REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 5, N.^{OS} 3 Y 4, 1989

Variational Characterization of Equations of Motion in Bundles of Embeddings

Hernán Cendra and Ernesto A. Lacomba

1. Introduction

In this paper we study variational principles for a general situation which includes free boundary problems with surface tension. Following [2], our main result concerns a variational principle in a infinite dimensional principal bundle of embeddings of a compact region D in a manifold M having the same dimension as D. By considering arbitrary variations, free boundary problems are included, while variations parallel to the boundary permit to consider fluid motion or flow of Hamitonian vector fields in non compact regions, generalizing [3], [4].

In Section 2 the main result is stated and proved. The Lagrangian includes a boundary term allowing us to include surface tension [5], or to remove it. Section 3 applies our result to Hamiltonian vector fields, while Section 4 concerns free boundary problems.

2. Variational Principle in Bundles of Embeddings

Let M be an n-manifold and Ω a given volume on M. Let $D \subseteq M$ a submanifold having dimension n with boundary $\partial D \subseteq M$.

The set

$$P = \{\eta: D \to M: \eta \text{ is an embedding}\}$$

is a principal bundle having structure group

$$G = \{g: D \to D: g \text{ is a diffeomorphism}\}$$

acting on P on the right, by composition of maps.

Similarly, we define P_{vol}, G_{vol}, by adding the further incompressibility condition, namely

$$\eta * \Omega = \Omega, \qquad g * \Omega = \Omega,$$

where the star means the pull-back operation.

A typical example of this situation to be considered afterwards with some more detail, is the liquid drop D moving freely in $M = \mathbb{R}^3$ with

$$\Omega = dx^1 \wedge dx^2 \wedge dx^3.$$

Thus, at each instant of time t, the element $\eta_t: D \to \mathbb{R}^3$ of P represents the position of the fluid particles at that instant namely, if $X = (X^1, X^2, X^3)$ are the coordinates of the position of a given particle at time $t = 0$, and $x = (x^1, x^2, x^3)$ the position of the same particle after the interval of time $[0, t]$ has passed, then $x = \eta_t(X)$. If the fluid is incompressible, then for each $X \in D$ and each t, we have

$$J_{\eta_t}(X) = 1$$

where J_{η_t} is the Jacobian of η_t. An equivalent condition is that $\eta_t * \Omega = \Omega$.

Now, back to the general situation, let $L: TM \to \mathbb{R}$ be a given lagrangian. This induces a Lagrangian $\mathcal{L}: TP \to \mathbb{R}$ defined by

$$\mathcal{L}(\eta, \dot\eta) = \int_D L(\eta(X), \dot\eta(X))\Omega(X).$$

It is sometimes useful to think of $(\eta, \dot\eta)$ as the derivative with respect to t of a curve $x = \eta_t(X)$, $X \in M$. Thus

$$\dot\eta_t[\eta_t^{-1}(x)] = \frac{\partial \eta_t(\eta_t^{-1}(x))}{\partial t}$$

represents a vector field on $D_t = \eta_t(D) \subseteq M$.

Of course, $\mathcal{L}$ has a restriction

$$\mathcal{L}: TP_{\mathrm{vol}} \to \mathbb{R}.$$

Let η_t; $t \in [t_0, t_1]$ be a curve in $P_{\text{vol}} \subseteq P$. Thus, for each $t \in [t_0, t_1]$, $\eta_t : D \to M$ is a volume preserving diffeomorphism. Now consider the following functional with $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$ fixed, defined on the curves (η_t, λ_t) where λ_t is a curve on the set $\mathfrak{F}(D)$ of real valued C^∞ functions on D

$$\mathcal{A}(\eta, \lambda) = \int_{t_0}^{t} \left[\mathcal{L}(\eta,\ \dot\eta) + \lambda_t \frac{dJ_{\eta_t}}{dt} \right] dt$$
$$= \int_{t_0}^{t_1} dt \int_D \left[L(\eta_t(X),\ \dot\eta_t(X)) + \lambda_t(X) \frac{d}{dt} J_{\eta_t}(X) \right] \Omega(X).$$

The constraint $J_{\eta_t} =$ constant, or equivalently $\dfrac{dJ_{\eta_t}}{dt} = 0$ with the Lagrange multiplier $\lambda_t \in \mathfrak{F}(D)$ together with the condition $J_{\eta_0} = 1$ gives the end this does not imply any loss of generality. Likewise, we can assume that

$$\Omega(X) = dX^1 \wedge \cdots \wedge dX^n.$$

This is because variational principles are essentially local in nature.

