

Explicit bounds of $|\zeta(1 + it)|$

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ABSTRACT – In this work, we study a bound of the form $|\zeta(1 + it)| \leq v \log t$ for $t \geq t_0$. We show that the exponential sum method with second-order derivatives can achieve any $v > \frac{1}{2}$ as long as t_0 is sufficiently large. Using the Riemann–Siegel formula and numerical computations, we show that when $t \geq e$, $|\zeta(1 + it)| \leq \frac{1}{2} \log t + 0.6633$. This allows us to show that $|\zeta(1 + it)| \leq 0.6443 \log t$ when $t \geq e$. This is the best possible result of the form $|\zeta(1 + it)| \leq v \log t$ that holds for all $t \geq e$, as the equality is achieved when $t = 17.7477$.

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1. Introduction

The Riemann zeta function $\zeta(s)$ is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

when $s = \sigma + it$ is a complex number with $\operatorname{Re} s > 1$. It has an analytic continuation to the complex plane $\mathbb{C}$ except for a simple pole at $s = 1$. For more details on the Riemann zeta function, one can refer to the books [7, 13, 24].

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It is a classical result that $\zeta(1 + it) \neq 0$ for any real number $t \neq 0$. In this work, we consider upper bounds for $|\zeta(1 + it)|$. This has been a problem of interest for more than 100 years. It was first shown by Mellin [19] that

$$\zeta(1 + it) = O(\log t).$$

Weyl [27] proved that

$$\zeta(1 + it) = O\left(\frac{\log t}{\log \log t}\right).$$

Using an improved form of his own mean value theorem, Vinogradov [26] showed that

$$\zeta(1 + it) = O((\log t)^{\frac{2}{3}}).$$

If we believe that the Riemann hypothesis is true, the order is much smaller. Littlewood [18] proved that

$$\zeta(1 + it) = O(\log \log t)$$

if the Riemann hypothesis is true (see also [24, Section 14]). For unconditional explicit upper bounds, Landau [15] showed that

$$|\zeta(1 + it)| \leq 2 \log t \quad \text{when } t \geq 10.$$

Later, Backlund [4] proved that

$$|\zeta(1 + it)| \leq \log t \quad \text{when } t \geq 50.$$

In [8], Ford refined Vinogradov's method and showed that

$$|\zeta(1 + it)| \leq 76.2(\log t)^{\frac{2}{3}} \quad \text{when } t \geq 3.$$

In fact, he obtained a bound for $|\zeta(\sigma + it)|$ for all $\frac{1}{2} \leq \sigma \leq 1$ and $t \geq 3$. Trudgian [25] noticed that the constant 76.2 can be reduced to 62.6 when $\sigma = 1$. In the same paper [25], Trudgian claimed that

$$|\zeta(1 + it)| \leq \frac{3}{4} \log t \quad \text{when } t \geq 3.$$

Unfortunately, the result was obtained using an incorrect lemma in [6]. In [21], Patel pointed out the error and showed that

$$|\zeta(1 + it)| \leq \min\left\{\log t, \frac{1}{2} \log t + 1.93, \frac{1}{5} \log t + 44.02\right\}.$$

Notice that

$$\log t \geq \frac{1}{2} \log t + 1.93$$

if and only if $t \geq 47.47$, and

$$\frac{1}{2} \log t + 1.93 \geq \frac{1}{5} \log t + 44.02$$

if and only if $t \geq 8.54 \times 10^{60}$. Thus, the bound

$$|\zeta(1 + it)| \leq \frac{1}{5} \log t + 44.02$$

might not be that useful from the point of view of applications.

In this work, we first consider the best upper bound we can obtain for $|\zeta(1 + it)|$ using only the representation of the Riemann zeta function (see Section 2)

$$\begin{aligned} \zeta(s) &= \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + B_1(\{x\})x^{-s} \\ &\quad + \frac{s}{2} B_2(\{x\})x^{-s-1} - \frac{s(s+1)}{2} \int_x^\infty \frac{B_2(\{u\})}{u^{s+2}} du \end{aligned}$$

and the exponential sum methods. We obtain bounds of the form

$$|\zeta(1 + it)| \leq v \log t \quad \text{when } t \geq t_0$$

for various v and t_0 , with $v > \frac{1}{2}$ and t_0 sufficiently large. When $t_0 = 10^6$, the best we can obtain is

$$|\zeta(1 + it)| \leq 0.7421 \log t \quad \text{when } t \geq 10^6.$$

Then numerical computations show that the same still holds when $e \leq t \leq 10^6$. In fact, numerical computations show that when $e \leq t \leq 10^6$, $|\zeta(1 + it)|/\log t$ achieves the maximum value 0.6443 when $t = 17.7477$. In principle, we can show that $|\zeta(1 + it)| \leq 0.6443 \log t$ holds for all $t \geq e$ if we can perform numerical computations of $\zeta(1 + it)$ for $t \leq 10^{10}$. Due to the limitation of our computer, we prefer to use a different approach. Using the representation of the Riemann zeta function provided by the Riemann–Siegel formula, we can show that

$$|\zeta(1 + it)| \leq \frac{1}{2} \log t + 0.6633 \quad \text{when } t \geq 10^6.$$

Numerical computations show that this inequality still holds when $e \leq t \leq 10^6$. In other words, we find that

$$|\zeta(1 + it)| \leq \frac{1}{2} \log t + 0.6633 \quad \text{when } t \geq e.$$

Using this result, together with numerical computations, we obtain

$$\begin{aligned} |\zeta(1+it)| &\leq 0.6443 \log t \quad \text{when } t \geq e, \\ |\zeta(1+it)| &\leq 0.5480 \log t \quad \text{when } t \geq 652.3704. \end{aligned}$$

The following theorem summarizes the main results of this work.

THEOREM 1.1. *For $t \geq e$, we have*

$$(1.1) \quad |\zeta(1+it)| \leq \frac{1}{2} \log t + 0.6633$$

and

$$(1.2) \quad |\zeta(1+it)| \leq 0.6443 \log t.$$

The result (1.2) is the best possible for a bound of the form $|\zeta(1+it)| \leq v \log t$ that holds for all $t \geq e$. When $t \geq 100$, the bound (1.1) is better than the bound (1.2). When $t \geq 652.3704$, we can obtain a better bound of the form $|\zeta(1+it)| \leq v \log t$, which says that

$$(1.3) \quad |\zeta(1+it)| \leq 0.5480 \log t \quad \text{when } t \geq 652.3704.$$

When $652.3704 \leq t \leq 10^6$, the bound (1.3) is better than the bound (1.1).

In [11], it was shown that

$$|\zeta(1+it)| \leq 1.7310 \frac{\log t}{\log \log t} \quad \text{for all } t \geq 3.$$

When $t \leq 6.05 \times 10^{11}$,

$$\frac{1}{2} \log t + 0.6633 \leq 1.7310 \frac{\log t}{\log \log t}.$$

Thus, for $t \leq 6.05 \times 10^{11}$, our results are still better than that obtained in [11].

2. Classical representation of the Riemann zeta function

In this section we refine a classical formula that can be used to compute $\zeta(1+it)$. Given a real number x , we let $\lfloor x \rfloor$ denote the floor of x , and let $\{x\} = x - \lfloor x \rfloor$. Then we have

$$x - 1 < \lfloor x \rfloor \leq x \quad \text{and} \quad 0 \leq \{x\} < 1.$$

The first and second Bernoulli polynomials are given respectively by

$$B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}.$$

Using the Euler–Maclaurin summation formula (see for example [13, 20]), one has the following integral representation of the Riemann zeta function $\zeta(s)$ for $\operatorname{Re} s > -1$:

$$\begin{aligned} \zeta(s) = & \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + B_1(\{x\})x^{-s} + \frac{s}{2}B_2(\{x\})x^{-s-1} \\ & - \frac{s(s+1)}{2} \int_x^\infty \frac{B_2(\{u\})}{u^{s+2}} du. \end{aligned}$$

By choosing x large enough, the first line can be used to give an approximation of $\zeta(1 + it)$ with the error given by the integral in the second line. To obtain a smaller error bound, we replace the polynomial $B_2(x) = x^2 - x + \frac{1}{6}$ with the polynomial $h(x) = x^2 - x + \frac{1}{8}$. Since $B_2(x) - h(x) = \frac{1}{24}$ is a constant, and

$$\frac{s(s+1)}{2} \int_x^\infty \frac{1}{u^{s+2}} du = \frac{s}{2}x^{-s-1},$$

we obtain the following integral representation.

THEOREM 2.1. *Let $s = \sigma + it$ and let $h(x) = x^2 - x + \frac{1}{8}$. If $s \neq 1$, $\sigma > -1$, then for any $x \geq 1$, we have*

$$\begin{aligned} \zeta(s) = & \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + \left(\{x\} - \frac{1}{2}\right)x^{-s} + \frac{s}{2}h(\{x\})x^{-s-1} \\ (2.1) \quad & - \frac{s(s+1)}{2} \int_x^\infty \frac{h(\{u\})}{u^{s+2}} du. \end{aligned}$$

The choice of $h(x)$ over $B_2(x)$ is due to the fact that

$$\max_{0 \leq x \leq 1} |B_2(x)| = \frac{1}{6},$$

while

$$\max_{0 \leq x \leq 1} |h(x)| = \frac{1}{8}.$$

Using formula (2.1), we obtain the following.

