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Finite groups with few average orders of subgroups

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Abstract – For a finite group G, we define $X(G) = \{o(H) \mid H \leq G\}$, where $o(H)$ is the average of all order elements in H. In this paper, we classify finite groups in which $|X(G)| \leq 4$.

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1. Introduction and results

The problem of detecting the structural properties of a finite group by looking at element orders has been considered by various authors, from many different points of view. Amiri, Jafarian Amiri, and Isaacs [2] introduced the function $\psi(G)$, which denotes the sum of element orders of a finite group G and proved that if G is a non-cyclic group of order n, then $\psi(G) < \psi(C_n)$, where C_n denotes the cyclic group of order n. Following this result, in several papers, it was proved that certain classes of finite groups G can be characterized using the function $\psi(G)$ (see for example [3, 4, 6–9, 13–15, 24, 27]).

Later, the average order of G was defined as $o(G) = \psi(G)/|G|$. Recently, many authors have studied the function o and more generally properties of finite groups determined by their average orders (see for example [1, 10–12, 16–18, 21, 22, 25, 26]). Jaikin-Zapirain [16] used the average order concept while determining a lower bound for the number of conjugacy classes of a finite p-group (nilpotent group) G. In the same paper, the author suggested the following question.

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Question 1. Let G be a finite (p-) group and N be a normal (abelian) subgroup of G. Is it true that $o(N)^{1/2} \leq o(G)$?

Recently, Khukhro, Moretó, and Zarrin [17] gave a strong negative answer to this question. They proved that "if c is a real number and $p \geq 3/c$ is a prime, then there exists a finite p-group with a normal abelian subgroup N such that $o(G) < o(N)^c$". Inspired by this question, in [22], we considered a strong condition and asked the following question.

Question 2. For which finite group G is $o(H) \leq o(G)$ for all (normal) subgroups H of G?

Throughout this paper all groups are finite. Our notation and terminology are standard and primarily derived from [20]. In particular, the order of a finite group G is shown by $|G|$. The set of all distinct prime divisors of $|G|$ is denoted by $\pi(G)$. The dihedral group of order 8, the quaternion group of order 8, and the elementary abelian p-group of order p^m are denoted by D_8, Q_8, and C_p^m, respectively. We denote the set of all non-isomorphic subgroups of G by $\mathcal{N}(G)$.

We note that if S_n is the symmetric group on n letters, then $o(S_3) = 13/6 < 7/3 = o(C_3)$. Therefore, if N is a normal subgroup of G, then $o(G)$ may be smaller than $o(N)$. However, there is the following result about the quotient groups.

Lemma 1.1 ([10, Lemma 3.1]). *Let G be a finite group containing a non-trivial normal subgroup N. Then $o(G/N) < o(G)$.*

We say that G satisfies the average condition if $o(H) \leq o(G)$ for every subgroup H of G. In [22] we studied the average condition on abelian groups. The function o is multiplicative, that is, $o(G \times H) = o(G)o(H)$ if and only if $\gcd(|G|, |H|) = 1$. The cyclic group C_n has the maximum value of o in the class of groups of order n, that is, if G is a finite group of order n, then $o(G) \leq o(C_n)$. We proved that every finite abelian group satisfies the average condition [22].

Let p be a prime. If $m \geq 2$ and $n \geq 1$, then

$$M_{m,n,p} = \left\langle a, b \;\middle|\; a^{p^m} = b^{p^n} = 1,\ a^b = a^{1+p^{m-1}} \right\rangle = \langle a \rangle \rtimes \langle b \rangle$$

is a p-group of order p^{m+n}, and if $m \geq n \geq 1$, then

$$\begin{aligned} N_{m,n,p} &= \left\langle a, b, c \;\middle|\; a^{p^m} = b^{p^n} = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1 \right\rangle \\ &= (\langle a \rangle \times \langle c \rangle) \rtimes \langle b \rangle = (\langle b \rangle \times \langle c \rangle) \rtimes \langle a \rangle \end{aligned}$$

is a p-group of order p^{m+n+1}. Recall that a finite non-abelian group G is said to be minimal non-abelian if every proper subgroup of G is abelian. It is well known that a minimal no-abelian p-group is isomorphic to Q_8, $M_{m,n,p}$, or $N_{m,n,p}$ [19].