Sometimes we will write $\Omega(X) = d^3X$, whenever computations are simpler in the case $M = \mathbb{R}^3$.

Now think of a variation

$$\delta\eta_t = \frac{d}{d\epsilon} \eta_{t\epsilon} \bigg|_{\epsilon = 0}$$

such that

$$\delta\eta_{t_0} = 0, \qquad \delta\eta_{t_1} = 0,$$

and $\eta_{t\epsilon}$ is a curve on P for each $\epsilon \in (-\epsilon_1, \epsilon_2)$. On the other hand assume that $\eta_t \equiv \eta_{t_0}$ is a curve on P_{vol}. If (η_t, λ_t) is a critical point of $\mathcal{A}$, then we have

$$\delta\mathcal{A} = \frac{d}{d\epsilon} \mathcal{A} \bigg|_{\epsilon = 0} = 0.$$

This means that

$$0 = \delta \int_0^{t_1} dt \int_D \left\{ L[\eta_{t\epsilon}(X),\ \dot\eta_{t\epsilon}(X)] + \frac{d}{dt} J_{t\epsilon}(X) \lambda_t(X) \right\} d^3X$$
$$= \int_{t_0}^{t_1} dt \int_D \left\{ \frac{\partial L}{\partial x} [\eta_t(X),\ \dot\eta_t(X)] \delta\eta_t + \frac{\partial L}{\partial \dot x} [\eta_t(X),\ \dot\eta_t(X)]\, \delta\eta_t \right.$$
$$\left. + \frac{d}{dt} [\nabla \cdot (\partial\eta_t \circ \eta_t^{-1}) \circ \eta_t(X)] \lambda_t(X) \right\} d^3X. \qquad (\delta)$$

Since

$$\frac{d}{d\epsilon} J_{\eta_{t\epsilon}}(X)\bigg|_{\epsilon=0} = \nabla \cdot (\delta\eta_t \circ \eta_t^{-1}) \circ \eta_t(X).$$

(To check this, let

$$v = \frac{d\eta_\epsilon}{d\epsilon}\bigg|_{\epsilon=0} \circ \eta^{-1}(x).$$

Then

$$(\nabla \cdot v) \circ \eta = \frac{dJ_{\eta_\epsilon}}{d\epsilon}\bigg|_{\epsilon=0}.$$

In fact we can assume without loss of generality that $\eta = \eta_0 = \textit{identity}$. Thus

$$\frac{d}{d\epsilon} \eta_\epsilon * (d^3x)\bigg|_{\epsilon=0} = \frac{dJ_{\eta_\epsilon}}{d\epsilon} d^3X = L_v(d^3x) = \nabla \cdot v\, d^3x.\Big)$$

Thus by applying integration by parts to (δ), we get

$$0 = \int_{t_0}^{t_1} dt \int_D \left\{ \left[\frac{\partial L}{\partial x}(\eta_t, \dot\eta_t) - \frac{d}{dt}\frac{\partial L}{\partial \dot x}(\eta_t, \dot\eta_t) \right] \delta\eta_t - \right.$$
$$\left. - [\nabla \cdot (\delta\eta_t \circ \eta_t^{-1})] \circ \eta_t \frac{d}{dt}\lambda_t \right\}(X)\, d^3X.$$

Since η_t is volume preserving we have $J_{\eta_t} = 1$, and then, by the change of variables formula for a multiple integral, we get

$$0 = \int_{t_0}^{t_1} dt \int_{\eta_t(D)} \left[\left(\frac{\partial L}{\partial x}(x, \dot x) - \frac{d}{dt}\frac{\partial L}{\partial \dot x}(x, \dot x) \right) \delta x_t - (\nabla \cdot \delta x_t)\eta_t(x) \right] d^3x.$$

where

$$\mu_t(x) = \frac{d}{dt}(\lambda_t \circ\ ) \circ \eta_t^{-1}(x).$$

But from Gauss' divergence theorem we have

$$\int_\Omega (\nabla \cdot Y)(x)\mu(x)\, d^3x = \int_\Omega [\nabla \cdot (\mu Y) - \nabla\mu \cdot Y]\, d^3x =$$
$$= \int_{\partial\Omega} \mu(Y, \bar n) - \int_\Omega \nabla\mu \cdot Y.$$

