Rend. Sem. Mat. Univ. Padova 156 (2026), 59–84
DOI 10.4171/RSMUP/180

The stalk formula for multi-microlocal Hom functors and the multi-microlocal Sato triangle

Ryosuke Sakamoto (*)

Abstract – The concept of "multi-microlocalization" was introduced to extend the usual microlocal sheaf theory to a more general scope. This paper aims to further extend this theory by exploring advanced topics. One is a stalk formula for multi-microlocalized Hom functors and we compute some examples in multi-microlocal settings. Secondly we construct the Sato triangle in the context of multi-microlocal analysis. Ultimately we get the Sato triangle for the multi-microlocalized Hom functors.

Mathematics Subject Classification 2020 – 32C35 (primary); 35A27 (secondary).

Keywords – multi-microlocalizations, sheaves on subanalytic sites.

Introduction

In this paper, we consider extending the usual microlocal sheaf theory, which holds in the more general scope of the multi-microlocal sheaf theory and subanalytic sheaf theory.

Sato, Kawai, and Kashiwara introduced a new method of analysis: algebraic analysis [10]. This theory concerns the microlocal properties of a function. Kashiwara and Schapira used a homological-geometric method based on the derived category of sheaves to reformalize the theory [5,6]. They introduced the notion of the microsupport $\mathrm{SS}(F)$ for a sheaf F on a derived category of sheaves. This is a formalization of the singularity of a function using the theory of sheaves, and it is the collection of directions in which the cohomology of the sheaf F varies. This work provided a powerful

(*) *Indirizzo dell'A.*: Department of Mathematics, Faculty of Science, Hokkaido University, Kita-10 Nishi-8, Kita-ku, Sapporo 060-0810, Japan; sakamoto.ryosuke.h0@elms.hokudai.ac.jp

new framework for studying singularities of solutions to differential equations. The microlocal sheaf theory can also be interpreted as "a construction that enables the treatment of Fourier transforms on sheaves on manifolds". It is well known that Fourier transforms have applications in differential equation theory. By using the Fourier transformation theory of sheaves, we can consider microlocalization μ_M. Microlocalization is the process of considering a sheaf F on X as a sheaf on the conormal bundle to a submanifold M. Based on such foundations, the study of differential equations has often been successful.

On the other hand, important analytical objects such as asymptotically developable functions and tempered distributions do not form classical sheaves, therefore conventional methods cannot be applied. New concepts such as Ind-sheaves and sheaves on subanalytic sites have been introduced and studied to handle these objects [7, 8]. Honda, Yamazaki, and Prelli [3] introduced a more extended framework called "multi-microlocalization". After introducing this multi-microlocalization, they further obtained multi-microlocalization along the universal family of submanifolds of Hom functors. Algebraic analysis is becoming applicable to differential equations with more complex singularities of a wider class of functions. Various types of microlocalization have been defined and applied to asymptotic analysis and boundary value problems [11, 12]. The theory of multi-microlocalization is constructed to unify these theories.

In Section 1, we assume knowledge of the derived category of sheaves and summarize the basic concepts and definitions related to sheaves on subanalytic sites and multi-microlocal analysis as explained in [8, 9] and [3]. Also, we would like to give supplementary facts which follow immediately from the past results but could not be found in the literature. These subsections provide a foundation for the rest of the paper, which builds upon this material to explore more advanced topics and ideas.

In Section 2, we derive a stalk formula for multi-microlocalized Hom functors in general multi-microlocal cases for subanalytic sheaves. A significant challenge in obtaining a stalk formula for subanalytic sheaves is the absence of a microlocal cut-off functor. We circumvent this issue by considering constructible sheaves as the first argument of the Hom functors. In the following section we prepare some lemmas to obtain a more explicit formula for an easy computation in particular in multi-microlocal settings. Finally, we compute some examples in specific multi-microlocal settings. As is well known in the classical theory, a stalk formula is crucial for the microlocal analysis of sheaves (see [6]). We believe that the stalk formula for multi-microlocalized Hom functors proved in this paper will be instrumental in the future.

In Section 3, our objective is to establish Sato's triangle within the framework of multi-microlocal analysis. To achieve this, we first prepare Uchida's triangle in subanalytic sheaf theory. Next we need to multi-microlocalize Uchida's triangle in subanalytic sheaf theory. The next step involves multi-microlocalizing Uchida's triangle, which poses a challenge because proper direct image functors do not generally map conic subanalytic sheaves to conic subanalytic sheaves. We address this difficulty by proving several lemmas, ultimately leading to the formulation of Sato's triangle for the μhom functor in multi-microlocalizations. We conclude this section with a simple example in a multi-microlocal setting to illustrate the theoretical developments. Experts recognize Uchida's triangle and Sato's triangle as fundamental tools for the microlocal analysis of sheaves. They are particularly suited for solving microlocal Cauchy problems, boundary value problems, and various other issues. Our results extend these concepts within the theory of multi-microlocalizations, opening up possibilities for addressing problems in more general settings.

1. Preliminaries

In this section we recall some basic notions. References are made to [7–9] for the theory of sheaves on subanalytic sites, and [1–3] for the theory of multi-microlocalizations.

1.1 – *Theory of subanalytic sheaves*

Here, X will be a real analytic manifold and k a field. We assume all manifolds are analytic, finite-dimensional, and countable at infinity. Denote by $\mathrm{Op}(X_{\mathrm{sa}})$ (resp. $\mathrm{Op}^c(X_{\mathrm{sa}})$) the category of open (resp. relatively compact open) subanalytic subsets of X. One endows $\mathrm{Op}(X_{\mathrm{sa}})$ with the following topology: $\mathrm{Cov}(U) \subset \mathrm{Op}(X_{\mathrm{sa}})$ is a covering of $U \in \mathrm{Op}(X_{\mathrm{sa}})$ if for any compact subset K of X there exists a finite subset S_0 of $\mathrm{Cov}(U)$ such that $K \cap \bigcup_{V \in S_0} V = K \cap U$. We will call X_{sa} the subanalytic site. Let $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$ denote the category of sheaves on X_{sa}.

Let $\mathrm{Mod}_{\mathbb{R}\text{-}c}(k_X)$ be the abelian category of $\mathbb{R}$-constructible sheaves on X, and consider its subcategory $\mathrm{Mod}^c_{\mathbb{R}\text{-}c}(k_X)$ consisting of sheaves whose support is compact. Denote by $\rho\colon X \to X_{\mathrm{sa}}$ the natural morphism of sites. The functor $\rho_*\colon \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_{X_{\mathrm{sa}}})$ is fully faithful and left exact.

Notation 1.1.1. Since the functor ρ_* is fully faithful and exact on $\mathrm{Mod}_{\mathbb{R}\text{-}c}(k_X)$ we identify $\mathrm{Mod}_{\mathbb{R}\text{-}c}(k_X)$ with the image in $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$. When there is no risk of confusion we will write F instead of $\rho_* F$, for $F \in \mathrm{Mod}_{\mathbb{R}\text{-}c}(k_X)$.

Let X, Y be two real analytic manifolds, and let $f\colon X \to Y$ be a real analytic map. We get external operations f_* and f^{-1} which are always defined for sheaves on Grothendieck topologies. Let Z be a locally closed subanalytic set. As in classical sheaf theory we may define Γ_Z and $(\cdot)_Z$. We may define the functor of proper direct image $f_{!!}$ for subanalytic sheaves. If f is proper on $\mathrm{supp}(F)$ then $f_* F \simeq f_{!!} F$; in this case $f_{!!}$ commutes with ρ_*.

We denote by $D(k_{X_{\mathrm{sa}}})$ the derived category of $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$ and its full subcategory consisting of bounded (resp. bounded below, resp. bounded above) complexes is denoted by $D^b(k_{X_{\mathrm{sa}}})$ (resp. $D^+(k_{X_{\mathrm{sa}}})$, resp. $D^-(k_{X_{\mathrm{sa}}})$). As usual, we denote by $D^b_{\mathbb{R}\text{-}c}(k_X)$ (resp. $D^b_{\mathbb{R}\text{-}c}(k_{X_{\mathrm{sa}}})$) the full subcategory of $D^b(k_X)$ (resp. $D^b(k_{X_{\mathrm{sa}}})$) consisting of objects with $\mathbb{R}$-constructible cohomology. We would like to state the result which immediately follows from [8, Proposition 2.4.8].

Proposition 1.1.2. *Let $F \in D^+(k_{Y_{\mathrm{sa}}})$, and let $f: X \to Y$ be a closed embedding. Then $Rf_{!!} f^{-1} F \simeq F_X$.*

Remark 1.1.3. Unlike the classical case, this isomorphism does not hold for an open embedding in general.

The functor $Rf_{!!}$ admits a right adjoint, denoted by $f^!$. A similar consequence to [7, Proposition 5.3.8] follows as shown below.

Proposition 1.1.4. *Let $G \in D^b(k_{X_{\mathrm{sa}}})$, $F \in D^+(k_{X_{\mathrm{sa}}})$. There is a natural isomorphism $f^! R\mathcal{H}om(G, F) \simeq R\mathcal{H}om(f^{-1}G, f^! F)$.*

1.2 – *Multi-microlocalization*

Let $\tau_i: E_i \to Z$ ($1 \le i \le \ell$) be vector bundles over Z, and let E_i^* be the dual bundle of E_i. We denote by $\wedge_i$ and $\vee_i$ the Fourier–Sato transformations and the inverse Fourier–Sato transformations on E_i respectively. Moreover, we denote by $\wedge_i^*$ and $\vee_i^*$ the Fourier–Sato and the inverse Fourier–Sato transformations on E_i^* respectively. Set $E := E_1 \times_Z \cdots \times_Z E_\ell$ and $E^* := E_1^* \times_Z \cdots \times_Z E_\ell^*$. Let $\tau: E \to Z$ be the canonical projection. Set $P_i' := \{(\eta, \xi) \in E_i \times_Z {E_i}^*; \langle \eta, \xi \rangle \le 0\}$:

$$P' := P_1' \underset{Z}{\times} \cdots \underset{Z}{\times} P_\ell', \quad P^+ := E \underset{Z}{\times} E^* \setminus P',$$

and denote by $p_1': P' \to E$ and $p_2': P' \to E^*$ the respective canonical projections. Set $P_i := \{(\eta, \xi) \in E_i \times_Z E_i^*; \langle \eta, \xi \rangle \ge 0\}$:

$$P := P_1 \underset{Z}{\times} \cdots \underset{Z}{\times} P_\ell, \quad P^- := E \underset{Z}{\times} E^* \setminus P,$$

and denote $p_1'': P \to E$ and $p_2'': P \to E^*$ the respective canonical projections. Let F and G be multi-conic objects on E and E^*, respectively. Then, for short, we set $\wedge_E$ (resp. $\vee_E^*$) to be the composition of the Fourier–Sato transformations $\wedge_i$ on E_i (resp. the composition of the inverse Fourier–Sato transformations $\vee_i^*$) for each $i \in \{1, \ldots, \ell\}$.

Following [3], we recall the notion of multi-microlocalization of sheaves. Let X be a real analytic manifold with $\dim X = n$, and let $\chi = \{M_1, \ldots, M_\ell\}$ be a family of closed submanifolds in X. We set, for $N \in \chi$ and $p \in N$,

$$\mathrm{NR}_p(N) := \left\{M_j \in \chi;\, p \in M_j,\ N \not\subseteq M_j, \text{ and } M_j \not\subseteq N\right\}.$$

Let us consider the following conditions for χ:

H1. Each $M_j \in \chi$ is connected and the submanifolds are mutually distinct, i.e., $M_j \neq M_{j'}$ for $j \neq j'$.