In [23] we classify minimal non-abelian groups which satisfy the average condition and prove the following theorem.

THEOREM 1.2 ([23]). *Let G be a finite minimal non-abelian group. Then G satisfies the average condition if and only if one of the following holds:*

(a) $G \ncong D_8$ *is a p-group, where p is a prime,*

(b) $G \cong C_3 \rtimes C_{2^\alpha}$, $\alpha \geq 2$,

(c) $G \cong C_q^\beta \rtimes C_{p^\alpha}$, *where $p > q \geq 3$ are distinct primes, $\beta \geq 2$, and $\alpha \geq 1$.*

As a consequence of the above theorems, we concluded the following useful results.

COROLLARY 1.3 ([22]). *Let G be a finite abelian group or a minimal non-abelian p-group which is not isomorphic to D_8. Then $o(H) < o(G)$, for all maximal subgroups H of G. In particular, $o(M_{m,n,p}) = o(C_{p^m} \times C_{p^n})$ and $o(N_{m,n,p}) = o(C_p \times C_{p^m} \times C_{p^n})$.*

COROLLARY 1.4 ([23]). *Let $G \cong C_q^\beta \rtimes C_{p^\alpha}$ be a finite minimal non-abelian group, where p and q are distinct primes, and let $G_1 \cong C_q^\beta \times C_{p^{\alpha-1}}$ be a maximal subgroup of G. If $\beta = 1$, $p = 2$, $q = 3$, $\alpha \geq 2$, or $\beta \geq 2$, $p > q$, $\alpha \geq 1$, then $o(G_1) < o(G)$. Otherwise $o(G_1) > o(G)$.*

In this paper we study finite groups in terms of the number of average order of its non-isomorphic subgroups. For a finite group G we set $X(G) := \{o(H) \mid H \leq G\}$. It is obvious that $X(H) \subseteq X(G)$ for all subgroups H of G. It is clear that if H and K are two isomorphic subgroups of G, then $o(H) = o(K)$. The converse is not true. For example, let

$$\begin{aligned} G &= \langle a,b,c,d \mid a^p = b^p = c^p = d^p = 1,\ dc = acd,\ bd = db,\ ad = da, \\ &\qquad\qquad bc = cb,\ ac = ca,\ ab = ba\rangle \\ &\cong ((C_p \times C_p) \rtimes C_p) \times C_p, \\ H &= \langle a,b,c \mid a^p = b^p = c^p = 1,\ bc = cb,\ ac = ca,\ ab = ba\rangle, \\ K &= \langle a,c,d \mid a^p = c^p = d^p = 1,\ dc = acd,\ ad = da,\ ac = ca\rangle. \end{aligned}$$

Then $H \ncong K$, $o(H) = o(K)$, by Corollary 1.3. Therefore, we have $X(G) = \{o(T) \mid T \in \mathcal{N}(G)\}$, $|\mathcal{N}(G)| \neq |X(G)|$.

In Section 2 we characterize finite groups G with $|X(G)| \leq 3$. In fact, $|X(G)| = k$, $k = 1,2,3$, if and only if $|G| = p^{k-1}$, for some prime p.

In Section 3 we characterize finite groups with $|X(G)| = 4$ and show the following result.

Theorem A. *Let G be a finite group . Then $|X(G)| = 4$ if and only if either $|G| = pq$, where p and q are different primes, or $G \cong Q_8$, or, for some prime p, G is isomorphic to one of the groups C_{p^3}, C_p^3, $N_{1,1,p}$.*

2. Finite groups with $|X(G)| \leq 3$

We begin with the following easy observation.