Applying this formula we finally get

$$0 = \int_{t_0}^{t_1} dt \int_{\eta_t(D)} \left[\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x, \dot{x}) + \nabla \mu_t(x) \right] \delta x_t \, d^3x - \int_{t_0}^{t_1} dt \int_{\partial \eta_t(D)} \mu(\delta x_t, \vec{n}).$$

At any event, we need the two separate integrals to be zero. From the first integral we have that

$$\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x, \dot{x}) = -\nabla \mu_t(x).$$

We consider two posibilities now.

(*a*) If the variations ∂x_t are arbitrary on the boundary, we need

$$\mu_t|_{\partial D} = 0.$$

(*b*) If the δx_t are parallel to the boundary, the second integral is automatically zero and there is no additional condition on μ_t.

Before we state our results, let us introduce some notation. The Euler-Lagrange operator will be denoted by $[L]_x$. In local coordinates

$$[L]_x = \left[\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x, \dot{x}) \right] dx.$$

This is a well defined 1-form on M, once a curve $\eta_t(X) = x$ on P has been chosen. Here

$$\dot{x} = \dot{\eta}_t(X).$$

Summarizing the previous calculations we have

Lemma 1. *Let η_t be a curve on P_{vol} with $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$ fixed, and λ_t a curve on $\mathfrak{F}(D)$, and let*

$$\mu_t = \frac{d\lambda_t}{dt} \circ \eta_t^{-1}.$$

Then the following statements are equivalent.

(i) *(η_t, λ_t) is a critical point of $\mathfrak{A}(\eta, \lambda)$ in the set of curves (η_t, λ_t) such that $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$, and ∂D_t fixed (i.e. $\delta\eta_t \parallel \partial D_t$).*

(ii) $[L]_x = d\mu_t(x)$, $x \in \eta_t(D)$.

We will need now the following lemma, which gives a particular version of the Lagrange Multipliers Theorem.

Lemma 2. *Let η_t be a curve on P_{vol}. The following statements are equivalent*

(i) *η_t is a critical point of $\int_{t_0}^{t_1} \mathcal{L}(\eta_t, \dot{\eta}_t)\, dt$ on curves η_t belonging to P_{vol} with fixed end points $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$ and ∂D_t fixed (i.e. $\partial\eta_t \parallel \partial D_t$).*

(ii) *There exists a curve $\lambda_t \in \mathfrak{F}(D)$ such that (η_t, λ_t) is a critical point of the functional $\mathfrak{A}$ on curves $\eta_t \in P$, $\lambda_t \in \mathfrak{F}(D)$ with the conditions $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$.*

PROOF. That (ii) implies (i) is easy to check.

To prove that (i) implies (ii), we must show the global existence of λ_t.

Let η_t be a curve on P_{vol} satisfying (i). A variation $\eta_{t\epsilon}$ of η_t on P_{vol} can be constructed as follows.

Let Z be a vector field on D which is divergence-free ($\operatorname{div} Z = 0$) and parallel to the boundary ($Z \parallel \partial D$). Then for each t, $\eta_t * Z = Z_t$ is a vector field on $D_t = \eta_t(D)$ such that $\operatorname{div} Z_t = 0$ and $Z_t \parallel \partial D_t$. Let $\varphi(t, \epsilon)$ be any real valued function defined for $t \in [t_0, t_1]$ and $\epsilon > 0$. For our particular purposes, φ will be taken to be a bump function in the variable t for each ϵ, approximating the Dirac Delta function at $T \in [t_0, t_1]$ and satisfying $\varphi(t_0, \epsilon) = \varphi(t_1, \epsilon) = 0$.

