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Monotonicity properties of hyperbolic projections in holomorphic iteration

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Abstract. We consider hyperbolic projections of orbits of holomorphic self-maps of the unit disc, onto curves landing on the unit circle with a given angle. We show that under certain necessary assumptions, the projections exhibit monotonicity properties akin to those present in continuous dynamics. Our techniques are purely hyperbolic-geometric in nature and provide the general framework for analysing projections of arbitrary sequences onto curves.

1. Introduction

This article explores the monotonicity properties of hyperbolic projections of holomorphic orbits in the unit disc of the complex plane, onto curves landing on the unit circle. Our analysis is inspired by the theory of continuous holomorphic semigroups, where examining the behaviour of projections of trajectories has been at the forefront of many advancements in the last decade [7, 8, 11, 16–18, 22]. In fact, our main result, Theorem 1.1, can be thought of as a strong, discrete version of the remarkable monotonicity property for the trajectories of continuous semigroups proved recently by Betsakos and Karamanlis [7]. Surprisingly, this strong monotonicity is unique both to discrete iteration and to the type of functions considered in Theorem 1.1, as shown by our examples in Section 3.

The results presented here are part of a host of recent advances in the theory of discrete iteration in the unit disc that aim to shed light into the precise nature with which a sequence of iterates converges to a boundary attracting fixed point. These include a new version of the Denjoy–Wolff theorem for arbitrary simply connected domains [5], an analysis of the slope problem in parabolic iteration [13, 14], and purely geometric versions of classical results [4], to name a few. Also, of particular interest are the novel connections between discrete iteration and other areas of holomorphic dynamics, such as continuous semigroups [12, 15], and dynamics of entire functions [2, 6].

Let us start by writing $\mathbb{D}$ for the open unit disc in the complex plane $\mathbb{C}$, which we endow with the hyperbolic distance $d_{\mathbb{D}}$. The classical Denjoy–Wolff theorem [21] states that if $f\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic function with no fixed points in $\mathbb{D}$, then there exists a

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unique point $\tau \in \partial\mathbb{D}$, called the *Denjoy–Wolff point of* f, so that the sequence of iterates of f,

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}},$$

converges uniformly on compact sets to the constant function τ. Moreover, the *angular derivative* $f'(\tau)$ of f at τ exists and satisfies $f'(\tau) \in (0, 1]$. When $f'(\tau) < 1$, the holomorphic function f is called *hyperbolic*, and *parabolic* otherwise.

We say that a curve $\gamma : [0, +\infty) \to \mathbb{D}$ *lands* at a point $\zeta \in \partial\mathbb{D}$, if $\gamma(t)$ converges to ζ as $t \to +\infty$, in the Euclidean metric of $\mathbb{C}$. Given $z_0 \in \mathbb{D}$, a point $\pi_\gamma(z_0) \in \mathbb{D}$ will be called *a projection of* z_0 *onto* γ if $\pi_\gamma(z_0) = \gamma(t_0)$, for some $t_0 \geq 0$, and

$$d_{\mathbb{D}}(z_0, \gamma(t_0)) = \inf\{d_{\mathbb{D}}(z_0, \gamma(t)) : t \in [0, +\infty)\}.$$

Note that when γ lands at a point on the boundary, projections of any point $z \in \mathbb{D}$ onto γ always exist, but are not necessarily unique. It is well known and easy to prove, however, that when γ is a geodesic for the hyperbolic distance, every point of the disc has a unique projection onto γ; see Proposition 3.2 in [8].

Our main objective is to establish the following theorem.

Theorem 1.1. *Let* $f : \mathbb{D} \to \mathbb{D}$ *be a hyperbolic map with Denjoy–Wolff point* $\tau \in \partial\mathbb{D}$*, and let* $\gamma : [0, +\infty) \to \mathbb{D}$ *be a curve landing at* τ*, satisfying*

$$\lim_{t \to +\infty} \arg(1 - \bar{\tau}\gamma(t)) = \theta \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

Fix $z \in \mathbb{D}$ *and let* $\pi_\gamma(f^n(z))$ *be a projection of* $f^n(z)$ *onto* γ*, for* $n \in \mathbb{N}$*. Then, for any* $w \in \mathbb{D}$*, the sequence* $\{d_{\mathbb{D}}(w, \pi_\gamma(f^n(z)))\}$ *is eventually strictly increasing.*

For a curve $\gamma : [0, +\infty) \to \mathbb{D}$ landing at $\tau \in \partial\mathbb{D}$, the cluster set of $\arg(1 - \bar{\tau}\gamma(t))$ is often called the *slope* of the curve. So in Theorem 1.1 we essentially assume that γ lands at τ non-tangentially (i.e., stays within a Stolz angle) and with a specified slope. The archetypal examples of such curves are the hyperbolic geodesics of $\mathbb{D}$ landing at τ, for which $\arg(1 - \bar{\tau}\gamma(t))$ tends to zero. Therefore, as a special case of Theorem 1.1, we have that the hyperbolic distance between any point $w \in \mathbb{D}$ and the projections of a hyperbolic orbit $\{f^n(z)\}$ onto any geodesic is eventually strictly increasing (see Corollary 5.2).

The main advantage of our result, however, is the fact that it allows for projections onto *any* curve landing at the Denjoy–Wolff point with a specified slope. This may even include behaviours typically considered “pathological”, such as the curve having (diminishing) oscillations or self-intersections.

Moreover, Theorem 1.1 allows for complete freedom for the choice of each projection $\pi_\gamma(f^n(z))$, of which there might be infinitely many. Finally, it is worth mentioning that previous monotonicity results of this type – to be described shortly – consider the distance of the projections $\pi_\gamma(f^n(z))$ from z, i.e., the case $w = z$ in Theorem 1.1. The fact that we can choose the base-point w arbitrarily encapsulates the intuitive fact of hyperbolic geometry that an “observer” positioned at $w \in \mathbb{D}$ will eventually perceive the sequence $\{\pi_\gamma(f^n(z))\}$ as moving away from them monotonically, regardless of the position of w.

Our proof is based on a thorough analysis of the hyperbolic geometry behind projections of sequences onto curves, carried out in Section 4. The main technical result of this work is Theorem 4.2, which roughly states that if two curves γ_1 and γ_2 land at the same boundary point, with the same slope, then the projections $\pi_{\gamma_1}(z_n)$ and $\pi_{\gamma_2}(z_n)$ of a sequence $\{z_n\} \subset \mathbb{D}$ onto γ_1 and γ_2, respectively, are asymptotically close to one another. This result might be of independent interest since it implies that the projections onto an arbitrary curve can be controlled by projections onto simple, well-behaved curves landing with the same slope. In fact, we employ this technique for the proof of Theorem 1.1 in Section 5. Also, this result allows us to asymptotically correlate the projections onto curves landing with different angles, as shown in Corollary 4.3.

In order to describe the connection between Theorem 1.1 and the theory of continuous semigroups of holomorphic functions, let us consider a semigroup $\phi_t : \mathbb{D} \to \mathbb{D}$, $t \geq 0$, with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ (see the discussion in Example 3.1 for the precise definition). Also take a geodesic $\gamma : [0, +\infty) \to \mathbb{D}$ for the hyperbolic distance, emanating from a point $z \in \mathbb{D}$ and landing at τ.

The continuous function $v^z : [0, +\infty) \to [0, +\infty)$ given by

$$v^z(t) = d_{\mathbb{D}}(z, \pi_\gamma(\phi_t(z))),$$

is called the *orthogonal speed* of the semigroup. This was first considered by Bracci in his seminal paper [8], where he proved several key properties of v^z and computed the rate with which $v^z(t)$ tends to infinity in various cases. His work has been the starting point for a plethora of modern results that investigate the properties of the speeds of convergence for the trajectories of semigroups, such as [11, 16–18, 22].

Recently, Betsakos and Karamanlis showed (see Theorem 1.1 in [7]) that the orthogonal speed is a strictly increasing function, for any semigroup with a boundary Denjoy–Wolff point. This result serves as the main motivation behind Theorem 1.1, since the sequence $\{d_{\mathbb{D}}(w, \pi_\gamma(f^n(z)))\}$ considered in Theorem 1.1 is an analogue of the orthogonal speed in discrete iteration. In fact, since our theorem allows for projections onto curves landing at τ non-tangentially, $\{d_{\mathbb{D}}(w, \pi_\gamma(f^n(z)))\}$ can be thought of as the *non-tangential speed* of the map f.

In Section 3, we provide examples which show that the strict monotonicity described by Theorem 1.1 may fail for semigroups and for parabolic maps, meaning that it is unique to hyperbolic self-maps of $\mathbb{D}$. Moreover, we show that the term "*eventually*" cannot be omitted, and that the landing slope θ of the curve γ cannot be considered to be $\pm\pi/2$. Hence, when considering the strict monotonicity of the non-tangential speed, our result is the best possible.

As an immediate corollary of Theorem 1.1, we obtain the following global monotonicity for hyperbolic maps.

Corollary 1.2. *Let $f : \mathbb{D} \to \mathbb{D}$ be a hyperbolic map with Denjoy–Wolff point τ. For any $z, w \in \mathbb{D}$, the sequence $\{d_{\mathbb{D}}(w, f^n(z))\}$ is eventually strictly increasing.*

In the language of continuous semigroups introduced by Bracci in [8], the sequence considered in Corollary 1.2 is called *total speed*, whenever $w = z$. In Theorem 1.2 of [7], it was shown that there exist semigroups for which the total speed is not eventually increasing. In Example 3.5, we describe how Theorem 1.2 in [7] can be used in order to show that the sequence $\{d_{\mathbb{D}}(w, f^n(z))\}$ may not be eventually increasing if f is parabolic, and so the assumption that f is hyperbolic in Corollary 1.2 is necessary.

In the sixth, and final, section of this article, we discuss questions that arise from our work, as well as possible generalisations of our results.

2. Preliminaries

2.1. Hyperbolic geometry

We start by presenting the basics of hyperbolic geometry. For an in depth treatise on the subject, we refer to the excellent books [1] (see Chapter 1), [3] (see Chapter 7) and [9] (see Chapter 5).

The *hyperbolic distance* in $\mathbb{D}$ is defined as

$$d_{\mathbb{D}}(z,w) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{1-|z|^2},$$

where the infimum is taken over all piecewise C^1-smooth curves $\gamma\colon[0,1]\to\mathbb{D}$ satisfying $\gamma(0)=z$ and $\gamma(1)=w$. It is easy to show that $d_{\mathbb{D}}$ is in fact a metric, and that the metric space $(\mathbb{D}, d_{\mathbb{D}})$ is complete.

The Riemann mapping theorem allows us to transfer the above notions to any simply connected domain as follows. Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and $f\colon\Omega\to\mathbb{D}$ a Riemann mapping. The hyperbolic distance in Ω is defined by

$$d_{\Omega}(z,w) := d_{\mathbb{D}}(f(z), f(w)), \quad z,w\in\Omega. \tag{2.1}$$

It turns out that this definition is independent of the choice of the Riemann mapping f. More importantly, this immediately shows that the hyperbolic distance is a conformally invariant quantity.