H2. For any $N \in \chi$ and $p \in N$ with $\mathrm{NR}_p(N) \neq \emptyset$, we have

$$\left(\bigcap_{M_j \in \mathrm{NR}_p(N)} T_p M_j\right) + T_p N = T_p X.$$

If χ satisfies condition H2, then for any $p \in X$, there exist a neighborhood V of p in X, a system of local coordinates $(x_1, \dots, x_n)$ in V, and a family of subsets $\{I_j\}_{j=1}^{\ell}$ of the set $\{1, 2, \dots, n\}$ for which the following conditions hold:

(1) Either $I_k \subset I_j$, $I_j \subset I_k$, or $I_k \cap I_j = \emptyset$ holds ($k, j \in \{1, 2, \dots, \ell\}$).

(2) A submanifold $M_j \in \chi$ with $p \in M_j$ ($j = 1, 2, \dots, \ell$) is defined by $\{x_i = 0; i \in I_j\}$ in V.

We set, for $N \in \chi$,

$$\iota_\chi(N) := \bigcap_{N \subsetneq M_j} M_j.$$

Here, $\iota_\chi(N) := X$ by convention if there exists no j with $N \subsetneq M_j$. When there is no risk of confusion, we write $\iota(N)$ for short. We also assume the condition

H3. $M_j \neq \iota(M_j)$ for any $j \in \{1, 2, \dots, \ell\}$.

In local coordinates let $I_1, \dots, I_\ell \subset \{1, \dots, \ell\}$ such that $M_i = \{x_k = 0; k \in I_i\}$. Note that the family χ satisfies conditions H1, H2, and H3 if and only if $I_1, \dots, I_\ell$ satisfy the corresponding conditions

$$\text{(1.1)}\qquad \begin{aligned} &\text{(i) either } I_j \subsetneq I_k,\ I_k \subsetneq I_j, \text{ or } I_j \cap I_k = \emptyset \text{ holds for any } j \neq k,\\ &\text{(ii) } \left(\bigcup_{I_k \subsetneq I_j} I_k\right) \subsetneq I_j \text{ for any } j. \end{aligned}$$

Hence, for any $j \in \{1, 2, \dots, \ell\}$, the set

$$\hat{I}_j := I_j \setminus \left(\bigcup_{I_k \subsetneq I_j} I_k\right) \tag{1.2}$$

is not empty by condition H3. By conditions H1, H2, and H3, we have for $j \in \{1, \dots, \ell\}$,

$$I_j = \bigsqcup_{I_k \subset I_j} \hat{I}_k.$$

In particular, $\bigcup_{1\le j\le \ell} I_j = \hat{I}_1 \sqcup \cdots \sqcup \hat{I}_\ell$. For convenience, set $I_0 = \hat{I}_0 := \{1, \dots, n\} \setminus \bigcup_{j=1}^{\ell} I_j$. Then, in local coordinates, we can write the coordinates $(x_1, \dots, x_n)$ as

$$(x^{(0)}, x^{(1)}, \dots, x^{(\ell)}),$$

where $x^{(j)}$ denotes the coordinates $(x_i)_{i\in \hat{I}_j}$ $(j = 0, \dots, \ell)$.

Under conditions H1–H3, we may obtain multi-normal deformation along χ (see [3, Section 1.2] for details). As usual we denote by $\tau_{M_1}\colon T_{M_1}X \to M_1$ the normal bundle. We first consider a normal deformation $\widetilde{X}_{M_1}$ along M_1 (see [6, Section 4.1]), i.e., an analytic manifold $\widetilde{X}_{M_1}$, a pair of mappings $p_{M_1}\colon \widetilde{X}_{M_1} \to X$, $t_1\colon \widetilde{X}_{M_1} \to \mathbb{R}$, and an action of $\mathbb{R}\setminus\{0\}$ on $\widetilde{X}_{M_1}$, $(\tilde{x}, r) \mapsto \tilde{x}\cdot r$ such that $p_{M_1}^{-1}(X\setminus M_1) \simeq X\setminus M_1 \times \mathbb{R}\setminus\{0\}$, $t_1^{-1}(c) \simeq X$ for each $c \ne 0$, and $t_1^{-1}(0) \simeq T_{M_1}X$. We denote by Ω_{M_1} the open subset of $\widetilde{X}_{M_1}$ obtained as the inverse image of $\mathbb{R}^+$ by t_1, by $i_{\Omega_{M_1}}$ the embedding $\Omega_{M_1} \to \widetilde{X}_{M_1}$, $\tilde{p}_{M_1} = p_{M_1} \circ i_{\Omega_{M_1}}$, and by $s_{M_1}\colon t_1^{-1}(0) \simeq T_{M_1}X \hookrightarrow \widetilde{X}_{M_1}$ the inclusion. We get the commutative diagram

$$\begin{tikzcd}
T_{M_1}X \arrow[r, "s_{M_1}"] \arrow[d, "\tau_{M_1}"'] & \widetilde{X}_{M_1} \arrow[d, "p_{M_1}"] & \Omega_{M_1} \arrow[l, "i_{\Omega_{M_1}}"'] \arrow[dl, "\tilde{p}_{M_1}"] \\
M_1 \arrow[r, "i_{M_1}"] & X. &
\end{tikzcd}$$

Set $\widetilde{\Omega}_{M_1} = \{(x_1, t_1); t_1 \ne 0\}$ and $\widetilde{M_2} := \overline{(p_{M_1}|_{\widetilde{\Omega}_{M_1}})^{-1}M_2}$. Then $\widetilde{M_2}$ is a closed smooth submanifold of $\widetilde{X}_{M_1}$. Now we can define the normal deformation along M_1, M_2 as

$$\widetilde{X}_{M_1,M_2} := \widetilde{(\widetilde{X}_{M_1})_{\widetilde{M_2}}}.$$

Then we can recursively define the normal deformation along χ as

$$\widetilde{X} := \widetilde{(\widetilde{X}_{M_1,\dots,M_{\ell-1}})_{\widetilde{M_\ell}}}.$$

Set $S_\chi = \{t_1, \dots, t_\ell = 0\}$, $M = \bigcap_{i=1}^{\ell} M_i$ and $\Omega_\chi = \{t_1, \dots, t_\ell > 0\}$. Then we have the commutative diagram

$$\begin{tikzcd}
S_\chi \arrow[r, "s"] \arrow[d, "\tau"] & \widetilde{X} \arrow[d, "p"] & \Omega_\chi \arrow[l, "i_\Omega"'] \arrow[dl, "\tilde{p}"] \\
M \arrow[r, "i_M"] & X. &
\end{tikzcd}$$

Let us consider the canonical map $T_{M_j}\iota(M_j) \to M_j \hookrightarrow X$ $(j = 1, \dots, \ell)$. We write, for short,

$$S_\chi \simeq \underset{X,1\le j\le \ell}{\times} T_{M_j}\iota(M_j) := T_{M_1}\iota(M_1) \underset{X}{\times} T_{M_2}\iota(M_2) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_\ell}\iota(M_\ell).$$

Definition 1.2.1. The multi-specialization along χ is the functor

$$\begin{array}{cccc} \nu_\chi^{\mathrm{sa}}\colon & D^b(k_{X_{\mathrm{sa}}}) & \longrightarrow & D^b(k_{S_{\chi\,\mathrm{sa}}}) \\ & \Cup & & \Cup \\ & F & \longmapsto & s^{-1}R\Gamma_{\Omega_\chi} p^{-1}F. \end{array}$$

We also define the functor $\nu_\chi = \rho^{-1}\nu_\chi^{\mathrm{sa}}\colon D^b(k_{X_{\mathrm{sa}}}) \to D^b(k_{S_\chi})$.

Let S_χ^* be the dual vector bundle of S_χ:

$$S_\chi^* := T^*_{M_1}\iota(M_1) \times_X \cdots \times_X T^*_{M_\ell}\iota(M_\ell).$$

Definition 1.2.2. The multi-microlocalization along χ is the functor

$$\begin{array}{cccc} \mu_\chi^{\mathrm{sa}}\colon & D^b(k_{X_{\mathrm{sa}}}) & \longrightarrow & D^b(k_{S^*_{\chi\,\mathrm{sa}}}) \\ & \Cup & & \Cup \\ & F & \longmapsto & \nu_\chi^{\mathrm{sa}}(F)^{\wedge S_\chi}. \end{array}$$

We set $\mu_\chi := \rho^{-1}\mu_\chi^{\mathrm{sa}}\colon D^b(k_{X_{\mathrm{sa}}}) \to D^b(k_{S_\chi^*})$.

Now we are going to restate a stalk formula for multi-microlocalization proved in [3, Theorem 2.11]. As the problem is local, we may assume that $X = \mathbb{R}^n$ and $q = 0$ with coordinates $(x_1, \dots, x_n)$, and that there exists a subset I_k $(k = 1, 2, \dots, \ell)$ in $\{1, 2, \dots, n\}$ with the conditions (1.1) (i), (ii) such that each submanifold M_k is given by $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n; x_i = 0\ (i \in I_k)\}$. Recall that $\hat{I}_k$ was defined by (1.2) and that we set $M = \bigcap_k M_k$ and $n_k = \sharp\hat{I}_k$. Then locally we have

$$X = M \times (N_1 \times N_2 \times \cdots \times N_\ell) = M \times N, \tag{1.3}$$

where N_k is $\mathbb{R}^{n_k}$ with coordinates $x^{(k)} = (x_i)_{i \in \hat{I}_k}$. Set, for $k \in \{1, \dots, \ell\}$,

$$\begin{aligned} J_{\prec k} &:= \big\{j \in \{1, \dots, \ell\},\ I_j \subsetneq I_k\big\}, \\ J_{\succ k} &:= \big\{j \in \{1, \dots, \ell\},\ I_j \supsetneq I_k\big\}, \\ J_{\not\sharp k} &:= \big\{j \in \{1, \dots, \ell\},\ I_j \cap I_k = \emptyset\big\}. \end{aligned}$$

Let $p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)}) \in T^*_{M_1}\iota(M_1) \times_X \cdots \times_X T^*_{M_\ell}\iota(M_\ell)$ and consider the following conic subset in N:

$$\begin{aligned} \gamma_k := \big\{(x^{(1)}, x^{(2)}, \dots, x^{(\ell)}) \in N \;\big|\; & x^{(j)} = 0\ (j \in J_{\prec k} \sqcup J_{\not\sharp k}), \\ & x^{(j)} \in \mathbb{R}^{n_j}\ (j \in J_{\succ k}), \\ & \langle x^{(j)}, \xi^{(k)} \rangle > 0\ (j = k)\big\}. \end{aligned} \tag{1.4}$$

Theorem 1.2.3. *Let* $p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \ldots, \xi^{(\ell)}) \in S^*_\chi$*, and let* $F \in D^b(k_{X_{\mathrm{sa}}})$*. Then we have*

$$H^k(\mu_\chi F)_p \simeq \varinjlim_{G,U} H^k_G(U; F).$$

Here, U *is an open subanalytic neighborhood of* q *in* X *and* G *is a closed subanalytic subset in the form* $M \times \sum_{k=1}^{\ell} G_k$*, with* G_k *being a closed subanalytic convex cone in* N *satisfying* $G_k \setminus \{0\} \subset \gamma_k$*, where* γ_k *is defined in* (1.4).