Define $Z_{t\epsilon} = \varphi(t, \epsilon) Z_t$. Thus for each t, $Z_{t\epsilon}$ satisfies $\operatorname{div} Z_{t\epsilon} = 0$, $Z_{t\epsilon} \parallel \partial D_t$. For each t, let $F_{t\epsilon}$ be the flow of $Z_{t\epsilon}$ for $\epsilon > 0$. So for each t and ϵ, $F_{t\epsilon}: D_t \to D_t$ is a diffeomorphism. Define

$$\eta_{t\epsilon} = F_{t\epsilon} \circ \eta_t,$$

then $\eta_{t\epsilon}$ is a variation of η_t on P_{vol} satisfying

$$\eta_{t_0\epsilon} = \eta_0, \qquad \eta_{t_1\epsilon} = \eta_1$$

and

$$\left(\frac{d\eta_{t\epsilon}}{d\epsilon}(X)\right) \Big\| \, \partial D_t \quad \text{for all} \quad \epsilon > 0.$$

In general, if we are given a vector field $Z_{t\epsilon}$ for each t, ϵ such that $\operatorname{div} Z_{t\epsilon} = 0$, $Z_{t\epsilon} \parallel \partial D_t$ depending smoothly on the parameters, then we can construct in a similar way, a variation $\eta_{t\epsilon}$ as before.

Now we must compute

$$\frac{d}{d\epsilon} \int_{t_0}^{t_1} \mathcal{L}[\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)]\, dt = \frac{d}{d\epsilon} \int_{t_0}^{t_1} dt \int_D L[\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)]\, d^3X$$

$$= \int_{t_0}^{t_1} dt \int_D \left\{ \frac{\partial L}{\partial x}[\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)] - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}[\eta_{t\epsilon}(X), \dot{\eta}_{t\epsilon}(X)] \right\} \frac{d}{d\epsilon}\eta_{t\epsilon}(X)\, d^3X.$$

Since $\eta_{t\epsilon}$ is volume preserving we can change variables $x = \eta_{t\epsilon}(X)$, so that the right hand side equals

$$\int_{t_0}^{t_1} dt \int_{D_t} [L]_x Z_{t\epsilon}(x)\, d^3x.$$

Now choose $\varphi(t, \epsilon)$ such that $\varphi(t, \epsilon) \to \delta(t - T)$ for $\epsilon \to 0$. Then we finally get the condition

$$\int_{D_T} [L]_x Z_T(x)\, d^3x = 0.$$

At this point, we should remark that $Z_T = \eta_T * Z$ can be chosen to be an arbitrary vector field on D_T except for the conditions $\operatorname{div} Z_T = 0$, $Z_T \parallel \partial D_T$. By Hodge theorem, we can conclude that there exists μ_T globally defined on D_T and such that

$$[L]_x = d\mu_T(x), \qquad x \in D_T.$$

We leave to the reader to check that even though $\mu_T(x)$ is determined up to a constant, however we can choose $\mu_T(x)$ to be a C^∞ function of x, T and satisfying the previous condition.

To finish the proof, define λ_t by

$$\lambda_t(X) = \int^t \mu_t \circ \eta_t(X)\, dt$$

and apply Lemma 1. □

In order to state our main result, we need some notation. Let $\eta_t : M \to M$ be a curve on Diff(M) such that $J_{\eta_t} = 1$, *i.e.* η_t is volume preserving. For each $D \subseteq M$, a compact submanifold of M with boundary ∂D, define

$$\eta_t^D = \eta_t|_D$$

and denote by P^D the principal bundle of embeddings of D into M and by $P^D_{\text{vol}} \subseteq P^D$ the principal bundle of volume preserving embeddings. Define $\mathcal{L}^D_{\text{vol}}: TP^D_{\text{vol}} \to \mathbb{R}$ by

$$\mathcal{L}^D_{\text{vol}}(\eta_t^D, \dot\eta_t^D) = \int_D L[\eta_t^D(X), \dot\eta_t^D(X)]\, d^3X.$$

We also define for a given C^∞ curve λ_t on $\mathfrak{F}(M)$, $\lambda_t^D = \lambda_t|_D$ and

$$\mathcal{L}^D[\eta_t^D, \dot\eta_t^D, \lambda_t^D, \dot\lambda_t^D] = \int_D \left[L(\eta_t^D(X), \dot\eta_t^D(X)] + \lambda_t(X) \frac{d}{dt} J_{\eta_t}(X) \right] d^3X.$$

