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On weakly N-subnormal subgroups of finite groups

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Abstract – Let G be a finite group and $A, N \leq G$. Then $A_{\mathrm{sn}\, G}$ is the subnormal core of A in G, that is, the subgroup of A generated by all subnormal subgroups of G, contained in A; $A^{\mathrm{sn}\, G}$ is the subnormal closure of A in G, that is, the intersection of all subnormal subgroups of G containing A. We say that a subgroup A of G is (i) N-subnormal in G if $N \cap A^{\mathrm{sn}\, G} = N \cap A_{\mathrm{sn}\, G}$; (ii) weakly N-subnormal in G if for some subnormal subgroup T of G we have $AT = G$ and $A \cap T \leq S \leq A$, where S is N-subnormal in G. In this paper, we consider some applications of these two concepts. In particular, we prove that a finite group G is soluble if and only if G has a normal subgroup N with soluble factor G/N such that in each maximal chain $M_3 < M_2 < M_1 < M_0 = G$ of G of length 3, at least one of the subgroups M_3, M_2 or M_1 is weakly N-subnormal in G.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group; $\mathcal{L}(G)$ is the lattice of all subgroups of G. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

Wielandt [14] proved that the set of all subnormal subgroups of G form a sublattice in the lattice of all subgroups of G and this important result allows us to associate with each subgroup A of G two subnormal subgroups of G: the *subnormal core* $A_{\mathrm{sn}\,G}$ *of* A in G, that is, the subgroup of A generated by all subnormal subgroups of G, contained in A, and the *subnormal closure* $A^{\mathrm{sn}\,G}$ *of* A in G, that is, the intersection of all subnormal subgroups of G containing A.

Based on these concepts, consider the following generalizations of subnormality.

Definition 1.1. Let A and N be subgroups of G. Then we say that A is

(i) N*-subnormal* in G if $N \cap A^{\mathrm{sn}\,G} = N \cap A_{\mathrm{sn}\,G}$;

(ii) *weakly* N*-subnormal* in G if for some subnormal subgroup T of G we have $AT = G$ and $A \cap T \leq S \leq A$, where S is N-subnormal in G.

Before continuing, consider the following examples.

Example 1.2. Let G be a group.

(i) If A is subnormal in G, then $A^{\mathrm{sn}\,G} = A = A_{\mathrm{sn}\,G}$ and so A is N-subnormal in G for every subgroup N of G.

(ii) Let p, q, r, s, t be pairwise distinct prime numbers, where r divides $q-1$ and t divides $s-1$. Let $H = C_q \rtimes C_r$ be a non-abelian group of order qr and $L = C_s \rtimes C_t$ a non-abelian group of order st.

Let $N = P \rtimes H$, where P is a faithful simple $\mathbb{F}_p H$-module, and $G = N \times L$. Let S be a subgroup of order p in P and $A = SC_t$. Then $S < P$ and A is not subnormal in G due to [2, Chapter A, Corollary 14.2] since $C_t = A \cap L$ is not subnormal in L. Therefore, $A_{\mathrm{sn}\,G} = S$. We have also that $A^{\mathrm{sn}\,G} = S^{\mathrm{sn}\,G} C_t^{\mathrm{sn}\,G} = SL$. Therefore,

$$N \cap A^{\mathrm{sn}\,G} = N \cap SL = S(N \cap L) = S = N \cap S = N \cap A_{\mathrm{sn}\,G}$$

and so A is N-subnormal in G.

(iii) If A is N-subnormal in G, then $GA = G$ and $A \cap G = A$ and so A is weakly N-subnormal in G.

(iv) For the subgroup C_r in part (ii) we have $N = N \cap C_r^{\mathrm{sn}\, G} \neq N \cap (C_r)_{\mathrm{sn}\, G} = 1$, so C_r is not N-subnormal in G. On the other hand, we have $C_r T = G$ and $C_r \cap T = 1$, where $T = PC_qL$, so C_r is weakly N-subnormal in G.

(v) If $N = 1$, then every subgroup of G is N-subnormal in G.

(vi) Recall that a subgroup A of G satisfies the *Ore supplement condition* in G [3,4,9] or A is c*-normal in* G [13] if G has normal subgroups T and S such that $G = AT$ and $A \cap T \leq S \leq A$.

A subgroup A of G is called c*-subnormal in* G [6] or *weakly* c*-normal in* G [16] if G has a subnormal subgroup T and a normal subgroup S such that $AT = G$ and $A \cap T \leq S \leq A$.

In view of part (i), every c-subnormal subgroup is weakly N-subnormal in G for every subgroup N of G.

Now we show that in the general case, a weakly N-subnormal subgroup is not c-subnormal in the group. Indeed, the subgroup A in part (ii) is N-subnormal and so weakly N-subnormal in G by part (iii). Assume that for some subnormal subgroup T of G we have $AT = G$ and $A \cap T \leq A_G = 1$. Then $|G : T| = pt$ and for the Sylow p-subgroup P of G and for the Sylow p-subgroup $T_p = P \cap T$ of T we have $P = ST_p$. On the other hand, $H \leq N \cap T < N$, where H is a maximal subgroup of N. Hence $H = N \cap T$, so $T_p = 1$, a contradiction. Therefore, A is not c-subnormal in G.

Various applications of N-subnormal subgroups have been discussed in the recent papers [7,8,12]. In this given paper, we consider applications of weakly N-subnormal subgroups and, in particular, we give two applications of such subgroups in the theory of groups with the Ore supplement condition for subgroups.

Recall that if

$$(*) \qquad M_n < M_{n-1} < \cdots < M_1 < M_0 = G,$$

where M_i is a maximal subgroup of M_{i-1} for all $i = 1, \ldots, n$, then the chain $(*)$ is called a *maximal chain of* G *of length* n and M_n is an n*-maximal* subgroup of G.

Our first goal here is to prove the following result.

THEOREM A. *A group G is soluble if and only if G has a normal subgroup N with soluble factor G/N such that in each maximal chain $M_3 < M_2 < M_1 < M_0 = G$ of G of length 3, at least one of the subgroups M_3, M_2 or M_1 is weakly N-subnormal in G.*

In view of Example 1.2 (i), (iii), we get from Theorem A the following corollary.

