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Equivariant algebraic enhanced de Rham functor

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Abstract – We propose a definition of an equivariant algebraic enhanced de Rham functor, which is defined by using the enhanced de Rham functor due to D'Agnolo and Kashiwara. Moreover, as a small application of this functor, we give an approach to the proof of the well-known fact that any equivariant algebraic coherent $\mathscr{D}$-module is regular holonomic in the case that the number of orbits is finite.

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1. Introduction

The irregular Riemann–Hilbert correspondence was established by D'Agnolo and Kashiwara [5, Thm. 9.5.3], and states that there exists a fully faithful functor called the enhanced de Rham functor,

$$\mathrm{DR}_X^{\mathrm{E}}\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X) \rightarrowtail \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X),$$

from the triangulated category of holonomic $\mathscr{D}_X$-modules on a complex manifold X to that of enhanced ind-sheaves. After that, there exist attempts to determine the essential image of the enhanced de Rham (or solution) functor [12, 21–23]. By these results, we can extract information about holonomic $\mathscr{D}$-modules from their associated enhanced

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ind-sheaves, which can be regarded as their topological counterparts. From this point of view, this article is motivated by the following question by M. Kitagawa:[1]

If the number of G-orbits is finite, any G-equivariant coherent $\mathcal{D}$-module on a G-space is regular holonomic in the algebraic setting, although it is not true in general in the analytic setting. How does this phenomenon appear on the enhanced ind-sheaf side?

In this article, we try to give an answer to this question. For this purpose, we will define an equivariant category of enhanced ind-sheaves on a bordered space $X_{\mathrm{cp}} := (X^{\mathrm{an}}, \widehat{X}^{\mathrm{an}})$, where $\widehat{X}$ is a complete smooth algebraic variety containing a smooth algebraic G-variety X as an open subset such that $\widehat{X}\backslash X$ is a normal crossing divisor, and X^{an} (resp. $\widehat{X}^{\mathrm{an}}$) is the analytification of X (resp. $\widehat{X}$). Then we will prove that the equivariant algebraic enhanced de Rham complex of any equivariant algebraic coherent $\mathcal{D}_X$-module is an object in such a category. More precisely, we prove the following theorem.

THEOREM 1.1. *Let G be a linear algebraic group over $\mathbb{C}$ and X a smooth algebraic G-variety. Then there exist a category $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ with a (forgetful) functor $\mathsf{U}\colon \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ and a functor $\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}$ from the equivariant derived category $\mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$ of coherent $\mathcal{D}_X$-modules to the category $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}} \arrow[r, "\mathsf{U}"] \arrow[d, "\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}"] & \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X) \arrow[d, "\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}"] \\ \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) \arrow[r, "\mathsf{U}"] & \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}), \end{tikzcd} \tag{1.1}$$

where $\mathsf{U}\colon \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ is the forgetful functor and $\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}$ is the algebraic enhanced de Rham functor (Definition 6.1).

Moreover, as a small application of this theorem, and as an answer to the question above by M. Kitagawa, we give an approach to the well-known fact that any equivariant algebraic coherent $\mathcal{D}$-module is regular holonomic in the case that the number of orbits is finite using the functor $\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}$ of Theorem 1.1.

COROLLARY 1.2 ([16, Thm. 9.3.1], [11, Thm. 11.6.1]). *Let G be a linear algebraic group, X a smooth algebraic G-variety, and $M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$. Then, if the number of*

(1) Private communication.

G-orbits on X is finite, we have $M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{rh}}$, where $\mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{rh}}$ is the equivariant derived category of regular holonomic $\mathcal{D}_X$-modules.

Remark 1.3. Note that the proof of Corollary 1.2 using $\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}$ is based on the methods of the proofs in [16, Thm. 9.3.1] and [11, Thm. 11.6.1]. Namely, we prove Corollary 1.2 by using the distinguished triangle associated to the closed embedding and induction on the number of orbits.

Remark 1.4. For an interpretation that Corollary 1.2 does not hold in the analytic setting from the viewpoint of enhanced ind-sheaves, see Remark 7.2.

This article is organized as follows. In Section 2 we briefly recall the theory of bordered spaces and enhanced ind-sheaves on them for fixing the notation. In Section 3 we briefly recall the definition of the equivariant derived category of sheaves [2] for emphasizing the similarity between the definition of it and that of the equivariant category of enhanced ind-sheaves on X_{cp}, which is given in Section 4. In Section 5 we also recall the definition of the equivariant derived category of coherent $\mathcal{D}$-modules, which will be used in Section 6 for defining the equivariant algebraic enhanced de Rham functor. Finally, we give an approach to the proof of the fact that any equivariant algebraic coherent $\mathcal{D}$-module is regular holonomic in the case that the number of orbits is finite by using the equivariant algebraic enhanced de Rham functor in Section 7. In the appendix we give proofs of some lemmas for completeness.

2. Preliminaries

In this section we briefly recall the theory of bordered spaces and enhanced ind-sheaves on them for fixing the notation. Although almost all the materials in this section are well known, we prove some results for convenience.

2.1 – *Bordered spaces*

Our main references are [5, Sect. 3.2], [6], and [7]. In this article, we use the following terminology.

Definition 2.1. A topological space is called good if it is Hausdorff, locally compact, countable at infinity, and has finite flabby dimension.

We write gTop for the category of good topological spaces.

2.1.1. Definitions.

Definition 2.2 ([5, Def. 3.2.1]). A bordered space $\mathsf{M} = (M, C)$ is a pair of a good topological space C and an open subset M of C.

For a bordered space $\mathsf{M} = (M, C)$, we write

$$(2.1) \qquad \underline{\mathsf{M}} := M, \quad \check{\mathsf{M}} := C, \quad \mathsf{M}^{\mathrm{b}} := C \backslash M \quad \in \mathsf{gTop}.$$

Definition 2.3 ([5, Def. 3.2.1]). A morphism $f : \mathsf{M} \to \mathsf{N}$ between bordered spaces is a continuous map $\underline{f} : \underline{\mathsf{M}} \to \underline{\mathsf{N}} \in \mathsf{Mor}(\mathsf{gTop})$ of good topological spaces such that the projection $\overline{\Gamma}_f \to \check{\mathsf{M}}$ is proper, where $\overline{\Gamma}_f$ is the closure of the graph $\Gamma_{\underline{f}} \subset \underline{\mathsf{M}} \times \underline{\mathsf{N}}$ of $\underline{f}$ in $\check{\mathsf{M}} \times \check{\mathsf{N}}$.

Thus, $\mathsf{M} \mapsto \underline{\mathsf{M}}$ is a functor, which is right adjoint to the fully faithful functor $M \mapsto (M, M)$ from gTop to the category Bod of bordered spaces [7, paragraph before (2.1)], from which we identify $M \in \mathsf{gTop}$ with $(M, M) \in \mathsf{Bod}$.

Remark 2.4 (Cf. [7, paragraph before (2.1)]). Note that neither of $\mathsf{M} \to \check{\mathsf{M}}$ nor $\mathsf{M} \to \mathsf{M}^{\mathrm{b}}$ is a functor.

Definition 2.5 ([6, Def. 2.3.6]). Let $\mathsf{M} \in \mathsf{Bod}$. For a subset $Z \subset \underline{\mathsf{M}}$, we say that Z is relatively compact in M if Z is relatively compact in $\check{\mathsf{M}}$.

2.1.2. Bordered space of compact type.

Definition 2.6. We say that $\mathsf{M} \in \mathsf{Bod}$ is of compact type if $\check{\mathsf{M}}$ is compact.

Lemma 2.7 ([5, Rem. 3.2.2]). *Let $\mathsf{M}, \mathsf{N} \in \mathsf{Bod}$. If N is of compact type, then any continuous map $f : \underline{\mathsf{M}} \to \underline{\mathsf{N}}$ induces a morphism $f : \mathsf{M} \to \mathsf{N}$ of bordered spaces.*

Proof. By the compactness of $\check{\mathsf{N}}$, this is clear from Definition 2.3. ∎

Lemma 2.8 (Cf. [14, Lem. 2.2]). *Let $\mathsf{M}, \mathsf{N} \in \mathsf{Bod}$ be of compact type with $\underline{\mathsf{M}} = \underline{\mathsf{N}}$. Then there exists a unique isomorphism $f : \mathsf{M} \xrightarrow{\sim} \mathsf{N} \in \mathsf{Mor}(\mathsf{Bod})$ satisfying $\underline{f} = \mathrm{id}_{\underline{\mathsf{M}}}$, where $\mathrm{id}_{\underline{\mathsf{M}}}$ is the identity map of $\underline{\mathsf{M}} = \underline{\mathsf{N}} \in \mathsf{gTop}$.*

Proof. This is also an immediate consequence of Definition 2.3. ∎

Lemma 2.9. *Let $M \in \mathsf{gTop}$. Then there exists a compact good topological space M^*, which contains M as an open subset.*

Proof. If M is non-compact, it is sufficient to take the one-point (Alexandroff) compactification M^* ([4, I, p. 67, Thm. 4]) of M (see Appendix A for $M^* \in \mathsf{gTop}$). If M is compact, it is sufficient to take M itself. ■

For the notational convenience, we sometimes write $\mathsf{u}\colon \mathsf{Bod} \to \mathsf{gTop}$ for the functor $\mathsf{M} \to \underline{\mathsf{M}}$ given in (2.1).

Lemma 2.10. *There exists a fully faithful functor* $\mathsf{c}\colon \mathsf{gTop} \to \mathsf{Bod}$ *satisfying* $\mathsf{u} \circ \mathsf{c} = \mathrm{id}_{\mathsf{gTop}}$ *such that* $\mathsf{c}(M) \in \mathsf{Bod}$ *is of compact type for any* $M \in \mathsf{gTop}$. *Moreover, if* $\mathsf{c}'\colon \mathsf{gTop} \to \mathsf{Bod}$ *is another functor satisfying the same conditions as* $\mathsf{c}\colon \mathsf{gTop} \to \mathsf{Bod}$, *there exists a unique isomorphism* $\alpha\colon \mathsf{c} \xrightarrow{\sim} \mathsf{c}'$ *of functors such that* $\mathsf{u}(\alpha_M) = \mathrm{id}_M$ *for any* $M \in \mathsf{gTop}$.

Proof. For the existence, it is sufficient to take $\mathsf{c}(M) := (M, M^*)$ (under the notation of Lemma 2.9). Lemma 2.7 implies that $\mathsf{c}\colon \mathsf{gTop} \to \mathsf{Bod}$ is fully faithful because $\mathsf{c}(M) \in \mathsf{Bod}$ is of compact type for any $M \in \mathsf{gTop}$. The uniqueness follows from Lemma 2.8. ■

Notation 2.11. Let $M \in \mathsf{gTop}$ and $f \in \mathsf{Mor}(\mathsf{gTop})$. Then we set $M_{\mathrm{cp}} := \mathsf{c}(M) \in \mathsf{Bod}$ and $f_{\mathrm{cp}} := \mathsf{c}(f) \in \mathsf{Mor}(\mathsf{Bod})$, where $\mathsf{c}\colon \mathsf{gTop} \to \mathsf{Bod}$ is the functor given in Lemma 2.10. Although these are only determined up to unique isomorphism, this does not matter in this article.

We sometimes abbreviate f_{cp} to f for the notational convenience.

Remark 2.12. The functor $\mathsf{c}\colon \mathsf{gTop} \to \mathsf{Bod}$ given in Lemma 2.10 is right adjoint to $\mathsf{u}\colon \mathsf{Bod} \to \mathsf{gTop}$ and thus we have adjoint pairs $\mathsf{i} \dashv \mathsf{u}$ and $\mathsf{u} \dashv \mathsf{c}$:

$$\begin{tikzcd} \mathsf{gTop} \arrow[r, shift left=2, "\mathsf{i}"] \arrow[r, shift right=2, "\mathsf{c}"'] & \mathsf{Bod}, \arrow[l, "\mathsf{u}"'] \end{tikzcd}$$

where we write

$$\mathsf{i}(M) := (M, M), \quad \mathsf{u}(\mathsf{M}) := \underline{\mathsf{M}}, \quad \mathsf{c}(M) := M_{\mathrm{cp}}.$$

2.1.3. Bordered space with extra variable. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ be the two-point compactification of $\mathbb{R}$ and set $\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}}) \in \mathsf{Bod}$. Note that $\mathbb{R}_\infty \simeq \mathbb{R}_{\mathrm{cp}} \in \mathsf{Bod}$ (see Notation 2.11 for $\mathbb{R}_{\mathrm{cp}}$).

Notation 2.13. For a morphism $f\colon \mathsf{M} \to \mathsf{N}$ of bordered spaces, we write

$$\tilde{f} := f \times \mathrm{id}\colon \mathsf{M} \times \mathbb{R}_\infty \to \mathsf{N} \times \mathbb{R}_\infty \in \mathsf{Mor}(\mathsf{Bod}). \tag{2.2}$$

2.2 – *Ind-sheaves on bordered spaces*

Let $\mathsf{M} \in \mathsf{Bod}$ and $\mathrm{Mod}_{\mathrm{cp}}(\mathbb{C}_{\mathsf{M}}) \subset \mathrm{Mod}(\mathbb{C}_{\underline{\mathsf{M}}})$ be the full subcategory of sheaves on $\underline{\mathsf{M}}$ whose support is relatively compact in M (Definition 2.5). Then we write

$$\mathrm{I}\mathbb{C}_{\mathsf{M}} := \mathrm{Ind}(\mathrm{Mod}_{\mathrm{cp}}(\mathbb{C}_{\mathsf{M}})) \tag{2.3}$$

for the category of ind-sheaves on M. We also write $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}})$ for the bounded derived category of $\mathrm{I}\mathbb{C}_{\mathsf{M}}$.

Fact 2.14 ([6, Prop. 2.4.1]). We have an equivalence of triangulated categories

$$\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{\mathsf{M}}})/\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}^{\mathrm{b}}}) \tag{2.4}$$

for $\mathsf{M} \in \mathsf{Bod}$, where M^{b} is defined in (2.1).

By the equivalence (2.4), we identify these categories. We also write

$$\begin{aligned} \mathsf{q} := \mathsf{q}_{\mathsf{M}} &\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{\mathsf{M}}}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}), \\ \mathsf{l} := \mathsf{l}_{\mathsf{M}} &\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{\mathsf{M}}}), \\ \mathsf{r} := \mathsf{r}_{\mathsf{M}} &\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{\mathsf{M}}}) \end{aligned}$$

for the quotient functor, its left and right adjoints, respectively [6, p. 197].

We also write $\mathbf{R}f_*$, f^{-1}, $\mathbf{R}f_{!!}$, $f^!$, $\otimes$, and $\mathbf{R}\mathcal{I}hom$ for the six Grothendieck operations of ind-sheaves [5, Defs. 3.3.1, 3.3.4], where $f\colon \mathsf{M} \to \mathsf{N} \in \mathsf{Mor}(\mathsf{Bod})$. Write $j_{\mathsf{M}}\colon \mathsf{M} \to \check{\mathsf{M}}$ for the morphism induced by the inclusion $\underline{j_{\mathsf{M}}}\colon \underline{\mathsf{M}} \to \check{\mathsf{M}}$. Then we have [6, Rem. 2.4.4]

$$j_{\mathsf{M}}^{-1} \simeq j_{\mathsf{M}}^! \simeq \mathsf{q}_{\mathsf{M}}, \quad \mathsf{l}_{\mathsf{M}} \simeq \mathbf{R}j_{\mathsf{M}!!}, \quad \mathsf{r}_{\mathsf{M}} \simeq \mathbf{R}j_{\mathsf{M}*}.$$

The t-structure of $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}})$ is given by [5, Sect. 3.4]

$$\begin{aligned} \mathrm{D}^{\leq 0}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) &:= \left\{F \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \;\middle|\; H^n(\mathbf{R}j_{\mathsf{M}!!}(F)) = 0 \text{ for } n > 0\right\}, \\ \mathrm{D}^{\geq 0}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) &:= \left\{F \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \;\middle|\; H^n(\mathbf{R}j_{\mathsf{M}!!}(F)) = 0 \text{ for } n < 0\right\}. \end{aligned}$$

We write $\tau^{\leq n}$ and $\tau^{\geq n}$ for the truncation functors. For $a \leq b$, set

$$\mathrm{D}^{[a,b]}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) := \left\{F \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \;\middle|\; H^n(\mathbf{R}j_{\mathsf{M}!!}(F)) = 0 \text{ for } n \notin [a,b]\right\}.$$

Moreover, there exists an exact fully faithful functor [7, p. 31]

$$\iota_{\mathsf{M}}\colon \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\underline{\mathsf{M}}}) \simeq \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{M}}) \rightarrowtail \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}),$$

by which we identify $F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\mathsf{M}})$ with $\iota_{\mathsf{M}}(F) \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}})$.

Remark 2.15. Let $\mathrm{Mod}_{\mathrm{cp}}(\mathbb{C}_{\mathsf{M}}) \subset \mathrm{Mod}(\mathbb{C}_{\underline{\mathsf{M}}})$ be the full subcategory of sheaves on $\underline{\mathsf{M}}$ whose support is relatively compact in M (Definition 2.5). Then the proof of [6, Prop. 2.4.1] implies that for "$\varinjlim$"$F_j \in \mathrm{I}\mathbb{C}_{\mathsf{M}}$ with $F_j \in \mathrm{Mod}_{\mathrm{cp}}(\mathbb{C}_{\mathsf{M}})$, we have

$$\mathsf{A}(\text{“}\varinjlim\text{”}\, F_j) = \mathsf{q}_{\mathsf{M}}(\text{“}\varinjlim\text{”}\, \underline{j_{\mathsf{M}}}_! F_j), \tag{2.5}$$

where $\mathsf{A}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{\mathsf{M}}})/\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}^{\flat}})$ is the functor inducing the equivalence (2.4), and $\mathsf{q}_{\mathsf{M}}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{\mathsf{M}}}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{\mathsf{M}}})/\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}^{\flat}})$ is the quotient functor.

Conversely, the proof of [6, Prop. 2.4.1] implies that for "$\varinjlim$" $F_j \in \mathrm{I}\mathbb{C}_{\check{\mathsf{M}}}$ with $F_j \in \mathrm{Mod}_{\mathrm{cp}}(\mathbb{C}_{\check{\mathsf{M}}})$, we have

$$\mathsf{B}(\mathsf{q}_{\mathsf{M}}(\text{“}\varinjlim\text{”}\, F_j)) = \text{“}\varinjlim\text{”}\, \underline{j_{\mathsf{M}}}^{-1} F_j, \tag{2.6}$$

where $\mathsf{B}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{\mathsf{M}}})/\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}^{\flat}}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}})$ is the functor inducing the equivalence (2.4).

Note that the derived category of ind-sheaves on a bordered space M is defined as the right-hand side of (2.4) in [5, Def. 3.2.6]. Therefore, the functors $\mathbf{R}f_*$, f^{-1}, $\mathbf{R}f_{!!}$, and $f^!$ for $f \in \mathsf{Mor}(\mathsf{Bod})$ are also defined for the category of the right-hand side of (2.4) in [5, Def. 3.3.4 and Lem. 3.3.6]. Moreover, although the derived category of ind-sheaves on a bordered space M is defined as the left-hand side of (2.4) in [6, p. 196], it seems that the above four functors are also defined for the category of the right-hand side of (2.4) (via the isomorphism (2.4)) in [6, p. 198].