As final pieces of notation, we use $D_{\Omega}(w,R)$ to describe the open hyperbolic disc of Ω, centred at w and of radius R; that is the set

$$D_{\Omega}(w,R) := \{z\in\Omega : d_{\Omega}(z,w) < R\}. \tag{2.2}$$

Also, if $\gamma\colon[0,+\infty)\to\Omega$ is a curve and $z\in\Omega$, for convenience we write

$$d_{\Omega}(z,\gamma) := \inf\{d_{\Omega}(z,\gamma(t)) : t\in[0,+\infty)\}. \tag{2.3}$$

2.2. The right half-plane

The conformal invariance of the hyperbolic distance allows us to transfer our arguments from the unit disc to any simply connected domain. In fact, using the Cayley transform $T\colon\mathbb{D}\to\mathbb{H}$, given by $T(z)=(1+z)/(1-z)$, we can pass to the right half-plane $\mathbb{H} = \{w\in\mathbb{C} : \operatorname{Re} w > 0\}$. Working in $\mathbb{H}$ instead of $\mathbb{D}$ simplifies many of the computations involving the hyperbolic distance, and we will do so for the rest of this article.

For any $z,w\in\mathbb{H}$, the hyperbolic distance in $\mathbb{H}$ is given by the formula

$$d_{\mathbb{H}}(z,w) = \frac{1}{2}\log\frac{1+\left|\frac{z-w}{z+\overline{w}}\right|}{1-\left|\frac{z-w}{z+\overline{w}}\right|}. \tag{2.4}$$

The quantity

$$\rho_{\mathbb{H}}(z, w) = \left|\frac{z - w}{z + \overline{w}}\right|, \quad \text{for } z, w \in \mathbb{H}, \tag{2.5}$$

is called the *pseudo-hyperbolic distance* in $\mathbb{H}$. The pseudo-hyperbolic distance is a metric on $\mathbb{H}$, with $\rho_{\mathbb{H}}(z, w) \in [0, 1)$ for all $z, w \in \mathbb{H}$. Simple calculations show that $\rho_{\mathbb{H}}$ satisfies the following handy formula:

$$1 - \rho_{\mathbb{H}}(z, w)^2 = \frac{4 \operatorname{Re} z \operatorname{Re} w}{|z + \overline{w}|^2}, \quad \text{for any } z, w \in \mathbb{H}. \tag{2.6}$$

The hyperbolic geodesics in $\mathbb{H}$ are the Euclidean straight half-lines and semicircles lying in $\mathbb{H}$ that are perpendicular to the imaginary axis.

Let us now describe how our main objects of study are modified when moving to the right half-plane. First, any holomorphic function $f : \mathbb{D} \to \mathbb{D}$ induces a holomorphic function $F : \mathbb{H} \to \mathbb{H}$, by defining $F = T \circ f \circ T^{-1}$, where T is the Cayley transform mentioned above. When f has Denjoy–Wolff point $\tau \in \partial\mathbb{D}$, we can pre-compose T with a rotation of $\mathbb{D}$ in order to obtain that the Denjoy–Wolff point of F is the point at infinity. That is, the sequence of iterates F^n of F converges uniformly on compact sets to the point at infinity, in the Euclidean metric. Moreover, F has an angular derivative at infinity, given by

$$F'(\infty) = \frac{1}{f'(\tau)} \in [1, +\infty).$$

Hence, the holomorphic map F is hyperbolic if $F'(\infty) > 1$, and parabolic otherwise. Furthermore, by the Julia–Carathéodory theorem (Proposition 2.3.1 in [1]), we have that

$$F'(\infty) = \angle \lim_{z \to \infty} \frac{F(z)}{z}, \tag{2.7}$$

where the angular notation in the limit simply means that z tends to ∞ through any sector $\{z \in \mathbb{C} : |\arg z| < \phi\}$, where $\phi \in (-\pi/2, \pi/2)$. For more information on the angular derivative, we refer to Chapter 4 of [20] and Section 2.3 of [1].

Suppose now that $\gamma : [0, +\infty) \to \mathbb{D}$ is a curve in $\mathbb{D}$, landing at a point $\zeta \in \partial\mathbb{D}$. Then, again using an appropriate version of T, we have that the induced curve $\Gamma : [0, +\infty) \to \mathbb{H}$ *lands at infinity*; i.e.,

$$\lim_{t \to +\infty} \Gamma(t) = \infty.$$

In this setting, we have that if

$$\lim_{t \to +\infty} \arg(1 - \overline{\tau}\gamma(t)) = \theta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right],$$

then

$$\lim_{t \to +\infty} \arg(\Gamma(t)) = \theta.$$

So, examining the angle with which curves land at the boundary is a far simpler endeavour in $\mathbb{H}$.

Since all the quantities considered in our main result are conformally invariant, we can restate Theorem 1.1 in the setting of right half-plane as follows.

Theorem 2.1. *Let $f: \mathbb{H} \to \mathbb{H}$ be a hyperbolic map with Denjoy–Wolff point ∞, and let $\gamma: [0, +\infty) \to \mathbb{H}$ be a curve landing at ∞, satisfying*

$$\lim_{t \to +\infty} \arg(\gamma(t)) = \theta \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

Fix $z \in \mathbb{D}$ and let $\pi_\gamma(f^n(z))$ be a projection of $f^n(z)$ onto γ, for $n \in \mathbb{N}$. Then, for any $w \in \mathbb{H}$, the sequence $\{d_{\mathbb{H}}(w, \pi_\gamma(f^n(z)))\}$ is eventually strictly increasing.

Similarly, the right half-plane version of Corollary 1.2 is:

Corollary 2.2. *Let $f: \mathbb{H} \to \mathbb{H}$ be a hyperbolic map with Denjoy–Wolff point ∞. For any $z, w \in \mathbb{H}$, the sequence $\{d_{\mathbb{H}}(w, f^n(z))\}$ is eventually strictly increasing.*

We end this subsection with an important property regarding the hyperbolic step and the slope of convergence of hyperbolic maps. For the proofs, we refer to Theorem 4.3.4, Lemma 4.6.2, and Corollary 4.6.9 in [1].

Theorem 2.3. *If $f: \mathbb{H} \to \mathbb{H}$ is a hyperbolic map, then for every $z \in \mathbb{H}$ there exist constants $\phi \in (-\pi/2, \pi/2)$ and $d \in (0, +\infty)$ so that*

1. $\lim_{n \to +\infty} d_{\mathbb{H}}(f^{n+1}(z), f^n(z)) = d$; *and*
2. $\lim_{n \to +\infty} \arg(f^n(z)) = \phi$.

2.3. Key lemmas

In this final preliminary subsection, we prove several results involving the hyperbolic geometry of $\mathbb{H}$ that are crucial to our analysis. The first is a restatement of Lemma 4.2 in [10]. A proof can be obtained easily from direct calculations on the formula (2.4).

Lemma 2.4 ([10]). *Let $\theta \in (-\pi/2, \pi/2)$.*

1. *For every $r_0 > 0$ and $\phi \in (-\pi/2, \pi/2)$ the function $r \mapsto d_{\mathbb{H}}(re^{i\phi}, r_0 e^{i\theta})$, $r \in (0, +\infty)$ has a minimum at $r = r_0$, is strictly decreasing for $r < r_0$, and strictly increasing for $r > r_0$.*
2. *For any $r > 0$ and $\phi \in (-\pi/2, \pi/2)$, we have $d_{\mathbb{H}}(re^{i\theta}, re^{i\phi}) = d_{\mathbb{H}}(e^{i\theta}, e^{i\phi})$. Also, the function $(-\pi/2, \pi/2) \ni \phi \mapsto d_{\mathbb{H}}(e^{i\theta}, e^{i\phi})$ is strictly decreasing in $(-\pi/2, \theta)$ and strictly increasing in $(\theta, \pi/2)$.*

Using Lemma 2.4, we can prove the following minor generalization of Lemma 4.4 in [10], whose proof is almost identical to that of Lemma 4.4 in [10], and is thus omitted.

Lemma 2.5 ([10]). *Let $\gamma: [0, +\infty) \to \mathbb{H}$ be the straight half-line with $\gamma([0, +\infty)) = \{re^{i\theta} : r \geq r_0\}$, for some $\theta \in (-\pi/2, \pi/2)$ and $r_0 > 0$. For every $R > 0$, there exist $\phi_1 \in (-\pi/2, \theta)$ and $\phi_2 \in (\theta, \pi/2)$ satisfying $d_{\mathbb{H}}(e^{i\theta}, e^{i\phi_1}) = d_{\mathbb{H}}(e^{i\theta}, e^{i\phi_2}) = R$, so that*

$$\{z \in \mathbb{H} : d_{\mathbb{H}}(z, \gamma) < R\} = D_{\mathbb{H}}(r_0 e^{i\theta}, R) \cup \{re^{i\phi} : r > r_0 \text{ and } \phi \in (\phi_1, \phi_2)\}.$$

Lemma 2.6. *Let $\{r_n\}, \{R_n\} \subset (0, +\infty)$ and $\{\theta_n\}, \{\phi_n\} \subset (-\pi/2, \pi/2)$ be sequences so that both $\{\theta_n\}$ and $\{\phi_n\}$ converge to some $\theta \in (-\pi/2, \pi/2)$. The limit*

$$\lim_{n \to +\infty} d_{\mathbb{H}}(r_n, R_n)$$

exists if and only if the limit

$$\lim_{n\to+\infty} d_{\mathbb{H}}(r_n e^{i\theta_n}, R_n e^{i\phi_n})$$

exists. Moreover, if any of the above limits is zero or $+\infty$*, so is the other.*

Proof. The triangle inequality for the hyperbolic metric yields

$$\begin{aligned}
&|d_{\mathbb{H}}(r_n e^{i\theta_n}, R_n e^{i\phi_n}) - d_{\mathbb{H}}(r_n e^{i\theta}, R_n e^{i\theta})| \\
&\le |d_{\mathbb{H}}(r_n e^{i\theta_n}, R_n e^{i\phi_n}) - d_{\mathbb{H}}(r_n e^{i\theta}, R_n e^{i\phi_n})| + |d_{\mathbb{H}}(r_n e^{i\theta}, R_n e^{i\phi_n}) - d_{\mathbb{H}}(r_n e^{i\theta}, R_n e^{i\theta})| \\
&\le d_{\mathbb{H}}(r_n e^{i\theta_n}, r_n e^{i\theta}) + d_{\mathbb{H}}(R_n e^{i\phi_n}, R_n e^{i\theta}) = d_{\mathbb{H}}(e^{i\theta_n}, e^{i\theta}) + d_{\mathbb{H}}(e^{i\phi_n}, e^{i\theta}),
\end{aligned}$$

where for the last equality we used Lemma 2.4(2). Since $\{\theta_n\}$ and $\{\phi_n\}$ both converge to θ, we obtain

$$\lim_{n\to+\infty} |d_{\mathbb{H}}(r_n e^{i\theta_n}, R_n e^{i\phi_n}) - d_{\mathbb{H}}(r_n e^{i\theta}, R_n e^{i\theta})| = 0,$$

and this means that it suffices to prove the lemma for the sequences $\{d_{\mathbb{H}}(r_n, R_n)\}$ and $\{d_{\mathbb{H}}(r_n e^{i\theta}, R_n e^{i\theta})\}$. Note that the Möbius transformations $z \mapsto z/r_n$ and $z \mapsto z/R_n$ are conformal automorphism of $\mathbb{H}$, for any $n \in \mathbb{N}$. Thus,

$$d_{\mathbb{H}}(r_n e^{i\theta}, R_n e^{i\theta}) = d_{\mathbb{H}}\Big(e^{i\theta}, \frac{R_n}{r_n} e^{i\theta}\Big) = d_{\mathbb{H}}\Big(\frac{r_n}{R_n} e^{i\theta}, e^{i\theta}\Big), \tag{2.8}$$

and

$$d_{\mathbb{H}}(r_n, R_n) = d_{\mathbb{H}}\Big(1, \frac{R_n}{r_n}\Big) = d_{\mathbb{H}}\Big(\frac{r_n}{R_n}, 1\Big). \tag{2.9}$$

It is now easy to see that $\{d_{\mathbb{H}}(r_n, R_n)\}$ converges to 0 if and only if $\{R_n/r_n\}$ converges to 1, which in turn is equivalent to $\{d_{\mathbb{H}}(r_n e^{i\theta}, R_n e^{i\theta})\}$ converging to 0. Similarly $\{d_{\mathbb{H}}(r_n, R_n)\}$ converges to $+\infty$ if and only if $\{R_n/r_n\}$ converges to 0 or $+\infty$, and that is if and only if $\{d_{\mathbb{H}}(r_n e^{i\theta}, R_n e^{i\theta})\}$ converges to $+\infty$.