Corollary 1.3 (See [11, Theorem 6]). *If in each maximal chain $M_3 < M_2 < M_1 < M_0 = G$ of group G of length 3 at least one of the subgroups M_3, M_2 or M_1 is subnormal in G, then G is soluble.*

Note that if M is a maximal subgroup of a soluble group G and H/M_G is a chief factor of G, then $MH = G$ and $H \cap M \leq M_G$, so H is c-normal in G. Therefore, in view of Example 1.2 (vi), we get from Theorem A the following results.

Corollary 1.4. *A group G is soluble if and only if in each maximal chain $M_3 < M_2 < M_1 < M_0 = G$ of G of length 3 at least one of the subgroups M_3, M_2 or M_1 is c-subnormal in G.*

Corollary 1.5 (See [13, Theorem 3.1]). *A group G is soluble if and only if every maximal subgroup of G is c-normal in G.*

Corollary 1.6 (See [6, Theorem 2]). *A group G is soluble if and only if every maximal subgroup M is c-subnormal in G.*

In view of Example 1.2 (iii), from Theorem A we get the following known result.

Corollary 1.7 (See [7, Theorem B]). *A group G is soluble if and only if G has a normal subgroup N with soluble factor G/N such that in each maximal chain $M_3 < M_2 < M_1 < M_0 = G$ of G of length 3 at least one of the subgroups M_3, M_2 or M_1 is N-subnormal in G.*

We also prove the following two results.

Theorem B. *A group G is soluble if and only if G has a normal subgroup N with soluble factor G/N and a soluble maximal subgroup M such that $\pi(N) \subseteq \pi(M/M_G)$ and in each maximal chain $M_3 < M_2 < M_1 = M < M_0 = G$ of G of length 3 at least one of the subgroups M_3, M_2 or M_1 is weakly N-subnormal in G.*

Theorem C. *A group G is soluble if and only if G has a normal subgroup N with soluble factor G/N and a soluble maximal subgroup M such that in each maximal chain $M_2 < M_1 = M < M_0 = G$ of G of length 2 at least one of the subgroups M_2 or M_1 is weakly N-subnormal in G.*

Corollary 1.8. *A group G is soluble if and only if G has a soluble maximal subgroup M and such that in each maximal chain $M_2 < M_1 = M < M_0 = G$ of G of length 2 at least one of the subgroups M_2 or M_1 is c-subnormal in G.*

Corollary 1.9 (See [13, Theorem 3.4]). *A group G is soluble if and only if G has a soluble c-normal maximal subgroup.*

COROLLARY 1.10 (See [16, Theorem 3.3]). *A group G is soluble if and only if G has a soluble weakly c-normal maximal subgroup.*

2. Proof of Theorem A

The first lemma is a corollary of Wielandt's theorem on the lattice of subnormal subgroups of a group and [2, Lemmas 14.1 and 14.2, Chapter A].

LEMMA 2.1. *Let $R \leq A \leq E \leq G$ and $R \trianglelefteq G$. Then the following statements are true:*

(1) $A_{\mathrm{sn}\,G}$ *and* $A^{\mathrm{sn}\,G}$ *are subnormal subgroups of* G.

(2) $A_{\mathrm{sn}\,G} \leq E_{\mathrm{sn}\,E}$.

(3) $(A/R)_{\mathrm{sn}(G/R)} = A_{\mathrm{sn}\,G}/R$.

(4) $A^{\mathrm{sn}\,E} \leq A^{\mathrm{sn}\,G}$.

(5) $(A/R)^{\mathrm{sn}(G/R)} = A^{\mathrm{sn}\,G}/R$.

(6) *The intersection of any two subnormal subgroups of G is a subnormal subgroup of G.*

LEMMA 2.2. *Let A and N be subgroups of G and for a normal subgroup R of G we have either $R \leq N$ or $R \leq A$. If A is N-subnormal in G, then AR/R is (NR/R)-subnormal in G/R.*

PROOF. Suppose that $R \leq N$. Then

$$\begin{aligned}(NR/R) \cap (AR/R)^{\mathrm{sn}(G/R)} &= (N/R) \cap (A^{\mathrm{sn}\,G}R/R) = (N \cap A^{\mathrm{sn}\,G}R)/R \\ &= R(N \cap A^{\mathrm{sn}\,G})/R = R(N \cap A_{\mathrm{sn}\,G})/R\end{aligned}$$

by Lemma 2.1.

On the other hand,

$$\begin{aligned}R(N \cap A_{\mathrm{sn}\,G})/R \leq (N \cap (AR)_{\mathrm{sn}\,G})/R &= (N/R) \cap (AR)_{\mathrm{sn}\,G}/R \\ &= (N/R) \cap (AR/R)_{\mathrm{sn}(G/R)}.\end{aligned}$$

Therefore,

$$(NR/R) \cap (AR/R)^{\mathrm{sn}(G/R)} = (NR/R) \cap (AR/R)_{\mathrm{sn}(G/R)}.$$

If $R \leq A$, then similarly it is shown that

$$(NR/R) \cap (AR/R)^{\mathrm{sn}(G/R)} = (NR/R) \cap (A/R)_{\mathrm{sn}(G/R)}.$$

Therefore, AR/R is (NR/R)-subnormal in G/R. The lemma is proved. ∎

Lemma 2.3. *Let $R \leq N$ and A be subgroups of G, where R is normal in G and $R \leq A$. If A is weakly N-subnormal in G, then A is weakly R-subnormal in G and A/R weakly (N/R)-subnormal in G/R.*

Proof. Let T be a subnormal subgroup and S an N-subnormal subgroup of G such that $AT = G$ and $A \cap T \leq S \leq A$.

From $N \cap S^{\mathrm{sn}\,G} = N \cap S_{\mathrm{sn}\,G}$ it follows that $R \cap S^{\mathrm{sn}\,G} = R \cap N \cap S^{\mathrm{sn}\,G} = R \cap N \cap S_{\mathrm{sn}\,G} = R \cap S_{\mathrm{sn}\,G}$, that is, S is R-subnormal in G and so A is weakly R-subnormal in G.

