Rend. Sem. Mat. Univ. Padova 156 (2026), 179–185
DOI 10.4171/RSMUP/179

Uncountable groups in which all large subgroups are permutably complemented

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Abstract – A subgroup H of a group G is permutably complemented if it admits a permutable complement, that is, a subgroup K of G such that $G = HK$ and $H \cap K = \{1\}$. A group G in which every subgroup is permutably complemented is said to be a C-group. The main purpose of this paper is to investigate the behaviour of uncountable groups of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are permutably complemented, with the particular aim of identifying sufficient conditions for such groups to be C-groups. Finally, in such a context, uncountable soluble groups of cardinality $\aleph$ are considered by restricting attention to only subnormal subgroups of cardinality $\aleph$ whose subnormal defect is at most 2.

Mathematics Subject Classification 2020 – 20F19 (primary); 20E15 (secondary).

Keywords – uncountable group, large subgroup, complemented subgroup.

1. Introduction

A subgroup H of a group G is *complemented* if it admits a complement, that is, a subgroup K of G such that $G = \langle H, K \rangle$ and $H \cap K = \{1\}$. If H is a complemented subgroup of a group G and there exists a complement K to H such that $HK = KH$, then H is said to be *permutably complemented*. A group G is called a C-group if all its subgroups are permutably complemented. The structure of C-groups has been completely described. Indeed, [7, Theorem 3.2.5] states that a C-group is the semidirect product of the group $A = \mathrm{Dr}_{i \in I}\, A_i$ by the group $B = \mathrm{Dr}_{j \in J}\, B_j$, where all factors A_i

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and B_j are finite of prime order and, for each $i \in I$, the subgroup A_i is G-invariant. In particular, an abelian C-group is the direct product of elementary abelian groups.

There are many papers addressing groups that admit a particular subset of permutably complemented subgroups (see, for instance, [3,4]), and the survey [5] describes these kinds of results, often generalizing some of them.

A group G is said to have finite rank if there exists a positive integer r such that every finitely generated subgroup of G can be generated by at most r elements; if such an r does not exist, we say that the group G has infinite rank. In a long series of papers, it has been shown that the structure of a (soluble) group of infinite rank is strongly influenced by that of its proper subgroups of infinite rank.

In [5], periodic locally soluble groups of infinite rank in which all subgroups of infinite rank are permutably complemented are analyzed. Although it remains unclear whether such groups are always C-groups, it has been proved that when the commutator subgroup has infinite rank, all subgroups are permutably complemented.

Results of the above type suggest that the behaviour of *small* subgroups in a *large* group can be neglected, at least for the right choice of the definition of largeness (and within a suitable universe). Indeed, this perspective has been adopted in a series of recent papers, where it has been shown that the behaviour of the subgroups of cardinality $\aleph$ of an uncountable group of cardinality $\aleph$ has a strong influence on the structure of the whole group (see, for instance, [1,2]). In both of these papers, the structure of uncountable groups of cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are normal was investigated.

The aim of this paper is to further contribute to the study of the influence of the behaviour of *large* subgroups on the structure of a group by considering uncountable groups of cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ (or a subset of them) are permutably complemented. Throughout this paper, $\aleph$ will denote an uncountable regular cardinal number, and if G is a group of cardinality $\aleph$, a subgroup H of G will be called *large* if it has likewise cardinality $\aleph$, and *small* otherwise. The class of all groups of cardinality $\aleph$ whose large subgroups are permutably complemented (and the trivial group) will be denoted with $\mathcal{C}(\aleph)$.

Most of our notation is standard and can be found in [6].

2. Groups with permutably complemented large subgroups

The first part of this section is devoted to the study of $\mathcal{C}(\aleph)$-groups under the specific condition of containing a large abelian normal subgroup. Here, the structure of K-groups, i.e., groups in which every subgroup is complemented, plays a key role. Most of the results regarding this class of groups can be found in [7].

Subsequently, we prove a result for uncountable groups that is analogous to the one presented in [5] and that we previously cited in the introduction. An essential step towards this objective will consist in examining the case of $\mathfrak{C}(\aleph)$-groups with small commutator subgroup.

In the last part of the section, $C(\aleph)$-groups with a large Hirsch–Plotkin radical are considered. In particular, we will show that such groups are actually C-groups since they have all subgroups permutably complemented.

Lemma 2.1. *Let G be an abelian $\mathfrak{C}(\aleph)$-group. Then G is a C-group.*

Proof. Since G contains a direct product of large subgroups $A_1 \times A_2$, it is isomorphic to a subgroup of $G/A_1 \times G/A_2$. The statement follows from [7, Lemma 3.2.1]. ∎

Lemma 2.2. *Let G be a group of cardinality $\aleph$ and let A be a large abelian normal subgroup of G whose large subgroups admit a permutable complement in G. Then A is generated by cyclic G-invariant groups of prime order.*

Proof. Since each large subgroup of A is permutably complemented in G, it has a complement in A, so Lemma 2.1 ensures that A is the direct product of elementary abelian groups. Let H be a subgroup of A which is maximal with respect to being generated by G-invariant cyclic subgroups. Suppose that $H < A$. Then there is a proper subgroup U of A containing H and having prime index in A. If V is a permutable complement of U in G, then $V \cap A$ is a cyclic G-invariant subgroup of prime order, so that $H \times (V \cap A)$ is generated by G-invariant cyclic subgroups, contradicting the maximality of H. So A is generated by cyclic G-invariant subgroups. If D is a cyclic G-invariant subgroup of A, it is clear that D is a direct product of G-invariant subgroups of prime order, so we have the statement. ∎

Now we are in the position to prove that a $\mathfrak{C}(\aleph)$-group is actually a C-group, provided that it contains a large abelian normal subgroup.

