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Inverse semialgebras and partial actions of Lie algebras

Mikhailo Dokuchaev, Farangis Johari and José L. Vilca-Rodríguez

Abstract. We introduce the concept of a nonassociative (i.e., not necessarily associative) inverse semialgebra over a field, the Lie version of which is inspired by the set of all partially defined derivations of a nonassociative algebra, whereas the associative case is based on such examples as the set of all partially defined linear maps of a vector space, the set of all sections of the structural sheaf of a scheme, the set of all regular functions defined on open subsets of an algebraic variety, and the set of all smooth real-valued functions defined on open subsets of a smooth manifold. Given a Lie algebra L, we define the notion of a partial action of L on a nonassociative algebra A as an appropriate premorphism and introduce a Lie inverse semialgebra $E(L)$, which is a Lie analogue of R. Exel's inverse semigroup $S(G)$ that governs the partial actions of a group G. We discuss how $E(L)$ controls the premorphisms from L to A, obtaining results on its total control. We define the concept of a Lie F-inverse semialgebra and obtain Lie theoretic analogues of some classical results of the theory of inverse semigroups, namely, we show that the category of partial representations of L in meet semilattices is equivalent to the category $\mathcal{F}$ of Lie F-inverse semialgebras with morphisms that preserve the greatest elements of σ-classes. In addition, we establish an adjunction between the category of Lie algebras and the category $\mathcal{F}$.

1. Introduction

It is well known that the partial symmetries of a set X form the notorious symmetric inverse semigroup $\mathcal{I}(X)$ under the composition of the partial bijections taken on the largest possible domain (see [37]). Moreover, partial group actions on a set are closely related to inverse semigroups, in particular, to $\mathcal{I}(X)$. Indeed, on the one hand, Exel's definition of a partial action of a group G on X, given in Definition 1.2 of [22], can be seen as a map $G \to \mathcal{I}(X)$, satisfying properties which characterize what we call a partial representation of G into an inverse semigroup, see Proposition 4.1 in [22]. In addition, a partial action of G on X can also be defined as a premorphism of the form $G \to \mathcal{I}(X)$ (see Proposition 2.1 in [32]). On the other hand, in the same paper [22] R. Exel associated to any group G an inverse semigroup $S(G)$, defined by generators and relations, such that the actions of $S(G)$ on a set X are in a one-to-one correspondence with the partial actions of G on X. Later,

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J. Kellendonk and M. V. Lawson [32] proved that $S(G)$ is isomorphic to the Szendrei expansion of G (see [48]), the latter being also isomorphic to the Birget–Rhodes [10] prefix expansion of G.

The above mentioned Exel's definition was given as an abstraction of the notion of a partial group action on a C^*-algebra, worked out in the theory of operator algebras in order to describe important classes of C^*-algebras as more general crossed products (see [20, 21, 39]). This approach has yielded remarkable results concerning representations, ideal structures, and K-theory of algebras under consideration. A closely related concept (also referred to as a "partial action") can be found in [28], where the authors explore graded modules over quotients of path algebras.

Kellendonk and Lawson [37] highlighted the importance of partial actions for $\mathbb{R}$-trees, model theory, the profinite topology of groups, Fuchsian groups, tilings of Euclidean space, graph immersions and inverse semigroups, as well as topology and group presentations. Subsequent developments have extended the theory to include partial actions of semigroups [16, 27, 29, 33, 34, 36, 40], groupoids [6, 12, 24, 38, 43], and categories [42], as well as to the setting of partial (co)actions of Hopf algebras (or weak Hopf algebras) [1,2,5,8,9,13,14,30]. In algebra the new concepts are useful to graded algebras, Hecke algebras, Leavitt path algebras, inverse semigroups, restriction semigroups and automata (see the survey article [18] and the references therein).

More recent applications include the use of partial actions to dynamical systems associated to separated graphs and related C^*-algebras [3, 4], to shifts [4], to paradoxical decompositions [3], to full or reduced C^*-algebras of E-unitary or strongly E^*-unitary inverse semigroups [41], to Matsumoto and Carlsen–Matsumoto C^*-algebras of arbitrary subshifts [19], to topological higher rank graphs [45], to ultragraph C^*-algebras [26], and to universal F-inverse monoids [35]. For more information around partial actions and their applications, we refer the reader to Exel's book [23] and the surveys [7, 17, 18].

In view of the numerous interesting developments around partial group actions and applications, it is natural to ask whether a similar approach can be applied to actions by derivations on algebras. The main motivation of this paper is to establish a suitable framework in which an analogue of the theory of partial group actions can be developed in the setting of Lie algebras and bring into consideration new interesting algebraic structures which naturally appear this way.

It is well known that Lie algebras act on algebras by derivations. This observation leads to the following question:

How can a Lie algebra act partially on a non-necessarily associative algebra?

Since partial group actions can be obtained by restricting global actions, it is reasonable to approach this question looking at restrictions of global (usual) actions of Lie algebras. Specifically, given an action of a Lie algebra L over a field $\mathbb{F}$ on a nonassociative (i.e., not necessarily associative) $\mathbb{F}$-algebra B, and a subalgebra A of B, we consider a natural restriction of this action to A. This way we obtain the so-called "*partial derivations*" of A, which are linear maps defined on subalgebras of A that satisfy the Leibniz identity. It is worth noting that partial derivations (more specifically, those defined on ideals) appear in the construction of the algebra of quotients of a Lie algebra. The latter was introduced by M. S. Molina in [47], followed by interesting developments and applications (see, for example, [11] and [46]).

In order to obtain an analogue of the inverse semigroup of all partial symmetries in our context, we consider the set $\mathsf{PDer}(A)$ of all partial derivations of the nonassociative algebra A. This set can be equipped with an addition $+ : \mathsf{PDer}(A) \times \mathsf{PDer}(A) \to \mathsf{PDer}(A)$ and a product (Lie bracket) $[\cdot,\cdot] : \mathsf{PDer}(A) \times \mathsf{PDer}(A) \to \mathsf{PDer}(A)$, as well as a scalar multiplication $\mathbb{F} \times \mathsf{PDer}(A) \to \mathsf{PDer}(A)$, which is compatible with the previous operations. For ϕ_1 and ϕ_2 in $\mathsf{PDer}(A)$, the partial derivations $\phi_1 + \phi_2$ and $[\phi_1, \phi_2]$ are defined on the largest subalgebra of A on which the expressions $\phi_1(x) + \phi_2(x)$ and $\phi_1\phi_2(x) - \phi_2\phi_1(x)$, respectively, make sense (see Section 2). By axiomatizing the properties of this structure, we arrive at the notion of what we call a "*Lie inverse semialgebras*" (see Definition 2.10). In particular, $(\mathsf{PDer}(A), +)$ is a commutative inverse semigroup. When combined with scalar multiplication, this gives rise to a relaxed version of the notion of a vector space, where the underlying abelian (additive) group is replaced by a commutative (additive) inverse semigroup. We refer to this structure as an "*inverse semivector space*" (see Definition 2.3).

On the other hand, if we focus exclusively on the vector space structure of A, then each partial derivation can be regarded simply as a linear map from a subspace of A into A. In this setting, it is natural to consider the set $\mathsf{PEnd}(V)$ of all partial (linear) endomorphisms of a vector space V. Equipped with addition and scalar multiplication as previously described, $\mathsf{PEnd}(V)$ forms an inverse semivector space. Furthermore, we can define a "*partial composition*" of two partial endomorphisms as their composition on the largest subspace where it is well defined. Axiomatizing the properties of this structure leads to the notion of an "*associative inverse semialgebra*". This structure arises naturally in many important contexts, including $\mathsf{PEnd}(V)$. Examples of associative inverse semialgebras (see Example 3.6 for details) include:

(1) all sections of the structural sheaf of a scheme over a field,

(2) all sections of the sheaf of regular functions of an algebraic variety,

(3) the set of all locally defined smooth real-valued functions on a smooth manifold,

(4) all sections of a sheaf of C^*-algebras of locally defined bounded continuous complex-valued functions on a topological space.

This work is devoted to develop the foundations of the notions mentioned above, including that of a *partial action of a Lie algebra* on a nonassociative algebra (see Definition 4.4), and to study the connection between them.

The structure of the paper is as follows. Section 2 addresses the preliminary notions. In particular, we introduce the concept of an *inverse semivector space* over a field $\mathbb{F}$. As mentioned earlier, this is a relaxed version of the notion of a vector space. It is important to note that this concept is also a weaker version of the known notion of a semimodule over a ring with a unity (in particular, over a field), as explained before Definition 2.3. We also define the notion of a *nonassociative inverse semialgebra*, as well as the concept of a *Lie inverse semialgebra* and that of an *associative inverse semialgebra*. We provide several examples and establish some properties needed for subsequent sections. Given a Lie algebra L, we introduce the Lie inverse semialgebra $E(L)$ (see Proposition 2.21), which is pivotal in our study. It is inspired by the above mentioned Exel's inverse semigroup $S(G)$, which governs the partial actions of a group G on sets and whose algebra rules the partial representations of G in algebras.

In Section 3, we study a special class of nonassociative inverse semialgebras, namely, *semilattices of nonassociative algebras*. Roughly speaking, these are nonassociative inverse semialgebras in which the (additive) idempotents have well-behaved interactions with the product (see Definition 3.1). More specifically, we focus on semilattices of associative (or Lie) algebras. An important feature of this structure is its close relationship with presheaves of associative (or Lie) algebras, as we show in Proposition 3.10. Furthermore, Corollary 3.14 establishes that any semilattice of Lie algebras can be embedded into a semilattice of associative algebras.

In Section 4, we introduce the notion of a premorphism between Lie inverse semialgebras. Inspired by the concept of partial group actions, we define a partial action of a Lie algebra L on a nonassociative algebra A as a premorphism from L to $\mathsf{PDer}(A)$. One of the main results in the section is Theorem 4.11, which essentially states that the above mentioned Lie inverse semialgebra $E(L)$ governs the premorphisms defined on L in a specific way. This result serves as a Lie-theoretic analogue of a well-known result by R. Exel, Proposition 2.2 in [22] (see also Theorem 2.4 in [32]). Moreover, Corollary 4.12 says that $E(L)$ controls perfectly the premorphisms from L to semilattices of algebras. Another important consequence is Corollary 4.13, which establishes a one-to-one correspondence between partial actions of L and actions of $E(L)$ when dealing with some important specific classes of algebras.

An interesting class of inverse semigroups close to groups, which keeps drawing attention of researchers (see, for instance, [35]) is formed by F-inverse semigroups. It is therefore natural to introduce an analogue of this notion in the context of Lie inverse semialgebras, naturally expecting that such Lie inverse semialgebras are closely related to Lie algebras. It is well known that the category of the so-called F-pairs (following the terminology of M. Petrich in [44]) and their morphisms is equivalent to the category of F-inverse semigroups with morphisms that preserve the greatest elements of σ-classes, see Section VII.6 of [44]. An important result due to M. Szendrei [48] states that there exists an adjunction between the latter category and the category of groups. In Lemma 5.4, we observe that an F-pair can be identified with what is now commonly referred to as a unital partial representation of a group in a meet semilattice, which can be also equivalently seen as a unital partial action of the group on the semilattice.

In Section 5, we introduce the notion of a *partial representation* of a Lie algebra L in a unital meet semilattice Λ, defined as a partial representation of the additive group $(L, +)$ in Λ that satisfies an additional compatibility condition. We also adapt the notions of a congruence and the minimum group congruence from semigroup theory to our setting. Subsequently, we present the concept of an *Lie F-inverse semialgebra* and establish several basic results concerning partial representations and Lie F-inverse semialgebras. A central fact of the section is Theorem 5.14, which establishes a categorical equivalence between partial representations of L in unital meet semilattices and Lie F-inverse semialgebras with morphisms that preserve the greatest elements of σ-classes. This can be regarded as a Lie-theoretic analogue of the categorical equivalence from Section VII.6 of [44], as mentioned in the previous paragraph. Furthermore, in Theorem 5.15, we prove the existence of an adjunction between the category of Lie algebras and the category of Lie F-inverse semialgebras with morphisms preserving the greatest elements of σ-classes. This result serves as a Lie-theoretic counterpart to the adjunction for groups and F-inverse semigroups established by M. Szendrei.

2. Notions and preliminaries: Lie and associative inverse semialgebras

In this paper, given a commutative additive inverse semigroup S, we denote by $\mathcal{E}(S)$ the set of all idempotents of S. Furthermore, if S is a monoid, then 0 denotes the identity element of S. For convenience, we use the notation $0_x := x + (-x)$ for any $x \in S$. The sum $x + (-x)$ will be simply written as $x - x$. It follows immediately that $\mathcal{E}(S) = \{0_x \mid x \in S\}$, $0_{x+y} = 0_x + 0_y$, and $0_e = e$ for any $e \in \mathcal{E}(S)$. We recall that in an inverse semigroup S, we have the following natural partial order:

$$x \preceq y \iff x = y + e \quad \text{for some } e \in \mathcal{E}(S).$$

It can be observed that

$$x \preceq y \iff x = y + 0_x \iff 0_x = y - x. \tag{2.1}$$

Basics on inverse semigroups can be found, for example, in [37] and [44].

It is well known that the partial symmetries of a given set form an inverse semigroup under the composition of the partial bijections (see [37]). Moreover, the study of partial group actions on a set is closely related to the semigroup of all partial symmetries of that set, as established by R. Exel in [22] (see also [32]). A natural question arises: can a similar approach be applied to actions by derivations on algebras? Dealing with this question requires us to consider what we call an "*inverse semivector space*", which is a relaxed version of the notion of a vector space, in which the abelian (additive) group is replaced by a commutative (additive) inverse semigroup. In order to compare this idea with a known concept of a semimodule over a semiring, we recall formal definitions.

Definition 2.1 (see [25,50]). A *semiring* is a nonempty set S on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

(i) $(S, +)$ is a commutative monoid with identity element 0_S;
(ii) $(S, \cdot)$ is a monoid with identity element 1_S;
(iii) multiplication distributes over addition from either side;
(iv) $0s = 0 = s0$ for all $s \in S$.

Thus, any associative ring with unity element is a semiring in the sense of the above definition.

Definition 2.2 (see [25,50]). Let $(S, \cdot, +)$ be a semiring. An additive commutative monoid M is called a *semimodule* over S if for all $x, y \in M$ and $\alpha, \beta \in S$, the following conditions hold:

(1) $(\alpha + \beta)x = \alpha x + \beta x$;
(2) $\alpha(x + y) = \alpha x + \alpha y$;
(3) $\alpha(\beta x) = (\alpha\beta)x$;
(4) $1_S x = x$;
(5) $0_S x = 0_M$.

It is worth noting that if in Definition 2.2, S is a ring with unity (in particular, if it is a field), then conditions (1), (4), and (5) imply that M must be a group: the inverse of x is $(-1)x$. As a result, we recover the usual notion of a module over a ring. This shows that considering semimodules over rings does not yield anything new. In fact, as the reader will see in our motivating examples, we need a generalization of a vector space whose additive structure is not necessarily a group, but only an inverse semigroup. Moreover, it will also be seen that our examples do not satisfy axiom (5) of Definition 2.2, while satisfying all the others. This is reflected in the following definition, where the choice of the term "*inverse semivector space*" seems very natural and avoids confusion with existing concepts in the literature.

Definition 2.3. An *inverse semivector space* over a field $\mathbb{F}$ is a commutative additive inverse semigroup $(V, +)$ endowed with a multiplication by scalars $\mathbb{F} \times V \to V$, denoted by $(\alpha, x) \mapsto \alpha x$, which for every $x, y \in V$ and $\alpha, \beta \in \mathbb{F}$ satisfies:

(1) $(\alpha + \beta)x = \alpha x + \beta x$;

(2) $\alpha(x + y) = \alpha x + \alpha y$;

(3) $\alpha(\beta x) = (\alpha\beta)x$;

(4) $1x = x$.

Lemma 2.4. *Let $(V, +)$ be an inverse semivector space over the field $\mathbb{F}$. Then, for any $x, y \in V$ and $\alpha \in \mathbb{F}$, we have that*

(1) $(-\alpha)x = \alpha(-x) = -\alpha x$;

(2) $0x = 0_{\alpha x}$, *and in particular,* $0_x = 0_{\alpha x}$;

(3) $\alpha 0_x = 0_x$, *that is* $\alpha e = e$ *for any* $e \in \mathcal{E}(V)$;

(4) $x \preceq y$ *implies* $\alpha x \preceq \alpha y$.

Proof. (1) The property $(-\alpha)x = -\alpha x$ follows from the equalities

$$\alpha x = (\alpha - \alpha + \alpha)x = \alpha x + (-\alpha)x + \alpha x$$

and

$$(-\alpha)x = (-\alpha + \alpha - \alpha)x = (-\alpha)x + \alpha x + (-\alpha)x,$$

as $(V, +)$ is an inverse semigroup. The other equality is proven analogously.

(2) By item (1), we have $0x = (\alpha - \alpha)x = \alpha x + (-\alpha)x = \alpha x - \alpha x = 0_{\alpha x}$. The remaining assertion follows immediately.

(3) Keeping in mind item (2), we see that $\alpha 0_x = \alpha(x - x) = \alpha x - \alpha x = 0_{\alpha x} = 0_x$.

(4) If $x \preceq y$, then $x = e + y$ for some $e \in \mathcal{E}(V)$. Thus, by item (3), $\alpha x = \alpha e + \alpha y = e + \alpha y$, yielding $\alpha x \preceq \alpha y$. ■

Note that by (4) of Definition 2.3 and (1) of Lemma 2.4, we have that

$$(-1) \cdot x = -(1 \cdot x) = -x$$

for all x in an inverse semivector space V.

An *inverse semivector subspace* of V is subsemigroup W of $(V, +)$, which is closed under the multiplication by the scalars. If V and W are inverse semivector spaces, then a mapping $\phi\colon V \to W$ is called *linear* if for all $x, y \in V$ and $\alpha \in \mathbb{F}$, we have $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(\alpha x) = \alpha\phi(x)$. Additionally, if $(V, +)$ and $(W, +)$ are monoids, we say that a linear map $\phi\colon S \to T$ *is zero-preserving* if $\phi(0) = 0$.

Remark 2.5. Observe that for the linearity of a mapping $\phi\colon V \to W$ between inverse semivector spaces, it is enough to require that

$$\phi(x + y) = \phi(x) + \phi(y) \quad \text{and} \quad \phi(\alpha x) = \alpha\phi(x)$$

for all $x, y \in V$ and $0 \neq \alpha \in \mathbb{F}$. Indeed, for $\alpha = 0$ we see, using Lemma 2.4, that

$$\phi(0_x) = \phi(x + (-1)x) = \phi(x) + \phi((-1)x) = \phi(x) + (-1)\phi(x) = \phi(x) - \phi(x) = 0_{\phi(x)},$$

and so $\phi(0x) = 0\phi(x)$.

2.1. Inverse semialgebras

Suppose now that an inverse semivector space S over the field $\mathbb{F}$ is endowed with a mapping $\cdot : S \times S \to S$, which we shall denote by $(x, y) \mapsto x \cdot y$ and call *a multiplication map*. We shall say that S is *right distributive* if

$$(y + z) \cdot x = y \cdot x + z \cdot x \tag{2.2}$$

for all $x, y, z \in S$. Symmetrically, we define the left distributive property, and we shall say that S is *distributive* if it is both left and right distributive.

Lemma 2.6. *Let S be an inverse semivector space S over the field $\mathbb{F}$ endowed with a multiplication map.*

(1) *If S is right distributive, then $0_x \cdot y = 0_{x\cdot y}$, for all $x, y \in S$.*
(2) *If S is left distributive, then $x \cdot 0_y = 0_{x\cdot y}$, for all $x, y \in S$.*
(3) *If S is distributive, then $0_x \cdot 0_y = 0_{x\cdot y}$, for all $x, y \in S$.*

Proof. (1) We see that $0_x \cdot y = (x - x) \cdot y = x \cdot y - x \cdot y = 0_{x\cdot y}$ and (2) follows symmetrically. (3) Using (1) and (2), we obtain $0_x \cdot 0_y = 0_{x\cdot 0_y} = 0_{0_{x\cdot y}} = 0_{x\cdot y}$, as desired. ■

Usually, "*nonassociative*" means "*not necessarily associative*". Keeping this in mind, we give the next definition.