Proposition 2.1. *Let G be a finite group. Then $|X(G)| = 1$ if and only if $G = 1$.*

Proof. If $G = 1$ then it is obvious that $X(G) = \{1\}$ and the result is true. Conversely, we suppose that $|X(G)| = 1$. Since $\{1, o(G)\} \subseteq X(G)$, $o(G) = 1$ and $\psi(G) = |G|$. If $G \neq 1$ then G contains an element x of order $o(x) > 1$. Therefore,

$$\psi(G) = o(x) + \sum_{x \neq y \in G} o(y) \geq o(x) + \sum_{x \neq y \in G} 1 = o(x) + |G| - 1 > |G|,$$

which is a contradiction. Hence, $G = 1$ and the proof is complete. ■

We need the following simple result on the sum of element orders of elementary abelian p-groups.

Lemma 2.2. *Let G be an elementary abelian p-group of order p^a. Then*

$$\psi(G) = p^{a+1} - p + 1.$$

We need the following easy lemma.

Lemma 2.3. *Let G be a finite group. If $|G| > p$, where p is the smallest prime divisor of $|G|$, then $o(C_p) < o(G)$.*

Proof. If $o(C_p) \geq o(G)$, then $|G|\psi(C_p) \geq p\psi(G)$. Using Lemma 2.2, we have

$$|G|(p^2 - p + 1) \geq p\left(1 + \sum_{1 \neq x \in G} o(x)\right) \geq p\left(1 + \sum_{1 \neq x \in G} p\right) = p^2|G| - p^2 + p,$$

from which it follows that $|G| \leq p$, a contradiction. Hence, $o(C_p) < o(G)$ and the proof is complete. ■

Remark 2.4. In Lemma 2.3, the condition on p is essential; for example $o(C_3) = 14/6$ is greater than $o(S_3) = 13/6$.

Proposition 2.5. *Let G be a finite group. Then $|X(G)| = 2$ if and only if $G \cong C_p$, for some prime p.*

Proof. If $G \cong C_p$, then $\mathcal{N}(G) = \{1, C_p\}$. Since $o(C_p) > 1$, $|X(G)| = 2$.

Conversely, suppose that $|X(G)| = 2$. Then, by Proposition 2.1, $|G| > 1$. Let p be the smallest prime divisor of $|G|$ and let an element $x \in G$ be such that $C_p \cong \langle x \rangle \leq G$. If $\langle x \rangle < G$ then, by Lemma 2.3, $o(C_p) < o(G)$. Hence, $|X(G)| \geq 3$, a contradiction. Therefore, $G \cong C_p$ and the proof is complete. ■

In what follows we characterize groups G with $|X(G)| = 3$. We need the following classification of non-abelian groups of order p^3.

Theorem 2.6 ([5, §112]). *Let G be a non-abelian group of order p^3, where p is a prime.*

(a) *If $p > 2$, then G is isomorphic to either $M_{2,1,p} \cong C_{p^2} \rtimes C_p$ or $N_{1,1,p} \cong (C_p \times C_p) \rtimes C_p$.*

(b) *If $p = 2$, then G is isomorphic to either Q_8 or D_8.*

Also, we use the following results.

Lemma 2.7. *If G is a finite group then $\psi(G)$ is an odd integer.*

Proof. Every non-trivial element of G whose order is not 2 is distinct from its inverse. Since the order of any element is equal to its inverse, $\sum_{x\in G, o(x)>2} o(x)$ is an even number. Since $\sum_{x\in G, o(x)=2} o(x)$ is even,

$$\psi(G) = 1 + \sum_{\substack{x\in G\\ o(x)>2}} o(x) + \sum_{\substack{x\in G\\ o(x)=2}} o(x)$$

is an odd integer. ■

Lemma 2.8. *Let p be a prime. If G is a finite p-group, then $\psi(G)$ is congruent to $1 \bmod p$.*

Proof. We have $\psi(G) = 1 + \sum_{x\in G\setminus\{1\}} o(x)$. Now p divides $o(x)$, for every $x \in G\setminus\{1\}$, and we have the result. ■