THEOREM 2.2. *For $t > 0$ and N a positive integer, let*

$$(2.2) \quad g_N(t) = \sum_{n=1}^N \frac{1}{n^{1+it}} + \frac{N^{-it}}{it} - \frac{1}{2}N^{-1-it} + \frac{1+it}{16}N^{-2-it}.$$

Then

$$|\zeta(1+it) - g_N(t)| \leq \frac{(1+t)(2+t)}{32N^2}.$$

PROOF. From (2.1), we find that when $t > 0$ and N is a positive integer, we have

$$\begin{aligned} |\zeta(1+it) - g_N(t)| &\leq \left| \frac{(1+it)(2+it)}{2} \right| \int_N^\infty \frac{|h(\{u\})|}{|u^{3+it}|} du \\ &\leq \frac{(1+t)(2+t)}{16} \int_N^\infty \frac{du}{u^3} \\ &= \frac{(1+t)(2+t)}{32N^2}. \end{aligned}$$

By Theorem 2.2, for $t > 0$, when N is a large enough positive integer, the function $g_N(t)$ (2.2) can be used to approximate $\zeta(1+it)$. The error is at most

$$\frac{(1+t)(2+t)}{32N^2}.$$

To compute $\zeta(1+it)$ for t in the interval $[t_0, T]$, choose a fixed spacing h and generate the points $t_k = t_0 + kh$ for $0 \leq k \leq K$, where $K = \lfloor (T - t_0)/h \rfloor$. The spacing h should be small enough so that the values $|\zeta(1+it_k)|$ computed can sufficiently reflect the behavior of $|\zeta(1+it)|$ for t in the interval $[t_0, T]$. Usually it is sufficient to take h to be 0.01. To control the accuracy of approximation, we fixed an error threshold r and let N be the smallest positive integer so that

$$\frac{(1+T)(2+T)}{32N^2} \leq r.$$

Then $g_N(t_k)$, $0 \leq k \leq K$ are computed and they gave values of $|\zeta(1+it_k)|$ with error at most r . The value of r we take depends on the accuracy of $|\zeta(1+it_k)|$ that we want to achieve. For plotting graphs over a large range of t , we take $r = 0.005$.

The following is the MATLAB code.

```
function zeta(t0,T,h,r)

t=t0:h:T;
N=ceil(sqrt((1+T)*(2+T)/(32*r)));
g=zeros(size(t));

for n=1:N
g=g+1./n.^(1+1i*t);
end

g=g+N.^(-1i*t)./(1i*t)-1/2*N.^(-1-1i*t)+(1+1i*t)/16.*N
.^( -2-1i*t); % g_N(t)
```

This algorithm has the advantage of being simple. However, the computations become very intensive when T is large. In this work, we limit our computations to $T \leq 10^6$ only. For $e \leq t \leq 10^6$, we find that $|\zeta(1 + it)|/\log t$ achieves its maximum value 0.6443 when $t = 17.7477$.

In the coming section, we want to obtain explicit upper bounds for $|\zeta(1 + it)|$ using the methods of exponential sums. Applying the triangle inequality to (2.1), together with the simple estimate

$$\left| \frac{(1 + it)(2 + it)}{2} \int_x^\infty \frac{h(\{u\})}{u^{3+it}} du \right| \leq \frac{(t + 1)(t + 2)}{32x^2},$$

we immediately obtain the following.

THEOREM 2.3. *For $x \geq 1$ and $t > 0$,*

$$(2.3) \quad |\zeta(1 + it)| \leq \left| \sum_{n \leq x} \frac{1}{n^{1+it}} \right| + \frac{1}{t} + \frac{1}{2x} + \frac{t^2 + 5t + 4}{32x^2}.$$

To find an upper bound for $|\zeta(1 + it)|$, it remains to bound the sum in (2.3). A crude estimate is given by

$$(2.4) \quad \left| \sum_{n \leq x} \frac{1}{n^{1+it}} \right| \leq \sum_{n \leq x} \frac{1}{n}.$$

The sum $\sum_{n \leq x} \frac{1}{n}$ is classical. In [9], it has been shown that for $x \geq 1$,

$$\sum_{n \leq x} \frac{1}{n} \leq \log x + \gamma + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{64x^4},$$

where γ is the Euler–Mascheroni constant. However, we can derive a simpler formula as follows.

LEMMA 2.4. *For $x \geq 1$,*

$$\sum_{n \leq x} \frac{1}{n} \leq \log x + \gamma + \frac{1}{x}.$$

PROOF. Using Euler’s summation formula, one can obtain the harmonic sum bound

$$\sum_{n \leq N} \frac{1}{n} \leq \log N + \gamma + \frac{1}{N}$$

when N is an integer (see for example [2]). Now, for any real number $x \geq 1$, let $N = \lfloor x \rfloor$. It is easy to check that the function $\log x + \frac{1}{x}$ is increasing if $x \geq 1$. By definition, $N \leq x$. Hence,

$$\sum_{n \leq x} \frac{1}{n} = \sum_{n \leq N} \frac{1}{n} \leq \log N + \gamma + \frac{1}{N} \leq \log x + \gamma + \frac{1}{x}. \quad \blacksquare$$

3. Bounds of $|\zeta(1 + it)|$ using exponential sums

To obtain an upper bound for $|\zeta(1 + it)|$, one of the elementary methods is to use (2.3). One then needs to find a bound for

$$\sum_{n \leq x} \frac{1}{n^{1+it}}.$$

Let a and b be real numbers with $a < b$. If $f(n)$ is real for all integers n in the interval $I = (a, b]$, a sum of the form

$$\sum_{a < n \leq b} e^{2\pi i f(n)}$$

is called an exponential sum. A good reference for exponential sums is the book [10]. A classical result by Kuzmin and Landau [14, 16] says that if f is twice continuously differentiable and $\lambda \leq |f''(x)| \leq \alpha\lambda$ for some positive constants λ and α , then

$$(3.1) \quad \sum_{a < n \leq b} e^{2\pi i f(n)} = O(\alpha |I| \lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}),$$

where $|I| = b - a$. To obtain explicit bounds for $|\zeta(1 + it)|$, one needs to compute the implied constants in (3.1). This has been done by several authors [12, 21, 22, 28] in the case where a and b are integers. For our applications, we consider the general case where a and b are any real numbers with $a < b$. Patel [21] made a remark that one can use an observation in [22] to improve his result to the following. If N and L are integers, $f: [N + 1, N + L] \rightarrow \mathbb{R}$ is a twice continuously differentiable function, V and W are positive numbers with $V < W$, and

$$\frac{1}{W} \leq |f''(x)| \leq \frac{1}{V} \quad \text{for all } x \in [N + 1, N + L],$$

then

$$(3.2) \quad \left| \sum_{n=N+1}^{N+L} e^{2\pi i f(n)} \right| \leq \frac{4(L-1)}{\sqrt{\pi}} \frac{\sqrt{W}}{V} + \frac{8\sqrt{W}}{\sqrt{\pi}} + \frac{L-1}{V} + 3.$$

In the following we consider arbitrary real numbers a and b with $a < b$. If $\lfloor b \rfloor = \lfloor a \rfloor = 0$, there is no integer n satisfying $a < n \leq b$ and so the sum is vacuous. If $\lfloor b \rfloor - \lfloor a \rfloor \geq 1$, take

$$N = \lfloor a \rfloor, \quad L = \lfloor b \rfloor - \lfloor a \rfloor, \quad W = \frac{1}{\lambda}, \quad V = \frac{1}{\alpha\lambda}$$

in (3.2). Since

$$L \leq b - a + 1,$$

we obtain the following.

THEOREM 3.1. *Let a and b be real numbers such that $a < b$, and let $I = (a, b]$. If $f: I \rightarrow \mathbb{R}$ is a twice continuously differentiable function, and there exist $\lambda > 0$ and $\alpha \geq 1$ such that*

$$\lambda \leq |f''(x)| \leq \alpha\lambda \quad \text{for all } x \in I,$$

then

$$(3.3) \quad \left| \sum_{n \in I} e^{2\pi i f(n)} \right| \leq \frac{4}{\sqrt{\pi}} \alpha |I| \lambda^{\frac{1}{2}} + \frac{8}{\sqrt{\pi}} \lambda^{-\frac{1}{2}} + \alpha |I| \lambda + 3,$$

where $|I| = b - a$.

Applying Theorem 3.1, we obtain the following.