Assume now that $\{d_{\mathbb{H}}(r_n, R_n)\}$ converges to some $d \in (0, +\infty)$. Then, by (2.9) the sequence $\{R_n/r_n\}$ can have at most two accumulation points, say $x, y \in (0,1) \cup (1, +\infty)$, that satisfy $xy = 1$. So (2.8) implies that $\{d_{\mathbb{H}}(r_n e^{i\theta}, R_n e^{i\theta})\}$ also has a finite and non-zero limit. A similar argument shows that if $\{d_{\mathbb{H}}(r_n e^{i\theta}, R_n e^{i\theta})\}$ has a finite, non-zero limit, then so does $\{d_{\mathbb{H}}(r_n, R_n)\}$. ∎

Lemma 2.7. *Let* $\{r_n\} \subset \mathbb{R}^+$ *be a sequence converging to* $+\infty$ *and let* $\theta \in (-\pi/2, \pi/2)$*. For any* $r_0 > 0$*, we have that*

$$\lim_{n\to+\infty} (d_{\mathbb{H}}(r_0 e^{i\theta}, r_n e^{i\theta}) - d_{\mathbb{H}}(r_0, r_n)) = -\log(\cos\theta) \in [0, +\infty).$$

Proof. Using the formulas (2.4) and (2.5), for $d_{\mathbb{H}}$ and $\rho_{\mathbb{H}}$, we obtain that for all $n \in \mathbb{N}$,

$$d_{\mathbb{H}}(r_0 e^{i\theta}, r_n e^{i\theta}) - d_{\mathbb{H}}(r_0, r_n) = \log \frac{1 + \rho_{\mathbb{H}}(r_0 e^{i\theta}, r_n e^{i\theta})}{1 + \rho_{\mathbb{H}}(r_0, r_n)} + \frac{1}{2} \log \frac{1 - \rho_{\mathbb{H}}(r_0, r_n)^2}{1 - \rho_{\mathbb{H}}(r_0 e^{i\theta}, r_n e^{i\theta})^2}.$$

Since r_n converges to $+\infty$ and r_0 is fixed, we have that the pseudo-hyperbolic distances $\rho_{\mathbb{H}}(r_0, r_n)$ and $\rho_{\mathbb{H}}(r_0e^{i\theta}, r_ne^{i\theta})$ both converge to 1. Therefore,

$$\lim_{n\to+\infty} \log \frac{1+\rho_{\mathbb{H}}(r_0e^{i\theta}, r_ne^{i\theta})}{1+\rho_{\mathbb{H}}(r_0, r_n)} = 0.$$

So it suffices to show that

$$\lim_{n\to+\infty} \frac{1-\rho_{\mathbb{H}}(r_0, r_n)^2}{1-\rho_{\mathbb{H}}(r_0e^{i\theta}, r_ne^{i\theta})^2} = \frac{1}{\cos^2\theta}.$$

This easily follows from the next string of equations, obtained by using the pseudo-hyperbolic formula (2.6):

$$\begin{aligned}\frac{1-\rho_{\mathbb{H}}(r_0, r_n)^2}{1-\rho_{\mathbb{H}}(r_0e^{i\theta}, r_ne^{i\theta})^2} &= \frac{|r_0e^{i\theta}+r_ne^{-i\theta}|^2}{\cos^2\theta\,(r_0+r_n)^2} = \frac{r_0^2+2r_0r_n\cos 2\theta+r_n^2}{\cos^2\theta\,(r_0+r_n)^2}\\ &= \frac{(r_0/r_n)^2+2r_0/r_n\cos(2\theta)+1}{(r_0/r_n+1)^2\cos^2\theta}.\end{aligned}$$

■

We end the section with a handy lemma involving hyperbolic projections, which can be found in Proposition 3.4 of [8].

Lemma 2.8. *Let $\gamma\colon[0,+\infty)\to\mathbb{H}$ be a hyperbolic geodesic, $w\in\gamma([0,+\infty))$ and $z\in\mathbb{H}$. Then*

$$d_{\mathbb{H}}(w,\pi_\gamma(z))+d_{\mathbb{H}}(z,\pi_\gamma(z))-\frac{1}{2}\log 2 \le d_{\mathbb{H}}(z,w) \le d_{\mathbb{H}}(w,\pi_\gamma(z))+d_{\mathbb{H}}(z,\pi_\gamma(z)),$$

where $\pi_\gamma(z)$ denotes the unique hyperbolic projection of z onto the geodesic γ.

3. Examples

Here we present the examples which show that Theorem 1.1 is best possible, in the sense described in the introduction. All our examples will be constructed in the right half-plane, and so we refer to Theorem 2.1 instead, which is the equivalent formulation of Theorem 1.1 in $\mathbb{H}$.

Example 3.1. We first show that the strong monotonicity of Theorem 2.1 fails for continuous semigroups of holomorphic self-maps of $\mathbb{H}$. Let us first properly define such a semigroup and discuss some of its basic properties. For a complete reference on continuous semigroups of holomorphic functions, see Chapter 8 of [9].

A *semigroup* (ϕ_t) *in* $\mathbb{H}$ is a family of holomorphic functions $\phi_t\colon\mathbb{H}\to\mathbb{H}$, for $t\ge 0$, satisfying the following properties:

(i) $\phi_0=\mathrm{id}_{\mathbb{H}}$;

(ii) $\phi_{t+s}=\phi_t\circ\phi_s$, for all $t,s\ge 0$; and

(iii) $\lim_{t\to 0}\phi_t(z)=z$, for all $z\in\mathbb{H}$.

Notice that conditions (ii) and (iii) imply that ϕ_t depends continuously on the real parameter t, in the topology of uniform convergence on compact sets; see Theorem 8.1.15 in [9].

The continuous version of the Denjoy–Wolff theorem (Theorem 8.3.1 in [9]) states that if the function ϕ_{t_0}, for some $t_0 > 0$, does not have any fixed points in $\mathbb{H}$, then there exists a point $\tau \in \partial\mathbb{H}$ such that ϕ_t converges to the constant τ, as $t \to +\infty$, uniformly on compact subsets of $\mathbb{H}$. Here, by $\partial\mathbb{H}$ we mean the imaginary axis along with the point infinity. The point τ is called the Denjoy–Wolff point of the semigroup (ϕ_t) and, up to conjugation with a conformal automorphism of $\mathbb{H}$, we can always assume that $\tau = \infty$. The angular derivative

$$\phi_t'(\infty) = \angle \lim_{z\to\infty} \frac{\phi_t(z)}{z}, \tag{3.1}$$

exists for all t and satisfies $\phi_t'(\infty) \in [1, +\infty)$. Just like for self-maps of $\mathbb{H}$, the semigroup (ϕ_t) is called *hyperbolic* if $\phi_t'(\infty) > 1$, for some (and hence any) $t > 0$.

Returning to our example, consider the functions $\phi_t(z) = e^t z$, $z \in \mathbb{H}$, for $t \geq 0$. The family (ϕ_t) is clearly a hyperbolic semigroup in $\mathbb{H}$, with Denjoy–Wolff point ∞.

Let $\delta\colon [0, +\infty) \to \mathbb{H}$ be a curve with trace $\delta([0, +\infty)) = H \cup (\bigcup_{n\in\mathbb{N}} C_n)$, where H is the horizontal half-line $\{t + i : t \geq 0\}$ and C_n is the hyperbolic geodesic segment of $\mathbb{H}$ joining e^n with $\sqrt{e^{2n} - 1} + i \in H$; see Figure 1.

We can parametrise δ so that $\lim_{t\to+\infty} \delta(t) = \infty$. Then, since the trace of the curve is bounded between two horizontal half-lines, we can easily see that $\lim_{t\to+\infty} \arg(\delta(t)) = 0$. As a result, all assumptions of Theorem 2.1 hold, but with a hyperbolic semigroup instead of a hyperbolic function.

However, because the points e^n all lie in $\delta([0, +\infty))$, by construction, we have that $\pi_\delta(e^n) = e^n$. Moreover, the real axis intersects each C_n orthogonally, meaning that the projection of any point $x > 0$ onto the geodesic C_n is e^n, for all $n \in \mathbb{N}$. So, for each $n \in \mathbb{N}$, there exists some $\varepsilon_n > 0$ small enough, so that $\pi_\delta(e^t) = e^n$, for all $t \in (n - \varepsilon_n, n + \varepsilon_n)$. In other words, the distance $d_{\mathbb{H}}(1, \pi_\delta(\phi_t(1))) = d_{\mathbb{H}}(1, \pi_\delta(e^t))$ remains constant on each interval $(n - \varepsilon_n, n + \varepsilon_n)$, $n \in \mathbb{N}$, which shows that (ϕ_t) does not satisfy the monotonicity result described in our main result.

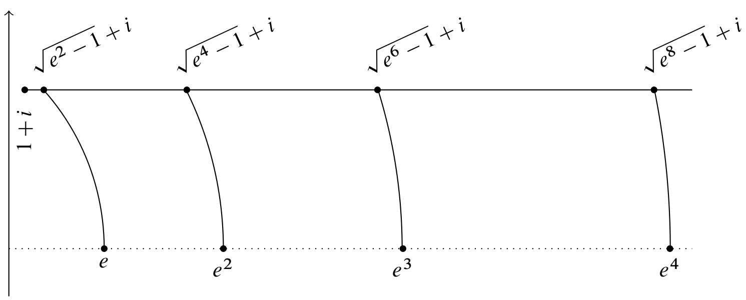

Figure 1. Example for a hyperbolic semigroup.

Example 3.2. In this second example, we are going to verify that Theorem 2.1 fails in general for parabolic self-maps of the right half-plane. We consider two examples, one for each main type of parabolic maps. It is known (see, for instance, Section 4.6 of [1]) that for a parabolic map $f\colon \mathbb{H} \to \mathbb{H}$, we have that the limit $\lim_{n\to+\infty} d_{\mathbb{H}}(f^n(1), f^{n+1}(1))$ exists and is either zero or positive. The former type of parabolic map is called *parabolic of zero hyperbolic step*, while the latter *parabolic of positive hyperbolic step*.