For the later use, we need formulas of those functors for the category of the left-hand side of (2.4). We distinguish the categories of (2.4) in the following lemma to avoid confusion.

Lemma 2.16. *Let $f\colon \mathsf{M} \to \mathsf{N} \in \mathsf{Mor}(\mathsf{Bod})$ and $k \in \mathbb{Z}$. Moreover, we write $\mathsf{A}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{\mathsf{M}}})/\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}^{\flat}})$ and $\mathsf{B}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{\mathsf{M}}})/\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}^{\flat}}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}})$ for the functors inducing the equivalence (2.4).*

(i) *For "$\varinjlim$" $G_j \in \mathrm{I}\mathbb{C}_{\mathsf{N}}$ with $G_j \in \mathrm{Mod}_{\mathrm{cp}}(\mathbb{C}_{\mathsf{N}})$, we have*

$$\mathsf{B}f^{-1}\mathsf{A}(\text{“}\varinjlim\text{”}\, G_j) \simeq \text{“}\varinjlim\text{”}\, \underline{f}^{-1} G_j,\quad \mathsf{B}H^k(f^!\mathsf{A}(\text{“}\varinjlim\text{”}\, G_j)) \simeq \text{“}\varinjlim\text{”}\, H^k(\underline{f}^! G_j).$$

(ii) *For "$\varinjlim$" $F_j \in \mathrm{I}\mathbb{C}_{\mathsf{M}}$ with $F_j \in \mathrm{Mod}_{\mathrm{cp}}(\mathbb{C}_{\mathsf{M}})$, we have*

$$\mathsf{B}H^k(\mathbf{R}f_{!!}\mathsf{A}(\text{“}\varinjlim\text{”}\, F_j)) \simeq \text{“}\varinjlim\text{”}\, H^k(\mathbf{R}\underline{f}_! F_j).$$

(iii) *Suppose that M is of compact type (Definition 2.6). Then, for "$\varinjlim$" $F_j \in \mathrm{I}\mathbb{C}_{\mathsf{M}}$ with $F_j \in \mathrm{Mod}_{\mathrm{cp}}(\mathbb{C}_{\mathsf{M}}) = \mathrm{Mod}(\mathbb{C}_{\underline{\mathsf{M}}})$, we have*

$$\mathsf{B}H^k(\mathbf{R}f_*\mathsf{A}(\text{“}\varinjlim\text{”}\, F_j)) \simeq \text{“}\varinjlim\text{”}\, H^k(\mathbf{R}\underline{f}_* F_j).$$

Proof. Because the proofs are similar, we only prove the push-forward case (iii).

By using [5, Lem. 3.2.5], we can assume that $\underline{f}\colon \underline{\mathsf{M}} \to \underline{\mathsf{N}} \in \mathsf{Mor}(\mathsf{gTop})$ is the restriction of $\check{f}\colon \check{\mathsf{M}} \to \check{\mathsf{N}} \in \mathsf{Mor}(\mathsf{gTop})$. Moreover, we set $j := \underline{j_{\mathsf{M}}}$ (where $\underline{j_{\mathsf{M}}}$ is the inclusion $\underline{\mathsf{M}} \to \check{\mathsf{M}} \in \mathsf{Mor}(\mathsf{gTop})$) and $j' := \underline{j_{\mathsf{N}}}$ for the notational simplicity. Then $\mathsf{A}(\text{“}\varinjlim\text{”}\, F_j)$ is equal to $j_{\mathsf{M}}^{-1}(\text{“}\varinjlim\text{”}\, j_! F_j)$ by (2.5) and the isomorphism [5, Lem. 3.3.7 (i)] of functors $j_{\mathsf{M}}^{-1} \simeq \mathsf{q}_{\mathsf{M}}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{\mathsf{M}}}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\check{\mathsf{M}}})/\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}^{\mathrm{b}}})$. Therefore, it is sufficient to prove

$$H^k(\mathbf{R}f_* j_{\mathsf{M}}^{-1}(\text{“}\varinjlim\text{”}\, j_! F_j)) \simeq j_{\mathsf{N}}^{-1}(\text{“}\varinjlim\text{”}\, H^k(\mathbf{R}j'_*\mathbf{R}\underline{f}_* F_j)) \tag{2.7}$$

because (2.6) and the isomorphism [5, Lem. 3.3.7 (i)] of functors $j_{\mathsf{N}}^{-1} \simeq \mathsf{q}_{\mathsf{N}}$ imply that the image of the right-hand side of (2.7) by B is isomorphic to

$$\text{“}\varinjlim\text{”}\, j'^{-1} H^k(\mathbf{R}j'_*\mathbf{R}\underline{f}_* F_j) \simeq \text{“}\varinjlim\text{”}\, H^k(j'^{-1}\mathbf{R}j'_*\mathbf{R}\underline{f}_* F_j) \simeq \text{“}\varinjlim\text{”}\, H^k(\mathbf{R}\underline{f}_* F_j)$$

by the exactness [19, p. 91] of the inverse image functor $j'^{-1} = \underline{j_{\mathsf{N}}}^{-1}$:

$$\begin{tikzcd}
\mathsf{M} \arrow[r, "j_{\mathsf{M}}"] \arrow[d, "f"] & \check{\mathsf{M}} \arrow[d, "\check{f}"] & \underline{\mathsf{M}} \arrow[l, "j := \underline{j_{\mathsf{M}}}"'] \arrow[d, "\underline{f}"] \\
\mathsf{N} \arrow[r, "j_{\mathsf{N}}"'] & \check{\mathsf{N}} & \underline{\mathsf{N}}. \arrow[l, "j' := \underline{j_{\mathsf{N}}}"]
\end{tikzcd}$$

From now on, we prove (2.7). By [5, Lems. 3.3.12 (i), 3.3.7 (i), (ii)], we have

$$\begin{aligned}
\mathbf{R}f_* j_{\mathsf{M}}^{-1}(\text{“}\varinjlim\text{”}\, j_! F_j) &\simeq j_{\mathsf{N}}^{-1}\mathbf{R}\check{f}_*\mathbf{R}j_{\mathsf{M}*} j_{\mathsf{M}}^{-1}(\text{“}\varinjlim\text{”}\, j_! F_j) \\
&\simeq j_{\mathsf{N}}^{-1}\mathbf{R}\check{f}_*\mathbf{R}\mathcal{I}hom(\mathbb{C}_{\underline{\mathsf{M}}}, \text{“}\varinjlim\text{”}\, j_! F_j) \\
&\simeq j_{\mathsf{N}}^{-1}\mathbf{R}\check{f}_{!!}\mathbf{R}\mathcal{I}hom(\mathbb{C}_{\underline{\mathsf{M}}}, \text{“}\varinjlim\text{”}\, j_! F_j),
\end{aligned} \tag{2.8}$$

where the last isomorphism follows from the assumption that $\check{\mathsf{M}}$ is compact and [20, Prop. 5.2.6 (ii)]. Moreover, we have

$$\mathbf{R}\check{f}_{!!}\mathbf{R}\mathcal{I}hom(\mathbb{C}_{\underline{\mathsf{M}}}, *) \simeq \mathbf{R}(\check{f}_{!!} \circ \mathcal{I}hom(\mathbb{C}_{\underline{\mathsf{M}}}, *)),$$

because $\mathcal{I}hom(\mathbb{C}_{\underline{\mathsf{M}}}, *)$ preserves quasi-injective objects [20, Lem. 5.1.2 (ii)]. Then [20, Thm. 1.5.6 (ii), (iii)] implies that

$$H^k\big(\mathbf{R}(\check{f}_{!!} \circ \mathcal{I}hom(\mathbb{C}_{\underline{\mathsf{M}}}, *))(\text{“}\varinjlim\text{”}\, j_! F_j)\big) \simeq \text{“}\varinjlim\text{”}\, H^k\big(\mathbf{R}(\check{f}_! \circ \mathcal{H}om(\mathbb{C}_{\underline{\mathsf{M}}}, *))(j_! F_j)\big)$$

because $\check{f}_{!!} \circ \mathcal{I}hom(\mathbb{C}_{\underline{\mathsf{M}}}, *)$ has the form $\mathrm{I}(\check{f}_! \circ \mathcal{H}om(\mathbb{C}_{\underline{\mathsf{M}}}, *))$ (under the notation of [20, p. 7]). Moreover, [19, (2.3.16) and Prop. 2.3.9 (iii)] imply

$$\mathbf{R}(\check{f}_! \circ \mathcal{H}om(\mathbb{C}_{\underline{\mathsf{M}}}, *)) \simeq \mathbf{R}(\check{f}_! j_* j^{-1}) \simeq \mathbf{R}(\check{f}_* j_* j^{-1}) \simeq \mathbf{R}(j'_* \underline{f}_* j^{-1}).$$

Thus, because j^{-1} preserves injective objects [19, Prop. 2.4.1 (i)], we have

$$\mathbf{R}(j'_* \underline{f}_* j^{-1})(j_! F_j) \simeq \mathbf{R}(j'_* \underline{f}_*)(j^{-1} j_! F_j) \simeq \mathbf{R}(j'_* \underline{f}_*)(F_j) \simeq \mathbf{R}j'_* \mathbf{R}\underline{f}_*(F_j),$$

where the last isomorphism follows from [19, (2.6.5)]. Thus, the exactness of j_{N}^{-1} [6, Prop. 2.4.6 (iv)] and (2.8) imply

$$H^k(\mathbf{R}f_* j_{\mathsf{M}}^{-1}(\text{“}\varinjlim\text{”}\, j_! F_j)) \simeq j_{\mathsf{N}}^{-1}(\text{“}\varinjlim\text{”}\, H^k(\mathbf{R}j'_* \mathbf{R}\underline{f}_* F_j)),$$

which completes the proof of (2.7). ∎

2.3 – *Enhanced ind-sheaves on bordered spaces*

Let $\mathsf{M} \in \mathsf{Bod}$ and $\pi := \pi_{\mathsf{M}} \colon \mathsf{M} \times \mathbb{R}_\infty \to \mathsf{M}$ be the first projection. Then we write

$$\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) := \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}\times\mathbb{R}_\infty})/\pi_{\mathsf{M}}^{-1}\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}})$$

for the category of enhanced ind-sheaves ([5, Def. 4.4.1] or [7, p. 33]).

We write

$$\begin{aligned} \mathsf{Q} &:= \mathsf{Q}_{\mathsf{M}} \colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}\times\mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}), \\ \mathsf{L} &:= \mathsf{L}_{\mathsf{M}} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}\times\mathbb{R}_\infty}), \\ \mathsf{R} &:= \mathsf{R}_{\mathsf{M}} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}\times\mathbb{R}_\infty}) \end{aligned} \tag{2.9}$$

for the quotient functor, its left and right adjoints, respectively [6, p. 204].

We also write $\mathrm{E}f_*$, $\mathrm{E}f^{-1}$, $\mathrm{E}f_{!!}$, $\mathrm{E}f^!$, $\overset{+}{\otimes}$, and $\mathbf{R}\mathcal{I}hom^+$ for the six Grothendieck operations of enhanced ind-sheaves [5, Defs. 4.5.1, 4.5.6], where $f \colon \mathsf{M} \to \mathsf{N} \in \mathsf{Mor}(\mathsf{Bod})$. Then we have [6, p. 204]

$$\begin{aligned} \mathsf{L}_{\mathsf{M}}\mathsf{Q}_{\mathsf{M}}(F) &\simeq (\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}) \overset{+}{\otimes} F, \\ \mathsf{R}_{\mathsf{M}}\mathsf{Q}_{\mathsf{M}}(F) &\simeq \mathbf{R}\mathcal{I}hom^+(\mathbb{C}_{\{t\geq 0\}} \oplus \mathbb{C}_{\{t\leq 0\}}, F), \end{aligned}$$

where we write $\{t \geq 0\} := \{(x,t) \in \underline{\mathsf{M}} \times \mathbb{R} \mid t \geq 0\}$ and $\{t \leq 0\} := \{(x,t) \in \underline{\mathsf{M}} \times \mathbb{R} \mid t \leq 0\}$. The t-structure of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}})$ is given by [5, Def. 4.6.1]

$$\begin{aligned} \mathrm{E}^{\leq 0}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) &:= \{K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \mid \mathsf{L}_{\mathsf{M}}(K) \in \mathrm{D}^{\leq 0}(\mathrm{I}\mathbb{C}_{\mathsf{M}\times\mathbb{R}_\infty})\}, \\ \mathrm{E}^{\geq 0}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) &:= \{K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \mid \mathsf{L}_{\mathsf{M}}(K) \in \mathrm{D}^{\geq 0}(\mathrm{I}\mathbb{C}_{\mathsf{M}\times\mathbb{R}_\infty})\}. \end{aligned} \tag{2.10}$$

We write $\tau^{\leq n}$ and $\tau^{\geq n}$ for the truncation functors. For $a \leq b$, set

$$\mathrm{E}^{[a,b]}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) := \{K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathsf{M}}) \mid \mathsf{L}_{\mathsf{M}}(K) \in \mathrm{D}^{[a,b]}(\mathrm{I}\mathbb{C}_{\mathsf{M}\times\mathbb{R}_\infty})\}.$$

2.3.1. Sheaf type. For $M \in \mathsf{gTop}$, we write

$$\mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{M_{\mathrm{cp}}}) \subset \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_{\mathrm{cp}}})$$

for the essential image of the fully faithful functor [5, Prop. 4.7.15]

$$\begin{aligned} \mathsf{e}_{M_{\mathrm{cp}}} \colon \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{M_{\mathrm{cp}}}) &\rightarrowtail \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_{\mathrm{cp}}}), \\ F &\mapsto \mathbb{C}^{\mathrm{E}}_{M_{\mathrm{cp}}} \otimes \pi^{-1}(F). \end{aligned}$$

Lemma 2.17. *Let $f \colon M \to N \in \mathsf{Mor}(\mathsf{gTop})$. Then we have*

$$\begin{aligned} \mathrm{E}f_{\mathrm{cp}}^{-1}(\mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}})) &\subset \mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{M_{\mathrm{cp}}}), \\ \mathrm{E}f_{\mathrm{cp}}^{!}(\mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}})) &\subset \mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{M_{\mathrm{cp}}}), \\ \mathrm{E}f_{\mathrm{cp}!!}(\mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{M_{\mathrm{cp}}})) &\subset \mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}}). \end{aligned}$$

Proof. This follows from the commutativity of the functor e and the above functors [14, p. 49]. (See Notation 2.11 for $f_{\mathrm{cp}} \colon M_{\mathrm{cp}} \to N_{\mathrm{cp}}$.) ∎

2.3.2. A distinguished triangle.

Lemma 2.18 ([6, Lem. 2.7.7]). *Let $Z \subset M$ be a closed subset of $M \in \mathsf{gTop}$ and set $U := M \backslash Z$. Then, for any $F \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{M_{\mathrm{cp}}})$, we have a distinguished triangle*

$$\mathrm{E}i_*\mathrm{E}i^!(F) \to F \to \mathrm{E}j_*\mathrm{E}j^!(F) \xrightarrow{+1},$$

where $i \colon Z_{\mathrm{cp}} \hookrightarrow M_{\mathrm{cp}}$ and $j \colon U_{\mathrm{cp}} \hookrightarrow M_{\mathrm{cp}}$ are induced by the closed and open embeddings, respectively.

Proof. Because $\check{M}_{\mathrm{cp}}$ is compact, the closure $\mathrm{cl}(U)$ of U in $\check{M}_{\mathrm{cp}}$ is also compact, which implies $U_{\mathrm{cp}} \simeq U_\infty := (U, \mathrm{cl}(U)) \in \mathsf{Bod}$. For the same reason, we have $Z_{\mathrm{cp}} \simeq Z_\infty := (Z, \mathrm{cl}(Z)) \in \mathsf{Bod}$. Thus, this lemma is an immediate consequence of [6, Lem. 2.7.7] (and [6, Notation 2.3.3]). ∎

Remark 2.19. Note that if we replace the maps $i \colon Z_{\mathrm{cp}} \hookrightarrow M_{\mathrm{cp}}$ and $j \colon U_{\mathrm{cp}} \hookrightarrow M_{\mathrm{cp}}$ in Lemma 2.18 with $k_Z \colon Z \to M \in \mathsf{Mor}(\mathsf{Bod})$ and $k_U \colon U \to M \in \mathsf{Mor}(\mathsf{Bod})$ which are induced by the closed and open embeddings, respectively, then the associated triangle

$$\mathrm{E}k_{Z*}\mathrm{E}k_Z^!(F) \to F \to \mathrm{E}k_{U*}\mathrm{E}k_U^!(F) \xrightarrow{+1} \tag{2.11}$$

is not distinguished in general. For example, a pair $M = \mathbb{R}$, $U = \mathbb{R}\backslash\{0\}$, and $F = \text{“}\varinjlim_{a\to\infty}\text{”}\, \mathbb{C}_{\{(x,t)\in U\times\mathbb{R} \mid t+x^{-1}\geq a\}}$ gives an example that (2.11) is not distinguished.

2.3.3. Algebraic $\mathbb{C}$-constructible enhanced ind-sheaves.

NOTATION 2.20. Let X be a smooth algebraic variety over $\mathbb{C}$, and $\widehat{X}$ a smooth complete algebraic variety containing X as an open subset such that $\widehat{X}\backslash X$ is a normal crossing divisor (whose existence is provided by Nagata's compactification theorem [24, Thm. 4.3] and Hironaka's desingularization theorem [9, 10]). In this setting, we set

$$X_{\mathrm{cp}} := (X^{\mathrm{an}}, \widehat{X}^{\mathrm{an}}) \in \mathsf{Bod},$$

where X^{an} and $\widehat{X}^{\mathrm{an}}$ are the analytifications of X and $\widehat{X}$, respectively. We also write

$$j_{X_{\mathrm{cp}}} \colon X_{\mathrm{cp}} \to \widehat{X}^{\mathrm{an}} \in \mathsf{Mor}(\mathsf{Bod})$$

for the morphism induced by the analytification $j_X^{\mathrm{an}} \colon X^{\mathrm{an}} \to \widehat{X}^{\mathrm{an}}$ of the inclusion $j_X \colon X \to \widehat{X}$.

In the setting in Notation 2.20, we also write

$$\mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$$

for the triangulated full subcategory of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ defined in [14, Def. 3.10].