Lemma 2.3. *Let G be a $\mathfrak{C}(\aleph)$-group. If G contains a large abelian normal subgroup, then G is a C-group.*

Proof. Let A be a large abelian G-invariant subgroup. It is clear that G/A is a C-group. Moreover, Lemma 2.2 yields that A is generated by G-invariant cyclic subgroups of prime order, and it follows from [7, Lemma 3.1.9] that G is a K-group. Since G/A is hypercyclic by [7, Theorem 3.2.5], the same result ensures that G is a C-group. ∎

A group is said to be a (Hall) FK-group if every finite subgroup is (permutably) complemented. In [3], these types of groups have been thoroughly studied and the same paper provides several characterizations of periodic Hall FK-groups (cf. Theorems 3.4 and 3.5). In the following lemma, it is proved that a $C(\aleph)$-group whose commutator subgroup is small is a periodic Hall FK-group and therefore it is residually a finite C-group (see [3, Theorem 3.5]).

Proposition 2.1. *Let G be a $\mathcal{C}(\aleph)$-group with small commutator subgroup G'. Then*

(a) *G is locally a C-group; in particular, G is a periodic Hall FK-group;*

(b) *G is residually finite;*

(c) *all locally nilpotent subgroups of G are (abelian) C-groups; specifically, the Fitting radical $F(G)$ of G is abelian;*

(d) *all subgroups of $Z(G)$ are (permutably) complemented in G.*

Proof. Since G' is small, we have that for every $x \in G$, $|G : C_G(x)| < \aleph$. Let $E = \langle x_1, x_2, \ldots, x_n \rangle$ be a finitely generated subgroup of G; then

$$C_G(E) = \bigcap_{i=1}^{n} C_G(x_i)$$

and $|G : C_G(E)| < \aleph$. Furthermore, G/G' contains $A_1/G' \times A_2/G'$, where each A_i is a G-invariant subgroup of cardinality $\aleph$. If B is a permutable complement to A_1, for instance, then B is a large abelian subgroup of G and $|B : C_B(E)| < \aleph$. It follows that $C_B(E)$ admits a large subgroup U such that $U \cap E = \{1\}$ and UE/U is isomorphic to E. Thus, E is a C-group and G is a locally finite metabelian group. Corollary 5.12 of [5] guarantees that all finite subgroups of G are permutably complemented and G is residually finite, so that (a) and (b) are proved. Recalling that, for any positive prime p, a p-group with all subgroups permutably complemented is abelian, we have (c). Finally, (d) stems from (a) and [5, Theorem 5.28]. ■

Theorem 2.1. *Let G be a $\mathcal{C}(\aleph)$-group with large commutator subgroup G'. Then, if G'' has no large simple sections, G is a C-group.*

Proof. Assume that G'' is large. We have that $G/G^{(3)}$ is a C-group, in particular it is metabelian and G'' is perfect. Let H be a large normal subgroup of G'', G''/H is metabelian and so $G'' = H$. It follows that G'' has no large proper normal subgroups. Let us consider the set $\mathcal{L}$ of all proper normal subgroups of G'' ordered by inclusion and let $\mathcal{E}$ be a totally ordered part of $\mathcal{L}$. Put

$$\bar{N} = \bigcup_{N \in \mathcal{E}} N.$$

Since, $\bar{N}$ is G''-invariant, we have that either $\bar{N} = G''$ or $\bar{N}$ is small. If $\bar{N} = G''$, for every $x \in G''$, there exists $N_x \in \mathcal{E}$ such that $x \in N_x$. It follows that $\langle x \rangle^{G''}$ is small and so also the index $|G'' : G_{G''}(x)|$ is strictly smaller than $\aleph$. Thus, for every finitely generated subgroup E of G'', a similar argument to the one used in Proposition 2.1 ensures that $|G'' : C_{G''}(E)| < \aleph$. Furthermore, the C-group $C_{G''}(E)E/C_{G''}(E)$ is isomorphic to $E/Z(E)$, so that $E^{(3)}$ is trivial and G is soluble of derived length at most 3, although G'' is perfect. Such a contradiction tells us that $\bar{N}$ is a small subgroup and the ordered set $\mathcal{L}$ is inductive. Now, if M is a maximal element of $\mathcal{L}$, then G''/M is a large simple section of G, contradicting the assumption on G''. As a consequence, we have that G'' must be small. By Proposition 2.1 (a), G is locally finite. Let F be a finite subgroup of G, and let us consider $\bar{F} = G'F$ in which G' is a finite index subgroup with small commutator subgroup. It follows from Proposition 2.1 (b) that $\bar{F}$ is residually finite, so that $\bar{F}''$ is trivial and, in particular, F' is abelian. Thus, G' is abelian and so G is a C-group by Lemma 2.3. ■

Recalling that in a locally soluble group all simple sections have prime order, Theorem 2.1 has the following consequences.