Moreover, $(A/R)(TR/R) = G/R$ and

$$(A/R) \cap TR/R = (A \cap TR)/R = R(A \cap T)/R \leq RS/R \leq A/R,$$

where RS/R is (N/R)-subnormal in G by Lemma 2.2 and TR/R is subnormal in G/R by [2, Chapter A, Lemma 14.1]. Therefore, A/R is weakly (N/R)-subnormal in G/R. The lemma is proved. ■

Lemma 2.4. *Let $R \leq A \leq B \leq G$, where R is normal in G. If T is a subnormal complement to A in G, then TR/R is a subnormal complement to A/R in G/R and $T \cap B$ is a subnormal complement to A in B.*

Proof. We have $G/R = (A/R)(TR/R)$ and $(A/R) \cap (TR/R) = R(A \cap T)/R = R/R$, where TR/T is subnormal in G/R by [2, Chapter A, Lemma 14.1], so TR/R is a subnormal complement to A/R in G/R. Finally, $B = A(T \cap B)$, where $T \cap B$ is subnormal in B by [2, Chapter A, Corollary 14.2], so $T \cap B$ is a subnormal complement to A in B. The lemma is proved. ■

Lemma 2.5. *Let M be a maximal subgroup of G. If M is a Sylow 2-subgroup of G and every second maximal subgroup of M has a subnormal complement in G, then G is soluble.*

Proof. Assume that this lemma is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G.

First we show that G/R is soluble. Indeed, if $R \nleq M$, then $G/R = RM/R \simeq M/(M \cap R)$ is a 2-group. On the other hand, if $R \leq M$ and $|M/R|$ is not a prime, then the hypothesis holds for G/R by Lemma 2.4, so G/R is soluble by the choice of G. Therefore, R is the only minimal normal subgroup of G and R is a non-abelian group. Moreover, $|M| > 2^3$ by [5, Kapitel IV, Satz 7.4] and for a Sylow 2-subgroup $R \cap M = R_2$ of R we have $|R_2| > 2$ by [5, Kapitel IV, Satz 2.8, Satz 7.4] and the Feit–Thompson theorem on solubility of groups of odd order. It is clear also that there is a second maximal subgroup L of M such that $R_2 \cap L \neq 1$ since $|R_2| > 2$ and $|M| > 2^3$.

Let T be a subnormal complement to L in G. Then $|G : T|$ is a power of 2, so $O^2(T) = O^2(G)$ by [1, Lemma 1.1.11] and hence $O^2(T)$ is normal in G. Since $|G| = |G : T||T|$ is not a power of 2, $O^2(T) \neq 1$. Hence $R \leq O^2(T)$. Therefore, $1 < R_2 \cap L \leq T \cap L = 1$, a contradiction. The lemma is proved. ∎

Lemma 2.6. *Let A be a subnormal subgroup of G. If A is a p'-group (respectively, a p-soluble group, a p-nilpotent group), then A^G is a p'-group (respectively, a p-soluble group, a p-nilpotent group).*

Proof. Let $A = A_n \trianglelefteq A_{n-1} \trianglelefteq \cdots \trianglelefteq A_0 = G$. We prove the lemma by induction on n.

Assume that A is a p'-group. If $n = 1$, then $A \trianglelefteq G$, so $A \leq O_{p'}(G)$ and hence $A = A^G$ is a p'-group. Now assume that $n > 1$. Then $A \leq O_{p'}(A_1)$ by induction, where $O_{p'}(A_1)$ is a characteristic subgroup of A_1 and so $O_{p'}(A_1) \leq O_{p'}(G)$. Hence again we get that A^G is a p'-group.

If A is a p-soluble (respectively, a p-nilpotent) group, then similarly we can show that A^G is a p-soluble (respectively, a p-nilpotent) group. The lemma is proved. ∎

Lemma 2.7. *Let G be a group.*

(i) *If $G \neq 1$ has no maximal chain of length 3, then $|G|$ divides pq for some primes p and q.*

(ii) *If $G \neq 1$ has no maximal chain of length 4, then G' is nilpotent.*

Proof. (i) Let C_p be a subgroup of G of prime order p. If $C_p < M$ for some maximal subgroup M of G, then G has a maximal chain of length 3, contrary to the hypothesis. Hence either $C_p = G$ or C_p is a maximal subgroup of G. Therefore, G is soluble by [5, Kapitel IV, Satz 7.4] and either $|G| = p$ or $|G| = pq$ for some (maybe equal) primes p and q.

(ii) First we prove that G is soluble. Assume that this is false and let G_p be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Then $G_p \leq M$ for some maximal subgroup M of G.

If $|G_p| = p$, then G_p has a normal complement V in G [5, Kapitel IV, Satz 2.8] and V has no maximal chain of length 3. It follows that V is soluble by part (i), so G is soluble, a contradiction. Therefore, $|G_p| > p$. If $G_p = M$, then $|G_p| > p^2$ and so G has a maximal chain of length 4. Hence $|G_p| = p^2$, then G is soluble by [5, Kapitel IV, Satz 7.4], a contradiction. Therefore, G is soluble.

Now we show that G' is nilpotent. Assume that this is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G. Then R is not a maximal subgroup of G and G/R has no maximal chain of length 3, so R is a second maximal subgroup of G. Hence $(G/R)'$ is nilpotent by part (i). The choice of G implies that R is the only minimal normal subgroup of G and $R \nleq \Phi(G)$. It follows that $R = C_G(R) = F(G) < G'$ by [2, Chapter A, Theorem 15.6], so R is not cyclic. Then for a maximal subgroup V of R we have $1 < V < R < G' < G$, so G has a maximal chain of length 4. This contradiction completes the proof of assertion.

The lemma is proved. ■

Proof of Theorem A. First assume that G has a normal subgroup N with soluble factor G/N such that in each maximal chain $M_3 < M_2 < M_1 < M_0 = G$ of G of length 3, at least one of the subgroups M_3, M_2 or M_1 is weakly N-subnormal in G. We show that in this case G is soluble. Assume that this is false and let G be a counterexample of minimal order. Then $N \neq 1$ and 2 divides $|G|$ by the Feit–Thompson theorem on solubility of groups of odd order. Let R be a minimal normal subgroup of G contained in N.

(1) G/R is soluble and R is a non-abelian group. In particular, R is not p-soluble for every prime p dividing $|R|$.