FACT 2.21 ([14, Cor. 3.13]). Let $f \colon X \to Y$ be a morphism of smooth algebraic varieties and $f^{\mathrm{an}} \colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ its analytification, which induces $f^{\mathrm{an}}_{\mathrm{cp}} \colon X_{\mathrm{cp}} \to Y_{\mathrm{cp}} \in \mathsf{Mor}(\mathsf{Bod})$. Then we have

$$\mathrm{E}(f^{\mathrm{an}}_{\mathrm{cp}})^{-1}(\mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})) \subset \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}), \quad \mathrm{E}(f^{\mathrm{an}}_{\mathrm{cp}})^{!}(\mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})) \subset \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}),$$
$$\mathrm{E}(f^{\mathrm{an}}_{\mathrm{cp}})_{!!}(\mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})) \subset \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}}), \quad \mathrm{E}(f^{\mathrm{an}}_{\mathrm{cp}})_{*}(\mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})) \subset \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}}).$$

LEMMA 2.22. *Let $f \colon X \to Y$ be a morphism of smooth algebraic varieties and $f^{\mathrm{an}} \colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ its analytification, which induces $f^{\mathrm{an}}_{\mathrm{cp}} \colon X_{\mathrm{cp}} \to Y_{\mathrm{cp}} \in \mathsf{Mor}(\mathsf{Bod})$. Then we have*

$$\mathrm{E}(f^{\mathrm{an}}_{\mathrm{cp}})_{*}(\mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) \cap \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})) \subset \mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}}) \cap \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}}).$$

PROOF. Let

$$F \in \mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) \cap \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}).$$

Then, by $F \in \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ and [14, Thm. 3.11], there exists $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ satisfying

$$F \simeq \mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(M), \tag{2.12}$$

where $\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}$ is the algebraic enhanced solution functor defined in [14, p. 61] (or see (D.1)). Moreover, $F \in \mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ and [13, Lem. 3.10] imply that M in (2.12) satisfies

$M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$, where $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ is the derived category of regular holonomic $\mathcal{D}_X$-modules. Let $\mathbf{D}f_*$ be the direct image functor for $\mathcal{D}$-modules and $\mathbb{D}_X$ the duality functor [11, Def. 2.6.1]. Then, setting $\mathbf{D}f_! := \mathbb{D}_Y \mathbf{D}f_* \mathbb{D}_X$, we have

$$\mathrm{E}(f^{\mathrm{an}}_{\mathrm{cp}})_*(F) \simeq \mathrm{E}(f^{\mathrm{an}}_{\mathrm{cp}})_*(\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(M)) \simeq \mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(\mathbf{D}f_! M)[d_Y - d_X]$$

by (2.12) and [14, Prop. 3.12 (3)], where d_X and d_Y are the dimensions of X and Y, respectively. Because the functor $\mathbf{D}f_!$ preserves $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ by [11, Thm. 6.1.5 (ii)], we have

$$\mathrm{E}(f^{\mathrm{an}}_{\mathrm{cp}})_*(F) \simeq \mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(\mathbf{D}f_! M)[d_Y - d_X] \in \mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}}) \cap \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-}\mathrm{c}}(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})$$

by [14, Prop. 3.14], which completes the proof. ∎

3. Equivariant derived category of sheaves

In this section we recall the definition of the equivariant derived category of sheaves [2], in order to fix the notation and emphasize the similarity between the definition of it and that of an equivariant category of enhanced ind-sheaves on a bordered space of compact type (Definition 2.6) given in Definition 4.7. Although almost all the materials in this section are also well known, we prove some results for convenience.

3.1 – *Preliminaries*

In this section we prepare the notation and recall some definitions needed to define the equivariant derived category of sheaves [2].

3.1.1. Notation I. Let $G \in \mathsf{gTop}$ be a group object and write gTop^G for the category of G-spaces and G-equivariant morphisms. Namely, $G \in \mathsf{gTop}$ is a topological group, objects of gTop^G are those of gTop with continuous G-action, and morphisms of gTop^G are those of gTop commuting with the G-action.

Notation 3.1. For $X \in \mathsf{gTop}^G$, we write $\mathsf{gTop}^G_{/X}$ for the overcategory (slice category) of X in gTop^G. Namely, objects are morphisms in gTop^G whose codomain is X. Then a morphism from $r\colon R \to X \in \mathsf{gTop}^G_{/X}$ to $s\colon S \to X \in \mathsf{gTop}^G_{/X}$ is given by a morphism $t\colon R \to S \in \mathsf{Mor}(\mathsf{gTop}^G)$ satisfying $r = s \circ t$.

Convention 3.2 (Cf. [2, Def. in Sect. 0.3]). In Sections 3 and 4 we say that an action G on $R \in \mathsf{gTop}^G$ is free if the quotient map $R \to \overline{R} := R/G$ makes R into a locally trivial principal G-bundle.

Recall that a resolution R of X is an object $r\colon R \to X \in \mathsf{gTop}^G_{/X}$ with free G-action on R. We write $\mathsf{Res}^G_X \subset \mathsf{gTop}^G_{/X}$ for the full subcategory of resolutions R of X satisfying $\bar{R} := R/G \in \mathsf{gTop}$. We sometimes abbreviate Res^G_X to Res_X. For $f\colon X \to Y$, we also write $\bar{f}\colon \bar{X} \to \bar{Y}$ for the map induced by f.

3.1.2. Index category.

Notation 3.3. In this article, we write $\mathcal{I}$ for the category defined as follows.

- Objects are finite sequences $I := (i_1, \dots, i_k) \in (\mathbb{Z}_{\geq 0})^k$ for some $k \in \mathbb{Z}_{\geq 1}$. Namely, $\mathsf{Obj}(\mathcal{I}) := \coprod_{1 \leq k} (\mathbb{Z}_{\geq 0})^k$.
- For $I = (i_1, \dots, i_k)$, $J = (j_1, \dots, j_l) \in \mathrm{Ob}(\mathcal{I})$, there exists a unique arrow $I \to J$ if and only if J is a subsequence of I, namely, there exists a sequence $1 \leq n_1 < \cdots < n_l \leq k$ such that $j_m = i_{n_m}$ for all $m \in \{1, \dots, l\}$.

3.1.3. Acyclic morphisms.

Definition 3.4 ([2, Def. 1.9.1]). Let $n \in \mathbb{Z}_{\geq 0}$. We say that a map $f\colon M \to N \in \mathsf{Mor}(\mathsf{gTop})$ is n-acyclic if it satisfies the following two conditions:

(Ac1)$_n$ For any $F \in \mathrm{Mod}(\mathbb{C}_N) \simeq \mathrm{D}^{[0,0]}(\mathbb{C}_N)$, the adjunction $f^{-1} \dashv \mathbf{R}f_*$ [19, Prop. 2.6.4 (ii)] induces

$$F \xrightarrow{\sim} \tau^{\leq n}\mathbf{R}f_* f^{-1}F \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_N).$$

(Ac2)$_n$ For any $N' \to N \in \mathsf{Mor}(\mathsf{gTop})$, the base change $f'\colon M' := M \times_N N' \to N'$ satisfies the property (Ac1)$_n$.

We also say that $f\colon M \to N \in \mathsf{Mor}(\mathsf{gTop})$ is n-preacyclic if it satisfies (Ac1)$_n$.

Definition 3.5 ([2, Sect. 1.9.4]). Let $n \in \mathbb{Z}_{\geq 0}$. We say that $A \in \mathsf{gTop}$ is n-acyclic if A satisfies the following two conditions:

(A1)$_n$ The unique map $a\colon A \to \{\mathrm{pt}\}$ from A to the one-point topological space $\{\mathrm{pt}\}$ is n-preacyclic.

(A2)$_n$ A is non-empty, connected, and locally connected.

Lemma 3.6 ([2, Criterion in Sect. 1.9.4]). *Let $A \in \mathsf{gTop}$ and $n \in \mathbb{Z}_{\geq 0}$. Then $a\colon A \to \{\mathrm{pt}\}$ is n-acyclic if A is n-acyclic.*

Proof. This is a immediate consequence of [2, Criterion in Sect. 1.9.4]. ■

3.2 – *Definition*

Definition 3.7. We say that a group object $G \in \mathsf{gTop}$ has enough acyclic objects if for any $n \in \mathbb{Z}_{\geq 1}$, there exists $A_n \in \mathsf{Res}^G_{\{\mathrm{pt}\}}$ which is n-acyclic.

Notation 3.8. Let $G \in \mathsf{gTop}$ be a group object having enough acyclic objects[2] and $X \in \mathsf{gTop}^G$. In this setting, we fix $A_n \in \mathsf{Res}^G_{\{\mathrm{pt}\}}$ which is n-acyclic for each $n \in \mathbb{Z}_{\geq 1}$, and set $A_0 := G$. Moreover, we write

$$X_I := A_{i_1} \times \cdots \times A_{i_k} \times X,$$

where $I = (i_1, \ldots, i_k) \in \mathcal{I}$ (Notation 3.3). If $\mathrm{Hom}_{\mathcal{I}}(I, J) \neq \emptyset$, we also write

$$p^X_{IJ} \colon X_I \to X_J, \quad \bar{p}^X_{IJ} \colon \bar{X}_I \to \bar{X}_J$$

for the projection and the morphism induced by p^X_{IJ}, respectively. We sometimes abbreviate $\bar{p}^X_{IJ}$ to $\bar{p}_{IJ}$.

Definition 3.9 ([2, Sect. 2.4]). Let $G \in \mathsf{gTop}$ be a group object having enough acyclic objects. Then, for $X \in \mathsf{gTop}^G$, we write $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ for the equivariant derived category of sheaves defined in the following way (under Notation 3.8):

- An object of $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ is a pair $F := (\{F_I\}_{I \in \mathcal{I}}, \{\phi_{IJ}\}_{(I \to J) \in \mathsf{Mor}(\mathcal{I})})$, where

  $$F_I \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{\bar{X}_I}), \quad \phi_{IJ} \colon \bar{p}^{-1}_{IJ} F_J \xrightarrow{\sim} F_I$$

  satisfying the compatibility conditions $\phi_{II} = \mathrm{id}$ for any $I \in \mathcal{I}$ and

  $$\phi_{IK} = \phi_{IJ} \circ \bar{p}^{-1}_{IJ}(\phi_{JK})$$

  for any composable morphisms $I \to J \in \mathsf{Mor}(\mathcal{I})$ and $J \to K \in \mathsf{Mor}(\mathcal{I})$.
- A morphism $\alpha \colon F \to F'$ in $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ is a set $\alpha = \{\alpha_I\}_{I \in \mathcal{I}}$ of morphisms $\alpha_I \colon F_I \to F'_I$ satisfying the compatibility condition

  $$\begin{tikzcd} \bar{p}^{-1}_{IJ} F_J \arrow[r, "\phi_{IJ}"] \arrow[d, "\bar{p}^{-1}_{IJ}(\alpha_J)"'] & F_I \arrow[d, "\alpha_I"] \\ \bar{p}^{-1}_{IJ} F'_J \arrow[r, "\phi'_{IJ}"'] & F'_I \end{tikzcd}$$

  for any $I \to J \in \mathsf{Mor}(\mathcal{I})$.

Although the category $\mathcal{I}$ and the functor $\mathcal{I} \ni I \mapsto X_I \in \mathsf{Res}^G_X$ (Notations 3.3 and 3.8) do not literally satisfy the conditions of [2, Prop. in Sect. 2.4.4] (see Remark 3.10

(2) More precisely, we also use this notation for similar but not identical situations to this setting.

given below), this definition is equivalent to that of [2], and the definition of $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ does not depend on the choice of $\{A_n\}_{n\in\mathbb{Z}_{\geq 1}}$.

Remark 3.10. The category $\mathcal{I}$ and the functor $\mathcal{I} \ni I \mapsto X_I \in \mathsf{Res}^G_X$ satisfy all the conditions except condition (a) in [2, Prop. in Sect. 2.4.4]. However, the proof of [2, Prop. in Sect. 2.4.4] shows that the following weaker condition is sufficient (cf. [1, Lem. 6.4.8]):

(a)′ If $R \to X \in \mathsf{Res}^G_X$ and $S \to X \in \mathsf{Res}^G_X$ are in the image of j, then the projections $R \times_X S \to R \in \mathsf{Mor}(\mathsf{Res}^G_X)$ and $R \times_X S \to S \in \mathsf{Mor}(\mathsf{Res}^G_X)$ are also in the image of j,

where we use the notation of [2, Prop. in Sect. 2.4.4]. The category $\mathcal{I}$ and the functor $\mathcal{I} \ni I \mapsto X_I \in \mathsf{Res}^G_X$ satisfy this condition (a)′.

4. Equivariant category of enhanced ind-sheaves on a bordered space of compact type

In this section, we propose a definition of an equivariant category of enhanced ind-sheaves on a bordered space of compact type (Definition 2.6) in Definition 4.7. The definition is similar to that of the equivariant derived category of sheaves [2] (Definition 3.9).

4.1 – *Definitions*

4.1.1. Enhanced acyclic morphisms of compact type.

Definition 4.1. Let $n \in \mathbb{Z}_{\geq 0}$. We say that a map $f\colon M \to N \in \mathsf{Mor}(\mathsf{gTop})$ is enhanced n-acyclic of compact type if it satisfies the following conditions:

(cEAc1)$_n$ For any $F \in \mathrm{E}^{[0,0]}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}})$, the adjunction $\mathrm{E}f_{\mathrm{cp}}^{-1} \dashv \mathrm{E}(f_{\mathrm{cp}})_*$ [5, Prop. 4.5.8 (ii)] induces

$$F \xrightarrow{\sim} \tau^{\leq n}\mathrm{E}(f_{\mathrm{cp}})_*\mathrm{E}f_{\mathrm{cp}}^{-1}F \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}}).$$

(cEAc2)$_n$ For any $N \to N' \in \mathsf{Mor}(\mathsf{gTop})$, the base change $f'\colon M' := M \times_N N' \to N'$ satisfies the property (cEAc1)$_n$.

We also say that $f\colon M \to N \in \mathsf{Mor}(\mathsf{gTop})$ is enhanced n-preacyclic of compact type if it satisfies (cEAc1)$_n$.

Lemma 4.2. *If $A \in \mathsf{gTop}$ is n-acyclic, then the unique map $a\colon A \to \{\mathrm{pt}\}$ from A to the one-point topological space $\{\mathrm{pt}\}$ is enhanced n-acyclic of compact type.*

Remark 4.3. Lemma 4.2 asserts that if a group object $G \in \mathsf{gTop}$ has enough acyclic objects (Definition 3.7), it has also enough "enhanced acyclic objects of compact type". This remark is used in Definition 4.7 given below.

For the proof of Lemma 4.2, we need some preparation.

Definition 4.4. Let $n \in \mathbb{Z}_{\geq 0}$. We say that $f : M \to N \in \mathsf{Mor}(\mathsf{gTop})$ is ind-n-preacyclic of compact type if it satisfies the following condition:

$(\mathrm{IAc1})_n$ For any $F \in \mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}} \simeq \mathrm{D}^{[0,0]}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}})$, the adjunction $f_{\mathrm{cp}}^{-1} \dashv \mathbf{R} f_{\mathrm{cp}*}$ [20, Prop. 5.2.3] induces

$$F \simeq \tau^{\leq n} \mathbf{R} f_{\mathrm{cp}*} f_{\mathrm{cp}}^{-1} F \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}}).$$

Lemma 4.5. *Let $n \in \mathbb{Z}_{\geq 0}$ and $A \in \mathsf{gTop}$ such that A is n-acyclic. Then the first projection $p : N \times A \to N$ is ind-n-preacyclic of compact type (Definition 4.4) for any $N \in \mathsf{gTop}$.*

Proof. Let $F \in \mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}}$ and $p_{\mathrm{cp}} : N_{\mathrm{cp}} \times A_{\mathrm{cp}} \to N_{\mathrm{cp}}$ be the first projection. It is sufficient to prove that

$$H^k(\mathbf{R} p_{\mathrm{cp}*} p_{\mathrm{cp}}^{-1} F) \simeq \begin{cases} F & (k = 0), \\ 0 & (0 \neq k \leq n). \end{cases}$$

For this purpose, we write $F \simeq \text{“}\varinjlim\text{”}\, F_j$ with $F_j \in \mathrm{Mod}_{\mathrm{cp}}(\mathbb{C}_{N_{\mathrm{cp}}}) = \mathrm{Mod}(\mathbb{C}_N)$ by (2.3). Then Lemma 2.16 (iii) implies that we have

$$H^k(\mathbf{R} p_{\mathrm{cp}*} p_{\mathrm{cp}}^{-1} F) \simeq H^k(\mathbf{R} p_{\mathrm{cp}*} p_{\mathrm{cp}}^{-1} \text{“}\varinjlim\text{”}\, F_j) \simeq \text{“}\varinjlim\text{”}\, H^k(\mathbf{R} p_* p^{-1} F_j).$$

We note that the assumption A being n-acyclic implies that $a : A \to \{\mathrm{pt}\}$ is n-acyclic by Lemma 3.6, and thus $p : N \times A \to A$ is also n-acyclic because this is a base change of $a : A \to \{\mathrm{pt}\}$. Therefore, we have

$$\text{“}\varinjlim\text{”}\, H^k(\mathbf{R} p_* p^{-1} F_j) \simeq \begin{cases} \text{“}\varinjlim\text{”}\, F_j & (k = 0), \\ 0 & (0 \neq k \leq n), \end{cases}$$

which completes the proof. ∎

Lemma 4.6. *Let $n \in \mathbb{Z}_{\geq 0}$ and $A \in \mathsf{gTop}$ such that A is n-acyclic. Then the first projection $p : N \times A \to N \in \mathsf{Mor}(\mathsf{gTop})$ is enhanced n-preacyclic of compact type (Definition 4.1) for any $N \in \mathsf{gTop}$.*

Proof. Let $F \in \mathrm{E}^{[0,0]}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}})$; then $\mathsf{L}_{N_{\mathrm{cp}}}(F) \in \mathrm{D}^{[0,0]}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}} \times \mathbb{R}_\infty})$ by (2.10). Moreover, the definitions of $\mathrm{E}p_{\mathrm{cp}*}$ and $\mathrm{E}p_{\mathrm{cp}}^{-1}$ [6, p. 206] imply

$$\mathrm{E}p_{\mathrm{cp}*}\mathrm{E}p_{\mathrm{cp}}^{-1}(F) \simeq \mathrm{E}p_{\mathrm{cp}*}\mathrm{E}p_{\mathrm{cp}}^{-1}(\mathsf{Q}_{N_{\mathrm{cp}}}\mathsf{L}_{N_{\mathrm{cp}}}F) \simeq \mathsf{Q}_{N_{\mathrm{cp}}}(\mathbf{R}\tilde{p}_{\mathrm{cp}*}\tilde{p}_{\mathrm{cp}}^{-1}\mathsf{L}_{N_{\mathrm{cp}}}(F)), \tag{4.1}$$

where we use the notation (2.2). On the other hand, we have a distinguished triangle

$$\begin{aligned} &\tau^{\leq n}(\mathbf{R}\tilde{p}_{\mathrm{cp}*}\tilde{p}_{\mathrm{cp}}^{-1}\mathsf{L}_{N_{\mathrm{cp}}}(F)) \to \mathbf{R}\tilde{p}_{\mathrm{cp}*}\tilde{p}_{\mathrm{cp}}^{-1}\mathsf{L}_{N_{\mathrm{cp}}}(F) \\ &\qquad \to \tau^{\geq n+1}(\mathbf{R}\tilde{p}_{\mathrm{cp}*}\tilde{p}_{\mathrm{cp}}^{-1}\mathsf{L}_{N_{\mathrm{cp}}}(F)) \xrightarrow{+1} . \end{aligned} \tag{4.2}$$