In [3, Theorem 4.1], the authors proved the existence of Uchida's triangle ([13, Appendix A]) in the context of multi-microlocal analysis. In this paper we are going to show that it holds for sheaves on subanalytic sites as well.

Theorem 1.2.4. *Let* $F \in D^b_{(\mathbb{R}^+)^\ell}(k_{E_{\mathrm{sa}}})$*. There exists a natural isomorphism*

$$\tau^! R\tau_{!!} F \simeq Rp_{1*} p_2^!(F^{\wedge E}),$$

and the natural morphism $F \to \tau^! R\tau_{!!} F$ *is embedded in the following distinguished triangle:*

$$F \longrightarrow \tau^! R\tau_{!!} F \longrightarrow Rp_{1*} R\Gamma_{P^+}(p_2^!(F^{\wedge E})) \xrightarrow{+1} .$$

Next we explain the notion of the μhom in multi-microlocalizations (see [2]). Let χ satisfy conditions H1–H3. Let A be an $\ell \times \ell$ matrix with

$$a_{ji} = \begin{cases} 1 & (i \in I_j), \\ 0 & (i \notin I_j). \end{cases}$$

We call A an associated matrix with χ. Then $X^2 = X \times X$ and $\Delta_X \subset X^2$ denotes its diagonal, i.e., $\Delta_X = \{(x, y) \in X^2; x = y\}$.

Define the ambient space $\widehat{X}$ of the universal family as

$$(X^2)^\ell = X^2 \times \cdots \times X^2.$$

Define the universal family $\hat{\chi} = \{\widehat{M}_1, \ldots, \widehat{M}_\ell\}$ of closed submanifolds in $\widehat{X}$ in the following manner: Each $\widehat{M}_j$ has the form

$$\widehat{M}_j = Z_{j1} \times Z_{j2} \times \cdots \times Z_{j\ell} \subset \widehat{X},$$

where $Z_{jk} \subset X^2$ is either X^2 or Δ_X determined by the rules

$$Z_{jk} := \begin{cases} X^2 & (a_{jk} = 0), \\ \Delta_X & (a_{jk} = 1). \end{cases}$$

For the universal family $\hat{\chi}$, we may consider the multi-normal deformation $\widetilde{(X^2)^\ell}_{\hat{\chi}}$ of $(X^2)^\ell$ along $\hat{\chi}$ and the multi-microlocalization $\mu^{\mathrm{sa}}_{\hat{\chi}}$ (see [2] for details.). And we have

$$S^*_{\hat{\chi}} = T^*X \times \cdots \times T^*X.$$

Definition 1.2.5 (Multi-microlocal Hom functors). Let $F_1, \ldots, F_\ell$ and G be in $D^b(X_{\mathrm{sa}})$. We define the functor

$$\mu hom^{\mathrm{sa}}_{\hat{\chi}}(F_1, \ldots, F_\ell; G) := \mu^{\mathrm{sa}}_{\hat{\chi}}\big(R\mathcal{H}om(p_2^{-1}(F_1 \boxtimes \cdots \boxtimes F_\ell), p_1^! Ri_{\Delta *}G)\big)$$

and we set the functor

$$\mu hom_{\hat{\chi}}(F_1, \ldots, F_\ell; G) := \rho^{-1}\mu hom^{\mathrm{sa}}_{\hat{\chi}}(F_1, \ldots, F_\ell; G),$$

where

$$X^\ell \xleftarrow{p_1} \widehat{X} \xrightarrow{p_2} X^\ell \xleftarrow{i_\Delta} X,$$

- $i_\Delta \colon X \hookrightarrow X^\ell$ is defined by $x \in X \to (x, x, \ldots, x) \in X^\ell$,
- $p_1 \colon (X^2)^\ell \to X^\ell$ is the canonical projection $(x_1, y_1) \times \cdots \times (x_\ell, y_\ell) \to (x_1, \ldots, x_\ell)$, and
- $p_2 \colon (X^2)^\ell \to X^\ell$ is the canonical projection $(x_1, y_1) \times \cdots \times (x_\ell, y_\ell) \to (y_1, \ldots, y_\ell)$.

2. A stalk formula for the multi-microlocal Hom functors

In this section we are going to prove a stalk formula for the multi-microlocal Hom functors. As the question is local, we may assume X is a vector space $\mathbb{R}^n$. Let γ be a closed convex cone with vertex at 0. One denotes by X_γ the set X endowed with the induced γ-topology (see [6, p. 161]), and by $\phi_\gamma \colon X \to X_\gamma$ the natural continuous map from X (with the usual topology) to X_γ. Set $Z(\gamma) := \{(x, y) \in X \times X; y - x \in \gamma\}$.

Lemma 2.0.1. *Let $F \in D^b_{\mathbb{R}\text{-}c}(k_X)$ and V be a relatively compact open subanalytic set. For a closed convex cone γ in X, we have*

$$Rp_{1!!}(p_2^{-1}(R\rho_* F)_V)_{Z(\gamma)} \simeq R\rho_*(\phi_{\bar{\gamma}}^{-1} R\phi_{\gamma *}(F_V)),$$

where p_1 and p_2 are the first and second projections from $X \times X$ to X.

Proof. This follows immediately from [8, Proposition 2.2.1], [8, Proposition 1.3.3], [8, Proposition 1.3.2], and [6, Proposition 3.5.4]. In fact,

$$Rp_{1!!}(p_2^{-1}(R\rho_* F)_V)_{Z(\gamma)} \simeq Rp_{1!!}\big(R\rho_*(p_2^{-1}(F_V)_{Z(\gamma)})\big) \tag{2.1}$$

$$\simeq R\rho_* Rp_{1!}(p_2^{-1}(F_V))_{Z(\gamma)} \tag{2.2}$$

$$\simeq R\rho_*(\phi_\gamma^{-1} R\phi_{\gamma *}(F_V)).$$

The isomorphism (2.1) follows since F is constructible. The isomorphism (2.2) follows since p_1 is proper on $\mathrm{supp}(p_2^{-1}(F_V))_{Z(\gamma)}$. ■

THEOREM 2.0.2. *Let*

$$p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)}) \in \overbrace{T^*X \times_X \cdots \times_X T^*X}^{\ell},$$

$G_1, \dots, G_\ell \in \mathrm{Ob}(D^b_{\mathbb{R}\text{-}c}(k_X))$, *and* $F \in \mathrm{Ob}(D^b(k_{X_{\mathrm{sa}}}))$. *We have*

$$\begin{aligned} &H^j(\mu hom_{\hat{\chi}}(R\rho_* G_1, R\rho_* G_2, \dots, R\rho_* G_\ell; F))_p \\ &\simeq \varinjlim_{V,\tilde{\gamma}} H^j\Big(R\Gamma\Big(i_\Delta^{-1} V; R\mathcal{H}om\Big(i_\Delta^{-1} R\rho_* \phi_{\tilde{\gamma}}^{-1} R\phi_{\tilde{\gamma}*}\Big(\Big(\boxtimes_{i=1}^{\ell} G_i\Big)_V\Big), F\Big)\Big)\Big) \\ &\simeq \varinjlim_{U,\tilde{\gamma}} H^j\Big(R\Gamma\Big(U; R\mathcal{H}om\big(R\rho_*\big(i_\Delta^{-1} \phi_{\tilde{\gamma}}^{-1} R\phi_{\tilde{\gamma}*}((G_1 \boxtimes \cdots \boxtimes G_\ell)_{U\times\cdots\times U})\big), F\big)\Big)\Big), \end{aligned}$$

where $\tilde{\gamma} = \widetilde{\gamma_1} + \cdots + \widetilde{\gamma_\ell} \subset X^\ell$, *each* $\widetilde{\gamma_k}$ *is a closed conic subanalytic proper convex subset of* $\gamma_k \cup \{0\}$,

$$\begin{aligned} \gamma_k := \Big\{(y^{(1)}, \dots, y^{(\ell)}) \in X^\ell \;\Big|\; & y^{(j)} = 0 \ (j \in J_{\prec k} \sqcup J_{\nparallel k}), \\ & y^{(j)} \in \mathbb{R}^n \ (j \in J_{\succ k}), \\ & \langle y^{(j)}, \xi^{(k)} \rangle < 0 \ (j = k)\Big\}, \end{aligned}$$

and U *ranges through the family of open subanalytic neighborhoods of* q. *Also,* V *runs through a family of open subanalytic neighborhoods of* $i_\Delta(q)$ *in* X^ℓ.

PROOF. As the question is local, we may assume X is a vector space $\mathbb{R}^n$ and $q = 0$ with coordinates $(x_1, \dots, x_n)$. Clearly (1.3) becomes $X^{2\ell} = X^\ell \times X^\ell$. Set

$$\begin{aligned} \delta_k := \Big\{(y^{(1)}, \dots, y^{(\ell)}) \in X^\ell \;\Big|\; & y^{(j)} = 0 \ (j \in J_{\prec k} \sqcup J_{\nparallel k}), \\ & y^{(j)} \in \mathbb{R}^n \ (j \in J_{\succ k}), \\ & \langle y^{(j)}, \xi^{(k)} \rangle > 0 \ (j = k)\Big\}. \end{aligned}$$

We propose the following identification φ:

$$\begin{array}{ccc} X^\ell \times X^\ell & \xleftrightarrow{\quad\varphi\quad} & X^\ell \times X^\ell \\ \cup & & \cup \\ (x_1, \dots, x_\ell; y_1, \dots, y_\ell) & \longleftrightarrow & (x_1, \dots, x_\ell; x_1 - y_1, \dots, x_\ell - y_\ell). \end{array} \tag{2.3}$$

We have the following claim.

Claim 2.0.3. Let G be a closed subanalytic subset in the form $X^\ell \times \sum_{k=1}^{\ell} G_k$ with G_k being a closed subanalytic proper convex cone in X^ℓ satisfying $G_k \setminus \{0\} \subset \delta_k$. Then for each such G there exists a closed conic subanalytic proper convex subset $\widetilde{\gamma_k} \setminus \{0\} \subset -\delta_k = \gamma_k$ such that

$$G \subset \varphi^{-1}(Z(\tilde{\gamma})),$$

where $\tilde{\gamma} = \widetilde{\gamma_1} + \widetilde{\gamma_2} + \cdots + \widetilde{\gamma_\ell}$ and

$$Z(\tilde{\gamma}) := \big\{(x_1, x_2, \ldots, x_\ell; y_1, y_2, \ldots, y_\ell) \in X^\ell \times X^\ell \;\big|\; (y_1 - x_1, \ldots, y_\ell - x_\ell) \in \tilde{\gamma}\big\}.$$

Moreover, for such $\tilde{\gamma}$ we may find a closed subanalytic subset $G' = X^\ell \times \sum_{k=1}^{\ell} G'_k$, with G'_k being a closed subanalytic proper convex cone in X^ℓ satisfying $G'_k \setminus \{0\} \subset \delta_k$, so that $\varphi^{-1}(Z(\tilde{\gamma})) \subset G'$.