Definition 2.7. A *nonassociative inverse semialgebra* over the field $\mathbb{F}$ is an inverse semivector space S over $\mathbb{F}$ together with a map $\cdot : S \times S \to S$, denoted by $(x, y) \mapsto x \cdot y$, such that for every $x, y, z \in S$ and $0 \neq \alpha \in \mathbb{F}$, we have

(1) $\alpha(x \cdot y) = (\alpha x) \cdot y = x \cdot (\alpha y)$;
(2) $x \cdot (y + z) \succeq x \cdot y + x \cdot z$;
(3) $(x + z) \cdot y \succeq x \cdot y + z \cdot y$;
(4) $x \cdot (e + z) = x \cdot e + x \cdot z$ for all $e \in \mathcal{E}(S)$;
(5) $(x + e) \cdot z = x \cdot z + e \cdot z$ for all $e \in \mathcal{E}(S)$;
(6) $e + f \preceq e \cdot f \preceq f$ for all $e, f \in \mathcal{E}(S)$.

We shall say that a nonassociative inverse semialgebra S over the field $\mathbb{F}$ is *associative* if

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

for all $x, y, z \in S$.

Lemma 2.8. *Let S be a nonassociative inverse semialgebra over the field $\mathbb{F}$. Then*

(1) $e \cdot f + f \cdot e = e + f$ *for all* $e, f \in \mathcal{E}(S)$;
(2) $x \cdot 0_y,\ 0_x \cdot y \in \mathcal{E}(S)$ *for all* $x, y \in S$;
(3) $0_{x \cdot y} \preceq 0_x \cdot y$, $0_{x \cdot y} \preceq x \cdot 0_y$ *and* $0_{x \cdot y} \preceq 0_x \cdot 0_y$ *for all* $x, y \in S$;
(4) $x \cdot (y + z) - (x \cdot y + x \cdot z) = 0_{x \cdot y} + 0_{x \cdot z}$ *for all* $x, y, z \in S$.

Proof. (1) By item (6) of Definition 2.7, we have $e + f \preceq e \cdot f \preceq f$ and $f + e \preceq f \cdot e \preceq e$, for any $e, f \in \mathcal{E}(S)$, which gives

$$e + f \preceq e \cdot f + f \cdot e \preceq e + f,$$

implying the desired equality.

(2) By item (4) of Definition 2.7, we obtain that

$$x \cdot 0_y + x \cdot 0_y = x \cdot (0_y + 0_y) = x \cdot 0_y$$

and, analogously, $0_x \cdot y \in \mathcal{E}(S)$.

(3) By (4) and (5) of Definition 2.7, we see that

$$\begin{aligned} 0_{x \cdot y} &= x \cdot y - x \cdot y = (x + 0_x) \cdot (y + 0_y) - x \cdot y \\ &= x \cdot y + x \cdot 0_y + 0_x \cdot y + 0_x \cdot 0_y - x \cdot y = 0_{x \cdot y} + x \cdot 0_y + 0_x \cdot y + 0_x \cdot 0_y, \end{aligned}$$

which implies the desired inequalities in view of item (2).

(4) Since, $x \cdot y + x \cdot z \preceq x \cdot (y + z)$ (see (3) of Definition 2.7), we obtain that

$$x \cdot (y + z) - (x \cdot y + x \cdot z) = 0_{x \cdot y + x \cdot z} = 0_{x \cdot y} + 0_{x \cdot z},$$

as required. ∎

Notice that by (2) of Lemma 2.8, we have that $ef \in \mathcal{E}(S)$ for any $e, f \in \mathcal{E}(S)$. In addition, keeping in mind (2) of Lemma 2.4, observe that (3) of Lemma 2.8 says that

$$0(x \cdot y) \preceq (0x) \cdot y, \quad 0(x \cdot y) \preceq x \cdot (0y) \quad \text{and} \quad 0(x \cdot y) \preceq (0x) \cdot (0y)$$

for all $x, y \in S$.

Let V be a vector space over $\mathbb{F}$. A *partial* (*linear*) *endomorphism* of V is a linear map $\phi: K \to V$, where K is a subspace of V. The set of all partial endomorphisms of V will be denoted by $\mathsf{PEnd}(V)$. Given $\phi_1: K_1 \to V$ and $\phi_2: K_2 \to V$ in $\mathsf{PEnd}(V)$ and $\alpha \in \mathbb{F}$, we define the following operations:

$$\begin{aligned} \phi_1 + \phi_2 &: K_1 \cap K_2 \to V, && x \mapsto \phi_1(x) + \phi_2(x); \\ \phi_1 \phi_2 &: \phi_2^{-1}(K_1) \to V, && x \mapsto \phi_1(\phi_2(x)); \\ \alpha \phi_1 &: K_1 \to V, && x \mapsto \alpha \phi_1(x). \end{aligned}$$

Observe that the sum and the product of ϕ_1 and ϕ_2 are defined on the largest possible domains where the expressions $\phi_1(x) + \phi_2(x)$ and $\phi_1(\phi_2(x))$, respectively, make sense. Note also that the zero map $0\colon V \to V$ is a partial endomorphism on V.

Proposition 2.9. *Let V be a vector space. Then, $\mathsf{PEnd}(V)$ with the above defined operations is a right distributive associative inverse semialgebra.*

Proof. First, we show that $(\mathsf{PEnd}(V), +)$ is a commutative inverse monoid. Evidently, the sum is associative and commutative, and the zero map $0\colon V \to V$ is the neutral element. Now, for $\phi\colon K \to V \in \mathsf{PEnd}(V)$ it is clear that its (additive) inverse is given by the mapping $-\phi\colon K \to V$, $x \mapsto -\phi(x)$.

For the rest of the proof, let $\phi_1\colon K_1 \to V$, $\phi_2\colon K_2 \to V$ and $\phi_3\colon K_3 \to V$ be in $\mathsf{PEnd}(V)$.

It is easy check that $\mathsf{PEnd}(V)$ is an inverse semivector space. Indeed, to see (1) of Definition 2.3, observe that $\operatorname{dom}(\alpha(\phi_1 + \phi_2)) = K_1 \cap K_2 = \operatorname{dom}(\alpha\phi_1 + \alpha\phi_2)$, and for x in $K_1 \cap K_2$, the equality $(\alpha(\phi_1 + \phi_2))(x) = (\alpha\phi_1)(x) + (\alpha\phi_2)(x)$ is immediate. Similarly, the equalities

$$\operatorname{dom}((\alpha+\beta)\phi_1) = K_1 = \operatorname{dom}(\alpha\phi_1 + \beta\phi_1) \quad \text{and} \quad \operatorname{dom}((\alpha\beta)\phi_1) = K_1 = \operatorname{dom}(\alpha(\beta\phi_1))$$

readily give us (2) and (3) of Definition 2.3, respectively.

Finally, it is obvious that $1_{\mathbb{F}}\phi_1 = \phi_1$, so that (4) of Definition 2.3 also holds.

Notice that 0_ϕ coincides with the restriction $0_{\operatorname{dom}\phi}$ of the zero map $0\colon V \to V$ to $\operatorname{dom}\phi$, and for subspaces W_1 and W_2 of V, we have $0_{W_1} + 0_{W_2} = 0_{W_1 \cap W_2}$, so that

$$\mathcal{E}(\mathsf{PEnd}(V)) = \{0_\phi \mid \phi \in \mathsf{PEnd}(V)\} = \{0_W \mid W \text{ is a subspace of } V\}.$$

Hence, $\phi_1 \preceq \phi_2 \iff \phi_1$ is a restriction of ϕ_2. Since the composition of partial endomorphisms is associative, it remains to verify that the inverse semivector space $\mathsf{PEnd}(V)$ satisfies Definition 2.7 and that $\mathsf{PEnd}(V)$ is right distributive.

(1) For $0 \neq \alpha \in \mathbb{F}$, we have

$$\operatorname{dom}(\alpha(\phi_1\phi_2)) = \operatorname{dom}((\alpha\phi_1)\phi_2) = \operatorname{dom}(\phi_1(\alpha\phi_2))) = \phi_2^{-1}(K_1),$$

and clearly $\alpha(\phi_1\phi_2)(x) = (\alpha\phi_1)(\phi_2(x)) = \phi_1((\alpha\phi_2)(x))$ for all $x \in \phi_2^{-1}(K_1)$.

(2) Note that $\operatorname{dom}(\phi_1(\phi_2 + \phi_3)) = (\phi_2 + \phi_3)^{-1}(K_1)$ and $\operatorname{dom}(\phi_1\phi_2) \cap \operatorname{dom}(\phi_1\phi_3) = \phi_2^{-1}(K_1) \cap \phi_3^{-1}(K_1)$. Then $\operatorname{dom}(\phi_1\phi_2 + \phi_1\phi_3) \subseteq \operatorname{dom}\phi_1(\phi_2 + \phi_3)$. Hence, for all x in $\phi_2^{-1}(K_1) \cap \phi_3^{-1}(K_1)$, we have

$$(\phi_1(\phi_2 + \phi_3))(x) = (\phi_1\phi_2)(x) + (\phi_1\phi_3)(x),$$

yielding $\phi_1(\phi_2 + \phi_3) \succeq \phi_1\phi_2 + \phi_2\phi_3$.

(3) Observe that

$$\operatorname{dom}((\phi_1 + \phi_2)\phi_3) = \phi_3^{-1}(K_1 \cap K_2) = \phi_3^{-1}(K_1) \cap \phi_3^{-1}(K_2) = \operatorname{dom}(\phi_1\phi_3 + \phi_2\phi_3),$$

and for all $x \in \phi_3^{-1}(K_1 \cap K_2)$, we have

$$((\phi_1 + \phi_2)\phi_3)(x) = (\phi_1\phi_3)(x) + (\phi_2\phi_3)(x).$$

Consequently, $(\phi_1 + \phi_2)\phi_3 = \phi_1\phi_3 + \phi_2\phi_3$, proving (2.2). In particular, (3) of Definition 2.7 holds.

(4) It is easily seen that

$$\begin{aligned}\mathrm{dom}(\phi_1(0_{\phi_3}+\phi_2)) &= (0_{\phi_3}+\phi_2)^{-1}(K_1) = \{x \in K_3 \cap K_2 \mid \phi_2(x) \in K_1\}\\ &= K_3 \cap \phi_2^{-1}(K_1),\end{aligned}$$

and

$$\mathrm{dom}(\phi_1 0_{\phi_3} + \phi_1\phi_2) = 0_{\phi_3}^{-1}(K_1) \cap (\phi_2)^{-1}(K_1) = K_3 \cap \phi_2^{-1}(K_1).$$

Thus

$$\mathrm{dom}(\phi_1(0_{\phi_3}+\phi_2)) = \mathrm{dom}(\phi_1 0_{\phi_3} + \phi_1\phi_2).$$

As

$$\phi_1((0_{\phi_3}+\phi_2)(x)) = \phi_1(\phi_2(x)) = \phi_1(0_{\phi_3}(x)) + \phi_1(\phi_2(x)),$$

we conclude that $\phi_1(0_{\phi_3}+\phi_2) = \phi_1 0_{\phi_3} + \phi_1\phi_2$.

(5) This item follows from the right distributivity property (2.2).

(6) Clearly, $\mathrm{dom}(0_{\phi_1}0_{\phi_2}) = 0_{\phi_2}^{-1}(K_1) = K_2$, so that

$$\mathrm{dom}(0_{\phi_1}+0_{\phi_2}) = K_1 \cap K_2 \subseteq K_2 = \mathrm{dom}(0_{\phi_1}0_{\phi_2}) = \mathrm{dom}(0_{\phi_2}),$$

resulting in

$$0_{\phi_1}+0_{\phi_2} \preceq 0_{\phi_1}0_{\phi_2} = 0_{\phi_2}.$$

In particular, (6) of Definition 2.7 holds.

This concludes the proof. ∎

Observe that $\mathsf{PEnd}(V)$ satisfies the following specific property:

$$\phi_1 0_{\phi_2} = 0_{\phi_2},$$

for all $\phi_1, \phi_2 \in \mathsf{PEnd}(V)$. Indeed, using the above notation,

$$\mathrm{dom}(\phi_1 0_{\phi_2}) = 0_{\phi_2}^{-1}(K_1) = K_2 = \mathrm{dom}(0_{\phi_2})$$

and

$$\phi_1 0_{\phi_2}(x) = \phi_1(0) = 0 = 0_{\phi_2}(x), \quad \text{for all } x \in K_2,$$

as desired.

2.2. Lie inverse semialgebras

Definition 2.10. A *Lie inverse semialgebra* over the field $\mathbb{F}$ is an inverse semivector space S over the field $\mathbb{F}$ with a map $[\cdot,\cdot]\colon S \times S \to S$ given by $(x,y) \mapsto [x,y]$ such that for every $x,y,z \in S$ and $0 \neq \alpha \in \mathbb{F}$

(1) $\alpha[x,y] = [\alpha x, y] = [x, \alpha y]$;

(2) $[x, y+z] \succeq [x,y] + [x,z]$;

(3) $[x,y] = -[y,x]$;

(4) $[x, e+z] = [x,e] + [x,z]$ for all $e \in \mathcal{E}(S)$;

(5) $J(x,y,z) := [[x,y],z] + [[y,z],x] + [[z,x],y] \preceq 0_{x+y+z}$;

(6) $e + f = [e,f]$ for all $e, f \in \mathcal{E}(S)$.

The operation $[\cdot,\cdot]$ in Definition 2.10 will be called the Lie bracket.

Remark 2.11. It might be interesting for the reader to compare our Definition 2.10 with the notion of a Lie $\preceq$-semialgebra with a preorder $\preceq$ given in Definition 3.4 of [15], where the authors deal with semialgebras equipped with a negation map.

Remark 2.12. Any Lie inverse semialgebra over the field $\mathbb{F}$ is a nonassociative inverse semialgebra over $\mathbb{F}$. Indeed, condition (2) of Definition 2.10, using items (1) and (4) of Lemma 2.4, implies that

$$-[x, y+z] \succeq -([x, y]+[x, z]) = -[x, y]-[x, z].$$

So, by (3) of Definition 2.10 we get $[y+z, x] \succeq [y, x]+[z, x]$, which gives (3) of Definition 2.7. Next, combining (3) and (4) of Definition 2.10, we see that

$$[z+e, x] = -[x, z+e] = -[x, z]-[x, e] = [z, x]+[e, x],$$

which shows (5) of Definition 2.7. Finally, (6) of Definition 2.7 immediately follows from (6) of Definition 2.10. Thus, it is clear that if a nonassociative inverse semialgebra S, whose operation is denoted by the Lie bracket, satisfies (3), (5) and (6) of Definition 2.10, then S is a Lie inverse semialgebra. In this case we shall also say that the nonassociative inverse semialgebra S is Lie.

A *Lie inverse subsemialgebra* of a Lie inverse semialgebra S is an inverse semivector subspace of S which is closed under the Lie bracket. A *homomorphism* between two Lie inverse semialgebras S and T is an $\mathbb{F}$-linear map $\phi: S \to T$ such that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in S$. Evidently, the Lie inverse semialgebras with their homomorphisms form a category. If $(S, +)$ and $(T, +)$ are monoids, we say that a premorphism $\phi: S \to T$ *preserves 0* if $\phi(0) = 0$.

The following lemma deals with some elementary properties of the Lie inverse semialgebras, which we will frequently use throughout the paper.

Lemma 2.13. *Let S be a Lie inverse semialgebra over the field $\mathbb{F}$. Then for all $x, y, z \in S$ and $e \in \mathcal{E}(S)$, we have*:

(1) $[x, -y] = -[x, y] = [-x, y]$;

(2) $[x, e] \in \mathcal{E}(S)$;

(3) $0_{[x,y]} \preceq [x, 0_y] + [0_x, y] + 0_{x+y}$;

(4) *If* $\operatorname{char} \mathbb{F} \neq 2$, *then* $[x, x] \preceq 0_x$.

Proof. (1) Using (1) of Definition 2.10 and Lemma 2.4, we obtain

$$[x, -y] = [x, (-1)y] = (-1)[x, y] = -[x, y].$$

Similarly, $[-x, y] = -[x, y]$.

(2) This directly follows from (2) of Lemma 2.8.

(3) Item (3) of Lemma 2.8 gives

$$0_{[x,y]} \preceq [0_x, y] + [x, 0_y] + [0_x, 0_y] = [0_x, y] + [x, 0_y] + 0_{x+y},$$

since $[0_x, 0_y] = 0_x + 0_y = 0_{x+y}$ thanks to condition (6) of Definition 2.10.

(4) By item (3) of Definition 2.10, we have that $[x,x]+[x,x]=[x,x]-[x,x]=0_{[x,x]}\in\mathcal{E}(L)$. Thus $[x,x]=\frac{1}{2}0_{[x,x]}=0_{[x,x]}$, which, by item (3), implies that $[x,x]\preceq 0_{x+x}=0_{2x}=0_x$. ∎

Remark 2.14. It follows from (2) of Lemma 2.13 and items (3) and (4) of Definition 2.10 that if S is a Lie inverse semialgebra over $\mathbb{F}$ and $x,y\in S$, then

$$x\preceq y\Rightarrow[x,z]\preceq[y,z]\ \text{ and }\ [z,x]\preceq[z,y]\quad\forall z\in S. \tag{2.3}$$

Indeed, $x\preceq y$ means that $x=e+y$ for some $e\in\mathcal{E}(S)$. Then $[x,z]=[e+y,z]=[e,z]+[y,z]\preceq[y,z]$, and the second inequality is an immediate consequence of the first one.

Example 2.15. Every inverse semivector space V has a trivial structure of a Lie inverse semialgebra, which is given by $[x,y]=0_{x+y}$ for any $x,y\in V$.

Remark 2.16. If S is an associative inverse semialgebra over the field $\mathbb{F}$, then

$$0_{xy-yx}\preceq 0_x+0_y, \tag{2.4}$$

for all $x,y\in S$. Indeed, using (3) of Lemma 2.8 and (6) of Definition 2.7, we see that

$$0_{xy-yx}=0_{xy}+0_{yx}\preceq 0_x0_y+0_y0_x\preceq 0_y+0_x,$$

as desired.

Proposition 2.17. *Let S be a right distributive associative inverse semialgebra over the field $\mathbb{F}$. Then, the inverse semivector space S endowed with the bracket $[x,y]=xy-yx$ is a Lie inverse semialgebra in which the following identity holds*

$$0_{[x,y]}=[0_x,y]+[x,0_y], \tag{2.5}$$

for all $x,y\in S$.

Proof. We will verify conditions (1)-(6) of Definition 2.10 for every $x,y,z\in S$, $e,f\in\mathcal{E}(S)$ and $0\neq\alpha\in\mathbb{F}$.

(1) The equalities $\alpha[x,y]=[\alpha x,y]=[x,\alpha y]$ follow immediately from condition (1) of Definition 2.7.

(2) By (2) and (3) of Definition 2.7 we have

$$\begin{aligned}[x,y]+[x,z]&=xy-yx+xz-zx=(xy+xz)-(yx+zx)\\&\preceq x(y+z)-(y+z)x=[x,y+z].\end{aligned}$$

(3) We see that $[x,y]=xy-yx=-(yx-xy)=-[y,x]$.