Assume that G/R is not soluble. Then G/R has a maximal chain of length 3 by Lemma 2.7 (i). Now let $M_3/R < M_2/R < M_1/R < M_0/R = G/R$ be any maximal chain of G/R of length 3. Then $M_3 < M_2 < M_1 < M_0 = G$ is a maximal chain of G of length 3 and so for some $i > 0$ the subgroup M_i is weakly N-subnormal in G by hypothesis. Hence M_i/R is weakly (N/R)-subnormal in G/R by Lemma 2.3, where $(G/R)/(N/R) \simeq G/N$ is soluble. Therefore, the hypothesis holds for G/R, so G/R is soluble and R is a non-abelian group by the choice of G.

The Frattini argument implies that there is a maximal subgroup M such that $RM = G$ and $R_2 \leq G_2 \leq N_G(R_2) \leq M$ for some Sylow 2-subgroup R_2 of R and some Sylow 2-subgroups G_2 of G. Then $|G : M|$ is a $2'$-number.

(2) If V/M_G is a non-identity subnormal subgroup of G/M_G, then V/M_G is not soluble. In particular, V/M_G is not a $2'$-group.

Assume that V/M_G is soluble. Then $(V/M_G)^{G/M_G}$ is soluble by Lemma 2.6, so G/M_G has a soluble minimal normal subgroup K/M_G. Therefore, $R \simeq RM_G/M_G \simeq K/M_G$ by [2, Chapter A, Theorem 15.2] and so R is soluble, contrary to claim (1). Hence we have (2).

(3) For any subnormal subgroup V of G contained in M we have $V^G = V^{RM} = V^M \leq M_G$ (see [2, Chapter A, Lemma 14.3]).

(4) If $L \leq M$ and $L \nleq M_G$, then L is not R-subnormal in G.

Assume that $R \cap L^{\mathrm{sn}\, G} = R \cap L_{\mathrm{sn}\, G}$. Then $R \cap L_{\mathrm{sn}\, G}$ is a subnormal subgroup of G by Lemma 2.1 and so $R \cap L_{\mathrm{sn}\, G} \leq M_G$ by claim (3). If $R \cap L_{\mathrm{sn}\, G} \neq 1$, then $R = (R \cap L_{\mathrm{sn}\, G})^G \leq M_G$ and so $G = RM = M$, a contradiction. Therefore, $R \cap L^{\mathrm{sn}\, G} = R \cap L_{\mathrm{sn}\, G} = 1$, so $L^{\mathrm{sn}\, G}/1 \simeq (R \rtimes L^{\mathrm{sn}\, G})/R$ is a soluble subnormal subgroup of G by claim (1). Moreover, $L^{\mathrm{sn}\, G} \nleq M_G$ since $L \leq M$ and $L \nleq M_G$, hence $L^{\mathrm{sn}\, G} M_G/M_G \simeq L^{\mathrm{sn}\, G}/(L^{\mathrm{sn}\, G} \cap M_G)$ is a non-identity soluble subnormal subgroup of G/M_G, contrary to claim (2). Hence we have (4).

(5) If $L \leq M$ and $L \nleq M_G$ and L is weakly R-subnormal in G, then for some subnormal subgroup T of G we have $LT = G$ and $L \cap T \leq M_G$.

Let T be a subnormal subgroup of G such that $LT = G$ and $L \cap T \leq S \leq L$ for some R-subnormal subgroup S of G. Then claim (4) implies that $S \leq M_G$, so we have (5).

(6) G/M_G is not soluble. In particular, $M_G < M$ (since $R \cap M_G = 1$, this follows from claim (1) and the choice of G).

(7) For a Sylow 2-subgroup G_2M_G/M_G of M/M_G we have $|G_2M_G/M_G| > 2$.

Assume that $|G_2M_G/M_G| = 2$. Then for a Sylow 2-subgroup R_2M_G/M_G of $RM_G/M_G \simeq R$ we have $|R_2M_G/M_G| = 2$, so R is 2-nilpotent by [5, Kapitel IV, Satz 2.8], contrary to claim (1).

(8) G/M_G has a subgroup L/M_G such that L is weakly N-subnormal in G, $M_G < L \leq M$ and either $|G/M_G : L/M_G| = |G : L|$ is a $2'$-number or L/M_G is a maximal subgroup of G_2M_G/M_G.

If M is weakly N-subnormal in G, we can take $L/M_G = M/M_G$ since $|G/M_G : M/M_G| = |G : M|$ is a $2'$-number and $M_G < M$ by claim (6).

Now assume that M is not weakly R-subnormal in G. Then in every maximal chain $M_2 < M_1 < M_0 = M$ of M of length 2, for some $i > 0$ the subgroup M_i is weakly N-subnormal in G by hypothesis.

First suppose that $G_2M_G/M_G = M/M_G$. Then $|M/M_G| > 2^3$ by [5, Kapitel IV, Satz 7.4] since G/M_G is not soluble by claim (6). Lemma 2.5 and claim (5) imply that M/M_G has a maximal chain $1 < M_2/M_G < M_1/M_G < M_0/M_G = M/M_G$ of length 2 such that M_2 is not weakly N-subnormal in G. Hence M_1 is weakly N-subnormal in G, $M_G < M_1$ and $L/M_G = M_1/M_G$ is a maximal subgroup of $G_2M_G/M_G = M/M_G$.

If $G_2M_G/M_G = M_1/M_G$ is a maximal subgroup of M/M_G and M_2/M_G is a maximal subgroup of G_2M_G/M_G, then at least one of the subgroups M_1 or M_2 is weakly N-subnormal in G, M_1 and M_2 are not contained in M_G and $|G/M_G : M_1/M_G| = |G : M_1|$ is a $2'$-number.

Finally, if G_2M_G/M_G is a second maximal subgroup of M/M_G, then as above we can show that G/M_G has a subgroup $L/M_G \leq M/M_G$ such that L is weakly N-subnormal in G, $M_G < L \leq M$ and $|G/M_G : L/M_G| = |G : L|$ is a $2'$-number.

The final contradiction. In view of claim (8), G/M_G has a subgroup L/M_G such that L is weakly N-subnormal in G, $M_G < L \leq M$ and either $|G/M_G : L/M_G|$ is a $2'$-number or L/M_G is a maximal subgroup of G_2M_G/M_G.

