leistner – P. Loth – L. Strüngmann, Abelian groups with partial decomposition bases in $L^\delta_{\infty\omega}$, Part I. In *Groups and model theory*, pp. 163–175, Contemp. Math. 576, American Mathematical Society, Providence, RI, 2012. Zbl 1274.03061 MR 2962883

[10] C. Jacoby – P. Loth, Abelian groups with partial decomposition bases in $L^{\delta}_{\infty\omega}$, Part II. In *Groups and model theory*, pp. 177–185, Contemp. Math. 576, American Mathematical Society, Providence, RI, 2012. Zbl 1274.03062 MR 2962884

[11] C. Jacoby – P. Loth, $\mathbb{Z}_p$-modules with partial decomposition bases in $L^{\delta}_{\infty\omega}$. *Houston J. Math.* **40** (2014), no. 4, 1007–1019. Zbl 1330.13010 MR 3298734

[12] C. Jacoby – P. Loth, Partial decomposition bases and Warfield modules. *Comm. Algebra* **42** (2014), no. 10, 4333–4349. Zbl 1352.13006 MR 3210378

[13] C. Jacoby – P. Loth, The classification of $\mathbb{Z}_p$-modules with partial decomposition bases in $L_{\infty\omega}$. *Arch. Math. Logic* **55** (2016), no. 7-8, 939–954. Zbl 1401.20061 MR 3555335

[14] C. Jacoby – P. Loth, Partial decomposition bases and global Warfield groups. *Comm. Algebra* **44** (2016), no. 8, 3262–3277. Zbl 1401.20060 MR 3492188

[15] C. Jacoby – P. Loth, The classification of infinite abelian groups with partial decomposition bases in $L_{\infty\omega}$. *Rocky Mountain J. Math.* **47** (2017), no. 2, 463–477. Zbl 1401.20062 MR 3635370

[16] C. Jacoby – P. Loth, *Abelian groups – Structures and classifications*. De Gruyter Stud. Math. 73, de Gruyter, Berlin, 2019. Zbl 1446.20003 MR 4319034

[17] T. Jech, *Set theory*. Springer Monogr. Math., Springer, Berlin, 2003. Zbl 1007.03002 MR 1940513

[18] P. Rothmaler, *Introduction to model theory*. Algebra, Logic and Applications 15, Gordon and Breach Science Publishers, Amsterdam, 2000. Zbl 0962.03023 MR 1800596

[19] R. O. Stanton, Decompostion bases and Ulm's theorem. In *Abelian group theory (Proc. Second New Mexico State Univ. Conf., Las Cruces, N.M., 1976)*, pp. 39–56, Lecture Notes in Math. 616, Springer, Berlin-New York, 1977. Zbl 0374.20063 MR 0498902

[20] W. Szmielew, Elementary properties of Abelian groups. *Fund. Math.* **41** (1955), 203–271. Zbl 0064.00803 MR 0072131

[21] H. Ulm, Zur Theorie der abzählbar-unendlichen Abelschen Gruppen. *Math. Ann.* **107** (1933), no. 1, 774–803. Zbl 0006.15003 MR 1512826

[22] E. A. Walker, Ulm's theorem for totally projective groups. *Proc. Amer. Math. Soc.* **37** (1973), 387–392. Zbl 0257.20039 MR 0311805

[23] R. B. Warfield, Jr., Classification theory of abelian groups. II. Local theory. In *Abelian group theory (Oberwolfach, 1981)*, pp. 322–349, Lecture Notes in Math. 874, Springer, Berlin-New York, 1981. Zbl 0469.20027 MR 0645938

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