(4) By condition (4) and (5) of Definition 2.7 we obtain

$$[x,e+z]=x(e+z)-(e+z)x=xe+xz-ex-zx=[x,e]+[x,z].$$

(5) We must to show that $J(x,y,z)\preceq 0_x+0_y+0_z$. Since S is right distributive we see that

$$\begin{aligned}J(x,y,z)=&(xy)z-(yx)z-z(xy-yx)+(yz)x-(zy)x-x(yz-zy)\\&+(zx)y-(xz)y-y(zx-xz).\end{aligned}\tag{$*$}$$

Keeping in mind that S is associative and applying (2) of Definition 2.7, (3) of Lemma 2.8, (6) of Definition 2.7 and then (2.4), we obtain that $(*)$ is equal to

$$\begin{aligned}&= x(yz) - x(zy) - x(yz - zy) + y(zx) - y(xz) - y(zx - xz) + z(xy) - z(yx)\\&\quad - z(xy - yx)\\&\preceq x[y,z] - x[y,z] + y[z,x] - y[z,x] + z[x,y] - z[x,y] = 0_{x[y,z]} + 0_{y[z,x]} + 0_{z[x,y]}\\&\preceq 0_x 0_{[y,z]} + 0_y 0_{[z,x]} + 0_z 0_{[x,y]} \preceq 0_{[y,z]} + 0_{[z,x]} + 0_{[x,y]} \preceq 0_x + 0_y + 0_z = 0_{x+y+z}.\end{aligned}$$

(6) By (2) and (1) of Lemma 2.8, we obtain

$$[e,f] = ef - fe = ef + fe = e + f,$$

for all $e, f \in \mathcal{E}(S)$.

For the last assertion, note that (3) of Lemma 2.8 and (1) of Lemma 2.6 imply

$$0_{xy} = 0_{xy} + x0_y = 0_x y + x0_y$$

and similarly, $0_{yx} = 0_y x + y0_x$. Consequently, using (2) of Lemma 2.8, we conclude that

$$\begin{aligned}0_{[x,y]} &= 0_{xy-yx} = 0_{xy} + 0_{yx} = 0_x y + x0_y + 0_y x + y0_x\\&= 0_x y - y0_x + x0_y - 0_y x = [0_x, y] + [x, 0_y],\end{aligned}$$

as required. ∎

The Lie inverse semialgebra of the Proposition 2.17 will be denoted by S^-. As an immediate consequence of Propositions 2.17 and 2.9, we have:

Corollary 2.18. *Let V be a vector space. Then the set $\mathsf{PEnd}(V)^-$ of all partial endomorphism of V is a Lie inverse semialgebra.*

Definition 2.19. Let A be a nonassociative algebra. A *partial derivation* of A is a linear map $\phi\colon K \to A$, where K is a subalgebra of A and $\phi(xy) = \phi(x)y + x\phi(y)$ for every $x, y \in K$. The set of all partial derivations of A will be denoted by $\mathsf{PDer}(A)$.

Proposition 2.20. *Let A be a nonassociative algebra. Then, $\mathsf{PDer}(A)$ is a Lie inverse subsemialgebra of $\mathsf{PEnd}(A)^-$.*

Proof. Given $\alpha \in \mathbb{F}$ and $\phi_i : K_i \to A \in \mathsf{PDer}(A)$, $i = 1,2$, we observe that $\operatorname{dom}(\alpha\phi_1) = K_1$, $\operatorname{dom}(\phi_1 + \phi_2) = K_1 \cap K_2$ and $\operatorname{dom}([\phi_1,\phi_2]) = \phi_1^{-1}(K_2) \cap \phi_2^{-1}(K_1) \subseteq K_1 \cap K_2$ are subalgebras of A. It is immediate to check that $\alpha\phi_1 \in \mathsf{PDer}(A)$ and $\phi_1 + \phi_2 \in \mathsf{PDer}(A)$, so that $\mathsf{PDer}(A)$ is an inverse semivector subspace of $\mathsf{PEnd}(A)$. It remains to show that $\mathsf{PDer}(A)$ is closed under the Lie bracket of $\mathsf{PEnd}(A)^-$, i.e., $[\phi_1,\phi_2] = \phi_1\phi_2 - \phi_2\phi_1 \in \mathsf{PDer}(A)$. Indeed, for all $x, y \in \phi_1^{-1}(K_2) \cap \phi_2^{-1}(K_1)$, we have

$$\begin{aligned}\phi_1\phi_2(xy) &= \phi_1\big(\phi_2(x)y + x\phi_2(y)\big)\\&= (\phi_1\phi_2(x))y + \phi_2(x)\phi_1(y) + \phi_1(x)\phi_2(y) + x(\phi_1\phi_2(y)),\end{aligned}$$

and interchanging ϕ_1 and ϕ_2,

$$\begin{aligned}\phi_2\phi_1(xy) &= \phi_2\big(\phi_1(x)y + x\phi_1(y)\big)\\&= (\phi_2\phi_1(x))y + \phi_1(x)\phi_2(y) + \phi_2(x)\phi_1(y) + x(\phi_2\phi_1(y)).\end{aligned}$$

Thus $[\phi_1,\phi_2](xy) = [\phi_1,\phi_2](x)y + x[\phi_1\phi_2](y)$, as required. ∎

We present another pivotal example of a Lie inverse semialgebra, which will be frequently used in the subsequent sections. It is inspired by the inverse semigroup $S(G)$ introduced by R. Exel in [22] to govern partial representations of a group G and partial actions of G on sets. It is known that $S(G)$ is isomorphic to the Szendrei expansion of G, which is isomorphic to the Birget–Rhodes prefix expansion of G (see [32, 48]). Our idea is to work with a "Szendrei-like" construction in the Lie context.

Let L be a Lie algebra. Let us consider the set

$$E(L) := \{(A, a) \mid A \text{ is a finite-dimensional subspace of } L, a \in A\}.$$

In $E(L)$, we define the sum and the Lie bracket of two elements (A, a) and (B, b) as follows:

$$(A, a) + (B, b) = (A + B, a + b) \quad \text{and} \quad [(A, a), (B, b)] = (A + B + \mathbb{F}[a, b], [a, b]).$$

Additionally, for $\alpha \in \mathbb{F}$, we define

$$\alpha(A, a) = (A, \alpha a).$$

It is straightforward to verify that $(E(L), +)$ is an inverse semivector space with zero element $(0, 0)$, in which the (additive) inverse of (A, a) is $(A, -a)$. Moreover, for any (A, a), we have that $0_{(A,a)} = (A, 0)$, and so

$$\mathcal{E}(E(L)) = \{(A, 0) \mid A \text{ is a finite-dimensional subspace of } L\}.$$

Consequently, $(A, a) \preceq (B, b)$ holds if and only if

$$(A, a) + (A, -a) + (B, b) = (A, a) \iff (A + B, b) = (A, a) \iff a = b \text{ and } B \subseteq A.$$

Proposition 2.21. *$E(L)$ endowed with the above operations is a Lie inverse semialgebra, such that*

$$[0_x, y] = [x, 0_y] = 0_{x+y}, \tag{2.6}$$

for all $x, y \in E(L)$.

Proof. We verify conditions (1)-(6) of Definition 2.10 for all $(A, a), (B, b), (C, c) \in E(L)$ and $0 \neq \alpha \in \mathbb{F}$.

(1) We have

$$\alpha[(A, a), (B, b)] = \alpha(A + B + \mathbb{F}[a, b], [a, b]) = (A + B + \mathbb{F}[a, b], \alpha[a, b]).$$

On the other hand,

$$\begin{aligned}[\alpha(A, a), (B, b)] &= [(A, \alpha a), (B, b)] = (A + B + \mathbb{F}[\alpha a, b], [\alpha a, b]) \\ &= (A + B + \mathbb{F}[a, b], \alpha[a, b]).\end{aligned}$$

Thus, we have shown that $\alpha[(A, a), (B, b)] = [\alpha(A, a), (B, b)]$. Similarly, we can check the equality $\alpha[(A, a), (B, b)] = [(A, a), \alpha(B, b)]$.

(2) We see that

$$[(A,a),(B,b)+(C,c)] = [(A,a),(B+C,b+c)] = (A+B+C+\mathbb{F}[a,b+c],[a,b+c]).$$

On the other hand,

$$\begin{aligned}[(A,a),(B,b)]+[(A,a),(C,c)] &= (A+B+\mathbb{F}[a,b],[a,b])+(A+C+\mathbb{F}[a,c],[a,c]) \\ &= (A+B+C+\mathbb{F}[a,b]+\mathbb{F}[a,c],[a,b+c]).\end{aligned}$$

As $\mathbb{F}[a,b+c] \subseteq \mathbb{F}[a,b]+\mathbb{F}[a,c]$, we obtain

$$[(A,a),(B,b)+(C,c)] \succeq [(A,a),(B,b)]+[(A,a),(C,c)].$$

(3) This is straightforward from the definition of the Lie bracket in $E(L)$.

(4) We compute that

$$\begin{aligned}[(A,a),(B,0)+(C,c)] &= [(A,a),(B+C,c)] = (A+B+C+\mathbb{F}[a,c],[a,c]) \\ &= (A+B,0)+(A+C+\mathbb{F}[a,c],[a,c]) \\ &= [(A,a),(B,0)]+[(A,a),(C,c)],\end{aligned}$$

as desired.

(5) Since

$$\begin{aligned}[[(A,a),(B,b)],(C,c)] &= [(A+B+\mathbb{F}[a,b],[a,b]),(C,c)] \\ &= (A+B+C+\mathbb{F}[a,b]+\mathbb{F}[[a,b],c],[[a,b],c])\end{aligned}$$

we have that $J((A,a),(B,b),(C,c))$ is equal to

$$\begin{aligned}&= (A+B+C+\mathbb{F}[a,b]+\mathbb{F}[b,c]+\mathbb{F}[c,a]+\mathbb{F}[[a,b],c]+\mathbb{F}[[b,c],a]+\mathbb{F}[[c,a],b],0) \\ &\preceq (A+B+C,0) = 0_{(A,a)+(B,b)+(C,c)}.\end{aligned}$$

(6) This immediately follows from the definition of the Lie bracket in $E(L)$.

Finally, for any $(A,a),(B,b) \in E(L)$, we have that

$$[0_{(A,a)},(B,b)] = [(A,0),(B,b)] = (A+B,0) = [(A,a),0_{(B,b)}],$$

showing (2.6). ■

Example 2.22. Let L be the Heisenberg 3-dimensional Lie algebra with basis $\{a,b,c\}$, where $[a,b]=c$ and $[a,c]=[b,c]=0$. Let $A=\mathbb{F}a$ and $B=\mathbb{F}b$. Then,

$$0_{[(A,a),(B,b)]} = (L,0) \neq (A+B,0) = [0_{(A,a)},(B,b)]+[(A,a),0_{(B,b)}].$$

Thus, $x=(A,a)$, $y=(B,b)$ do not satisfy (2.5) and by Proposition 2.17, $E(L)$ cannot be embedded in S^- for a right distributive associative inverse semialgebra S. Note also that

$$0_{[x,y]} = (L,0) \neq (A+B,0) = 0_{x+y}.$$

In the next section, we will see that certain types of Lie inverse semialgebras can be embedded into some S^- for an appropriate distributive associative inverse semialgebra.

3. Semilattices of algebras

Let Λ be a poset. If λ and μ are elements of Λ, then their *meet* is an element $\lambda \wedge \mu$ such that

(1) $\lambda \geq \lambda \wedge \mu$ and $\mu \geq \lambda \wedge \mu$,
(2) if $\nu \leq \lambda$ and $\nu \leq \mu$, then $\nu \leq \lambda \wedge \mu$.

A poset Λ is called a *meet semilattice* if every finite subset of Λ has a meet. Equivalently, a meet semilattice is a commutative semigroup $(\Lambda, \wedge)$ in which every element is idempotent. For instance, the set of idempotents of an inverse semigroup is a meet semilattice. In particular, $\mathcal{E}(V)$ is a meet semilattice, where V is an inverse semivector space.

A meet semilattice Λ can be also viewed as a category, where the objects are the elements of Λ, and for $\lambda, \mu \in \Lambda$, there is exactly one arrow $\lambda \to \mu$ whenever $\lambda \leq \mu$ and no arrow otherwise. A presheaf on Λ with values in the category $\mathcal{C}$ is a contravariant functor $F\colon \Lambda \to \mathcal{C}$. The main objective of this section is to show that the presheaves on a meet semilattice, with values in the category of nonassociative algebras, correspond to a certain subclass of nonassociative inverse semialgebras.

Definition 3.1. A *semilattice of nonassociative algebras* over the field $\mathbb{F}$ is a nonassociative inverse semialgebra S over $\mathbb{F}$ such that

$$0_{xy} = 0_{x+y} \tag{3.1}$$

for any $x, y \in S$. If, in addition, S is associative (or Lie) then S will be called a *semilattice of associative* (*or Lie*) *algebras*.

Condition (3.1) implies distributivity:

Lemma 3.2. *If S is a semilattice of nonassociative algebras over $\mathbb{F}$, then S is distributive.*

Proof. Let $x, y, z \in S$. Since $xy + xz \preceq x(y + z)$, we have, using (4) of Lemma 2.8 and (3.1), that

$$\begin{aligned} xy + xz &= 0_{xy} + 0_{xz} + x(y+z) = 0_{x+y} + 0_{x+z} + x(y+z) \\ &= 0_{x+(y+z)} + x(y+z) = 0_{x(y+z)} + x(y+z) = x(y+z). \end{aligned}$$

Analogously, we check that $(y + z)x = yx + zx$. ∎

The behavior of the (additive) idempotents also becomes better. Indeed, Lemma 3.2, Lemma 2.6 and (3.1) directly imply:

Lemma 3.3. *Let S be a semilattice of nonassociative algebras over $\mathbb{F}$. Then*

$$x0_y = 0_x y = 0_x 0_y = 0_{xy} = 0_x + 0_y$$

for all $x, y \in S$. In particular,

$$ef = e + f = fe$$

for all $e, f \in \mathcal{E}(S)$.

To be aligned with our notation used in the case of Lie inverse semialgebras, we use brackets for the product in semilattices of Lie algebras. In this case, we have the following.

Lemma 3.4. *Let S be a semilattice of Lie algebras over $\mathbb{F}$. Then for all $x, y, z \in S$,*

$$J(x,y,z) = 0_{J(x,y,z)} = 0_{x+y+z}.$$

Moreover, if char $\mathbb{F} \neq 2$, *then*

$$[x,x] = 0_x.$$

Proof. In the Lie case, (3.1) takes the form

$$0_{[x,y]} = 0_{x+y},$$

which readily implies that

$$0_{[[x,y],z]} = 0_{[x,y]} + 0_z = 0_x + 0_y + 0_z$$

for all $x, y, z \in S$. Then, since $J(x,y,z) \preceq 0_{x+y+z}$, we obtain

$$\begin{aligned} J(x,y,z) = J(x,y,z) + 0_{x+y+z} &= 0_{[[x,y],z]} + 0_{[[y,z],x]} + 0_{[[z,x],y]} + 0_{x+y+z} \\ &= 0_x + 0_y + 0_z + 0_{x+y+z} = 0_{x+y+z} \end{aligned}$$

proving $J(x,y,z) = 0_{x+y+z}$.

Suppose that char $\mathbb{F} \neq 2$. Then by (4) of Lemma 2.13, $[x,x] \preceq 0_x$ and, consequently,

$$[x,x] = 0_{[x,x]} + 0_x = 0_x,$$

as desired. ■

Remark 3.5. Let S be a semilattice of associative algebras. Then (3.1) readily implies that $0_{[x,y]} = 0_{x+y}$, so by Lemma 3.2 and Proposition 2.17, we obtain that S^- is a semilattice of Lie algebras.

Evidently, the collection of semilattices of associative (or Lie) algebras constitutes a subcategory within the category of associative (or Lie) inverse semialgebras.

Example 3.6. Let $\mathcal{F}$ be a presheaf of associative $\mathbb{F}$-algebras over a topological space (X,τ). Given $U, V \in \tau$, with $V \subseteq U$, denote by $\operatorname{res}^U_V \colon \mathcal{F}(U) \to \mathcal{F}(V)$ the restriction homomorphism. Take the disjoint union

$$S_{\mathcal{F}} = \dot{\bigcup}_{U\in\tau} \mathcal{F}(U)$$

and define the following operations on $S_{\mathcal{F}}$: for $U, V \in \tau$, $x \in \mathcal{F}(U)$, $y \in \mathcal{F}(V)$ and $\alpha \in \mathbb{F}$, put

$$\begin{aligned} \alpha \cdot x &:= \alpha x \in \mathcal{F}(U), \\ x + y &:= \operatorname{res}^U_{U\cap V}(x) + \operatorname{res}^V_{U\cap V}(y) \in \mathcal{F}(U\cap V), \\ x \cdot y &:= \operatorname{res}^U_{U\cap V}(x) \cdot \operatorname{res}^V_{U\cap V}(y) \in \mathcal{F}(U\cap V). \end{aligned}$$

Then it is easy to see that $S_{\mathcal{F}}$ is a semilattice of associative $\mathbb{F}$-algebras with the above defined operations. The following are particular cases of this construction:

(1) $S_{\mathcal{F}}$ is the set of all sections of the structural sheaf $\mathcal{F}$ of a scheme X over $\mathbb{F}$.

(2) $S_{\mathcal{F}}$ is the set of all regular functions $U \to \mathbb{F}$, where U runs over all open subsets of an algebraic variety X over $\mathbb{F}$; $\mathcal{F}$ is the sheaf of regular functions whose restriction maps are the usual restrictions of functions.

(3) $S_{\mathcal{F}} = \mathcal{F}_{\mathrm{par}}(X, \mathbb{R})$ is the set of all continuous functions $f : U \to \mathbb{R}$, where U runs over all open subsets of a topological space X. Such functions form a sheaf on X whose restriction maps are the usual restrictions of functions. Replacing $\mathbb{R}$ by $\mathbb{C}$, we obtain a similar semilattice $\mathcal{F}_{\mathrm{par}}(X, \mathbb{C})$ of associative algebras over $\mathbb{C}$. In particular, if X is discrete, then $\mathcal{F}_{\mathrm{par}}(X, \mathbb{R})$ (or $\mathcal{F}_{\mathrm{par}}(X, \mathbb{C})$) consists of all partial functions (functions defined on subsets of X) with values in $\mathbb{R}$ (or $\mathbb{C}$).

(4) Let X be a smooth manifold. Then the set of all smooth functions $U \to \mathbb{R}$, where U runs over the open subsets of X (considered as embedded open submanifolds), forms a semilattice of associative algebras over $\mathbb{R}$ with the operations defined using the corresponding sheaf of functions.

(5) Let X be a complex manifold. Then the set of all holomorphic functions $U \to \mathbb{C}$, where U runs over the open subsets of X, forms a semilattice of associative algebras over $\mathbb{C}$ with the operations defined using the corresponding sheaf of functions.

(6) Let X be a topological space. For any open subset U of X, let $\mathcal{F}(U)$ be the C^*-algebra $C_b(U)$ of the bounded continuous functions $U \to \mathbb{C}$, and let $\mathcal{F}$ be the corresponding sheaf with the usual restriction of functions. Then $S_{\mathcal{F}}$ is a semilattice of associative algebras over $\mathbb{C}$ (this is an example of what could be called a semilattice of C^*-algebras).

Example 3.7. Let $\mathcal{F}$ be a presheaf of associative $\mathbb{F}$-algebras over a topological space (X, τ) and let $S_{\mathcal{F}}$ be the semilattice of associative algebras from Example 3.6. Then by Remark 3.5, $S_{\mathcal{F}}^-$ is a semilattice of Lie algebras. Clearly,

$$S_{\mathcal{F}}^- = \dot{\bigcup}_{U \in \tau} \mathcal{F}(U)^-,$$

where each $\mathcal{F}(U)^-$ is the Lie algebra obtained by endowing $\mathcal{F}(U)$ with the Lie bracket $[x, y] = xy - yx$. In particular, items (1)–(7) from Example 3.6 produce semilattices of Lie algebras.