Lemma 2.9. *Let G be a finite p-group of order p^α. Then $|X(G)| \geq \alpha + 1$.*

Proof. Since $|G| = p^\alpha$, there exist normal subgroups G_i of G of order p^i, $0 \leq i \leq \alpha$, such that

$$1 = G_0 < G_1 < G_2 < \cdots < G_\alpha = G.$$

If $o(G_i) = o(G_j)$, for some $0 \leq j < i \leq \alpha$, then $\psi(G_i) = p^{i-j}\psi(G_j)$. Since $i - j > 0$, p divides $\psi(G_i)$, contradicting Lemma 2.8. Therefore, $o(G_i) \neq o(G_j)$ for all $0 \leq j < i \leq \alpha$, and $X(G)$ contains $\alpha + 1$ distinct elements $1, o(G_1), o(G_2), \ldots, o(G_\alpha)$. Hence, $|X(G)| \geq \alpha + 1$, and the proof is complete. ■

Lemma 2.10. *Let p and q be distinct primes. If $p < q$, then $o(C_p) < o(C_q)$.*

Proof. If $o(C_p) \geq o(C_q)$, then, by Lemma 2.2, $q(p^2 - p + 1) \geq p(q^2 - q + 1)$. Simplifying this inequality we obtain that $pq \leq 1$, which is a contradiction. ■

Corollary 2.11. *If $p_1 < p_2 < \cdots < p_k$ are distinct primes then $o(C_{p_1}) < o(C_{p_2}) < \cdots < o(C_{p_k})$.*

Lemma 2.12. *Let p, q be different primes. If G and G_1 are two groups with $|G| = p^\alpha$ and $|G_1| = q^\beta$, then $o(G) \neq o(G_1)$.*

Proof. Assume $o(G) = o(G_1)$. Then $\psi(G)/p^\alpha = \psi(G_1)/q^\beta$. It follows that $\psi(G)q^\beta = \psi(G_1)p^\alpha$, and p divides $q^\beta\psi(G)$, a contradiction, since p does not divide $\psi(G)$, by Lemma 2.8, and $p \neq q$. ■

Proposition 2.13. *Let G be a finite group. Then $|X(G)| = 3$ if and only if either $G \cong C_p^2$ or $G \cong C_{p^2}$, for some prime p.*

Proof. If $G \cong C_p^2$ or $G \cong C_{p^2}$, then $\mathcal{N}(G) = \{1, C_p, G\}$. By Lemma 2.3, $1 < o(C_p) < o(G)$. Hence, $|X(G)| = 3$.

Conversely, suppose that $|X(G)| = 3$. If $|\pi(G)| \geq 3$ then, by Corollary 2.11, $|X(G)| \geq 4$, a contradiction. Therefore, $|\pi(G)| \leq 2$. Now we divide the rest of the proof into two steps:

(a) $|\pi(G)| = 1$. Then $|G| = p^\alpha$, where α is a positive integer. By Lemma 2.9, $\alpha \leq 2$, and, by Propositions 2.1 and 2.5, $\alpha > 1$. Hence $\alpha = 2$, $|G| = p^2$, and we have the result.

(b) $|\pi(G)| = 2$. Then $|G| = p^\alpha q^\beta$, where p, q are different primes and α, β are positive integers. Then $X(G) = \{1, o(C_p), o(C_q)\}$, by Lemma 2.10. Assume that one of the integers α, β is > 1, for example $\alpha > 1$. Then there exists a subgroup H of G of order p^2. We have $1 < o(H) \in X(G)$, and $o(H) \neq o(C_p)$, by Lemma 2.3, $o(H) \neq o(C_q)$, by Lemma 2.12, a contradiction.