PROPOSITION 3.2. *For any $t > 0$, and any real numbers a and b such that $0 < a < b$, we have*

$$\begin{aligned} \left| \sum_{a < n \leq b} \frac{1}{n^{1+it}} \right| &\leq \frac{4\sqrt{2}(b-a)}{\pi a^2} \sqrt{t} - \frac{2\sqrt{2}}{\pi a} \sqrt{t} \log \frac{b}{a} + \frac{8\sqrt{2}}{\sqrt{t}} \log \frac{b}{a} + \frac{8\sqrt{2}}{\sqrt{t}} \\ &\quad + \frac{t}{2\pi a^2} \log \frac{b}{a} + \frac{3}{a}. \end{aligned}$$

PROOF. For $u \in [a, b]$, let

$$S(u) = \sum_{a < n \leq u} \frac{1}{n^{it}}.$$

Then $S(a) = 0$. For fixed $t > 0$, we take

$$f(x) = -\frac{t}{2\pi} \log x, \quad x > 0.$$

Then

$$\frac{t}{2\pi u^2} \leq f''(x) \leq \frac{t}{2\pi a^2} \quad \text{for } a \leq x \leq u.$$

Applying Theorem 3.1 with

$$\lambda = \frac{t}{2\pi u^2}, \quad \alpha = \frac{u^2}{a^2},$$

we find that

$$|S(u)| \leq \frac{2\sqrt{2} u(u-a)}{\pi a^2} \sqrt{t} + \frac{8\sqrt{2}u}{\sqrt{t}} + \frac{t(u-a)}{2\pi a^2} + 3.$$

Using Riemann–Stieltjes integration, we have

$$\sum_{a < n \leq b} \frac{1}{n^{1+it}} = \int_a^b \frac{1}{u} dS(u) = \frac{S(b)}{b} + \int_a^b \frac{S(u)}{u^2} du.$$

Now,

$$\left| \frac{S(b)}{b} \right| \leq \frac{|S(b)|}{b} \leq \frac{2\sqrt{2}(b-a)}{\pi a^2} \sqrt{t} + \frac{8\sqrt{2}}{\sqrt{t}} + \frac{t(b-a)}{2\pi a^2 b} + \frac{3}{b}.$$

On the other hand,

$$\begin{aligned} \left| \int_a^b \frac{S(u)}{u^2} du \right| &\leq \int_a^b \frac{|S(u)|}{u^2} du \\ &\leq \frac{2\sqrt{2}}{\pi a^2} \sqrt{t} \int_a^b \frac{u-a}{u} du + \frac{8\sqrt{2}}{\sqrt{t}} \int_a^b \frac{1}{u} du + \frac{t}{2\pi a^2} \int_a^b \frac{u-a}{u^2} du \\ &\quad + 3 \int_a^b \frac{1}{u^2} du \\ &= \frac{2\sqrt{2}(b-a)}{\pi a^2} \sqrt{t} - \frac{2\sqrt{2}}{\pi a} \sqrt{t} \log \frac{b}{a} + \frac{8\sqrt{2}}{\sqrt{t}} \log \frac{b}{a} + \frac{t}{2\pi a^2} \log \frac{b}{a} \\ &\quad + \left(3 - \frac{t}{2\pi a} \right) \frac{(b-a)}{ab}. \end{aligned}$$

It follows that

$$\begin{aligned} \left| \sum_{a < n \leq b} \frac{1}{n^{1+it}} \right| &\leq \left| \frac{S(b)}{b} \right| + \left| \int_a^b \frac{S(u)}{u^2} du \right| \\ &\leq \frac{4\sqrt{2}(b-a)}{\pi a^2} \sqrt{t} - \frac{2\sqrt{2}}{\pi a} \sqrt{t} \log \frac{b}{a} + \frac{8\sqrt{2}}{\sqrt{t}} \log \frac{b}{a} + \frac{8\sqrt{2}}{\sqrt{t}} \\ &\quad + \frac{t}{2\pi a^2} \log \frac{b}{a} + \frac{3}{a}. \end{aligned} \quad \blacksquare$$

Now we can use (2.3) and Proposition 3.2 to give a better estimate of $|\zeta(1+it)|$. Given $x \geq 1$, let x_0 be such that $1 \leq x_0 \leq x$, and let k be the positive integer such that

$$\frac{x}{2^k} \leq x_0 < \frac{x}{2^{k-1}}.$$

This implies that

$$(3.4) \quad \frac{2^k}{x} < \frac{2}{x_0} \quad \text{and} \quad k < \frac{\log x - \log x_0 + \log 2}{\log 2}.$$

Split the set of integers n with $n \leq x$ into $k+1$ sets $S_0, S_1, S_2, \dots, S_k$, where

$$S_0 = \{n \mid n \leq x_0\},$$

and for $1 \leq j \leq k$,

$$S_j = \{n \mid \frac{x}{2^j} < n \leq \frac{x}{2^{j-1}}\}.$$

For the sum over S_0 , the crude estimate (2.4) and Lemma 2.4 give

$$\left| \sum_{n \leq x_0} \frac{1}{n^{1+it}} \right| \leq \sum_{n \leq x_0} \frac{1}{n} \leq \log x_0 + \gamma + \frac{1}{x_0}.$$

For the sum over n in S_j , $1 \leq j \leq k$, Proposition 3.2 gives

$$\begin{aligned} \left| \sum_{\frac{x}{2^j} < n \leq \frac{x}{2^{j-1}}} \frac{1}{n^{1+it}} \right| &\leq \frac{2^{j+1}\sqrt{2}}{\pi x} \sqrt{t}(2 - \log 2) + \frac{8\sqrt{2}}{\sqrt{t}}(1 + \log 2) + \frac{2^{2j-1}t}{\pi x^2} \log 2 \\ &\quad + \frac{3 \times 2^j}{x}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \sum_{x_0 < n \leq x} \frac{1}{n^{1+it}} \right| &\leq \sum_{j=1}^k \left\{ \frac{2^{j+1}\sqrt{2}}{\pi x} \sqrt{t}(2 - \log 2) + \frac{8\sqrt{2}}{\sqrt{t}}(1 + \log 2) + \frac{2^{2j-1}t}{\pi x^2} \log 2 \right. \\ &\quad \left. + \frac{3 \times 2^j}{x} \right\} \\ &= \frac{4(2^k - 1)\sqrt{2}}{\pi x} \sqrt{t}(2 - \log 2) + \frac{8\sqrt{2}k}{\sqrt{t}}(1 + \log 2) \\ &\quad + \frac{t}{\pi x^2} \times \frac{2(4^k - 1)}{3} \log 2 + \frac{6 \times (2^k - 1)}{x}. \end{aligned}$$

Using (3.4), we have

$$\begin{aligned} \left| \sum_{x_0 < n \leq x} \frac{1}{n^{1+it}} \right| &\leq \frac{8\sqrt{2}}{\pi x_0} \sqrt{t}(2 - \log 2) + \frac{8t}{3\pi x_0^2} \log 2 + \frac{12}{x_0} - \frac{6}{x} \\ &\quad + \frac{8\sqrt{2}(1 + \log 2)}{\sqrt{t} \log 2} (\log x - \log x_0 + \log 2). \end{aligned}$$

It follows from (2.3) that

$$\begin{aligned}
 |\zeta(1+it)| &\leq \left| \sum_{n \leq x_0} \frac{1}{n^{1+it}} \right| + \left| \sum_{x_0 < n \leq x} \frac{1}{n^{1+it}} \right| + \frac{1}{t} + \frac{1}{2x} + \frac{t^2 + 5t + 4}{32x^2} \\
 &\leq \log x_0 + \gamma + \frac{8\sqrt{2}}{\pi x_0} \sqrt{t}(2 - \log 2) + \frac{13}{x_0} + \frac{8t}{3\pi x_0^2} \log 2 \\
 (3.5) \quad &+ \frac{8\sqrt{2}(1 + \log 2)}{\sqrt{t} \log 2} (\log x - \log x_0 + \log 2) + \frac{1}{t} + \frac{t^2 + 5t + 4}{32x^2}.
 \end{aligned}$$

For any fixed t , we can use elementary calculus to find x_0 and x that minimize the expression on the right. More precisely, let

$$\begin{aligned}
 e_0 &= \frac{8\sqrt{2}(1 + \log 2)}{\log 2}, \quad e_1 = \frac{8\sqrt{2}}{\pi}(2 - \log 2), \quad e_2 = \frac{8}{3\pi} \log 2, \\
 (3.6) \quad E_0(t) &= 1 - \frac{e_0}{\sqrt{t}}, \quad E_1(t) = e_1 \sqrt{t} + 13, \quad E_2(t) = e_2 t,
 \end{aligned}$$

$$(3.7) \quad G_0(t) = \frac{e_0}{\sqrt{t}}, \quad G_2(t) = \frac{t^2 + 5t + 4}{32},$$

$$(3.8) \quad Q_0(t) = \gamma + \frac{e_0 \log 2}{\sqrt{t}} + \frac{1}{t}.$$

Then (3.5) says that

$$(3.9) \quad |\zeta(1+it)| \leq E_0(t) \log x_0 + \frac{E_1(t)}{x_0} + \frac{E_2(t)}{x_0^2} + G_0(t) \log x + \frac{G_2(t)}{x^2} + Q_0(t)$$

for any x_0 and x satisfying $1 \leq x_0 \leq x$. Notice that $E_1(t)$, $E_2(t)$, $G_0(t)$, $G_2(t)$, $Q_0(t)$ are positive for any $t > 0$, while $E_0(t) > 0$ if and only if

$$t > e_0^2 = 763.75.$$

For fixed $t > 0$, consider the functions

$$\begin{aligned}
 (3.10) \quad h_E(y) &= E_0(t) \log y + \frac{E_1(t)}{y} + \frac{E_2(t)}{y^2}, \quad y \geq 1, \\
 h_G(y) &= G_0(t) \log y + \frac{G_2(t)}{y^2}, \quad y \geq 1.
 \end{aligned}$$

Note that the functions $h_E(y)$ and $h_G(y)$ are functions depending on t . For fixed t , the derivative of $h_E(y)$ with respect to y is

$$h'_E(y) = \frac{E_0(t)}{y} - \frac{E_1(t)}{y^2} - \frac{2E_2(t)}{y^3} = \frac{E_0(t)y^2 - E_1(t)y - 2E_2(t)}{y^3}.$$