Theorem. *The following conditions on a curve $\eta_t \in \mathrm{Diff}_{\mathrm{vol}}(M)$ are equivalent.*

(i) *There exists* $\mu_t : M \to \mathbb{R}$, *a* C^∞ *curve on* $\mathfrak{F}(M)$ *such that*

$$[L]_{\eta_t(X)} = d\mu_t(\eta_t(X)).$$

(ii) *For each* D, η_t^D *is a critical point of*

$$\int_{t_0}^{t_1} \mathcal{L}_{\text{vol}}^D[\eta_t(X),\ \dot{\eta}_t(X)]\,dt$$

on the set of curves η_t *on* P_{vol}^D *with fixed end point conditions* $\eta_{t_0} = \eta_{t_0}^D$, $\eta_{t_1} = \eta_{t_1}^D$ *and* ∂D_t *fixed (i.e.* $\delta\eta_t \parallel \partial D_t$*).*

(iii) *There exists a* C^∞ *curve* $\lambda_t \in \mathfrak{F}(M)$ *such that for each* D, (η_t^D, λ_t^D) *is a critical point of*

$$\int_{t_0}^{t_1} \mathcal{L}_{\text{vol}}^D(\eta_t,\ \dot{\eta}_t, \lambda_t,\ \dot{\lambda}_t)\,dt$$

on the set of curves $(\eta_t, \lambda_t) \in P^D \times \mathfrak{F}(D)$ *with conditions* $\eta_{t_0} = \eta_{t_0}^D$, $\eta_{t_1} = \eta_{t_1}^D$ *and* ∂D_t *fixed (i.e.* $\delta\eta_t \parallel \partial D_t$*).*

Notice that η_t *and* λ_t *are related by*

$$\mu_t \circ \eta_t = \frac{d\lambda_t}{dt}.$$

PROOF. We first prove that (i) implies (ii). Let $D \subseteq M$ as before,

$$\mu_t^D = \mu_t\,|_{\eta_t(D)}$$

and

$$\lambda_t^D = \int^t \mu_t^D \circ \eta_t^D.$$

Thus by Lemma 1, (η_t^D, λ_t^D) is a critical point of $\mathfrak{A}(\eta^D, \lambda^D)$. By Lemma 2, we conclude that (ii) holds.

We now prove that (ii) implies (i). Using Lemma 2 and Lemma 1 we get for each D a function $\mu_t^D : \eta_t^D(D) \to \mathbb{R}$ such that

$$[L]_{\eta_t^D(X)} = d\mu_t^D[\eta_t^D(X)].$$

This shows that the Euler-Lagrange operator $[L]_{\eta_t^D(X)}$ is exact on $\eta_t^D(D)$. Since $\eta_t^D(D)$ can be chosen to be any given compact submanifold with boundary of M (having the same dimension as M), this immediately implies that μ_t^D can be taken as being the restriction to D of a globally defined 0-form μ_t.

Similarly, we can easily prove the equivalence between (iii) and (i) or (ii) by using Lemmas 1 and 2 if we define λ_t by

$$\mu_t \circ \eta_t = \frac{d\lambda_t}{dt}.$$

3. Hamiltonian Vector Fields

Let us consider a symplectic manifold M with volume element $\Omega = \omega^n$ where $\omega = d\alpha$ is its canonical 2-form and ω^n is the exterior power of order n. This problem was studied by Lacomba and Losco [4] for the case where M is a compact manifold with boundary.

We construct the principal fiber bundles P with structure group G and P_{vol} with structure group G_{vol} as in the general theory.

In this case we define, for a given curve η_t in $P_{\text{vol}} \subseteq P$ and the corresponding curve $Z_t = \dot{\eta}_t \cdot \eta_t^{-1}$ of vector fields on M, the Lagrangian

$$L[\eta_t(X),\ \dot{\eta}_t(X)] = i_{Z_t(X)}\alpha = \alpha[Z_t(X)].$$

For any compact submanifold with boundary $D \subseteq M$ as in Section 2, this induces a Lagrangian $\mathcal{L}^D_{\text{vol}}: TP^D_{\text{vol}} \to \mathbb{R}$ by

$$\mathcal{L}^D_{\text{vol}}(\eta^D_t,\ \dot{\eta}^D_t) = \int_D L[\eta^D_t(X),\ \dot{\eta}^D_t(X)]d^{2n}x.$$