Claim (5) implies that for some subnormal subgroup T of G we have $LT = G$ and $L \cap T \leq M_G$. Therefore, TM_G/M_G is a subnormal complement to L/M_G in G/M_G. Therefore, $|G/M_G : L/M_G| = |TM_G/M_G|$.

If $|G/M_G : L/M_G|$ is a $2'$-number, then TM_G/M_G is a non-identity subnormal $2'$-subgroup of G/M_G, contrary to claim (2). Hence for a Sylow 2-subgroup $(T/M_G)_2$ of T/M_G we have $|(T/M_G)_2| = 2$. Then T/M_G is 2-nilpotent by [5, Kapitel IV, Satz 2.8], contrary to claim (2). Therefore, G is soluble.

Finally, note that if G is soluble and $N = 1$, then every subgroup of G is weakly N-subnormal by Example 1.2 (v).

The theorem is proved. ■

3. Proofs of Theorems B and C

Lemma 3.1 (See [15, Lemma 2.8]). *If $G = R \rtimes M$, where M is a soluble maximal subgroup of G, then G is soluble.*

Lemma 3.2. *If L is a non-abelian minimal subnormal subgroup of G, then L^G is a minimal normal subgroup of G.*

Proof. Since every two perfect subnormal subgroups are permutable by [10, Chapter II, Theorem 7.9], for some $x_1, \ldots, x_t \in G$ we have $L^G = L^{x_1} \cdots L^{x_t}$. Now let R be a minimal normal subgroup of G contained in L^G. In view of [2, Chapter A, Lemma 14.3], $R \leq N_G(L^{x_i})$, so for some j we have $L^{x_j} \leq R$ since L^G and R are non-abelian groups. But then $L^G \leq R \leq L^G$, so $L^G \leq R$. The lemma is proved. ■

Lemma 3.3. *Let M be a nilpotent maximal subgroup of G. If in each maximal chain $M_2 < M_1 < M_0 = M$ of M of length 2 at least one of the subgroups M_2 or M_1 has a subnormal complement in G, then G is soluble.*

Proof. Assume that this lemma is false and let G be a counterexample of minimal order. Then $|M|$ is not odd by [5, Kapitel IV, Satz 7.4].

Let R be a minimal normal subgroup of G. Suppose first that $R \leq M$. Then $|M/R|$ is not a prime by [5, Kapitel IV, Satz 7.4]. By Lemma 2.4, the hypothesis holds for G/R, so G/R is soluble by the choice of G and, as R is also soluble, G is soluble, against the choice of G. We conclude that $R \not\leq M$. Then $G/R = RM/R \simeq M/(M \cap R)$ is nilpotent. Therefore, R is the only minimal normal subgroup of G and R is non-abelian. Hence $M_G = 1$ and 2 divides $|R|$ by the Feit–Thompson theorem on solubility of groups of odd order. It is also clear that M is a Hall subgroup of G.

First assume that M is not a 2-group. Then M has a second maximal subgroup L such that $|M : L| = 2p$ for some odd prime p dividing $|M|$. It follows that G has a subnormal subgroup T such that for a Sylow 2-subgroup T_2 of T we have $|T_2| \leq 2$. It follows that T is 2-nilpotent by [5, Kapitel IV, Satz 2.8], so T is soluble by the Feit–Thompson theorem on solubility of groups of odd order. Therefore, $1 < T^G$ is soluble by Lemma 2.6, so R is soluble, a contradiction.

Hence M is a Sylow 2-subgroup of G, so $|M| > 2^3$ by [5, Kapitel IV, Satz 7.4]. Hence for every second maximal subgroup V of M we have $|V| > 2$, so there is a maximal or a second maximal subgroup V of M such that $V \cap R \neq 1$. Let T be a subnormal complement to V in G. Then $|G : T|$ is a power of 2 and for a Sylow 2-subgroup T_2 of T we have $|T_2| \leq 4$. Moreover, in view of [1, Lemma 1.1.11], $1 \neq O^2(T) = O^2(G)$ is normal in G and so $R \leq O^2(T)$. It follows that $R \cap V = 1$, a contradiction. The lemma is proved. ∎

Lemma 3.4. *If in each maximal chain $M_2 < M_1 < M_0 = G$ of G of length 2 at least one of the subgroups M_2 or M_1 has a subnormal complement in G, then G is soluble and $|G : F(G)|$ divides a prime.*

Proof. Assume that this lemma is false and let G be a counterexample of minimal order. In view of Theorem A, G is soluble.

Assume that $|G : F(G)|$ is not a prime. Then G has a maximal chain $M_2 < M_1 < M_0 = G$ of length 2 such that $F(G) \leq M_2$, so for some $i > 0$ the subgroup M_i has a subnormal complement $T \neq 1$ in G. Then $1 < F(T) \leq F(G)$ by Lemma 2.6, so $1 < F(T) \leq M_i \cap T = 1$, a contradiction. The lemma is proved. ∎

Proof of Theorem B. It is enough to show that if G has a normal subgroup N with soluble factor G/N and a soluble maximal subgroup M such that $\pi(N) \subseteq \pi(M/M_G)$ and in each maximal chain $M_3 < M_2 < M_1 = M < M_0 = G$ of G of length 3 at least one of the subgroups M_3, M_2 or M_1 is weakly N-subnormal in G, then G is soluble.

Assume that this is false and let G be a counterexample of minimal order. Then $|\pi(N)| > 2$ by Burnside's $p^a q^b$-theorem and 2 divides $|G|$ by the Feit–Thompson theorem on solubility of groups of odd order. Then $|\pi(M/M_G)| > 2$, hence M/M_G has non-identity 2-maximal subgroups.

Let R be a minimal normal subgroup of G contained in N.

(1) $G/R = RM/R \simeq M/(M \cap R)$ is soluble and R is non-abelian group. In particular, 2 divides $|R|$.

First we show that $R \not\leq M$. Indeed, assume that $R \leq M$. Then $(G/R)/(N/R) \simeq G/R$ is soluble and

$$\begin{aligned}\pi(N/R) \subseteq \pi(N) \subseteq \pi(M/M_G) &= \pi((M/R)/(M_G/R)) \\ &= \pi((M/R)/(M/R)_{G/M_G}).\end{aligned}$$

Hence the hypothesis holds for G/R by Lemma 2.3, so G/R is soluble by the choice of G. But then G is soluble, a contradiction.