Example 3.8. If L is an abelian Lie algebra, then it is clear that $E(L)$ is a semilattice of Lie algebras. Similarly, if L is a 2-dimensional solvable Lie algebra, $E(L)$ is also a semilattice of Lie algebras. Indeed, for all (A, a) and (B, b) in $E(L)$, we need to check that

$$A + B + \mathbb{F}[a, b] = A + B.$$

If either A or B is a trivial subspace of L, the equality is immediate. Now, assume that A and B are non-trivial subspaces of L. If $A \neq B$, we have $A + B = L$, and the equality still holds. Finally, if $A = B$, then $[a, b] = 0$ for any $a, b \in A$, as A is 1-dimensional. Thus, the equality holds for every $(A, a), (B, b) \in E(L)$.

On the other hand, if L is the 3-dimensional Heisenberg algebra, then it follows from Example 2.22 that $E(L)$ is not a semilattice of Lie algebras.

To provide more examples of semilattices of Lie algebras, we introduce the following notation. Let $\mathcal{A}$ be a class of nonassociative algebras, and let $A \in \mathcal{A}$. Let us consider the set

$$\mathsf{PDer}_{\mathcal{A}}(A) = \{\phi : I \to A \in \mathsf{PDer}(A) \mid I \trianglelefteq A \text{ and } I \in \mathcal{A}\},$$

where $I \trianglelefteq A$ denotes the fact that I is a two-sided ideal of A. Before stating the following proposition, which offers examples of semilattices of Lie algebras, we first recall that a Lie algebra L is called *sympathetic* if it is perfect, centerless, and has no outer derivations, i.e., $[L, L] = L$, $Z(L) = 0$ and $\operatorname{Der}(L) = \operatorname{ad}(L)$. Every semisimple Lie algebra is sympathetic, but there are sympathetic Lie algebras that are not semisimple.

Proposition 3.9. *Suppose that $\mathcal{A}$ is any of the following classes: unital associative algebras, unital Jordan algebras, sympathetic Lie algebras or semisimple Lie algebras. Let $A \in \mathcal{A}$. Then, for $\phi_i\colon K_i \to A$ in $\mathsf{PDer}_{\mathcal{A}}(A)$, $i = 1, 2$, we have*

$$\operatorname{dom}(\phi_1 + \phi_2) = K_1 \cap K_2 = \phi_1^{-1}(K_2) \cap \phi_2^{-1}(K_1) = \operatorname{dom}([\phi_1, \phi_2]).$$

Consequently, $\mathsf{PDer}_{\mathcal{A}}(A)$ is a Lie inverse subsemialgebra of $\mathsf{PDer}(A)$ which is a semilattice of Lie algebras.

Proof. First observe that the intersection of two unital ideals of an associative algebra is unital and that every ideal of a semisimple Lie algebra is itself semisimple. Furthermore, if $\mathcal{A}$ is the class of sympathetic Lie algebras or unital Jordan algebras, then Corollary 2.11 in [49] and Proposition 2.14(1) in [49] imply that $K_1 \cap K_2$ belongs to $\mathcal{A}$. Therefore, in all cases, $K_1 \cap K_2$ is idempotent. Let $x \in K_1 \cap K_2$. As $x = yz$ for some $y, z \in K_1 \cap K_2$, we have

$$\phi_i(x) = \phi_i(yz) = \phi_i(y)z + y\phi_i(z) \in K_1 \cap K_2 \subseteq K_i,$$

for $i = 1, 2$. Consequently, $\phi_1^{-1}(K_2) \cap \phi_2^{-1}(K_1) \supseteq K_1 \cap K_2$, and since the opposite inclusion is obvious, we obtain the equality $K_1 \cap K_2 = \phi_1^{-1}(K_2) \cap \phi_2^{-1}(K_1)$. Moreover, as

$$\operatorname{dom}(\phi_1 + \phi_2) = \operatorname{dom}([\phi_1, \phi_2]) \in \mathcal{A},$$

it follows that $\mathsf{PDer}_{\mathcal{A}}(A)$ is a Lie inverse subsemialgebra of $\mathsf{PDer}(A)$. Additionally, $0_{\phi_1+\phi_2} = 0_{[\phi_1,\phi_2]}$. So, by Definition 3.1, $\mathsf{PDer}_{\mathcal{A}}(A)$ is a semilattice of Lie algebras. ∎

Let Λ be a meet semilattice, let $\mathcal{C}$ be a category of nonassociative algebras, and let $F\colon \Lambda \to \mathcal{C}$ be a presheaf. For each object λ in Λ, let $F(\lambda) = S_\lambda$, and for any morphism $f\colon \mu \to \lambda$ in Λ, let $F(f) = \eta_{\lambda,\mu} : S_\lambda \to S_\mu$. The collection $\{\eta_{\lambda,\mu} \mid \lambda \geq \mu\}$ satisfies

(1) $\eta_{\lambda,\lambda} = \operatorname{id}_{V_\lambda}$, and

(2) $\eta_{\mu,\nu}\eta_{\lambda,\mu} = \eta_{\lambda,\nu}$ for $\lambda \geq \mu \geq \nu$.

It is straightforward to verify that $S_F = \dot{\bigcup}_{\lambda\in\Lambda} S_\lambda$, endowed with the operations

$$\begin{aligned} x_\nu + x_\mu &= \eta_{\nu,\lambda}(x_\nu) + \eta_{\mu,\lambda}(x_\mu), \\ x_\nu x_\mu &= \eta_{\nu,\lambda}(x_\nu)\eta_{\mu,\lambda}(x_\mu), \end{aligned}$$

where $x_\nu \in S_\nu$, $x_\mu \in S_\mu$, and $\lambda = \nu \wedge \mu$, is a semilattice of nonassociative algebras. Thus, for a given presheaf $F\colon \Lambda \to \mathcal{C}$, we obtain a semilattice of nonassociative algebras S_F.

We shall say that S_F is a *semilattice of algebras in* $\mathcal{C}$. This establishes a correspondence $F \to S_F$ from presheaves with values in $\mathcal{C}$ to semilattices of algebras in $\mathcal{C}$. For a fixed Λ, a straightforward verification shows that this correspondence gives a functor from the category of presheaves on Λ with values in $\mathcal{C}$ to the category of semilattices of algebras in $\mathcal{C}$. Observe that if $\mathcal{C}$ is the category of associative (or Lie) algebras, then it is readily seen that S_F is an associative (or Lie) inverse semialgebra, so that it is a semilattice of associative (or Lie) algebras in the sense of Definition 3.1.

Proposition 3.10. *Let S be a semilattice of nonassociative algebras. Then there exists a presheaf $F(S)$ on a meet semilattice Λ with values in the category of nonassociative algebras, such that S is isomorphic to $S_{F(S)} = \dot{\bigcup}_{\lambda\in\Lambda} S_\lambda$.*

Proof. As $(S,+)$ is a commutative inverse semigroup, the set $\Lambda = \mathcal{E}(S)$ is a meet semilattice. For each $\lambda \in \Lambda$, let us consider $S_\lambda = \{x \in S \mid 0_x = \lambda\}$. Since S is a semilattice of nonassociative algebras, for any $x, y \in S$ and $\alpha \in \mathbb{F}$, we have $0_{xy} = 0_{x+y} = 0_x + 0_y$ and $0_{\alpha x} = 0_x$, which implies that each S_λ is a nonassociative algebra, whose zero element is λ. For $\lambda \geq \mu$, the translation map $\eta_{\lambda,\mu}\colon S_\lambda \to S_\mu$, defined by $x \mapsto \mu + x$, is linear. In order to see that $\eta_{\lambda,\mu}$ is also an algebra homomorphism, let $x, y \in S_\lambda$. Then

$$(\mu+x)(\mu+y) = \mu\mu + x\mu + \mu y + xy \overset{\text{Lemma 3.3}}{=} \mu + \lambda + \mu + \lambda + \mu + xy = \mu + xy,$$

showing that $\eta_{\lambda,\mu}$ preserves the product. Moreover, if $\lambda \geq \mu \geq \nu$, then $\nu = \mu + \nu$ implies $\eta_{\mu,\nu}\eta_{\lambda,\mu} = \eta_{\lambda,\nu}$. This way we obtain a contravariant functor F from Λ to the category of nonassociative algebras defined by $F(\lambda) = S_\lambda$ and $F(f) = \eta_{\lambda,\mu}$, for a morphism $f : \mu \to \lambda$ in Λ, which is the claimed presheaf $F(S)$. It is easy to see that S is isomorphic to $S_{F(S)} = \dot{\bigcup}_{\lambda\in\Lambda} S_\lambda$. ■

Given a semilattice of nonassociative algebras S, let S_λ be as in the proof of Proposition 3.10, i.e., $S_\lambda = \{x \in S \mid 0_x = \lambda\}$, $(\lambda \in \Lambda)$. Then obviously S is a disjoint union of the S_λ's, and identifying it with the external disjoint union of the S_λ's, we may assume that $S = S_{F(S)}$. Moreover, let F be a presheaf on a meet semilattice Λ with values in the category of nonassociative algebras. Then obviously $\mathcal{E}(S_F)$ is isomorphic with Λ, and identifying them, we may assume that $F(S_F) = F$. Then the correspondence $S \to F(S)$ is the inverse of the correspondence $F \to S_F$ defined above, so that there is a one-to-one correspondence between presheaves on meet semilattices with values in the category of nonassociative algebras and semilattices of nonassociative algebras. In particular, we have a one-to-one correspondence between presheaves on meet semilattices with values in the category of associative (or Lie) algebras and semilattices of nonassociative (or Lie) algebras.

More generally, let $\mathcal{C}$ be a category of nonassociative algebras and suppose that S is a semilattice of nonassociative algebras such that, for each $\lambda \in \Lambda = \mathcal{E}(S)$, the algebra S_λ is in $\mathcal{C}$. Then by Proposition 3.10, S is isomorphic to $S_{F(S)}$ and $S_{F(S)}$ is a semilattice of algebras in $\mathcal{C}$ as defined above. This allows us to extend the term "semilattice of algebras in $\mathcal{C}$" also for such an S. Then we have a one-to-one correspondence between presheaves on meet semilattices with values in the category $\mathcal{C}$ and semilattices of algebras in $\mathcal{C}$.

It is worth noting that there are homomorphisms of semilattices of associative (or Lie) algebras which do not correspond to morphisms of presheaves.

Example 3.11. Let $\varphi: A \to B$ be a homomorphism of associative (or Lie) algebras and let Λ be the semilattice $\{\lambda, \mu\}$ in which $\lambda \geq \mu$. Write $S_\lambda = A$, $S_\mu = B$ and $\nu_{\lambda,\mu} = \varphi$, $\nu_{\lambda,\lambda} = \mathrm{id}_A$, $\nu_{\mu,\mu} = \mathrm{id}_B$. Then $S = S_\lambda \dot{\cup} S_\mu$ is a semilattice of associative (or Lie) algebras. Consider the map $\phi: S \to S$ defined by $\phi(a) = \varphi(a)$ and $\phi(b) = b$ for any $a \in A$ and $b \in B$. Then evidently ϕ is a homomorphism of inverse semialgebras, which does not come from a morphism of presheaves.

Let Λ be a fixed meet semilattice and for each semilattice of nonassociative algebras S such that $\mathcal{E}(S) \cong \Lambda$ fix an isomorphism $\zeta_S: \Lambda \to \mathcal{E}(S)$. Let $\mathcal{D}_\Lambda$ be the category whose objects are the pairs (S, ζ_S), where S and ζ_S are as above. By a morphism $(S, \zeta_S) \to (T, \zeta_T)$ in $\mathcal{D}_\Lambda$ we mean a homomorphism $\varphi: S \to T$ such that $\varphi|_{\mathcal{E}(S)} \circ \zeta_S = \zeta_T$ (if we identify $\mathcal{E}(S)$ with Λ, then the morphism are those homomophisms $S \to T$ which restrict to the identity map on Λ).

Proposition 3.12. *Let Λ be a meet semilattice. Then the category of presheaves on Λ with values in the category of nonassociative algebras is equivalent to the category $\mathcal{D}_\Lambda$.*

Proof. The above constructed correspondence $S \to F(S)$, restricted to the semilattices of nonassociative algebras S with $\mathcal{E}(S) \cong \Lambda$, gives rise to a functor from $\mathcal{D}_\Lambda$ to the category of presheaves of nonassociative algebras on Λ, which sends an object (S, ζ_S) of $\mathcal{D}_\Lambda$ to the presheaf $F(S) \circ \zeta_S$ on Λ, and if φ is a morphism $(S, \zeta_S) \to (T, \zeta_T)$ in $\mathcal{D}_\Lambda$, then for each $\lambda \in \Lambda$, it is easy to see that φ restricts to a homomorphism $S_\lambda \to T_\lambda$, so that we have a morphism of presheaves $F(S) \circ \zeta_S \to F(T) \circ \zeta_T$. This, together with the functor $F \to S_F$, results in a category equivalence. ∎

Remark 3.13. Evidently, the category equivalence in Proposition 3.12 restricts to an equivalence between the category of presheaves of associative (or Lie) algebras on Λ and the full subcategory of $\mathcal{D}_\Lambda$, whose objects are the pairs (S, ζ_S), where S is a semilattice of associative (or Lie) algebras.

Corollary 3.14. *Every semilattice S of Lie algebras can be embedded into T^- for some semilattice of associative algebras T.*

Proof. Let U be the covariant functor that sends a Lie algebra to its universal enveloping algebra. As, by Proposition 3.10, $S \cong S_F$, for some contravariant functor F from a semilattice Λ to the category of Lie algebras, the composition UF is a contravariant functor from Λ to the category of associative algebras. For each $\lambda \in \Lambda$, let $\iota_\lambda: F(\lambda) \to U(F(\lambda))^-$ be the canonical injection. Then, setting $T = \dot{\bigcup}_{\lambda \in \Lambda} U(F(\lambda))$, it is readily seen that the map $\iota: S_F \to T^-$, defined by $\iota|_{F(\lambda)} = \iota_\lambda$, is an injective homomorphism of semilattices of Lie algebras. ∎

Remark 3.15. Considering an inverse semivector space V as a semilattice of associative (or Lie) algebras, where the product for all x and y is defined by $xy = 0_{x+y}$, we obtain by Proposition 3.10 a presheaf $F(V)$ of vector spaces on Λ. Then $V = \dot{\bigcup}_{\lambda \in \Lambda} F(\lambda)$, with $F(\lambda) = V_\lambda = \{x \in V \mid 0_x = \lambda\}$, which we call a *semilattice of vector spaces*. Thus, any inverse vector space V is a semilattice of vector spaces. Obviously, $(V, +)$ is a semilattice of abelian groups, in the sense which is well established in the theory of semigroups. Clearly, any nonassociative inverse semialgebra, with respect to the addition and multiplication by the scalars from the field $\mathbb{F}$, is a semilattice of vector spaces.

4. Partial actions and premorphisms

The main goal of this section is to introduce the notion of a premorphism between Lie inverse semialgebras, that of a partial action of a Lie algebra L on a nonassociative algebra, and to explore their relationship with the Lie inverse semialgebra $E(L)$ introduced in Section 2.2.

Partial group actions on sets can be defined using the concept of a premorphism of inverse semigroups (see [32]). For the Lie case, we give the following definition.

Definition 4.1. Let S and T be two Lie inverse semialgebras. A map $\rho\colon S \to T$ is called a *premorphism* if for all $x, y \in S$ and $0 \neq \alpha \in \mathbb{F}$, we have

(1) $\rho(x + y) \succeq \rho(x) + \rho(y)$,

(2) $\rho([x, y]) \succeq [\rho(x), \rho(y)]$,

(3) $\rho(\alpha x) = \alpha\rho(x)$.

If $(S, +)$ and $(T, +)$ are monoids, we say that a premorphism $\rho\colon S \to T$ *is zero-preserving* if $\rho(0) = 0$.

Notice that (1) and (3) of Definition 4.1 imply that

$$\rho(0_x) \succeq 0_{\rho(x)},$$

for any $x \in S$. Indeed,

$$\rho(0_x) = \rho(x + (-1)x) \succeq \rho(x) + \rho((-1)x) = \rho(x) - \rho(x) = 0_{\rho(x)}.$$

Suppose that S is a Lie algebra and T is a Lie inverse semialgebra. If $\rho\colon S \to T$ is a not necessarily zero-preserving premorphism of Lie inverse semialgebras, then, in particular, ρ is a premorphism of the additive group $(S, +)$ to the additive semigroup $(T, +)$. Hence, by the proof of Proposition 2.1 in [32], ρ satisfies

$$\rho(x) + \rho(y) = \rho(x + y) + 0_{\rho(y)}, \tag{4.1}$$

for all $x, y \in S$. Observe that, despite Proposition 2.1 in [32] is stated for a unital premorphism of a group into a monoid, the part of its proof that we need does not use the facts that the target semigroup is a monoid and that the premorphism is unital. We appreciate the comment of one of the referees pointing this out.

Definition 4.2. Let S and T be Lie inverse semialgebras. A premorphism $\rho\colon S \to T$ is called *strong* if

$$[\rho(x), \rho(y)] = \rho([x, y]) + 0_{\rho(x)+\rho(y)},$$

for all $x, y \in L$.

One can easily see that a premorphism $\rho\colon S \to T$ is strong if and only if

$$0_{\rho([x,y])} + 0_{\rho(x)+\rho(y)} = 0_{[\rho(x),\rho(y)]}, \tag{4.2}$$

for all $x, y \in S$. For instance, if T is a semilattice of Lie algebras, every premorphism $\rho\colon S \to T$ is strong. The next fact gives an important example of a strong premorphism.

Lemma 4.3. *Let L be a Lie algebra. Then, the mapping $\tau\colon L \to E(L)$, given by $a \mapsto (\mathbb{F}a, a)$, is a zero-preserving strong premorphism.*

Proof. It is clear that $\tau(0) = (0,0)$. Let $a, b \in L$. Then

$$\tau(a) + \tau(b) = (\mathbb{F}a + \mathbb{F}b, a+b) \preceq (\mathbb{F}(a+b), a+b) = \tau(a+b).$$

Moreover,

$$[\tau(a), \tau(b)] = [(\mathbb{F}a, a), (\mathbb{F}b, b)] = (\mathbb{F}a + \mathbb{F}b + \mathbb{F}[a,b], [a,b]) \preceq (\mathbb{F}[a,b], [a,b]) = \tau([a,b])$$

and

$$\begin{aligned}[\tau(a), \tau(b)] &= (\mathbb{F}a, 0) + (\mathbb{F}b, 0) + (\mathbb{F}[a,b], [a,b]) = 0_{\tau(a)} + 0_{\tau(b)} + \tau([a,b]) \\ &= 0_{\tau(a)+\tau(b)} + \tau([a,b]).\end{aligned}$$

Finally, for $\alpha \neq 0$, we have that

$$\tau(\alpha a) = (\mathbb{F}(\alpha a), \alpha a) = (\mathbb{F}a, \alpha a) = \alpha(\mathbb{F}a, a) = \alpha\tau(a). \qquad \blacksquare$$

In analogy with the case of groups, it is natural to give the following definition.

Definition 4.4. Let L be a Lie algebra and let A be a nonassociative algebra. A collection

$$\theta = \{\theta_x : D_x \to A \mid D_x \trianglelefteq A, x \in L\} \subseteq \mathsf{PDer}(A)$$

of partial derivations of A is called *a partial action* of L on A, if the mapping

$$\theta : L \to \mathsf{PDer}(A), \quad x \mapsto \theta_x, \quad (x \in L),$$

is a zero-preserving premorphism.

The next fact gives an alternative definition of a partial action of a Lie algebra.