In fact, let G be such a conic set. Then there exists $G_k \setminus \{0\} \subset \delta_k$. We may find $\widetilde{\gamma_k}$ such that $G_k \subset -\widetilde{\gamma_k}$ and $-\widetilde{\gamma_k} \setminus \{0\} \subset -\delta_k$. By the identification (2.3), $\varphi^{-1}(Z(\tilde{\gamma})) = \{(x; y) \in X^\ell \times X^\ell \mid -y \in \tilde{\gamma}\}$. If $(x, y) \in G$ then $y \in \sum_{k=1}^{\ell} G_k$. And $y \in -\tilde{\gamma}$ thus $(x, y) \in \varphi^{-1}(Z(\tilde{\gamma}))$. Finally, take a closed subanalytic proper convex cone G'_k in X^ℓ so that $-\widetilde{\gamma_k} \subset G'_k$ and $G'_k \setminus \{0\} \subset \delta_k$ for each k. It is easy to see $\varphi^{-1}(Z(\tilde{\gamma})) \subset G' = X^\ell \times \sum_{k=1}^{\ell} G'_k$. Therefore by Theorem 1.2.3 and Lemma 2.0.1 we obtain the chain of isomorphisms (put $\boxtimes_{i=1}^{\ell} G_i := G_1 \boxtimes \cdots \boxtimes G_\ell$)

$$H^k \mu hom_{\hat{\chi}}(R\rho_* G_1, R\rho_* G_2, \ldots, R\rho_* G_\ell; F)_p$$

$$\simeq H^k \mu_{\hat{\chi}}\Big(R\mathcal{H}om\Big(p_2^{-1}\Big(R\rho_* \boxtimes_{i=1}^{\ell} G_i\Big), p_1^! R i_{\Delta *} F\Big)\Big)_p$$

$$\simeq \varinjlim_{G,U,V} H^k\Big(R\Gamma_G\Big(U \times V; R\mathcal{H}om\Big(p_2^{-1}\Big(R\rho_* \boxtimes_{i=1}^{\ell} G_i\Big), p_1^! R i_{\Delta *} F\Big)\Big)\Big) \tag{2.4}$$

$$\simeq \varinjlim_{\tilde{\gamma},U,V} H^k\Big(R\Gamma_{Z(\tilde{\gamma})}\Big(U \times V; R\mathcal{H}om\Big(p_2^{-1}\Big(R\rho_* \boxtimes_{i=1}^{\ell} G_i\Big), p_1^! R i_{\Delta *} F\Big)\Big)\Big) \tag{2.5}$$

$$\simeq \varinjlim_{\tilde{\gamma},U,V} H^k\Big(R\Gamma\Big(U \times X; R\mathcal{H}om\Big(\Big(p_2^{-1}\Big(R\rho_* \boxtimes_{i=1}^{\ell} G_i\Big)_V\Big)_{Z(\tilde{\gamma})}, p_1^! R i_{\Delta *} F\Big)\Big)\Big)$$

$$\simeq \varinjlim_{\tilde{\gamma},U,V} H^k\Big(R\Gamma\Big(U; R\mathcal{H}om\Big(R p_{1!!}\Big(\Big(p_2^{-1}\Big(R\rho_* \boxtimes_{i=1}^{\ell} G_i\Big)_V\Big)_{Z(\tilde{\gamma})}\Big), R i_{\Delta *} F\Big)\Big)\Big)$$

$$\simeq \varinjlim_{\tilde{\gamma},U,V} H^k\Big(R\Gamma\Big(i_\Delta^{-1} U; R\mathcal{H}om\Big(i_\Delta^{-1} R p_{1!!}\Big(p_2^{-1}\Big(R\rho_* \boxtimes_{i=1}^{\ell} G_i\Big)_V\Big)_{Z(\tilde{\gamma})}, F\Big)\Big)\Big)$$

$$\simeq \varinjlim_{\tilde{\gamma},U} H^k\Big(R\Gamma\Big(i_\Delta^{-1} U; R\mathcal{H}om\Big(i_\Delta^{-1} R\rho_* \phi_{\tilde{\gamma}}^{-1} R\phi_{\tilde{\gamma} *}\Big(\Big(\boxtimes_{i=1}^{\ell} G_i\Big)_U\Big), F\Big)\Big)\Big). \tag{2.6}$$

The set G runs through a family of closed subanalytic subsets in the form $X^\ell \times \sum_{k=1}^{\ell} G_k$ with G_k being a closed subanalytic proper convex cone in X^ℓ satisfying $G_k \setminus \{0\} \subset \delta_k$, U and V runs through a family of open subanalytic neighborhoods of $i_\Delta(q)$ in X^ℓ. Notice that U is different from the one in the statement of theorem. The first isomorphism follows from [8, Proposition 1.3.3] and [9, Proposition 1.3.1], (2.4) follows from Theorem 1.2.3, and (2.5) is a consequence of Claim 2.0.3. Finally, Lemma 2.0.1 implies the isomorphism (2.6). We have used adjunctions many times throughout the chain of isomorphisms. ■

Lemma 2.0.4. *Let γ be a closed proper convex cone (with vertex at 0) in a vector space V, G be a closed convex subset. If $\gamma^a + G$ is closed then*

$$\phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_G \simeq \mathbb{C}_{\gamma^a+G}.$$

Here we denote the antipodal map $x \mapsto -x$ by "a".

Proof. Let us first prove $\phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_{G+\gamma^a} \xrightarrow{\sim} \mathbb{C}_{G+\gamma^a}$. By adjunction, there exists a morphism $\phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_{G+\gamma^a} \to \mathbb{C}_{G+\gamma^a}$. Then taking a stalk of the cohomology proves the assertion. In fact, by [6, Theorem 3.5.3] for any $x \in X$,

$$\varinjlim_{x\in U} H^k(U;\phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_{G+\gamma^a}) \simeq \varinjlim_{x\in U} H^k(U+\gamma;\mathbb{C}_{G+\gamma^a})$$

and since $(x+\gamma)\cap G+\gamma^a$ is convex when $(x+\gamma)\cap G+\gamma^a \neq \emptyset$ we have $R\Gamma((x+\gamma)\cap G+\gamma^a;\mathbb{C}_{G+\gamma^a}) \simeq \mathbb{C}$. Thus it is enough to prove that if $x \notin G+\gamma^a$ then $(x+\gamma)\cap G+\gamma^a = \emptyset$. By contradiction, if we can find $g_1, g_2 \in \gamma$ and $g_3 \in G$ so that $x+g_1 = g_3 - g_2$ then $x = g_3 - g_1 - g_2$. This implies $x \in G+\gamma^a$ which is a contradiction.

Now we prove the general assertion. We have a morphism $\mathbb{C}_{G+\gamma^a} \to \mathbb{C}_G$. Therefore, by the first assertion we get a morphism

$$\mathbb{C}_{G+\gamma^a} \simeq \phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_{G+\gamma^a} \to \phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_G.$$

Then taking a stalk of the cohomology proves the assertion. In fact, by [6, Theorem 3.5.3],

$$\varinjlim_{x\in U} H^k(U;\phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_G) \simeq \varinjlim_{x\in U} H^k(U+\gamma;\mathbb{C}_G)$$

and since $(x+\gamma)\cap G$ is convex when $(x+\gamma)\cap G \neq \emptyset$ we have $R\Gamma((x+\gamma)\cap G;\mathbb{C}_G) \simeq \mathbb{C}$. Thus it is enough to prove that if $x \notin G+\gamma^a$ then $(x+\gamma)\cap G = \emptyset$. By contradiction, if we can find $g_1 \in \gamma$ and $g_2 \in G$ so that $x+g_1 = g_2$ then $x = g_2 - g_1$. This implies $x \in G+\gamma^a$ which is a contradiction. ■

Corollary 2.0.5 (Normal crossing type). *Assume X is a vector space. Let $\hat{\chi} = \{\widehat{D}_1, \dots, \widehat{D}_\ell\}$ where*

$$\begin{aligned}\widehat{D}_j &= Z_{j1} \times Z_{j2} \times \cdots \times Z_{j\ell} \subset \widehat{X},\\ Z_{jk} &:= \begin{cases} \Delta_X & (j = k),\\ X^2 & (\text{otherwise}).\end{cases}\end{aligned}$$

Let $M_1, \dots, M_\ell$ be ℓ closed subanalytic convex subsets of X. Then for $p = (q; \xi_1, \dots, \xi_\ell) \in T^*X \times_X \cdots \times_X T^*X$ with $M_k \subset \{x \in X; \langle x - q, \xi_k\rangle \geq 0\}$ $(k = 1, \dots, \ell)$, we have

$$\begin{aligned}&H^j(\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}, \dots, \mathbb{C}_{M_\ell}; F))_p\\ &\quad\simeq \varinjlim_{U, \widetilde{\delta_1}, \dots, \widetilde{\delta_\ell}} H^j_{(M_1+\widetilde{\delta_1})\cap(M_2+\widetilde{\delta_2})\cap\cdots\cap(M_\ell+\widetilde{\delta_\ell})\cap U}(U; F),\end{aligned}$$

where each $\widetilde{\delta_k}$ is a closed subanalytic proper convex cone of $\delta_k \cup \{0\}$, $\delta_k = \{x \in X; \langle x, \xi_k\rangle > 0\}$ and U is an open subanalytic neighborhood of q in X.

Proof. By Theorem 2.0.2 we have

$$\begin{aligned}&H^j(\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1}, \dots, \mathbb{C}_{M_\ell}; F))_p\\ &\simeq \varinjlim_{U_1, \dots, U_\ell, \tilde{\gamma}} H^j\Big(\bigcap_{j=1}^{\ell} U_j; R\mathcal{H}om\big(R\rho_* i_\Delta^{-1}(\phi_{\tilde{\gamma}}^{-1} R\phi_{\tilde{\gamma}*}(\mathbb{C}_{M_1\cap U_1 \times M_2 \cap U_2 \times \cdots \times M_\ell \cap U_\ell})), F\big)\Big),\end{aligned}$$

where $\tilde{\gamma} = \widetilde{\gamma_1} + \cdots + \widetilde{\gamma_\ell} \subset X^\ell$, each $\widetilde{\gamma_k}$ is a closed conic subanalytic proper convex subset of $\gamma_k \cup \{0\}$,

$$\begin{aligned}\gamma_k := \big\{(x^{(1)}, \dots, x^{(\ell)}) \in X^\ell \;\big|\; &\langle x^{(j)}, \xi_k\rangle < 0 \ (j = k),\\ &x^{(j)} = 0 \ (\text{otherwise})\big\},\end{aligned}$$

and each U_j ranges through the family of open subanalytic neighborhoods of q in X.

Let $\tilde{\gamma}$ be a cone as above. We may find open neighborhoods $U_1, U_2, \dots, U_\ell$ of q such that

$$M_1 \cap U_1 \times M_2 \cap U_2 \times \cdots \times M_\ell \cap U_\ell = M_1 \times M_2 \times \cdots \times M_\ell \cap (U_1 \times \cdots \times U_\ell + \tilde{\gamma}).$$

In fact, for each j we may take an open neighborhood U_j of q which is an intersection of a $\widetilde{\gamma_j}$-open set in M_j and $\{x \in X; \langle x - q, \xi_j\rangle > -\varepsilon\}$ $(\varepsilon > 0)$.