Corollary 2.1. *Let G be a locally soluble $\mathfrak{C}(\aleph)$-group with large commutator subgroup G'. Then G is a C-group.*

Corollary 2.2. *Let G be a locally nilpotent $\mathfrak{C}(\aleph)$-group. Then G is an abelian C-group.*

Proof. Theorem 2.1 tells us that G' must be small so that we can apply Proposition 2.1 (c) yielding that G is an abelian C-group. ■

Corollary 2.3. *Let G be a $\mathfrak{C}(\aleph)$-group with large Hirsch–Plotkin radical. Then G is a C-group. In particular, in a $\mathfrak{C}(\aleph)$-group containing a large ascendant finite-by-(locally nilpotent) subgroup, all subgroups are permutably complemented.*

Proof. The first part of the statement is a direct consequence of Corollary 2.2 and Lemma 2.3. Now, if N is a large finite-by-(locally nilpotent) ascendant subgroup of G, actually it is finite-by-abelian by Corollary 2.2, so N' is finite. By Hall's theorem, in addition the index $|N : Z_2(N)|$ is finite, so that Corollary 2.2 ensures that $Z_2(N)$ is a large abelian ascendant subgroup of G. Thus, the Hirsch–Plotkin radical of G is large. ■

Corollary 2.4. *Let G be a $\mathfrak{C}(\aleph)$-group with finite commutator subgroup G'. Then G is a C-group.*

A group G is said to be an FC-group if any element of G has finitely many conjugates in G, or equivalently, if for every element x of G, the index $|G : C_G(x)|$ is finite. Many fundamental results on the class of FC-groups are collected in [8].

Recall that a group G is called *locally normal* if every finitely generated subgroup is contained in a finite normal subgroup. Theorem 1.8 of [8] guarantees that locally normal groups are precisely periodic FC-groups.

Proposition 2.2. *Let G be a $\mathfrak{C}(\aleph)$-group. If G is an FC-group, then G is a C-group.*

Proof. Let T be the torsion subgroup of G. We have that G/T is torsion-free and abelian, so that G/T has to be both torsion-free and a C-group. It follows that G is periodic so, in particular, it is locally normal. Assume first that G' is small, thus G is residually a C-group by Proposition 2.1 (a), so that [5, Theorem 2.1] ensures that G is a C-group. Let G' be large. If G'' is large, we have that it is also perfect by a similar argument to the one used in Theorem 2.1 and, for each finite normal subgroup N of G, $G'' = C_G(N)''$. It follows that G'' is a central subgroup, which is a contradiction. So G'' has to be small and the statement stems from an application of Theorem 2.1. ■

3. Soluble groups with some permutably complemented large subgroups

In this short section, it will turn out that, as for soluble groups with all subgroups permutably complemented (see, for instance, [7, Theorem 3.2.6], due to Kochendörfer), to prove that a soluble group G of cardinality $\aleph$ admitting a large abelian normal subgroup is a C-group, it is not necessary to require that every large subgroup of G is permutably complemented; it suffices to consider all large subnormal subgroups of defect at most 2.

Theorem 3.1. *Let G be a soluble group of cardinality $\aleph$ such that every large subnormal subgroup of defect at most* 2 *is permutably complemented. If G contains a large abelian normal subgroup, then it is a C-group.*

Proof. Let A be a large abelian G-invariant subgroup. It is clear that G/A is a C-group. Since each large subgroup of A is subnormal in G with defect at most 2, Lemma 2.1 yields that A is generated by G-invariant cyclic subgroups of prime order. It follows from [7, Lemma 3.1.9] that G is a K-group. Since G/A is hypercyclic by [7, Theorem 3.2.5], the same result ensures that G is a C-group. ■

In the universe of nilpotent groups, we will show that in order to prove that an uncountable group of cardinality $\aleph$ is a C-group, it is enough to look at large subgroups whose defect of subnormality is at most 2, even without the assumption of containing a large abelian normal subgroup.

THEOREM 3.2. *Let G be a nilpotent group of cardinality $\aleph$ such that every large subnormal subgroup of defect at most 2 is permutably complemented. Then G is a C-group. In particular, G is abelian.*

PROOF. Assuming first that the centre $Z(G)$ is large, the statement holds as a consequence of Theorem 3.1. Now, let $Z(G)$ be small and, by induction on the nilpotency class of G, assume that $G/Z(G)$ is a C-group. We have that G is nilpotent of class at most 2 so that each large subgroup of G is permutably complemented and G' is small. The statement follows from Proposition 2.1 (c). ∎

ACKNOWLEDGEMENTS – The author is a member of GNSAGA (INdAM) and AGTA – Advances in Group Theory and Applications (http://www.advgrouptheory.com).

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Manoscritto pervenuto in redazione il 5 gennaio 2025.