First, consider the holomorphic function $f\colon \mathbb{H} \to \mathbb{H}$, with $f(z) = z + 1$. It is clear that $f'(\infty) = 1$, and thus f is parabolic. Also,

$$d_{\mathbb{H}}(f^{n+1}(1), f^n(1)) = \frac{1}{2}\log\frac{n+2}{n+1},$$

due to (2.4), meaning that f is parabolic of zero hyperbolic step.

Let $\delta\colon [0, +\infty) \to \mathbb{H}$ be a curve with trace $\delta([0, +\infty)) = H \cup (\bigcup_{n\in\mathbb{N}} C_n)$, where H is the horizontal half-line $\{t + i : t \geq 1\}$ and C_n is the hyperbolic geodesic segment of $\mathbb{H}$ joining $3n$ with $\sqrt{(3n)^2 - 1} + i$; see Figure 2. Similarly to the previous example, we may write $\lim_{t\to+\infty} \delta(t) = \infty$ and $\lim_{t\to+\infty} \arg(\delta(t)) = 0$. Thus, δ satisfies the assumptions of Theorem 2.1.

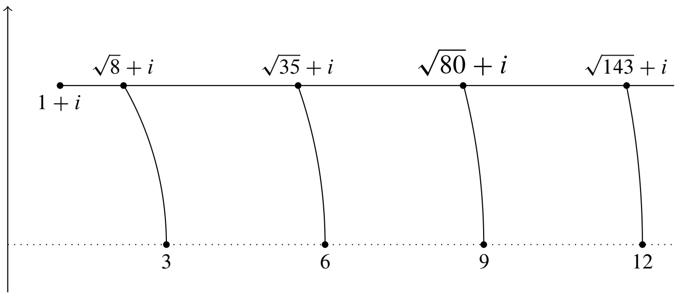

Figure 2. Example for a parabolic map with zero hyperbolic step.

Now, consider the points $f^{3n-2}(1) = 3n - 1$, for $n \in \mathbb{N}$, which form a subsequence of the orbit $\{f^n(1)\}$. By construction, the projection of $3n - 1$ onto δ is either its projection onto the half-line H, or its projection onto one of the circular arcs C_n. Note that H and all the C_n are hyperbolic geodesic segments of $\mathbb{H}$, and thus the aforementioned projections are unique. Given that the real axis is orthogonal to every C_k, we can see that the projection of $3n - 1$ onto each C_k is the point $3k$. Therefore, the point of the set $\bigcup_{k\in\mathbb{N}} C_k$ that is closest, in hyperbolic terms, to $3n - 1$ is either $3n$ or $3n - 3$. Using (2.4), we easily check that

$$d_{\mathbb{H}}(3n - 3, 3n - 1) > d_{\mathbb{H}}(3n - 1, 3n).$$

So, the point of the set $\bigcup_{k\in\mathbb{N}} C_k$ closest to $3n - 1$ is indeed $3n$.

We claim that $\pi_\delta(3n - 1) = 3n$. Note that for any $x > 0$, the Euclidean circle centred at i and of radius $|x - i|$ is a hyperbolic geodesic of $\mathbb{H}$ perpendicular to the half-line H. So, the projection of x onto H is the point of intersection of the aforementioned circle and H. Simple calculations show that this point is $\sqrt{x^2 + 1} + i$. Consequently, the projection of $3n - 1$ onto the half-line H is $\sqrt{(3n - 1)^2 + 1} + i$. Therefore, our goal is to show that

$$d_{\mathbb{H}}\big(\sqrt{(3n-1)^2+1}+i, 3n-1\big) > d_{\mathbb{H}}(3n, 3n-1). \tag{3.2}$$

Due to the connection between the hyperbolic distance $d_{\mathbb{H}}$ and the pseudo-hyperbolic distance $\rho_{\mathbb{H}}$ evident in the formula (2.4), it suffices to show that

$$\rho_{\mathbb{H}}\big(\sqrt{(3n-1)^2+1}+i, 3n-1\big) > \rho_{\mathbb{H}}(3n, 3n-1). \tag{3.3}$$

Now, simple computations using the formula (2.5) of $\rho_{\mathbb{H}}$ yield

$$\begin{aligned}\rho_{\mathbb{H}}\big(3n-1, \sqrt{9n^2+1}+i\big) &= \frac{1}{\sqrt{(3n-1)^2+1}+3n-1}\\ &> \frac{1}{2(3n-1)+1} = \rho_{\mathbb{H}}(3n-1, 3n),\end{aligned}$$

which is exactly the desired inequality (3.3).

So far we have shown that $\pi_\delta(f^{3n-2}(1)) = 3n$, for all $n \in \mathbb{N}$. Moreover, since, by construction, we have that $3n \in \delta([0,+\infty))$, for all $n \in \mathbb{N}$, we immediately get that

$$\pi_\delta(f^{3n-1}(1)) = \pi_\delta(3n) = 3n.$$

To sum up, we have that

$$\pi_\delta(f^{3n-2}(1)) = \pi_\delta(f^{3n-1}(1)), \quad \text{for all } n \in \mathbb{N},$$

and thus the sequence $\{d_{\mathbb{H}}(1, \pi_\delta(f^k(1)))\}$ is not eventually strictly increasing.

The example for parabolic maps of positive hyperbolic step is similar, so we provide a rough sketch of the construction. Let $g\colon \mathbb{H} \to \mathbb{H}$ be the map with $g(z) = z + i$. It is easy to see that g is a parabolic automorphism of $\mathbb{H}$, and thus has positive hyperbolic step (see Lemma 4.6.3 in [1]). Consider the curve $\gamma\colon [0,+\infty) \to \mathbb{H}$, with trace $\gamma([0,+\infty)) = H' \cup (\bigcup_{n\in\mathbb{N}} C'_n)$, where H' is the horizontal half-line $\{t - i : t \geq 3\}$ and C'_n is the hyperbolic geodesic joining $\sqrt{1+(2n+1)^2}$ and $(2n+1) - i$, as in Figure 3.

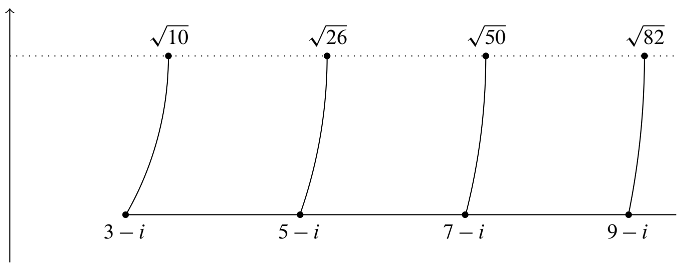

Figure 3. Example for a parabolic map with positive hyperbolic step.

Note that $|g^n(1)| = |1 + ni| = \sqrt{1+n^2}$, and so following arguments similar to those presented in the example for the zero hyperbolic step we have that the projection of $g^{2n+1}(1)$ onto γ is the point $\sqrt{1+(2n+1)^2}$.

We now need to evaluate the projection of $g^{2n+2}(1)$ onto γ. Firstly, the projection of $g^{2n+2}(1)$ onto the half-line H' is $\sqrt{1+(2n+2)^2}-i$. Also, observe that the segment C_n' lies in the geodesic

$$L_n = \left\{z \in \mathbb{H} : |z| = \sqrt{1+(2n+1)^2}\right\}.$$

Simple Euclidean arguments show that the hyperbolic projection of the point $g^{2n+2}(1) = 1+(2n+2)i$ onto L_n lies in the upper half-plane. This means that the point of C_n' closest (in hyperbolic terms) to $g^{2n+2}(1)$ is $\sqrt{1+(2n+1)^2}$. Similarly the point of C_{n+1}' closest to $g^{2n+2}(1)$ is $\sqrt{1+(2n+3)^2}$. We therefore have that the projection of $g^{2n+2}(1)$ onto γ is one of the points $\sqrt{1+(2n+2)^2}-i$, $\sqrt{1+(2n+1)^2}$ or $\sqrt{1+(2n+3)^2}$. By performing long, but elementary computations using the pseudo-hyperbolic formula (2.6), one can verify that

$$\rho_{\mathbb{H}}\left(g^{2n+2}(1), \sqrt{1+(2n+1)^2}\right) < \rho_{\mathbb{H}}\left(g^{2n+2}(1), \sqrt{1+(2n+2)^2}-i\right),$$

and

$$\rho_{\mathbb{H}}\left(g^{2n+2}(1), \sqrt{1+(2n+1)^2}\right) < \rho_{\mathbb{H}}\left(g^{2n+2}(1), \sqrt{1+(2n+3)^2}\right).$$

We conclude that

$$\pi_\gamma(g^{2n+2}(1)) = \sqrt{1+(2n+1)^2} = \pi_\gamma(g^{2n+1}(1)), \quad \text{for all } n \in \mathbb{N},$$

implying that the sequence $\{d_{\mathbb{H}}(1, \pi_\gamma(g^n(1)))\}$ is not eventually strictly increasing.

Example 3.3. Here, we present an example demonstrating that the "eventually" part of the monotonicity in Theorem 2.1 cannot be omitted. Let $f\colon \mathbb{H} \to \mathbb{H}$ be the hyperbolic function with $f(z) = 2z$.

Consider the curve $\delta\colon [0, +\infty) \to \mathbb{H}$, with trace

$$\delta([0,+\infty)) = \partial D_{\mathbb{H}}\left(2, \log \sqrt[4]{2}\right) \cup \partial D_{\mathbb{H}}\left(4, \log \sqrt[4]{2}\right) \cup \left\{t \in \mathbb{R} : t \geq 4\sqrt{2}\right\},$$

where $\partial D_{\mathbb{H}}(z, r)$ is the hyperbolic circle centred at $z \in \mathbb{H}$ and of radius $r > 0$ (see Figure 4).

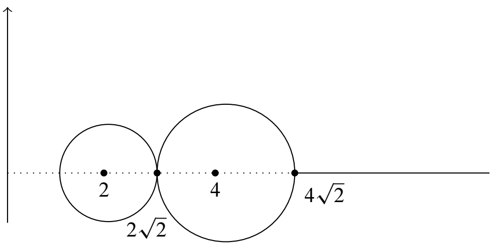

Figure 4. Example for a hyperbolic map.

The curve can be parametrised so that $\lim_{t\to+\infty} \delta(t) = \infty$, and so by construction there exists some $t_0 > 0$ so that $\arg(\delta(t)) = 0$, for all $t \geq t_0$. Therefore, δ satisfies the assumptions of Theorem 2.1.

Using the formula (2.4) for $d_{\mathbb{H}}$, we can see that the circles $\partial D_{\mathbb{H}}(2, \log \sqrt[4]{2})$ and $\partial D_{\mathbb{H}}(4, \log \sqrt[4]{2})$ intersect at the point $2\sqrt{2}$. Moreover, we trivially have that all the points of $\partial D_{\mathbb{H}}(2, \log \sqrt[4]{2})$ are equidistant from 2 and so the projection of $f(1) = 2$ onto δ can be chosen to be $2\sqrt{2}$. Similarly, the projection of $f^2(1) = 4$ can be, again, chosen to be $2\sqrt{2}$, meaning that $\pi_\delta(f(1)) = \pi_\delta(f^2(2))$. So, the sequence $d_{\mathbb{H}}(1, \pi_\delta(f^n(1))$ is strictly increasing only for $n \geq 3$.