Therefore, $R \not\leq M$, so $G/R = RM/R \simeq M/(M \cap R)$ is soluble and R is a non-abelian group.

Let $\bar{G} = G/M_G$, $\bar{M} = M/M_G$, $\bar{R} = RM_G/M_G$ and $\bar{D} = \bar{R} \cap \bar{M}$.

(2) $\bar{G}$ is not soluble. In particular, $\bar{D} \neq 1$.

Since M is soluble and G is not soluble, $\bar{G} = G/M_G$ is not soluble.

If $\bar{D} = 1$, then $\bar{G} = \bar{R} \rtimes \bar{M}$, where $\bar{M} = M/M_G$ is a soluble maximal subgroup of $\bar{G}$, so $\bar{G}$ is soluble by Lemma 3.1, a contradiction. Hence $\bar{D} \neq 1$.

(3) For some prime p and for some minimal normal subgroup $\bar{N}$ of $\bar{M}$ we have $\bar{N} \leq O_p(\bar{M})$ and $\bar{M} = N_{\bar{G}}(\bar{N})$ (since $\bar{M}_{\bar{G}} = 1$, this follows from claims (1) and (2)).

(4) If V/M_G is a non-identity subnormal subgroup of G/M_G, then V/M_G is not soluble. In particular, V/M_G is not a $2'$-group (see (2) in the proof of Theorem A).

(5) For any subnormal subgroup V of G contained in M we have $V^G = V^{RM} = V^M \leq M_G$ (see (3) in the proof of Theorem A).

(6) If $L \leq M$ and $L \not\leq M_G$, then L is not R-subnormal in G (see (4) in the proof of Theorem A).

(7) If $L \leq M$ and $L \not\leq M_G$ and L is weakly R-subnormal in G, then for some subnormal subgroup T of G we have $LT = G$ and $L \cap T \leq M_G$. In particular, TM_G/M_G is a subnormal complement to L/M_G in G/M_G (see (5) in the proof of Theorem A).

(8) M is not weakly R-subnormal in G.

Assume that M is weakly R-subnormal in G, then $\bar{M}$ has a subnormal complement T in $\bar{G}$ by claim (7) since $M_G \neq M$ by claim (2).

Let A be a minimal subnormal subgroup of T. Then A is a minimal normal subgroup of $\bar{R} \simeq R$ by Lemma 3.2 and $\pi(A) = \pi(R) = \pi(\bar{R})$ since R is the direct product of isomorphic non-abelian simple groups. It follows that $N \leq N_{\bar{G}}(A)$ (see claim (3)). Let $E = A \rtimes N$. Then, in view of the Frattini argument, for some Sylow p-subgroup A_p

of A we have $A_pN = NA_p$, where $A_p \neq 1$ since $\pi(A) = \pi(R)$. Hence $N < N_E(N)$ and so $N_E(N) \nleq \bar{M}$ since $E \cap \bar{M} = N$, contrary to claim (3). Hence we have (8).

(9) If $1 < M_2/M_G < M_1/M_G < M_0/M_G = \bar{M}$ is a maximal chain of $\bar{M}$ of length 2, then at least one of the subgroups M_2/M_G or M_1/M_G has a subnormal complement in $\bar{G}$.

Since $M_2 < M_1 < M < M_0 = G$ is a maximal chain of G of length 3 and M is not weakly R-subnormal in G by claim (8), then for some $i > 0$ the subgroup M_i is weakly R-subnormal in G, so M_i/M_G has a subnormal complement in $\bar{G}$ by claim (7) since $M_G < M_i$.

(10) $\bar{M}$ is not nilpotent (this follows from claims (2), (9) and Lemma 3.3).

(11) $|\bar{M} : F(\bar{M})| = p$ is a prime and $F(\bar{M})$ has no subnormal complement in $\bar{G}$. Hence every maximal subgroup of $F(\bar{M})$ has a subnormal complement in $\bar{G}$.

From claims (9), (10) and Lemma 3.4, it follows that $F(\bar{M})$ is a normal maximal subgroup of $\bar{M}$, so $|\bar{M} : F(\bar{M})| = p$ for some prime p.

Now assume that $F(\bar{M})$ has a subnormal complement T in $\bar{G}$. It follows that $\bar{M} = F(\bar{M})(T \cap \bar{M})$, where $T \cap \bar{M}$ is a subnormal subgroup of prime order in $\bar{M}$ by Lemma 2.4, so $\bar{M}$ is nilpotent by Lemma 2.6, contrary to claim (10).

(12) $F(\bar{M})$ is a $2'$-group and $F(\bar{M})$ is a Hall subgroup of $\bar{G}$.

Assume that $p \in \pi(F(\bar{M}))$ and let V be a maximal subgroup of $F(\bar{M})$ such that $|F(\bar{M}) : V| = p$. Then V has a subnormal complement T in $\bar{G}$ by claim (11). Therefore, $T \cap \bar{M}$ is subnormal in $\bar{M}$ and $\bar{M} = V(\bar{M} \cap T)$, so $|\bar{M} \cap T| = |\bar{M} : V| = p^2$. Hence $\bar{M} \cap T$ is a nilpotent subnormal subgroup of $\bar{M}$ and so $\bar{M}$ is nilpotent by Lemma 2.6, contrary to claim (10). Hence $F(\bar{M})$ is a p'-group, so $F(\bar{M})$ is a Hall subgroup of $\bar{M}$.

Finally, assume that 2 divides $|F(\bar{M})|$. Then $2 \neq p$. Let V be a maximal subgroup of $F(\bar{M})$ such that $|F(\bar{M}) : V| = 2$. Then V has a subnormal complement T in $\bar{G}$, $|T \cap \bar{M}| = 2p$ and for a Sylow 2-subgroup $(T \cap \bar{M})_2$ of $T \cap \bar{M}$ we have $(T \cap \bar{M})_2 = F(\bar{M}) \cap T \cap \bar{M}$, so $T \cap \bar{M}$ is 2-closed and so it is nilpotent since $p > 2$. It follows that $\bar{M}$ is nilpotent, a contradiction. Hence $F(\bar{M})$ is a $2'$-group. Hence we have (12).