Its sign is determined by the sign of the quadratic function

$$(3.11) \quad Q_E(y) = E_0(t)y^2 - E_1(t)y - 2E_2(t).$$

When $t \leq 763.75$, $E_0(t) \leq 0$ and so $h'_E(y) \leq 0$ for all $y > 0$. This implies that when $t \leq 763.75$, $h_E(y)$ is a strictly decreasing function of y . When $t > 763.75$, $E_0(t) > 0$. The discriminant of the quadratic function $Q(y)$ (3.11) is

$$\Delta_E(t) = E_1(t)^2 + 8E_0(t)E_2(t),$$

which is positive. Therefore, when $t > 763.75$, $Q_E(y)$ has two distinct real roots y_1 and y_2 . Assume that $y_1 < y_2$. Then we must have $y_1 < 0 < y_2$. For $0 < y < y_2$, $Q_E(y) < 0$. For $y > y_2$, $Q_E(y) > 0$. Hence, the function $h_E(y)$ has minimum value at the point

$$(3.12) \quad y = y_E(t) = y_2 = \frac{E_1(t) + \sqrt{E_1^2(t) + 8E_0(t)E_2(t)}}{2E_0(t)}.$$

For the function $h_G(y)$, its derivative with respect to y is

$$h'_G(y) = \frac{G_0(t)}{y} - \frac{2G_2(t)}{y^3} = \frac{G_0(t)y^2 - 2G_2(t)}{y^3}.$$

Its sign is determined by the quadratic function $Q_G(y) = G_0(t)y^2 - 2G_2(t)$. Since $G_0(t)$ is positive for all $t > 0$, a similar argument shows that the function $h_G(y)$ has minimum value at the point

$$(3.13) \quad y = y_G(t) = \sqrt{\frac{2G_2(t)}{G_0(t)}}.$$

For fixed t , the expression (3.9) is minimized when we take $x_0 = y_E(t)$ and $x = y_G(t)$. Since we must have $x_0 \leq x$, we want to give some simple bounds to $y_E(t)$ and $y_G(t)$ to determine a value t_0 so that $y_E(t) \leq y_G(t)$ when $t \geq t_0$.

When $t > 763.75$, we obtain from (3.12) and (3.6) that

$$y_E(t) \leq \frac{(e_1 + \frac{13}{\sqrt{t}} + \sqrt{e_1^2 + 8e_2 + \frac{26e_1}{\sqrt{t}} + \frac{169}{t}})\sqrt{t}}{2(1 - \frac{e_0}{\sqrt{t}})}.$$

On the other hand, (3.13) and (3.7) give

$$y_G(t) \geq \frac{1}{\sqrt{16e_0}}t^{\frac{5}{4}}.$$

When $t > 763.75$, the function

$$\xi_E(t) = \frac{\left(e_1 + \frac{13}{\sqrt{t}} + \sqrt{e_1^2 + 8e_2 + \frac{26e_1}{\sqrt{t}} + \frac{169}{t}}\right)}{2\left(1 - \frac{e_0}{\sqrt{t}}\right)}$$

is decreasing in t , while the function

$$\xi_G(t) = \frac{1}{\sqrt{16e_0}} t^{\frac{3}{4}}$$

is increasing in t . Since $\xi_E(2000) \leq \xi_G(2000)$, we find that when $t \geq 2000$, $\xi_E(t) \leq \xi_G(t)$. Hence, when $t \geq 2000$, we find that

$$y_E(t) \leq \frac{\left(e_1 + \frac{13}{\sqrt{t}} + \sqrt{e_1^2 + 8e_2 + \frac{26e_1}{\sqrt{t}} + \frac{169}{t}}\right)\sqrt{t}}{2\left(1 - \frac{e_0}{\sqrt{t}}\right)} \leq \frac{1}{\sqrt{16e_0}} t^{\frac{5}{4}} \leq y_G(t).$$

Therefore, when $t \geq 2000$, we have

$$|\zeta(1 + it)| \leq h_E(y_E(t)) + h_G(y_G(t)) + Q_0(t).$$

When t is large,

$$(3.14) \quad y_E(t) \sim \lambda_1 \sqrt{t}, \quad y_G(t) = O(t^{\frac{5}{4}}),$$

where

$$(3.15) \quad \lambda_1 = \frac{e_1 + \sqrt{e_1^2 + 8e_2}}{2} = 4.9443.$$

Hence,

$$(3.16) \quad h_E(y_E(t)) \sim \frac{1}{2} \log t + \lambda_2, \quad h_G(y_G(t)) = O(t^{-\frac{1}{2}} \log t),$$

where

$$\lambda_2 = \log \lambda_1 + \frac{e_1}{\lambda_1} + \frac{e_2}{\lambda_1^2} = 2.5742.$$

Therefore, when t is large,

$$h_E(y_E(t)) + h_G(y_G(t)) + Q_0(t) \sim \frac{1}{2} \log t + \lambda_2 + \gamma = \frac{1}{2} \log t + 3.1514.$$

These show that using exponential sums with second-order derivatives only, we cannot achieve a bound for $|\zeta(1 + it)|$ that is better than $\frac{1}{2} \log t$.

Now we turn to bounds of the form

$$(3.17) \quad |\zeta(1+it)| \leq v \log t \quad \text{when } t \geq t_0.$$

To achieve this, we take

$$x = t^u, \quad x_0 = \beta t^v,$$

where β, u, v are positive constants satisfying

$$v \leq u, \quad \beta \leq 1.$$

These conditions ensure that $x_0 \leq x$. Substituting into (3.5), we find that

$$(3.18) \quad |\zeta(1+it)| \leq h_E(\beta t^v) + h_G(t^u) + Q_0(t) = v \log t + A + \omega(t),$$

where

$$\begin{aligned} A &= \log \beta + \gamma, \\ \omega(t) &= \frac{8\sqrt{2}}{\pi\beta} t^{\frac{1}{2}-v} (2 - \log 2) + \frac{13}{\beta t^v} + \frac{8t^{1-2v}}{3\pi\beta^2} \log 2 + \frac{1}{t} + \frac{t^2 + 5t + 4}{32t^{2u}} \\ &\quad + \frac{8\sqrt{2}(1 + \log 2)}{\log 2} t^{-\frac{1}{2}} [(u-v) \log t - \log \beta + \log 2]. \end{aligned}$$

To make $v \log t$ the leading term in (3.18), we must have

$$u \geq 1, \quad v \geq \frac{1}{2}.$$

For fixed u and v with $\frac{1}{2} \leq v \leq u$, the function $\omega(t)$ is decreasing on $[e^2, \infty)$. If we can find $\beta \in (0, 1]$ and $t_0 \geq e^2$ so that

$$A + \omega(t_0) = 0,$$

then

$$A + \omega(t) \leq 0 \quad \text{for all } t \geq t_0.$$

This will imply that

$$|\zeta(1+it)| \leq v \log t \quad \text{when } t \geq t_0.$$

When $v = \frac{1}{2}$ and $u > 1$,

$$\begin{aligned} \lim_{t \rightarrow \infty} (A + \omega(t)) &= \gamma + \log \beta + \frac{8\sqrt{2}}{\pi\beta} (2 - \log 2) + \frac{8}{3\pi\beta^2} \log 2 \\ &= \gamma + \log \beta + \frac{e_1}{\beta} + \frac{e_2}{\beta^2}. \end{aligned}$$

The function

$$h_C(\beta) = \gamma + \log \beta + \frac{e_1}{\beta} + \frac{e_2}{\beta^2}, \quad \beta > 0$$

has a minimum at

$$\beta = \frac{e_1 + \sqrt{e_1^2 + 8e_2}}{2} = \lambda_1 = 4.9443,$$

with minimum value

$$h_C(4.9443) = \gamma + \log \lambda_1 + \frac{e_1}{\lambda_1} + \frac{e_2}{\lambda_1^2} = \gamma + 2.5742 = 3.1514.$$

Therefore, using this method, we cannot achieve a result better than

$$|\zeta(1 + it)| \leq \frac{1}{2} \log t + 3.1514,$$

agreeing with our earlier asymptotic analysis.

Now consider the case where $v > \frac{1}{2}$. When u is also greater than 1, we have

$$\lim_{t \rightarrow \infty} \omega(t) = 0.$$

Hence, if $\beta < e^{-v} = 0.5615$, there exists $t_0 > e^2$ such that

$$A + \omega(t_0) = 0.$$

Instead of fixing $v > \frac{1}{2}$ and looking for t_0 , we can fix t_0 and look for u , v , and β that can achieve (3.17). As discussed above, we want to have $A + \omega(t_0) = 0$. By (3.18), this is achieved if

$$(3.19) \quad h_E(\beta t_0^v) + h_G(t_0^u) + Q_0(t_0) = v \log t_0.$$

To make v the smallest possible for the given t_0 , we need to choose β , u , and v that will minimize the left-hand side. From our earlier analysis, when $t_0 \geq 2000$, the minimum of the left-hand side is achieved when $\beta t_0^v = y_E(t_0)$ and $t_0^u = y_G(t_0)$, with $y_E(t_0) \leq y_G(t_0)$. Hence, for given $t_0 \geq 2000$, we first compute $y_E(t_0)$ and $y_G(t_0)$ from (3.12) and (3.13) respectively. By (3.19), we should let v be the positive number such that

$$(3.20) \quad v = \frac{h_E(y_E(t_0)) + h_G(y_G(t_0)) + Q_0(t_0)}{\log t_0}.$$

Then u and β should be defined so that

$$(3.21) \quad \beta t_0^v = y_E(t_0) \quad \text{and} \quad t_0^u = y_G(t_0).$$

TABLE 1. The values of β , v , and u satisfying (3.20) and (3.21) for given t_0 .

t_0	β	v	u	t_0	β	v	u
10^5	0.1474	0.8134	0.9854	10^{20}	0.2116	0.5684	1.1839
10^6	0.1796	0.7421	1.0295	10^{30}	0.2116	0.5456	1.2059
10^7	0.1978	0.7003	1.0610	10^{40}	0.2116	0.5342	1.2169
10^8	0.2061	0.6726	1.0847	10^{50}	0.2116	0.5274	1.2235
10^9	0.2095	0.6526	1.1030	10^{60}	0.2116	0.5228	1.2280
10^{10}	0.2108	0.6370	1.1177	10^{70}	0.2116	0.5196	1.2311
10^{11}	0.2113	0.6245	1.1297	10^{80}	0.2116	0.5171	1.2335
10^{12}	0.2115	0.6141	1.1398	10^{90}	0.2116	0.5152	1.2353
10^{13}	0.2115	0.6053	1.1482	10^{100}	0.2116	0.5137	1.2368
10^{14}	0.2116	0.5978	1.1555	10^{200}	0.2116	0.5068	1.2434
10^{15}	0.2116	0.5912	1.1618	10^{300}	0.2116	0.5046	1.2456

For v , β , and u defined in this way, we must have $A + \omega(t_0) = 0$. From what we have discussed above, this implies that

$$(3.22) \quad |\zeta(1 + it)| \leq v \log t \quad \text{when } t \geq t_0.$$

For a fixed $t_0 \geq 2000$, the v obtained in this way is the minimum possible v such that (3.22) holds.