If η^D_t is a critical point of $\int_{t_0}^{t_1} \mathcal{L}^D_{\text{vol}}[\eta_t(X),\ \dot{\eta}_t(X)]\,dt$ on the set of curves η_t on P^D_{vol} with fixed endpoint conditions $\eta_{t_0} = \eta^D_{t_0}$, $\eta_{t_1} = \eta^D_{t_1}$ and ∂D_t fixed, the main result implies the existence of a C^∞ curve $\mu_t: M \to \mathbb{R}$. From [4] and considering each $D \subseteq M$, we conclude that $\mu_t = H_t$ is a Hamiltonian function and Z_t is the associated Hamiltonian vector field. This means that the critical curves preserve not only the volume Ω, but also the symplectic from ω.

Notice that the arbitrariness of H_t permits to get any given Hamiltonian vector field.

We remark that this construction is still valid if (M, ω) is a non exact symplectic manifold. Since ω is closed, consider two different local expressions $\omega = d\alpha$ and $\omega = d\tilde{\alpha}$. Hence, $\tilde{\alpha} = \alpha + \gamma$ where γ is a closed form.

They produce two different but equivalent Lagrangians L, $\tilde{L}$ such that $\tilde{L} = L + \gamma$. It can be proved that the corresponding integrals

$$\int_{t_0}^{t_1} \mathcal{L}^D_{\text{vol}}[\eta_t(X),\ \dot{\eta}_t(X)]\,dt$$

give the same variational principle.

Indeed, we can write in local coordinates

$$\begin{aligned}\delta \int_{t_0}^{t_1} \mathcal{L}^D_{\text{vol}}[\eta_t(X),\ \dot{\eta}_t(X)]\,dt &= \int_{t_0}^{t_1} dt \int_D \{\omega[\dot{\eta}_t(X), \delta\eta_t(X)] - d\mu[\dot{\eta}_t(X), \delta\eta_t(X)]\}\, d^{2n}x \\ &= \int_{t_0}^{t_1} dt \int_{D_t} [\omega(\dot{x}, \delta x) - dH(\dot{x}, \delta x)]\, d^{2n}x.\end{aligned}$$

A related result for non exact symplectic manifolds appears in [1].

4. Free Boundary Problems

Free boundary problems, like a liquid incompressible homogeneous drop with surface tension, or a free elastic body, can be studied by using methods like those described in the previous sections. In this paper we will concentrate on the example of the liquid drop. A setting for this, from the Hamiltonian point of view can be found in [5]. However we may use part of that framework for our purposes, within the variational approach.

Let us denote P_{vol} the principal bundle of embeddings of the unit closed ball $D \subseteq \mathbb{R}^3$ into $\mathbb{R}^3$. Thus a curve $\eta_t \in P_{\text{vol}}$ represents a motion of the liquid drop. Note that the base B of the bundle P_{vol} consists of the set of all $\Sigma \subseteq \mathbb{R}^3$ where Σ is a 2-submanifold of $\mathbb{R}^3$ diffeomorphic to ∂D. Obviously every $\Sigma \in B$ can be written $\Sigma = \eta(\partial D)$ for some $\eta \in P_{\text{vol}}$. The group acting on P_{vol} on the right, is $G_{\text{vol}} = \text{Diff}_{\text{vol}}(D)$.

The surface tension coefficient being τ and the density being 1 and assuming that gravitational forces are absent, we can write the Lagrangian for the liquid drop as follows

$$\mathcal{L}(\eta_t, \dot{\eta}_t) = \int_D \frac{1}{2}[\dot{\eta}_t(X)]^2 d^3X - \tau \int_{\Sigma_t} d\Sigma_t$$

where $d\Sigma_t$ represents the area element on $\Sigma_t = \eta_t(\partial D)$.

Now suppose that η_t is a curve on P_{vol} which is a critical point of the functional $\int_{t_0}^{t_1} \mathcal{L}(\eta_t, \dot{\eta}_t)\, dt$ on the set of curves η_t such that $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$ fixed (note that we are not imposing here the condition ∂D_t fixed; thus variations $\delta\nu_t$ are allowed such that they are not necessarily assumed to be parallel to ∂D_t).