Thus since for such $U_1, U_2, \dots, U_\ell$ the direct and inverse images of $\phi_{\tilde{\gamma}}$ commute with $(\cdot)_{U_1\times\cdots\times U_\ell + \tilde{\gamma}}$, we have

$$(2.7)\quad (\phi_{\tilde{\gamma}}^{-1} R\phi_{\tilde{\gamma}*}\mathbb{C}_{M_1\cap U_1\times\cdots\times M_\ell\cap U_\ell})|_{U_1\times\cdots\times U_\ell} \simeq (\phi_{\tilde{\gamma}}^{-1} R\phi_{\tilde{\gamma}*}\mathbb{C}_{M_1\times\cdots\times M_\ell})|_{U_1\times\cdots\times U_\ell}.$$

Therefore, by Lemma 2.0.4 we have
$H^j(\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1},\dots,\mathbb{C}_{M_\ell};F))_p$

$$\begin{aligned}
&\simeq \varinjlim_{U_1,\dots,U_\ell,\tilde{\gamma}} H^j\Big(\bigcap_{j=1}^{\ell} U_j; R\mathcal{H}om\big(R\rho_* i_\Delta^{-1}(\phi_{\tilde{\gamma}}^{-1} R\phi_{\tilde{\gamma}*}(\mathbb{C}_{M_1\cap U_1\times M_2\cap U_2\times\cdots\times M_\ell\cap U_\ell})), F\big)\Big)\\
&\simeq \varinjlim_{U_1,\dots,U_\ell,\tilde{\gamma}} H^j\Big(\bigcap_{j=1}^{\ell} U_j; R\mathcal{H}om\big(R\rho_* i_\Delta^{-1}((\phi_{\tilde{\gamma}}^{-1} R\phi_{\tilde{\gamma}*}(\mathbb{C}_{M_1\times M_2\times\cdots\times M_\ell}))_{U_1\times\cdots\times U_\ell}), F\big)\Big)\\
&\simeq \varinjlim_{U_1,\dots,U_\ell,\tilde{\gamma}} H^j\Big(\bigcap_{j=1}^{\ell} U_j; R\mathcal{H}om(R\rho_* i_\Delta^{-1}(\mathbb{C}_{M_1\times M_2\times\cdots\times M_\ell+\tilde{\gamma}^a\cap U_1\times\cdots\times U_\ell}), F)\Big)\\
&\simeq \varinjlim_{U_1,\dots,U_\ell,\tilde{\gamma}} H^j\Big(\bigcap_{j=1}^{\ell} U_j; R\mathcal{H}om(R\rho_*(i_\Delta^{-1}(\mathbb{C}_{\prod_{j=1}^{\ell}(M_j+q_j(\widetilde{\gamma_j}^a))\cap\prod_{j=1}^{\ell}U_j})), F)\Big)\\
&\simeq \varinjlim_{U_1,\dots,U_\ell,\tilde{\gamma}} H^j\Big(\bigcap_{j=1}^{\ell} U_j; R\mathcal{H}om(R\rho_*\mathbb{C}_{(M_1+q_1(\widetilde{\gamma_1}^a))\cap\cdots\cap(M_\ell+q_\ell(\widetilde{\gamma_\ell}^a))\cap\bigcap_{j=1}^{\ell}U_j}, F)\Big)\\
&\simeq \varinjlim_{U,\tilde{\gamma}} H^j_{(M_1+q_1(\widetilde{\gamma_1}^a))\cap(M_2+q_2(\widetilde{\gamma_2}^a))\cap\cdots\cap(M_\ell+q_\ell(\widetilde{\gamma_\ell}^a))\cap U}(U;F),
\end{aligned}$$

where q_k is kth projection and U ranges through the family of open subanalytic neighborhoods of q in X. The second isomorphism follows from (2.7) and the third one follows from Lemma 2.0.4. The fourth isomorphism follows from the fact that $M_1\times M_2\times\cdots\times M_\ell+\tilde{\gamma}=(M_1+q_1(\widetilde{\gamma_1}^a))\times(M_2+q_2(\widetilde{\gamma_2}^a))\times\cdots\times(M_\ell+q_\ell(\widetilde{\gamma_\ell}^a))$. The fifth one follows since $i_\Delta^{-1}\mathbb{C}_Z\simeq\mathbb{C}_{i_\Delta^{-1}(Z)}$ for a locally closed subset Z. ∎

Example 2.0.6. Let $X=\mathbb{C}^2$ and $\hat{\chi}=\{\Delta\times\mathbb{C}^2,\ \mathbb{C}^2\times\Delta\}$. Consider closed subsets $M_1=\{y_1=0,\ x_1\geq 0\}=\mathbb{R}_{\geq 0}\times\mathbb{C}$ and $M_2=\{y_2=0,\ x_2\geq 0\}=\mathbb{C}\times\mathbb{R}_{\geq 0}$.

By Corollary 2.0.2 for any subanalytic sheaf $F\in D^b(\mathbb{C}_{X_{\mathrm{sa}}})$ and

$$p=(q;\zeta_1,\zeta_2)\in T^*X\times_X T^*X=T^*\mathbb{C}^2\times_{\mathbb{C}^2}T^*\mathbb{C}^2=\overbrace{\underbrace{\mathbb{C}^2}_{\text{base space}}\times\underbrace{\mathbb{C}^2\times\mathbb{C}^2}_{\text{fiber space}}}^{\text{total space}},$$

with $M_k\subset\{z\in X;\mathrm{Re}\langle z-q,\zeta_k\rangle\geq 0\}$ $(k=1,2)$,

$$H^j(\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1},\mathbb{C}_{M_2};F))_p\simeq\varinjlim_{U,\widetilde{\delta_1},\widetilde{\delta_2}} H^j_{(M_1+\widetilde{\delta_1})\cap(M_2+\widetilde{\delta_2})\cap U}(U;F),$$

where each $\widetilde{\delta_k}$ is a closed subanalytic proper convex cone of $\delta_k\cup\{0\}$, $\delta_k=\{z\in\mathbb{C}^2;\ \mathrm{Re}\langle z,\zeta_k\rangle>0\}$, and U ranges through the family of open subanalytic neighborhoods of q. Here, $\langle(z_1,z_2),(z_3,z_4)\rangle:=z_1z_3+z_2z_4$.

Let $p = ((1,1); (\sqrt{-1}, 0), (0, \sqrt{-1}))$. We have

$$H^j(\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}; F))_p \simeq \varinjlim_{U} H^j_{H\times H\cap U}(U; F),$$

where U ranges through the family of open subanalytic neighborhoods of $(1,1)$ and $H := \{x + iy; y \le 0\}$. If $F = R\rho_* \mathcal{O}_X$ then by [4, Theorem 2.2.2] or the useful criterion [6, (2.9.14)] for $j \neq 2$ we have

$$H^j(\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}; R\rho_* \mathcal{O}_X))_p \simeq \varinjlim_{U} H^j_{H\times H\cap U}(U; R\rho_* \mathcal{O}_X) = 0.$$

When $p = ((1,0); (\sqrt{-1}, 0), (0, \sqrt{-1}))$,

$$H^j(\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}; F))_p \simeq \varinjlim_{\lambda, U} H^j_{G_\lambda\times H\cap U}(U, F),$$

where $G_\lambda = \lambda + \mathbb{R}_{\ge 0} \subset \mathbb{C}$ and λ runs through the family of closed conic subanalytic proper convex subsets in $\mathbb{C}$ with its vertex being 0 such that $\lambda \subset \{y_1 < 0\} \cup \{0\}$ and U ranges through the family of open subanalytic neighborhoods of $(1,0)$. If $F = R\rho_* \mathcal{O}_X$ then by the useful criterion [6, (2.9.14)] for $j \neq 2$ we have

$$H^j(\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}; R\rho_* \mathcal{O}_X))_p = 0.$$

Finally, consider $p = ((0,0); (\sqrt{-1}, 0), (0, \sqrt{-1}))$:

$$H^j(\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}; F))_p \simeq \varinjlim_{\lambda_1, \lambda_2, U} H^j_{G_{\lambda_1}\times G_{\lambda_2}\cap U}(U, F),$$

where each $G_{\lambda_k} = \lambda_k + \mathbb{R}_{\ge 0} \subset \mathbb{C}$ and λ_k runs through the family of closed conic subanalytic proper convex subsets in $\mathbb{C}$ with its vertex being 0 such that $\lambda_k \subset \{y_k < 0\} \cup \{0\}$ and U ranges through the family of open subanalytic neighborhoods of $(0,0)$. If $F = R\rho_* \mathcal{O}_X$ then by the useful criterion [6, (2.9.14)] for $j \neq 2$ we have

$$H^j(\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}; R\rho_* \mathcal{O}_X))_p = 0.$$

Summing up the computations above, $H^2(\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}; R\rho_* \mathcal{O}_X))_p$ is isomorphic to

$$\begin{cases} \varinjlim_{U} H^j_{H\times H\cap U_1}(U; F) & \text{if } p = ((1,1); (\sqrt{-1}, 0), (0, \sqrt{-1})), \\ \varinjlim_{\lambda, V} H^j_{G_\lambda\times H\cap V}(V, F) & \text{if } p = ((1,0); (\sqrt{-1}, 0), (0, \sqrt{-1})), \\ \varinjlim_{\lambda_1, \lambda_2, W} H^j_{G_{\lambda_1}\times G_{\lambda_2}\cap W}(W, F) & \text{if } p = ((0,0); (\sqrt{-1}, 0), (0, \sqrt{-1})). \end{cases}$$

Here, U_1 ranges through the family of open subanalytic neighborhoods of $(1,1)$ and $H := \{x+iy; y \le 0\}$. Also, $G_\lambda = \lambda + \mathbb{R}_{\ge 0} \subset \mathbb{C}$ and λ runs through the family of closed conic subanalytic proper convex subsets in $\mathbb{C}$ with its vertex being 0 such that $\lambda \subset \{y_1 < 0\} \cup \{0\}$ and V ranges through the family of open subanalytic neighborhoods of $(1,0)$. Each $G_{\lambda_k} = \lambda_k + \mathbb{R}_{\ge 0} \subset \mathbb{C}$ and λ_k runs through the family of closed conic subanalytic proper convex subsets in $\mathbb{C}$ with its vertex being 0 such that $\lambda_k \subset \{y_k < 0\} \cup \{0\}$ and W ranges through the family of open subanalytic neighborhoods of $(0,0)$. If $j \neq 2$, $H^j(\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}; R\rho_*\mathcal{O}_X))_p \simeq 0$.

3. The multi-microlocal Sato triangle

In this section we construct the multi-microlocal Sato triangle. Suppose $\chi = \{M_1, M_2, \dots, M_\ell\}$ satisfies the conditions H1, H2, and H3.

PROPOSITION 3.0.1. *Let $i\colon M \to X$ be the canonical embedding and $\pi\colon S_\chi^* \to M$ be the canonical projection where $M = \bigcap_{j=1}^{\ell} M_j$. For $F \in D^b(k_{X_{\mathrm{sa}}})$,*

$$\mu_\chi^{\mathrm{sa}}(F)|_M \simeq R\pi_*\mu_\chi^{\mathrm{sa}}(F) \simeq R\Gamma_M(F)|_M \simeq i^!F.$$

PROOF. It is enough to prove the second isomorphism. By Theorem 1.2.4, for $H = \nu_\chi^{\mathrm{sa}}(F)$ we have

$$\tau^! R\tau_{!!} H \simeq Rp_{1*}p_2^! H^{\wedge S_\chi}.$$

By the base change formula ([8, Proposition 2.2.9]) on the Cartesian square

$$\begin{tikzcd} S_\chi \times_M S_\chi^* \arrow[r, "p_2"] \arrow[d, "p_1"'] & S_\chi^* \arrow[d, "\pi"] \\ S_\chi \arrow[r, "\tau"'] & M, \end{tikzcd}$$

we have $Rp_{1*}p_2^! H^{\wedge S_\chi} \simeq \tau^! R\pi_* H^{\wedge S_\chi}$. Thus

$$\tau^! R\tau_{!!} H \simeq \tau^! R\pi_* H^{\wedge S_\chi}.$$

Applying $i^!$ to both sides,

$$R\tau_{!!} H \simeq R\pi_* H^{\wedge S_\chi}.$$

Since $R\tau_{!!}\nu_\chi^{\mathrm{sa}}(F) \simeq R\Gamma_M(F)$ by [3, Proposition 2.6], we get

$$R\Gamma_M(F) \simeq R\pi_*\mu_\chi^{\mathrm{sa}}(F).$$ ■

Next we extend the result in [13, Corollary A.2] to subanalytic sheaf theory.