Example 3.4. Our next example shows that our main theorem fails if we assume that the curve $\gamma\colon [0, +\infty) \to \mathbb{H}$ satisfies

$$\lim_{t\to+\infty} \arg(\gamma(t)) = \pm\frac{\pi}{2}\cdot$$

For this, we let γ be such that $\gamma(t) = 1 + it$, for all $t \geq 0$. It is easy to see that $\arg(\gamma(t))$ converges to $\pi/2$. Now consider the hyperbolic map $f\colon \mathbb{H} \to \mathbb{H}$, with $f(z) = 2z$, we used in Example 3.3. Then, simple computations show that the projection of $f^n(1) = 2^n$ onto γ is the point 1, for all $n \in \mathbb{N}$, and thus the sequence $\pi_\gamma(f^n(z))$ is constant.

For the case where $\arg(\gamma(t))$ tends to $-\pi/2$, one can use the curve $\gamma(t) = 1 - it$.

Example 3.5. Finally, we describe how the example constructed in [7] can be used to show that Corollary 1.2 may fail completely for parabolic functions.

In Section 4 of [7], the authors construct a simply connected domain $\Omega \subsetneq \mathbb{C}$ with the following properties:

(1) $z + t \in \Omega$, for all $t \geq 0$ and any $z \in \Omega$;

(2) $\mathbb{R} \subset \Omega$;

(3) the family of holomorphic functions $\phi_t\colon \Omega \to \Omega$, with $\phi_t(z) = z + t$ is a parabolic semigroup in Ω.

Let us remark that even though the definition of a semigroup presented in Example 3.1 was stated only for the right half-plane, the definition can easily be extended to any simply connected domain via a Riemann map. Similarly, all the notation introduced for self-maps of $\mathbb{H}$ (or $\mathbb{D}$) can be easily transferred to Ω. With this in mind, we will construct our example in Ω.

A key point in their construction is that there exist real sequences $\{x_n\}, \{y_n\} \subset \mathbb{R}$, tending to $+\infty$ such that $x_n < y_n < x_{n+1}$, for any $n \in \mathbb{N}$, and

$$d_\Omega(0, x_n) > d_\Omega(0, y_n). \tag{3.4}$$

In fact, x_n and y_n can be chosen to be positive integers (it is evident from Lemma 4.1 in [7] that one may chose $x_n = 3^n - 2^n$ and $y_n = 3^n + 2^n$).

Now consider the function $\phi_1\colon \Omega \to \Omega$ from the semigroup (ϕ_t); that is, $\phi_1(z) = z + 1$. Since the semigroup (ϕ_t) is parabolic, every function ϕ_t, for $t > 0$, is also parabolic as a self-map of Ω (see the angular derivative condition (3.1) in Example 3.1). Note that $\phi_1^{x_n}(0) = x_n$ and $\phi_1^{y_n}(0) = y_n$. Thus, (3.4) can be rewritten as

$$d_\Omega(0, \phi_1^{x_n}(0)) > d_\Omega(0, \phi_1^{y_n}(0)),$$

meaning that the sequence $\{d_\Omega(0, \phi_1^n(0))\}$ is not eventually increasing.

4. Projections of sequences onto curves

This section contains the bulk of our technical results about projections of arbitrary sequences onto curves. As usual, our proofs will be carried out in the right half-plane.

We start our analysis by proving that in the setting of Theorem 2.1 (the half-plane version of Theorem 1.1), the projections tend to the point at infinity.

Lemma 4.1. *Let $\gamma\colon[0,+\infty)\to\mathbb{H}$ be a curve landing at infinity with*

$$|\arg(\gamma(t))|<\theta,\quad \textit{for all } t\geq 0, \textit{ for some } \theta\in\left(-\tfrac{\pi}{2},\tfrac{\pi}{2}\right).$$

If $\{z_n\}\subset\mathbb{H}$ is a sequence converging to infinity and $\pi_\gamma(z_n)$ is a projection of z_n onto γ, for $n\in\mathbb{N}$, then the sequence $\{d_{\mathbb{H}}(\pi_\gamma(z_n),|z_n|)\}$ is bounded. In particular,

$$\lim_{n\to+\infty}\pi_\gamma(z_n)=\infty.$$

Proof. For simplicity, we write $\pi_n:=\pi_\gamma(z_n)$, $n\in\mathbb{N}$. If $V_\theta:=\{w\in\mathbb{H}:|\arg w|\leq\theta\}$, then by hypothesis we have that $\gamma([0,+\infty))\subset V_\theta$.

First, assume that $\{z_n\}\subset V_\theta$. Possibly disregarding finitely many terms, for each $n\in\mathbb{N}$ we can find $t_n\geq 0$ such that $|\gamma(t_n)|=|z_n|$. By the definition of hyperbolic projections and the triangle inequality, we get

$$d_{\mathbb{H}}(z_n,\pi_n)\leq d_{\mathbb{H}}(z_n,\gamma(t_n))\leq d_{\mathbb{H}}(z_n,|z_n|)+d_{\mathbb{H}}(|\gamma(t_n)|,\gamma(t_n)). \tag{4.1}$$

However, for any point $p\in V_\theta$, we have

$$d_{\mathbb{H}}(p,|p|)=d_{\mathbb{H}}(e^{i\arg p},1)\leq d_{\mathbb{H}}(e^{i\theta},1), \tag{4.2}$$

where we have used the fact that $z\mapsto z/|p|$ is a conformal automorphism of the right half-plane and Lemma 2.4(2). Using (4.1) and (4.2), we obtain that

$$d_{\mathbb{H}}(\pi_n,|z_n|)\leq d_{\mathbb{H}}(\pi_n,z_n)+d_{\mathbb{H}}(z_n,|z_n|)\leq 3\,d_{\mathbb{H}}(e^{i\theta},1),$$

as desired.

Next, assume that $\arg z_n\in(\theta,\pi/2)$, for all $n\in\mathbb{N}$. For each $n\in\mathbb{N}$, we write $z_n=|z_n|e^{i\phi_n}$ and, as before, we may find $t_n\geq 0$ with $|\gamma(t_n)|=|z_n|$. Under this notation, we have

$$\begin{aligned} d_{\mathbb{H}}(z_n,\pi_n)\leq d_{\mathbb{H}}(z_n,\gamma(t_n))&\leq d_{\mathbb{H}}(z_n,|z_n|e^{i\theta})+d_{\mathbb{H}}(|z_n|e^{i\theta},\gamma(t_n))\\ &\leq d_{\mathbb{H}}(e^{i\phi_n},e^{i\theta})+d_{\mathbb{H}}(e^{i\theta},1), \end{aligned} \tag{4.3}$$

where the last inequality again follows from Lemma 2.4(2). By the triangle inequality, (4.2), and Lemma 2.8 applied on the hyperbolic geodesic of $\mathbb{H}$ with trace $(0,+\infty)$, we obtain

$$\begin{aligned} d_{\mathbb{H}}(\pi_n,z_n)&\geq d_{\mathbb{H}}(|\pi_n|,z_n)-d_{\mathbb{H}}(\pi_n,|\pi_n|)\geq d_{\mathbb{H}}(|\pi_n|,z_n)-d_{\mathbb{H}}(e^{i\theta},1)\\ &\geq d_{\mathbb{H}}(|\pi_n|,|z_n|)+d_{\mathbb{H}}(|z_n|,z_n)-d_{\mathbb{H}}(e^{i\theta},1)-\frac{1}{2}\log 2\\ &\geq d_{\mathbb{H}}(|\pi_n|,|z_n|)+d_{\mathbb{H}}(|z_n|e^{i\theta},z_n)-d_{\mathbb{H}}(|z_n|,|z_n|e^{i\theta})-d_{\mathbb{H}}(e^{i\theta},1)-\frac{1}{2}\log 2\\ &= d_{\mathbb{H}}(|\pi_n|,|z_n|)+d_{\mathbb{H}}(e^{i\theta},e^{i\phi_n})-2d_{\mathbb{H}}(e^{i\theta},1)-\frac{1}{2}\log 2. \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4) leads to

$$d_{\mathbb{H}}(|\pi_n|, |z_n|) \leq 3 d_{\mathbb{H}}(e^{i\theta}, 1) + \frac{1}{2}\log 2,$$

and thus $\{d_{\mathbb{H}}(|\pi_n|, |z_n|)\}$ is bounded. Given that $\{d_{\mathbb{H}}(\pi_n, |\pi_n|)\}$ is also bounded due to (4.2), we can see that the sequence $\{d_{\mathbb{H}}(\pi_n, |z_n|)\}$ is bounded by some constant $C > 0$ depending only on θ.

For the case where $\arg z_n \in (-\pi/2, -\theta)$, for all $n \in \mathbb{N}$, similar arguments show that the sequence $\{d_{\mathbb{H}}(\pi_n, |z_n|)\}$ is bounded by the same constant $C > 0$.

Finally, if $\{z_n\}$ does not satisfy any of the aforementioned conditions, by separating it into three disjoint subsequences (one for each case investigated above), we get that $d_{\mathbb{H}}(\pi_n, |z_n|) \leq \max\{3 d_{\mathbb{H}}(e^{i\theta}, 1), C\}$, for all $n \in \mathbb{N}$.

To conclude the proof, we note that since $\{|z_n|\}$ converges to infinity, the boundedness of $\{d_{\mathbb{H}}(\pi_n, |z_n|)\}$ implies that the same is true for $\{\pi_n\}$; see Lemma 1.8.6 in [9]. ■

The next theorem is the main result of this section and shows that projections of a sequence onto different curves landing with the same angle are asymptotically close.

Theorem 4.2. *Let $\gamma_j\colon [0, +\infty) \to \mathbb{H}$, $j = 1, 2$, be two curves landing at ∞, with*

$$\lim_{t\to+\infty} \arg(\gamma_j(t)) = \theta,$$

for some $\theta \in (-\pi/2, \pi/2)$ and $j = 1, 2$. Let $\{z_n\} \subset \mathbb{H}$ be a sequence converging to ∞. If $\pi_{\gamma_j}(z_n)$ is a projection of z_n on γ_j, for $j = 1, 2$ and $n \in \mathbb{N}$, then

$$\lim_{n\to+\infty} d_{\mathbb{H}}(\pi_{\gamma_1}(z_n), \pi_{\gamma_2}(z_n)) = 0.$$

Proof. In order to simplify notation, we write $\pi_n^j := \pi_{\gamma_j}(z_n)$ for $j = 1, 2$ and all $n \in \mathbb{N}$. Note that Lemma 4.1 is applicable and yields that both sequences $\{\pi_n^1\}$ and $\{\pi_n^2\}$ converge to ∞, as $n \to +\infty$.