For the first term of (4.2), applying Lemma 4.5 to the projection $N \times \mathbb{R} \times A \to N \times \mathbb{R}$ and $\mathsf{L}_{N_{\mathrm{cp}}}(F) \in \mathrm{D}^{[0,0]}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}} \times \mathbb{R}_\infty})$, we have

$$\mathsf{L}_{N_{\mathrm{cp}}}(F) \xrightarrow{\sim} \tau^{\leq n}(\mathbf{R}\tilde{p}_{\mathrm{cp}*}\tilde{p}_{\mathrm{cp}}^{-1}\mathsf{L}_{N_{\mathrm{cp}}}(F)).$$

Thus, (4.2) can be rewritten as

$$\mathsf{L}_{N_{\mathrm{cp}}}(F) \to \mathbf{R}\tilde{p}_{\mathrm{cp}*}\tilde{p}_{\mathrm{cp}}^{-1}\mathsf{L}_{N_{\mathrm{cp}}}(F) \to \tau^{\geq n+1}(\mathbf{R}\tilde{p}_{\mathrm{cp}*}\tilde{p}_{\mathrm{cp}}^{-1}\mathsf{L}_{N_{\mathrm{cp}}}(F)) \xrightarrow{+1} .$$

Applying the functor $\mathsf{Q}_{N_{\mathrm{cp}}}$, we have

$$\mathsf{Q}_{N_{\mathrm{cp}}}\mathsf{L}_{N_{\mathrm{cp}}}(F) \to \mathrm{E}p_{\mathrm{cp}*}\mathrm{E}p_{\mathrm{cp}}^{-1}(F) \to \mathsf{Q}_{N_{\mathrm{cp}}}\big(\tau^{\geq n+1}(\mathbf{R}\tilde{p}_{\mathrm{cp}*}\tilde{p}_{\mathrm{cp}}^{-1}\mathsf{L}_{N_{\mathrm{cp}}}(F))\big) \xrightarrow{+1} \tag{4.3}$$

by (4.1). For the first term of (4.3), we have

$$\mathsf{Q}_{N_{\mathrm{cp}}}\mathsf{L}_{N_{\mathrm{cp}}}(F) \simeq F \in \mathrm{E}^{[0,0]}(\mathrm{I}\mathbb{C}_{\mathsf{N}}) \tag{4.4}$$

by the assumption. For the third term of (4.3), we have

$$\mathsf{Q}_{N_{\mathrm{cp}}}\big(\tau^{\geq n+1}(\mathbf{R}\tilde{p}_{\mathrm{cp}*}\tilde{p}_{\mathrm{cp}}^{-1}\mathsf{L}_{N_{\mathrm{cp}}}(F))\big) \in \mathrm{E}^{\geq n+1}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}}) := \mathrm{E}^{\geq 0}(\mathrm{I}\mathbb{C}_{N_{\mathrm{cp}}})[-(n+1)] \tag{4.5}$$

by [6, Lem. 2.6.5]. Then (4.4) and (4.5) imply

$$F \xrightarrow{\sim} \tau^{\leq n}(\mathrm{E}p_{\mathrm{cp}*}\mathrm{E}p_{\mathrm{cp}}^{-1}(F))$$

by (4.3), which completes the proof. ∎

Proof of Lemma 4.2. Let $a\colon A \to \{\mathrm{pt}\}$ be the unique map. By Definition 4.1, it is sufficient to prove that for any $N \to \{\mathrm{pt}\}$, the base change $N \times A \to N$ of $a\colon A \to \{\mathrm{pt}\}$ is enhanced n-preacyclic of compact type, which follows from Lemma 4.6. ∎

4.1.2. Main definition.

Definition 4.7. Let $G \in \mathsf{gTop}$ be a group object having enough acyclic objects (Definition 3.7) and $X \in \mathsf{gTop}^G$. Then we define a category $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ called an equivariant category of enhanced ind-sheaves on X_{cp} as follows (where we use Notation 3.8):

- An object of $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ is a pair $F := (\{F_I\}_{I\in\mathcal{I}}, \{\phi_{IJ}\}_{(I\to J)\in\mathsf{Mor}(\mathcal{I})})$, where

$$F_I \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(\bar{X}_I)_{\mathrm{cp}}}), \quad \phi_{IJ}\colon \mathrm{E}(\bar{p}_{IJ})^{-1}_{\mathrm{cp}} F_J \xrightarrow{\sim} F_I$$

satisfying the compatibility conditions $\phi_{II} = \mathrm{id}$ for any $I \in \mathcal{I}$ and

$$\phi_{IK} = \phi_{IJ} \circ \mathrm{E}(\bar{p}_{IJ})^{-1}_{\mathrm{cp}}(\phi_{JK})$$

for any composable morphisms $I \to J \in \mathsf{Mor}(\mathcal{I})$ and $J \to K \in \mathsf{Mor}(\mathcal{I})$.

- A morphism $\alpha\colon F \to F'$ in $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ is a set $\alpha = \{\alpha_I\}_{I\in\mathcal{I}}$ of morphisms $\alpha_I\colon F_I \to F'_I$ satisfying the compatibility condition

$$\begin{tikzcd}
\mathrm{E}(\bar{p}_{IJ})^{-1}_{\mathrm{cp}} F_J \arrow[r, "\phi_{IJ}"] \arrow[d, "\mathrm{E}(\bar{p}_{IJ})^{-1}_{\mathrm{cp}}(\alpha_J)"'] & F_I \arrow[d, "\alpha_I"] \\
\mathrm{E}(\bar{p}_{IJ})^{-1}_{\mathrm{cp}} F'_J \arrow[r, "\phi'_{IJ}"'] & F'_I
\end{tikzcd}$$

for any $I \to J \in \mathsf{Mor}(\mathcal{I})$.

Remark 4.8. This definition also does not depend on the choice of $\{A_n\}_{n\in\mathbb{Z}_{\geq 1}}$ for the same reason as Definition 3.9.

4.1.3. Distinguished triangle.

Setting 4.9. From now on, we write $G \in \mathsf{gTop}$ for a group object having enough acyclic objects (Definition 3.7) and $X \in \mathsf{gTop}^G$ until the end of Section 4.2.3.

Definition 4.10 (Cf. [2, Sect. 2.5]). The shift functor $[1]\colon \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) \to \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ is defined pointwise, namely,

$$(F[1])_I := F_I[1].$$

Definition 4.11 (Cf. [2, Def. 2.5.2]). The distinguished triangles of $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ are defined pointwise, namely,

$$F \to F' \to F'' \xrightarrow{+1}$$

is a distinguished triangle in $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ if and only if

$$F_I \to F'_I \to F''_I \xrightarrow{+1}$$

is a distinguished triangle in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(\bar{X}_I)_{\mathrm{cp}}})$ for any $I \in \mathcal{I}$.

Lemma 4.12. *The above definitions make* $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ *into a triangulated category.*

Proof. Exactly the same way as the case of $\mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X)$ [2, Prop. in Sect. 2.5.2]. ∎

4.2 – *Functors*

We continue with Setting 4.9.

Notation 4.13. Recall Notation 3.8. Then, for $f\colon X \to Y \in \mathsf{Mor}(\mathsf{gTop}^G)$, we also use the following notation:

$$\begin{tikzcd} X_I \simeq X \times_Y Y_I \arrow[r, "f_I"] \arrow[d] & Y_I \arrow[d] \\ X \arrow[r, "f"] & Y, \end{tikzcd} \qquad \bar{X}_I \xrightarrow{\bar{f}_I} \bar{Y}_I := Y_I/G,$$

with obvious meaning. Moreover, we sometimes abbreviate $(\bar{f}_I)_{\mathrm{cp}}$ to $\bar{f}_I$ for notational convenience.

4.2.1. Inverse image and proper direct image.

Definition 4.14 (Cf. [2, Sect. 2.4.6 (ii)]). Let $f\colon X \to Y \in \mathsf{Mor}(\mathsf{gTop}^G)$. Then, for $F \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})$, we define $\mathrm{E}f^{-1}(F) \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ by

$$\mathrm{E}f^{-1}(F)_I := \mathrm{E}\bar{f}_I^{-1}(F_I).$$

Definition 4.15 (Cf. [2, Sect. 3.3]). Let $f\colon X \to Y \in \mathsf{Mor}(\mathsf{gTop}^G)$. Then, for $F \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$, we define $\mathrm{E}f_{!!}(F) \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})$ by

$$\mathrm{E}f_{!!}(F)_I := \mathrm{E}\bar{f}_{I!!}(F_I).$$

Remark 4.16 (Cf. [2, Sect. 3.3]). In Definition 4.15, the definition of the transition isomorphism

$$\mathrm{E}(\bar{p}^Y_{IJ})^{-1}\mathrm{E}\bar{f}_{J!!}(F_J) \xrightarrow{\sim} \mathrm{E}\bar{f}_{I!!}(F_I)$$

uses the proper base change (cf. [5, Prop. 4.5.11])

$$\mathrm{E}(\bar{p}^Y_{IJ})^{-1}\mathrm{E}\bar{f}_{J!!} \simeq \mathrm{E}\bar{f}_{I!!}\mathrm{E}(\bar{p}^X_{IJ})^{-1},$$

where

$$\begin{tikzcd}
(\bar{X}_I)_{\mathrm{cp}} \arrow[r, "\bar{f}_I"] \arrow[d, "\bar{p}^X_{IJ}"] & (\bar{Y}_I)_{\mathrm{cp}} \arrow[d, "\bar{p}^Y_{IJ}"] \\
(\bar{X}_J)_{\mathrm{cp}} \arrow[r, "\bar{f}_J"'] & (\bar{Y}_J)_{\mathrm{cp}}.
\end{tikzcd}$$

4.2.2. Forgetful functor.

Definition 4.17 (Cf. [2, Sect. 2.4.6 (i)]). For $F \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$, we define the forgetful functor by

$$\begin{aligned}
\mathsf{U}\colon \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) &\to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}),\\
F &\mapsto F_{I_0},
\end{aligned}$$

where $I_0 := (0) \in \mathcal{I}$ and we identify $\bar{X}_{I_0} \simeq X$ (Notation 3.8).

4.2.3. Sheaf type.

Definition 4.18. We define a functor

$$\mathsf{e}_{X_{\mathrm{cp}}}\colon \mathrm{D}^{\mathrm{b}}_G(\mathbb{C}_X) \to \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$$

by

$$\mathsf{e}_{X_{\mathrm{cp}}}(F)_I := \mathsf{e}_{(\bar{X}_I)_{\mathrm{cp}}}(F_I) \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(\bar{X}_I)_{\mathrm{cp}}})$$

for $I \in \mathcal{I}$. We write

$$\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathsf{sh}} \subset \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$$

for the essential image of $\mathsf{e}_{X_{\mathrm{cp}}}$.

Lemma 4.19. *The functor $\mathsf{e}_{X_{\mathrm{cp}}}$ is fully faithful and $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathsf{sh}}$ is a triangulated subcategory of $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$.*

Proof. This follows from the corresponding property of (usual) e [6, Lem. 2.8.2 (i)] and the fact that the triangulated structure of $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ is defined pointwise. ∎

4.2.4. Upper shriek and direct image.

Definition 4.20. We say that a group object $G \in \mathsf{gTop}$ has enough smooth acyclic objects if G is a Lie group, and for any $n \in \mathbb{Z}_{\geq 1}$, there exists $A_n \in \mathsf{Res}^G_{\{\mathrm{pt}\}}$ such that the unique map $A_n \to \{\mathrm{pt}\}$ is n-acyclic and A_n is a smooth manifold.

Setting 4.21. From now on, we write $G \in \mathsf{gTop}$ for a group object having enough smooth acyclic objects and $X \in \mathsf{gTop}^G$ until the end of Section 4.2.5.

Definition 4.22 (Cf. [2, Sect. 3.3]). Let $f\colon X \to Y \in \mathsf{Mor}(\mathsf{gTop}^G)$. Then, for $F \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})$, we define $\mathrm{E}f^!(F) \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ by (under Notation 4.13)

$$\mathrm{E}f^!(F)_I := \mathrm{E}\bar{f}_I^!(F_I).$$

Remark 4.23 (Cf. [2, Sects. 1.8 and 3.3]). In Definition 4.22, the definition of the transition isomorphism uses the smooth base change

$$\mathrm{E}(\bar{p}^X_{IJ})^{-1}\mathrm{E}\bar{f}_J^! \simeq \mathrm{E}\bar{f}_I^!\mathrm{E}(\bar{p}^Y_{IJ})^{-1}, \tag{4.6}$$

which we prove from now on. First, by the definition [6, p. 206] of functors, (4.6) is equivalent to

$$\mathsf{Q}p_{\mathrm{cp}}^{-1}g_{\mathrm{cp}}^! \simeq \mathsf{Q}f_{\mathrm{cp}}^!q_{\mathrm{cp}}^{-1},$$

where $\mathsf{Q}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(\bar{X}_I)_{\mathrm{cp}}\times\mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(\bar{X}_I)_{\mathrm{cp}}})$ is the quotient functor (2.9) and we use the following notation for the notational simplicity:

$$\begin{tikzcd}
(\bar{X}_I)_{\mathrm{cp}}\times\mathbb{R}_\infty \arrow[r, "f_{\mathrm{cp}}"] \arrow[d, "p_{\mathrm{cp}}"] & (\bar{Y}_I)_{\mathrm{cp}}\times\mathbb{R}_\infty \arrow[d, "q_{\mathrm{cp}}"] \\
(\bar{X}_J)_{\mathrm{cp}}\times\mathbb{R}_\infty \arrow[r, "g_{\mathrm{cp}}"'] & (\bar{Y}_J)_{\mathrm{cp}}\times\mathbb{R}_\infty,
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{X}_I\times\mathbb{R} \arrow[r, "f:=\bar{f}_I\times\mathrm{id}"] \arrow[d, "p:=\bar{p}^X_{IJ}\times\mathrm{id}"] & \bar{Y}_I\times\mathbb{R} \arrow[d, "q:=\bar{p}^Y_{IJ}\times\mathrm{id}"] \\
\bar{X}_J\times\mathbb{R} \arrow[r, "g:=\bar{f}_J\times\mathrm{id}"'] & \bar{Y}_J\times\mathbb{R}.
\end{tikzcd}$$

Therefore, it is sufficient to prove that we have

$$p_{\mathrm{cp}}^{-1}g_{\mathrm{cp}}^!F \simeq f_{\mathrm{cp}}^!q_{\mathrm{cp}}^{-1}F \tag{4.7}$$

for any $F \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(\bar{X}_I)_{\mathrm{cp}}\times\mathbb{R}_\infty})$. Moreover, we can assume $F \in \mathrm{I}\mathbb{C}_{(\bar{X}_I)_{\mathrm{cp}}\times\mathbb{R}_\infty}$ by dévissage. In this case, we can write $F \simeq \text{“}\varinjlim\text{”}\, F_j$ with $F_j \in \mathrm{Mod}(\mathbb{C}_{\bar{X}_I\times\mathbb{R}})$ by the definition (2.3). Thus, for (4.7), it is sufficient to prove that we have

$$p^{-1}g^!F_j \simeq f^!q^{-1}F_j, \tag{4.8}$$

for any $F_j \in \mathrm{Mod}(\mathbb{C}_{\bar{X}_I\times\mathbb{R}})$ because we can obtain (4.7) by applying the kth cohomology functor and “$\varinjlim$” to (4.8) by Lemma 2.16 (i). Moreover, (4.8) follows from the smooth base change for sheaves [2, Sect. 1.8] because p and q are smooth. This completes the proof of the smooth base change (4.6) for enhanced ind-sheaves in the present case.

Definition 4.24 (Cf. [2, Sect. 3.3]). Let $f\colon X \to Y \in \mathsf{Mor}(\mathsf{gTop}^G)$. Then, for $F \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$, we define $\mathrm{E}f_*(F) \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})$ by (under Notation 4.13)

$$\mathrm{E}f_*(F)_I := \mathrm{E}\bar{f}_{I*}(F_I).$$

Remark 4.25 (Cf. [2, Sects. 1.8 and 3.3]). In Definition 4.24, the definition of the transition isomorphism uses the smooth base change $\mathrm{E}(\bar{p}^Y_{IJ})^{-1}\mathrm{E}\bar{f}_{J*} \simeq \mathrm{E}\bar{f}_{I*}\mathrm{E}(\bar{p}^X_{IJ})^{-1}$, which follows from the sheaf case [2, Sect. 1.8] as in Remark 4.23 by using Lemma 2.16.

4.2.5. A distinguished triangle. We continue with Setting 4.21.

Lemma 4.26. *Let $Z \subset X$ be a closed G-invariant subset of $X \in \mathsf{gTop}^G$ and set $U := X\backslash Z$. Then, for any $F \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$, we have a distinguished triangle*

$$\mathrm{E}i_*\mathrm{E}i^!(F) \to F \to \mathrm{E}j_*\mathrm{E}j^!(F) \xrightarrow{+1},$$

where $i\colon Z \hookrightarrow X$ and $j\colon U \hookrightarrow X$ are the closed and open embeddings, respectively.

Proof. By Definition 4.11, it is sufficient to prove that

$$(\mathrm{E}i_*\mathrm{E}i^!(F))_I \to F_I \to (\mathrm{E}j_*\mathrm{E}j^!(F))_I \xrightarrow{+1}$$

is a distinguished triangle in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{(\bar{X}_I)_{\mathrm{cp}}})$ for any $I \in \mathcal{I}$. By the definitions of functors, the above triangle can be rewritten as

$$\mathrm{E}\bar{i}_{I*}\mathrm{E}\bar{i}^!_I(F_I) \to F_I \to \mathrm{E}\bar{j}_{I*}\mathrm{E}\bar{j}^!_I(F_I) \xrightarrow{+1}, \tag{4.9}$$

where

$$\begin{tikzcd} Z_I \arrow[r, hook, "i_I"] \arrow[d] & X_I \arrow[d] & U_I \arrow[l, hook', "j_I"'] \arrow[d] \\ Z \arrow[r, hook, "i"'] & X & U, \arrow[l, hook', "j"] \end{tikzcd} \qquad \begin{tikzcd} (\bar{Z}_I)_{\mathrm{cp}} \arrow[r, hook, "\bar{i}_I"] & (\bar{X}_I)_{\mathrm{cp}} & (\bar{U}_I)_{\mathrm{cp}}. \arrow[l, hook', "\bar{j}_I"'] \end{tikzcd}$$

Note that $\bar{Z}_I$ is closed in $\bar{X}_I$ because Z_I is closed in X_I. This implies that the above triangle (4.9) is exact by Lemma 2.18, which completes the proof. ∎

5. Equivariant algebraic $\mathcal{D}$-modules

In this section we recall the definition of the equivariant derived category of algebraic $\mathcal{D}$-modules for the later use. Although almost all the materials in this section also seem to be well known, we prove some results for convenience. Our main references are [2, Sect. 4], [18, Sect. 4.5], and [1, Chap. 6].

Convention 5.1. In this article we always work over $\mathbb{C}$ and algebraic varieties are all quasi-projective.