Proposition 4.5. *Let L be a Lie algebra and let A be a nonassociative algebra. A collection*

$$\theta = \{\theta_x : D_x \to A \mid D_x \trianglelefteq A, x \in L\} \subseteq \mathsf{PDer}(A)$$

of partial derivations of A is a partial action of L on A if and only if the following properties are satisfied:

(1) *$D_0 = A$, and θ_0 is the zero derivation in A.*

(2) *$D_x \cap D_y \subseteq D_{x+y}$ and $\theta_{x+y}(a) = \theta_x(a) + \theta_y(a)$ for all $x, y \in L$ and $a \in D_x \cap D_y$.*

(3) *$\theta_x^{-1}(D_y) \cap \theta_y^{-1}(D_x) \subseteq D_{[x,y]}$ for all $x, y \in L$, and*

$$\theta_{[x,y]}(a) = \theta_x\theta_y(a) - \theta_y\theta_x(a) \quad \textit{for all } a \in \theta_x^{-1}(D_y) \cap \theta_y^{-1}(D_x).$$

(4) *$D_{\alpha x} = D_x$ and $\theta_{\alpha x}(a) = \alpha\theta_x(a)$ for all $x \in L$, $a \in D_x$ and $0 \neq \alpha \in \mathbb{F}$.*

Note that if every D_x is equal to A, then this concept coincides with the traditional notion of an action (by derivations) of L on A, and we shall call it a *global action*.

The inclusion $D_x \cap D_y \subseteq D_{x+y}$ in Proposition 4.5 can be replaced by the equality

$$D_x \cap D_y = D_{x+y} \cap D_x, \quad \forall x, y \in L.$$

Indeed, this inclusion together with condition (4) imply $D_{x+y} \cap D_x = D_{x+y} \cap D_{-x} \subseteq D_y$, which yields the desired equality.

A partial action $\theta = \{\theta_x : D_x \to A \mid D_x \trianglelefteq A, x \in L\}$ of L on A will be called *strong* if the premorphism $\theta : L \to \mathsf{PDer}(A)$ is strong. Equivalently, θ is strong if

$$\theta_x^{-1}(D_y) \cap \theta_y^{-1}(D_x) = D_x \cap D_y \cap D_{[x,y]}.$$

for all $x, y \in L$. This implies that

$$\theta_x^{-1}(D_y) \cap D_y \cap D_{[x,y]} = D_x \cap \theta_y^{-1}(D_x) \cap D_{[x,y]} = D_x \cap D_y \cap D_{[x,y]},$$

for all $x, y \in L$. Indeed,

$$\begin{aligned} D_x \cap D_y \cap D_{[x,y]} &= \theta_x^{-1}(D_y) \cap \theta_y^{-1}(D_x) \cap D_{[x,y]} \\ &\subseteq \theta_x^{-1}(D_y) \cap D_y \cap D_{[x,y]} \subseteq D_x \cap D_y \cap D_{[x,y]}, \end{aligned}$$

yielding $\theta_x^{-1}(D_y) \cap D_y \cap D_{[x,y]} = D_x \cap D_y \cap D_{[x,y]}$, and the other equality is obtained analogously.

Definition 4.6. Assume that $\mathcal{A}$ is a class of nonassociative algebras. Let L be a Lie algebra and let $A \in \mathcal{A}$. We say that a partial action $\theta = \{\theta_x : D_x \to A \mid D_x \trianglelefteq A, x \in L\}$ of L on A is an $\mathcal{A}$-*partial action* if each $D_x \in \mathcal{A}$.

Suppose that B is a nonassociative algebra and that $\eta : L \to \mathrm{Der}(B)$ is a Lie algebra homomorphism. If A is an ideal of B, then for each $x \in L$, we set $D_x = A \cap \eta_x^{-1}(A)$ and $\theta_x = \eta_x|_{D_x}$. It is straightforward to verify that the collection $\theta = \{\theta_x : D_x \to A \mid x \in L\}$ is a partial action of L on A such that

$$\theta_x^{-1}(D_y) \cap \theta_y^{-1}(D_x) = D_{[x,y]} \cap \theta_x^{-1}(D_y) \cap D_y = D_{[x,y]} \cap D_x \cap \theta_y^{-1}(D_x),$$

for all $x, y \in L$. This means that the premorphism $\theta : L \to \mathsf{PDer}(A)$, $x \mapsto \theta_x$ is such that

$$[\theta(x), \theta(y)] = \theta([x, y]) + 0_{\theta(y)\theta(x)} + 0_{\theta(y)} = \theta([x, y]) + 0_{\theta(x)\theta(y)} + 0_{\theta(x)},$$

for all $x, y \in L$. This partial action is called the *restriction* of η to A. If A is invariant under the action η, then the restriction of η to A constitutes a global action on A. For instance, if L is a Lie algebra that contains an ideal I that is not invariant under $\mathrm{Der}(L)$ (see, e.g., p. 75 of [31]), then the resulting restriction of the action of $\mathrm{Der}(L)$ to I is not a global partial action.

Example 4.7. Let L be a Lie algebra and let A be a nonassociative idempotent algebra, meaning $A^2 = A$. Define $D_0 = A$ and $D_x = 0$ for $x \neq 0$, and let θ_x be the zero map on D_x for every $x \in L$. It is straightforward to check that $\theta = \{\theta_x : D_x \to A \mid x \in L\}$ is a strong partial action of L on A. Furthermore, θ is not a restriction of a global action of L. To see why, assume by contradiction that there exist a nonassociative algebra B with $A \trianglelefteq B$, and an action $\eta : L \to \mathrm{Der}(B)$, such that $D_x = \eta_x^{-1}(A) \cap A$ for every $x \in L$. Then

$$\eta_x(A) = \eta_x(A^2) = A\eta_x(A) + \eta_x(A)A \subseteq A,$$

yielding $A \subseteq \eta_x^{-1}(A)$. Consequently, $D_x = A$ for every $x \in L$, contradicting our definition of D_x. Thus, θ cannot be a restriction of a global action of L.

Lemma 4.8. *Let L be a Lie algebra, T a Lie inverse semialgebra, and $\rho\colon L \to T$ a premorphism. Assume that A is a finite-dimensional subspace of L and that β is a finite set of generators of A. Then*

$$\inf \mathcal{E}(\rho(A)) := \inf\{0_{\rho(a)} \mid a \in A\} = \sum_{x_i \in \beta} 0_{\rho(x_i)}.$$

In particular, the sum $\sum_{x_i\in\beta} 0_{\rho(x_i)}$ does not depend of the finite set of generators of A.

Proof. Let $a \in A$. Then $a = \sum_{x_i\in\beta} \alpha_i x_i$, for some $\alpha_i \in \mathbb{F}$. As ρ is a premorphism, we have $\rho(a) = \rho(\sum_{x_i\in\beta} \alpha_i x_i) \succeq \sum_{x_i\in\beta} \alpha_i \rho(x_i)$, which implies

$$0_{\rho(a)} \succeq 0_{\sum_{x_i\in\beta} \alpha_i \rho(x_i)} = \sum_{x_i\in\beta} 0_{\alpha_i \rho(x_i)} = \sum_{x_i\in\beta} 0_{\rho(x_i)}.$$

Thus $\sum_{x_i\in\beta} 0_{\rho(x_i)}$ is a lower bound of $\mathcal{E}(\rho(A))$. Now, suppose that $\lambda \in \mathcal{E}(\rho(L))$ is another lower bound of $\mathcal{E}(\rho(A))$. Then, in particular, $\lambda \preceq 0_{\rho(x_i)}$ for all $x_i \in \beta$, and hence $|\beta|\lambda \preceq \sum_{x_i\in\beta} 0_{\rho(x_i)}$, where $|\beta|$ denotes the number of elements of β. Since $\lambda \in \mathcal{E}(\rho(L))$, we have $|\beta|\lambda = \lambda$, which implies $\lambda \preceq \sum_{x_i\in\beta} 0_{\rho(x_i)}$. Therefore, $\sum_{x_i\in\beta} 0_{\rho(x_i)}$ is the infimum of $\mathcal{E}(\rho(A))$. ∎

We derive the following result as a direct consequence of the Lemma 4.8.

Corollary 4.9. *Let L, ρ and T be as defined as in Lemma 4.8. Then, for any finite-dimensional subspaces A and B of L, we have*

$$\inf \mathcal{E}(\rho(A+B)) = \inf \mathcal{E}(\rho(A)) + \inf \mathcal{E}(\rho(B)).$$

The following result was inspired by a well-known result due to R. Exel, Proposition 2.2 in [22] (see also Theorem 2.4 in [32]), which establishes a correspondence between premorphisms from a group G into a inverse monoid and homomorphism from the Exel semigroup into the same inverse monoid.

Lemma 4.10. *Let L a Lie algebra and let S be a Lie inverse semialgebra satisfying (2.5). Suppose that $\rho\colon L \to S$ is a premorphism. Then, the mapping $\tilde{\rho}\colon E(L) \to S$, defined by $(A,a) \mapsto \inf \mathcal{E}(\rho(A)) + \rho(a)$, satisfies*

$$[\tilde{\rho}(A,a), \tilde{\rho}(B,b)] = \tilde{\rho}([(A,a),(B,b)]) + [\rho(a), \inf \mathcal{E}(\rho(B))] + [\inf \mathcal{E}(\rho(A)), \rho(b)], \tag{4.3}$$

for all $(A,a), (B,b) \in E(L)$.

Proof. Note, keeping in mind (4) and (6) of Definition 2.10, that

$$\begin{aligned}
[\tilde{\rho}(A,a), \tilde{\rho}(B,b)] &= [\inf \mathcal{E}(\rho(A)) + \rho(a), \inf \mathcal{E}(\rho(B)) + \rho(b)] \\
&= \inf \mathcal{E}(\rho(A)) + \inf \mathcal{E}(\rho(B)) + [\rho(a), \inf \mathcal{E}(\rho(B))] \\
&\quad + [\inf \mathcal{E}(\rho(A)), \rho(b)] + [\rho(a), \rho(b)].
\end{aligned} \tag{4.4}$$

Now, since ρ is a premorphism and S satisfies (2.5), we have that

$$[\rho(a), \rho(b)] = [\rho(a), 0_{\rho(b)}] + [0_{\rho(a)}, \rho(b)] + \rho([a,b]).$$

Additionally, by (2.3) the inequalities $\inf \mathcal{E}(\rho(A)) \preceq 0_{\rho(a)}$ and $\inf \mathcal{E}(\rho(B)) \preceq 0_{\rho(b)}$ imply that

$$[\inf \mathcal{E}(\rho(A)), \rho(b)] \preceq [0_{\rho(a)}, \rho(b)] \quad \text{and} \quad [\rho(a), \inf \mathcal{E}(\rho(B))] \preceq [\rho(a), 0_{\rho(b)}].$$

Therefore, by Corollary 4.9, (4.4) is equal to

$$\begin{aligned}
&\inf \mathcal{E}(\rho(A+B)) + [\rho(a), \inf \mathcal{E}(\rho(B))] + [\inf \mathcal{E}(\rho(A)), \rho(b)] \\
&\qquad + [\rho(a), 0_{\rho(b)}] + [0_{\rho(a)}, \rho(b)] + \rho([a,b]) \\
&\qquad = \inf \mathcal{E}(\rho(A+B)) + [\rho(a), \inf \mathcal{E}(\rho(B))] + [\inf \mathcal{E}(\rho(A)), \rho(b)] + \rho([a,b]).
\end{aligned} \tag{4.5}$$

Observe that, using again Corollary 4.9, we have that

$$\begin{aligned}
\inf \mathcal{E}(\rho(A+B)) + \rho([a,b]) &= \inf \mathcal{E}(\rho(A+B)) + 0_{\rho([a,b])} + \rho([a,b]) \\
&= \inf \mathcal{E}(\rho(A+B+\mathbb{F}[a,b])) + \rho([a,b]).
\end{aligned}$$

Hence, (4.5) is equal to

$$\begin{aligned}
\inf \mathcal{E}(\rho(A+B+\mathbb{F}[a,b])) &+ \rho([a,b]) + [\rho(a), \inf \mathcal{E}(\rho(B))] + [\inf \mathcal{E}(\rho(A)), \rho(b)] \\
&= \tilde{\rho}([(A,a),(B,b)]) + [\rho(a), \inf \mathcal{E}(\rho(B))] + [\inf \mathcal{E}(\rho(A)), \rho(b)],
\end{aligned}$$

completing the proof of (4.3). ∎

Theorem 4.11. *Let L be a Lie algebra and let $\tau\colon L \to E(L)$ be as in Lemma 4.3. Let, furthermore, S be a Lie inverse semialgebra. Suppose that $\rho\colon L \to S$ is a premorphism. Then, the mapping $\tilde{\rho}\colon E(L) \to S$, defined by $(A,a) \mapsto \inf \mathcal{E}(\rho(A)) + \rho(a)$, is the unique linear map of inverse semivector spaces such that*

$$\tilde{\rho} \circ \tau = \rho. \tag{4.6}$$

Consequently, ρ is zero-preserving if and only if so is $\tilde{\rho}$.

Proof. For any $a \in L$, we see that

$$\tilde{\rho} \circ \tau(a) = \tilde{\rho}(\mathbb{F}a, a) = \inf(\mathcal{E}(\rho(\mathbb{F}a))) + \rho(a) = 0_{\rho(a)} + \rho(a) = \rho(a),$$

which shows (4.6).

We shall check next that $\tilde{\rho}$ is linear. Let $(A,a), (B,b) \in E(L)$ and $0 \neq \alpha \in \mathbb{F}$. As ρ is a premorphism, by (4.1) we have $\rho(a) + \rho(b) = \rho(a+b) + 0_{\rho(b)}$, which implies

$$\inf \mathcal{E}(\rho(B)) + \rho(a+b) = \inf \mathcal{E}(\rho(B)) + \rho(a) + \rho(b),$$

and thus, using Corollary 4.9,

$$\begin{aligned}
\tilde{\rho}((A,a)+(B,b)) = \tilde{\rho}(A+B, a+b) &= \inf \mathcal{E}(\rho(A+B)) + \rho(a+b) \\
&= \inf \mathcal{E}(\rho(A)) + \inf \mathcal{E}(\rho(B)) + \rho(a+b) \\
&= \inf \mathcal{E}(\rho(A)) + \inf \mathcal{E}(\rho(B)) + \rho(a) + \rho(b) \\
&= \tilde{\rho}(A,a) + \tilde{\rho}(B,b).
\end{aligned}$$

Furthermore,

$$\begin{aligned}\tilde{\rho}((A,\alpha a)&=\inf\mathcal{E}(\rho(A))+\rho(\alpha a)=\alpha\inf\mathcal{E}(\rho(A))+\alpha\rho(a)\\&=\alpha(\inf\mathcal{E}(\rho(A))+\rho(a))=\alpha(\tilde{\rho}(A,a)),\end{aligned}$$

as $\alpha\neq 0$. It follows by Remark 2.5 that $\tilde{\rho}$ is linear.

Now, let us address the assertion of uniqueness. For this suppose that $\eta\colon E(L)\to S$ is a linear mapping such that $\eta\circ\tau=\rho$. In particular, we have that $\eta(0_{(A,a)})=\eta(0(A,a))=0\eta(A,a)=0_{\eta(A,a)}$ for all $(A,a)\in E(L)$. Since

$$\eta(A,a)=\eta(A,0)+\eta(\mathbb{F}a,a)=\eta(A,0)+\rho(a)=\eta(A,0)+\tilde{\rho}(\mathbb{F}a,a),$$

it is enough to verify that $\eta(A,0)=\tilde{\rho}(A,0)$. Let β be a finite set of generators of A. Then we obtain

$$\begin{aligned}\eta(A,0)&=\eta\Big(\sum_{x\in\beta}\mathbb{F}x,0\Big)=\sum_{x\in\beta}\eta(\mathbb{F}x,0)=\sum_{x\in\beta}\eta(0_{(\mathbb{F}x,x)})\\&=\sum_{x\in\beta}0_{\eta(\mathbb{F}x,x)}=\sum_{x\in\beta}0_{\rho(x)}=\inf\mathcal{E}(\rho(A))=\tilde{\rho}(A,0),\end{aligned}$$

as required. ∎

Corollary 4.12. *Suppose that L is a Lie algebra, S is a semilattice of Lie algebras, $\rho\colon L\to S$ is a premorphism, and let $\tau\colon L\to E(L)$ be as defined in Lemma 4.3. Then $\tilde{\rho}\colon E(L)\to S$ is a homomorphism of Lie inverse semialgebras, and it is the unique homomorphism satisfying $\tilde{\rho}\circ\tau=\rho$.*

Proof. By Theorem 4.11, it remains only to show that $\tilde{\rho}$ satisfies $[\tilde{\rho}(A,a),\tilde{\rho}(B,b)]=\tilde{\rho}([(A,a),(B,b)])$. Since S is a semilattice of Lie algebras, it follows by Lemma 3.3 that

$$\begin{aligned}[\rho(a),\inf\mathcal{E}(\rho(B))]+[\inf\mathcal{E}(\rho(A)),\rho(b)]&=0_{\rho(a)}+\inf\mathcal{E}(\rho(B))+\inf\mathcal{E}(\rho(A))+0_{\rho(b)}\\&=\inf\mathcal{E}(\rho(A))+\inf\mathcal{E}(\rho(B)).\end{aligned}$$

Then (4.3) and Corollary 4.9 imply

$$\begin{aligned}[\tilde{\rho}((A,a)),\tilde{\rho}((B,b))]&=\tilde{\rho}([(A,a),(B,b)])+[\rho(a),\inf\mathcal{E}(\rho(B))]\\&\quad+[\inf\mathcal{E}(\rho(A)),\rho(b)]\\&=\inf\mathcal{E}(\rho(A+B))+\rho([a,b])+\inf\mathcal{E}(\rho(A))+\inf\mathcal{E}(\rho(B))\\&=\inf\mathcal{E}(\rho(A+B))+0_{\rho([a,b])}+\rho([a,b])\\&=\inf\mathcal{E}(\rho(A+B+\mathbb{F}[a,b]))+\rho([a,b])=\tilde{\rho}([(A,a),(B,b)]),\end{aligned}$$

as required. ∎

Corollary 4.13. *Assume that $\mathcal{A}$ is any of the following classes of nonassociative algebras: unital associative algebras, unital Jordan algebras, sympathetic Lie algebras, or semisimple Lie algebras. Let $A\in\mathcal{A}$ and let L be a Lie algebra. Then, there exists a one-to-one correspondence between*

(1) *$\mathcal{A}$-partial actions of L on A, and*

(2) *zero-preserving homomorphisms of Lie inverse semialgebras $E(L)\to\mathsf{PDer}_{\mathcal{A}}(A)$.*

Proof. By definition, the $\mathcal{A}$-partial actions of L on A are exactly the zero-preserving premorphisms of the form $\rho\colon L \to \mathsf{PDer}_{\mathcal{A}}(A)$. By Proposition 3.9, $\mathsf{PDer}_{\mathcal{A}}(A)$ is a semilattice of Lie algebras, so that given such a ρ, Corollary 4.12 gives us the unique homomorphism of Lie inverse semialgebras $\tilde{\rho}\colon E(L) \to \mathsf{PDer}_{\mathcal{A}}(A)$ such that $\tilde{\rho} \circ \tau = \rho$. Moreover, $\tilde{\rho}$ is zero-preserving. On the other hand, for any zero-preserving homomorphism of Lie inverse semialgebras $\varphi\colon E(L) \to \mathsf{PDer}_{\mathcal{A}}(A)$, the composition $\rho = \varphi \circ \tau$ is obviously a zero-preserving premorphism. Hence by the uniqueness property, $\varphi = \tilde{\rho}$, and we obtain the desired one-to-one correspondence. ■

Example 4.14. We will see that, in general, the map $\tilde{\rho}\colon E(L) \to S$ in Theorem 4.11 is not a homomorphism of Lie inverse semialgebras. To see this, we refer to the well-known example by Jacobson involving a Lie algebra with an ideal that is not invariant under derivations, see p. 75 of [31]. Let us consider a field $\mathbb{F}$ with characteristic p greater than 2, and let A be the commutative associative algebra with basis $\{1, z \dots, z^{p-1}\}$, where $z^p = 0$. Now, let S be any simple Lie algebra, and put $L = S \otimes A$, where the bracket in L is given by $[s \otimes a, t \otimes b] = [s,t] \otimes ab$. Consider the ideal $I = S \otimes \operatorname{span}\{z, \dots, z^{p-1}\}$ of L and the derivation $d\colon L \to L$ given by $d(s \otimes z^i) = s \otimes i z^{i-1}$, for $i = 1, \dots, p-1$. Then, we have that

$$I_d := I \cap d^{-1}(I) = \{s \otimes a \in I \mid d(s \otimes a) \in I\} = S \otimes \operatorname{span}\{z^2, \dots, z^{p-1}\}.$$

The restriction $\rho = \{\rho_h : I_h \to I \mid h \in \operatorname{Der}(L)\}$ of the action of $\operatorname{Der}(L)$ on L to I is a non-global partial action of $\operatorname{Der}(L)$ on I. Consequently, the mapping $\rho\colon \operatorname{Der}(L) \to \mathsf{PDer}(I)$, defined by $h \mapsto \rho_h$, is a premorphism. We claim that the mapping $\tilde{\rho}\colon E(\operatorname{Der}(L)) \to \mathsf{PDer}(I)$, given by $\tilde{\rho}(A, a) = \inf \mathcal{E}(\rho(A)) + \rho(a)$, is not a homomorphism. Indeed, if $\tilde{\rho}$ were a homomorphism, then

$$[d|_{I_d}, d|_{I_d}] = [\rho_d, \rho_d] = [\tilde{\rho}(\mathbb{F}d, d), \tilde{\rho}(\mathbb{F}d, d)] = \tilde{\rho}(\mathbb{F}d, 0) = 0_{\rho_d},$$

which implies that the domains of the maps $[d|_{I_d}, d|_{I_d}]$ and $d|_{I_d}$ must be equal, that is, $d^{-1}(I_d) \cap I_d = I_d$. However, since $d(s \otimes z^2) = 2s \otimes z \notin I_d$ for $s \neq 0$, we find $s \otimes z^2$ in $I_d \backslash d^{-1}(I_d)$. Therefore, $\tilde{\rho}\colon E(\operatorname{Der}(L)) \to \mathsf{PDer}(I)$ is not a homomorphism of Lie inverse semialgebras.