Consider the straight half-line $\gamma\colon [1, +\infty) \to \mathbb{H}$, with $\gamma(t) = te^{i\theta}$, and recall from Lemma 2.5 that for all $R > 0$, we have that

$$S_R := \{z \in \mathbb{H} : d_{\mathbb{H}}(z, \gamma) < R\} = D_{\mathbb{H}}(e^{i\theta}, R) \cup \{re^{i\phi} : r > 1,\ \phi \in (\phi_1, \phi_2)\},$$

for some $\phi_1 \in (-\pi/2, \theta)$ and $\phi_2 \in (\theta, \pi/2)$ satisfying $d_{\mathbb{H}}(e^{i\theta}, e^{i\phi_1}) = d_{\mathbb{H}}(e^{i\theta}, e^{i\phi_2}) = R$. So, since γ_1 lands at ∞ with $\lim_{t\to+\infty} \arg(\gamma_1(t)) = \theta$, we get that for every $R > 0$, there exists a $t_0 \geq 0$ so that $\gamma_1([t_0, +\infty)) \subset S_R$. So, for any sequence $\{\zeta_n\} \subset \gamma_1([0, +\infty))$, converging to ∞, we have that

$$\lim_{n\to+\infty} d_{\mathbb{H}}(\zeta_n, \gamma) = 0. \tag{4.5}$$

Observe that due to Lemma 2.4(1), the unique projections of z_n and π_n^1 onto γ are $|z_n|e^{i\theta}$ and $|\pi_n^1|e^{i\theta}$, respectively, for sufficiently large $n \in \mathbb{N}$ (see Figure 5). Since $\{\pi_n^1\} \subset \gamma_1([0, +\infty))$ and $\pi_n^1 \to \infty$ as $n \to +\infty$, (4.5) yields

$$\lim_{n\to+\infty} d_{\mathbb{H}}(\pi_n^1, |\pi_n^1|e^{i\theta}) = 0. \tag{4.6}$$

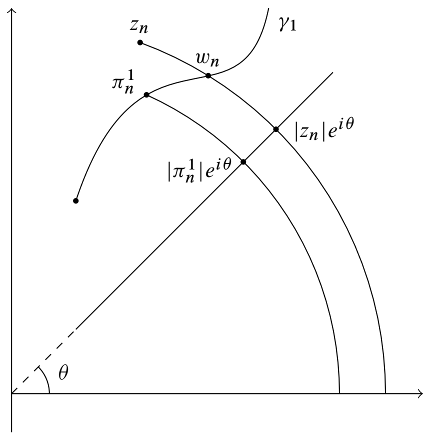

Figure 5. The projections in Theorem 4.2.

Now, since, for all $n \in \mathbb{N}$ large, $|z_n|e^{i\theta}$ is the projection of z_n onto γ we get that

$$d_{\mathbb{H}}(z_n, |z_n|e^{i\theta}) - d_{\mathbb{H}}(z_n, |\pi_n^1|e^{i\theta}) \leq 0. \tag{4.7}$$

Let w_n be a point of intersection of the geodesic $\{z \in \mathbb{H} : |z| = |z_n|\}$ with the trace $\gamma_1([0, +\infty))$, which is well defined for all $n \in \mathbb{N}$ large. Note that w_n tends to ∞, as $n \to +\infty$, and the projection of w_n onto γ is $|z_n|e^{i\theta}$. So, applying (4.5) once more yields

$$\lim_{n\to+\infty} d_{\mathbb{H}}(w_n, |z_n|e^{i\theta}) = 0. \tag{4.8}$$

Using the triangle inequality, we have

$$\begin{aligned} d_{\mathbb{H}}(z_n, |z_n|e^{i\theta}) &\geq d_{\mathbb{H}}(z_n, w_n) - d_{\mathbb{H}}(w_n, |z_n|e^{i\theta}) \geq d_{\mathbb{H}}(z_n, \pi_n^1) - d_{\mathbb{H}}(w_n, |z_n|e^{i\theta}) \\ &\geq d_{\mathbb{H}}(z_n, |\pi_n^1|e^{i\theta}) - d_{\mathbb{H}}(|\pi_n^1|e^{i\theta}, \pi_n^1) - d_{\mathbb{H}}(w_n, |z_n|e^{i\theta}), \end{aligned}$$

which can be rewritten as

$$d_{\mathbb{H}}(z_n, |z_n|e^{i\theta}) - d_{\mathbb{H}}(z_n, |\pi_n^1|e^{i\theta}) \geq -d_{\mathbb{H}}(|\pi_n^1|e^{i\theta}, \pi_n^1) - d_{\mathbb{H}}(w_n, |z_n|e^{i\theta}). \tag{4.9}$$

Combining (4.6), (4.7), (4.8) and (4.9) yields

$$\lim_{n\to+\infty} \left(d_{\mathbb{H}}(z_n, |z_n|e^{i\theta}) - d_{\mathbb{H}}(z_n, |\pi_n^1|e^{i\theta})\right) = 0. \tag{4.10}$$

Our goal now is to show that

$$\lim_{n\to+\infty} d_{\mathbb{H}}(|\pi_n^1|e^{i\theta}, |z_n|e^{i\theta}) = 0. \tag{4.11}$$

Since $z \mapsto z/|z_n|$ is a conformal automorphism of $\mathbb{H}$, for any $n \in \mathbb{N}$, (4.11) is equivalent to

$$\lim_{n\to+\infty} d_{\mathbb{H}}\left(\left|\frac{\pi_n^1}{z_n}\right| e^{i\theta}, e^{i\theta}\right) = 0.$$

Therefore, in order to prove (4.11) it suffices to prove

$$\lim_{n\to+\infty}\left|\frac{\pi_n^1}{z_n}\right| = 1. \tag{4.12}$$

Note that Lemma 4.1 is applicable to the curve γ_1 and the sequence $\{z_n\}$, and yields that the sequence $\{d_{\mathbb{H}}(\pi_n^1, |z_n|)\}$ is bounded. Therefore, using (4.2) we have

$$d_{\mathbb{H}}\left(\left|\frac{\pi_n^1}{z_n}\right|, 1\right) = d_{\mathbb{H}}(|\pi_n^1|, |z_n|) \le d_{\mathbb{H}}(\pi_n^1, |\pi_n^1|) + d_{\mathbb{H}}(\pi_n^1, |z_n|) < C,$$

for all $n \in \mathbb{N}$ and some constant $C > 0$. As a result, $\{|\pi_n^1/z_n|\}$ is relatively bounded in $\mathbb{H}$, meaning that there exists $c \in (1, +\infty)$ such that

$$\frac{1}{c} \le \left|\frac{\pi_n^1}{z_n}\right| \le c, \quad \text{for all } n \in \mathbb{N}.$$

For the sake of simplicity, write

$$v_n := \left|\frac{\pi_n^1}{z_n}\right|$$

and let $v \in [1/c, c]$ be an accumulation point of $\{v_n\}$. Passing to a subsequence, if necessary, we assume that $\lim_{n\to+\infty} v_n = v$. Note that

$$\left|d_{\mathbb{H}}\left(\frac{z_n}{|z_n|}, v_n e^{i\theta}\right) - d_{\mathbb{H}}\left(\frac{z_n}{|z_n|}, v e^{i\theta}\right)\right| \le d_{\mathbb{H}}(v_n e^{i\theta}, v e^{i\theta})$$

and thus

$$\lim_{n\to+\infty}\left[d_{\mathbb{H}}\left(\frac{z_n}{|z_n|}, v_n e^{i\theta}\right) - d_{\mathbb{H}}\left(\frac{z_n}{|z_n|}, v e^{i\theta}\right)\right] = 0. \tag{4.13}$$

Using the map $z \mapsto z/|z_n|$ on (4.10) and combining with (4.13) yields

$$\lim_{n\to+\infty}\left[d_{\mathbb{H}}\left(\frac{z_n}{|z_n|}, e^{i\theta}\right) - d_{\mathbb{H}}\left(\frac{z_n}{|z_n|}, v e^{i\theta}\right)\right] = 0. \tag{4.14}$$

If the sequence $\{z_n/|z_n|\}$ has any accumulation point $z_0 \in \mathbb{H}$, then we directly get $v = 1$ due to (4.14). If not, up to a further subsequence, we may assume that $\{z_n/|z_n|\}$ converges to i (if it converges to $-i$, the proof is similar). In such a case, elementary computations with the help of formula (2.4) yield

$$\lim_{z\to i}\left[d_{\mathbb{H}}(e^{i\theta}, z) - d_{\mathbb{H}}(v e^{i\theta}, z)\right] = \frac{1}{2}\log\frac{(2-2\sin\theta)v}{v^2 - 2v\sin\theta + 1}. \tag{4.15}$$

Since $\{z_n/|z_n|\}$ converges to i, by (4.14) we have

$$\lim_{z\to i}\left[d_{\mathbb{H}}(e^{i\theta}, z) - d_{\mathbb{H}}(v e^{i\theta}, z)\right] = \lim_{n\to+\infty}\left[d_{\mathbb{H}}\left(e^{i\theta}, \frac{z_n}{|z_n|}\right) - d_{\mathbb{H}}\left(v e^{i\theta}, \frac{z_n}{|z_n|}\right)\right] = 0. \tag{4.16}$$

As a consequence of relations (4.15) and (4.16), we once again get $v = 1$. So, the only accumulation point of $\{v_n\}$ is 1 and thus (4.12) is proved. Hence (4.11) holds as well. With (4.11) proved, we use the triangle inequality to obtain

$$d_{\mathbb{H}}(|z_n|e^{i\theta}, \pi_n^1) \le d_{\mathbb{H}}(|z_n|e^{i\theta}, |\pi_n^1|e^{i\theta}) + d_{\mathbb{H}}(|\pi_n^1|e^{i\theta}, \pi_n^1). \tag{4.17}$$

But according to (4.6) and (4.11), the right-hand side term in (4.17) tends to zero, as $n \to +\infty$. Hence,

$$\lim_{n\to+\infty} d_{\mathbb{H}}(|z_n|e^{i\theta}, \pi_n^1) = 0. \tag{4.18}$$

All of the above arguments can be applied verbatim to the curve γ_2 and the projections $\{\pi_n^2\}$ in order to show that

$$\lim_{n\to+\infty} d_{\mathbb{H}}(|z_n|e^{i\theta}, \pi_n^2) = 0. \tag{4.19}$$

Finally, the limits (4.18) and (4.19), along with a simple use of the triangle inequality, yield

$$\lim_{n\to+\infty} d_{\mathbb{H}}(\pi_n^1, \pi_n^2) = 0,$$

as required. ■

Using Theorem 4.2, we can correlate the projections of a sequence onto curves that land with different angles.