Proposition 3.0.2. *Let $\tau\colon E \to Z$ be a vector bundle and E^* be its dual. Set $P = \{(x, y) \in E \times_Z E^*; \langle x, y\rangle \geq 0\}$. For a conic object $G \in D^b_{\mathbb{R}+}(k_{E_{\mathrm{sa}}})$, there is a natural isomorphism*

$$\tau^{-1} R\tau_* G \otimes \omega_{E/Z} \simeq Rp_{1!!} p_2^{-1} G^\wedge$$

and a commutative diagram

$$\begin{tikzcd}
G & Rp_{1!!}(p_2^{-1}G^\wedge)_P \otimes \omega_{E/Z} \arrow[l] \\
\tau^{-1}R\tau_*G \arrow[u] & Rp_{1!!}p_2^{-1}G^\wedge \otimes \omega_{E/Z}, \arrow[l] \arrow[u]
\end{tikzcd}$$

where every horizontal arrow is an isomorphism.

Proof. We have a commutative diagram

$$\begin{tikzcd}
G & Rp_{1!!}(p_2^!G^\wedge)_P \arrow[l, "\sim"'] & Rp_{1!!}p_2^!G^\wedge \arrow[l] \\
\tau^{-1}R\tau_*G \arrow[u] & \tau^{-1}R\tau_*Rp_{1!!}(p_2^!G^\wedge)_P \arrow[l, "\sim"'] \arrow[u] & \tau^{-1}R\tau_*Rp_{1!!}p_2^!G^\wedge, \arrow[l, "\alpha"'] \arrow[u, "\beta"']
\end{tikzcd}$$

where each horizontal arrow of the left square is an isomorphism by [9, Theorem 3.2.6]. To show α is an isomorphism, it is enough to prove

$$R\tau_* Rp_{1!!} H \to R\tau_* Rp_{1!!} H_P$$

is an isomorphism for a conic object $H \in D^b_{\mathbb{R}+}(k_{E\times_M E^*{}_{\mathrm{sa}}})$ on $E \times_M E^*{}_{\mathrm{sa}}$. We make use of the following Cartesian diagram:

$$\begin{tikzcd}
E \times_Z E^* \arrow[r, "\tau'"] \arrow[d, "p_1"'] & Z \times E^* \arrow[r, hook, "i'"] \arrow[d, "p_1'"'] & E \times_Z E^* \arrow[r, "p_2"] \arrow[d, "p_1"'] & E^* \\
E \arrow[r, "\tau"'] & Z \arrow[r, hook, "i"'] & E. &
\end{tikzcd}$$

We have the commutative diagram

$$\begin{tikzcd}
R\tau_*Rp_{1!!}H_P \arrow[r, "\sim"] & i^{-1}Rp_{1!!}H_P \arrow[r, "\sim"] & Rp'_{1!!}i'^{-1}H_P \\
R\tau_*Rp_{1!!}H \arrow[r, "\sim"] \arrow[u] & i^{-1}Rp_{1!!}H \arrow[r, "\sim"] \arrow[u] & Rp'_{1!!}i'^{-1}H, \arrow[u]
\end{tikzcd}$$

where horizontal isomorphisms follow from $R\tau_* \simeq i^{-1}$ ([9, Lemma 3.1.5]) and the base change formula ([8, Proposition 2.2.8]). Since

$$i'(Z \times_Z E^*) \subset P,$$

the third vertical arrow is an isomorphism. Thus α is an isomorphism. We will prove β is an isomorphism. Since $p_2 = p_2 \circ i' \circ \tau'$ and by the base change formula([8, Proposition 2.2.9]), we have

$$\begin{aligned} Rp_{1!!}p_2^{-1}G^\wedge \otimes \omega_{E/Z} &\simeq Rp_{1!!}\tau'^{-1}i'^{-1}p_2^{-1}G^\wedge \otimes \omega_{E/Z} \\ &\simeq \tau^{-1}i^{-1}Rp_{1!!}p_2^{-1}G^\wedge \otimes \omega_{E/Z} \\ &\simeq \tau^{-1}R\tau_* Rp_{1!!}p_2^!G^\wedge. \end{aligned}$$ ■

Proposition 3.0.3. *Let G be a multi-conic object on E^*_{sa}. We have*

$$G^{\vee^*_E} \simeq Rp''_{1!!}p''^{-1}_2 G \otimes \omega_{E/Z} \simeq Rp_{1!!}(p_2^! G)_P.$$

(For notation see Section 1.2.)

Proof. By induction assume $\ell = 2$. We have the following diagram:

$$\begin{tikzcd} & P \arrow[r, "p''_{2,2}"] \arrow[d, "p''_{1,1}"] \arrow[dl, bend right, "p''_1"'] \arrow[rr, bend left, "p''_2"] \arrow[dr, phantom, "\square"] & P_1 \underset{Z}{\times} E_2^* \arrow[r, "p''_{1,2}"'] \arrow[d, "p''_{1,1}"] & E^* \\ E & E_1 \underset{Z}{\times} P_2 \arrow[l, "p''_{2,1}"] \arrow[r, "p''_{2,2}"'] & E_1 \underset{Z}{\times} E_2^*. & \end{tikzcd}$$

Then by Proposition 1.1.2 we have

$$\begin{aligned} G^{\vee_1^*\vee_2^*} &\simeq Rp_{2,1!!}(p_{2,2}^{-1}Rp_{1,1!!}(p_{1,2}^{-1}G)_{P_1})_{P_2} \otimes \omega_{E/Z} \\ &\simeq Rp''_{2,1!!}p''^{-1}_{2,2}Rp''_{1,1!!}p''^{-1}_{1,2}G \otimes \omega_{E/Z} \\ &\simeq Rp''_{2,1!!}Rp''_{1,1!!}p''^{-1}_{2,2}p''^{-1}_{1,2}G \otimes \omega_{E/Z} \\ &\simeq Rp''_{1!!}p''^{-1}_2 G \otimes \omega_{E/Z}. \end{aligned}$$

Assume $\ell > 2$. Set

$$E' := \mathop{\times}_{Z,i=2}^{\ell} E_i, \quad E'^* := \mathop{\times}_{Z,i=2}^{\ell} E_i'^*, \quad P_{E'} := \mathop{\times}_{Z,i=2}^{\ell} P_i.$$

We have the following commutative diagram:

$$\begin{tikzcd}
& P \arrow[r, "p''_{E',2}"] \arrow[d, "p''_{1,1}"'] \arrow[dl, bend right, "p''_1"'] \arrow[rr, bend left, "p''_2"] \arrow[dr, phantom, "\square"] & P_1 \underset{Z}{\times} E'^* \arrow[r, "p''_{1,2}"'] \arrow[d, "p''_{1,1}"] & E^* \\
E & E_1 \underset{Z}{\times} P_{E'} \arrow[l, "p''_{E',1}"] \arrow[r, "p''_{E',2}"'] & E_1 \underset{Z}{\times} E'^*. &
\end{tikzcd}$$

By the induction hypothesis, we have

$$(G^{\vee_1^*})^{\vee_{E'}^*} \simeq Rp''_{1!!}p_2''^{-1}G \otimes \omega_{E/Z}.$$

Therefore, the induction proceeds. ∎

We need to extend the result [9, Proposition 3.2.9]:

LEMMA 3.0.4. *Let E_1, E_2 be vector bundles over Z, a biconic object $F \in D^b_{\mathbb{R}^+\times\mathbb{R}^+}(k_{E_1\times_Z E_{2\mathrm{sa}}})$ on $E_1 \times_Z E_{2\mathrm{sa}}$, and $p\colon E_1 \times_Z E_2 \to E_1$. Then $Rp_{!!}F \in D^b_{\mathbb{R}^+}(k_{E_{1\mathrm{sa}}})$. Here, the first $\mathbb{R}^+$-action on $E_1 \times_Z E_2$ is either $((z;\xi_1,\xi_2),t_1) \mapsto (z;t_1\xi_1,t_1\xi_2)$ or $((z;\xi_1,\xi_2),t_1) \mapsto (z;t_1\xi_1,\xi_2)$, and the second one is $((z;\xi_1,\xi_2),t_2) \mapsto (z;\xi_1,t_2\xi_2)$. Moreover,*

$$(Rp_{!!}F)^{\wedge E_1} \simeq Rp'_{!!}F^{\wedge E_1},$$

where $p'\colon E_1^* \times_Z E_2 \to E_1^*$.

PROOF. The natural embedding $i\colon E_1 \to E_1 \times_Z E_2$ is a conic morphism with respect to the first action on $E_1 \times_Z E_2$. By [9, Proposition 2.5.13] we get $i^!F \in D^b_{\mathbb{R}^+}(k_{E_{1\mathrm{sa}}})$. Since F is a conic object on $E_1 \times_Z E_{2\mathrm{sa}}$ with respect to the second action, by [9, Lemma 3.1.5] we obtain $i^!F \simeq Rp_{!!}F$. Therefore, $Rp_{!!}F \in D^b_{\mathbb{R}^+}(k_{E_{1\mathrm{sa}}})$. Consider the diagram below where the squares are Cartesian:

$$\begin{tikzcd}
E_1^* \underset{Z}{\times} E_2 \arrow[r, "p'"] \arrow[dr, phantom, "\square"] & E_1^* & \\
P' \underset{Z}{\times} E_2 \arrow[u, "q'_2"] \arrow[d, "q'_1"'] \arrow[r, "\tilde{p}"] \arrow[dr, phantom, "\square"] & P' \arrow[u, "q_2"'] \arrow[d, "q_1"] \arrow[r, hook, "j"'] & E_1 \underset{Z}{\times} E_1^* \arrow[ul, "p_2"'] \arrow[dl, "p_1"] \\
E_1 \underset{Z}{\times} E_2 \arrow[r, "p"] & E_1. &
\end{tikzcd}$$

Here, $P' := \{(\eta,\xi) \in E_1 \times_Z E_1^*; \langle \eta,\xi\rangle \leq 0\}$. Notice that $Rq_{2!!}q_1^{-1}G \simeq G^{\wedge E_1}$ for a conic object G on $E_{1\mathrm{sa}}$. In fact,

$$\begin{aligned} Rp_{2!!}(p_1^{-1}G)_{P'} &\simeq Rp_{2!!}Rj_{!!}j^{-1}p_1^{-1}G \\ &\simeq Rq_{2!!}q_1^{-1}G. \end{aligned}$$

Therefore,

$$\begin{aligned} (Rp_{!!}F)^{\wedge E_1} &\simeq Rq_{2!!}q_1^{-1}Rp_{!!}F \\ &\simeq Rq_{2!!}R\tilde{p}_{!!}q_1'^{-1}F \\ &\simeq Rp'_{!!}Rq'_{2!!}q_1'^{-1}F \\ &\simeq Rp'_{!!}(F^{\wedge E_1}). \end{aligned}$$

■

Proposition 3.0.5. *Let F be a multi-conic object on E_{sa}. We have a natural isomorphism*

$$\tau^{-1}R\tau_*F \simeq Rp_{1!!}p_2^!F^{\wedge E},$$

and the natural morphism $\tau^{-1}R\tau_*F \to F$ is embedded in the following distinguished triangle:

$$Rp_{1!!}(p_2^!F^{\wedge E})_{P^-} \longrightarrow \tau^{-1}R\tau_*F \longrightarrow F \xrightarrow{+1}.$$

(For notation see Section 1.2.)