Therefore, $|G| = pq$. Then $X(G) = \{1, o(C_p), o(C_q), o(G)\}$ has size 4, by Corollaries 1.3 and 1.4, a contradiction. ■

3. Proof of Theorem A

Lemma 3.1. *Let G be an abelian group of order p^α, where p is a prime and α is a positive integer. Then $|X(G)| = 4$ if and only if $G \cong C_{p^3}$ or $G \cong C_p^3$.*

Proof. First, suppose that $G \cong C_{p^3}$. Then $\mathcal{N}(G) = \{1, C_p, C_{p^2}, G\}$. By Corollary 1.3, $1 < o(C_p) < o(C_{p^2}) < o(G)$. Hence, $|X(G)| = 4$. Similarly, if $G \cong C_p^3$ then $\mathcal{N}(G) = \{1, C_p, C_p^2, G\}$. By Corollary 1.3, $1 < o(C_p) < o(C_p^2) < o(G)$. Hence, $|X(G)| = 4$.

Conversely, suppose that $|X(G)| = 4$. Since G is a p-group, $4 = |X(G)| \geq \alpha + 1$, by Lemma 2.9. It follows that $\alpha \leq 3$. If $\alpha = 0$, 1, or 2 then, by Propositions 2.1, 2.5, and 2.13, we obtain that $|X(G)| \leq 3$, a contradiction. Thus we have $\alpha = 3$ and $|G| = p^3$. If $G \cong C_{p^2} \times C_p$ then $\mathcal{N}(G) = \{1, C_p, C_p \times C_p, C_{p^2}, G\}$. By Corollary 1.3, $1 < o(C_p) < o(C_p \times C_p) < o(C_{p^2}) < o(G)$. Therefore, $|X(G)| = 5$, a contradiction. Hence, by the first part of the proof, we have $G \cong C_{p^3}$ or $G \cong C_p^3$ and the proof is complete. ■

Lemma 3.2. *Let G be a non-abelian group of order 2^α. Then $|X(G)| = 4$ if and only if $G \cong Q_8$.*

Proof. If $G \cong Q_8$, then $\mathcal{N}(G) = \{1, C_2, C_4, G\}$ and $X(G) = \{1, 3/2, 11/4, 27/8\}$. Hence $|X(Q_8)| = 4$.

Conversely, suppose that $|G| = 2^\alpha$ and $|X(G)| = 4$. Then, by Lemma 2.9, $4 = |X(G)| \geq \alpha + 1$. It follows that $\alpha \leq 3$. If $\alpha = 0$, 1, or 2 then, by Propositions 2.1, 2.5, and 2.13, we obtain that $|X(G)| \leq 3$, a contradiction. So $\alpha = 3$ and $|G| = 8$. If $G \cong D_8$, then $\mathcal{N}(G) = \{1, C_2, C_2 \times C_2, C_4, G\}$ and $X(G) = \{1, 3/2, 7/4, 11/4, 19/8\}$. Hence, $|X(D_8)| = 5$, a contradiction. Therefore, by the first part of the proof $G \cong Q_8$ and the proof is complete. ■

As a consequence of Lemmas 3.1 and 3.2 we can prove the following corollary.

Corollary 3.3. *Let G be a finite p-group, where p is a prime. Then $|X(G)| = 4$ if and only if G is isomorphic to one of the following groups:*

(a) C_{p^3},

(b) C_p^3,

(c) Q_8,

(d) $N_{1,1,p}$.

Proof. If G is isomorphic to one of the groups in the corollary, then by Lemmas 3.1 and 3.2 and Corollary 1.3, we have $|X(G)| = 4$.

Now suppose that $|G| = p^\alpha$ and $|X(G)| = 4$. Then $\alpha + 1 \le 4$ and $\alpha \le 3$, by Lemma 2.9. If G is abelian or non-abelian with $p = 2$ then, by Lemmas 3.1 and 3.2, $G \cong C_{p^3}$, $G \cong C_p^3$, or $G \cong Q_8$. So we suppose that G is a non-abelian group of order p^α, p is an odd prime. If $\alpha = 0$, 1, or 2 then, by Propositions 2.1, 2.5, and 2.13, $|X(G)| \le 3$. This is a contradiction. Hence, we have $\alpha = 3$ and G is a non-abelian group of order p^3, $p > 2$. Using Theorem 2.6, $G \cong M_{2,1,p}$ or $G \cong N_{1,1,p}$. Since $|X(M_{2,1,p})| = 5$ and $|X(N_{1,1,p})| = 4$, by Corollary 1.3, we have $G \cong N_{1,1,p}$ and the proof is complete. ■

Now we are in position to prove Theorem A.