Corollary 4.3. *Let $\gamma_j\colon [0,+\infty) \to \mathbb{H}$, $j = 1,2$, be two curves landing at ∞, with*

$$\lim_{t\to+\infty} \arg(\gamma_j(t)) = \theta_j,$$

for some $\theta_j \in (-\pi/2, \pi/2)$ and $j = 1,2$. Suppose that $\{z_n\} \subset \mathbb{H}$ is a sequence converging to ∞ and let $\pi_{\gamma_j}(z_n)$ be a projection of z_n in γ_j, for $j = 1,2$ and $n \in \mathbb{N}$. Then,

$$\lim_{n\to+\infty} d_{\mathbb{H}}(\pi_{\gamma_1}(z_n), \pi_{\gamma_2}(z_n)) = d_{\mathbb{H}}(e^{i\theta_1}, e^{i\theta_2}).$$

Proof. Consider the straight half-lines $\eta_1\colon [1,+\infty) \to \mathbb{H}$ and $\eta_2\colon [1,+\infty) \to \mathbb{H}$, with

$$\eta_1(t) = te^{i\theta_1} \quad \text{and} \quad \eta_2(t) = te^{i\theta_2}, \quad \text{for } t \in [1,+\infty).$$

Note that the projections of z_n onto η_1 and η_2 are $|z_n|e^{i\theta_1}$ and $|z_n|e^{i\theta_2}$, respectively, for all n large. Moreover, from Lemma 2.4(2), for all $n \in \mathbb{N}$ we have

$$d_{\mathbb{H}}(|z_n|e^{i\theta_1}, |z_n|e^{i\theta_2}) = d_{\mathbb{H}}(e^{i\theta_1}, e^{i\theta_2}).$$

Then, applying Theorem 4.2 to the pairs of curves γ_1, η_1 and γ_2, η_2, we have that

$$\lim_{n\to+\infty} d_{\mathbb{H}}(\pi_{\gamma_1}(z_n), |z_n|e^{i\theta_1}) = \lim_{n\to+\infty} d_{\mathbb{H}}(\pi_{\gamma_2}(z_n), |z_n|e^{i\theta_2}) = 0. \tag{4.20}$$

Using the triangle inequality, we obtain

$$\begin{aligned} d_{\mathbb{H}}(|z_n|e^{i\theta_2}, \pi_{\gamma_1}(z_n)) &\geq d_{\mathbb{H}}(|z_n|e^{i\theta_2}, |z_n|e^{i\theta_1}) - d_{\mathbb{H}}(|z_n|e^{i\theta_1}, \pi_{\gamma_1}(z_n)) \\ &= d_{\mathbb{H}}(e^{i\theta_2}, e^{i\theta_1}) - d_{\mathbb{H}}(|z_n|e^{i\theta_1}, \pi_{\gamma_1}(z_n)), \end{aligned}$$

and similarly,

$$d_{\mathbb{H}}(|z_n|e^{i\theta_2}, \pi_{\gamma_1}(z_n)) \leq d_{\mathbb{H}}(e^{i\theta_2}, e^{i\theta_1}) + d_{\mathbb{H}}(|z_n|e^{i\theta_1}, \pi_{\gamma_1}(z_n)).$$

So, (4.20) implies that

$$\lim_{n\to+\infty} d_{\mathbb{H}}(|z_n|e^{i\theta_2}, \pi_{\gamma_1}(z_n)) = d_{\mathbb{H}}(e^{i\theta_1}, e^{i\theta_2}). \tag{4.21}$$

Another use of the triangle inequality yields

$$d_{\mathbb{H}}(\pi_{\gamma_1}(z_n), \pi_{\gamma_2}(z_n)) \geq d_{\mathbb{H}}(\pi_{\gamma_1}(z_n), |z_n|e^{i\theta_2}) - d_{\mathbb{H}}(|z_n|e^{i\theta_2}, \pi_{\gamma_2}(z_n))$$

and

$$d_{\mathbb{H}}(\pi_{\gamma_1}(z_n), \pi_{\gamma_2}(z_n)) \leq d_{\mathbb{H}}(\pi_{\gamma_1}(z_n), |z_n|e^{i\theta_2}) + d_{\mathbb{H}}(|z_n|e^{i\theta_2}, \pi_{\gamma_2}(z_n)).$$

Using (4.20) and (4.21), we get

$$\lim_{n\to\infty} d_{\mathbb{H}}(\pi_{\gamma_1}(z_n), \pi_{\gamma_2}(z_n)) = d_{\mathbb{H}}(e^{i\theta_1}, e^{i\theta_2}),$$

and the proof is complete. ■

5. Proofs of the monotonicity properties

This section contains the proofs of our main result, Theorem 1.1, and its corollary, Corollary 1.2. We start with a simple lemma that uses the angular derivative.

Lemma 5.1. *Let $f\colon \mathbb{H} \to \mathbb{H}$ be a hyperbolic map with Denjoy–Wolff point ∞. For any $z \in \mathbb{H}$ and $w \in \mathbb{C}$, the sequence $\{|f^n(z) - w|\}$ is eventually strictly increasing.*

Proof. Because f is hyperbolic, the angular derivative condition (2.7) implies that

$$f'(\infty) = \angle \lim_{z\to\infty} \frac{f(z)}{z} > 1.$$

Now, fix $z \in \mathbb{H}$ and $w \in \mathbb{C}$. Theorem 2.3(2) implies that the sequence $\{f^n(z)\}$ tends to infinity through a sector $\{z \in \mathbb{C} : |\arg z| < \phi\}$, for some $\phi \in (-\pi/2, \pi/2)$. So, the above limit can be rewritten as

$$\lim_{n\to+\infty} \left|\frac{f(f^n(z)) - w}{f^n(z) - w}\right| = \lim_{n\to+\infty} \left|\frac{f(f^n(z))}{f^n(z)}\right| = \angle \lim_{z\to\infty} \left|\frac{f(z)}{z}\right| > 1,$$

which completes the proof. ■

Using Lemma 5.1, we can prove that projections of hyperbolic orbits onto geodesics are eventually strictly increasing, as stated in the following corollary. Note that this result is merely a special case of our main theorem. We include a proof, however, to highlight the fact that the monotonicity of the orthogonal speed of hyperbolic maps does not require the heavy machinery of Section 4.

Corollary 5.2. *Let $f\colon \mathbb{H} \to \mathbb{H}$ be a hyperbolic map with Denjoy–Wolff point ∞. For any $w \in \mathbb{H}$, consider the hyperbolic geodesic γ of $\mathbb{H}$ emanating from w and landing at ∞. If $z \in \mathbb{H}$ and $\{\pi_\gamma(f^n(z))\}$ denotes the sequence of projections of $f^n(z)$ onto γ, then the sequence $\{d_{\mathbb{H}}(w, \pi_\gamma(f^n(z)))\}$ is eventually strictly increasing.*

Proof. Fix $z, w \in \mathbb{H}$. Note that the geodesic $\gamma\colon [0, +\infty) \to \mathbb{H}$ joining w and ∞ is the horizontal half-line $\gamma(t) = w + t$. Lemma 5.1 is applicable and yields that the sequence $\{|f^n(z) - i \operatorname{Im} w|\}$ is eventually strictly increasing. Notice that the projection of $f^n(z)$ onto γ is the point

$$\pi_\gamma(f^n(z)) = |f^n(z) - i \operatorname{Im} w| + i \operatorname{Im} w,$$

for all large $n \in \mathbb{N}$. So, using a vertical translation and the invariance of the hyperbolic distance under Möbius maps, we obtain

$$d_{\mathbb{H}}(w, \pi_\gamma(f^n(z))) = d_{\mathbb{H}}(w, |f^n(z) - i \operatorname{Im} w| + i \operatorname{Im} w) = d_{\mathbb{H}}(\operatorname{Re} w, |f^n(z) - i \operatorname{Im} w|).$$

So the result follows from the monotonicity of $\{|f^n(z) - i \operatorname{Im} w|\}$ and Lemma 2.4(1). ∎

Using Lemma 5.1 and the main result of Section 4, we can now prove Theorem 1.1. We are going to prove its equivalent formulation in $\mathbb{H}$, stated in Theorem 2.1 and restated below for the convenience of the reader.

Theorem. *Let $f\colon \mathbb{H} \to \mathbb{H}$ be a hyperbolic map with Denjoy–Wolff point ∞, and let $\gamma\colon [0, +\infty) \to \mathbb{H}$ be a curve landing at ∞, satisfying*

$$\lim_{t \to +\infty} \arg(\gamma(t)) = \theta \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

Fix $z \in \mathbb{D}$ and let $\pi_\gamma(f^n(z))$ be a projection of $f^n(z)$ onto γ, for $n \in \mathbb{N}$. Then, for any $w \in \mathbb{H}$, the sequence $\{d_{\mathbb{H}}(w, \pi_\gamma(f^n(z)))\}$ is eventually strictly increasing.

Proof. Fix $z, w \in \mathbb{H}$, and consider the straight half-line $\eta\colon [0, +\infty) \to \mathbb{H}$ given by $\eta(t) = w + te^{i\theta}$. It is easy to see that the vertical translation $z \mapsto z - i\,(\operatorname{Im} w - \tan\theta \operatorname{Re} w)$ maps $\eta([0, +\infty))$ to the half-line

$$\left\{te^{i\theta} : t \geq \frac{\operatorname{Re} w}{\cos\theta}\right\}.$$

Recall that by Lemma 2.4 the projection of a point $z_0 \in \mathbb{H}$ onto the half-line $\{te^{i\theta} : t > 0\}$ is $|z_0|e^{i\theta}$. So, the projection of $f^n(z)$ onto η is the point

$$\pi_\eta(f^n(z)) = |f^n(z) - i\,(\operatorname{Im} w - \tan\theta \operatorname{Re} w)|e^{i\theta} + i\,(\operatorname{Im} w - \tan\theta \operatorname{Re} w). \tag{5.1}$$

To simplify our notation, we write

$$\pi_\eta(n) = \pi_\eta(f^n(z)) \quad \text{and} \quad \pi_\gamma(n) = \pi_\gamma(f^n(z)),$$

for all $n \in \mathbb{N}$. Our goal is to show that $\{d_{\mathbb{H}}(w, \pi_\gamma(n)\}$ is eventually strictly increasing. Observe that is suffices to show that

$$\liminf_{n \to +\infty} \left(d_{\mathbb{H}}(w, \pi_\gamma(n+1)) - d_{\mathbb{H}}(w, \pi_\gamma(n))\right) > 0. \tag{5.2}$$

By the triangle inequality, we get that

$$\begin{aligned}
&d_{\mathbb{H}}(w, \pi_\gamma(n+1)) - d_{\mathbb{H}}(w, \pi_\gamma(n)) \\
&\geq d_{\mathbb{H}}(w, \pi_\eta(n+1)) - d_{\mathbb{H}}(\pi_\eta(n+1), \pi_\gamma(n+1)) - d_{\mathbb{H}}(w, \pi_\eta(n)) - d_{\mathbb{H}}(\pi_\eta(n), \pi_\gamma(n)) \\
&= d_{\mathbb{H}}(w, \pi_\eta(n+1)) - d_{\mathbb{H}}(w, \pi_\eta(n)) - d_{\mathbb{H}}(\pi_\eta(n+1), \pi_\gamma(n+1)) - d_{\mathbb{H}}(\pi_\eta(n), \pi_\gamma(n)).
\end{aligned}$$

Because

$$\lim_{t\to+\infty} \arg(\eta(t)) = \lim_{t\to+\infty} \arg(\gamma(t)) = \theta,$$

Theorem 4.2 is applicable and yields that

$$\lim_{n\to+\infty} d_{\mathbb{H}}(\pi_\eta(n), \pi_\gamma(n)) = \lim_{n\to+\infty} d_{\mathbb{H}}(\pi_\eta(n+1), \pi_\gamma(n+1)) = 0.$$

So, in order to prove our goal, (5.2), it suffices to show that

$$\liminf_{n\to+\infty} \left(d_{\mathbb{H}}(w, \pi_\eta(n+1)) - d_{\mathbb{H}}(w, \pi_\eta(n))\right) > 0. \tag{5.3}$$

Write

$$f^n(z) - i\,(\operatorname{Im} w - \tan\theta \operatorname{Re} w) = r_n e^{i\theta_n}, \tag{5.4}$$

and

$$w_0 = w - i\,(\operatorname{Im} w - \tan\theta \operatorname{Re} w) = \operatorname{Re} w + i \tan\theta \operatorname{Re} w. \tag{5.5}$$

See Figure 6 for the construction. Using the translation $z \mapsto z - i\,(\operatorname{Im} w - \tan\theta \operatorname{Re} w)$, the invariance of the hyperbolic distance under Möbius maps and (5.1), we obtain

$$d_{\mathbb{H}}(w, \pi_\eta(f^n(z))) = d_{\mathbb{H}}(w_0, r_n e^{i\theta}).$$

So, (5.3) can be rewritten as

$$\liminf_{n\to+\infty} \left(d_{\mathbb{H}}(w_0, r_{n+1}e^{i\theta}) - d_{\mathbb{H}}(w_0, r_n e^{i\theta})\right) > 0. \tag{5.6}$$

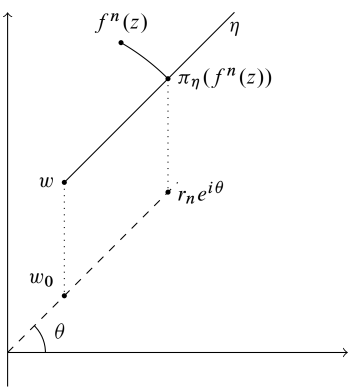

Figure 6. The framework for the proof of Theorem 2.1.