(13) $\bar{D}$ is not nilpotent.

Assume that $\bar{D}$ is nilpotent. Then $\bar{D} \leq F(\bar{M})$, so $\bar{D}$ is a non-identity $2'$-group by claims (2) and (12).

Let p be any prime dividing $|\bar{D}|$. Then the Sylow p-subgroup $\bar{D}_p$ of $\bar{D}$ is normal in $\bar{M}$ and $\bar{D}_p$ is a Sylow p-subgroup of $\bar{R}$ since $\bar{M}_{\bar{G}} = 1$. Therefore, $Z(J(\bar{D}_p))$ is normal in $\bar{M}$. Hence $N_{\bar{G}}(Z(J(\bar{D}_p))) = \bar{M}$ and so $N_{\bar{G}}(Z(J(\bar{D}_p))) \cap \bar{R} = \bar{M} \cap \bar{R} = \bar{D}$ is nilpotent. It follows that $R \simeq \bar{R}$ is p-nilpotent by the Glauberman–Thompson theorem about normal p-complements. Hence R is a p-group, contrary to claim (1). Thus we have (13).

The final contradiction. In view of claims (3), (11) and (13), $\bar{D} \cap F(\bar{M}) \neq 1$ and $D \nleq F(\bar{M})$.

Assume that $\bar{D} \cap F(\bar{M}) = F(\bar{M})$, so $F(\bar{M}) \leq \bar{D} \leq \bar{M}$, where $F(\bar{M})$ is maximal in $\bar{M}$. It follows, in view of claim (13), that $\bar{D} = \bar{M}$ and hence $\bar{M} \leq \bar{R}$, where $\bar{M}$ is not normal in $\bar{G}$, so $\bar{R} = \bar{G}$ is a simple group. Since $|\pi(\bar{M})| > 2$, $\bar{M}$ has a maximal chain $\overline{M_2} < \overline{M_1} < \overline{M_0} = \bar{M}$ of length 2, where $\overline{M_2} \neq 1$, so at least one of the subgroups $\overline{M_2}$ or $\overline{M_1}$ has a subnormal complement T in $\bar{G}$ by claim (9). But then $1 < T < \bar{G}$, so $\bar{G}$ is not simple. This contradiction shows that $\bar{D} \cap F(\bar{M}) < F(\bar{M})$.

Let V be a maximal subgroup of $F(\bar{M})$ containing $\bar{D} \cap F(\bar{M})$ and let T be a subnormal complement to V in $\bar{G}$. Then $(\bar{D} \cap F(\bar{M})) \cap T = 1$.

From claim (12) it follows that V is a $2'$-group, so $|\bar{R} : \bar{R} \cap T| = |\bar{R}T : T|$ is a $2'$-number. In view of [2, Chapter A, Corollary 14.2], $\bar{R} \cap T$ is subnormal and so normal in $\bar{R}$. Since $\bar{D} \cap F(\bar{M}) = \bar{R} \cap \bar{M} \cap F(\bar{M}) \neq 1$ and $T \cap V = 1$, $\bar{R} \nleq T$. Hence $\bar{R}/(\bar{R} \cap T)$ is a non-identity $2'$-group and so $\bar{R}' < \bar{R}$, where $\bar{R}'$ is characteristic in $\bar{R}$, so $\bar{R}'$ is normal in $\bar{G}$. Hence $\bar{R}' = 1$, so $R \simeq \bar{R}$ is abelian. This contradiction complements the proof of the result. ■

Proof of Theorem C. See the proof of Theorem B. ■

4. Final remarks, some open questions

(1) By the nilpotent length (denoted by $l(G)$) of a soluble group G we mean the length of the shortest normal chain of G with nilpotent factors. In particular, $l(G) = 1$ if $G \neq 1$ is nilpotent and $l(G) = 0$ if $G = 1$.

We do not know whether a group G in which all of its Sylow subgroups are weakly N-subnormal, where G/N is metanilpotent, is metanilpotent. This question is partially answered by the following result

Theorem 4.1. *A group G is soluble (respectively, G is a soluble group with the nilpotent length $l(G) \leq r$, $r \in \mathbb{N}$) if and only if the following two conditions hold:*

(i) *G has a normal subgroup N such that the factor G/N is soluble (respectively, a soluble group with $l(G) \leq r$) and*

(ii) *every Sylow subgroup of G is N-subnormal in G.*

First we prove the following lemma.

Lemma 4.2. *Let N be a normal subgroup of G with p-soluble factor G/N. If every Sylow p-subgroup of G is N-subnormal in G, then G is p-soluble.*

Proof. Assume that this lemma is false and let G be a counterexample of minimal order. Then p divides $|G|$ and $N \neq 1$.

Let R be a minimal normal subgroup of G contained in N and P a Sylow p-subgroup of G. Then $R \cap P^{\mathrm{sn}\,G} = R \cap P_{\mathrm{sn}\,G}$ by Lemma 2.3 and $R \cap P_{\mathrm{sn}\,G}$ is a subnormal p-subgroup of G by Lemma 2.1, so $V := R \cap P_{\mathrm{sn}\,G} \leq O_p(G)$ by Lemma 2.6. Assume that $V \neq 1$, so $R \leq O_p(G)$. In view of Lemma 2.3, the hypothesis holds for G/R, so G/R is p-soluble and hence G is p-soluble, a contradiction. Therefore, $R \cap P^{\mathrm{sn}\,G} = V = 1$, so $P^{\mathrm{sn}} \simeq P^{\mathrm{sn}\,G}R/R$ is p-soluble. Hence $(P^{\mathrm{sn}})^G$ is p-soluble by Lemma 2.6, where $G/(P^{\mathrm{sn}})^G$ is a p'-group, and so G is p-soluble, a contradiction. The lemma is proved. ■

Proof of Theorem 4.1. In view of Lemma 4.2, G is soluble and so it is enough to show that if G has a normal subgroup N such that the factor G/N is a soluble group with $l(G) \leq r$ and every Sylow subgroup of G is N-subnormal in G, then $l(G) \leq r$. Assume that this is false and let G be a counterexample of minimal order. Then $N \neq 1$. Let R be a minimal normal subgroup of G contained in N.