Proof of Theorem A. One direction follows from the previous results.

Conversely, let G be a finite group with $|X(G)| = 4$. Then $|\pi(G)| \le 3$, by Corollary 2.11. Suppose that $|\pi(G)| = 3$. Then $|G| = p^\alpha q^\beta r^\gamma$, where p, q, r are different primes, α, β, γ are positive integers, and $X(G) = \{1, o(C_p), o(C_q), o(C_r)\}$.

Assume that one of the integers α, β, γ is > 1, for example $\alpha > 1$. Then there exists a subgroup H of G of order p^2. We have $1 < o(H) \in X(G)$, and $o(H) \ne o(C_p)$, by Lemma 2.3, $o(H) \ne o(C_q), o(C_r)$, by Lemma 2.12, a contradiction. Therefore $|G| = pqr$. Now suppose that $p > q > r$. Thus $1 < o(C_r) < o(C_q) < o(C_p)$. By Sylow theorems, there exists a normal subgroup of G of order p, and hence there exists a subgroup H of G of order pq. We have $1 < o(C_r) < o(C_q) < o(H)$, by Lemma 1.1. If H is cyclic then $o(C_p) < o(H)$, and $o(H) < o(C_p)$, by Corollary 1.4, if H is not cyclic. In any case $o(H) \ne o(C_p)$, and $|X(G)| \ge 5$, a contradiction. Hence $|\pi(G)| \le 2$.

If G is a p-group, p a prime, the result follows from Corollary 3.3. Thus we can assume that $|G| = p^\alpha q^\beta$, where p, q are different primes and α, β are positive integers. If $\alpha = \beta = 1$, we have the result.

Now assume, by contradiction, that one of α, β, say α, is greater than 1. Then there exists a subgroup P of G of order p^2. We have $X(G) = \{1, o(C_p), o(C_q), o(P)\}$. If $\beta > 1$, there exists a subgroup K of order q^2. But then we have the contradiction $o(K) \in X(G)$, by Lemmas 2.9, 2.12. Similarly we get that $\alpha = 2$.

Thus $|G| = p^2 q$. If there exists a subgroup M of order pq, then we have $o(M) \in X(G)$. We know that $o(M)$ is different from 1, $o(C_p)$, $o(C_q)$. Thus $o(M) = o(P)$. But this is not possible, otherwise $\psi(M)/pq = \psi(P)/p^2$, $\psi(M)p = q\psi(P)$, and p divides $q\psi(P)$, a contradiction. Thus G does not have a subgroup of order pq. Let Q be a subgroup of order q. Then by Sylow theorems either P is normal in G or Q is normal in G. Since G does not have a subgroup of order pq, we infer that P is normal in G, $G = P \rtimes Q$, and every element of $G \backslash P$ has order q. Then we have $o(G) = (\psi(P) + q(q-1)|P|)/p^2 q$. It is easy to see that $o(G) > 1$ and that $o(G)$ is different from $o(C_p)$, $o(C_q)$, by Lemmas 2.12, 1.1. The only possibility is

$o(G) = o(P) = \psi(P)/p^2$. But in this case, $\psi(P) + q(q-1)|P| = \psi(P)q$, and we have $q(q-1)|P| = (q-1)\psi(P)$, $q|P| = \psi(P)$, and again the contradiction that p divides $\psi(P)$. ■

We conclude the paper with the following question.

Question 3. Is there a finite group G such that

(i) $o(G) = o(C_p)$, for some $p \in \pi(G)$?

(ii) $o(G) = o(H)$, for some subgroup H of G?

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