Now, recalling (5.4), Lemma 5.1 implies that the sequence

$$\{r_n\} = \{|f^n(z) - i\,(\operatorname{Im} w - \tan\theta \operatorname{Re} w)|\}$$

is eventually strictly increasing, and $r_n \to +\infty$ as $n \to +\infty$.

Observe that due to (5.5), the point w_0 lies in the half-line $\{te^{i\theta} : t > 0\}$. So, $w_0 = |w_0|e^{i\theta}$. This means that we can apply Lemma 2.7 to the sequence $\{r_n\}$ to obtain that

$$\lim_{n\to+\infty}\big(d_{\mathbb{H}}(w_0, r_n e^{i\theta}) - d_{\mathbb{H}}(|w_0|, r_n)\big) = -\log(\cos\theta) \in [0, +\infty). \tag{5.7}$$

We also have that

$$\begin{aligned} d_{\mathbb{H}}(w_0, r_{n+1}e^{i\theta}) - d_{\mathbb{H}}(w_0, r_n e^{i\theta}) &\\ = d_{\mathbb{H}}(w_0, r_{n+1}e^{i\theta}) - d_{\mathbb{H}}(|w_0|, r_{n+1}) - d_{\mathbb{H}}(w_0, r_n e^{i\theta}) + d_{\mathbb{H}}(|w_0|, r_n)&\\ + d_{\mathbb{H}}(|w_0|, r_{n+1}) - d_{\mathbb{H}}(|w_0|, r_n).& \end{aligned}$$

Therefore, applying (5.7) to the above equation, we get

$$\liminf_{n\to+\infty}\big(d_{\mathbb{H}}(w_0, r_{n+1}e^{i\theta}) - d_{\mathbb{H}}(w_0, r_n e^{i\theta})\big) = \liminf_{n\to+\infty}\big(d_{\mathbb{H}}(|w_0|, r_{n+1}) - d_{\mathbb{H}}(|w_0|, r_n)\big).$$

So, (5.6) is equivalent to

$$\liminf_{n\to+\infty}\big(d_{\mathbb{H}}(|w_0|, r_{n+1}) - d_{\mathbb{H}}(|w_0|, r_n)\big) > 0. \tag{5.8}$$

Summarizing the work in this proof so far, we have that in order to show the desired result, it suffices to establish (5.8).

Because $\{r_n\}$ is eventually strictly increasing and the points $|w_0|$, r_n lie on a geodesic of $\mathbb{H}$ (namely, the real axis), we have that

$$d_{\mathbb{H}}(|w_0|, r_{n+1}) - d_{\mathbb{H}}(|w_0|, r_n) = d_{\mathbb{H}}(r_{n+1}, r_n), \quad \text{for all } n \text{ large.} \tag{5.9}$$

Moreover, since f is hyperbolic, Theorem 2.3 implies that there exist $d_0 > 0$ and $\phi \in (-\pi/2, \pi/2)$ so that

$$\lim_{n\to+\infty} d_{\mathbb{H}}(r_{n+1}e^{i\theta_{n+1}}, r_n e^{i\theta_n}) = \lim_{n\to+\infty} d_{\mathbb{H}}(f^{n+1}(z), f^n(z)) = d_0,$$

and

$$\lim_{n\to+\infty}\theta_n = \lim_{n\to+\infty}\arg(r_n e^{i\theta_n}) = \lim_{n\to+\infty}\arg(f^n(z)) = \phi.$$

These last two limits imply that Lemma 2.6 is applicable to the sequence $\{r_n e^{i\theta}\}$ and yields a constant $d > 0$ so that

$$\lim_{n\to+\infty} d_{\mathbb{H}}(r_{n+1}, r_n) = d. \tag{5.10}$$

Combining (5.9) with (5.10) yields (5.8), as required. ∎

With our main result proved, we move on to its corollary, Corollary 1.2. As usual, we prove its right half-plane version, Corollary 2.2, restated below.

Corollary. *Let $f\colon \mathbb{H} \to \mathbb{H}$ be a hyperbolic map with Denjoy–Wolff point ∞. For any $z, w \in \mathbb{H}$, the sequence $\{d_{\mathbb{H}}(w, f^n(z))\}$ is eventually strictly increasing.*

Proof. Fix $z, w \in \mathbb{H}$. Since f is hyperbolic, Theorem 2.3(2) implies that there exists $\phi \in (-\pi/2, \pi/2)$, depending only on f and z, so that

$$\lim_{n\to+\infty} \arg(f^n(z)) = \phi. \tag{5.11}$$

Let $\ell_n = [f^n(z), f^{n+1}(z)]$ be the Euclidean line segment joining $f^n(z)$ and $f^{n+1}(z)$, for $n \in \mathbb{N}$. We obviously have that $\ell_n \subset \mathbb{H}$, for all $n \in \mathbb{N}$. Let $\gamma\colon [0, +\infty) \to \mathbb{H}$ be a curve with trace $\gamma([0, +\infty)) = \bigcup_{n\in\mathbb{N}} \ell_n$. Since the sequence of iterates $\{f^n(z)\}$ converges to ∞, we can parametrise γ so that $\lim_{t\to+\infty} \gamma(t) = \infty$. Moreover, (5.11) implies that $\lim_{t\to+\infty} \arg(\gamma(t)) = \phi$. Therefore, γ satisfies the assumptions of Theorem 2.1 and yields that the sequence $\{d_{\mathbb{H}}(w, \pi_\gamma(f^n(z)))\}$ is eventually strictly increasing. But, by construction $f^n(z) \in \gamma([0, +\infty))$, for all $n \in \mathbb{N}$, meaning that $\pi_\gamma(f^n(z)) = f^n(z)$, and the proof is complete. ∎

6. Concluding remarks

Recall that in Example 3.2 we showed that Theorem 1.1 does not hold if one replaces the hyperbolic map with a parabolic. However, we can see that in this particular counter-example, the sequences $\{d_{\mathbb{H}}(1, \pi_\delta(f^n(1)))\}$ and $\{d_{\mathbb{H}}(1, \pi_\gamma(g^n(1)))\}$ we constructed for the zero and positive hyperbolic step, respectively, are eventually increasing (but not strictly). It would be interesting to examine whether this behaviour is the norm for parabolic maps, or merely a product of our construction. A first step would be to consider projections of parabolic orbits onto hyperbolic geodesics. In other words, we ask if the orthogonal speed of parabolic maps is eventually increasing. We now present simple arguments that provide an affirmative answer to this question for parabolic maps of positive hyperbolic step.

It is known (see Theorem 1 in [19]) that for a parabolic map of positive hyperbolic step, we have

$$\lim_{n\to+\infty} \frac{\operatorname{Im}(f^{n+1}(1)) - \operatorname{Im}(f^n(1))}{\operatorname{Re}(f^n(1))} = b \in \mathbb{R}^*. \tag{6.1}$$

Moreover, the sequence $\{\operatorname{Im}(f^n(1))\}$ either converges to $+\infty$, or converges to $-\infty$ (see Remark 2.5 in [14]), whereas the sequence $\{\operatorname{Re}(f^n(1))\}$ is non-decreasing by Julia's lemma (see Section 2.1 of [1]). Combining these two facts with (6.1), we obtain that the sequence $\{|\operatorname{Im}(f^n(1))|\}$ is eventually strictly increasing, and so the same holds for $\{|f^n(1)|\}$. Hence, if $\gamma\colon [0, +\infty) \to \mathbb{H}$ is the hyperbolic geodesic with $\gamma(t) = 1 + t$, then $\pi_\gamma(f^n(1)) = |f^n(1)|$, meaning that $\{d_{\mathbb{H}}(1, \pi_\gamma(f^n(1)))\}$ is eventually strictly increasing. Similar arguments show that the sequence $\{d_{\mathbb{H}}(z, \pi_{\gamma_z}(f^n(z)))\}$ is eventually strictly increasing for any $z \in \mathbb{H}$, where $\gamma_z\colon [0, +\infty) \to \mathbb{H}$ is the hyperbolic geodesic with $\gamma_z(t) = z + t$.

Similarly, one could check that in Example 3.1, the function $t \mapsto d_{\mathbb{H}}(1, \pi_\delta(\phi_t(1)))$ is also eventually increasing. Therefore, it might be possible to show that Theorem 1.1 holds with a hyperbolic semigroup in place of a the hyperbolic map f, and with the conclusion "eventually increasing" instead of "eventually strictly increasing".

To discuss a final question that arises from our work, let us first formally define the slope of a curve in $\mathbb{D}$.

Let $\gamma: [0, +\infty) \to \mathbb{D}$ be a curve landing at a boundary point $\zeta \in \partial\mathbb{D}$. The *slope of* γ is the set Slope(γ) consisting of all $\theta \in [-\pi/2, \pi/2]$ for which there exists a sequence $\{t_n\} \subset [0, +\infty)$, converging to $+\infty$, such that $\arg(1 - \bar{\zeta}\gamma(t_n)) \to \theta$.

Consider a hyperbolic map $f: \mathbb{D} \to \mathbb{D}$. With the above terminology, Theorem 1.1 yields the eventual monotonicity of the sequence $\{d_{\mathbb{D}}(w, \pi_\gamma(f^n(z)))\}$ under the assumption that Slope(γ) $= \{\theta\}$, for some $\theta \in (-\pi/2, \pi/2)$. Also, Example 3.4 shows that the monotonicity of projections described in Theorem 1.1 fails if we assume that Slope(γ) is either $\{-\pi/2\}$ or $\{\pi/2\}$.

However, Lemma 4.1 implies that if there exists $\theta \in (0, \pi/2)$ so that Slope(γ) $\subseteq [-\theta, \theta]$, then the projections $\pi_\gamma(f^n(z))$ will still converge to τ. Therefore, it makes sense to study the monotonicity of $\{d_{\mathbb{D}}(w, \pi_\gamma(f^n(z)))\}$ in this broader setting. The key points of our analysis in Section 5 revolved around the fact that each orbit $\{f^n(z)\}$ tends to τ with a fixed slope, and that hyperbolic maps have positive hyperbolic step, described in Theorem 2.3. These remain intact even in the setting of curves with "wild" slope. Our main technical result, however, Theorem 4.2, may require modifications in order to be used to substitute the projections on such a γ with projections on a simpler curve. Finally, preliminary examples also show that the monotonicity of projections in this case, if it indeed holds, will no longer be strict.

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