Proof. If $\ell = 1$, the result follows by Proposition 3.0.2. Assume $\ell > 1$, and $E' := \bigtimes_{Z,i=2}^{\ell} E_i$, $E'^* := \bigtimes_{Z,i=2}^{\ell} E_i^*$, and $P'_{E'} := \bigtimes_{Z,i=2}^{\ell} P'_i$. Moreover, let $\wedge_{E'}$ (resp. $\vee_{E'}^*$) be the composition of $\wedge_i$ (resp. $\vee_i^*$) for $i = 2,\dots,\ell$. Consider the following diagram:

$$\begin{tikzcd} & E \underset{Z}{\times} E^* \arrow[r, "p_{E',2}"] \arrow[dl, bend right, "p_1"'] \arrow[rr, bend left, "p_2"] \arrow[d, "p_{1,1}"'] & E_1 \underset{Z}{\times} E^* \arrow[r, "p_{1,2}"'] \arrow[d, "p_{1,1}"] & E^* \\ E \arrow[r, "p_{E',1}"'] & E_1 \underset{Z}{\times} E' \underset{Z}{\times} E'^* \arrow[r, "p_{E',2}"'] \arrow[ur, phantom, "\square"] & E_1 \underset{Z}{\times} E'^*. & \end{tikzcd}$$

By the commutative diagram

$$\begin{tikzcd} E \arrow[r, "\tau_1"] \arrow[d, "\tau_{E'}"'] \arrow[dr, phantom, "\square"] & E' \arrow[d, "\tau_{E'}"] \\ E_1 \arrow[r, "\tau_1"'] & Z \end{tikzcd}$$

and [8, Proposition 2.4.8] and the dual base change formula, we have

$$\begin{aligned}\tau_1^{-1} R\tau_{1*}\tau_{E'}^{-1} R\tau_{E'*} F &\simeq \tau_1^{-1}\tau_{E'}^{-1} R\tau_{1*} R\tau_{E'*} F \\ &\simeq \tau^{-1} R\tau_* F. \end{aligned} \tag{3.1}$$

Hence, by Proposition 3.0.2 and the induction hypothesis, we obtain the following commutative diagram:

$$\begin{tikzcd}
F & F^{\wedge_{E'}\vee_{E'}{}^*} \arrow[l, "\sim"'] & F^{\wedge_{E'}\vee^*_{E'}\wedge_1\vee_1{}^*} \arrow[l, "\sim"'] \\
\tau_{E'}^{-1} R\tau_{E'*} F \arrow[u] & & Rp_{E',1!!}p^!_{E',2}(F^{\wedge_{E'}}) \arrow[ll, "\sim"', "\beta_1"] \arrow[u] \\
\tau_1^{-1} R\tau_{1*}\tau_{E'}^{-1} R\tau_{E'*} F \arrow[u] & & \tau_1^{-1} R\tau_{1*} Rp_{E',1!!}p^!_{E',2}(F^{\wedge_{E'}}) \arrow[ll, "\sim"', "\beta_2"] \arrow[u] \\
\tau^{-1} R\tau_* F \arrow[u, "\alpha", "\sim"'] & & Rp_{1,1!!}p^!_{1,2}((Rp_{E',1!!}p^!_{E',2}(F^{\wedge_{E'}}))^{\wedge_1}). \arrow[ll, "\sim"', "\beta_3"] \arrow[u, "\sim", "\gamma"']
\end{tikzcd}$$

The isomorphism α comes from (3.1), β_1 is an isomorphism by the induction hypothesis, β_2 is the isomorphism $\tau_1^{-1} R\tau_{1*}(\beta_1)$, γ follows from Proposition 3.0.2, and β_3 is a composition of isomorphisms.

By Lemma 3.0.6 below for a multi-conic object H on $E_1 \times_Z E'^*_{\mathrm{sa}}$, we have $H^{\vee^*_{E'}\vee_1} \simeq H^{\vee_1\vee^*_{E'}}$. Therefore, we have

$$\begin{aligned} H^{\vee^*_{E'}\wedge_1} &\simeq (H^{\vee^*_{E'}\vee_1})^{\mathrm{id}\times a} \otimes \omega^{\otimes -1}_{E_1/Z} \\ &\simeq (H^{\vee_1\vee^*_{E'}})^{\mathrm{id}\times a} \otimes \omega^{\otimes -1}_{E_1/Z} \\ &\simeq H^{\wedge_1\vee^*_{E'}}. \end{aligned}$$

Thus, we get the following chain of isomorphisms by Proposition 3.0.3 and [3, Proposition 2.2]:

$$\begin{aligned} F^{\wedge_{E'}\vee^*_{E'}\wedge_1\vee_1{}^*} &\simeq F^{\wedge_{E'}\wedge_1\vee^*_{E'}\vee_1{}^*} \\ &\simeq Rp''_{1!!}p''^{-1}_2 F^{\wedge_E} \otimes \omega_{E/Z}. \end{aligned}$$

Lastly, by Lemma 3.0.4 and [9, Proposition 3.2.9] we have

$$\begin{aligned} & Rp_{1,1!!}p^{-1}_{1,2}\big((Rp_{E',1!!}p^{-1}_{E',2}(F^{\wedge_{E'}}))^{\wedge_1}\big) \\ \simeq\ & Rp_{1,1!!}p^{-1}_{1,2} Rp'_{E',1!!}p'^{-1}_{E',2}(F^{\wedge_E}) \qquad (3.2) \\ \simeq\ & Rp_{1,1!!} Rp_{E',1!!}p'^{-1}_{1,2}\,p'^{-1}_{E',2}(F^{\wedge_E}) \qquad (3.3) \end{aligned}$$

$$
\begin{aligned}
&\simeq Rp_{1,1!!} Rp_{E',1!!} p_2^{-1}(F^{\wedge_E}) \\
&\simeq Rp_{1!!} p_2^{-1}(F^{\wedge_E}),
\end{aligned}
$$

where morphisms on (3.2) are $p'_{E',2}\colon E' \times_Z E^* \to E^*$ and $p'_{E',1}\colon E_1^* \times_Z E' \times_Z E'^* \to E_1^* \times_Z E'$. On (3.3) we used the base change formula [8, Proposition 2.2.9] on the Cartesian square as shown below:

$$
\begin{tikzcd}
E \underset{Z}{\times} E^* \arrow[r, "p'_{1,2}"] \arrow[d, "p_{E',1}"'] & E' \underset{Z}{\times} E^* \arrow[d, "p'_{E',1}"] \\
E_1 \underset{Z}{\times} E_1^* \underset{Z}{\times} E' \arrow[r, "p_{1,2}"'] & E_1^* \underset{Z}{\times} E'
\end{tikzcd}
$$

By the above isomorphisms,

$$
\begin{tikzcd}
 & \tau^{-1} R\tau_* F \arrow[r] & F \\
 & Rp_{1,1!!} p^!_{1,2}(Rp_{E',1!!} p^!_{E',2} F^{\wedge_{E'}})^{\wedge_1} \arrow[r] \arrow[u, "\sim"'] & F^{\wedge_{E'} \vee^*_{E'} \wedge_1 \vee_1 {}^*} \arrow[u, "\sim"'] \\
Rp_{1!!}(p_2^!(F^{\wedge_E}))_{P-} \arrow[r] \arrow[d, equal] & Rp_{1!!} p_2^! F^{\wedge_E} \arrow[r] \arrow[u, "\sim"'] \arrow[d, equal] & Rp''_{1!!} p''^{-1}_2(F^{\wedge_E}) \otimes \omega_{E/Z} \arrow[r, "+1"] \arrow[u, "\sim"'] \arrow[d, "\sim"] & {} \\
Rp_{1!!}(p_2^!(F^{\wedge_E}))_{P-} \arrow[r] & Rp_{1!!} p_2^! F^{\wedge_E} \arrow[r] & Rp_{1!!}(p_2^! F^{\wedge_E})_P \arrow[r, "+1"] & {} .
\end{tikzcd}
$$

■

Lemma 3.0.6. *Let E_1 and E'^* be as in the proof of Proposition 3.0.5. For a multi-conic object H on $E_1 \times_Z E'^*_{\mathrm{sa}}$ we have $H^{\vee^*_{E'} \vee_1} \simeq H^{\vee_1 \vee^*_{E'}}$.*

Proof. Consider the following commutative diagram:

$$
\begin{tikzcd}
 & P'_1 \underset{Z}{\times} P'_{E'} \arrow[r, "p'_{E',2}"] \arrow[d, "p'_{1,2}"] \arrow[dl, bend right, "p'_1"'] \arrow[rr, bend left, "p'_2"] & P'_1 \underset{Z}{\times} E'^* \arrow[r, "p'_{1,1}"] \arrow[d, "p'_{1,2}"] & E_1 \underset{Z}{\times} E'^* \\
E_1^* \underset{Z}{\times} E' & E_1^* \underset{Z}{\times} P'_{E'} \arrow[l, "p'_{E',1}"] \arrow[r, "p'_{E',2}"'] & E_1^* \underset{Z}{\times} E'^*, &
\end{tikzcd}
$$

where the square is Cartesian. Then we have

$$
\begin{aligned}
H^{\vee^*_{E'} \vee_1} \simeq Rp'_{E',1*} p'^!_{E',2} Rp'_{1,2*} p'^!_{1,1} H &\simeq Rp'_{E',1*} Rp'_{1,2*} p'^!_{E',2} p'^!_{1,1} H \\
&\simeq Rp'_{1*} p'^!_2 H.
\end{aligned}
$$

For the same reason, we obtain $H^{\vee_1 \vee^*_{E'}} \simeq Rp'_{1*} p'^{!}_2 H$. ■

Proposition 3.0.7. *Let $F \in D^b(k_{X_{\mathrm{sa}}})$. Let i and M be as in Proposition 3.0.1. Then*

$$R\pi_{!!}\mu^{\mathrm{sa}}_\chi(F) \simeq R\Gamma_M(\mu^{\mathrm{sa}}_\chi(F)) \simeq i^{-1}F \otimes \omega_{M/X}.$$

Proof. Set $H := \nu^{\mathrm{sa}}_\chi(F)$ and $E := S_\chi$. Then by Proposition 3.0.5 we have

$$\tau^{-1}R\tau_* H \otimes \omega_{E/M} \simeq Rp_{1!!}p_2^{-1}H^{\wedge_E}.$$

By the base change formula ([8, Proposition 1.4.6]) on the Cartesian square,

$$\begin{tikzcd} E \times_M E^* \arrow[r, "p_2"] \arrow[d, "p_1"'] & E^* \arrow[d, "\pi"] \\ E \arrow[r, "\tau"'] & M, \end{tikzcd}$$

we have

$$\tau^{-1}R\pi_{!!}H^{\wedge_E} \simeq Rp_{1!!}p_2^{-1}H^{\wedge_E}.$$

Thus

$$\tau^{-1}R\pi_{!!}H^{\wedge_E} \simeq \tau^{-1}R\tau_* H \otimes \omega_{E/M}.$$

We apply i^{-1} to get

$$R\pi_{!!}H^{\wedge_E} \simeq R\tau_* H \otimes \omega_{M/X}.$$

Since $R\tau_* H \simeq F|_M$ ([3, Proposition 2.6]) we have

$$R\pi_{!!}\nu^{\mathrm{sa}}_\chi(F)^{\wedge_E} \simeq F|_M \otimes \omega_{M/X}.$$ ■

Corollary 3.0.8 (Multi-microlocal Sato triangle). *We get the distinguished triangle*

$$F|_M \otimes \omega_{M/X} \longrightarrow R\Gamma_M(F)|_M \longrightarrow R\dot{\pi}_*\mu^{\mathrm{sa}}_\chi(F) \xrightarrow{+1},$$

where $\dot{\pi}$ is the restriction of $\pi\colon S^*_\chi \to M$ to $S^*_\chi \setminus M$.