In Table 1, we list some values of β , v , and u computed using (3.20) and (3.21) with given t_0 . We can see the trend that as t_0 gets large, v approaches the limit $\frac{1}{2} = 0.5$, while u approaches the limit $\frac{5}{4} = 1.25$. As for β , equations (3.21) and (3.20) say that

$$(3.23) \quad -\log \beta = (h_E(y_E(t_0)) - \log y_E(t_0)) + h_G(y_G(t_0)) + Q_0(t_0).$$

By (3.10) and (3.14), we find that when t is large,

$$h_E(y_E(t_0)) - \log y_E(t_0) \sim \frac{E_1(t_0)}{y_E(t_0)} + \frac{E_2(t_0)}{y_E(t_0)^2} \sim \frac{e_1}{\lambda_1} + \frac{e_2}{\lambda_1^2},$$

where λ_1 is given by (3.15). From (3.16), we have $h_G(y_G(t_0)) \rightarrow 0$ when $t_0 \rightarrow \infty$. From the definition of $Q_0(t)$ (3.8), $Q_0(t_0) \rightarrow \gamma$ when $t_0 \rightarrow \infty$. Hence, when t_0 is large, (3.23) shows that β should approach the limiting value

$$\exp\left(-\frac{e_1}{\lambda_1} - \frac{e_2}{\lambda_1^2} - \gamma\right) = 0.2116.$$

This is indeed the case, as shown in Table 1.

Once we have shown that the inequality $|\zeta(1 + it)| \leq v \log t$ holds for $t \geq t_0$, we can extend this to $t \geq t_1$ for some $0 < t_1 < t_0$, by numerically computing $\zeta(1 + it)$ for $t_1 \leq t < t_0$. For example, we have shown that

$$|\zeta(1 + it)| \leq 0.7421 \log t \quad \text{when } t \geq 10^6.$$

When $e \leq t < 10^6$, numerical calculations of $|\zeta(1 + it)|$ using the code in Section 2 show that we still have

$$|\zeta(1 + it)| \leq 0.7421 \log t.$$

Hence, we conclude that

$$|\zeta(1 + it)| \leq 0.7421 \log t \quad \text{when } t \geq e.$$

This implies the result in [25] which says that

$$|\zeta(1 + it)| \leq 0.75 \log t \quad \text{when } t \geq e.$$

From Table 1, we find that

$$|\zeta(1 + it)| \leq 0.6370 \log t \quad \text{when } t \geq 10^{10}.$$

If we can numerically compute $\zeta(1 + it)$ for t up to 10^{10} , we can then verify that

$$|\zeta(1 + it)| \leq 0.6443 \log t \quad \text{when } t \geq e.$$

However, ordinary computers cannot handle computations based on the simple algorithm given in Section 2 up to this value of t . To get around this, we use the Riemann–Siegel formula.

4. Bounds of $|\zeta(1 + it)|$ using the Riemann–Siegel formula

When t is large, Theorem 2.2 does not give an efficient way to compute $\zeta(1 + it)$. For a more efficient way, we can use the Riemann–Siegel formula.

In 1932, Siegel presented in his paper [23] an unpublished result of Riemann's and gave derivations of the formula that is now known as the Riemann–Siegel formula. In this section we are mainly concerned with using the Riemann–Siegel formula to obtain a better bound for $|\zeta(1 + it)|$. For efficient computations of $\zeta(s)$ using the Riemann–Siegel formula, one can refer to the work [3].

The starting point of the Riemann–Siegel formula is the following representation of the Riemann zeta function given by Riemann.

THEOREM 4.1 (Riemann). For $s \in \mathbb{C}$,

$$(4.1) \quad \zeta(s) = \mathcal{R}(s) + \chi(s)\bar{\mathcal{R}}(1-s),$$

where

$$\mathcal{R}(s) = \int_{0 \swarrow 1} \frac{w^{-s} e^{i\pi w^2}}{e^{i\pi w} - e^{-i\pi w}} dw, \quad \bar{\mathcal{R}}(s) = \overline{\mathcal{R}(\bar{s})}$$

and

$$(4.2) \quad \chi(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}.$$

The integration contour $0 \swarrow 1$ in the definition of $\mathcal{R}(s)$ is the line $a + te^{-\frac{3\pi i}{4}}$, $t \in \mathbb{R}$ that passes through a point a between 0 and 1, with southwest direction as indicated by the arrow.

The main ingredient of the Riemann–Siegel formula is the asymptotic expansion for the function

$$\mathcal{R}(s) = \int_{0 \swarrow 1} \frac{w^{-s} e^{i\pi w^2}}{e^{i\pi w} - e^{-i\pi w}} dw.$$

Let τ and z be complex numbers. For a real number σ , consider the function

$$(4.3) \quad g(\tau, z) = \exp\left\{-\left(\sigma + \frac{i}{8\tau^2}\right) \log(1 + 2i\tau z) - \frac{z}{4\tau} + i\frac{z^2}{4}\right\}.$$

When $|2\tau z| < 1$, Taylor series expansion about the point $z = 0$ gives

$$\begin{aligned} & -\left(\sigma + \frac{i}{8\tau^2}\right) \log(1 + 2i\tau z) - \frac{z}{4\tau} + i\frac{z^2}{4} \\ &= \sum_{k=1}^{\infty} (-1)^k \left(\sigma \frac{(2i)^k}{k} z^k + \frac{(2i)^{k-1}}{k+2} z^{k+2} \right) \tau^k. \end{aligned}$$

Hence, $g(\tau, z)$ has an expansion of the form

$$(4.4) \quad g(\tau, z) = \sum_{k=0}^{\infty} P_k(z) \tau^k,$$

where $P_k(z)$ is a polynomial in z and σ . In particular, $P_0(z)$ and $P_1(z)$ are given respectively by

$$(4.5) \quad P_0(z) = 1, \quad P_1(z) = -\frac{1}{3}z^3 - 2i\sigma z.$$

The following expansion of $\mathcal{R}(s)$ given in [3] is often attributed to Lehmer [17]. For a proof, see [3].

THEOREM 4.2. Given $s = \sigma + it$ with $t > 0$, let

$$(4.6) \quad a = \sqrt{\frac{t}{2\pi}}, \quad N = \lfloor a \rfloor, \quad p = 1 - 2a + 2N, \quad \tau = \frac{1}{\sqrt{8t}},$$

$$U = \exp\left\{-i\left[\frac{t}{2} \log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8}\right]\right\}.$$

Then $-1 \leq p \leq 1$. For any nonnegative integer K , the function

$$\mathcal{R}(s) = \int_{0 \swarrow 1} \frac{w^{-s} e^{i\pi w^2}}{e^{i\pi w} - e^{-i\pi w}} dw$$

has an expansion of the form

$$(4.7) \quad \mathcal{R}(s) = \sum_{n=1}^N \frac{1}{n^s} + (-1)^{N-1} U a^{-\sigma} \left\{ \sum_{k=0}^K \frac{C_k(p)}{a^k} + RS_K(p) \right\},$$

where

$$(4.8) \quad C_k(p) = \frac{e^{-\frac{i\pi}{8}}}{4} \frac{1}{(4\sqrt{\pi})^k} \int_{\searrow ip} \frac{e^{-\frac{i\pi}{2}(v-ip)^2}}{\cosh \frac{\pi}{2}v} P_k(\sqrt{\pi}(v-ip)) dv,$$

$$(4.9) \quad RS_K(p) = \frac{e^{-\frac{i\pi}{8}}}{4} \int_{\searrow ip} \frac{e^{-\frac{i\pi}{2}(v-ip)^2}}{\cosh \frac{\pi}{2}v} Rg_K(\tau, \sqrt{\pi}(v-ip)) dv,$$

$g(\tau, z)$ and $P_k(z)$ are defined in (4.3) and (4.4),

$$Rg_K(\tau, z) = g(\tau, z) - \sum_{k=0}^K P_k(z) \tau^k,$$

and the integration path $\searrow ip$ is the line $ip + te^{-\frac{i\pi}{4}}$, $t \in \mathbb{R}$ that passes through the point ip and points in the southeast direction as indicated by the arrow.