5.1 – *$\mathcal{D}$-acyclicity*

Recall a well-known fact:

Fact 5.2 (Cf. [17, Thm. 4.40]). Let $f\colon X \to Y$ be a smooth map of algebraic smooth varieties. Then we have an adjunction

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathbf{D}f^*N, M)[d_X] \simeq \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(N, \mathbf{D}f_*M)[d_Y] \tag{5.1}$$

for $M \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$ and $N \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y)$, where d_X and d_Y are the dimensions of X and Y, respectively.

Proof. The same proof as for the analytic setting [17, Thm. 4.40] also works in the algebraic setting. ∎

Definition 5.3. Let $n \in \mathbb{Z}_{\geq 0}$. We say that a smooth map $f\colon X \to Y$ between smooth algebraic varieties is $\mathcal{D}$-n-acyclic if it satisfies the following conditions:

(DAc1)$_n$ For any $M \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X) \simeq \mathrm{D}^{[0,0]}_{\mathrm{coh}}(\mathcal{D}_Y)$, the adjunction $\mathbf{D}f^* \dashv \mathbf{D}f_*[d_Y - d_X]$ (5.1) induces

$$M \xrightarrow{\sim} \tau^{\leq n}(\mathbf{D}f_*\mathbf{D}f^*M[d_Y - d_X]) \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y).$$

(DAc2)$_n$ For any morphism $Y' \to Y$ from a smooth algebraic variety Y' such that the fiber product $X' := X \times_Y Y'$ is smooth, the base change $f'\colon X' \to Y'$ satisfies the property (DAc1)$_n$.

Lemma 5.4 (Cf. [18, (4.5.2)]). *Let A be a smooth algebraic variety and $a\colon A \to \{\mathrm{pt}\}$ the unique map. Then $a\colon A \to \{\mathrm{pt}\}$ is $\mathcal{D}$-n-acyclic if $A^{\mathrm{an}} \in \mathsf{gTop}$ is n-acyclic, where A^{an} is the analytification of A.*

Although this seems to be well known, we prove this lemma for the sake of completeness.

Proof of Lemma 5.4. Let $a\colon A \to \{\mathrm{pt}\}$ be the unique map from a smooth algebraic variety A to the one-point space and $a'\colon A \times B \to B$ the base change of $a\colon A \to \{\mathrm{pt}\}$ by the unique map $B \to \{\mathrm{pt}\}$ from a smooth algebraic variety B to the one-point space:

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d, "a' := a \times \mathrm{id}"] & A \arrow[d, "a"] \\ \{\mathrm{pt}\} \times B \arrow[r] & \{\mathrm{pt}\}. \end{tikzcd}$$

Using [11, Prop. 1.5.18 (i) and Prop. 1.5.30], we have

$$\mathbf{D}a'_*\mathbf{D}a'^*M \simeq \mathbf{D}(a\times \mathrm{id})_*\mathbf{D}(a\times \mathrm{id})^*(\mathbb{C}\boxtimes M) \simeq \mathbf{D}a_*\mathbf{D}a^*(\mathbb{C})\boxtimes M,$$

where $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_B)$ and $\boxtimes$ is the external tensor product. Moreover, by [11, Prop. 1.5.28 (i)], we have

$$\mathbf{D}a_*\mathbf{D}a^*(\mathbb{C}) \simeq \mathbf{D}a_*(\mathscr{O}_A) \simeq \mathbf{R}a_*\mathrm{DR}_{A/\{\mathrm{pt}\}}(\mathscr{O}_A) \simeq \mathbf{R}\Gamma(A,\Omega_A^\bullet)[d_A],$$

where $\mathrm{DR}_{A/\{\mathrm{pt}\}}$ is the relative de Rham complex [11, p. 45], $\Omega_A^\bullet$ is the algebraic de Rham complex on A, and $\Gamma(A,\cdot)$ is the global sections functor. Then we have

$$\mathbf{R}\Gamma(A,\Omega_A^\bullet)[d_A] \simeq \mathbf{R}\Gamma(A^{\mathrm{an}},\mathbb{C})[d_A] \simeq \mathbf{R}a^{\mathrm{an}}_*(a^{\mathrm{an}})^{-1}(\mathbb{C})[d_A],$$

because the algebraic de Rham cohomology is isomorphic to the usual cohomology of A^{an} by [8, Thm. 1']. Thus, we have

$$\mathbf{D}a'_*\mathbf{D}a'^*M[-d_A] \simeq \mathbf{R}a^{\mathrm{an}}_*(a^{\mathrm{an}})^{-1}(\mathbb{C})\boxtimes M. \tag{5.2}$$

We note that the assumption that A^{an} is n-acyclic implies

$$\mathbb{C} \xrightarrow{\sim} \tau^{\leq n}(\mathbf{R}a^{\mathrm{an}}_*(a^{\mathrm{an}})^{-1}(\mathbb{C})).$$

Thus, the exactness of the exterior tensor product $\boxtimes$ [11, p. 39] implies

$$\tau^{\leq n}(\mathbf{R}a^{\mathrm{an}}_*(a^{\mathrm{an}})^{-1}(\mathbb{C})\boxtimes M) \simeq \tau^{\leq n}(\mathbf{R}a^{\mathrm{an}}_*(a^{\mathrm{an}})^{-1}(\mathbb{C}))\boxtimes M \simeq \mathbb{C}\boxtimes M \simeq M.$$

Therefore, (5.2) implies that

$$M \xrightarrow{\sim} \tau^{\leq n}(\mathbf{D}a'_*\mathbf{D}a'^*(M)[d_B - d_{A\times B}]),$$

because $d_B - d_{A\times B} = -d_A$, which completes the proof. ∎

5.2 – *Definition*

Convention 5.5 (Cf. [1, Rem. 6.1.12]). In this section, we say that an action of a linear algebraic group G on a smooth algebraic variety A is free if the quotient map $A \to A/G$ makes A into a principal G-bundle.[3]

(3) In this setting, $A \to A/G$ is locally trivial in the ètale topology by [1, Rem. 6.1.12]. Moreover, if the action of G on A is free in the sense of Convention 5.5, the action of G^{an} on A^{an} is free in the sense of Convention 3.2 by [1, Prop. 6.1.11], where G^{an} and A^{an} are analytifications of G and A, respectively.

Definition 5.6. Let G be a linear algebraic group. We say that G has enough $\mathcal{D}$-acyclic objects if for any $n \in \mathbb{Z}_{\geq 1}$, there exists a smooth algebraic G-variety A_n with free G-action such that the unique map $A_n \to \{\mathrm{pt}\}$ is $\mathcal{D}$-n-acyclic (Definition 5.3), and that $(A_n \times X)/G$ is a smooth algebraic variety[4] for any smooth algebraic G-variety X.

Setting 5.7. From now on, we write G for a linear algebraic group having enough $\mathcal{D}$-acyclic objects, until the end of Section 5.

Definition 5.8 ([2, Sect. 4.2]). For a smooth G-variety X, we write $\mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$ for the equivariant derived category of coherent $\mathcal{D}$-modules defined in the following way (under Notation 3.8):

- An object of $\mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$ is a pair $M := (\{M_I\}_{I\in\mathcal{I}}, \{\phi_{IJ}\}_{(I\to J)\in\mathsf{Mor}(\mathcal{I})})$, where
$$M_I \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{\bar{X}_I}), \quad \phi_{IJ}\colon \mathbf{D}\bar{p}_{IJ}^{*} M_J \xrightarrow{\sim} M_I,$$
satisfying the compatibility conditions $\phi_{II} = \mathrm{id}$ for any $I \in \mathcal{I}$ and
$$\phi_{IK} = \phi_{IJ} \circ \mathbf{D}\bar{p}_{IJ}^{*}(\phi_{JK})$$
for any composable morphisms $I \to J \in \mathsf{Mor}(\mathcal{I})$ and $J \to K \in \mathsf{Mor}(\mathcal{I})$.
- A morphism $\alpha\colon M \to M'$ in $\mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)$ is a set $\alpha = \{\alpha_I\}_{I\in\mathcal{I}}$ of morphisms $\alpha_I\colon M_I \to M'_I$ satisfying the compatibility condition
$$\begin{tikzcd}
\mathbf{D}\bar{p}_{IJ}^{*} M_J \arrow[r, "\phi_{IJ}"] \arrow[d, "\mathbf{D}\bar{p}_{IJ}^{*}(\alpha_J)"'] & M_I \arrow[d, "\alpha_I"] \\
\mathbf{D}\bar{p}_{IJ}^{*} M'_J \arrow[r, "\phi'_{IJ}"'] & M'_I
\end{tikzcd}$$
for any $I \to J \in \mathsf{Mor}(\mathcal{I})$.

Remark 5.9. This definition also does not depend on the choice of $\{A_n\}_{n\in\mathbb{Z}_{\geq 1}}$ for the same reason as Definition 3.9.

(4) This condition is enough to ensure that $(A_{n_1} \times A_{n_2} \times \cdots \times A_{n_k} \times X)/G$ is smooth for any $n_1, \ldots, n_k \in \mathbb{Z}_{\geq 1}$. In fact, it is sufficient to apply this condition to a smooth algebraic G-variety $A_{n_2} \times \cdots \times A_{n_k} \times X$.

5.2.1. Inverse image.

Definition 5.10 (Cf. [2, Sect. 2.4.6 (ii)]). Let $f\colon X \to Y$ be a G-equivariant morphism of smooth algebraic G-varieties, and $M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_Y)_{\mathrm{coh}}$. Then the inverse image $\mathbf{D}f^*(M) \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$ of M is given by (under Notation 4.13)

$$\mathbf{D}f^*(M)_I := \mathbf{D}\bar{f}_I^*(M_I).$$

5.2.2. Forgetful functor.

Definition 5.11 ([2, Sect. 2.4.6 (i)]). Let $M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$. Then the forgetful functor is given by

$$\begin{aligned} \mathsf{U}\colon \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}} &\to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X), \\ M &\mapsto M_{I_0}, \end{aligned}$$

where $I_0 := (0) \in \mathcal{I}$ and we identify $\bar{X}_{I_0} \simeq X$ (Notation 3.8).

5.3 – *A lemma*

Let

$$\mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{rh}} := \left\{ M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}} \;\middle|\; M_I \in \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_{\bar{X}_I}) \text{ for any } I \in \mathcal{I} \right\}$$

be the full subcategory of $\mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$ consisting of objects $M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$ satisfying $M_I \in \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_{\bar{X}_I})$ for any $I \in \mathcal{I}$, where $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_{\bar{X}_I})$ is the derived category of regular holonomic $\mathcal{D}$-modules [11, Def. 6.1.1]. The following lemma also seems to be well known.

Lemma 5.12. *Let G be a linear algebraic group and H a closed algebraic subgroup of G. Then we have*

$$\mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_{G/H})_{\mathrm{coh}} \simeq \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_{G/H})_{\mathrm{rh}}.$$

Proof. We note that the induction equivalence ([2, Thm. in Sect. 2.6.3] or [18, (4.5.7)]) preserves the coherence and regular holonomicity (which follows from the definition of the induction functor). Thus, applying this equivalence, we have

$$\mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_{G/H})_{\mathrm{coh}} \simeq \mathrm{D}^{\mathrm{b}}_H(\mathcal{D}_{\{\mathrm{pt}\}})_{\mathrm{coh}}.$$

Because

$$\mathrm{D}^{\mathrm{b}}_H(\mathcal{D}_{\{\mathrm{pt}\}})_{\mathrm{coh}} \simeq \mathrm{D}^{\mathrm{b}}_H(\mathcal{D}_{\{\mathrm{pt}\}})_{\mathrm{rh}}$$

is obvious, we have the lemma by the induction equivalence again. ∎

6. Equivariant algebraic enhanced de Rham functor

In this section we define an equivariant algebraic enhanced de Rham functor in Definition 6.11.

6.1 – *Algebraic enhanced de Rham functor*

Definition 6.1 (Cf. [14, Sect. 3.2]). Let X be a smooth algebraic variety, and $\widehat{X}$ a smooth complete algebraic variety containing X as an open subset such that $\widehat{X} \setminus X$ is a normal crossing divisor. Then, under Notation 2.20, we define

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}} \colon \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$$

by

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(M) := \mathrm{E} j^{-1}_{X_{\mathrm{cp}}} \mathrm{DR}^{\mathrm{E}}_{\widehat{X}}\big((\mathbf{D} j_{X*}(M))^{\mathrm{an}}\big)$$

for $M \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$, where $\mathrm{DR}^{\mathrm{E}}_{\widehat{X}}$ is the enhanced de Rham functor on $\widehat{X}^{\mathrm{an}}$ [5, Def. 9.1.1] and $(\mathbf{D} j_{X*}(M))^{\mathrm{an}}$ is the analytification of $\mathbf{D} j_{X*}(M)$.

Lemma 6.2. *The definition of* $\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}$ *in Definition 6.1 does not depend on the choice of* $\widehat{X}$.

Proof. We prove this lemma[5] in Appendix B. ∎

Lemma 6.3 (Cf. [14, Prop. 3.12 (2)]). *Let $f \colon X \to Y$ be a morphism of smooth algebraic varieties, and $f^{\mathrm{an}} \colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ its analytification, which induces the morphism $f^{\mathrm{an}}_{\mathrm{cp}} \colon X_{\mathrm{cp}} \to Y_{\mathrm{cp}} \in \mathsf{Mor}(\mathsf{Bod})$. Then, for any $M \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y)$, we have*

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(\mathbf{D} f^* M)[d_X] \simeq \mathrm{E}(f^{\mathrm{an}}_{\mathrm{cp}})^! \mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(M)[d_Y]. \tag{6.1}$$

Proof. We prove this lemma in Appendix C. ∎

Lemma 6.4. *Let X be a smooth algebraic variety. Then we have*

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)) \subset \mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) \cap \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}).$$

([5]) Because the independence of the algebraic enhanced solution functor $\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}$ (Definition D.1) from the choice of $\widehat{X}$ is not explicitly proved in [14], we prove this lemma in Appendix B.

Proof. Because the duality functor $\mathbb{D}_X$ is an anti-autoequivalence on $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ [11, Prop. 2.6.5 (ii) and Thm. 6.1.5 (i)], it is sufficient to prove

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}\big(\mathbb{D}_X(\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X))\big) \subset \mathrm{E}^{\mathrm{b}}_{\mathrm{sh}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) \cap \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}),$$

which follows from the corresponding property of the enhanced algebraic solution functor $\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}$ [14, Prop. 3.14] by using the identity $\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(\mathbb{D}_X M)[d_X] \simeq \mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(M)$ of Lemma D.4 for $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$. ∎

Lemma 6.5 (Cf. [13, Lem. 3.10]). *Let X be a smooth algebraic variety and $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$. If $\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(M) \in \mathrm{E}^{\mathrm{b}}_{\mathrm{sh}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) \cap \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$, then we have $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$.*

Proof. Although this also seems to be well known, we prove this lemma in Appendix D for completeness.[6] ∎

6.2 – *Equivariant algebraic enhanced de Rham functor*

Definition 6.6. Let G be a linear algebraic group. We say that G has enough compatible $\mathcal{D}$-acyclic objects if for any $n \in \mathbb{Z}_{\geq 1}$, there exists a smooth algebraic G-variety A_n with free G-action such that $A_n^{\mathrm{an}} \in \mathsf{gTop}$ is n-acyclic (Definition 3.5), and that $(A_n \times X)/G$ is a smooth algebraic variety for any smooth algebraic G-variety X.

Lemma 6.7. *Let G be a linear algebraic group. Then G has enough compatible $\mathcal{D}$-acyclic objects.*

Although this seems to be well known, we prove this lemma for the sake of completeness. We need a lemma.

Lemma 6.8. *Let $n \in \mathbb{Z}_{\geq 0}$ and suppose that $A \in \mathsf{gTop}$ is a non-empty smooth manifold. Then A is n-acyclic if and only if $a\colon A \to \{\mathrm{pt}\}$ is n-preacyclic.*

Proof. The "only if" part is obvious by Lemma 3.6. For the "if" part, let $a\colon A \to \{\mathrm{pt}\}$ be n-preacyclic, which means that A satisfies condition $(\mathrm{A1})_n$ in Definition 3.5. In particular, we have $a_* a^{-1}(\mathbb{C}) \simeq \mathbb{C}$, which implies that A is connected. Moreover, it is clear that A is locally connected because A is a manifold. Thus, A satisfies condition $(\mathrm{A2})_n$, which completes the proof by Definition 3.5. ∎

[6] In [13, Lem. 3.10], this lemma (for $\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}$) is proved under the assumption $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ (instead of $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$).

Proof of Lemma 6.7. Let $n \in \mathbb{Z}_{\geq 1}$. By Definition 6.6, it is sufficient to prove that there exists a smooth algebraic G-variety A_n with free G-action such that A_n^{an} is n-acyclic, and that $(A_n \times X)/G$ is a smooth algebraic variety for any smooth G-variety X.

Let k be an integer such that G can be realized as a closed subgroup of $\mathrm{GL}(k, \mathbb{C})$, which we regard as a subgroup of $\mathrm{GL}(k+n, \mathbb{C})$ by

$$\mathrm{GL}(k, \mathbb{C}) \simeq \mathrm{GL}(k, \mathbb{C}) \times 1 \subset \mathrm{GL}(k, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(k+n, \mathbb{C}).$$

Moreover, we write H for a subgroup of $\mathrm{GL}(k+n, \mathbb{C})$ defined by

$$H := \left\{ \left(\begin{smallmatrix} 1 & u \\ 0 & g \end{smallmatrix} \right) \in \mathrm{GL}(k+n, \mathbb{C}) \;\middle|\; u \in \mathrm{M}_{k,n}(\mathbb{C}),\, g \in \mathrm{GL}(n, \mathbb{C}) \right\},$$

where $\mathrm{M}_{k,n}(\mathbb{C})$ is the space of $k \times n$ matrices. Then we set $A_n := \mathrm{GL}(n+k, \mathbb{C})/H$ (the non-compact complex Stiefel variety of k-frames in $\mathbb{C}^{k+n}$) with right G-action, which is well defined because $\mathrm{GL}(k, \mathbb{C})$ normalizes H.

By [25, Thm. 2.2] (with Convention 5.1) for any smooth G-variety X, the quotient $(\mathrm{GL}(k+n, \mathbb{C}) \times X)/(G \times H) \simeq (A_n \times X)/G$ is an algebraic variety (which is equal to $\mathrm{GL}(k+n, \mathbb{C}) *_{G \times H} X$ under the notation of [25, Def. 2.1]). The smoothness of $(A_n \times X)/G$ follows from [1, Prop. 6.1.16] (applying to the second projection $A_n \times X \to A_n$) because A_n/G is smooth.