As we did in Section 2 where we assumed the condition $\delta\eta_t \parallel \partial D_t$, we can show that the problem of finding a critical curve η_t as stated above is equivalent to the problem at finding a critical curve (η_t, λ_t) of the functional

$$\mathfrak{A}(\eta, \lambda) = \int_{t_0}^{t_1} \left[\mathcal{L}(\eta, \dot{\eta}) + \lambda_t \frac{dJ_{\eta_t}}{dt} \right] dt$$

on curves (η_t, λ_t) with $\lambda \in \mathfrak{F}(D)$, $\eta_t \in P$, $\eta_{t_0} = \eta_0$, $\eta_{t_1} = \eta_1$ fixed.

By a similar computation to the one performed before the statement of Lemma 1 in Section 2, we can find that for a variation $\delta\eta_t$ with $\delta\eta_{t_0} = 0$, $\delta\eta_{t_1} = 0$, we have

$$0 = \int_{t_0}^{t_1} \int \left[\frac{\partial L}{\partial x}(x, \dot{x}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x, \dot{x}) + \nabla\mu_t(x) \right] \delta x_t\, d^3x - \\ \int_{t_0}^{t_1} dt \int_{\partial D_t} \mu_t(x)(\delta x_t, \vec{n})\, d\Sigma - \tau \int_{t_t}^{t_1} dt \int_{\partial D_t} K(x)(\delta x_t, \vec{n})\, d\Sigma$$

where K is the mean curvature of ∂D_t and integrals on ∂D_t are both surface integrals (here, we are implicitly assuming the standard Riemannian metric given on $\mathbb{R}^3$).

The last term comes out as follows (see [5] for more details). A given variation $\eta_{t\epsilon}$ induces a variation $\eta_{t\epsilon}(\partial D) = D_{t\epsilon}$. Thus

$$\frac{d}{d\epsilon}\int_{D_{t\epsilon}} d\Sigma \bigg|_{\epsilon=0} = \int_{D_t} K(\delta x_t, \bar{n})\, d\Sigma.$$

A simple argument shows that equality to 0 for arbitrary variations δx_t will imply

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\nabla \mu_t, \qquad \text{for} \quad x \in D_t,$$

$$\mu_t(x) = -\tau K(x), \quad \text{for} \quad x \in \partial D_t,$$

the incompressibility condition $J_{\eta_t} = 1$ comes out after variations $\delta\lambda_t$ are considered, as usual. Putting all this together and taking into account that

$$L(x, \dot{x}) = \frac{1}{2}\dot{x}^2,$$

we finally get

$$\frac{\partial^2 x}{\partial t^2} = -\nabla \mu_t \circ \eta_t, \quad \text{on} \quad D$$

$$\mu_t = \tau K, \qquad \text{on} \quad \partial D_t$$

$$J_{\eta_t} = 1, \qquad \text{on} \quad D.$$

We can write these equations in Eulerian (rather than Lagrangian) variables. Namely let

$$v = \frac{\partial x}{\partial t} \circ \eta_t^{-1}$$

be the Eulerian velocity.

Then we get

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial v}{\partial t} + (v \cdot \nabla)v$$

and $J_{\eta_t} = 1$ implies $\nabla \cdot v = 0$. Thus

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla \mu, \quad \text{on} \quad D_t,$$

$$\nabla \cdot v = 0, \quad \text{on} \quad D_t,$$
$$\mu_t = 2K, \quad \text{on} \quad \partial D_t.$$

These are precisely the equations of motion of the liquid drop with surface tension.

Acknowledgement. The second author acknowledges the warm hospitality of the University of Bahía Blanca, where this work was carried out.

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Recibido: 26 de junio de 1989.

Hernán Cendra†
Departamento de Matemática
Universidad Nacional del Sur
Ave. Alem 1253,
8000 Bahía Blanca
ARGENTINA

Ernesto A. Lacomba*†
Departamento de Matemática
Universidad Autónoma Metropolitana, Iztapalapa
Apdo. Postal 55-534
09340 México, D.F.
MEXICO

* Member of CIFMA (México).
† Research partially supported by CONICET (Argentina).