5. Lie F-inverse semialgebras and partial representations

5.1. Recalling F-inverse semigroups

An interesting class of inverse semigroups close to groups is formed by F-inverse semigroups, i.e., inverse semigroups each σ-class of which contains a greatest element. An important fact due to M. Szendrei states that there is an adjunction between the category of groups and the category of F-inverse semigroups, with those morphisms which preserve greatest elements of σ-classes [48]. It is also known that the latter category is equivalent with the category of the so-called F-pairs and their morphisms (see Section VII.6 of [44]). It is easy to see that an F-pair can be identified with what we call a unital partial action of a group on a meet semilattice. For the reader's convenience, we recall the following.

Definition 5.1. (see [22] and [23]) Let G be a group with identity element 1_G and let $\mathcal{X}$ be a set. A *partial action* ϑ of G on $\mathcal{X}$ is a collection of subsets $\mathcal{D}_g \subseteq \mathcal{X}$ $(g \in G)$ and bijections $\vartheta_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$ such that for all $g, h \in G$, the following conditions are satisfied:

(i) $\mathcal{D}_{1_G} = \mathcal{X}$ and ϑ_{1_G} is the identity map of $\mathcal{X}$;

(ii) $\mathcal{D}_{(gh)^{-1}} \supseteq \vartheta_h^{-1}(\mathcal{D}_h \cap \mathcal{D}_{g^{-1}})$;

(iii) $\vartheta_g \circ \vartheta_h(x) = \vartheta_{gh}(x)$ for each $x \in \vartheta_h^{-1}(\mathcal{D}_h \cap \mathcal{D}_{g^{-1}})$.

Note that conditions (ii) and (iii) mean that the function ϑ_{gh} is an extension of the function $\vartheta_g \circ \vartheta_h$. Moreover, $\vartheta_g^{-1} = \vartheta_{g^{-1}}$ and it is easily seen that (ii) can be replaced by a "stronger looking" condition:

$$\vartheta_h^{-1}(\mathcal{D}_h \cap \mathcal{D}_{g^{-1}}) = \mathcal{D}_{h^{-1}} \cap \mathcal{D}_{h^{-1}g^{-1}}, \tag{5.1}$$

for all $g, h \in G$ (see [23]). If $\mathcal{X}$ is a semigroup, then we assume that each $\mathcal{D}_g$ is an ideal in $\mathcal{X}$ and each ϑ_g is an isomorphism of semigroups. We say that the partial action of G on a semigroup $\mathcal{X}$ is *unital* if each $\mathcal{D}_g$ is a monoid (note that we use the term "unital" for partial actions in a different sense than the authors of [32]).

Let now ϑ be a unital partial action of a group G on a meet semilattice $\Lambda = (\Lambda, \leq)$. Denote by e_g the identity element of the monoid $\mathcal{D}_g$, $(g \in G)$. In particular, e_{1_G} is the identity element of $\mathcal{D}_{1_G} = \Lambda$, which we denote by ε. Observe that (5.1) can be rewritten as $\vartheta_g(\mathcal{D}_{g^{-1}} \cap \mathcal{D}_h) = \mathcal{D}_g \cap \mathcal{D}_{gh}$, which implies that

$$\vartheta_g(e_{g^{-1}} \wedge e_h) = e_g \wedge e_{gh}, \tag{5.2}$$

for all $g, h \in G$. Let

$$\vartheta = \{\vartheta_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g \mid g \in G, \mathcal{D}_g \subseteq \Lambda\} \quad \text{and} \quad \tilde{\vartheta} = \{\tilde{\vartheta}_h : \tilde{\mathcal{D}}_{h^{-1}} \to \tilde{\mathcal{D}}_h \mid h \in H, \tilde{\mathcal{D}}_h \subseteq \tilde{\Lambda}\}$$

be unital partial actions of the groups G and H on the semilattices Λ and $\tilde{\Lambda}$, respectively. A *morphism* $\vartheta \to \tilde{\vartheta}$ is a pair (φ, ϕ), where $\varphi : \Lambda \to \tilde{\Lambda}$ is a monoid homomorphism and $\phi : G \to H$ is a group homomorphism, which for every $g \in G$ satisfy

$$\varphi(\mathcal{D}_g) \subseteq \tilde{\mathcal{D}}_{\phi(g)} \quad \text{and} \quad \varphi \circ \vartheta_g = \tilde{\vartheta}_{\phi(g)} \circ \varphi \ \text{ on } \mathcal{D}_{g^{-1}}.$$

We say that the morphism (φ, ϕ) is *unital* if the restriction of φ to $\mathcal{D}_g$ is a homomorphism of monoids $\mathcal{D}_g \to \tilde{\mathcal{D}}_{\phi(g)}$ for each $g \in G$. Evidently, the collection of unital partial actions of groups on semilattices, together with their unital morphisms, form a category.

Definition 5.2. Let G be a group and let Λ be semilattice with unity ε. A *unital partial representation* of G on Λ is a map $\pi : G \to \operatorname{End}(\Lambda)$ such that for all $g, h \in G$, the following conditions are satisfied:

(1) $\operatorname{im} \pi_g$ is an ideal of Λ;

(2) π_{1_G} is the identity map on Λ;

(3) $\pi_{g^{-1}} \pi_g \pi_h = \pi_{g^{-1}} \pi_{gh}$;

(4) $\pi_g \pi_h \pi_{h^{-1}} = \pi_{gh} \pi_{h^{-1}}$.

Note that it follows from the above definition that $\pi_g(\Lambda)$ is a monoid with unity $\pi_g(\varepsilon)$, so that $\operatorname{im} \pi_g$ is a unital ideal of Λ.

Given a unital partial representation $\pi\colon G \to \operatorname{End}(\Lambda)$, for the sake of simplicity, we will denote $\pi_g\pi_{g^{-1}}$ by χ_g, for $g \in G$. Note that χ_g is idempotent, which satisfies $\chi_g\pi_g = \pi_g\chi_{g^{-1}} = \pi_g$. Also, an element $\lambda \in \Lambda$ lies in $\operatorname{im}\pi_g$ if and only if $\chi_g(\lambda) = \lambda$. Conditions (3) and (4) in the definition above can be rewritten as:

$$\pi_g\pi_h = \pi_{gh}\chi_{h^{-1}} \quad \text{and} \quad \pi_g\pi_h = \chi_g\pi_{gh}.$$

Moreover, for all $\lambda, \mu \in \Lambda$ and $g, h \in G$, the following useful relation holds:

$$\pi_g(\lambda) \wedge \pi_h(\mu) = \pi_g(\lambda \wedge \pi_{g^{-1}h}(\mu)). \tag{5.3}$$

To verify this, observe that

$$\begin{aligned}\pi_g(\lambda \wedge \pi_{g^{-1}h}(\mu)) &= \pi_g(\lambda) \wedge \pi_g(\pi_{g^{-1}h}(\mu)) = \chi_g\pi_g(\lambda) \wedge \chi_g(\pi_h(\mu))\\ &= \chi_g(\pi_g(\lambda) \wedge \pi_h(\mu)) = \pi_g(\lambda) \wedge \pi_h(\mu).\end{aligned}$$

Let $\pi\colon G \to \operatorname{End}(\Lambda)$ and $\widetilde{\pi}\colon H \to \operatorname{End}(\tilde{\Lambda})$ be unital partial representations of the groups G and H on the unital semilattices Λ and $\tilde{\Lambda}$, respectively. A pair (φ, ϕ), where $\varphi\colon \Lambda \to \tilde{\Lambda}$ is a monoid homomorphism and $\phi\colon G \to H$ is a group homomorphism, is a *morphism* $\pi \to \widetilde{\pi}$ if for all $g \in G$ and $\lambda \in \Lambda$, we have

$$\varphi(\pi_g(\lambda)) = \widetilde{\pi}_{\phi(g)}(\varphi(\lambda)).$$

Observe that, since $\varphi(\varepsilon) = \tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is the unity element of $\tilde{\Lambda}$, we have that

$$\varphi(\pi_g(\varepsilon)) = \widetilde{\pi}_{\phi(g)}(\varphi(\varepsilon)) = \widetilde{\pi}_{\phi(g)}(\tilde{\varepsilon}),$$

which is the unity of $\widetilde{\pi}_{\phi(g)}(\widetilde{\Lambda})$. Thus the restriction of φ to $\pi_g(\Lambda)$ is a homomorphism of monoids $\pi_g(\Lambda) \to \widetilde{\pi}_{\phi(g)}(\widetilde{\Lambda})$ for each $g \in G$. Clearly, unital partial representations of groups with their morphisms form a category.

Given a unital partial representation $\pi\colon G \to \operatorname{End}(\Lambda)$ as above, we can define a unital partial action ϑ by setting $\mathcal{D}_g = \operatorname{im}\pi_g$ and $\vartheta_g = \pi_g|_{\mathcal{D}_{g^{-1}}}$. Now, let $\widetilde{\pi}\colon H \to \operatorname{End}(\tilde{\Lambda})$ be a unital partial representation of the group H on the unital semilattice $\tilde{\Lambda}$, let $\tilde{\vartheta} = \{\tilde{\vartheta}_h : \widetilde{\mathcal{D}}_{h^{-1}} \to \widetilde{\mathcal{D}}_h \mid h \in H, \widetilde{\mathcal{D}}_h \subseteq \tilde{\Lambda}\}$ be the corresponding unital partial action of H on $\tilde{\Lambda}$, and let (φ, ϕ) be a morphism $\pi \to \widetilde{\pi}$. Then for all $g \in G$,

$$\varphi(\mathcal{D}_g) = \varphi(\pi_g(\Lambda)) = \widetilde{\pi}_{\phi(g)}(\varphi(\Lambda)) \subseteq \widetilde{\pi}_{\phi(g)}(\tilde{\Lambda}) = \widetilde{\mathcal{D}}_{\phi(g)},$$

and for any $g \in G$, $\lambda \in \mathcal{D}_{g^{-1}}$,

$$\varphi(\vartheta_g(\lambda)) = \varphi(\pi_g(\lambda)) = \widetilde{\pi}_{\phi(g)}(\varphi(\lambda)) = \widetilde{\vartheta}_{\phi(g)}(\varphi(\lambda)),$$

showing that (φ, ϕ) is a morphism $\vartheta \to \tilde{\vartheta}$ of partial actions, which is clearly unital. Thus, we have a functor from the category of unital partial representations of groups on semilattices and their morphisms to the category of unital partial group actions on semilattices with unital morphisms.

In fact, we have the following result.

Proposition 5.3. *The category of unital partial actions of groups on semilattices and unital morphisms is isomorphic to the category of unital partial group representations on semilattices and their morphisms.*

Proof. It remains to construct the inverse functor. If ϑ is a unital partial action of G on Λ, then the mapping $\pi : G \to \operatorname{End}(\Lambda)$ given by

$$\pi_g(\lambda) = \vartheta_g(e_{g^{-1}} \wedge \lambda)$$

defines a unital partial representation. Let $\tilde{\vartheta} = \{\tilde{\vartheta}_g : \widetilde{\mathcal{D}}_{h^{-1}} \to \widetilde{\mathcal{D}}_h \mid h \in H, \widetilde{\mathcal{D}}_h \subseteq \tilde{\Lambda}\}$ be a unital partial action of a group H on a semilattice $\tilde{\Lambda}$, let $\tilde{\pi} : H \to \operatorname{End}(\tilde{\Lambda})$ be the corresponding unital partial representation, and let (φ, ϕ) be a unital morphism $\vartheta \to \tilde{\vartheta}$. Using the symbol $\tilde{e}_g$ for the unity element of $\widetilde{\mathcal{D}}_g$, we have that $\varphi(e_g) = \tilde{e}_{\phi(g)}$ for each $g \in G$, and for any $\lambda \in \Lambda$, we obtain

$$\begin{aligned}\varphi(\pi_g(\lambda)) &= \varphi(\vartheta_g(e_{g^{-1}} \wedge \lambda)) = \tilde{\vartheta}_{\phi(g)}(\varphi(e_{g^{-1}} \wedge \lambda)) \\ &= \tilde{\vartheta}_{\phi(g)}(\tilde{e}_{\phi(g^{-1})} \wedge \varphi(\lambda)) = \tilde{\pi}_{\phi(g)}(\varphi(\lambda)),\end{aligned}$$

showing that (φ, ϕ) is a morphism $\pi \to \tilde{\pi}$ of unital partial representations. It is readily seen that the constructed functors are inverse to each other. ■

Lemma 5.4. *Let G be a group and let $(\Lambda, \leq)$ be a semilattice with unity ε. A map $\pi : G \to \operatorname{End}(\Lambda)$ is a unital partial representation if and only if for any $g, h \in G$ and $\lambda \in \Lambda$, we have*:

(1) π_{1_G} *is the identity map on* Λ; *and*

(2) $\pi_g \pi_h(\lambda) = \pi_g(\varepsilon) \wedge \pi_{gh}(\lambda)$.

Proof. First suppose that $\pi : G \to \operatorname{End}(\Lambda)$ is a unital partial representation. We need to verify condition (2) above. From (5.3), we have that

$$\pi_g(\varepsilon) \wedge \pi_{gh}(\lambda) = \pi_g(\varepsilon \wedge \pi_h(\lambda)) = \pi_g \pi_h(\lambda).$$

Conversely, suppose that $\pi : G \to \operatorname{End}(\Lambda)$ verifies conditions (1) and (2). We first check that $\operatorname{im} \pi_g = \{\lambda \in \Lambda \mid \lambda \leq \pi_g(\varepsilon)\}$. Indeed, taking $h = 1$ in (2) immediately gives the inclusion $\operatorname{im} \pi_g \subseteq \{\lambda \in \Lambda \mid \lambda \leq \pi_g(\varepsilon)\}$. Now, taking $\lambda \in \Lambda$ such that $\lambda \leq \pi_g(\varepsilon)$, we have that

$$\lambda = \pi_g(\varepsilon) \wedge \lambda = \pi_g(\varepsilon) \wedge \pi_{gg^{-1}}(\lambda) = \pi_g(\pi_{g^{-1}}(\lambda)) \in \operatorname{im} \pi_g,$$

showing the converse inclusion. It follows immediately that $\operatorname{im} \pi_g$ is an ideal of Λ with unity $\pi_g(\varepsilon)$.

Now, we see that

$$\begin{aligned}\pi_{g^{-1}} \pi_{gh}(\lambda) &= \chi_{g^{-1}}(\varepsilon \wedge \pi_{g^{-1}} \pi_{gh}(\lambda)) = \chi_{g^{-1}}(\varepsilon) \wedge \pi_{g^{-1}} \pi_{gh}(\lambda) \\ &= \pi_{g^{-1}}(\pi_g(\varepsilon) \wedge \pi_{gh}(\lambda)) = \pi_{g^{-1}} \pi_g \pi_h(\lambda),\end{aligned}$$

showing (3) of Definition 5.2. Finally, keeping in mind equation (5.3), we obtain

$$\pi_{gh} \pi_{h^{-1}}(\lambda) = \pi_{gh}(\varepsilon) \wedge \pi_g(\lambda) = \pi_g(\lambda \wedge \pi_h(\varepsilon)) = \pi_g(\pi_h \pi_{h^{-1}}(\lambda)),$$

which shows (4) of Definition 5.2. ■

Lemma 5.4 states that unital partial representations of groups in unital meet semilattices are precisely the F-pairs as defined by M. Petrich in Section VII.6 of [44]. For the reader's convenience, we recall an important result from [44], which establishes an equivalence between the category of F-pairs (unital partial representations) and the category of F-inverse semigroups whose morphisms are homomorphisms which preserve the greatest elements of σ-classes. To this end, we first recall the notion of an F-inverse semigroup. Given an inverse semigroup S, the congruence σ is defined by setting $s\,\sigma\,t$ if and only if there exists an idempotent e such that $se = te$. An inverse semigroup S is called *F-inverse* if every σ-class $\bar{s} \in S/\sigma$ has a greatest element.

Theorem 5.5 ([44]). *The category of F-pairs (unital partial representations) is equivalent to the category of F-inverse semigroups and their homomorphisms which preserve the greatest elements of σ-classes.*

As an immediate consequence of Theorem 5.5 and Proposition 5.3, we have the following.

Corollary 5.6. *The category of unital partial actions of groups on unital semilattices with unital morphisms is equivalent to the category of F-inverse semigroups and their homomorphisms which preserve the greatest elements of σ-classes.*

One of the main objectives in the next subsections is to prove a Lie analogue of the Theorem 5.5. To this end, we adapt the notions of unital partial representation and F-inverse semigroup to the Lie context.

5.2. Partial representations

To avoid certain anomalies, we assume for the remainder of the paper that the base field $\mathbb{F}$ is of *characteristic different from* 2.

We start introducing the notion of a partial representation of a vector space V on a unital semilattice Λ. This is defined as a unital partial representation $\pi\colon V \to \operatorname{End}(\Lambda)$ of the additive group $(V, +)$ on Λ together with an additional condition. More precisely, we give the following.

Definition 5.7. Let V be a vector space and let $\Lambda = (\Lambda, \leq)$ be a meet semilattice with unity ε. A *partial representation* of V on Λ is a mapping $V \times \Lambda \to \Lambda$, $(a, \lambda) \mapsto a \cdot \lambda$ which verifies the following axioms for all $a, b \in V$, $0 \neq \alpha \in \mathbb{F}$ and $\lambda, \mu \in \Lambda$:

(1) $0 \cdot \lambda = \lambda$,

(2) $a \cdot (\lambda \wedge \mu) = a \cdot \lambda \wedge a \cdot \mu$,

(3) $a \cdot (b \cdot \lambda) = a \cdot \varepsilon \wedge (a + b) \cdot \lambda$,

(4) $(\alpha a) \cdot \lambda = a \cdot \lambda$.