The hypothesis holds for G/R by Lemma 2.3, so $l(G/R) \leq r$. Moreover, $R \leq O_p(G) \leq P$ for some prime p, where P is a Sylow p-subgroup of G, hence $O_p(G/C_G(R)) = 1$ by [2, Chapter A, Lemma 13.6 (b)].

Now we show that $Q^x \leq C_G(R)$ for every $x \in G$ and every Sylow subgroup Q of G with $(|Q|, |P|) = 1$. First note that Q^x is R-subnormal in G by Lemma 2.3. Therefore, $R \cap (Q^x)^{\mathrm{sn}\,G} = R \cap (Q^x)_{\mathrm{sn}\,G} = 1$. On the other hand, since $(Q^x)^{\mathrm{sn}\,G}$ is subnormal in G by Lemma 2.1, we have $R \leq N_G((Q^x)^{\mathrm{sn}\,G})$ by [2, Chapter A, Lemma 14.3], Hence $R(Q^x)^{\mathrm{sn}\,G} = R \times (Q^x)^{\mathrm{sn}\,G}$, so $Q^x \leq C_G(R)$. Therefore, $Q^G \leq C_G(R)$. It follows that $G/C_G(R)$ is a p-group, so $C_G(R) = G$.

Let $F/R = F(G/R)$. Then $l(G/F) = l((G/R)(F/R)) \leq r - 1$ since $l(G/R) \leq r$. On the other hand, $F \leq F(G)$ since $R \leq Z(G) \cap F \leq Z(F)$. Therefore, $l(G/F(G)) \leq r - 1$ and hence $l(G) \leq r$, contrary to the choice of G. The theorem is proved. ■

(2) Theorems A, B and some fragments in their proofs are motivations for the following questions.

Question 4.3. What is the exact structure of G under the condition that in each maximal chain $M_3 < M_2 < M_1 = M < M_0 = G$ of G of length 3 at least one of the subgroups M_3, M_2 or M_1 has a subnormal complement in G?

Question 4.4. What is the exact structure of G under the condition that either every maximal subgroup or every second maximal subgroup or every third maximal subgroup of G has a subnormal complement in G?

Question 4.5. What is the exact structure of a non-nilpotent group G under the condition that every third maximal subgroup of G has a subnormal complement in G?

In the case when we are dealing either with maximal subgroups or with second maximal subgroups of a non-nilpotent groups, the answer to Question 4.4 gives the following result.

Theorem 4.6. *Let G be a group.*

(i) *Every maximal subgroup of G has a subnormal complement in G if and only if G is a nilpotent group with $\Omega_1(G_p) = G_p$ for every non-identity Sylow subgroup G_p of G.*

(ii) *Every second maximal subgroup of a non-nilpotent group G has a subnormal complement in G if and only if G is a non-abelian group of order $|G| = pq$ for some primes $p \neq q$.*

Proof. (i) First we prove that if every maximal subgroup of G has a subnormal complement in G, then G is nilpotent. Assume that this is false. Then G is soluble and $|G : F(G)|$ is a prime by Lemma 3.4. Hence the subgroup $F(G)$ has a subnormal complement $T \neq 1$ in G. Then $1 < F(T) \leq F(G)$ by Lemma 2.6, so $1 < F(T) \leq F(G) = F(G) \cap T = 1$, a contradiction. Hence G is nilpotent.

Now we show that $\Omega_1(G_p) = G_p$ for every non-identity Sylow subgroup G_p of G. Let V be a maximal subgroup of G_p and let $M = VG_q \times \cdots \times G_q$. Then M has a complement T in G, so $G_p \cap T$ is a complement to V in G_p by Lemma 2.4. Assume that $\Omega_1(G_p) \leq M$ for some maximal subgroup M of G_p. Then M has no complement in G_p, a contradiction. Hence $\Omega_1(G_p) = G_p$.

Finally, assume that G is a nilpotent group with $\Omega_1(G_p) = G_p$ for every non-identity Sylow subgroup G_p of G, and let M be a maximal subgroup of G such that $|G : M| = p$. Then $M = V \times G_q \times \cdots \times G_r$ for some maximal subgroup V of G_p, so $\Omega_1(G_p) \nleq V$, therefore for some subgroup C_p of G_p of order p we have $G_p = VC_p$ and so C_p is a complement to M in G. Hence every maximal subgroup of G has a complement in G. The assertion is proved.

(ii) First observe that if $L \leq M \leq G$ and T is a subnormal complement to L in G, then $M \cap T$ is a subnormal complement to L in M by Lemma 2.4. Therefore, every maximal subgroup of G is nilpotent by part (i). Therefore, in view of [3, Chapter 1, Proposition 1.9], $G = P \rtimes Q$, where $P = G^{\mathfrak{N}} = G'$ is a Sylow p-subgroup of G and $Q = \langle x \rangle$ is a cyclic Sylow q-subgroup of G.

Assume that $|Q| > q$. Let V be a second maximal subgroup of Q and $M = PV$. Then $|G : M| = |PQ : PV| = |Q : V| = q^2$, so for some subnormal subgroup T of G we have $G = VT$ and $V \cap T = 1$. Then $T \neq 1$ and $Q = V \times (Q \cap T)$. It follows that $V = 1$, so P is a second maximal subgroup of G, so Q is subnormal in G by hypothesis. But then $G = PQ$ is nilpotent, a contradiction. Hence $|Q| = q$. Then P is a maximal subgroup of G, so a maximal subgroup V of P has a subnormal complement T in G. Assume that $V \neq 1$, so $T < G$ and hence T is nilpotent. Then $G = PT$ is nilpotent by Lemma 2.6 since G is the product of two nilpotent subnormal subgroups P and T, a contradiction. Therefore, $V = 1$, so $|G| = pq$. The assertion is proved.

The theorem is proved. ■

From Theorem A we can also obtain some information about Question 4.3.

Corollary 4.7. *Suppose that in each maximal chain $M_3 < M_2 < M_1 = M < M_0 = G$ of G of length 3, at least one of the subgroups M_3, M_2 or M_1 has a subnormal complement in G. Then G is soluble and $|G : F(G)|$ divides pq for some primes p, q.*

Proof. In view of Theorem A, G is soluble. Now, using Lemma 2.7 and the proof of Lemma 3.4, we can show that $|G : F(G)|$ divides pq for some primes p, q. ■

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