For the n-acyclicity of A_n^{an}, it is sufficient to prove that $a\colon A_n^{\mathrm{an}} \to \{\mathrm{pt}\}^{\mathrm{an}}$ is n-preacyclic by Lemma 6.8. For this purpose, we write A'_n for the compact complex Stiefel manifold of orthonormal k-frames in $\mathbb{C}^{k+n}$. Then A'_n is homotopy equivalent to A_n^{an} [15, p. 15]. Thus, writing $a'\colon A'_n \to \{\mathrm{pt}\}$, we have

$$\mathbf{R}a_* a^{-1} F \simeq \mathbf{R}a'_* (a')^{-1} F$$

for any $F \in \mathrm{Mod}(\mathbb{C}_{\{\mathrm{pt}\}})$ by the homotopy invariance of cohomology [19, Cor. 2.7.7 (i)], which implies that it is sufficient to prove that $a'\colon A'_n \to \{\mathrm{pt}\}$ is n-preacyclic. This follows from Lemma 6.9 given below because we have

$$\mathbb{Z} \xrightarrow{\sim} \tau^{\leq n}(\mathbf{R}a'_* (a')^{-1}(\mathbb{Z}))$$

by [3, Prop. 9.1]. ■

Lemma 6.9. *Let $n \in \mathbb{Z}_{\geq 0}$ and $A \in$ gTop. Suppose that A is a compact smooth manifold satisfying*

$$\mathbb{Z} \xrightarrow{\sim} \tau^{\leq n}(\mathbf{R}a_* a^{-1}(\mathbb{Z})).$$

Then $a\colon A \to \{\mathrm{pt}\}$ is n-preacyclic.

Proof. The compactness of A implies $\mathbf{R}a_! = \mathbf{R}a_*$. Therefore, we have

$$\begin{aligned}\mathbf{R}a_* a^{-1} F &\simeq \mathbf{R}a_! a^{-1} F \\ &\simeq \mathbf{R}a_!(a^{-1}(\mathbb{Z}) \otimes_{a^{-1}(\mathbb{Z})} a^{-1}(F)) \\ &\simeq \mathbf{R}a_!(a^{-1}(\mathbb{Z})) \otimes_{\mathbb{Z}} F \\ &\simeq \mathbf{R}a_*(a^{-1}(\mathbb{Z})) \otimes_{\mathbb{Z}} F\end{aligned}$$

for any $F \in \mathrm{Mod}(\mathbb{C}_{\{\mathrm{pt}\}^{\mathrm{an}}})$ by [20, Prop. 2.6.6], which completes the proof. ∎

Remark 6.10. Let G be a linear algebraic group. Then Lemma 6.7 implies that G has enough compatible $\mathcal{D}$-acyclic objects, and thus there exists a sequence $\{A_n\}_{n\in\mathbb{Z}_{\geq 1}}$ satisfying the conditions of Definition 6.6. In this setting, we can define $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ by using this $\{A_n^{\mathrm{an}}\}_{n\in\mathbb{Z}_{\geq 1}}$ (Definition 4.7) because $A_n^{\mathrm{an}} \in \mathsf{gTop}$ is n-acyclic by Definition 6.6. Moreover, we can define $\mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)$ by using the same $\{A_n\}_{n\in\mathbb{Z}_{\geq 1}}$ (Definition 5.8) because the unique map $a\colon A_n \to \{\mathrm{pt}\}$ is $\mathcal{D}$-n-acyclic by Lemma 5.4. This remark is used in Definition 6.11 given below. Moreover, the definition of the transition isomorphism of Definition 6.11 uses Lemma 6.3 and the isomorphism of functors $\mathrm{E}(\bar{p}^X_{IJ})^!_{\mathrm{cp}}[d_{\bar{X}_I}] \simeq \mathrm{E}(\bar{p}^X_{IJ})^{-1}_{\mathrm{cp}}[d_{\bar{X}_J}]$ (which follows from [7, (2.3)] and [19, Prop. 3.3.6 (iii)]), where we use Notation 3.8.

Definition 6.11. Let G be a linear algebraic group. We define an equivariant algebraic enhanced de Rham functor

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}\colon \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}} \to \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$$

by

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)_I := \mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(M_I),$$

where $I \in \mathcal{I}$ and $M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$.

Proof of Theorem 1.1. We check the commutativity of (1.1). Let $M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$. Then, by Definition 5.11, we have

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(\mathsf{U}(M)) = \mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(M_{I_0}).$$

On the other hand, by Definitions 4.17 and 6.11, we have

$$\mathsf{U}(\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)) \simeq \mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)_{I_0} \simeq \mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(M_{I_0}),$$

which completes the proof. ∎

Lemma 6.12. *Let $f\colon X \to Y$ be a G-equivariant morphism of smooth algebraic G-varieties, $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ its analytification, and $M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_Y)_{\mathrm{coh}}$. Then we have*

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(\mathbf{D} f^* M)[d_X] \simeq \mathrm{E}(f^{\mathrm{an}})^! \mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)[d_Y].$$

Proof. By Definitions 4.10, 6.11, and 5.10, we have

$$\begin{aligned}(\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(\mathbf{D} f^* M)[d_X])_I &\simeq \mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(\mathbf{D} f^* M)_I[d_X]\\ &\simeq \mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}((\mathbf{D} f^* M)_I)[d_X]\\ &\simeq \mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(\mathbf{D} \bar{f}_I^*(M_I))[d_X].\end{aligned}$$

Lemma 6.3 and Definition 6.11 imply that

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(\mathbf{D} \bar{f}_I^*(M_I))[d_X] \simeq \mathrm{E}(\bar{f}_I^{\mathrm{an}})^! \mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(M_I)[d_Y] \simeq \mathrm{E}(\bar{f}_I^{\mathrm{an}})^! (\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)_I)[d_Y].$$

Moreover, we have

$$\mathrm{E}(\bar{f}_I^{\mathrm{an}})^! (\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)_I)[d_Y] \simeq (\mathrm{E}(f^{\mathrm{an}})^! \mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M))_I[d_Y] \simeq (\mathrm{E}(f^{\mathrm{an}})^! \mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)[d_Y])_I$$

by Definitions 4.22 and 4.10. This completes the proof. ∎

6.3 – *Equivariant algebraic $\mathbb{C}$-constructible enhanced ind-sheaves*

In this section, G always denotes a linear algebraic group.

Definition 6.13. Let X be a smooth algebraic G-variety. Then we define a full subcategory $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-c}}$ of $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ by

$$\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-c}} := \left\{F \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) \;\middle|\; F_I \in \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{(\bar{X}_I)_{\mathrm{cp}}}) \text{ for any } I \in \mathcal{I}\right\}.$$

Lemma 6.14. *Let X be a smooth algebraic G-variety. Then $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-c}}$ is a triangulated full subcategory of $\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$.*

Proof. This follows from the corresponding property of $\mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) \subset \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$ (cf. [14, paragraph after Def. 3.10]). ∎

Lemma 6.15. *Let $f\colon X \to Y$ be a G-equivariant morphism of smooth algebraic G-varieties and $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ its analytification. Then we have*

$$\begin{gathered}\mathrm{E}(f^{\mathrm{an}})^{-1}(\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})_{\mathbb{C}\text{-c}}) \subset \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-c}}, \quad \mathrm{E}(f^{\mathrm{an}})^!(\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})_{\mathbb{C}\text{-c}}) \subset \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-c}},\\ \mathrm{E} f^{\mathrm{an}}_{!!}(\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-c}}) \subset \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})_{\mathbb{C}\text{-c}}, \quad \mathrm{E} f^{\mathrm{an}}_{*}(\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-c}}) \subset \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})_{\mathbb{C}\text{-c}}.\end{gathered}$$

Proof. This follows from Fact 2.21. ∎

Lemma 6.16. *Let $f\colon X \to Y$ be a G-equivariant morphism of smooth algebraic G-varieties and $f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ its analytification. Then we have*

$$\mathrm{E}f^{\mathrm{an}}_*(\mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathsf{sh}} \cap \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-}\mathrm{c}}) \subset \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})_{\mathsf{sh}} \cap \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{Y_{\mathrm{cp}}})_{\mathbb{C}\text{-}\mathrm{c}}.$$

Proof. This follows from Lemma 2.22. ∎

Lemma 6.17. *Let H be a closed algebraic subgroup of G. Then we have*

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(\mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_{G/H})_{\mathrm{coh}}) \subset \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{(G/H)_{\mathrm{cp}}})_{\mathsf{sh}} \cap \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{(G/H)_{\mathrm{cp}}})_{\mathbb{C}\text{-}\mathrm{c}}.$$

Proof. By Lemma 5.12, we have

$$\mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_{G/H})_{\mathrm{coh}} \simeq \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_{G/H})_{\mathrm{rh}},$$

which completes the proof by Lemma 6.4 and the definitions. ∎

Lemma 6.18. *Let X be a smooth algebraic G-variety and $M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$. If $\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M) \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathsf{sh}} \cap \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-}\mathrm{c}}$, then we have $M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{rh}}$.*

Proof. This easily follows from Lemma 6.5. ∎

7. Regular holonomicity in the finite orbits case in the algebraic setting

In this section, as an answer to the question in Section 1, we give an approach to the well-known fact (Corollary 1.2) that any equivariant algebraic coherent $\mathcal{D}$-module is regular holonomic if the number of orbits is finite. We need a lemma.

Lemma 7.1. *Let G be a linear algebraic group, X a smooth algebraic G-variety, and $M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$. Then, if the number of G-orbits on X is finite, we have*

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M) \in \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathsf{sh}} \cap \mathrm{E}^{\mathrm{b}}_G(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-}\mathrm{c}}.$$

Proof. We use induction on the number of orbits. Thus, we first consider the case that there exists only one orbit. In this case, we have an isomorphism $X \simeq G/H$ for some closed algebraic subgroup H of G. Thus, the lemma follows from Lemma 6.17 in this case.

Suppose that the number of orbits is $n > 1$. Take a closed orbit Z in X and set $U := X \backslash Z$. Then the number of orbits on U is $n-1$. Let $M \in \mathrm{D}^{\mathrm{b}}_G(\mathcal{D}_X)_{\mathrm{coh}}$. By Lemma 4.26, we have a distinguished triangle

$$\mathrm{E}i^{\mathrm{an}}_*\mathrm{E}(i^{\mathrm{an}})^!(\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)) \to \mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M) \to \mathrm{E}j^{\mathrm{an}}_*\mathrm{E}(j^{\mathrm{an}})^!(\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)) \xrightarrow{+1}, \tag{7.1}$$

where $i\colon Z \hookrightarrow X$ is the closed embedding and $j\colon U \hookrightarrow X$ is the open embedding. Moreover, Lemma 6.12 and the induction hypothesis imply

$$\begin{aligned}\mathrm{E}(i^{\mathrm{an}})^{!}\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)[d_X] \simeq \mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(\mathbf{D}i^{*}M)[d_Z] \in \mathrm{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{Z_{\mathrm{cp}}})_{\mathsf{sh}} \cap \mathrm{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{Z_{\mathrm{cp}}})_{\mathbb{C}\text{-}\mathrm{c}},\\ \mathrm{E}(j^{\mathrm{an}})^{!}(\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)) \simeq \mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(\mathbf{D}j^{*}M) \in \mathrm{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{U_{\mathrm{cp}}})_{\mathsf{sh}} \cap \mathrm{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{U_{\mathrm{cp}}})_{\mathbb{C}\text{-}\mathrm{c}}.\end{aligned}$$

Thus, Lemma 6.16 implies

$$\begin{aligned}\mathrm{E}j^{\mathrm{an}}_{*}\mathrm{E}(j^{\mathrm{an}})^{!}(\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)) \in \mathrm{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathsf{sh}} \cap \mathrm{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-}\mathrm{c}},\\ \mathrm{E}i^{\mathrm{an}}_{*}\mathrm{E}(i^{\mathrm{an}})^{!}(\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M)) \in \mathrm{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathsf{sh}} \cap \mathrm{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-}\mathrm{c}}.\end{aligned}$$

This and (7.1) imply

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M) \in \mathrm{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathsf{sh}} \cap \mathrm{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-}\mathrm{c}}$$

by Lemmas 4.19 and 6.14, which completes the proof. ■

Then we prove Corollary 1.2 from Section 1.

Proof of Corollary 1.2. By Lemma 7.1, we have

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{eq}}(M) \in \mathrm{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathsf{sh}} \cap \mathrm{E}^{\mathrm{b}}_{G}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})_{\mathbb{C}\text{-}\mathrm{c}},$$

which completes the proof by Lemma 6.18. ■

Remark 7.2. In the proof of Lemma 7.1, we use the distinguished triangle (7.1) associated to $U_{\mathrm{cp}} \hookrightarrow X_{\mathrm{cp}}$, which is induced by the open embedding $U \hookrightarrow X$. (The existence of this distinguished triangle is provided by Lemma 4.26, which is based on Lemma 2.18.) This can be regarded as a reason (from the viewpoint of enhanced ind-sheaves) why Corollary 1.2 is not valid in the analytic setting. In fact, the target category of the enhanced de Rham functor in the analytic setting is $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ (instead of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$). Thus, if we want to imitate the proof of Lemma 7.1, we must use a distinguished triangle associated to the morphism $U \hookrightarrow X$ (instead of $U_{\mathrm{cp}} \hookrightarrow X_{\mathrm{cp}}$) of bordered spaces. However, the associated triangle

$$\mathrm{E}k_{Z*}\mathrm{E}k_Z^{!}(F) \to F \to \mathrm{E}k_{U*}\mathrm{E}k_U^{!}(F) \xrightarrow{+1}$$

is not distinguished in general (see Remark 2.19), where $k_Z\colon Z \to M \in \mathsf{Mor}(\mathsf{Bod})$ and $k_U\colon U \to M \in \mathsf{Mor}(\mathsf{Bod})$ are induced by the closed and open embeddings, respectively. Thus, the analogous proof of Lemma 7.1 in the analytic setting does not work.

A. Compactification of a good topological space

In this section we prove that the one-point (Alexandroff) compactification of $M \in \mathsf{gTop}$ is good (Definition 2.1).

Lemma A.1. *Let M be a non-compact good topological space and M^* the one-point (Alexandroff) compactification of M. Then M^* is good.*

Proof. We only show that M^* has finite flabby dimension because it is clear that M^* is Hausdorff, locally compact, and countable at infinity.

Recall that for any locally compact space, it has finite flabby dimension if and only if it has finite c-soft dimension [19, Ex. II.9 (c)]. Therefore, it is sufficient to prove that M^* has finite c-soft dimension. This is a consequence of Lemma A.2 given below. ∎

Lemma A.2. *Let M be a non-compact good topological space and M^* be the one-point (Alexandroff) compactification of M. Then the c-soft dimension of M^* is equal to that of M.*

Proof. Let c_M and c_{M*} be the c-soft dimensions of M and M^*, respectively. Then, by definition (cf. [20, p. 83] or [19, Ex. II.9 (b)]), c_{M*} is smaller than or equal to c_M if and only if

$$\mathbf{R}a'_!(F) \in \mathrm{D}^{[0,c_M]}(\mathbb{C}) \tag{A.1}$$

for any $F \in \mathrm{Mod}(\mathbb{C}_{M^*})$, where $a'\colon M^* \to \{\mathrm{pt}\}$ is the unique map from M^* to the one-point topological space $\{\mathrm{pt}\}$ and $\mathrm{Mod}(\mathbb{C}_{M^*})$ is the category of sheaves on M^*. First, we will prove (A.1).

Recall that there exists an exact sequence ([19, Prop. 2.3.6 (v)]) for any $F \in \mathrm{Mod}(\mathbb{C}_{M^*})$,

$$0 \to F_M \to F \to F_{\{\infty\}} \to 0, \tag{A.2}$$

where $\infty \in M^*$ is the unique point in $M^*\backslash M$. Applying $\mathbf{R}a'_!$ to (A.2), we have a distinguished triangle

$$\mathbf{R}a'_!(F_M) \to \mathbf{R}a'_!(F) \to \mathbf{R}a'_!(F_{\{\infty\}}) \xrightarrow{+1} . \tag{A.3}$$

It is clear that

$$\mathbf{R}a'_!(F_{\{\infty\}}) \simeq F_\infty \in \mathrm{D}^{[0,0]}(\mathbb{C}), \tag{A.4}$$

where F_∞ is the stalk of F at $\infty \in M^*$. On the other hand, by [19, Prop. 2.5.4 (ii)], we have

$$F_M \simeq j_! j^{-1}(F) \simeq \mathbf{R}j_! j^{-1}(F),$$

where j is the open embedding $j\colon M \to M^*$. Then we have

$$\mathbf{R}a'_!(F_M) \simeq \mathbf{R}a'_!(\mathbf{R}j_! j^{-1}(F)) \simeq \mathbf{R}a_! j^{-1}(F),$$

where $a := a' \circ j\colon M \to \{\mathrm{pt}\}$. Thus, we have

$$\mathbf{R}a_! j^{-1}(F) \in \mathrm{D}^{[0,c_M]}(\mathbb{C}) \tag{A.5}$$

by the definition of c_M and because $j^{-1}(F) \in \mathrm{Mod}(\mathbb{C}_M)$. Combining (A.3), (A.4), and (A.5) we have (A.1), which implies

$$c_{M^*} \leq c_M. \tag{A.6}$$

Moreover, because c_M is the c-soft dimension of M, there exists $F \in \mathrm{Mod}(\mathbb{C}_M)$ satisfying

$$H^{c_M}(\mathbf{R}a_!(F)) \neq 0,$$

where H^{c_M} is the c_Mth cohomology functor. Then $j_! F = \mathbf{R}j_! F \in \mathrm{Mod}(\mathbb{C}_{M^*})$ satisfies

$$H^{c_M}(\mathbf{R}a'_!(\mathbf{R}j_! F)) = H^{c_M}(\mathbf{R}a_! F) \neq 0$$

by $a' \circ j = a\colon M \to \{\mathrm{pt}\}$, which implies

$$c_{M^*} \geq c_M. \tag{A.7}$$

By (A.6) and (A.7), we have the lemma. ■

B. Proof of Lemma 6.2

In this section we prove Lemma 6.2. Therefore, we do not use the notation $\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}$ in this section. We need four lemmas.

Lemma B.1. *Let X be a smooth algebraic variety, and $\widehat{X}$ a smooth complete algebraic variety containing X as an open subset such that $\widehat{X}\backslash X$ is a normal crossing divisor. Then, under Notation 2.20, if the support* $\mathrm{supp}(M)$ *of $M \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\widehat{X}})$ is contained in $X^{\mathsf{b}}_{\mathrm{cp}} = \widehat{X}^{\mathrm{an}}\backslash X^{\mathrm{an}}$, we have*

$$\mathrm{E}j^{-1}_{X_{\mathrm{cp}}} \mathrm{DR}^{\mathrm{E}}_{\widehat{X}}(M^{\mathrm{an}}) \simeq 0, \tag{B.1}$$

where $\mathrm{DR}^{\mathrm{E}}_{\widehat{X}}$ is the enhanced de Rham functor on $\widehat{X}^{\mathrm{an}}$ [5, Def. 9.1.1].