By definition (4.6), $|U| = 1$. Hence, (4.7) gives the following bound for $\mathcal{R}(\sigma + it)$:

$$(4.10) \quad |\mathcal{R}(\sigma + it)| \leq \left| \sum_{n=1}^N \frac{1}{n^{\sigma+it}} \right| + \left(\frac{2\pi}{t} \right)^{\frac{\sigma}{2}} \left\{ \sum_{k=0}^K |C_k(p)| \left(\frac{2\pi}{t} \right)^{\frac{k}{2}} + |RS_K(p)| \right\}.$$

Since $P_0(z) = 1$, $C_0(p)$ is independent of σ , but $C_k(p)$ depends on σ for all $k \geq 1$. Similarly, $RS_K(p)$ depends on σ for all $K \geq 0$.

Specializing to $\sigma = 1$, (4.1) gives

$$\zeta(1+it) = \mathcal{R}(1+it) + \chi(1+it) \bar{\mathcal{R}}(-it).$$

As in [21], we take $K = 1$ in (4.10) and obtain

$$(4.11) \quad |\zeta(1+it)| \leq \left| \sum_{n \leq \sqrt{t/(2\pi)}} \frac{1}{n^{1+it}} \right| + |\chi(1+it)| \left| \sum_{n \leq \sqrt{t/(2\pi)}} \frac{1}{n^{-it}} \right| + \kappa(t),$$

where

$$\begin{aligned} \kappa(t) = & \sqrt{\frac{2\pi}{t}} \left(b_0 + b_1(1) \sqrt{\frac{2\pi}{t}} + \frac{c(1)}{t} \right) \\ & + |\chi(1+it)| \left(b_0 + b_1(0) \sqrt{\frac{2\pi}{t}} + \frac{c(0)}{t} \right), \end{aligned}$$

with

$$(4.12) \quad b_0 = \max_{-1 \leq p \leq 1} |C_0(p)|,$$

$$(4.13) \quad b_1(\sigma) = \max_{-1 \leq p \leq 1} |C_1(p)|,$$

$$(4.14) \quad c(\sigma) = \max_{-1 \leq p \leq 1} |tRS_1(p)|.$$

In the following, we find upper bounds for $|\chi(1+it)|$ and b_0 , as well as $b_1(\sigma)$ and $c(\sigma)$ when $\sigma = 0$ and $\sigma = 1$, and compare our results to [21].

We start with $|\chi(1+it)|$. By Euler's reflection formula and Legendre's duplication formula, we find that

$$\chi(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})}{\Gamma(\frac{s}{2})\Gamma(\frac{1+s}{2})} = 2^{s-1} \pi^s \frac{1}{\sin \frac{\pi(1-s)}{2}} \frac{1}{\Gamma(s)}.$$

Hence, when $t > 0$,

$$\begin{aligned} \chi(1+it) &= -2^{it} \pi^{1+it} \frac{1}{\Gamma(1+it)} \frac{1}{\sin \frac{\pi it}{2}} \\ (4.15) \quad &= 2^{1+it} i \pi^{1+it} \frac{1}{\Gamma(1+it)} \frac{1}{(e^{\frac{\pi it}{2}} - e^{-\frac{\pi it}{2}})}. \end{aligned}$$

By [1, Theorem 1.4.2], we find that when $\operatorname{Re} s > 0$,

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log(2\pi) + \frac{1}{12s} - \frac{1}{2} \int_0^\infty \frac{B_2(\{x\})}{(x+s)^2} dx,$$

where $B_2(x) = x^2 - x + \frac{1}{6}$ is the second Bernoulli polynomial. As explained in Section 2, we can replace $B_2(x)$ with the function $h(x) = x^2 - x + \frac{1}{8}$ and obtain

$$(4.16) \quad \log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log(2\pi) + \frac{1}{16s} - \frac{1}{2} \int_0^\infty \frac{h(\{x\})}{(x+s)^2} dx.$$

In the following theorem, we use this formula to obtain an upper bound for $|\chi(1+it)|$.

THEOREM 4.3. For $t > 0$, we have

$$(4.17) \quad |\chi(1 + it)| \leq \sqrt{\frac{2\pi}{t}} \exp\left(\frac{\pi}{32t} - \frac{1}{24t^2} + \frac{5}{24t^4}\right) \frac{1}{1 - e^{-\pi t}}.$$

PROOF. When $t > 0$, (4.15) gives

$$(4.18) \quad |\chi(1 + it)| = 2\pi \left| \frac{1}{\Gamma(1 + it)} \right| \frac{e^{-\frac{\pi t}{2}}}{(1 - e^{-\pi t})}.$$

When $t > 0$,

$$\log(1 + it) = \frac{1}{2} \log(1 + t^2) + i \tan^{-1} t.$$

Hence (4.16) gives

$$(4.19) \quad \begin{aligned} \log \frac{1}{|\Gamma(1 + it)|} &= -\operatorname{Re} \log \Gamma(1 + it) \\ &= -\frac{1}{4} \log(1 + t^2) + t \tan^{-1} t + 1 - \frac{1}{2} \log(2\pi) - \frac{1}{16(1 + t^2)} \\ &\quad + \frac{1}{2} \operatorname{Re} \int_0^\infty \frac{h(\{x\})}{(x + 1 + it)^2} dx. \end{aligned}$$

By Taylor's remainder theorem,

$$\log\left(1 + \frac{1}{t^2}\right) \geq \frac{1}{t^2} - \frac{1}{2t^4}.$$

Therefore,

$$\log(1 + t^2) = 2 \log t + \log\left(1 + \frac{1}{t^2}\right) \geq 2 \log t + \frac{1}{t^2} - \frac{1}{2t^4}.$$

It is easy to see that

$$\frac{1}{1 + u^2} \geq 1 - u^2 \quad \text{for all } u \in \mathbb{R}.$$

Therefore,

$$\frac{1}{1 + t^2} = \frac{1}{t^2(1 + \frac{1}{t^2})} \geq \frac{1}{t^2} \left(1 - \frac{1}{t^2}\right) \geq \frac{1}{t^2} - \frac{1}{t^4}.$$

On the other hand, when $x \geq 0$,

$$\tan^{-1} x = \int_0^x \frac{1}{1 + u^2} du \geq \int_0^x (1 - u^2) du = x - \frac{x^3}{3}.$$

This gives

$$t \tan^{-1} t + 1 = t \left(\frac{\pi}{2} - \tan^{-1} \frac{1}{t} \right) + 1 \leq \frac{\pi t}{2} + \frac{1}{3t^2}.$$

For the last term in (4.19), we find that

$$\begin{aligned} \frac{1}{2} \operatorname{Re} \int_0^\infty \frac{h(\{x\})}{(x+1+it)^2} dx &\leq \frac{1}{2} \left| \int_0^\infty \frac{h(\{x\})}{(x+1+it)^2} dx \right| \\ &\leq \frac{1}{16} \int_0^\infty \frac{1}{(x+1)^2 + t^2} dx = \frac{1}{16} \int_1^\infty \frac{dx}{x^2 + t^2} \\ &= \frac{1}{16t} \left(\frac{\pi}{2} - \tan^{-1} \frac{1}{t} \right) \leq \frac{1}{16} \left(\frac{\pi}{2t} - \frac{1}{t^2} + \frac{1}{3t^4} \right). \end{aligned}$$

Collecting the estimates together, we find that

$$\log \left| \frac{1}{\Gamma(1+it)} \right| \leq -\frac{1}{2} \log(2\pi t) + \frac{\pi t}{2} + \frac{\pi}{32t} - \frac{1}{24t^2} + \frac{5}{24t^4}.$$

From this, we conclude that

$$\left| \frac{1}{\Gamma(1+it)} \right| \leq \frac{1}{\sqrt{2\pi t}} \exp \left(\frac{\pi t}{2} + \frac{\pi}{32t} - \frac{1}{24t^2} + \frac{5}{24t^4} \right).$$

Then (4.17) follows from (4.18). ■

Our result (4.17) is better than the bound

$$|\chi(1+it)| \leq \sqrt{\frac{2\pi}{t}} \exp \left(\frac{\pi}{6t} + \frac{5}{3t^2} \right)$$

obtained in [21]. The reason behind this is we use the alternate expression (4.15) to bound $|\chi(1+it)|$ while Patel [21] obtained bounds for the terms $|\Gamma(\frac{-it}{2})|$ and $|\Gamma(\frac{1+it}{2})|$ in the numerator and denominator of (4.2) separately.

Next we turn to b_0 (4.12). By [3, Theorem 6.1], we have the following.

THEOREM 4.4. *For $p \in [-1, 1]$, let*

$$(4.20) \quad C_0(p) = \frac{e^{-\frac{i\pi}{8}}}{4} \int_{\searrow ip} \frac{e^{-\frac{\pi i}{2}(v-ip)^2}}{\cosh \frac{\pi v}{2}} dv.$$

Then

$$b_0 = \max_{-1 \leq p \leq 1} |C_0(p)| = \frac{1}{2}.$$

PROOF. The proof given in [3] is for the more general case. Here we give a straightforward proof. As mentioned in [3], the integral (4.20) can be evaluated explicitly, and it is given by (see for example [5])

$$(4.21) \quad C_0(p) = \frac{1}{2 \cos \pi p} \left(\exp \left\{ \pi i \left(\frac{p^2}{2} + \frac{3}{8} \right) \right\} - i \sqrt{2} \cos \frac{\pi p}{2} \right).$$

This is an entire function, and $C_0(-p) = C_0(p)$. Hence, to determine b_0 , it is sufficient to consider $C_0(p)$ for $0 \leq p \leq 1$. Figure 1 shows the function $|C_0(p)|$ when $p \in [0, 1]$.