Proof. We finish the proof by applying the above results (Propositions 3.0.1, 3.0.7) to the following distinguished triangle:

$$R\pi_{!!}\mu^{\mathrm{sa}}_\chi(F) \longrightarrow R\pi_*\mu^{\mathrm{sa}}_\chi(F) \longrightarrow R\dot{\pi}_*\mu^{\mathrm{sa}}_\chi(F) \xrightarrow{+1}.$$ ■

Now we construct Sato's triangle for multi-microlocal Hom functors. We replace X and χ with an ambient space $\widehat{X}$ and the universal family $\hat{\chi}$.

Convention 3.0.9. Since, if we let $i\colon T^*X\times_X\cdots\times_X T^*X\to S^*_{\hat{\chi}}$ be a canonical embedding, by [2, Lemma 2.18] we get

$$Ri_*i^{-1}\mu hom^{\mathrm{sa}}_{\hat{\chi}}(G_1,\dots,G_\ell;F)\simeq \mu hom^{\mathrm{sa}}_{\hat{\chi}}(G_1,\dots,G_\ell;F),$$

$\mu hom^{\mathrm{sa}}_{\hat{\chi}}(G_1,\dots,G_\ell;F)$ may be seen as an object on $T^*X\times_X\cdots\times_X T^*X$. We apply this convention throughout this paper.

Proposition 3.0.10. *Let $\pi\colon T^*X\times_X\cdots\times_X T^*X\to X$ be a canonical projection and $G_1,\dots,G_\ell,F\in\mathrm{Ob}(D^b(k_{X_{\mathrm{sa}}}))$. Then*

$$\begin{aligned}R\pi_*\mu hom^{\mathrm{sa}}_{\hat{\chi}}(G_1,\dots,G_\ell;F)&\simeq R\mathcal{H}om(G_1\boxtimes\cdots\boxtimes G_\ell,Ri_{\Delta *}F)\\&\simeq Ri_{\Delta *}R\mathcal{H}om(G_1\otimes\cdots\otimes G_\ell,F).\end{aligned}$$

In particular,

$$\mathrm{Hom}_{D^b(k_{X_{\mathrm{sa}}})}\left(\bigotimes_{i=1}^{\ell}G_i,F\right)\simeq H^0(T^*X\times_X\cdots\times_X T^*X;\mu hom^{\mathrm{sa}}_{\hat{\chi}}(G_1,\dots,G_\ell;F)).$$

Proof. We have the following chain of isomorphisms:

$$\begin{aligned}&R\pi_*\mu^{\mathrm{sa}}_{\hat{\chi}}R\mathcal{H}om(q_2^{-1}(G_1\boxtimes\cdots\boxtimes G_\ell),q_1^!Ri_{\Delta *}F)\\&\simeq\delta^!R\mathcal{H}om(q_2^{-1}(G_1\boxtimes\cdots\boxtimes G_\ell),q_1^!Ri_{\Delta *}F)\\&\simeq R\mathcal{H}om(\delta^{-1}q_2^{-1}(G_1\boxtimes\cdots\boxtimes G_\ell),\delta^!q_1^!Ri_{\Delta *}F)\\&\simeq R\mathcal{H}om(G_1\boxtimes\cdots\boxtimes G_\ell,Ri_{\Delta *}F)\\&\simeq Ri_{\Delta *}R\mathcal{H}om(G_1\otimes\cdots\otimes G_\ell,F),\end{aligned}$$

where δ is a diagonal embedding $X^\ell\to(X^2)^\ell$, $(x_1,\dots,x_\ell)\mapsto(x_1,x_1,x_2,x_2,\dots,x_\ell,x_\ell)$. The first isomorphism follows from Proposition 3.0.1. The second isomorphism follows from Proposition 1.1.4. ∎

Proposition 3.0.11. *Let $G_1,\dots,G_\ell\in D^b_{\mathbb{R}\text{-}c}(k_X)$ and $F\in D^b(k_{X_{\mathrm{sa}}})$. We obtain the following isomorphisms:*

$$\begin{aligned}R\pi_{!!}\mu hom^{\mathrm{sa}}_{\hat{\chi}}(G_1,G_2,\dots,G_\ell,F)&\simeq R\mathcal{H}om(G_1\boxtimes\cdots\boxtimes G_\ell,k_{X^\ell})\otimes Ri_{\Delta *}F\\&\simeq Ri_{\Delta *}(D'G_1\otimes\cdots\otimes D'G_\ell\otimes F),\end{aligned}$$

where we set $D'F:=R\mathcal{H}om(F,k_X)$ for $F\in D^b(k_{X_{\mathrm{sa}}})$.

Proof. We have the following isomorphisms by Proposition 3.0.7:

$$
\begin{aligned}
& R\pi_{!!}\mu hom^{\mathrm{sa}}_{\hat{\chi}}(G_1, G_2, \dots, G_\ell, F) \\
& \quad \simeq \delta^{-1}\big(R\mathcal{H}om(p_2^{-1}(G_1 \boxtimes \cdots \boxtimes G_\ell), p_1^! Ri_{\Delta *}F)\big) \otimes \omega_{X^\ell/\hat{X}} \\
& \quad \simeq \delta^{-1}(R\mathcal{H}om(G_1 \boxtimes \cdots \boxtimes G_\ell, k_{X^\ell}) \boxtimes Ri_{\Delta *}F) \quad (3.4) \\
& \quad \simeq R\mathcal{H}om(G_1 \boxtimes \cdots \boxtimes G_\ell, k_{X^\ell}) \otimes Ri_{\Delta *}F \\
& \quad \simeq Ri_{\Delta *}(D'G_1 \otimes \cdots \otimes D'G_\ell \otimes F). \quad (3.5)
\end{aligned}
$$

Here, $\delta\colon X^\ell \hookrightarrow (X^2)^\ell$ is a diagonal embedding. The isomorphism (3.4) follows from [9, Lemma 5.3.9], and (3.5) follows from [6, Proposition 3.4.4] and the projection formula. ∎

Corollary 3.0.12. *For $G_1, G_2, \dots, G_\ell \in D^b_{\mathbb{R}\text{-}c}(k_X)$ and $F \in D^b(k_{X_{\mathrm{sa}}})$, we have the following distinguished triangle:*

$$
\bigotimes_{i=1}^{\ell} D'G_i \otimes F \longrightarrow R\mathcal{H}om\left(\bigotimes_{i=1}^{\ell} G_i, F\right) \longrightarrow R\dot{\pi}_*\mu hom^{\mathrm{sa}}_{\hat{\chi}}(G_1, \dots, G_\ell; F) \xrightarrow{+1},
$$

where $\dot{\pi}$ is the restriction of $\pi\colon T^*X \times_X \cdots \times_X T^*X \to X$ to $T^*X \times_X \cdots \times_X T^*X \setminus X$.

Example 3.0.13. Let $X = \mathbb{C}^2$ and $\hat{\chi} = \{\Delta \times \mathbb{C}^2,\ \mathbb{C}^2 \times \Delta\}$. Consider closed subsets $M_1 = \{y_1 = 0,\ x_1 \geq 0\} = \mathbb{R}_{\geq 0} \times \mathbb{C}$ and $M_2 = \{y_2 = 0,\ x_2 \geq 0\} = \mathbb{C} \times \mathbb{R}_{\geq 0}$.

Since $D'(\mathbb{C}_{M_1}) \simeq \mathbb{C}_{\{y_1=0, x_1>0\}}[-1]$, $D'(\mathbb{C}_{M_2}) \simeq \mathbb{C}_{\{y_2=0, x_2>0\}}[-1]$ we obtain

$$
(R\mathcal{H}om(\mathbb{C}_{M_1 \cap M_2}, R\rho_*\mathcal{O}_X))_{(0,0)} \simeq (R\dot{\pi}_*\mu hom_{\hat{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}; R\rho_*\mathcal{O}_X))_{(0,0)}.
$$

Acknowledgments – We would like to express our greatest appreciation to Prof. Naofumi Honda for the valuable advice and generous support.

Funding – This work was partially supported by JST SPRING, Grant Number JPMJSP2119.

References

[1] N. Honda – L. Prelli, Multi-specialization and multi-asymptotic expansions. *Adv. Math.* **232** (2013), 432–498. Zbl 1262.32012 MR 2989990

[2] N. Honda – L. Prelli, *μhom* and multi-microlocal operators. [v1] 2024, [v2] 2025, arXiv:2404.06123v2.

[3] N. Honda – L. Prelli – S. Yamazaki, Multi-microlocalization and microsupport. *Bull. Soc. Math. France* **144** (2016), no. 3, 569–611. Zbl 1370.35023 MR 3558433

[4] M. Kashiwara – T. Kawai – T. Kimura, *Foundations of algebraic analysis*. Princeton Math. Ser. 37, Princeton University Press, Princeton, NJ, 1986. Zbl 0605.35001 MR 0855641

[5] M. Kashiwara – P. Schapira, Microlocal study of sheaves. *Astérisque* (1985), no. 128, 235 pp. Zbl 0589.32019 MR 0794557

[6] M. Kashiwara – P. Schapira, *Sheaves on manifolds*. Grundlehren Math. Wiss. 292, Springer, Berlin, 1990. Zbl 0709.18001 MR 1074006

[7] M. Kashiwara – P. Schapira, Ind-sheaves. *Astérisque* (2001), no. 271, 136 pp. Zbl 0993.32009 MR 1827714

[8] L. Prelli, Sheaves on subanalytic sites. *Rend. Semin. Mat. Univ. Padova* **120** (2008), 167–216. Zbl 1171.32002 MR 2492657

[9] L. Prelli, Microlocalization of subanalytic sheaves. *Mém. Soc. Math. Fr. (N.S.)* (2013), no. 135, vi+91 pp. Zbl 1295.32018 MR 3157166

[10] M. Sato – T. Kawai – M. Kashiwara, Microfunctions and pseudo-differential equations. In *Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau)*, pp. 265–529, Lecture Notes in Math. 287, Springer, Berlin-New York, 1973. Zbl 0277.46039 MR 0420735

[11] P. Schapira – K. Takeuchi, Déformation binormale et bispécialisation. *C. R. Acad. Sci. Paris Sér. I Math.* **319** (1994), no. 7, 707–712. Zbl 0817.58039 MR 1300074

[12] K. Takeuchi, Binormal deformation and bimicrolocalization. *Publ. Res. Inst. Math. Sci.* **32** (1996), no. 2, 277–322. Zbl 0873.58061 MR 1382804

[13] M. Uchida, A generalization of Bochner's tube theorem in elliptic boundary value problems. *Publ. Res. Inst. Math. Sci.* **31** (1995), no. 6, 1065–1077. Zbl 0874.58085 MR 1382567

Manoscritto pervenuto in redazione il 13 agosto 2024.