Proof. Let $i\colon X_{\mathrm{cp}}^{\mathrm{b}} \hookrightarrow \widehat{X}^{\mathrm{an}}$ be the closed embedding. Then the definition [5, Def. 9.1.1] of $\mathrm{DR}_{\widehat{X}}^{\mathrm{E}}$ and the assumption $\mathrm{supp}(M) \subset X_{\mathrm{cp}}^{\mathrm{b}}$ imply that we have

$$\mathrm{DR}_{\widehat{X}}^{\mathrm{E}}(M^{\mathrm{an}}) \in \mathsf{Q}(\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}^{\mathrm{b}}\times\mathbb{R}_\infty})) \subset \mathsf{Q}(\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widehat{X}^{\mathrm{an}}\times\mathbb{R}_\infty})) = \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widehat{X}^{\mathrm{an}}}),$$

where $\mathsf{Q}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widehat{X}^{\mathrm{an}}\times\mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widehat{X}^{\mathrm{an}}})$ is the quotient functor and where we regard $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}^{\mathrm{b}}\times\mathbb{R}_\infty})$ as a full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widehat{X}^{\mathrm{an}}\times\mathbb{R}_\infty})$ by the fully faithful functor $\mathbf{R}\tilde{i}_{!!}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}^{\mathrm{b}}\times\mathbb{R}_\infty}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widehat{X}^{\mathrm{an}}\times\mathbb{R}_\infty})$ (under the notation (2.2)). Thus, we have (B.1) because

$$\mathrm{E}j_{X_{\mathrm{cp}}}^{-1}\colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widehat{X}^{\mathrm{an}}}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$$

is induced by

$$\tilde{j}_{X_{\mathrm{cp}}}^{-1}\colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widehat{X}^{\mathrm{an}}\times\mathbb{R}_\infty}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}\times\mathbb{R}_\infty}),$$

which sends objects in $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}^{\mathrm{b}}\times\mathbb{R}_\infty}) \subset \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widehat{X}^{\mathrm{an}}\times\mathbb{R}_\infty})$ to zero. ■

Lemma B.2. *Let X be a smooth algebraic variety, and $\widehat{X}$ a smooth complete algebraic variety containing X as an open subset such that $\widehat{X}\setminus X$ is a normal crossing divisor. Then, under Notation 2.20, if there exists a morphism $M_1 \to M_2$ in $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\widehat{X}})$ whose restriction to X is an isomorphism, we have*

$$\mathrm{E}j_{X_{\mathrm{cp}}}^{-1}\,\mathrm{DR}_{\widehat{X}}^{\mathrm{E}}(M_1^{\mathrm{an}}) \simeq \mathrm{E}j_{X_{\mathrm{cp}}}^{-1}\,\mathrm{DR}_{\widehat{X}}^{\mathrm{E}}(M_2^{\mathrm{an}}).$$

Proof. By the assumption, there exists a morphism $M_1 \to M_2$ whose restriction to X is an isomorphism. Thus, there exists a distinguished triangle

$$M_1 \to M_2 \to N \xrightarrow{+1} \tag{B.2}$$

for some $N \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\widehat{X}})$ whose support is contained in $X_{\mathrm{cp}}^{\mathrm{b}} = \widehat{X}^{\mathrm{an}}\setminus X^{\mathrm{an}}$. By applying the functor $\mathrm{E}j_{X_{\mathrm{cp}}}^{-1}\,\mathrm{DR}_{\widehat{X}}^{\mathrm{E}}((\cdot)^{\mathrm{an}})$ to (B.2), we have

$$\mathrm{E}j_{X_{\mathrm{cp}}}^{-1}\,\mathrm{DR}_{\widehat{X}}^{\mathrm{E}}(M_1^{\mathrm{an}}) \to \mathrm{E}j_{X_{\mathrm{cp}}}^{-1}\,\mathrm{DR}_{\widehat{X}}^{\mathrm{E}}(M_2^{\mathrm{an}}) \to \mathrm{E}j_{X_{\mathrm{cp}}}^{-1}\,\mathrm{DR}_{\widehat{X}}^{\mathrm{E}}(N^{\mathrm{an}}) \xrightarrow{+1},$$

whose third term is zero by Lemma B.1 because $\mathrm{supp}(N) \subset X_{\mathrm{cp}}^{\mathrm{b}}$. This completes the proof. ■

Lemma B.3. *Let X be a smooth algebraic variety and $j\colon U \to X$ an open embedding. Then we have an adjunction $\mathbf{D}j^* \dashv \mathbf{D}j_*\colon \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X) \leftrightarrows \mathrm{D}^{\mathrm{b}}(\mathcal{D}_U)$.*

Proof. Because j is an open embedding, we have isomorphisms of functors

$$\mathbf{D}j_X^* \simeq j_X^{-1}, \quad \mathbf{D}j_{X*} \simeq \mathbf{R}j_{X*}$$

by [11, Ex. 1.5.12] and [11, Ex. 1.5.22], respectively. Thus, by the adjunction $j_X^{-1} \dashv \mathbf{R}j_{X*}$ [19, Prop. 2.6.4 (ii)], we have the lemma. ∎

LEMMA B.4. *Let $f\colon X \to Y$ be a morphism of smooth algebraic varieties, and $\widehat{X}$ (resp. $\widehat{Y}$) a smooth complete algebraic variety containing X (resp. Y) as an open subset such that $\widehat{X}\backslash X$ (resp. $\widehat{Y}\backslash Y$) is a normal crossing divisor. Then there exist another smooth complete algebraic variety $\widehat{X}'$ satisfying the same conditions as $\widehat{X}$ (namely, $\widehat{X}'$ contains X as an open subset and $\widehat{X}'\backslash X$ is a normal crossing divisor), and a morphism $\hat{f}\colon \widehat{X}' \to \widehat{Y}$ (resp. $g\colon \widehat{X}' \to \widehat{X}$) whose restriction to X coincides with $f\colon X \to Y$ (resp. $\mathrm{id}_X\colon X \to X$).*

PROOF. Let $j_Y\colon Y \to \widehat{Y}$ be the inclusion, and $\Gamma_{j_Y\circ f} \subset X \times Y$ the graph of $j_Y \circ f$. Take the closure $\overline{\Gamma_{j_Y\circ f}}$ of $\Gamma_{j_Y\circ f}$ in $\widehat{X} \times \widehat{Y}$ and write $\mathrm{pr}_2\colon \overline{\Gamma_{j_Y\circ f}} \to \widehat{Y}$ for the second projection. Let $\pi\colon \widehat{X}' \to \overline{\Gamma_{j_Y\circ f}}$ be a resolution of singularities. We summarize the situation in the following diagram, where "RS" means that $\pi\colon \widehat{X}' \to \overline{\Gamma_{j_Y\circ f}}$ is a resolution of singularities:

$$\begin{tikzcd}
 & \pi^{-1}(\Gamma_{j_Y\circ f}) \arrow[r, "\mathrm{open}"] \arrow[d, "\pi"', "\wr"] & \widehat{X}' \arrow[d, "\pi"', "\mathrm{RS}"] & \\
X \arrow[r, "\sim"] \arrow[d, "f"] & \Gamma_{j_Y\circ f} \arrow[r, "\mathrm{closure}"] \arrow[dl, "\mathrm{pr}_2"] & \overline{\Gamma_{j_Y\circ f}} \arrow[r] \arrow[d, "\mathrm{pr}_2"] \arrow[dr, "\mathrm{pr}_1"] & \widehat{X} \times \widehat{Y} \\
Y \arrow[rr, "j_Y"] & & \widehat{Y} & \widehat{X}.
\end{tikzcd}$$

Then it is sufficient to set $\hat{f} := \mathrm{pr}_2 \circ \pi\colon \widehat{X}' \to \widehat{Y}$ and $g := \mathrm{pr}_1 \circ \pi\colon \widehat{X}' \to \widehat{X}$. ∎

Then we prove Lemma 6.2. Namely, we prove the following.

LEMMA B.5. *Let X be a smooth algebraic variety, $\widehat{X}_1$ and $\widehat{X}_2$ two smooth complete algebraic varieties containing X as an open subset such that $\widehat{X}_1\backslash X$ and $\widehat{X}_2\backslash X$ are normal crossing divisors. Set $X_{\mathrm{cp}}^k := (X^{\mathrm{an}}, \widehat{X}_k^{\mathrm{an}}) \in \mathsf{Bod}$ and write $j_{X_{\mathrm{cp}}^k}\colon X_{\mathrm{cp}}^k \to \widehat{X}_k^{\mathrm{an}} \in \mathsf{Mor}(\mathsf{Bod})$ for the morphism induced by the analytification $j_{X_k}^{\mathrm{an}}\colon X_k^{\mathrm{an}} \to \widehat{X}_k^{\mathrm{an}}$ of the inclusion $j_{X_k}\colon X_k \to \widehat{X}_k$ for $k = 1, 2$. Then, for any $M \in \mathrm{D}^{\mathrm{b}}(\mathscr{D}_X)$, we have*

$$\mathrm{E}j_{X_{\mathrm{cp}}^1}^{-1}\,\mathrm{DR}_{\widehat{X}_1}^{\mathrm{E}}\big((\mathbf{D}j_{X_1*}(M))^{\mathrm{an}}\big) \simeq \mathrm{E}f^{!}\mathrm{E}j_{X_{\mathrm{cp}}^2}^{-1}\,\mathrm{DR}_{\widehat{X}_2}^{\mathrm{E}}((\mathbf{D}j_{X_2*}(M))^{\mathrm{an}}),$$

where $f\colon X_{\mathrm{cp}}^1 \xrightarrow{\sim} X_{\mathrm{cp}}^2 \in \mathsf{Mor}(\mathsf{Bod})$ is the unique isomorphism of Lemma 2.8.

Proof. (i) First we will prove the lemma in the case that there exists a morphism $\hat{f}\colon \widehat{X}_1 \to \widehat{X}_2$ which is an extension of $\underline{f} = \mathrm{id}_X\colon X \to X$:

$$\text{(B.3)}\qquad \begin{tikzcd} X \arrow[r, "j_{X_1}"] \arrow[d, "\underline{f}=\mathrm{id}_X"', "\wr"] & \widehat{X}_1 \arrow[d, "\hat{f}"] \\ X \arrow[r, "j_{X_2}"'] & \widehat{X}_2, \end{tikzcd} \qquad \begin{tikzcd} X^1_{\mathrm{cp}} \arrow[r, "j_{X^1_{\mathrm{cp}}}"] \arrow[d, "f"', "\wr"] & \widehat{X}^{\mathrm{an}}_1 \arrow[d, "\hat{f}^{\mathrm{an}}"] \\ X^2_{\mathrm{cp}} \arrow[r, "j_{X^2_{\mathrm{cp}}}"'] & \widehat{X}^{\mathrm{an}}_2. \end{tikzcd}$$

By the commutative diagram on the left-hand side in (B.3), we have

$$\mathbf{D}j^*_{X_1}\mathbf{D}\hat{f}^*\mathbf{D}j_{X_2*}(M) \simeq \mathbf{D}j^*_{X_2}\mathbf{D}j_{X_2*}(M) \simeq M.$$

Moreover, we have the adjunction $\mathbf{D}j^*_{X_1} \dashv \mathbf{D}j_{X_1*}$ by Lemma B.3 because j_{X_1} is the open embedding. Thus, there exists a morphism

$$\text{(B.4)}\qquad \mathbf{D}\hat{f}^*\mathbf{D}j_{X_2*}(M) \to \mathbf{D}j_{X_1*}(M).$$

Because $\mathbf{D}j^*_{X_1}$ is isomorphic to the restriction functor [11, Ex. 1.5.12], the restriction to X of the morphism (B.4) is given by

$$\mathbf{D}j^*_{X_1}\mathbf{D}\hat{f}^*\mathbf{D}j_{X_2*}(M) \to \mathbf{D}j^*_{X_1}\mathbf{D}j_{X_1*}(M).$$

This is an isomorphism because

$$\mathbf{D}j^*_{X_1}\mathbf{D}\hat{f}^*\mathbf{D}j_{X_2*}(M) \simeq \mathbf{D}j^*_{X_2}\mathbf{D}j_{X_2*}(M) \simeq M, \quad \mathbf{D}j^*_{X_1}\mathbf{D}j_{X_1*}(M) \simeq M$$

by the commutative diagram on the left-hand side in (B.3). Thus, applying Lemma B.2 to the morphism (B.4), we have an isomorphism

$$\text{(B.5)}\qquad \mathrm{E}j^{-1}_{X^1_{\mathrm{cp}}}\,\mathrm{DR}^{\mathrm{E}}_{\widehat{X}_1}\big((\mathbf{D}\hat{f}^*\mathbf{D}j_{X_2*}(M))^{\mathrm{an}}\big) \simeq \mathrm{E}j^{-1}_{X^1_{\mathrm{cp}}}\,\mathrm{DR}^{\mathrm{E}}_{\widehat{X}_1}\big((\mathbf{D}j_{X_1*}(M))^{\mathrm{an}}\big).$$

Moreover, the left-hand side of (B.5) is isomorphic to

$$\text{(B.6)}\quad \mathrm{E}j^{-1}_{X^1_{\mathrm{cp}}}\,\mathrm{DR}^{\mathrm{E}}_{\widehat{X}_1}\big(\mathbf{D}(\hat{f}^{\mathrm{an}})^*(\mathbf{D}j_{X_2*}(M))^{\mathrm{an}}\big) \simeq \mathrm{E}j^{-1}_{X^1_{\mathrm{cp}}}\mathrm{E}(\hat{f}^{\mathrm{an}})^!\,\mathrm{DR}^{\mathrm{E}}_{\widehat{X}_2}\big((\mathbf{D}j_{X_2*}(M))^{\mathrm{an}}\big)$$

by [11, Prop. 4.7.2 (i)] and [5, Thm. 9.1.2 (ii)]. The right-hand side of (B.6) is isomorphic to

$$\text{(B.7)}\quad \mathrm{E}j^!_{X^1_{\mathrm{cp}}}\mathrm{E}(\hat{f}^{\mathrm{an}})^!\,\mathrm{DR}^{\mathrm{E}}_{\widehat{X}_2}\big((\mathbf{D}j_{X_2*}(M))^{\mathrm{an}}\big) \simeq \mathrm{E}f^!\mathrm{E}j^!_{X^2_{\mathrm{cp}}}\,\mathrm{DR}^{\mathrm{E}}_{\widehat{X}_2}\big((\mathbf{D}j_{X_2*}(M))^{\mathrm{an}}\big)$$

by the isomorphism $j^{-1}_{X^1_{\mathrm{cp}}} \simeq j^!_{X^1_{\mathrm{cp}}}$ of functors [5, 3.3.7 (i)] and the commutative diagram on the right-hand side in (B.3).

Thus, (B.5), (B.6), and (B.7) imply the lemma in the case that there exists a morphism $\hat{f}\colon \widehat{X}_1 \to \widehat{X}_2$ which is an extension of the identity morphism $\mathrm{id}_X\colon X \to X$.

(ii) Then we will prove the lemma in the general case. By Lemma B.4, there exist a smooth complete algebraic variety $\widehat{X}_0$ satisfying the same conditions as $\widehat{X}_1$, and a morphism $\hat{f}_2 \colon \widehat{X}_0 \to \widehat{X}_2$ (resp. $\hat{f}_1 \colon \widehat{X}_0 \to \widehat{X}_1$) whose restriction to X coincides with id_X (resp. id_X):

$$\begin{tikzcd}
X \arrow[r, "j_{X_0}"] \arrow[d, "\underline{f_k}=\mathrm{id}_X"', "\wr"] & \widehat{X}_0 \arrow[d, "\hat{f}_k"] \\
X \arrow[r, "j_{X_k}"'] & \widehat{X}_k,
\end{tikzcd}
\quad
\begin{tikzcd}
X^0_{\mathrm{cp}} \arrow[r, "j_{X^0_{\mathrm{cp}}}"] \arrow[d, "f_k"', "\wr"] & \widehat{X}^{\mathrm{an}}_0 \arrow[d, "\hat{f}^{\mathrm{an}}_k"] \\
X^k_{\mathrm{cp}} \arrow[r, "j_{X^k_{\mathrm{cp}}}"'] & \widehat{X}^{\mathrm{an}}_k,
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[r, "\underline{f_1}^{-1}=\mathrm{id}_X", "\sim"'] \arrow[d, "j_{X_1}"'] & X \arrow[r, "\underline{f_2}=\mathrm{id}_X", "\sim"'] \arrow[d, "j_{X_0}"] & X \arrow[d, "j_{X_2}"] \\
X^1_{\mathrm{cp}} \arrow[r, "f_1^{-1}", "\sim"'] \arrow[rr, bend right, "f"'] & X^0_{\mathrm{cp}} \arrow[r, "f_2", "\sim"'] & X^2_{\mathrm{cp}}.
\end{tikzcd}$$

Thus, by the first part of the proof of this lemma, we have two isomorphisms

$$\mathrm{E}j^{-1}_{X^0_{\mathrm{cp}}}\, \mathrm{DR}^{\mathrm{E}}_{\widehat{X}_0}\big((\mathbf{D}j_{X_0*}(M))^{\mathrm{an}}\big) \simeq \mathrm{E}f_1^{!}\mathrm{E}j^{-1}_{X^1_{\mathrm{cp}}}\, \mathrm{DR}^{\mathrm{E}}_{\widehat{X}_1}\big((\mathbf{D}j_{X_1*}(M))^{\mathrm{an}}\big), \tag{B.8}$$

$$\mathrm{E}j^{-1}_{X^0_{\mathrm{cp}}}\, \mathrm{DR}^{\mathrm{E}}_{\widehat{X}_0}\big((\mathbf{D}j_{X_0*}(M))^{\mathrm{an}}\big) \simeq \mathrm{E}f_2^{!}\mathrm{E}j^{-1}_{X^2_{\mathrm{cp}}}\, \mathrm{DR}^{\mathrm{E}}_{\widehat{X}_2}\big((\mathbf{D}j_{X_2*}(M))^{\mathrm{an}}\big), \tag{B.9}$$

where $f_k \colon X^0_{\mathrm{cp}} \to X^k_{\mathrm{cp}}$ is the unique morphism of Lemma 2.8 satisfying $\underline{f_k} = \mathrm{id}_X$ for $k = 1, 2$. Because $f_1 \colon X^0_{\mathrm{cp}} \xrightarrow{\sim} X^1_{\mathrm{cp}}$ has the inverse $f_1^{-1} \colon X^1_{\mathrm{cp}} \xrightarrow{\sim} X^0_{\mathrm{cp}}$, we have

$$\mathrm{E}j^{-1}_{X^1_{\mathrm{cp}}}\, \mathrm{DR}^{\mathrm{E}}_{\widehat{X}_1}\big((\mathbf{D}j_{X_1*}(M))^{\mathrm{an}}\big) \simeq \mathrm{E}(f_1^{-1})^{!}\mathrm{E}f_1^{!}\mathrm{E}j^{-1}_{X^1_{\mathrm{cp}}}\, \mathrm{DR}^{\mathrm{E}}_{\widehat{X}_1}\big((\mathbf{D}j_{X_1*}(M))^{\mathrm{an}}\big).$$

By (B.8) and (B.9), this is isomorphic to

$$\mathrm{E}(f_1^{-1})^{!}\mathrm{E}f_2^{!}\mathrm{E}j^{-1}_{X^2_{\mathrm{cp}}}\, \mathrm{DR}^{\mathrm{E}}_{\widehat{X}_2}\big((\mathbf{D}j_{X_2*}(M))^{\mathrm{an}}\big).$$

Note that $f \colon X^1_{\mathrm{cp}} \to X^2_{\mathrm{cp}}$ has a factorization $f = f_2 \circ f_1^{-1} \in \mathsf{Mor}(\mathsf{Bod})$. Thus, this is isomorphic to

$$\mathrm{E}f^{!}\mathrm{E}j^{-1}_{X^2_{\mathrm{cp}}}\, \mathrm{DR}^{\mathrm{E}}_{\widehat{X}_2}\big((\mathbf{D}j_{X_2*}(M))^{\mathrm{an}}\big),$$

which completes the proof of the general case. ∎

C. Proof of Lemma 6.3

In this section we prove Lemma 6.3.