From Figure 1, we find that $|C_0(p)|$ is increasing on $[0, 1]$. Since

$$C_0(1) = -\frac{1}{2}e^{\frac{7\pi}{8}i},$$

we find that

$$b_0 = |C_0(1)| = \frac{1}{2}. \quad \blacksquare$$

Now we consider $b_1(\sigma)$ (4.13). By (4.8), we find that

$$C_1(p) = \frac{e^{-\frac{i\pi}{8}}}{16\sqrt{\pi}} \int_{\searrow ip} \frac{e^{-\frac{\pi i}{2}(v-ip)^2}}{\cosh \frac{\pi v}{2}} P_1(\sqrt{\pi}(v-ip)) dv,$$

where

$$P_1(z) = -\frac{1}{3}z^3 - 2i\sigma z$$

is the polynomial given in (4.5).

THEOREM 4.5. *Let*

$$b_1(\sigma) = \max_{-1 \leq p \leq 1} |C_1(p)|.$$

Then

$$(4.22) \quad b_1(0) = 0.0173 \quad \text{and} \quad b_1(1) = 0.0932.$$

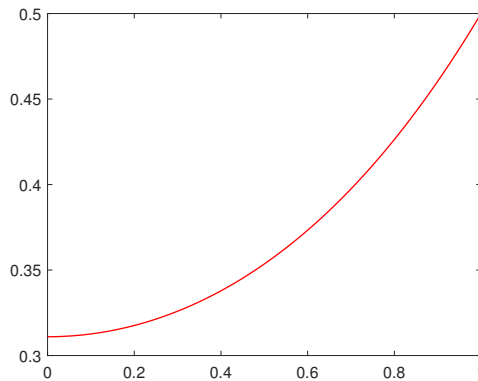
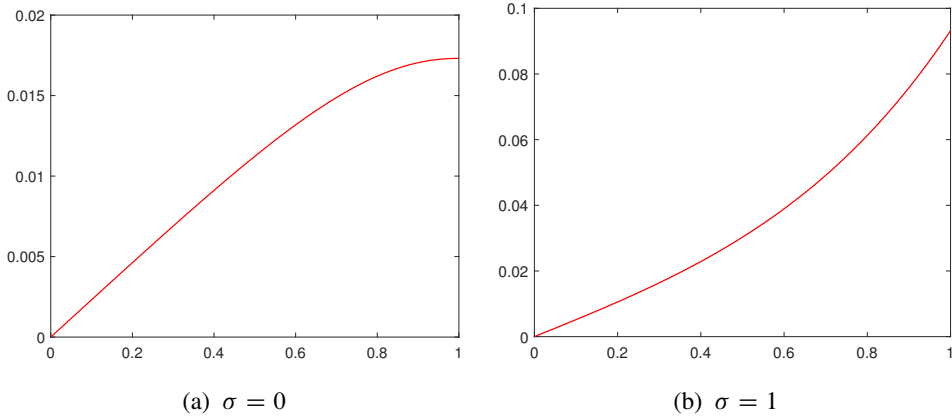


FIGURE 1. The function $|C_0(p)|$ when $0 \leq p \leq 1$.

FIGURE 2. The function $|C_1(p)|$ when $0 \leq p \leq 1$.

PROOF. By definition,

$$C_1(p) = \frac{e^{-\frac{i\pi}{8}}}{16\sqrt{\pi}} \int_{\searrow ip} \frac{e^{-\frac{\pi i}{2}(v-ip)^2}}{\cosh \frac{\pi v}{2}} \left(-\frac{1}{3} [\sqrt{\pi}(v-ip)]^3 - 2i\sigma \sqrt{\pi}(v-ip) \right) dv.$$

Using the definition (4.20) for $C_0(p)$, we find that

$$(4.23) \quad C_1(p) = \frac{1}{12\pi^2} C_0'''(p) + \frac{(1-2\sigma)}{4i\pi} C_0'(p).$$

Since $C_0(p)$ is an even function, $C_1(p)$ is an odd function. Hence, to find a bound for $|C_1(p)|$ for $p \in [-1, 1]$, it is sufficient to consider $|C_1(p)|$ when $p \in [0, 1]$. Using the explicit expression for $C_0(p)$ given by (4.21), we can compute $C_1(p)$ by (4.23). Figure 2 shows the graphs of $|C_1(p)|$, $p \in [0, 1]$ when $\sigma = 0$ and $\sigma = 1$ respectively.

From Figure 2(a), we find that when $\sigma = 0$, $|C_1(p)|$ is increasing on $[0, 1]$. Hence,

$$b_1(0) = |C_1(1)_{\sigma=0}| = 0.0173.$$

From Figure 2(b), we find that when $\sigma = 1$, $|C_1(p)|$ is increasing on $[0, 1]$. Hence,

$$b_1(1) = |C_1(1)_{\sigma=1}| = 0.0932.$$

This completes the proof. ■

In [21], [3, Theorem 4.1] is quoted directly to give the bounds

$$b_1(0) \leq \frac{1}{\pi \sqrt{2(3-2\log 2)}} = 0.1772, \quad b_1(1) \leq \frac{9}{2\sqrt{2}\pi} = 1.7952.$$

Obviously, our exact results (4.22) are much better.

Finally, we consider

$$c(\sigma) = \max_{-1 \leq p \leq 1} |tRS_1(p)|,$$

where $RS_1(p)$ is defined by (4.9). Arias de Reyna [3] has obtained a bound $RS_K(p)$ for all $K \geq 1$. We specialize to the case where $K = 1$ and obtain a better upper bound here.

THEOREM 4.6. *Let*

$$c(\sigma) = \max_{-1 \leq p \leq 1} |tRS_1(p)|.$$

Then

$$c(0) \leq 0.9704, \quad c(1) \leq 1.0450.$$

PROOF. As in [3], let

$$f(u) = -\frac{1}{2} - \frac{1}{u} - \frac{1}{u^2} \log(1-u).$$

Then by [3, equation (4.9)], we have

$$|tRS_1(p)| \leq \frac{1}{\pi^2} \int_{-\infty}^{\infty} H(\sigma, y) dy,$$

where

$$H(\sigma, y) = |1 - u(y)|^{-\sigma} |u(y)|^{-2} \frac{1}{1 + V(u(y))},$$

$$u(y) = \frac{1}{2} + ye^{\frac{i\pi}{4}}, \quad V(u) = \operatorname{Re} f(u).$$

Using numerical calculations, we obtain

$$\frac{1}{\pi^2} \int_{-\infty}^{\infty} H(0, y) dy = 0.9704, \quad \frac{1}{\pi^2} \int_{-\infty}^{\infty} H(1, y) dy = 1.0450. \quad \blacksquare$$

Patel [21], cited [3, Theorem 4.2] with $K = 1$ to obtain

$$c(0) \leq \frac{121\pi}{350} = 1.0861, \quad c(1) \leq \frac{242\sqrt{2}\pi}{350} = 3.0719.$$

Our explicit computations in Theorem 4.6 give smaller upper bounds.

Now we return to the estimate of $|\zeta(1 + it)|$. From (4.11), Theorems 4.3, 4.4, 4.5, 4.6, we find that when $t > 0$,

$$|\zeta(1 + it)| \leq \left| \sum_{n \leq \sqrt{t/(2\pi)}} \frac{1}{n^{1+it}} \right| + \kappa_1(t) \left| \sum_{n \leq \sqrt{t/(2\pi)}} \frac{1}{n^{-it}} \right| + \kappa_2(t),$$

where

$$\begin{aligned} \kappa_1(t) &= \sqrt{\frac{2\pi}{t}} \exp\left(\frac{\pi}{32t} - \frac{1}{24t^2} + \frac{5}{24t^4}\right) \frac{1}{1 - e^{-\pi t}}, \\ \kappa_2(t) &= \sqrt{\frac{2\pi}{t}} \left(\frac{1}{2} + b(1)\sqrt{\frac{2\pi}{t}} + \frac{c(1)}{t}\right) + \kappa_1(t) \left(\frac{1}{2} + b(0)\sqrt{\frac{2\pi}{t}} + \frac{c(0)}{t}\right), \end{aligned}$$

with

$$b(0) = 0.0173, \quad b(1) = 0.0932, \quad c(0) = 0.9704, \quad c(1) = 1.0450.$$

Notice that

$$\kappa_2(t) = O\left(\frac{1}{\sqrt{t}}\right).$$

For the two sums over $n \leq a = \sqrt{t/(2\pi)}$, using exponential sums with second-order derivatives (3.3), one would not be able to obtain estimates that are better than the crude estimates

$$\left| \sum_{n \leq a} \frac{1}{n^{1+it}} \right| \leq \sum_{n \leq a} \frac{1}{n} \leq \log a + \gamma + \frac{1}{a} \quad \text{and} \quad \left| \sum_{n \leq a} \frac{1}{n^{it}} \right| \leq a.$$

Thus, we use these crude estimates and obtain

$$|\zeta(1 + it)| \leq \frac{1}{2} \log t + \gamma - \frac{1}{2} \log(2\pi) + \vartheta(t),$$

where

$$\vartheta(t) = \sqrt{\frac{2\pi}{t}} \exp\left(\frac{\pi}{32t} - \frac{1}{24t^2} + \frac{5}{24t^4}\right) \frac{1}{1 - e^{-\pi t}} + \kappa_2(t).$$

It is easy to see that $\vartheta(t)$ is decreasing when $t \geq 1$. Hence, for any $t_0 \geq 1$, if we let

$$(4.24) \quad C = \gamma - \frac{1}{2} \log(2\pi) + \vartheta(t_0),$$

then

$$|\zeta(1 + it)| \leq \frac{1}{2} \log t + C \quad \text{when } t \geq t_0.$$

In Table 2, we list the values of C for different t_0 .