Notice that Lemma 5.4 implies that in the above definition $V \times \Lambda \to \Lambda$ is indeed a unital partial representation of the additive group $(V, +)$ on the semilattice Λ. In particular, $a \cdot \Lambda = \{a \cdot \lambda \mid \lambda \in \Lambda\}$ is a unital ideal with unity $a \cdot \varepsilon$ for every $a \in V$.

A *partial representation* of a Lie algebra L on a meet semilattice Λ, with unity ε, is a partial representation of the vector space L on Λ. We shall denote this partial representation by (Λ, L).

Given two partial representations (Λ, L) and (Π, H), a morphism from (Λ, L) to (Π, H) is defined as a pair (φ, ϕ), where $\varphi\colon \Lambda \to \Pi$ is a monoid homomorphism and $\phi\colon L \to H$ is a Lie algebra homomorphism satisfying

$$\varphi(a \cdot \lambda) = \phi(a) \cdot \varphi(\lambda)$$

for every $a \in L$ and $\lambda \in \Lambda$. It is easy to see that the partial representations of Lie algebras on unital meet semilattices form a category, which we denote by $\mathcal{I}$.

The following result addresses some elementary properties of partial representations that will be used throughout this section.

Lemma 5.8. *Let* (Λ, L) *be a partial representation. Then, for any* $\lambda, \mu \in \Lambda$ *and* $a, b \in L$, *the following holds*:

(1) $a \cdot (a \cdot \lambda) = a \cdot \lambda$.

(2) $a \cdot \lambda \wedge a \cdot \mu = a \cdot \lambda \wedge \mu = \lambda \wedge a \cdot \mu$.

(3) $a \cdot (b \cdot \lambda) = b \cdot (a \cdot \lambda)$.

(4) $a \cdot \lambda \leq \lambda$ *for all* $\lambda \in \Lambda$ *and* $a \in L$.

(5) *The action of* L *on* Λ *is compatible with the order of* Λ, *that is,* $\lambda \leq \mu$ *implies* $a \cdot \lambda \leq a \cdot \mu$ *for all* $\lambda, \mu \in \Lambda$. *In particular,* $a \cdot \lambda \leq a \cdot \varepsilon$ *for all* $\lambda \in \Lambda$.

(6) $\lambda \leq a \cdot \varepsilon$ *if and only if* $a \cdot \lambda = \lambda$.

(7) *If* $a \in \operatorname{span}\{x_1, \ldots, x_k\}$, *then*

$$\begin{aligned} x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon) \cdots) &= a \cdot \varepsilon \wedge x_1 \cdot \varepsilon \wedge \cdots \wedge x_k \cdot \varepsilon \\ &= a \cdot (x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon) \cdots)). \end{aligned} \tag{5.4}$$

In particular, $x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon) \cdots)$ *is invariant under any permutation of* $x_1, \ldots, x_k$.

Proof. To see (1), since $a \cdot \varepsilon$ is the unity of the ideal $a \cdot \Lambda := \{a \cdot \lambda \mid \lambda \in \Lambda\}$, we have from (3) and (4) of Definition 5.7 that

$$a \cdot (a \cdot \lambda) = a \cdot \varepsilon \wedge (2a) \cdot \lambda = a \cdot \varepsilon \wedge a \cdot \lambda = a \cdot \lambda,$$

For item (2), observe that, since $a \cdot \lambda \wedge \mu$ belongs to the ideal $a \cdot \Lambda$, the previous item implies

$$a \cdot \lambda \wedge \mu = a \cdot (a \cdot \lambda \wedge \mu) = (a \cdot a \cdot \lambda) \wedge (a \cdot \mu) = a \cdot \lambda \wedge a \cdot \mu.$$

The second equality follows analogously.

For item (3), note that $a \cdot (b \cdot \lambda) = a \cdot \varepsilon \wedge (a + b) \cdot \lambda \in (a + b) \cdot \Lambda$. Using item (1) and then (4) and (3) from Definition 5.7, we have

$$\begin{aligned} a \cdot \varepsilon \wedge (a + b) \cdot \lambda &= (a + b) \cdot (a \cdot \varepsilon \wedge (a + b) \cdot \lambda) = (a + b) \cdot ((-a) \cdot \varepsilon) \wedge (a + b) \cdot \lambda) \\ &= (a + b) \cdot \varepsilon \wedge b \cdot \varepsilon \wedge (a + b) \cdot \lambda = b \cdot \varepsilon \wedge (a + b) \cdot \lambda; \end{aligned}$$

the last equality follows from the fact that $(a + b)\varepsilon$ is the unity of the ideal $(a + b) \cdot \Lambda$. Therefore $a \cdot (b \cdot \lambda) = b \cdot (a \cdot \lambda)$.

Statements (4)–(6) follow directly from items (1)–(3).

To see (7), we will first show that for any $x_1, \dots, x_k \in V$, we have

$$\begin{aligned} x_1 \cdot \varepsilon \wedge \cdots \wedge x_k \cdot \varepsilon &= x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon) \cdots) \\ &= x_1 \cdot \varepsilon \wedge (x_1 + x_2) \cdot \varepsilon \wedge \cdots \wedge (x_1 + \cdots + x_k) \cdot \varepsilon. \end{aligned} \tag{5.5}$$

For the first equality in (5.5), observe that

$$\begin{aligned} x_1 \cdot (x_2 \cdot \varepsilon) &= x_1 \cdot \varepsilon \ \wedge \ (x_1 + x_2) \cdot \varepsilon \overset{\text{item (2)}}{=} \varepsilon \ \wedge \ x_1 \cdot ((x_1 + x_2) \cdot \varepsilon) \\ &= (-x_1) \cdot ((x_1 + x_2) \cdot \varepsilon) = (-x_1) \cdot \varepsilon \ \wedge \ x_2 \cdot \varepsilon = x_1 \cdot \varepsilon \ \wedge \ x_2 \cdot \varepsilon. \end{aligned}$$

Arguing by induction, suppose that the first equality in (5.5) holds for $k - 1$ elements. Then we see that

$$\begin{aligned} x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon) \cdots) &= x_1 \cdot (x_2 \cdot \varepsilon \wedge \cdots \wedge x_k \cdot \varepsilon) = x_1 \cdot (x_2 \cdot \varepsilon) \wedge \cdots \wedge x_1 \cdot (x_k \cdot \varepsilon) \\ &= x_1 \cdot \varepsilon \wedge x_2 \cdot \varepsilon \wedge \cdots \wedge x_1 \cdot \varepsilon \wedge x_k \cdot \varepsilon \\ &= x_1 \cdot \varepsilon \wedge x_2 \cdot \varepsilon \wedge \cdots \wedge x_k \cdot \varepsilon. \end{aligned}$$

For the second equality, observe that the case $k = 2$ is (3) of Definition 5.7. Suppose that the second equality in (5.5) holds for $k - 1$ elements. Then

$$\begin{aligned} x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon) \cdots) &= x_1 \cdot (x_2 \cdot \varepsilon \wedge (x_2 + x_3) \cdot \varepsilon \wedge \cdots \wedge (x_2 + \cdots + x_k) \cdot \varepsilon) \\ &= x_1 \cdot (x_2 \cdot \varepsilon) \wedge x_1 \cdot ((x_2 + x_3) \cdot \varepsilon) \\ &\quad \wedge \cdots \wedge x_1 \cdot ((x_2 + \cdots + x_k) \cdot \varepsilon) \\ &= x_1 \cdot \varepsilon \wedge (x_1 + x_2) \cdot \varepsilon \wedge \cdots \wedge x_1 \cdot \varepsilon \wedge ((x_1 + x_2 + \cdots + x_k) \cdot \varepsilon) \\ &= x_1 \cdot \varepsilon \wedge (x_1 + x_2) \cdot \varepsilon \wedge \cdots \wedge (x_1 + \cdots + x_k) \cdot \varepsilon. \end{aligned}$$

Suppose now that $a = \alpha_1 x_1 + \cdots + \alpha_k x_k$ for some $\alpha_i \in \mathbb{F}$. Then using $\alpha_i x_i$ instead of x_i in (5.5) and applying (2), we obtain

$$\begin{aligned} x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon) \cdots) &= (\alpha_1 x_1) \cdot ((\alpha_2 x_2) \cdots ((\alpha_k x_k) \cdot \varepsilon) \cdots) \\ &= (\alpha_1 x_1) \cdot \varepsilon \wedge (\alpha_1 x_1 + \alpha_2 x_2) \cdot \varepsilon \wedge \cdots \wedge (\alpha_1 x_1 + \cdots + \alpha_k x_k) \cdot \varepsilon \\ &= a \cdot ((\alpha_1 x_1) \cdot \varepsilon \wedge (\alpha_1 x_1 + \alpha_2 x_2) \cdot \varepsilon \\ &\quad \wedge \cdots \wedge (\alpha_1 x_1 + \cdots + \alpha_{k-1} x_{k-1}) \cdot \varepsilon \wedge \varepsilon), \end{aligned}$$

so that $x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon) \cdots) \in a\Lambda$. Then, using (1), (2) and (5.5), we conclude that

$$\begin{aligned} x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon) \cdots) &= a \cdot (x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon) \cdots)) = a \cdot (x_1 \cdot \varepsilon \wedge \cdots \wedge x_k \cdot \varepsilon) \\ &= a \cdot (\varepsilon \wedge x_1 \cdot \varepsilon \wedge \cdots \wedge x_k \cdot \varepsilon) = a \cdot \varepsilon \wedge x_1 \cdot \varepsilon \wedge \cdots \wedge x_k \cdot \varepsilon, \end{aligned}$$

as desired. ■

Let (Λ, L) be a partial representation, and let A be a finite-dimensional subspace of L. We use $A \cdot \varepsilon$ to denote the set $\{a \cdot \varepsilon \mid a \in A\}$. For a finite set of generators $\beta = \{x_1, \dots, x_k\}$ of A, we define

$$\beta \circ \varepsilon := x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon) \cdots).$$

Note that from (5.5), we have that $\beta \circ \varepsilon = x_1 \cdot \varepsilon \wedge \cdots \wedge x_k \cdot \varepsilon$.

Lemma 5.9. *Let (Λ, L) be a partial representation, let A be a finite-dimensional subspace of L, and let $\beta = \{x_1, \dots, x_k\}$ be a finite set of generators of A. Then*

$$\inf(A \cdot \varepsilon) = \beta \circ \varepsilon = x_1 \cdot \varepsilon \wedge \cdots \wedge x_k \cdot \varepsilon.$$

In particular, $\beta \circ \varepsilon := x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon) \cdots)$ does not depend on the choice of the finite set of generators β of A.

Proof. Let $a \in A$. Then, by equation (5.4), we have

$$\beta \circ \varepsilon = a \cdot \varepsilon \wedge x_1 \cdot \varepsilon \wedge \cdots \wedge x_k \cdot \varepsilon \leq a \cdot \varepsilon,$$

which implies that $\beta \circ \varepsilon$ is a lower bound of $A \cdot \varepsilon$. Now, assume that $d \in \Lambda$ is another lower bound of $A \cdot \varepsilon$. In particular, $d \leq x_i \cdot \varepsilon$ for all $i = 1, \dots, k$; and this implies $d \leq \beta \circ \varepsilon$. Hence $\beta \circ \varepsilon$ is the greatest lower bound of $A \cdot \varepsilon$. ∎

Proposition 5.10. *The following holds.*

(1) *If A and B are finite-dimensional subspaces of L, then*

$$\inf((A + B) \cdot \varepsilon) = \inf(A \cdot \varepsilon) \wedge \inf(B \cdot \varepsilon).$$

(2) *For any $a \in L$, we have that*

$$a \cdot \varepsilon \wedge \inf(A \cdot \varepsilon) = \inf((A + \mathbb{F}a) \cdot \varepsilon) = a \cdot \inf(A \cdot \varepsilon).$$

Proof. (1) Suppose that $\beta = \{x_1, \dots, x_k\}$ generates A and that $\beta' = \{y_1, \dots, y_l\}$ generates B. Then $\beta \cup \beta'$ generates $A + B$ and, using Lemma 5.9, we obtain

$$\inf(A \cdot \varepsilon) \wedge \inf(B \cdot \varepsilon) = x_1 \cdot \varepsilon \wedge \cdots \wedge x_k \cdot \varepsilon \wedge y_1 \cdot \varepsilon \wedge \cdots \wedge y_l \cdot \varepsilon = \inf((A + B) \cdot \varepsilon).$$

(2) The first equality follows from item (1). For the second equality, suppose that $\{x_1, \dots, x_k\}$ generates A. Then $\{a, x_1, \dots, x_k\}$ generates $A + \mathbb{F}a$, and thus, using again Lemma 5.9, we conclude that

$$\inf((A + \mathbb{F}a) \cdot \varepsilon) = a \cdot (x_1 \cdot (x_2 \cdots (x_k \cdot \varepsilon)) \cdots)) = a \cdot \inf(A \cdot \varepsilon),$$

as required. ∎

5.3. Lie F-inverse semialgebras

Let S be a Lie inverse semialgebra. Inspired by the concept of a congruence on semigroups (see [37, 44]), we define a *congruence* on S as an equivalence relation ω such that $(s, t) \in \omega$ implies that $(r + s, r + t)$, $([r, s], [r, t])$ and $(\alpha s, \alpha t)$ also belong to ω for all $r \in S$ and $\alpha \in \mathbb{F}$. It is straightforward to see that if ω is a congruence on S, then the quotient S/ω is a Lie inverse semialgebra with the natural operations, and the natural projection $\omega^{\#}\colon S \to S/\omega$ is a homomorphism of Lie inverse semialgebras.

On S we define the relation σ by setting $s \,\sigma\, t$ if and only if there exists $e \in \mathcal{E}(S)$ such that $s + e = t + e$. It is easy to see that σ is a congruence and that S/σ is a Lie algebra. Moreover, if ω is another congruence on S such that S/ω is a Lie algebra, then $\sigma \subseteq \omega$. We will refer to σ as the *minimum Lie congruence* on S. If $\phi\colon S \to T$ is a homomorphism between Lie inverse semialgebras, there exists a unique Lie algebra homomorphism ϕ^σ : $S/\sigma \to T/\sigma$ such that $\sigma^{\#} \circ \phi = \phi^\sigma \circ \sigma^{\#}$. This means that $\phi^\sigma(\bar{s}) = \overline{\phi(s)}$, where $\bar{s}$ stands for the σ-class of $s \in S$ and $\overline{\phi(s)}$ for that of $\phi(s)$. Furthermore, clearly, $(\phi \circ \psi)^\sigma = \phi^\sigma \circ \psi^\sigma$.

Adapting the notion of an F-inverse semigroup to the Lie context, we say that a Lie inverse semialgebra S is *F-inverse* if every σ-class $\bar{s} \in S/\sigma$ has a greatest element $m_{\bar{s}}$ and for every $s, t \in S$, the following condition holds:

$$[s,t] = m_{\overline{[s,t]}} + 0_{s+t}. \tag{5.6}$$

It follows, in particular, that $(S, +)$ is an F-inverse semigroup and, consequently, it is a monoid (see Proposition 7.1.3 in [37]), so that S has 0. Observe that the fact that $m_{\bar{s}}$ is the greatest element of the σ-class $\bar{s}$ means that $\bar{s} = \mathcal{E}(S) + m_{\bar{s}}$, i.e., $\bar{s}$ is the principal order ideal generated by $m_{\bar{s}}$ in the additive inverse semigroup $(S, +)$ considered as a partially ordered set. Moreover,

$$m_{\overline{\alpha s}} = \alpha m_{\bar{s}} \quad \text{and} \quad m_{\bar{s}} + m_{\bar{t}} \leq m_{\overline{s+t}},$$

for all $s, t \in S$ and $\alpha \in \mathbb{F}$. Since $m_{\bar{0}} = 0$, it follows that the map $(S/\sigma, +) \to (S, +)$ given by $\bar{s} \mapsto m_{\bar{s}}$ is a premorphism from an additive group into an additive inverse monoid. In particular, from (4.1) we have that

$$m_{\bar{s}} + m_{\bar{t}} = 0_{m_{\bar{s}}} + m_{\overline{s+t}} = 0_{m_{\bar{t}}} + m_{\overline{s+t}}, \tag{5.7}$$

for all $s, t \in S$.

Example 5.11. A pivotal example of a Lie F-inverse semialgebra is the Lie inverse semialgebra $E(L)$ associated with any Lie algebra L (see Section 2). Specifically, for (A, a) and (B, b) in $E(L)$, it can be verified that $(A, a)\sigma(B, b)$ if and only if $a = b$. Consequently, $(\mathbb{F}a, a)$ is the greatest element for the σ-class of (A, a), and the equality

$$(A + B + \mathbb{F}[a,b], [a,b]) = (\mathbb{F}[a,b], [a,b]) + (A + B, 0)$$

implies that $E(L)$ is an Lie F-inverse semialgebra.

Let (Λ, L) be a partial representation. We define

$$F(\Lambda, L) = \{(\lambda, a) \in \Lambda \times L \mid \lambda \leq a \cdot \varepsilon\},$$

endowed with the following operations:

$$\begin{aligned} (\lambda, a) + (\mu, b) &= (\lambda \wedge \mu, a + b), \\ \alpha(\lambda, a) &= (\lambda, \alpha a), \\ [(\lambda, a), (\mu, b)] &= ([a,b] \cdot (\lambda \wedge \mu), [a,b]), \end{aligned}$$

for all $\alpha \in \mathbb{F}$, $\lambda, \mu \in \Lambda$ and $a, b \in L$.

Proposition 5.12. *$F(\Lambda, L)$ is a Lie F-inverse semialgebra.*

Proof. First, we will see that the above operations are well defined. Let (λ, a) and (μ, b) be in $F(\Lambda, L)$. Then, using (6) of Lemma 5.8, we have

$$\begin{aligned} \lambda \wedge \mu = \lambda \wedge \mu \wedge \varepsilon &= a \cdot \lambda \wedge b \cdot \mu \wedge \varepsilon = a \cdot \lambda \wedge \mu \wedge b \cdot \varepsilon \\ &= \lambda \wedge \mu \wedge a \cdot (b \cdot \varepsilon) = \lambda \wedge \mu \wedge a \cdot \varepsilon \wedge (a + b) \cdot \varepsilon \leq (a + b) \cdot \varepsilon, \end{aligned}$$

which implies $(\lambda, a) + (\mu, b) \in F(\Lambda, L)$. Next, as $\lambda \wedge \mu \leq \varepsilon$, by (5) of Lemma 5.8, we have $[a,b] \cdot (\lambda \wedge \mu) \leq [a,b] \cdot \varepsilon$, which ensures that $[(\lambda, a), (\mu, b)] \in F(\Lambda, L)$. Furthermore, for any $\alpha \in \mathbb{F}$, we have $(\alpha a) \cdot \varepsilon = a \cdot \varepsilon \geq \lambda$, showing that $\alpha(\lambda, a) \in F(\Lambda, L)$. Hence, the operations defined on $F(\Lambda, L)$ are indeed well defined. It is also straightforward to verify that $F(\Lambda, L)$ is a commutative inverse semigroup, whose set of (additive) idempotents is given by $\mathcal{E}(F(\Lambda, L)) = \{(\lambda, 0) \mid \lambda \in \Lambda\}$. Note that $(\lambda, a) \preceq (\mu, b)$ if and only if $a = b$ and $\lambda \leq \mu$.