Proof. Take a smooth complete algebraic variety $\widehat{X}$ (resp. $\widehat{Y}$) containing X (resp. Y) as an open subset such that $\widehat{X} \setminus X$ (resp. $\widehat{Y} \setminus Y$) is a normal crossing divisor (see Notation 2.20). Then Lemma B.4 implies that there exist another smooth complete algebraic variety $\widehat{X}'$ satisfying the same conditions as $\widehat{X}$, and a morphism $\hat{f} \colon \widehat{X}' \to \widehat{Y}$

whose restriction to X coincides with $f\colon X \to Y$. Thus, by changing the notation, we can assume that there exists a morphism $\hat{f}\colon \widehat{X} \to \widehat{Y}$ whose restriction to X coincides with $f\colon X \to Y$ from the beginning. Then we use those $\widehat{X}$ and $\widehat{Y}$ for the definition of $\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}$ (Definition 6.1) in this proof (where we use the independence of $\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}$ from the choice of $\widehat{X}$ (Lemma 6.2)):

$$
\text{(C.1)} \qquad
\begin{tikzcd}
X \arrow[r, "j_X"] \arrow[d, "f"'] & \widehat{X} \arrow[d, "\hat{f}"] \\
Y \arrow[r, "j_Y"'] & \widehat{Y},
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{\mathrm{cp}} \arrow[r, "j_{X_{\mathrm{cp}}}"] \arrow[d, "f^{\mathrm{an}}_{\mathrm{cp}}"'] & \widehat{X}^{\mathrm{an}} \arrow[d, "\hat{f}^{\mathrm{an}}"] \\
Y_{\mathrm{cp}} \arrow[r, "j_{Y_{\mathrm{cp}}}"'] & \widehat{Y}^{\mathrm{an}}.
\end{tikzcd}
$$

Thus, the right-hand side of (6.1) is isomorphic to the leftmost term of the following chain of isomorphisms:

$$
\begin{aligned}
\mathrm{E}(\hat{f}^{\mathrm{an}}_{\mathrm{cp}})^{!}\mathrm{E}j^{-1}_{Y_{\mathrm{cp}}}\,\mathrm{DR}^{\mathrm{E}}_{\widehat{Y}}\big((\mathbf{D}j_{Y*}(M))^{\mathrm{an}}\big)[d_Y] &\simeq \mathrm{E}(\hat{f}^{\mathrm{an}}_{\mathrm{cp}})^{!}\mathrm{E}j^{!}_{Y_{\mathrm{cp}}}\,\mathrm{DR}^{\mathrm{E}}_{\widehat{Y}}\big((\mathbf{D}j_{Y*}(M))^{\mathrm{an}}\big)[d_Y] \\
&\simeq \mathrm{E}j^{!}_{X_{\mathrm{cp}}}\mathrm{E}(\hat{f}^{\mathrm{an}})^{!}\,\mathrm{DR}^{\mathrm{E}}_{\widehat{Y}}\big((\mathbf{D}j_{Y*}(M))^{\mathrm{an}}\big)[d_Y],
\end{aligned}
$$

where the first and second isomorphisms follow from the isomorphism $j^{-1}_{Y_{\mathrm{cp}}} \simeq j^{!}_{Y_{\mathrm{cp}}}$ of functors [5, 3.3.7 (i)] and the diagram (C.1), respectively. Then [5, Thm. 9.1.2 (ii)] implies that the rightmost term in the last equation is isomorphic to the left-hand side of the following equation:

$$
\mathrm{E}j^{!}_{X_{\mathrm{cp}}}\,\mathrm{DR}^{\mathrm{E}}_{\widehat{X}}\big((\mathbf{D}\hat{f}^{\mathrm{an}})^{*}(\mathbf{D}j_{Y*}(M))^{\mathrm{an}}\big)[d_X] \simeq \mathrm{E}j^{!}_{X_{\mathrm{cp}}}\,\mathrm{DR}^{\mathrm{E}}_{\widehat{X}}\big((\mathbf{D}\hat{f}^{*}\mathbf{D}j_{Y*}(M))^{\mathrm{an}}\big)[d_X],
$$

where we use [11, Prop. 4.7.2 (i)]. Moreover, the right-hand side of the last equation is isomorphic to

$$
\text{(C.2)} \qquad \mathrm{E}j^{!}_{X_{\mathrm{cp}}}\,\mathrm{DR}^{\mathrm{E}}_{\widehat{X}}\big((\mathbf{D}j_{X*}\mathbf{D}j^{*}_{X}\mathbf{D}\hat{f}^{*}\mathbf{D}j_{Y*}(M))^{\mathrm{an}}\big)[d_X]
$$

by Lemma B.2 because the adjunction $\mathbf{D}j^{*}_{X} \dashv \mathbf{D}j_{X*}$ (Lemma B.3) implies that there exists a morphism

$$
\mathbf{D}\hat{f}^{*}\mathbf{D}j_{Y*}(M) \to \mathbf{D}j_{X*}\mathbf{D}j^{*}_{X}\mathbf{D}\hat{f}^{*}\mathbf{D}j_{Y*}(M)
$$

whose restriction to X is an isomorphism. Moreover, we have

$$
\mathbf{D}j_{X*}\mathbf{D}j^{*}_{X}\mathbf{D}\hat{f}^{*}\mathbf{D}j_{Y*}(M) \simeq \mathbf{D}j_{X*}\mathbf{D}f^{*}\mathbf{D}j^{*}_{Y}\mathbf{D}j_{Y*}(M) \simeq \mathbf{D}j_{X*}\mathbf{D}f^{*}(M)
$$

by the diagram on the left-hand side of (C.1). Therefore, (C.2) is isomorphic to the left-hand side of (6.1) by the definition of $\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}$ (Definition 6.1) and the isomorphism $j^{-1}_{X_{\mathrm{cp}}} \simeq j^{!}_{X_{\mathrm{cp}}}$ of functors [5, 3.3.7 (i)], which completes the proof. ■

D. Proof of Lemma 6.5

In this section we prove Lemma 6.5. We need some preparation.

Definition D.1 ([14, p. 61]). Let X be a smooth algebraic variety, and $\widehat{X}$ a smooth complete algebraic variety containing X as an open subset such that $\widehat{X}\backslash X$ is a normal crossing divisor. Then, under Notation 2.20, we define

$$\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})$$

by

$$\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(M) := \mathrm{E}j^{-1}_{X_{\mathrm{cp}}}\,\mathrm{Sol}^{\mathrm{E}}_{\widehat{X}}\big((\mathbf{D}j_{X*}(M))^{\mathrm{an}}\big) \tag{D.1}$$

for $M\in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, where $\mathrm{Sol}^{\mathrm{E}}_{\widehat{X}}$ is the enhanced solution functor on $\widehat{X}^{\mathrm{an}}$ [5, Def. 9.1.1] and $(\mathbf{D}j_{X*}(M))^{\mathrm{an}}$ is the analytification of $\mathbf{D}j_{X*}(M)$. This definition does not depend on the choice of $\widehat{X}$ (which follows from[7] Lemmas 6.2 and D.4).

Remark D.2. The analogous statements to Lemmas B.1 and B.2 also hold for $\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}$ because we can apply the same proof mutatis mutandis to this case.

Lemma D.3 (Cf. [14, p. 61][8]). *Let X be a smooth algebraic variety, and $\widehat{X}$ a smooth complete algebraic variety containing X as an open subset such that $\widehat{X}\backslash X$ is a normal crossing divisor. Then, under Notation* 2.20, *we have*

$$\mathrm{E}j^{-1}_{X_{\mathrm{cp}}}\,\mathrm{Sol}^{\mathrm{E}}_{\widehat{X}}\big((\mathbf{D}j_{X!}(M))^{\mathrm{an}}\big) \simeq \mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(M) \tag{D.2}$$

for any $M\in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, where we set $\mathbf{D}j_{X!} := \mathbb{D}_{\widehat{X}}\mathbf{D}j_{X*}\mathbb{D}_X$ and $\mathbb{D}_X$ is the duality functor [11, Def. 2.6.1].

Proof. First, we will construct a morphism

$$\mathbf{D}j_{X!}M \to \mathbf{D}j_{X*}M. \tag{D.3}$$

By the definition $\mathbf{D}j_{X!} := \mathbb{D}_{\widehat{X}}\mathbf{D}j_{X*}\mathbb{D}_X$ and the adjunction $\mathbf{D}j^*_X \dashv \mathbf{D}j_{X*}$ (Lemma B.3), it is sufficient to construct a morphism

$$\mathbf{D}j^*_X\mathbb{D}_{\widehat{X}}\mathbf{D}j_{X*}\mathbb{D}_X M \to M \tag{D.4}$$

(7) The proof of Lemma D.4 does not use this independence.
(8) In [14, p. 61], this result is proved under the assumption $M\in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$.

for (D.3). By [11, Thm. 2.7.1 (ii) and Prop. 2.6.5 (ii)], we have

$$\mathbf{D}j_X^*\mathbb{D}_{\widehat{X}}\mathbf{D}j_{X*}\mathbb{D}_X M \simeq \mathbb{D}_X\mathbf{D}j_X^*\mathbf{D}j_{X*}\mathbb{D}_X M \simeq \mathbb{D}_X\mathbb{D}_X M \simeq M, \tag{D.5}$$

which implies the existence of (D.4). Thus, we have a morphism (D.3).

Second, we will show that the restriction of (D.3) to X is an isomorphism. Because $\mathbf{D}j_X^*$ is isomorphic to the restriction functor [11, Ex. 1.5.12], it is sufficient to show that

$$\mathbf{D}j_X^*\mathbf{D}j_{X!}M \to \mathbf{D}j_X^*\mathbf{D}j_{X*}M \simeq M$$

is an isomorphism, which follows from the definition $\mathbf{D}j_{X!} := \mathbb{D}_{\widehat{X}}\mathbf{D}j_{X*}\mathbb{D}_X$ and (D.5). Thus, the restriction of (D.3) to X is an isomorphism, which completes the proof of the lemma by Lemma B.2 (and Remark D.2) applying to (D.3) and the definition (D.1) of $\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}$. ∎

Lemma D.4. *Let X be a smooth algebraic variety, and $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Then we have*

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(M) \simeq \mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(\mathbb{D}_X M)[d_X]. \tag{D.6}$$

Proof. We have an isomorphism

$$M \simeq \mathbb{D}_X\mathbb{D}_X M \tag{D.7}$$

by [11, Prop. 2.6.5 (ii)]. Take a smooth complete algebraic variety $\widehat{X}$ containing X as an open subset such that $\widehat{X}\backslash X$ is a normal crossing divisor. Applying the functor $\mathbb{D}_{\widehat{X}}\mathbf{D}j_{X*}$ (under Notation 2.20) to (D.7), we have

$$\mathbb{D}_{\widehat{X}}\mathbf{D}j_{X*}M \simeq \mathbf{D}j_{X!}\mathbb{D}_X M, \tag{D.8}$$

by $\mathbf{D}j_{X!} := \mathbb{D}_{\widehat{X}}\mathbf{D}j_{X*}\mathbb{D}_X$. Moreover, applying the functor $\mathrm{E}j_{X_{\mathrm{cp}}}^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widehat{X}}((\cdot)^{\mathrm{an}})$ to (D.8), we have

$$\begin{aligned} \mathrm{E}j_{X_{\mathrm{cp}}}^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widehat{X}}((\mathbb{D}_{\widehat{X}}\mathbf{D}j_{X*}M)^{\mathrm{an}}) &\simeq \mathrm{E}j_{X_{\mathrm{cp}}}^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widehat{X}}((\mathbf{D}j_{X!}\mathbb{D}_X M)^{\mathrm{an}}) \\ &\simeq \mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(\mathbb{D}_X M) \end{aligned} \tag{D.9}$$

by (D.2). On the other hand, we have a morphism

$$(\mathbb{D}_{\widehat{X}}\mathbf{D}j_{X*}M)^{\mathrm{an}} \to \mathbb{D}_{\widehat{X}^{\mathrm{an}}}(\mathbf{D}j_{X*}M)^{\mathrm{an}}, \tag{D.10}$$

whose restriction to X is an isomorphism by [11, Prop. 4.7.1] and the assumption $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$. Thus, applying Lemma B.2 (and Remark D.2) for (D.10), we have

$$\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(\mathbb{D}_X M) \simeq \mathrm{E}j_{X_{\mathrm{cp}}}^{-1}\mathrm{Sol}^{\mathrm{E}}_{\widehat{X}}(\mathbb{D}_{\widehat{X}^{\mathrm{an}}}(\mathbf{D}j_{X*}M)^{\mathrm{an}})$$

by (D.9). This completes the proof by Lemma D.5 given below. ∎

Lemma D.5. *Let X be a smooth algebraic variety, and $\widehat{X}$ a smooth complete algebraic variety containing X as an open subset such that $\widehat{X}\backslash X$ is a normal crossing divisor. Then, under Notation 2.20, we have*

$$\mathrm{E} j_{X_{\mathrm{cp}}}^{-1} \mathrm{Sol}_{\widehat{X}}^{\mathrm{E}}(\mathbb{D}_{\widehat{X}^{\mathrm{an}}}(\mathbf{D} j_{X*}M)^{\mathrm{an}}) \simeq \mathrm{DR}_{\mathrm{alg}}^{\mathrm{E}}(M)[-d_X]$$

for any $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$.

Proof. By induction of the cohomological length of $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$, it is sufficient to prove the lemma in the case that M is a coherent $\mathcal{D}_X$-module. In this case, there exists a $\mathcal{D}_{\widehat{X}}$-submodule

$$N \hookrightarrow H^0(\mathbf{D} j_{X*}M)$$

such that N is coherent and the restriction $N|_X$ of N to X is isomorphic to $M \simeq \mathbf{D} j_{X*}M|_X$ by [11, Cor. 1.4.17 (ii)] (which is applicable because the assumption that M is a coherent $\mathcal{D}_X$-module implies $H^0(\mathbf{D} j_{X*}M)$ is a quasi-coherent $\mathcal{D}_{\widehat{X}}$-module [11, Prop. 1.5.29]). Thus, there exists a distinguished triangle

$$N \to \mathbf{D} j_{X*}M \to N' \xrightarrow{+1} \tag{D.11}$$

for some $N' \in \mathrm{D}^{\mathrm{b}}(\mathcal{D}_{\widehat{X}})$ whose support is contained in $\widehat{X}\backslash X$. Note that we have $N^{\mathrm{an}} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{\widehat{X}^{\mathrm{an}}})$ (cf. [11, p. 120]). Applying the functors $\mathrm{E} j_{X_{\mathrm{cp}}}^{-1} \mathrm{DR}_{\widehat{X}}^{\mathrm{E}}((\cdot)^{\mathrm{an}})[-d_X]$ and $\mathrm{E} j_{X_{\mathrm{cp}}}^{-1} \mathrm{Sol}_{\widehat{X}}^{\mathrm{E}}(\mathbb{D}_{\widehat{X}^{\mathrm{an}}}(\cdot)^{\mathrm{an}})$ to (D.11), we have a morphism of distinguished triangles

$$\begin{array}{ccccccc}
\mathrm{E} j_{X_{\mathrm{cp}}}^{-1} \mathrm{DR}_{\widehat{X}}^{\mathrm{E}}(N^{\mathrm{an}})[-d_X] & \to & \mathrm{E} j_{X_{\mathrm{cp}}}^{-1} \mathrm{DR}_{\widehat{X}}^{\mathrm{E}}((\mathbf{D} j_{X*}M)^{\mathrm{an}})[-d_X] & \to & \mathrm{E} j_{X_{\mathrm{cp}}}^{-1} \mathrm{DR}_{\widehat{X}}^{\mathrm{E}}(N'^{\mathrm{an}})[-d_X] & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{E} j_{X_{\mathrm{cp}}}^{-1} \mathrm{Sol}_{\widehat{X}}^{\mathrm{E}}(\mathbb{D}_{\widehat{X}^{\mathrm{an}}} N^{\mathrm{an}}) & \longrightarrow & \mathrm{E} j_{X_{\mathrm{cp}}}^{-1} \mathrm{Sol}_{\widehat{X}}^{\mathrm{E}}(\mathbb{D}_{\widehat{X}^{\mathrm{an}}}(\mathbf{D} j_{X*}M)^{\mathrm{an}}) & \longrightarrow & \mathrm{E} j_{X_{\mathrm{cp}}}^{-1} \mathrm{Sol}_{\widehat{X}}^{\mathrm{E}}(\mathbb{D}_{\widehat{X}^{\mathrm{an}}} N'^{\mathrm{an}}) & \xrightarrow{+1} & ,
\end{array}$$

whose leftmost vertical map is an isomorphism by [5, equation after Def. 9.1.1] and $\mathbb{D}^2_{\widehat{X}^{\mathrm{an}}} \simeq \mathrm{id}$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_{\widehat{X}^{\mathrm{an}}})$ [17, (3.9)], and both of the rightmost terms are isomorphic to zero by Lemma B.1 (and Remark D.2) because $\mathrm{supp}(N') \subset \widehat{X}\backslash X$. Thus, the middle vertical map is also an isomorphism, which completes the proof of the lemma by the definition (6.1) of $\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}$. ∎

Proof of Lemma 6.5. By (D.6) and the assumption of the lemma, we have

$$\mathrm{DR}^{\mathrm{E}}_{\mathrm{alg}}(M) \simeq \mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(\mathbb{D}_X M)[d_X] \in \mathrm{E}^{\mathrm{b}}_{\mathrm{sh}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}) \cap \mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}).$$

Thus, by [14, Lem. 3.15[9] and Prop. 3.16], we have

$$\mathsf{sh}_{X_{\mathrm{cp}}}(\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(\mathbb{D}_X M))[d_X] \simeq \mathrm{Sol}_X(\mathbb{D}_X M)[d_X] \in \mathsf{sh}_{X_{\mathrm{cp}}}(\mathrm{E}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}})) \subset \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X),$$

where $\mathsf{sh}_{X_{\mathrm{cp}}}$ is the sheafification functor on X_{cp} ([14, §2.2]) and Sol_X is the (usual) solution functor. Then [19, Thm. 11.3.3] implies $\mathbb{D}_X(M) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$. Therefore, we can apply [13, Lem. 3.10] to

$$\mathrm{Sol}^{\mathrm{E}}_{\mathrm{alg}}(\mathbb{D}_X M)[d_X] \in \mathrm{E}^{\mathrm{b}}_{\mathsf{sh}}(\mathrm{I}\mathbb{C}_{X_{\mathrm{cp}}}),$$

which implies

$$\mathbb{D}_X M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X). \tag{D.12}$$

Thus, applying $\mathbb{D}_X$ to (D.12), we have the lemma because $\mathbb{D}_X$ preserves $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ [11, Thm. 6.1.5 (i)] and $\mathbb{D}_X^2 \simeq \mathrm{id}$ [11, Prop. 2.6.5 (ii)]. ∎

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