TABLE 2. The values of C (4.24) for different t_0 .

t_0	C
10^1	2.4868
10^2	1.1727
10^3	0.8178
10^4	0.7085
10^5	0.6741
10^6	0.6633
10^7	0.6599
10^8	0.6588
10^9	0.6584
10^{10}	0.6583

Since

$$\lim_{t \rightarrow \infty} \vartheta(t) = 1 \quad \text{and} \quad \gamma - \frac{1}{2} \log(2\pi) = -0.3417,$$

we find that

$$C \geq \gamma - \frac{1}{2} \log(2\pi) + 1 = 0.6583.$$

In other words, we cannot use this method to yield a bound for $|\zeta(1 + it)|$ that is better than

$$\frac{1}{2} \log t + 1 - 0.3417 = \frac{1}{2} \log t + 0.6583.$$

Since we only perform numerical calculations of $\zeta(1 + it)$ for $t \leq 10^6$, we take $t_0 = 10^6$. In this case, $C = 0.6633$, and we have

$$|\zeta(1 + it)| \leq \frac{1}{2} \log t + 0.6633 \quad \text{for } t \geq 10^6.$$

The same code as in Section 2 is used for the numerical calculations of $|\zeta(1 + it)|$ and for plotting Figure 3. The graphs show that for $e \leq t \leq 10^6$, $|\zeta(1 + it)|$ is bounded above by $\frac{1}{2} \log t + 0.6633$. Thus,

$$|\zeta(1 + it)| \leq \frac{1}{2} \log t + 0.6633 \quad \text{for } e \leq t \leq 10^6.$$

Hence, we obtain the following theorem.

THEOREM 4.7. *For $t \geq e$,*

$$|\zeta(1 + it)| \leq \frac{1}{2} \log t + 0.6633.$$

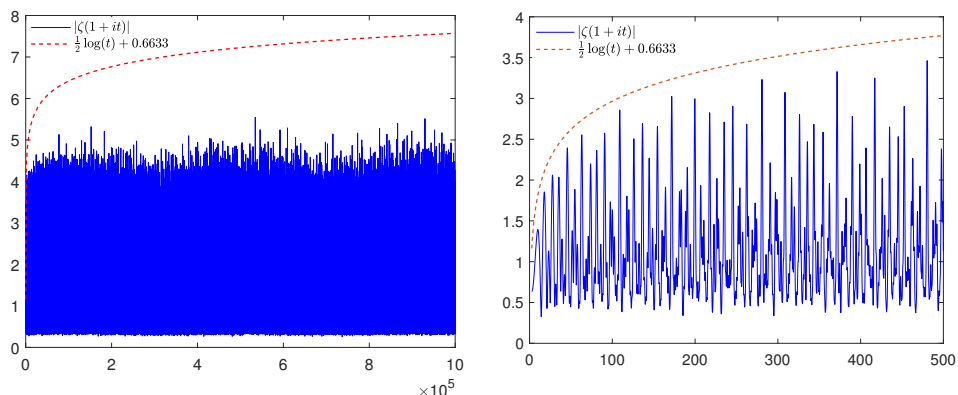


FIGURE 3. The figures show comparisons of $|\zeta(1 + it)|$ with $\frac{1}{2} \log t + 0.6633$ when $e \leq t \leq 10^6$ and when $e \leq t \leq 500$.

REMARK 4.8. Patel [21] directly cited the results of [3] for bounds of $b(0)$, $b(1)$, $c(0)$, and $c(1)$. We have obtained exact values for $b(0)$, $b(1)$, and obtained better bounds for $c(0)$ and $c(1)$ using direct computations. These improvements are not that significant for improving the bounds for $|\zeta(1 + it)|$ when t is large, as they only affect terms of order $t^{-\frac{1}{2}}$. The reason Patel only obtained the bound $\frac{1}{2} \log t + 1.93$ is that he has only used $t_0 = 47.47$, as this is the point where the bound $\frac{1}{2} \log t + 1.93$ is better than Backlund's bound $\log t$. If we take $t_0 = 47.47$, we will get the bound $\frac{1}{2} \log t + 1.4184$, which is still better than Patel's result.

5. Concluding the proof of Theorem 1.1

As in Section 3, we are interested in a bound of the form $|\zeta(1 + it)| \leq v \log t$ for all $t \geq e$. As mentioned in Section 2, for $e \leq t \leq 10^6$, numerical calculations show that $|\zeta(1 + it)|/\log t$ achieves its maximum value 0.6443 when $t = 17.7477$. To show that $|\zeta(1 + it)| \leq 0.6443 \log t$ holds for all $t \geq e$, we can use Theorem 4.7. Notice that when $t \geq 100$,

$$\frac{1}{2} \log t + 0.6633 \leq 0.6443 \log t.$$

In Figure 4, the graphs of $|\zeta(1 + it)|/\log t$ when $e \leq t \leq 100$ and when $10 \leq t \leq 20$ are shown. The values of $|\zeta(1 + it)|$ are computed using the same code as in Section 2. The graphs show that for $e \leq t \leq 100$, $|\zeta(1 + it)|/\log t$ has a maximum value of 0.6443. Thus, we obtain the following theorem.

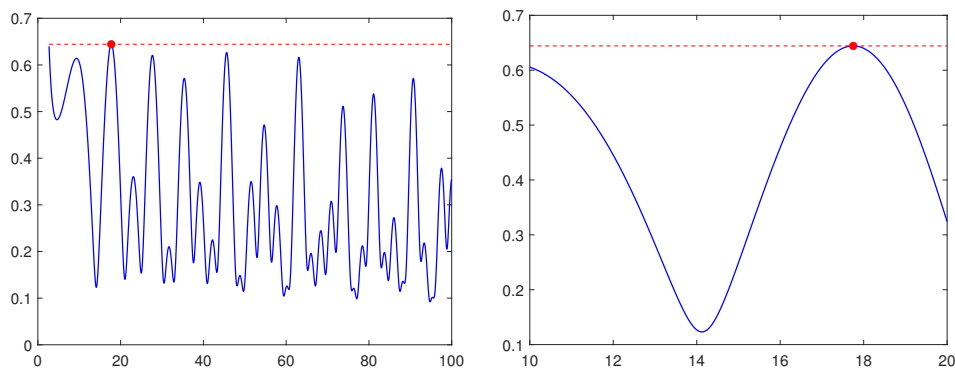


FIGURE 4. The figures show $|\zeta(1+it)|/\log t$ when $e \leq t \leq 100$ and when $10 \leq t \leq 20$.

THEOREM 5.1. When $t \geq e$,

$$|\zeta(1+it)| \leq 0.6443 \log t.$$

Equality is achieved when $t = 17.7477$.

Finally, we use Theorem 4.7 to refine our results in Section 3 in the following way. For $t \geq t_0 \geq e$,

$$\frac{1}{2} + \frac{0.6633}{\log t} \leq \frac{1}{2} + \frac{0.6633}{\log t_0}.$$

Hence, if

$$(5.1) \quad \tilde{v} = \frac{1}{2} + \frac{0.6633}{\log t_0},$$

then

$$|\zeta(1+it)| \leq \frac{1}{2} \log t + 0.6633 \leq \tilde{v} \log t \quad \text{when } t \geq t_0.$$

In Table 3, we list the values of $\tilde{v}$ and compare to the values of v obtained in Section 3 for various values of t_0 . We see that the values of $\tilde{v}$ are always smaller than the values of v . For $t_0 = 10^6$, we find that

$$|\zeta(1+it)| \leq 0.5480 \log t \quad \text{when } t \geq 10^6.$$

Using the code in Section 2 to numerically compute $|\zeta(1+it)|$ for $e \leq t \leq 10^6$, we find that when $652.3704 \leq t \leq 10^6$, we also have $|\zeta(1+it)| \leq 0.5480 \log t$. This is sharp as the equality is achieved when $t = 652.3704$. Therefore,

$$|\zeta(1+it)| \leq 0.5480 \log t \quad \text{when } t \geq 652.3704.$$

TABLE 3. The values of v (3.20) and $\tilde{v}$ (5.1) for given t_0 .

t_0	v	$\tilde{v}$	t_0	v	$\tilde{v}$
10^5	0.8134	0.5576	10^{20}	0.5684	0.5144
10^6	0.7421	0.5480	10^{30}	0.5456	0.5096
10^7	0.7003	0.5412	10^{40}	0.5342	0.5072
10^8	0.6726	0.5360	10^{50}	0.5274	0.5058
10^9	0.6526	0.5320	10^{60}	0.5228	0.5048
10^{10}	0.6370	0.5288	10^{70}	0.5196	0.5041
10^{11}	0.6245	0.5262	10^{80}	0.5171	0.5036
10^{12}	0.6141	0.5240	10^{90}	0.5152	0.5032
10^{13}	0.6053	0.5222	10^{100}	0.5137	0.5029
10^{14}	0.5978	0.5206	10^{200}	0.5068	0.5014
10^{15}	0.5912	0.5192	10^{300}	0.5046	0.5010

This result is better than $\frac{1}{2} \log t + 0.6633$ when $652.3704 \leq t \leq 10^6$. This concludes the proof of our main results given in Theorem 1.1.

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