Next, we need to verify that conditions (1)–(6) of Definition 2.10 are satisfied. However, since the other conditions are straightforward, we will focus only on conditions (2) and (5). To see (2) let $(\lambda, a), (\mu, b), (\nu, c) \in F(\Lambda, L)$. Then we have

$$[(\nu, c), (\lambda \wedge \mu, a + b)] = ([c, a + b] \cdot \nu \wedge \lambda \wedge \mu, [c, a + b]),$$

and

$$\begin{aligned} [(\nu, c), (\lambda, a)] + [(\nu, c), (\mu, b)] &= ([c, a] \cdot \nu \wedge \lambda \wedge [c, b] \cdot \nu \wedge \mu, [c, a + b]) \\ &= (\nu \wedge [c, a] \cdot \lambda \wedge [c, b] \cdot \mu, [c, a + b]). \end{aligned}$$

Now, we see that

$$\begin{aligned} ([c, a] + [c, b]) \cdot \nu \wedge \lambda \wedge \mu &\geq [c, a] \cdot \varepsilon \wedge ([c, a] + [c, b]) \cdot \nu \wedge \lambda \wedge \mu \\ &= [c, a] \cdot ([c, b] \cdot \nu \wedge \lambda \wedge \mu) = \nu \wedge [c, a] \cdot \lambda \wedge [c, b] \cdot \mu. \end{aligned}$$

Hence, we conclude that $[(\nu, c), (\lambda, a) + (\mu, b)] \succeq [(\nu, c), (\lambda, a)] + [(\nu, c), (\mu, b)]$, which gives (2).

To prove (5), note that using (4) of Lemma 5.8, we have

$$[[a, b], c] \cdot ([a, b] \cdot (\lambda \wedge \mu) \wedge \nu) \leq \lambda \wedge \mu \wedge \nu,$$

and then

$$\begin{aligned} [[(\lambda, a), (\mu, b)], (\nu, c)] &= ([[a, b], c] \cdot ([a, b] \cdot (\lambda \wedge \mu) \wedge \nu), [[a, b], c]) \\ &\preceq (\lambda \wedge \mu \wedge \nu, [a, [b, c]]). \end{aligned}$$

Thus

$$J((\lambda, a), (\mu, b), (\nu, c)) \preceq (\lambda \wedge \mu \wedge \nu, J(a, b, c)) = (\lambda \wedge \mu \wedge \nu, 0) = 0_{(\lambda,a)+(\mu,b)+(\nu,c)}.$$

Therefore, $F(\Lambda, L)$ is a Lie inverse semialgebra.

It remains to prove that $F(\Lambda, L)$ is F-inverse. Notice that $(\lambda, a)\sigma(\mu, b)$ if and only if $a = b$. Hence $F(\Lambda, L)/\sigma = \{\overline{(\cdot, a)} \mid a \in L\} \cong L$. Since $\lambda \leq a \cdot \varepsilon$ for every $(\lambda, a) \in F(\Lambda, L)$, it follows that $\overline{(\cdot, a)} = ((a \cdot \varepsilon) \wedge \Lambda, a)$ and that $(a \cdot \varepsilon, a)$ is the greatest element of the σ-class $\overline{(\cdot, a)}$. Finally, the equality

$$\begin{aligned} ([a, b] \cdot \varepsilon, [a, b]) + (\lambda \wedge \mu, 0) &= ([a, b] \cdot \varepsilon \wedge \lambda \wedge \mu, [a, b]) \\ &= ([a, b] \cdot (\lambda \wedge \mu), [a, b]) = [(\lambda, a), (\mu, b)] \end{aligned}$$

shows that the equation (5.6) holds for $F(\Lambda, L)$. So, $F(\Lambda, L)$ is indeed an Lie F-inverse semialgebra. ∎

Example 5.13. Let L be a Lie algebra, and denote by $P_f(L)$ the set of all finite-dimensional subspaces of L. Then $(P_f, +)$ is a meet semilattice whose unity is the subspace 0. It is straightforward to see that the map $L \times P_f(L) \to P_f(L)$ defined by $a \cdot A = A + \mathbb{F}a$ gives a partial representation $(P_f(L), L)$. A simple calculation shows that $F(P_f(L), L)$ coincides with the Lie F-inverse semialgebra $E(L)$.

In what follows, $\mathcal{F}$ denotes the category whose objects are the Lie F-inverse semialgebras, and whose morphisms are the homomorphisms which map the greatest element of a σ-class to the greatest element of a σ-class.

Suppose that $(\theta, \phi): (\Lambda, L) \to (\Pi, H)$ is a morphism of partial representations. Then, it is straightforward to verify that the mapping $F(\theta, \phi): F(\Lambda, L) \to F(\Pi, H)$, defined by $(\lambda, a) \mapsto (\theta(\lambda), \phi(a))$, is a homomorphism of Lie F-inverse semialgebras. Furthermore, for any $a \in L$, we have

$$F(\theta, \phi)(a \cdot \varepsilon, a) = (\theta(a \cdot \varepsilon), \phi(a)) = (\phi(a) \cdot \varepsilon, \phi(a)),$$

which shows that $F(\theta, \phi)$ sends the greatest element of a σ-class to the greatest element of a corresponding σ-class. Thus we obtain a functor $F: \mathcal{I} \to \mathcal{F}$ sending (Λ, L) to $F(\Lambda, L)$ and (θ, ϕ) to $F(\theta, \phi)$. Our objective is to show that the category $\mathcal{I}$, of partial representations, is equivalent to the category $\mathcal{F}$. To achieve this, we need to construct a functor in the opposite direction.

Let S be a Lie F-inverse semialgebra, and let $m_{\bar{s}}$ denote the greatest element of the σ-class $\bar{s}$. Consider the Lie algebra S/σ, and the meet semilattice $\mathcal{E}(S)$ of (additive) idempotents of S. Then, it is easy to see that a mapping $S/\sigma \times \mathcal{E}(S) \to \mathcal{E}(S)$ defined by

$$\bar{s} \cdot \lambda = 0_{m_{\bar{s}}} + \lambda,$$

gives a partial representation $(\mathcal{E}(S), S/\sigma)$. Indeed, conditions (1), (2) and (4) of Definition 5.7 are straightforward, whereas (3) is a consequence of (5.7). Moreover, if $\varphi: S \to T$ is a homomorphism of Lie F-inverse semialgebras which maps the greatest element of a σ-class to the greatest element of a σ-class, then the pair (θ, φ^σ), where $\theta = \varphi|_{\mathcal{E}(S)}$ and $\varphi^\sigma: S/\sigma \to T/\sigma$ is as defined above, is a morphism in $\mathcal{I}$. Thus, we obtain a functor $K: \mathcal{F} \to \mathcal{I}$ setting $K(S) = (\mathcal{E}(S), S/\sigma)$ and $K(\varphi) = (\theta, \varphi^\sigma)$.

For every $(\Lambda, L) \in \mathcal{I}$, we define the mapping $(\xi_\Lambda, \eta_L): (\Lambda, L) \to KF(\Lambda, L)$, where $\xi_\Lambda(\lambda) = (\lambda, 0)$ and $\eta_L(a) = \overline{(\cdot, a)}$. On the opposite direction, for $S \in \mathcal{F}$, we define the mapping $\gamma_S: S \to FK(S)$ by $\gamma_S(s) = (0_s, \bar{s})$. The following theorem shows that the mappings $(\xi_\Lambda, \eta_L): (\Lambda, L) \to KF(\Lambda, L)$ and $\gamma_S: S \to FK(S)$ establish an equivalence between the categories $\mathcal{I}$ and $\mathcal{F}$. The proof is essentially the same as that of Lemmas VII.6.7 and VII.6.8 in [44], with some minor adaptations. However, to keep our exposition self-contained, we provide the details of the proof.

Theorem 5.14. *The mapping (ξ_Λ, η_L) establishes an equivalence between the functors $\mathrm{id}_{\mathcal{I}}$ and KF, while the mapping γ_S establishes an equivalence between the functors $\mathrm{id}_{\mathcal{F}}$ and FK. Consequently, the categories $\mathcal{I}$ and $\mathcal{F}$ are equivalent.*

Proof. First note that for any $(\Lambda, L) \in \mathcal{I}$, we have that ξ_Λ is an isomorphism between Λ and $\mathcal{E}(F(\Lambda, L))$, and η_L is an isomorphism between L and $F(\Lambda, L)/\sigma$. Furthermore, for any $a \in L$ and $\lambda \in \Lambda$, we have

$$\begin{aligned}\xi_\Lambda(a \cdot \lambda) &= (a \cdot \lambda, 0) = (\varepsilon \wedge a \cdot \lambda, 0) = (a \cdot \varepsilon \wedge \lambda, 0) \\ &= (a \cdot \varepsilon, 0) + (\lambda, 0) = 0_{m_{\overline{(\cdot,a)}}} + (\lambda, 0) = \eta_L(a) \cdot \xi_\Lambda(\lambda),\end{aligned}$$

which shows that (ξ_Λ, η_L) is an isomorphism in the category $\mathcal{I}$. In order to show that the correspondence $(\Lambda, L) \mapsto (\xi_\Lambda, \eta_L)$ is natural, let $(\theta, \phi): (\Lambda, L) \to (\Pi, H)$ be a morphism of partial representations, and for convenience, set $(\theta', \phi') = KF(\theta, \phi)$. Then, for every $\lambda \in \Lambda$ and $a \in L$, we have

$$\begin{aligned}(\theta', \phi') \circ (\xi_\Lambda, \eta_L)(\lambda, a) &= (\theta', \phi')((\lambda, 0), \overline{(\cdot, a)}) = ((\theta(\lambda), 0), \overline{(\cdot, \phi(a))} \\ &= (\xi_\Pi, \eta_H) \circ (\theta, \phi)(\lambda, a),\end{aligned}$$

showing that the correspondence $(\Lambda, L) \mapsto (\xi_\Lambda, \eta_L)$ is a natural equivalence.

We now address the second assertion. First note that for any $S \in \mathcal{F}$, the mapping γ_S is a bijection. Indeed, if $s, t \in S$ are such that $0_s = 0_t$ and $\bar{s} = \bar{t}$, then $s = m_{\bar{s}} + 0_s = m_{\bar{t}} + 0_t = t$, showing that γ_S in injective. Now, consider any $(\mu, \bar{s}) \in FK(S)$. Since $\mu \leq 0_{m_{\bar{s}}}$, setting $t = \mu + m_{\bar{s}}$ we obtain

$$\gamma_S(t) = (\mu + 0_{m_{\bar{s}}}, \overline{(\mu + m_{\bar{s}})}) = (\mu, \overline{m_{\bar{s}}}) = (\mu, \bar{s}),$$

which shows that γ_S is onto. Furthermore, it is clear that for any $\alpha \in \mathbb{F}$ and $s, t \in S$, we have $\gamma_S(s + t) = \gamma_S(s) + \gamma_S(t)$ and $\gamma_S(\alpha s) = \alpha \gamma_S(s)$. In addition, using (5.6), we see that

$$\begin{aligned}[\gamma_S(s), \gamma_S(t)] &= [(0_s, \bar{s}), (0_t, \bar{t})] = (\overline{[s,t]} \cdot 0_{s+t}, \overline{[s,t]}) \\ &= (0_{m_{\overline{[s,t]}}} + 0_{s+t}, \overline{[s,t]}) = (0_{[s,t]}, \overline{[s,t]}) = \gamma_S([s,t]),\end{aligned}$$

which implies that γ_S is an isomorphism in the category $\mathcal{F}$. It remains to show that the correspondence $S \mapsto \gamma_S$ is natural. For this suppose that $\varphi: S \to T$ is a homomorphism in $\mathcal{F}$. Then, for any $s \in S$, we have that

$$FK(\varphi) \circ \gamma_S(s) = FK(\varphi)(0_s, \bar{s}) = (\varphi(0_s), \overline{\varphi(s)}) = (0_{\varphi(s)}, \overline{\varphi(s)}) = \gamma_T \circ \varphi(s),$$

which gives the naturality of the correspondence $S \mapsto \gamma_S$. ■

5.4. Lie analogue of Szendrei's theorem

Suppose that $\phi: L \to H$ is a Lie algebra homomorphism. Then, ϕ induces a homomorphism $\tilde{\phi}: E(L) \to E(H)$ given by $\tilde{\phi}(A, a) = (\phi(A), \phi(a))$. This homomorphism maps the greatest elements of the σ-class of (A, a) to the greatest element of the σ-class of $\tilde{\phi}(A, a)$. Additionally, it is important to note that $\widetilde{\phi \circ \psi} = \tilde{\phi} \circ \tilde{\psi}$.

Let $\mathcal{L}$ denote the category of Lie algebras, and let $\mathcal{F}$ denote the category of Lie F-inverse semialgebras and their homomorphisms which map the greatest element of a σ-class to the greatest element of a σ-class. By the previous paragraph, the correspondence $L \mapsto E(L), \phi \mapsto \tilde{\phi}$ defines a functor $E: \mathcal{L} \to \mathcal{F}$. Similarly, the correspondence $S \mapsto S/\sigma$, $\phi \mapsto \phi^\sigma$, defines a functor $Q: \mathcal{F} \to \mathcal{L}$. The main result of this section is the following theorem, which is a Lie analogue of a well-known result by M. Szendrei, see Theorem 2 in [48].

Theorem 5.15. *The functor $E\colon \mathcal{L} \to \mathcal{F}$ is left adjoint to the functor $Q\colon \mathcal{F} \to \mathcal{L}$.*

The remainder of this section is devoted to prove Theorem 5.15. The proof follows the approach from [48], with additional ingredients needed for our Lie context. More specifically, instead of proving the theorem directly, we factorize the functors $E\colon \mathcal{L} \to \mathcal{F}$ and $Q\colon \mathcal{Q} \to \mathcal{L}$ in Theorem 5.15 through the functors F and K described in Subsection 5.3. To do this, we first recall from Example 5.13, that for a given Lie algebra L we have the meet semilattice $P_f(L)$ of all finite-dimensional subspaces of L, and the mapping $L \times P_f(L) \to P_f(L)$ defined by $a \cdot A = A + \mathbb{F}a$, gives a partial representation $(P_f(L), L)$. Furthermore, if $\phi\colon L \to H$ is a Lie algebra homomorphism, then the mapping $\widehat{\phi}\colon P_f(L) \to P_f(H)$, defined by $\widehat{\phi}(A) = \phi(A)$, is a homomorphism of monoids; and, clearly, the mapping $(\widehat{\phi}, \phi)\colon (P_f(L), L) \to (P_f(H), H)$ is a morphism of partial representations. Moreover, we have that $\widehat{\phi \circ \psi} = \widehat{\phi} \circ \widehat{\psi}$. Therefore, we can define a functor $\overline{E}\colon \mathcal{L} \to \mathcal{I}$ by setting $\overline{E}(L) = (P_f(L), L)$ and $\overline{E}(\phi) = (\widehat{\phi}, \phi)$. Similarly, we can define a functor $\overline{Q}\colon \mathcal{I} \to \mathcal{L}$ by setting $\overline{Q}(\Lambda, L) = L$ and $\overline{Q}(\theta, \phi) = \phi$. The functors $\overline{E}$ and $\overline{Q}$ satisfy $E = F\overline{E}$ and $Q = \overline{Q}K$, where E and Q are the functors in Theorem 5.15. By virtue of Theorem 5.14, to prove Theorem 5.15 it suffices to show the following theorem.

Theorem 5.16. *The functor $\overline{E}$ is left adjoint to the functor $\overline{Q}$.*

Proof. Let L be a Lie algebra and (Π, H) a partial representation. Then, for each homomorphism $\phi\colon L \to H = \overline{Q}(\Pi, H)$, let $\beta(\phi)\colon \overline{E}(L) \to (\Pi, H)$ be defined by $\phi \mapsto (\Theta_\phi, \phi)$, where $\Theta_\phi\colon P_f(L) \to \Pi$ is defined by $\Theta_\phi(A) = \inf(\phi(A) \cdot \varepsilon)$. It follows from Proposition 5.10, that the pair (Θ_ϕ, ϕ) is a homomorphism of partial representations. The mapping $\beta\colon \mathrm{Hom}(L, \overline{Q}(\Pi, H)) \to \mathrm{Hom}(\overline{E}(L), (\Pi, H))$, defined by $\phi \mapsto \beta(\phi)$, is clearly injective. To see that β is onto, note that for $(\Theta, \phi) \in \mathrm{Hom}(\overline{E}(L), (\Pi, H))$ and for any $a \in L$, we have that

$$\Theta(\mathbb{F}a) = \Theta(a \cdot \varepsilon) = \phi(a) \cdot \varepsilon = \Theta_\phi(\mathbb{F}a),$$

which implies that $\Theta = \Theta_\phi$. Therefore the mapping β is bijective. To complete the proof, we need to establish the naturality of $\beta = \beta_{L,(\Pi,H)}$ in both components.

(Naturality of β in the first component). Let $\gamma\colon L \to K$ be a Lie algebra homomorphism. For $\phi \in \mathrm{Hom}(K, H)$, we have that

$$\beta(\mathrm{Hom}(\gamma, H)(\phi)) = \beta(\phi\gamma) = (\Theta_{\phi\gamma}, \phi\gamma),$$

and on the other hand,

$$\mathrm{Hom}((\hat{\gamma}, \gamma), (\Pi, H))(\beta(\phi)) = \mathrm{Hom}((\hat{\gamma}, \gamma), (\Pi, H))(\Theta_\phi, \phi) = (\Theta_\phi \hat{\gamma}, \phi\gamma).$$

Now, for any $A \in P_f(L)$, we see that

$$\Theta_\phi \hat{\gamma}(A) = \Theta_\phi \gamma(A) = \inf(\phi\gamma(A) \cdot \varepsilon) = \Theta_{\phi\gamma}(A),$$

which implies $\Theta_\phi \hat{\gamma} = \Theta_{\phi\gamma}$. Therefore, β is natural in the first component.

(Naturality of β in the second component). Let $(\Theta, \phi)\colon (\Pi, H) \to (\Lambda, M)$ be a homomorphism of partial representations. Then, for $\psi \in \mathrm{Hom}(L, H)$, we have that

$$\mathrm{Hom}(\overline{E}(L), (\Theta, \phi))(\beta(\psi)) = \mathrm{Hom}(\overline{E}(L), (\Theta, \phi))(\Theta_\psi, \psi) = (\Theta\Theta_\psi, \phi\psi)$$

and

$$\beta(\mathrm{Hom}(L,\phi)(\psi)) = \beta(\phi\psi) = (\Theta_{\phi\psi}, \phi\psi).$$

We need to verify the equality $\Theta\Theta_\psi = \Theta_{\phi\psi}$. For $A \in P_f(A)$, suppose that $\{x_1, \dots, x_k\}$ is a finite set of generators for A. Using Lemma 5.9 and the fact that (Θ, ϕ) is a homomorphism of partial representations, that is, it satisfies $\Theta(a \cdot \lambda) = \phi(a) \cdot \Theta(\lambda)$ for $a \in H$ and $\lambda \in \Pi$, we have

$$\begin{aligned}\Theta\Theta_\psi(A) &= \Theta((\inf \psi(A) \cdot \varepsilon)) = \Theta(\psi(x_1) \cdot \varepsilon \wedge \cdots \wedge \psi(x_k) \cdot \varepsilon)\\ &= \phi(\psi(x_1)) \cdot \varepsilon \wedge \cdots \wedge \phi(\psi(x_k)) \cdot \varepsilon = \inf(\phi\psi(A) \cdot \varepsilon) = \Theta_{\phi\psi}(A).\end{aligned}$$

Hence $\Theta\Theta_\psi = \Theta_{\phi\psi}$, which establishes the naturality of β in the second component. ∎

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Mikhailo Dokuchaev
Department of Mathematics, University of São Paulo
Rua do Matão 1010, SP 05508-090 São Paulo, Brazil;
dokucha@ime.usp.br
Author IDs: zbMATH dokuchaev.mikhailo MR 231275 ORCID 0000-0003-1250-4831

Farangis Johari
Center for Mathematics, Computing and Cognition, Federal University of ABC
Av. dos Estados 5001, SP 09280-560 Santo André, Brazil;
farangis.johari@ufabc.edu.br
Author IDs: zbMATH johari.farangis MR 1171290 ORCID 0000-0002-2173-0323

José L. Vilca-Rodríguez
Department of Mathematics, University of São Paulo
Rua do Matão 1010, SP 05508-090 São Paulo, Brazil;
jvilca@ime.usp.br
Author IDs: zbMATH vilca-rodriguez.jose-1 MR 1494073 ORCID 0000-0002-1921-5058